

**Opera Mathematica,
Tomus Primus —
Commentaria in Euclidis
Elementa Geometrica, in
Sphaerica Theodosii, ac
Sinuum, Tangentium &
Secantium doctrinam**

*[Mathematical Works, Volume One —
Commentaries on Euclid's Elements and
Theodosius's Spherics, with the doctrine of Sines,
Tangents, and Secants]*

(1611)

by Christoph Clavius (1538–1612), S.J.

Translation by Scott Weisman



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Translator Version 2

Upright reason dictates that the recipients of the good, of whatever type and level of beneficence it may be, must show gratitude and blessing to the beneficent person in every way possible, commensurate with the value of the beneficence. And one who has benefited all the people of the world — for example, one who invented a new instrument for the good of the world, or a good book — it is fitting for every discerning person, out of the obligation of love of fellow beings, to at least purchase it, so that the man will profit and his heart will be encouraged thereby to invent yet more good instruments in the world. And similarly, all other wise-hearted people will likewise strive and exert themselves to invent good things and instruments needed for the repair of the world and its perfection.

And therefore, whoever says: "What do I need this new instrument for?" — he does not act well towards the world. For if not for the man who invented it, where would you be? And what would your city do? And more than this, if the man did not exert himself, what would the world do?

— R. Pinchas Eliyahu Hurwitz, *Sefer HaBris HaShalem* (1797)

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A note on the work

Tomus I of Clavius's collected mathematical works gathers his geometry and trigonometry. Its core is his celebrated commentary on Euclid's *Elements* (Books I–XV, with a sixteenth added) — the standard Renaissance Euclid, dense with labelled construction diagrams and scholia. It is followed by his commentary on Theodosius's *Sphaerica* (the geometry of the sphere) and by a treatise on the doctrine of sines, tangents, and secants with the theory of plane and spherical triangles, accompanied by extensive numeric trigonometric tables. Together they form the geometric and trigonometric foundation on which the astronomical volumes build.

A note on the translation process

This translation was produced with the [translation-pipeline](#). See the linked repository for details on the process.

This text is the product of a highly refined automated translation process. It has undergone only cursory human review and should not be treated as an authoritative or scholarly rendering. It is offered solely as a reference and reading aid — a way to make an otherwise inaccessible work approachable — and any passage of real consequence should be checked against the original source.

Acknowledgements

Translated with the assistance of Claude. The translator thanks the [Anthropic](#) team for making this work possible.

[The two leaves following the cover are blank flyleaves bearing only handwriting and library markings; no printed text of the work appears yet.]

[Margin: handwritten shelfmark, upper left of the front flyleaf: "2. Math. m. 10" over "71" (or "/1") — a Bayerische Staatsbibliothek pressmark.]

[Margin: handwritten author name, centre of the front flyleaf: "Clavius".]

[On the following flyleaf, a printed library barcode label:]

[Approval stamp: "<36616685880014>" (repeated on a second line) / "Bayer. Staatsbibliothek" (Bavarian State Library)]

[Margin: handwritten "33" in pencil at the lower left of the label, and a red-pencil mark to its upper right.]

Front matter — engraved title page

Front flyleaf (recto)

Blank flyleaf. At the upper left, in brown ink, the Bayerische Staatsbibliothek shelfmark handwritten: "2 Math. m. 10 – 1" [handwritten]. A few faint pencil marks and offset show-through elsewhere; no printed text.

Engraved title page

Diagram. An engraved architectural title page (frontispiece) in the form of a monumental portal, signed by the engraver Johann Leybold. A central oval cartouche carries the title lettering. The whole is framed by an ornamented architrave above and a plinth below, with allegorical figures at the sides and small narrative roundels and panels at the base. Handwritten ownership annotations in brown ink are overlaid across the middle of the title.

Transcription of the engraved and printed lettering, top to bottom:

Upper pediment / entablature. Two winged putti sit at the outer corners: the one at left holds up a radiant sun or sundial; the one at right holds a pair of crescent moons. Between them, in the broken pediment, are set two standing figures of the patron saints of Bamberg, each labelled on the ledge beneath:

- **S. HEINRIC.** [St. Henry — Emperor Henry II, patron of Bamberg] — mitred figure at left of centre.

- **S. KVNIGVND.** [St. Kunigunde — Empress Cunigunde, his consort] — mitred/crowned figure at right of centre.

Between the two saints, a central shield bears the arms of the See of Bamberg.

Side cartouches flanking the title.

- Left roundel: **IHS** [the Jesuit Christogram], with a small cross and three nails beneath.
- Right roundel: **MAR** [Maria — the monogram of the Virgin Mary].

Central title cartouche (large engraved capitals):

CHRISTOPHORI CLAVII BAMBERGENSIS. E Societate IESV OPERA
MATHE- MATICA V Tomis distributa. Ab auctore nunc denuo correcta, et
plurimis locis aucta. Ad Reverendiss. et Illustriss. Principem ac
Dominum, D. IOANNEM GODEFRIDVM EPISCOpum Bambergensem &c.
MOGVNTIAE, Sumptibus Antonij Hierat Excudebat Reinhard Eltz Cum
gratia et privilegio Sac. Cæs. Maiest. Anno M. DCXII.

Translation of the title cartouche:

The Mathematical Works of Christoph Clavius [Christophorus Clavius] of Bamberg, of the Society of Jesus, distributed in five volumes. Now once more corrected by the author, and enlarged in very many places. To the most reverend and most illustrious Prince and Lord, Lord Johann Gottfried [Johann Gottfried von Aschhausen], Bishop of Bamberg, etc. At Mainz [Moguntiae], at the expense of Antonius Hierat; printed by Reinhard Eltz. With the grace and privilege of His Sacred Imperial Majesty. In the year 1612.

Handwritten ownership annotations overlaid across the middle of the title (brown ink): to the left of the word "MATICA," a signature reading "Almerij" or "Alberti" [?]; to the right, a signature "Scheiner" [Christoph Scheiner?] with a flourish [?]; a further line of cursive beneath ("Ab auctore..." region) and two more short inscriptions ("Monac[ensis]" [?] and "Scheiner" [?]) lower down. These are later hands, not part of the printed sheet.

Engraver's signature on the socle beneath the side figures:

- Left: **Ioan. Leypolt** [Johann Leypolt, the engraver] ... | right: **sculp.** [sculpsit — "engraved (this)"].

Allegorical side figures standing on the plinth:

- Left: a woman drawing with a compass/dividers on a tablet, labelled on the pedestal band **GEOMETRIA** [Geometry].
- Right: a woman holding an armillary sphere, labelled on the pedestal band **ASTRONOMIA** [Astronomy].

Base register (three roundels and flanking panels around a central portrait):

- Central medallion: a bust-length portrait of an elderly bearded scholar — Clavius himself — seated with drawing compasses over instruments, an armillary sphere and geometrical solids about him. Faint inscriptions "IESV" (above left), "Christophorus" (above right), and beneath, partly legible, "CLAVIVS" [?] and other lettering [?].
- Small circular and rectangular scenes at the four corners depict biblical/astronomical episodes (figures observing the heavens, a battle scene at right, an ark/vessel at lower right) [subjects only partly legible].

Scroll beneath the portrait (printed motto):

Dedit mihi Deus ut sciam anni cursus et stellarum dispositiones. Sap. 7.

Translation: "God gave me to know the courses of the year and the dispositions of the stars. Wisdom 7." [Wisdom 7:19]

Title-page verso / flyleaf

Blank leaf (verso of the engraved title, showing offset of the plate through the paper). Near the centre, an inked oval library stamp:

BIBLIOTHECA / REGIA / MONACENSIS.

[Stamp: "Royal Library of Munich" — the Bayerische Staatsbibliothek's earlier institutional stamp.]

Below and to either side of the stamp, two handwritten cursive inscriptions in brown ink [illegible — later ownership hands, "...Scheiner" [?] at right] and a few faint marks. No printed text.

To the Most Reverend and Most Illustrious
Prince and Lord, Lord Johann Gottfried [Johann
Gottfried von Aschhausen, Bishop of Bamberg],
Bishop of Bamberg and Provost of the
Metropolitan Church of Würzburg
[Herbipolensis], etc., his most gracious Lord.

Christoph Clavius [Christoph Clavius] of Bamberg, of the Society of Jesus.

Since it has pleased almighty God, Most Reverend and Most Illustrious Prince, of his singular favor to grant that I should live to reach an age which we see not very many attain in these times — whether through the decay of an aging world (so that, just as the primal force and vigor of all things, so too that of Nature itself gradually grows faint), or rather, as I am more inclined to believe, by the sole will and dispensation of God — this above all has been most welcome to me: that whereas it commonly befalls other mortals that, as they sail this stormy sea, all the calamities and miseries of life flow together into its final stretch as into a kind of bilge, I, by contrast, find the very opposite to be true — that no part of my life has been heaped with greater joy and delight than this my decrepit old age. Not that I have been blessed beyond the common lot of humanity, or ever set apart and exempt from those evils that beset the rest of one's life, and above all this bloodless and worn-out age I now live through. But whatever it is that we all suffer alike, I now bear it so easily that I seem to have lost all sense of pain entirely.

For whereas I had once desired nothing more than to see the Church of Bamberg — in which I had been born and raised — as secure and flourishing as it could possibly be, I would often recall to mind its first origins and cradle-days from the auspicious founding by Saint Otto [Otto of Bamberg], and, passing through the whole memory of former ages, I traced its growth and maturity down to these present years. And although I found that it had always advanced, keeping pace with the age, toward immortal glory and the dignity of its own name — so that nothing seemed less to be feared than that it should ever fall away from the excellent institutions of its forefathers — yet the wickedness of these times brought it about that, while the neighboring peoples round about were being infected with the deadly plague of heresy, the Church of Bamberg indeed most

steadfastly maintained its religion; and yet all good men greatly feared that, just as we see that under a diseased sky diseases can scarcely be guarded against, so, if some pestilent breath of that license which the heretics devise should waft in from somewhere, then, even though the people might most firmly turn away from the [heretical] doctrine itself, they might nonetheless — innocent though they are — grow accustomed to the corruption of morals and of loosened discipline.

For these reasons it came about that I, who held my homeland dearer than my own life, when I learned that the Most Reverend and Most Illustrious Prince Johann Philipp [Johann Philipp von Gebsattel], your predecessor of blessed memory, had been called by God out of this life, thought it altogether necessary to commend to God, with all the zeal I could, the widowhood of the Church now bereft of its shepherd — praying that in these most difficult times it might obtain the best possible leader, one endowed with the greatest courage and virtue for driving off the onrushing diseases both of heresy and of every vice. While this concern was weighing on me as gravely as it deserved, behold — word of your most happy installation was brought to me, not through vague rumors at first, as usually happens, but with the common applause of the whole Christian world. At this news I suddenly began to exult with such joy that, casting off the burden of age that most oppressed me, I seemed to myself to have returned, like Aeson [Aeson], to the first bloom of youth. And so, raising my hands to heaven, I gave thanks to God, that He had not willed me to meet my final day before I had obtained so great a good — so that now, with the safety of my homeland brought to security, nearly the same thing befalls me as befell that old man of the Gospel who, having beheld the Savior of the world, joyfully awaited his last day.

I would be lying if I spoke anything otherwise than I feel; nor do I fear that anyone should ever charge me with that fault. For, besides the fact that I am now of an age at which I ought to appear unwilling to affirm anything lightly, the opinion of Your Virtue is so illustrious and so attested by every rank and age that what I might seem to have said out of private affection is shared with me by all, bad and good alike. For there is, believe me, there is a kind of virtue that towers so high above all the malice of detractors that — just as the loftiest mountains rise with their whole summit above the clouds and never feel the storms of wind and rain — so this virtue, forever serene and untroubled, rises as far as it possibly can above all the storm-clouds of slander. In this way it came about that, when news first spread that you had been raised to the episcopal dignity, to the very great good of the Christian commonwealth, no one was found [recte: inventus] either so estranged from the truth or so hostile to the public good as not to confess that

this honor had been most deservedly conferred upon you. Indeed, all were of one mind and one voice: that ever since the days of Saint Otto no happier day had dawned upon the Church of Bamberg than the one on which it obtained you as its bishop; that in you alone lay everything; that under your auspices above all the state could be happily governed; and that vows should be offered to God, praying that He might will this benefit of His to endure as long as possible [recte: diuturnum].

In this common joy and rejoicing, no one could rightly deny that some place ought to belong to me as well. For who would take it ill if I openly declare that I have, by God's supreme favor, obtained what I earnestly prayed of almighty God — namely, that He would set at the head of affairs above all the man who could best come to the aid of my struggling homeland? If what is celebrated in the talk of all peoples, I alone do not pass over in silence? If, now made certain of the most happy condition of my homeland, I cannot restrain the force of my joy? And even if I had passed my whole earlier life in silence and obscurity, known to the public by no writings at all, pardon would still have to be granted me if, stirred by that acclamation and applause of the whole world, I could not hold myself back, but sang out something — to mark the celebration and to attest my goodwill — in a swan-like voice tuned by nature rather than by art. But now that I have earned some name among the learned — a slight one, perhaps, yet one I trust will pass down to posterity — I could not fail to leave behind, by some illustrious testimony, as full a proof as I could of my love for my homeland and of my zeal to honor your name.

And to do this the more freely, both the greatness of the matter demands it, and I feel myself spurred on by the example and authority of Basil the Great [Basil of Caesarea]. When Basil had learned about Saint Ambrose [Ambrose of Milan] — that he had been elected Bishop of Milan — he began to love ardently a man he had never before seen, commended to him by the fame of his virtue alone, not so much for his own sake as for the sake of the Christian commonwealth, and to proclaim him, to the great good of that age, as one dropped, so to speak, from heaven:

"We have glorified our God," he says, "who chooses, out of every generation of mortals, those who have pleased Him. He who once raised up a leader for His people from among the shepherds of sheep, and who took Amos from the herd of goats and instructed him by His Spirit and raised him to prophetic dignity — now He has drawn to the pasturing of Christ's flock a man from the royal city, to whom the governance of the whole nation had been entrusted: a man lofty in wisdom,

distinguished in the nobility of his lineage, in the splendor of his life, in his gift of eloquence, and in worldly affairs; a man who, casting away all the wealth of this life and counting it as loss so that he might gain Christ, has taken up the helm of the great and noble ship committed to him — namely, the Church of God — in the faith of Christ. Come, then, O man of God: since it was not from men that you received or were taught the Gospel of Christ, but you yourself..."

Dedicatory Epistle (continued)

[The Basil quotation, begun on the preceding page, concludes:]

...whom the Lord, taking him from the midst of the judges of the earth, has raised to the Apostolic presidency. Fight the good fight without wavering; correct and temper the weakness of the people; and if any part of it has been touched by the taint of Arian madness, renew the ancient footsteps of the fathers, and so on.

You understand, I am sure, most Reverend and most Illustrious Prince, how perfectly all of this applies to you. Yet if, out of that modesty for which you are so highly commended, you feel otherwise than I do, I will not offend your modesty, nor will I ask for anything more beyond the testimony of those who know you more intimately.

Allow me to say somewhat more openly what St. Basil [Basil of Caesarea] urged in his closing words: he wished the ancient morals and religion to be restored, and the Arian heresy then raging to be rooted out of men's minds. Everyone promised themselves that both would come about, as soon as they had learned of your Election. Now we see it happily begun and, with incredible speed, already in great part accomplished. With the flood of vices banished, ancient piety returns as though from exile; integrity of life is restored, and the examples of the old discipline are renewed.

There is one thing more that deserves great praise — one I would gladly have mentioned in the first place, had I not held the consideration of the public good to be more important. Yet even this is of such a kind that it does not seem far separated from the common benefit. Among the deeds gloriously accomplished by St. Otto [Otto of Bamberg], this stands out as exceptional and singular: that at the very outset — when the fortunes of churches are usually too slender to meet great expenditures — he nevertheless erected very ample dwellings for religious men in many places, and endowed them with yearly revenues. Nothing mattered more to you than that this one thing should not be wanting: that there be nothing among his glorious deeds you had not imitated. And so, as soon as you were able

to overcome the difficulties that stood in the way, you willed that the Society of Jesus [Societas Iesu] — which from boyhood you had always embraced with your whole heart — should have a seat at Bamberg, and be enlarged with perpetual revenues. Although you are certain how ample the rewards are that God has appointed for this beneficence of yours — so that you desire nothing at all from us — the Society, I hope, will nonetheless see to it that you may never at any time regret this benefaction.

As for myself: certain Fathers of our Society from the College of Mainz [collegio Moguntino], out of their remarkable zeal and love for the mathematical disciplines, offered me their effort and labor. My works, which had been produced and published by me, were circulating scattered into many parts; they undertook to gather them all, to arrange each in its own order and place, and to set them in type. Judging that there should be no delay, I praised their willingness and urged them to undertake the task with great spirit. Indeed, examining myself, since I understood that I owed everything on your account, and believing that I could do everything, I set my own hand to the work, so that what I intended to dedicate to your Name might be completed as soon as possible. And although now heavily burdened by age, yet with the greatest speed that could be managed, all...

Dedicatory Epistle (conclusion) — p. 11

the books I reviewed from beginning to end, and — with the typographers' errors removed — enriched them by the addition of many things, wherever it seemed most opportune. These, then, most Reverend and most Illustrious Prince, are the fruits of my studies, bound together as in a single bundle. They are in no way equal to your great merits, but they are the richest witnesses to posterity of the devotion and love that I and the whole Society bear you; and by my last will I consecrate and dedicate them to your Name. If I learn that they have been pleasing to you, I shall reckon that I have received a great reward and recompense for my life and labors. As for what remains, most Reverend and most Illustrious Prince, I pray God that — since you were born not for yourself alone, but for the highest good of the whole Christian Commonwealth — He may will that you survive as long as possible, no more for your own sake than for hers. Rome, the Kalends of May, in the year 1611.

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Christoph Clavius, Opera Mathematica, Tomus I

[Title page]

CHRISTOPH CLAVIUS OF BAMBERG, OF THE SOCIETY OF JESUS.

THE MATHEMATICAL WORKS.

VOLUME ONE.

Comprising

Commentaries { On Euclid's Elements of Geometry, / On the Spherics of Theodosius;

The account and canons of Sines, Tangents, and Secants:

The treatment of triangles, both rectilinear and spherical.

MAINZ [MOGVNTIAE],

At the expense of Antonius Hierat [Antonius Hierat], printed by Reinhard Eltz [Reinhardus Eltz].

With the grace and privilege of His Sacred Imperial Majesty.

In the year 1611 [M.DC.XI].

[Verso]

CHRISTOPH CLAVIUS OF BAMBERG, OF THE SOCIETY OF JESUS. PROLEGOMENA TO THE MATHEMATICAL DISCIPLINES.

Why the Mathematical Disciplines Are So Called

The mathematical disciplines, all of which are concerned with quantity, took their name from the Greek word μάθημα [Greek: mathēma, "learning, that which is learned"], or μάθησις [Greek: mathēsis, "learning"], which means a discipline or body of teaching. As to why these arts that deal with quantity alone, among all the rest, obtained the name of "discipline" or "teaching," I find two chief reasons in reputable authors.

The Pythagoreans and Platonists believed that rational souls are contained within a certain fixed and determinate number, and that they migrate from body to body—though the Christian faith plainly teaches this to be false. They held that these arts obtain the name of teaching or discipline because it is chiefly through them that we acquire the recollection and remembrance of that knowledge with which our soul (so runs their error) was endowed before it gave form to the body. Plato tries to prove this readily, by a familiar example, in the dialogue titled *Meno* [Plato, *Meno*], where he brings in Socrates questioning a certain young boy about some geometrical matters concerning the measurement of a square. Although at first the boy answers like a child, yet by advancing step by step he is led to the point of answering what he would in the end have been able to say, had he thoroughly learned geometry over a very long time.

Others hold that these arts claim for themselves the name of "knowledge" and "teaching" above the rest because they alone preserve the method and rational procedure of a science. For they always proceed from certain foreknown principles to the demonstration of conclusions—which is the proper task and function of teaching or discipline, as Aristotle too testifies in the first book of the *Posterior Analytics* (*Posterior Analytics* I). Nor do mathematicians ever assume anything that has not been proved; but whenever they wish to teach something, if anything relevant belongs to what they have already taught, they take it as granted and proved, and then explain only that about which nothing had previously been written. This, we see, other arts and disciplines do not always

observe, since for the most part, in confirmation of the things they wish to show, they bring in as demonstrated things that have not yet been explained.

The Division of the Mathematical Disciplines

The Pythagoreans—whom afterwards nearly all mathematicians and not a few philosophers followed—divided the whole of the mathematical disciplines into four parts: arithmetic, music, geometry, and astronomy. For since all quantity with which they are concerned is either discrete, under which all numbers fall, or continuous, under which all magnitudes are comprehended, and since each can be considered both in itself and in comparison with the other, it seemed fitting to them to establish the four aforesaid faculties, which would carefully examine each kind of quantity under its twofold consideration. Thus arithmetic deals with discrete quantity in itself, investigating and accurately explaining all the properties and affections of numbers. Music treats the same discrete quantity—that is, number compared with another—insofar as it regards the concord and harmony of sounds. Geometry disputes about magnitude, or continuous quantity, likewise in itself, as it exists unmoved. Finally, astronomy considers the same magnitude as it is in motion, such as the heavenly bodies are, inasmuch as they are set in motion by continuous movement. To these four mathematical sciences—of which arithmetic and geometry are called pure, while music and astronomy are called mixed—the others that deal with quantity in any way, such as optics [Perspectiva], geography, and the rest of this kind, can readily be reduced, as to the heads on which they depend.

On another basis the mathematical disciplines are divided by Geminus [Geminus] the ancient geometer, and by others, as Proclus [Proclus] reports in the commentaries he published on the first book of Euclid (Proclus, *Commentary on Euclid, Book I*). Since he teaches elegantly and fully how far the mathematical disciplines extend, I have decided to append here that division, taken almost word for word from Proclus according to the translation of Francesco Barozzi [Francisci Barocij], a patrician of Venice. The aforesaid authors hold that some of the mathematical sciences are concerned only with intelligibles separated from all matter, while others are concerned with sensibles, so that they touch upon matter subject to the senses. Of the former kind they set the two by far first and foremost sciences, arithmetic and geometry; while in the latter kind they establish six: astro-

astronomy, optics, geodesy, music or canonics, computation, and mechanics.

They say that astronomy [astrologia] is the discipline that treats of the motions of the heavens, of the magnitudes, shapes, and illuminations of the celestial bodies, of their distances from the earth, and of other matters of this kind. It in turn is divided into three parts. The first is gnomonics [gnomonica], which is concerned with the measurement of the hours by means of the gnomon set up as an instrument. The second is meteoroscopy [meteoroscopica], which determines the differences of altitude and the distances of the stars, and teaches many other and various astronomical theorems besides. The third is dioptrics [dioptrica], which ascertains the distances of the planets and of the other stars by means of dioptric instruments of this sort.

They say that optics [perspectiva] is the offspring of geometry, and that it uses lines of sight as though they were geometrical lines, together with the angles formed by these visual rays. It is divided as follows. One part, which properly bears the name optics, gives the reason why things regularly present themselves to us otherwise than they actually are — on account of the differing positions and distances of the objects that fall under sight — for example, why parallel lines appear to converge, or why squares are seen as though they were circles. Another part is the whole science of mirror-images [specularia], which deals with the various and manifold kinds of reflection. And there is a further part called scenography [sciographice], that is, the art of representing shadows, which shows how it can come about that the things appearing in images do not look ungraceful or distorted on account of the distances and heights of the objects depicted. They call geodesy [geodaesia] the science that measures magnitudes as quantities — measuring heaps of material objects as though they were cones, and wells as though they were cylinders. This it does not accomplish through intelligible straight lines, as geometry does, but only through sensible ones — sometimes indeed by more reliable means, such as the rays of the Sun, but at other times by cruder means, such as cords and the plumb line. Geodesy, like geometry, is divided into the part that measures plane figures and the part that measures solid ones. They call canonics or music [canonica, sive musica] the science that considers the ratios of audible harmonies, everywhere relying upon the aid of the senses; it is such a science that, as Plato [Plato] says, it seems to have set the ears themselves before the mind. Computation [supputatrix] is among them what practical arithmetic is among us: for it considers numbers not as they exist in intelligible things, but as they are in sensible things themselves. Finally, mechanics [mechanica], which consists in the knowledge of sensible things joined to matter, is among them of several kinds. One kind produces instruments — this is called ὀργανοποιητικὴ [Greek: organopoiētikē, "instrument-making"] — namely those instruments suited to the waging of war, such as Archimedes [Archimedes]

is indeed reported to have built, which resisted the besiegers of Syracuse both by land and by sea. Another kind produces altogether marvelous effects and is called θαυματοποιητική [Greek: thaumatopoiētikē, "wonder-working"]: for it constructs some of its devices by means of air and breath with the greatest artifice — just as Ctesibius [Ctesibius] and Heron [Heron of Alexandria] also worked — and others by means of weights, whose disequilibrium is to be reckoned the cause of their motion and whose equilibrium the cause of their rest, as Timaeus [Timaeus] also determined; and still others by means of sinews and cords that imitate living coilings and motions. Another kind is entirely the science of balancing bodies and of what are called centers of gravity [centropanderantia]. And a final kind produces spheres and is called σφαιροποιία [Greek: sphairopoiia, "sphere-making"], for the imitation of the revolutions of the heavens — such a sphere as Archimedes also constructed. And, to say it in a word, mechanics is the whole art that has the power of moving matter.

These, then, are the mathematical disciplines as the ancients had them. As for the military art — I mean the one concerned with drawing up and arranging battle-lines, which the Greeks call τακτική [Greek: taktik[?]ō, "tactics"] — they do not hold that it should be counted as one of the parts of mathematics, as certain others have wished; rather, they hold that it makes use of mathematics in this way: sometimes it employs the art of computation, as in numbering the legions, and sometimes geodesy, as in laying out and measuring the spaces for an encampment in the field. In the same way they say that neither history nor the art of healing is any part of mathematics, even though historians and physicians alike very often make use of mathematical theorems. Writers of history do so in reporting the situations of climatic zones, or the sizes and diameters of cities, or their circuits and circumferences; and physicians do so in elucidating a great many matters of their art by methods of this kind. For the usefulness that has come to medicine from astronomy is shown by Hippocrates [Hippocrates] himself, and by nearly everyone who has said anything about the favorable times and places. By the same reasoning, the man who devotes his effort to drawing up battle-lines does indeed make use of mathematical theorems, yet he will not on that account be a mathematician — even though he may sometimes deliberately, when he wishes to make a numerous host appear as small as possible, form his camp and his armies into the figure of a circle, and at other times, when he wishes it to appear as large as possible, into the figure of a square, or a pentagon, or some other polygon. These, then, are more or less the things the ancient mathematicians have left us concerning the division of these sciences.

INVENTORS OF THE MATHEMATICAL DISCIPLINES

The histories clearly testify that all the mathematical disciplines took their rise and origin from various and different authors; indeed, it has also been handed down to memory that the several disciplines by no means attained their full perfection at the very outset, but advanced little by little from imperfect beginnings toward greater perfection. The first inventors of arithmetic are believed to have been the Phoenicians, on account of their frequent trading and commerce, as Proclus [Proclus] reports; and afterward Pythagoras [Pythagoras] and his successors, as well as the Egyptians, and finally the Greeks and the Arabs, wonderfully enlarged it and adorned it with various problems and theorems. Many writers hand down that music was first invented by Mercury [Mercury], who afterward entrusted and committed it to the famous musician Orpheus [Orpheus]; he in turn passed it to Thamyras and Linus [Linus], and Linus to Hercules [Hercules], and so by continuous succession, through other distinguished musicians, it has flowed down to our own times. Geometry, on the authority of Proclus, was discovered by the Egyptians and had its origin in the measurement of fields. For when the yearly inundation of the Nile so confounded and laid waste the boundaries and limits of the fields that no one could recognize his own, the Egyptians began to apply their minds to a method of measuring the fields, so that in this way each man might be given back what was his own. This method of measuring fields, although at that time it was still quite rough and unpolished, nevertheless took the name "geometry" from that very office; for γεωμετρέομαι, or γεωμετρέω [Greek: *geōmetreō*, "I measure the earth"], means the same as "I measure the land." But little by little geometry then began to be refined; and, not content with its own boundaries, it turned to measuring the heavenly bodies as well, and handed down the principles of the whole of astronomy, of cosmography, and of very many other disciplines that depend upon it as upon a root. Thales of Miletus [Thales] is said to have been the first to carry it over from Egypt into Greece; and thereafter many most distinguished and most acute philosophers and mathematicians enriched and adorned it with demon-

—strations. Among these the chief of the ancients are: Pythagoras, Anaxagoras of Clazomenae [Anaxagoras], Hippocrates of Chios [Hippocrates of Chios], Plato, Oenopides [Oenopides of Chios], Zenodorus, Brito [Bryson (?)], Antipho [Antiphon], Theodorus [Theodorus of Cyrene], Theaetetus [Theaetetus], Aristarchus [Aristarchus of Samos], Eratosthenes [Eratosthenes], Archytas of Tarentum [Archytas], Euclid [Euclid], Serenus [Serenus of Antinoöpolis], Hypsicles of Alexandria [Hypsicles], Archimedes of Syracuse, Apollonius of Perga

[Apollonius], Theodosius of Tripolis [Theodosius], Milaeus the Roman, who is also Menelaus [Menelaus of Alexandria], Theon of Alexandria [Theon], Ptolemy [Ptolemy], Eutocius of Ascalon [Eutocius], Pappus [Pappus of Alexandria], Proclus, and others almost beyond number, whom it would be tedious to recount in full. As for astronomy, not a few maintain that it was first invented by Atlas [Atlas]; and hence, on account of the extraordinary knowledge of astronomy with which he was endowed as the first among mortals, they hold that the fable arose that he supports the heavens upon his shoulders. Others think that the Chaldaeans discovered the science of the stars by long observation — which Cicero [Cicero] too affirms in his book *On Divination* [*De Divinatione*]. Others make the Egyptians the first inventors of this science; others the Assyrians; and others, finally, judge that this glory and praise should be given to the Babylonians. But this pursuit, or science, since it is most excellent, has also been made illustrious by most distinguished authors — which it is not the place here to set forth. Now, once these four chief mathematical disciplines had been invented, all the rest, which deal in any way with quantity, were easily derived and drawn from them, like streams from a spring.

THE NOBILITY AND EXCELLENCE OF THE MATHEMATICAL SCIENCES

Since the mathematical disciplines treat of things that are considered apart from all sensible matter — although in reality they are immersed in matter — it is plain that they hold a middle place between metaphysics and natural science, if we consider their subject, as is rightly proved by Proclus. For the subject of metaphysics is separated from all matter both in reality and in reasoning; the subject of physics, by contrast, is joined to sensible matter both in reality and in reasoning; and so, since the subject of the mathematical disciplines is considered apart from all matter, though in reality it is found within matter, it clearly follows that this middle stands between the other two. And if the nobility and excellence of a science is to be judged by the certainty of the demonstrations it employs, then without doubt the mathematical disciplines will hold the first place among all the rest. For they demonstrate everything they take up for discussion by the firmest reasonings, and confirm it so as truly to beget knowledge in the mind of the hearer and to remove every last doubt. This is what we can scarcely grant to the other sciences, since in them the mind often hangs in suspense and remains uncertain, amid a multitude of opinions and a variety of views, when it must judge the truth of the conclusions. The sects of the Peripatetics [Peripatetici] plainly show this to be so — to pass over the other philosophers in silence for

now — for, having sprung from Aristotle [Aristotle] like branches from some trunk, they so disagree among themselves, and sometimes even with Aristotle their very source, that one is left utterly at a loss to know what Aristotle means, and whether he is conducting his dispute about words or rather about things. Hence it comes about that some follow the Greek interpreters, some the Latin, others the Arabs, others the Nominalists, and others, finally, those they call the Realists — all of whom nevertheless boast that they are Peripatetics — as their leaders. How far this is from mathematical demonstrations, I think escapes no one. For the theorems of Euclid and of the other mathematicians retain in the schools today the same purity of truth, the same certainty of their matter, the same strength and firmness of demonstration, as they did so many years ago. To this may be added what Plato says in the *Philebus*, the dialogue entitled *On the Highest Good [Philebus]*: that the more worthy and excellent science is the one that is more devoted to sincerity and truth. Since, then, the mathematical disciplines so seek out, cherish, and cultivate truth that they admit nothing false, nothing merely probable — indeed nothing at all that they do not confirm and corroborate by the most certain demonstrations — there can be no doubt that the first place among all the other sciences must be granted to them.

THE VARIOUS USES OF THE MATHEMATICAL DISCIPLINES

The mathematical disciplines ought to be reckoned not only useful but even quite necessary — both for thoroughly learning the other arts and for rightly establishing and administering the commonwealth. For, as Proclus elegantly shows, there is no access to mathematics except through the mathematical disciplines. Indeed, if we should try to raise the powers and edge of our intellect from the sensible things that the physicist considers up to the things wholly separated and set apart from sensible matter, which the metaphysician contemplates — and this without any intermediate step — we should blind ourselves, no differently than happens to a man who is brought out from some dark prison, in which he has long lain hidden, into the most brilliant light of the Sun. For this reason, before the intellect ascends from physical things, which are bound and joined to matter subject to the senses, to metaphysical things, which are torn away from that same matter, it is necessary — lest it be dazzled by their brightness — that it first be accustomed to less abstract things, such as the mathematicians consider, so that it may more easily grasp those higher things. And so the divine Plato rightly affirms that the mathematical disciplines lift up the mind and sharpen its edge for the contemplation of divine things. As for how much benefit these disciplines confer toward rightly perceiving and interpreting the sacred Scriptures, Blessed Augustine [Augustine] most beautifully sets forth to

us in Book 2, chapter 16, of *On Christian Doctrine* [*De Doctrina Christiana*], showing that, through ignorance of numbers, many things are not understood by many people that are set down in the Scriptures figuratively and mystically. Of this he brings forward not a few examples, and he repeats the same view at much greater length, in the same book, in chapter 37. Saint Jerome [Jerome] teaches the same in Volume 1, Letter 5, asserting that there is great force in numbers for understanding many mysteries in the Scriptures; and in the same passage he declares that geometry too affords great usefulness to theologians. Again, Blessed Augustine, in the passage already cited, chapter 16, which I mentioned a little earlier, testifies that music is thoroughly necessary to the Christian teacher, adding a little later, in chapter 19, that theologians ought also to be diligently instructed in geography. Not unaware of this, the divine Gregory of Nazianzus [Gregory of Nazianzus] extols with the highest praises the divine Basil [Basil the Great], his teacher, because in astronomy, geometry, the knowledge of numbers, and the other mathe-

—matical sciences he was no ordinary master. These arts also contribute not a little toward thoroughly acquiring natural philosophy, moral philosophy, dialectic, and the other doctrines and arts of that kind, as Proclus clearly teaches. Add to this that all the volumes of the ancient philosophers — especially of Aristotle and Plato, whom we deservedly set before ourselves as the guides to be followed for sound and correct philosophizing — and indeed almost all the volumes of the Greeks and of the Latins as well, are filled with mathematical examples, chiefly for this reason: that things which otherwise seemed to be beset by many difficulties might, through examples of this kind, be made clearer and more evident. And this the man wholly ignorant of the mathematical sciences will surely be unable in any way to perceive. What of the fact that in former times no one dared to frequent the celebrated school of the divine Plato who had not first been well adorned with the mathematical disciplines? Hence he is said to have inscribed over the doors of the Academy this motto: ἀγεωμέτρητος οὐδεὶς εἰσὶτω [Greek: ageōmetrētos oudeis eisitō, "let no one ignorant of geometry enter"]. Indeed, this same Plato did not hesitate to assert, in the *Philebus*, that without mathematics the disciplines are worthless. For this reason he prescribes, in Book 7 of the *Republic* [*De Republica*], that the mathematical disciplines are to be learned first of all, on account of their various and manifold uses — as he writes at length — not only for perceiving the other arts more correctly, but also for administering the commonwealth well. Of this I could bring forward many examples, both from times past and from our own age, if it were necessary. In the same place he affirms in the clearest words that arithmeticians especially are by nature apt and suited to all branches of learning; so much so that, even if these

sciences brought us no other usefulness — though we do gain from them countless other advantages besides — he would still hold that they must be thoroughly learned with all diligence, because they render the mind and intellect keener and sharper for grasping all the other arts. And this is easily confirmed by experience herself as our teacher. For we see that those whose minds readily and effortlessly adapt themselves to these disciplines reap no small fruit from the other sciences; whereas, on the contrary, those found least suited to these pursuits are altogether unfit for the rest. For this reason Plato with the very best justification, again and again in his works, impresses upon us and commends the usefulness of these disciplines — especially in Book 7 of the *Republic*, or the *Philosopher*, and in the *Timaeus* [*Timaeus*], where he calls the mathematical disciplines the road to all noble learning — and in very many other places besides, which, mindful of brevity, I now diligently forbear to enumerate. To all these uses is added the greatest pleasantness and delight, with which anyone's mind is filled in cultivating and practicing these arts. For these are the chief of the seven liberal arts, in which not only well-born young men but also nobles, princes, kings, and emperors used to spend much time long and often — deriving from them a most honorable and most liberal delight of the mind, joined with the highest usefulness. And we see many in our own age still imitating their example. The divine Plato testifies, in Book 7 of the *Republic*, that great pleasure of the mind is gained from these arts, where he boldly says, and confidently affirms, that the eye of the soul, which is blinded and buried by other studies, is by the mathematical disciplines alone restored and roused once more to the contemplation of that which truly is. I pass over very many other testimonies of Plato and of other most weighty philosophers, by which the usefulness of these disciplines — joined with their necessity and delight — and their excellence can be abundantly proved.

COMMENDATION OF EUCLID AND OF GEOMETRY

Who Euclid was, the founder of these Elements — so that we may bring forward something about the author whom we have proposed to expound, and about geometry as a whole — and at what time he flourished, is not sufficiently agreed among the writers. For many, as the received edition of Euclid's Elements according to Campanus [Campanus of Novara] and Theon [Theon of Alexandria] testifies, together with the heading of that edition, suppose that Euclid was the philosopher born at Megara — a town lying by the Isthmus — and a pupil of Socrates [Socrates], who established the Megarian school, called after himself, which by another name was called dialectic, because his followers wrote books by questioning and answering (which is the proper office of the dialecticians).

Concerning him there is much in Diogenes Laertius [Diogenes Laertius], in his *Lives of the Philosophers*; and Cicero too writes of him in his second book, where he says: "After him, Euclid, the Megarian pupil of Socrates, from whom those same Megarians were named, who held that the only good is that which is one, and alike, and the same, and everlasting." These authors are supported in no small degree by what Valerius Maximus [Valerius Maximus] writes in his eighth book: namely, that the contractors of a sacred altar, who tried to confer with Plato — himself also a pupil of Socrates — about its manner and form, were sent by him to Euclid the geometer. But if we are to believe Proclus, that noble writer, and other ancient authors, this Euclid of ours was younger than the Megarian, and flourished in the time of Ptolemy the First [Ptolemy I], who began to rule over Egypt after the death of Alexander the Great [Alexander the Great], in the 115th Olympiad and in the year 319 before the birth of Christ, as Ioannes Lucidus [Giovanni Lucido] reports. And this I would believe to be the truer view, chiefly persuaded by this argument: that Diogenes Laertius, most carefully enumerating all the works of that Megarian Euclid, makes no mention whatever of this most celebrated volume on the elements of geometry, in which Euclid won for himself lasting and never-dying fame and glory. For it is not to be thought that Diogenes, so thoroughly practiced in the records of the philosophers, would either have wished knowingly to pass over so remarkable a work, or have been ignorant that it was composed by his own Euclid. And so our Euclid, that most acute geometer, is far different from the Megarian philosopher: for, having been engaged with the highest distinction in the teaching of the Academics, he turned his whole mind to the mathematical disciplines, in which he so excelled that, by the agreed judgment of all, he rightly claimed for himself the first place among mathematicians. He wrote, moreover, not a few other volumes, in which his extraordinary diligence and admirable learning readily shine forth — such as his *Optics*, his *Catoptrics*, his *Elementary Institutions*, and works pertaining to the study of music, his *Phenomena*, and the book of *Data* [Data], the work *On Divisions* (which some suspect to be that little book *On the Divisions of Surfaces* ascribed to Machometus of Baghdad [Muhammad al-Baghdadi], which was lately published through the labors of John Dee of London [John Dee] and Federico Commandino of Urbino [Federico Commandino]). He also composed *Elements of Conics*, according to Proclus, which, however, have not yet reached us, and other small works of that kind. But above all this volume of the *Geometrical Elements* — never-

—which, by the universal consensus, can never be praised enough, he composed with such marvelous order and such great learning that no one who has ever written comparable Elements (and there have been not a few, as Proclus

[Proclus] tells us) has equaled him, much less surpassed him. In this work he not only displayed the highest sharpness of intellect, but also declined to publish everything that pertains to geometry. He chose to make public only what seemed genuinely necessary and useful for the common good, and he resolved to establish it by the firmest arguments and reasoning.

As for how great the excellence and usefulness of these Geometrical Elements of Euclid [Euclid]—and therefore of geometry as a whole—may be, this can be seen clearly enough partly from what we have written already and partly from what we shall now say. They are called *geometrical elements* for this reason: that without them we can undertake no mathematical work at all, let alone gather any fruit from it. Indeed, all the writers on mathematical subjects—Archimedes [Archimedes], Apollonius [Apollonius], Theodosius [Theodosius], and the rest—make use of these Euclidean elements in their demonstrations, treating them as first principles long since examined and proved by everyone.

And so, just as anyone who wishes to read first learns the letters of the alphabet and, by constant repetition, uses them to spell out every word, in the same way whoever wants to make the other mathematical disciplines familiar to himself must first have a full and perfect command of these geometrical elements. For from these elements, as from a most abundant spring, flows all measurement of breadths, lengths, heights, and depths; all measuring and dividing of fields, mountains, and islands; all observation of the stars in the heavens by means of instruments; all construction of sundials [horologiorum sciotericorum]; all the power of machines and the calculation of weights; and all the variety of the various appearances such as we see in mirrors, in paintings, in water, and in air variously illuminated.

By these elements, I say, the middle and center of this whole world-machine has been discovered; its poles [cardines] have been found, about which it perpetually turns; and finally the shape and size of the entire globe have been determined. By the power of this single science it is shown and demonstrated how the whole heaven and the stars perform their endless revolution—their rising and setting, their departure and return, their ascent and descent—and how, throughout the whole year and across every region of the earth and every inclination of the sphere, the variation of day and night and of the seasons comes about. Likewise the conjunctions of the planets, their oppositions, and their various aspects are known so readily that the positions of the planets in the sky and the eclipses—the failures of the Sun and Moon—can be predicted by mathematicians with the utmost certainty, for all time to come, before they occur. In short, we behold this

vast work of God and Nature—the whole world, I mean—laid open before the eyes of our mind by the gift and service of geometry.

Consider further that geometry makes plain, and shows to be credible, many things that seem utterly incredible and beyond all belief. Such is the story told of Archimedes of Syracuse. When Hiero [Hiero], king of Syracuse, had built a ship of such enormous bulk—one he intended to send to Ptolemy [Ptolemy], king of Egypt—that the Syracusans could not all together budge it from its place, Archimedes, that most skilled geometer, relying on the resources of geometry alone, promised the king that he would arrange for the king himself, alone and without any effort, to draw it down to the sea. When he had made good on this, the astonished king is reported to have exclaimed before them all: "From this day on, whatever Archimedes says must be believed." Not unlike this, in my view, is that most beautiful feat which the same Archimedes accomplished at Syracuse by the help of geometry: when a crown had been made of gold and silver mixed together—one the king had ordered fashioned with the greatest care—he detected with the utmost subtlety, without breaking it apart, the separate weights of gold and of silver that had been fraudulently blended by the goldsmith's deceit. Nor should it be passed over in silence that this same Archimedes, relying on the strength and force of geometrical demonstrations, often boasted that if he had another earth on which to set his foot, he could move this earth of ours, on which we dwell, from its place—and by the same reasoning, given any force whatever, he could move any weight whatever.

Other feats of this kind, achieved not only by Archimedes but by other renowned and illustrious geometers as well, have been handed down to memory. In the end, this one science of geometry won Archimedes so great a name that Marcellus [Marcellus], commander of the Roman army—against whom Archimedes had long defended the city of Syracuse with certain engines devised and built through geometrical demonstrations—provided by public edict, in the sack of the captured city and the slaughter of its citizens, for the safety of Archimedes alone. When Marcellus learned that Archimedes had been killed, against his command and his will, by some common soldier, he grieved deeply, and paid to the dead man the honor he had been unable to pay the living. Cicero [Cicero] boasts, in a wonderful way, that he himself discovered Archimedes' tomb while serving as quaestor in Sicily. No one, then, should be surprised that geometry was always held in the highest honor among the Greeks, no less than among the Romans.

Yet another point adds to the excellence and usefulness of geometry: since geometrical demonstrations are supremely clear, no one without them will

adequately grasp what the force of demonstration is, and no one lacking them will be a perfect master of method. This is what Galen [Galen], the distinguished philosopher and prince of physicians, frankly admits in the book he entitled *On His Own Books*. For in matters of dialectic, most closely reasoned as they are, he testifies that although he had gone through all the schools of the Peripatetics and Stoics of his time, and had seized upon their precepts with wonderful eagerness and zeal of mind, he heard almost nothing from them that bore on the understanding of demonstration. On the contrary, he found that most of what they had handed down was left in dispute by them, and that some of it even conflicted with natural reason. So much so that he would have ended up in the hesitancy of the Pyrrhonists (those philosophers who decided nothing but doubted everything) had he not been called back by his knowledge and mastery of arithmetic, geometry, and dialectic—arts in which he had been trained by his grandfather and his father. Hence he urges that we should follow those arithmetical characters and the demonstrations of lines. Plato [Plato] too said that geometry must be learned—among other reasons, for this one especially: that knowledge of it is most useful for grasping the other arts more easily and correctly.

Finally, this is the crowning praise of geometry, to be proclaimed in every way: that it did not remain stuck in the small and lowly devices from which it took its origin, but flew up all the way to the heavens; it lifted our human minds, cast down upon the ground, back up to that heavenly seat, and laid open to our understanding the marvelous fabric of this world, together with its ordering and governance.

DIVISION OF GEOMETRY AND OF EUCLID'S ELEMENTS

Geometry is divided into the study of planes—which by a general term is called *geometry* [Geometria]—and into the doctrine of solids, which mathematicians call by the proper and special name *stereometry* [Stereometria].

For geometry, taken in general, sets itself this aim: either to construct planes or solids, or, when they have been constructed, to compare them with one another and to divide them. And no one should think it strange that, although there are three kinds of magnitude—line, surface, and solid—proper studies exist, as we have said, only for the two latter, and not for lines or even points. It should not, I say, seem strange, since (as Proclus says) geometry is concerned above all with figures, and all figures consist only of planes or solids. For points and lines constitute no figure without planes or solids; and so there was no need to establish a separate science of points and lines, whereas for surfaces (that is,

planes) and for bodies (that is, solids) it was most fitting that they should each receive their own treatment.

Wishing, then, to hand down in these elements a perfect knowledge of geometrical matters, complete in every respect, Euclid, that supreme master of the subject, treats of planes in the first six books, and of solids most acutely and fully in the following five, investigating their most notable properties.

But since geometrical matters generally—and in particular those five regular solids commonly called the Platonic bodies [corpora Platonica]—could not be perfectly treated without a knowledge of commensurable and incommensurable lines (indeed, a great many magnitudes can by no means be brought under measure without knowledge of those same lines, since their sides are very often such that no common and known given measure can measure them, as is plainly evident to those who have ever put geometrical demonstrations into practice and use), Euclid, in order to embrace in these geometrical elements all the material required for the understanding and measuring of magnitudes, placed before his stereometry the tenth book, in which he treats subtly and fully of lines of this kind.

Understanding, again, that this treatment of commensurable and incommensurable lines could not stand without a knowledge of numbers, before the tenth book he treats of the properties of numbers, and he pursued them fully and carefully in the three books that precede it.

For this reason the whole of this volume of geometrical elements, comprised in fifteen books (of which the first thirteen are without any dispute ascribed to Euclid by everyone, while the last two are believed by some to be the work of Hypsicles of Alexandria [Hypsicles]), can rightly be divided into four parts. The first part, contained in the first six books, deals with planes; the second, comprising the three following books, examines the properties of numbers; the third, which the tenth book alone constitutes, discusses commensurable and incommensurable lines; and the fourth, made up of the remaining five books, embraces the science of solids, or bodies.

The first part is itself threefold. In the first four books planes are treated absolutely, by investigating their equality and inequality; in the fifth book proportions of magnitudes in general are discussed; and in the sixth, finally, the proportions of plane figures are examined. What Euclid treats in each of the other books we shall explain in their proper places.

WHAT A PROBLEM, WHAT A THEOREM, WHAT A PROPOSITION, AND WHAT A LEMMA ARE AMONG MATHEMATICIANS

Every mathematical demonstration is divided by the ancient writers into *problem* and *theorem*. They call a *problem* a demonstration that bids and teaches us to construct something. For example, if someone should try to demonstrate that an equilateral triangle can be constructed upon a given finite straight line, such a demonstration will be called a *problem*, because it teaches by what method an equilateral triangle is to be constructed upon a finite straight line.

This kind of demonstration was called a *problem* by analogy with the dialectical problem. For just as among dialecticians a *problem* is called that question whose two sides of contradiction (as they put it) are each probable—such as this question: "Is a whole really distinct from its parts taken together?"—so too among mathematicians the inquiry by which they bid us construct something, and whose contrary can also be effected, is called a *problem*. Thus if someone proposes to demonstrate that an equilateral triangle can be constructed upon a given finite straight line, he will certainly succeed—and yet a non-equilateral triangle (an isosceles or a scalene) can also be constructed upon the same line. By the same reasoning, whoever sets out to bisect a rectilinear angle presents us with a problem, because the same angle can also be divided into unequal parts.

There is, however, no small difference between the dialectical and the mathematical problem. For in a dialectical problem, whichever side of the contradiction is taken up is confirmed only probably, so that anyone's intellect remains in doubt which of its two sides is true. But in a mathematical problem, whichever side one chooses, one will prove it by firm demonstration, so that nothing doubtful at all remains. For if a geometer resolves to draw a perpendicular line from any given point of a proposed straight line, he will accomplish this very thing by fixed and evident reasoning; and the same must be said if he wishes to draw a non-perpendicular line from the same point.

A *theorem*, on the other hand, is the name they give to a demonstration that merely examines some property or attribute of one or more quantities together. Thus if someone chooses to demonstrate that in every triangle the three angles are equal to two right angles, they will call such a demonstration a *theorem*, because it does not bid or teach us to construct a triangle or anything else, but only contemplates this property of any given triangle: that its angles are equal to two right angles. Hence, from that very contemplation, this demonstration is

called a *theorem*. In a theorem it is in no way possible for both sides of the contradiction to be true. For if someone demonstrates that the angles of any triangle are equal to two right angles, it can in no way come about that they are also unequal to two right angles. The same reasoning is to be understood in the case of other theorems.

And so, to put it in a word: that mathematical inquiry which, in teaching us to construct something, also admits of an opposite that can be effected, is called a *Problem*; but that which teaches us to construct nothing, and whose opposite side is always false, is called a *Theorem*. Hence if someone were to propose, in the manner of a problem, that he wishes to construct a right angle in a semicircle, he would deserve to be wholly laughed at and judged utterly ignorant of geometry, since all angles constructed in a semicircle are right, as will be demonstrated in Book 3, Proposition 31. For this reason it must be called a theorem, and not a problem. Both problem and theorem, however, are customarily called by mathematicians a *Proposition*, because each of them proposes something to us, as is clear from the examples adduced. I have said all this so that the diligent reader may not be surprised when he finds

in Euclid, Apollonius, and the other mathematicians that some propositions are called problems and others theorems. For the Geometrical Elements of Euclid, and the *Conics* of Apollonius (to say nothing of the volumes of others for now), consist partly of problems and partly of theorems. The demonstrations of problems always close with roughly these words: "which was to be done"; those of theorems with these: "which was to be shown," or "which was to be demonstrated"—with due regard, of course, to the aim of each. For in any given problem and theorem several demonstrations are contained, and not just one, even though the final demonstrative syllogism alone concludes what was proposed for demonstration at the outset, as we shall make clear in the first proposition of Euclid, and as will be evident in all the rest.

But since, for the demonstrations of problems and theorems, there are very often required certain other, less principal theorems or problems—ones that can easily be understood from what has been proved before—geometers sometimes insert theorems and problems of this kind among the problems and theorems that are their chief concern, so that these may be demonstrated more briefly. They call these *Lemmas*, because they are assumed solely for the sake of other demonstrations, and are not made the subject of principal discussion as the others are. And so a *Lemma* may be described as a demonstration, or a construction, of something that is assumed for the demonstration of some

principal theorem or problem, so that the demonstration may be made quicker and briefer.

WHAT THE PRINCIPLES ARE AMONG MATHEMATICIANS

Since every branch of learning and every discipline is generated from pre-existing knowledge—as Aristotle [Aristotle] is our authority—and demonstrates its conclusions from certain assumed and conceded principles, while no science (by the judgment of the same philosophers) demonstrates its own principles, the mathematical disciplines too will certainly have their own principles, from which, once laid down and granted, they may confirm their problems and theorems.

Now only three kinds of these are found among mathematicians. In the first are found all the *definitions*, which some, following Aristotle, call *suppositions* (as Proclus prefers). By these the technical terms are explained, so that in the discussion itself we may not, deceived by the ambiguity or obscurity of terms, fall into fallacious reasoning.

The second kind embraces the *Petitions*, or *Postulates*, which are so clear and evident in the science under consideration that they require no confirmation, but demand only the assent of the hearer, lest there be any hesitation or difficulty in demonstrating. To the third kind are referred the *Axioms*, or common notions of the mind, which are manifest and evident not only in the science proposed but in all others as well, so that no one who has rightly grasped the very terms could by any reasoning dissent from them. And to these principles, it seems to me, can rightly be applied what Aristotle writes about first principles in the *Metaphysics*: "Who could miss a barn door?"—so that, as Cicero splendidly puts it, they may be called *Pronouncements* or *Dicta*.

Euclid, then, in this volume of the Geometrical Elements, sets out all these principles before the demonstrations of his conclusions, so that from them—easily understood by anyone—he may deduce those admirable theorems to which no one would ever assent unless they were confirmed by sure and evident reasoning. On this account too geometry deserves to be exalted with the highest praise and proclaimed through every age: that from such slender beginnings, most familiar and perfectly easy to anyone however untrained and ignorant, it advances to theorems at first sight far removed from all human sense and understanding—and yet confirms them all, by a marvelous order and an easy method and the most certain demonstrations, so that nothing doubtful at all is left in them.

Moreover, in setting out principles of this kind, Euclid observes this order: at the very entrance to the science he proposes the principles common to all of geometry, while in the other books that follow, where the subject requires it, he sets out those principles that seem to pertain, in a proper and special way, to the particular matter treated there. Not, however, that all geometrical principles have been explained by Euclid in these elements; he has left much for the reader to investigate—things that can nonetheless be grasped and understood, without great labor and study, from what he has handed down. Yet, so that we may not seem to have failed the students of mathematics in this respect as well, we have added in various places, to the principles laid down by Euclid, several others drawn from approved authors, whose ignorance we judged would most delay the course of the demonstrations.

PREFACE TO EUCLID'S ELEMENTS

Should anyone happen to wonder why, after so many distinguished commentaries on Euclid's geometrical *Elements* — published by outstanding writers, expert above all in mathematical matters — I have myself composed yet new expositions, he will readily convince himself that I have not done so rashly, once he understands the reasoning behind my plan. For long and continued experience had shown and proved to me that these *Elements* are so useful, indeed so necessary, that no one can hope, without their support, to grasp the extremely acute and subtle demonstrations of Archimedes, Apollonius, Theodosius, Menelaus [Menelaus of Alexandria], Ptolemy, and the other illustrious mathematicians. I therefore deeply regretted that so remarkable and distinguished an author should be almost wholly neglected by most, and treated as he deserves by very few — so much so that in our own age one can scarcely find anyone who devotes diligent effort and study to mastering these *Elements*. The chief reason for this, I believe, is that people are deterred by the difficulty and obscurity of the matters treated, and have no guide in this field whom they might set before themselves to follow without risk of error.

There do exist, to be sure, the genuinely learned commentaries of Campanus [Campanus of Novara] and of Theon [Theon of Alexandria] on the several books of Euclid, which might well suffice for anyone to attain the doctrine of these *Elements* easily. But the one followed the tradition of the Arabs throughout, and they for the most part perverted Euclid's order and method and altered the wording of his propositions with more than a few substitute terms, so that the author's true and genuine sense can be understood only with great difficulty — as

is most apparent in Book 10. The other (I mean Theon) has been so corrupted by nearly countless errors and faults through the carelessness of copyists, and so encumbered by the Greek notes used in his demonstrations — obscure and badly rendered as they are — that he breeds great difficulty and perplexity even in trained minds. The result is that no one grasps Euclid himself except by the greatest labor and study.

And if others down to our own day have devoted much study and effort to the task of expounding Euclid's *Elements*, these have set out only the first six books; or if any have published commentaries on the whole of Euclid, they have very often abandoned the wholly reliable demonstrations of the ancients and fabricated other, new ones of their own — which for the most part are not so sound, nor do they settle the matter simply and completely. This is especially so in that they now wrongly strip certain words from the propositions, now ineptly add others, and now rashly change still others, so that one may with good reason come to doubt Euclid's true and proper meaning. In this matter, however, I make an exception of Federico Commandino of Urbino [Federico Commandino], no ordinary geometer, who recently restored Euclid — rendered into Latin — to his original luster, save in a few places where he strayed not a little from the truth, as I shall point out in the proper place.

Since this is so, and since a subject so distinguished and a science so worthy deserve that effort, study, and industry — and whatever else can contribute anything to it — be brought to bear on it, I decided, after devoting a long stretch of time to work in mathematics, that I should publish my labors and long night-watches, so as to bring students of these matters some help (unless I am mistaken). Now hear briefly by what method, path, and plan the whole is handled here, and what has been achieved in this exposition. The demonstrations of others — above all of Theon, which, on no slight grounds, I had come to believe, with most people, to be Euclid's own (and Proclus too attests this) — I have tried to render more concise, so far as the difficulty of the matter allowed, or at least plainer and clearer where I could not make them briefer. For I have not set them out bare, and in exactly as many words as they were written.

Indeed their brevity is sometimes such that what the most elegant poet said comes about: "I labor to be brief; I become obscure." [Horace, *Ars Poetica* 25–26] And sometimes, on the other hand, when things could be expressed more briefly and succinctly than is enough, a longer discourse brings annoyance to the reader. Avoiding both faults, I have handed them down as it were [Greek: [?]] — a term for paraphrase/restatement I cannot resolve on the page, and in roughly the

manner I observed when I taught Euclid publicly — in this too wishing, so far as lies in me, to satisfy the desire and wish of my hearers. For thus, in my judgment, Euclid will be learned more easily and successfully by students — especially by those who, as beginners, are now first taking up these mathematical studies — and with greater pleasure and profit.

Besides this, I have added in many places various problems and theorems, not unpleasant to know and not foreign to the aim of geometry, which I have partly culled from Proclus, Campanus, and other authors, and partly worked out myself by my own effort (as they say) and by constant reflection. I have moreover taken diligent care in this work that Euclid's definitions — especially the more obscure ones, and those that seemed to hold some difficulty (upon which very many, striking as it were on certain rocks, have veered from the straight course and slipped into various absurd errors wholly at odds with the design of the discipline) — should be explained clearly and plainly, so far as could be done; and students of these arts will readily judge this.

WHAT THE CHIEF THINGS ARE THAT HAVE BEEN ADDED TO THIS FIRST EDITION OF EUCLID

I.

A geometrical demonstration of Euclid's eleventh axiom [*If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, those two straight lines, produced to infinity, meet one another on that side where the angles are less than two right angles*], far more evident than the one Proclus offers. (*To prop. 28, bk. 1.*)

II.

The division of a straight line into any number of equal parts, by the first 40 propositions of Book 1 alone, without the aid of the proportions of Book 6. (*To prop. 40, bk. 1.*)

III.

How obtuse-angled [amblygonia] and acute-angled [oxygonia] triangles are to be constructed, so that they can be exhibited in numbers — which Euclid demonstrated in props. 12 and 13 of Book 2. (*To props. 12 and 13, bk. 2.*)

III.

A reply to the *Apology* of Jacques Peletier [Jacques Peletier], on the angle of contact. (*To prop. 16, bk. 3.*)

V.

A most beautiful consideration of plane equilateral and equiangular figures inscribed within a circle and described outside a circle: namely, whether every equilateral figure is also equiangular, and every equiangular one is likewise equilateral. (*To the end of bk. 4.*)

VI.

A most abundant treatment of proportions, and of the three chief proportionalities — arithmetic, geometric, and harmonic. (*To def. 4, bk. 5.*)

VII.

A beautiful reflection on why Euclid, in defs. 6 and 8 of Book 5, defined four proportional and non-proportional magnitudes by their equimultiples. (*To def. 8, bk. 5.*)

VIII.

A most easy description of a certain curved line, by which both a figure of any number of equal sides is inscribed in a circle; the circle is squared; a rectilinear angle is divided into any number of equal parts; a straight line equal to any arc of a circle is produced, and conversely; and finally other most pleasant things are accomplished. (*To the end of bk. 6.*)

IX.

How most of the theorems of Book 7 are demonstrated also of fractional numbers, whether or not whole numbers are attached to them. (*To the propositions of bk. 7.*)

X.

Most clear demonstrations of *minutiae*, that is, of fractional numbers. (*To the end of bk. 7.*)

XI.

On the composition of proportions: namely, in what way addition, subtraction, multiplication, and division ought to be carried out — against certain more recent writers. *(To the end of bk. 9.)*

XII.

A description of the five regular solids in a given sphere, from Pappus of Alexandria [Pappus of Alexandria] — and indeed easier than the one Euclid handed down to us. *(To the end of bk. 16.)*

XIII.

A comparison of the solidities and convex surfaces of these same five regular solids with one another. *(To the end of bk. 16.)*

The other things, not few, that have been added here and there throughout these commentaries and are lacking in other editions — such as, in particular, the more accurate finding of the last points of the quadratrix by perpendicular lines, and not by oblique intersections of lines, and so on — the diligent reader will note without any trouble.

[The third leaf of this opening is blank; only offset show-through from the facing page is visible. The Definitions of Book I begin on the following page.]

Euclid's Elements, Book One

Definitions

I.

(Euclid) **A point is that of which there is no part.**

(Clavius) This entire first book is devoted to conveying to us the origin and properties of triangles — both as regards their angles and as regards their sides, which it sometimes compares with one another, and sometimes examines and considers each on its own. For sometimes it investigates the angles from the sides of a triangle, and sometimes the sides from the angles, testing them for equality and inequality; and it pursues the same inquiry in various ways when two triangles are compared with each other. It then opens up for us the properties of parallels and takes up the study of parallelograms, both among themselves and also as they are compared with triangles set between the same parallels. And so that Euclid may carry all this out more correctly and conveniently, he teaches the

bisection of a rectilinear angle and of a straight line, the construction of a perpendicular, how one angle is made equal to another, and other such things. To sum up the whole matter in a word: in the first book — following Proclus's view — are treated the foremost and principal of the rectilinear figures, namely triangles and parallelograms.

But before all else Euclid, in the manner of mathematicians, begins the proposed subject from first principles, starting with the definitions, the first of which explains the point, teaching that in continuous quantity that is called a point which has no parts. This definition will be grasped more plainly and easily if we first understand that continuous quantity has three kinds of parts: some according to length, others according to breadth, and others according to depth or height. Not every quantity, however, has all these parts. Some have only one, according to length; some have two, adding to these the parts of breadth as well; and some, finally, besides these two kinds of parts, contain a third also, that of height or depth. For every continuous quantity is either long only, or at once long and broad, or long, broad, and deep. No magnitude can have any further dimension, as Ptolemy [Ptolemy] rightly demonstrated in his little book *On the Analemma* — recently restored to its former dignity by the work of Federico Commandino of Urbino [Federico Commandino] — and also, as Simplicius [Simplicius] reports, in his little book *On Dimension*, which, so far as I know, has not yet been printed. Therefore, whatever exists in continuous quantity or magnitude and is conceived without any part, so that it is thought of as neither long nor broad nor deep — thereby excluding, of course, the rational soul, the Now or instant of time, and unity, which likewise have no parts — this is what Euclid and the geometers call a point.

[Margin: *punctum* · (a point)]

No example of it can be found among material things, unless you wish to take the tip of some very sharp needle as representing the likeness of a point. Yet even that is not wholly true, since that tip can be divided and cut without end, whereas a point must be regarded as altogether indivisible. In short, in magnitude that must be conceived to be a point which in number is the unit, and in time the instant; for these too must be conceived as indivisible.

II.

(Euclid) **A line is length without breadth.**

(Clavius) Here he defines the line, the first species of magnitude, which he says is a quantity that is long only, not broad — understand, nor deep either. For from

whatever quantity breadth is excluded, from that same quantity depth too is necessarily removed; but not conversely. We can conceive and understand this line — that is, length without breadth — not absurdly, from the boundary of some place that is partly lit and partly in shadow. For the end, or common boundary, of the lit region and the shadowed one is a certain length, extended along the length of the light and the shadow themselves and lacking all breadth, since it is the limit of both. Mathematicians too, to impress on us the true understanding of a line, imagine the point already described in the previous definition to be moved from place to place. For since a point is utterly indivisible, there will be left from that imaginary motion a certain long trace devoid of all breadth. Thus if the point A is understood to flow from A to B, the trace produced, AB, will be called a line —

Diagram. Two lines are drawn, one above the other, illustrating a straight line and a curved line with the same endpoints.

- A (upper left) — left endpoint of the upper, straight line AB.
- B (upper right) — right endpoint of the upper, straight line AB.
- A (lower left) — left endpoint of the lower line, an arc bowing upward from A to B.
- B (lower right) — right endpoint of the lower, curved line AB.

— since the interval comprehended between the two points A and B is indeed a certain length lacking all breadth, because the point A, being deprived of every dimension, could in no way have produced [breadth]. Hence it has come about that some have said a line is nothing other than the flowing of a point, while others have said it is a magnitude contained by a single interval. For a line can be cut and divided in one way only, namely according to length.

III.

(Euclid) **The extremities of a line are points.**

(Clavius) He teaches what the extremities or boundaries of any line are, saying that a line is bounded, or closed off at either end, by points — not that every line has boundaries; for how could we assign boundaries to an infinite line? and on the same reasoning, how could we detect any extremity in a circular line? Rather, that every line which has extremities receives points at its extremities. Thus the line AB above has as its extremities the points A and B. The same is to be understood of all bounded and finite lines, so that we conceive their extremities to be points only.

IV.

(Euclid) **A straight line is one which lies evenly with the points on itself.**

(Clavius) Among mathematicians a line is of three kinds in all: straight, circular (which they also call curved), and mixed, or composed of both. Of these, Euclid here describes the straight line, which he says is that which extends evenly between its points — that is, in which no intermediate point leaps up above or drops down below the extremities, or swerves this way and that by bending, and in which, in short, nothing bent is found. A fine thread stretched taut with the greatest force represents this vividly for us; for in it all the middle parts keep an equal position with the extremities, and none is higher or lower, but all run on uniformly between the two ends. Proclus, expounding this definition, says that a line then lies evenly with its points when it occupies a space equal to that which lies between its extreme points. Thus the line ACB will be called straight, because it occupies precisely as much space as is the distance of point A from point B;

Diagram. Over a horizontal base line, three arcs of increasing height are drawn between the same two endpoints; the base line and each arc share the endpoints A and B.

- A (bottom left) — left endpoint shared by the base and all three arcs.
- B (bottom right) — right endpoint shared by the base and all three arcs.
- C (bottom center) — the midpoint of the straight base line ACB.
- D (top, highest) — apex of the highest arc, ADB.
- E (below D) — apex of the middle arc, AEB.
- F (below E, just above the base) — apex of the lowest arc, AFB.

whereas the lines ADB, AEB, AFB will not be called straight, since they occupy greater spaces than the distance of the extreme points A and B. In the same way you see that all the points of the line ACB, [with] the point C, lie evenly between the extremities A and B, in accordance with Euclid's definition — which is not observed in the other lines, since the points D, E, F leap up away from the extremities A and B. Plato [Plato] defines the straight line very elegantly thus: a straight line is one whose middle points hide its extremities. Thus in the line ACB, if the point C, or any other middle point, had the power of hiding, and A, the extremity, had the power of illuminating, the interposed point C would certainly be an obstacle to the other extremity B being illuminated from A; and again, an eye placed at the extremity A would not see the other extremity B, because of the interposed point C — which does not happen in lines that are not straight, as is plain in the lines ADB, AEB, AFB. Archimedes [Archimedes], for his part, says that

the straight line is the least of those which have the same endpoints; such is ACB compared with ADB, AEB, AFB. For if ACB were not the least of those which have the same endpoints A and B, it would not lie evenly with its points; rather, that line would do so which would be said to be smaller than ACB. Campanus [Campanus of Novara], finally, describing the straight line, calls it the shortest extension from one point to another.

Now, just as mathematicians conceive a line to be described by the imaginary flowing of a point, so from the character of the point's flowing they understand the character of the line described. For if a point is conceived to flow straight through the shortest space, so as to swerve neither to this side nor to that but to keep a certain even motion and progress, that line described will be called straight. But if the flowing point is thought to waver in its motion and stagger this way and that, the line described will be called mixed. And if, finally, the flowing point does not waver in its motion but is carried round in a circle with a certain uniform motion and at a fixed distance from some point about which it is carried, that line described will be called circular. And so, if two points move with wholly similar motions, so that they always remain equally distant from each other, two similar lines will be described by them: that is, if one of them is straight, the other too will be straight; but if one is curved, the other too will be curved in exactly the same way, and so on. The lines that are not straight — all of which may be called oblique — Euclid does not define here, but he will set out the circular one in the fifteenth definition, omitting the mixed line altogether, because it has no use in these geometrical Elements. There are, however, very many kinds of mixed lines: some are uniform, some non-uniform. Of the uniform, again, some lie in a plane, some in a solid. In a plane are the hyperbola, the parabola, and the ellipse, which Apollonius [Apollonius of Perga] treats most fully in his *Elements of Conics*; the conchoid line, treated by Nicomedes [Nicomedes]; the helical line [linea helica], on which Archimedes set up a treatment in his book *On Spiral Lines*; and others of this sort. In a solid, or on a curved surface, there are helical lines of a different kind from the one described by Archimedes — such as the one that winds about some cylinder, and also the one that lies about a cone, or again the one about a sphere, of which sort are those spirals that the Sun describes from its rising to its setting, as we have taught in *the Sphere*. Of the non-uniform ones the number is infinite, and there is no need to review them here. From all this it is established that there are only two simple lines, the straight and the circular, while all the others, whatever they are, are called mixed, because they are composed of these. Hence Aristotle [Aristotle] ingeniously concludes, in his book *On the Heavens*, that corresponding to the threefold line there are only three motions: two simple, the straight and the circular, and a third mixed, or composed of those two.

But since we are accustomed to draw straight lines with a ruler [regula], let us show by what method we may test a given ruler, to see whether the line drawn by it is straight or not.

Diagram. A long, thin straight-edge (a ruler) is drawn as a narrow rectangle with hatched lower edge. Its lower corners are lettered A (left) and B (right); the upper edge used for ruling carries C (left) and D (right).

- A (lower left) — one end of the ruler AB.
- B (lower right) — the other end of the ruler AB.
- C (upper left) — one end of the ruling edge CD.
- D (upper right) — the other end of the ruling edge CD.

Let AB, then, be a ruler; along its edge CD let the straight line CD be drawn from point C to point D. Then let the ruler be turned about, so that — the same side remaining uppermost — the point C is set at D and the point D at C, and along the same edge CD of the ruler let a straight line be drawn from the same point C to point D. For if this second line agrees at every point with the first, there can be no doubt that we may rely on the ruler AB for drawing straight lines. But if it does not agree at every point, that edge CD will not be perfectly straight, and must be corrected more carefully.

V.

(Euclid) **A surface is that which has length and breadth only.**

(Clavius) After the line, which is the first species of continuous quantity and has a single dimension, he defines the surface, which constitutes the second species of magnitude and adds to the first dimension, that of length, a second, that of breadth. For in a surface is found not only length, as in a line, but also breadth, yet without any depth.

Diagram. A square, its interior filled with fine horizontal hatching, representing a plane surface. Its four corners are lettered clockwise from the top left.

- A (top left) — corner of the surface.
- B (top right) — corner of the surface.
- C (bottom right) — corner of the surface.
- D (bottom left) — corner of the surface.

Thus the quantity ABCD, comprehended between the lines AB, BC, CD, DA, and considered according to its length AB (or DC) and according to its breadth AD (or

BC), being wholly without depth, is called a surface. The outermost breadth of any body presents this to us, if all solidity is removed from it by the understanding. Not inaptly too, as Proclus says, the shadows of bodies exhibit to us as it were an expressed image of a surface; for these, since they cannot penetrate the inner part of the earth, will be only long and broad. Mathematicians, however, to set it before our eyes, bid us understand some line to be moved crosswise: the trace left by that motion will indeed be long, on account of the length of the line, and broad too, on account of the motion made crosswise; but it can in no way be deep, since the line describing it lacks all depth — and so it will be called a surface. Thus if the line AB flows toward DC, the surface ABCD will be produced. Others, in describing a surface, say that it is the boundary of a body; others, that it is a magnitude consisting of two intervals. For a surface can be divided and cut in two ways: in one according to length, in the other according to breadth.

VI.

(Euclid) **The extremities of a surface are lines.**

(Clavius) This definition is not unlike the previous one, in which the boundaries of a line were explained. For he means that the extremities of a surface are lines, just as the boundaries of a line were points. Thus of the surface ABCD above, the extremities are the lines AB, BC, CD, DA. And in the same way, in any other surface that has extremities, one must conceive lines at its extremities — but not in an infinite surface, or in a spherical one that surrounds a spherical body. A surface can also be closed and bounded by a single line only, such as is the circular surface, as we shall say in the definition of the circle.

VII.

(Euclid) **A plane surface is one which lies evenly with the lines on itself.**

(Clavius) This definition too bears a certain resemblance to the description of the straight line. For a surface that lies evenly with its lines — so that its middle parts do not depart from the extremities by leaping up or dropping down — will be called plane: such as the surface of some well-polished marble, in which all the parts are set in a level arrangement, so that it has nothing cut into it by angles, nothing broken by unevenness, nothing projecting, nothing hollowed out. For in it the intermediate parts have obtained an equal position with the extremities, and none is higher or lower, but all extend uniformly. Others define a plane surface by saying that it is one whose middle parts hide its extremities; or that it is the least, or shortest, of all those that have the same extremities; or one to all of

whose parts a straight line can be fitted, as Hero the ancient geometer [Hero of Alexandria] holds.

Diagram. A plane surface ABCD (roughly a rectangle) with a bundle of straight lines radiating from the fixed corner A, illustrating a straight line swept round about A across the surface.

- A (top left) — the fixed point about which the straight line is carried round; corner of the surface ABCD.
- B (top, to the right of A) — upper-right corner of the surface ABCD; the ray AB continues to E.
- C (right, below B) — lower-right corner of the surface ABCD.
- D (left, below A) — lower-left corner of the surface ABCD; the ray AD continues to H.
- E (far right, top) — endpoint of the swept ray AE (the extension of AB).
- F (far right, below E) — endpoint of the swept ray AF.
- G (bottom center) — endpoint of the swept ray AG.
- H (bottom left) — endpoint of the swept ray AH (the extension of AD).

Thus the surface ABCD is to be called plane only when a straight line AE, carried round about the fixed point A — so that now it is the same as AF, now the same as AG, and now the same as AH — [finds] nothing on the surface depressed or raised, but all the points of the surface are touched and in a manner grazed by the straight line. But if some least particle of the surface, lying lower, were not touched by the straight line, or if the straight line itself could not be freely carried round because of some swelling or prominence met with on the surface, it could no longer be called plane. And so, for it to be plane, it is required that a straight line can be made to measure it in every way — that is, that a straight line can be applied to it along AB and AF, and finally along all its parts. This surface, moreover, will be the only one that we can imagine and understand to be described by the motion of a straight line crosswise, carried over two other straight lines. Thus if the straight line AB is carried over the two straight lines AD, BC, a perfectly plane surface will be produced, in accordance with all the definitions; for it will not be difficult to fit the descriptions handed down to this surface. Mathematicians commonly call a plane surface simply a "plane," so that when they speak of a plane, a plane surface is always to be understood. All other surfaces, to which a straight line cannot be fitted at every point — such as the interior surface of some vault [fornix], or the exterior of some globe, or of a round column, or of a cone, and so on — are called curved and not plane.

Diagram. An engraved woodcut of a semicircular vault or arch [fornix], shown in perspective with its curved band hatched, illustrating a curved (concave/convex) surface. No lettered points.

For although a straight line can be fitted to the surface of a round column, or cylinder, along its length, yet along its breadth it cannot at all. And the same is to be said of the others. A curved surface, moreover, is of two kinds: convex — namely, the exterior surface of a sphere or cylinder — and concave, as the interior of a vault or of some arch. But since the study of all these belongs to stereometry, Euclid in this first book has explained to us the plane surface only, which he is going to discuss in the first six books.

VIII.

(Euclid) **A plane angle is the inclination to one another of two lines in a plane which touch one another and do not lie in a straight line.**

(Clavius) He declares what a plane angle is, saying: whenever two lines in some plane surface meet one another and are not set in a straight line, there is produced from such a meeting — or from the inclination of the one to the other — an angle that is called plane, because it is formed in a plane surface. For example, because the two lines AB, AC meet at A and do not lie in a straight line, they therefore form the angle A, plane, existing in the same surface in which those two lines are set.

Diagram. Three separate figures side by side, each showing two lines meeting at an apex A and descending to feet B (left) and C (right), illustrating plane angles formed by straight and by curved lines.

- A (apex of each figure) — the vertex where the two lines meet.
- B (lower left of each figure) — the foot of the left-hand line.
- C (lower right of each figure) — the foot of the right-hand line.
- In the first (left) figure the left arm AB is curved and the right arm AC is nearly straight.
- In the second (middle) figure the left arm AB is curved and the right arm AC is straight.
- In the third (right) figure both arms, AB and AC, are straight, forming a rectilinear angle.

Now, two lines are said not to lie in a straight line when one of them, extended toward the meeting-point, does not coincide with the other, but either cuts it, or at any rate immediately after the point of meeting recedes from it. This I say on

account of the angle of contact, which arises when two circles touch one another, or also when a straight line touches a circle. For a straight line drawn out beyond the point of contact, although it does not cut the circle, nevertheless immediately after that point separates from it. In the same way that circular line, extended according to its own disposition and form, recedes from the straight tangent line, although it does not cut it. Hence there truly is an angle formed...

(Continuing the commentary on Definition VIII, the plane angle — the angle of contact between a tangent and a circle.)

formed in that point of contact — a matter on which we shall write more at Proposition 16 of Book III, against Jacques Peletier [Jacques Peletier du Mans], who contends that it is not an angle.

But if two lines touch one another while lying in a straight line, so that either one, when produced, coincides with the whole of the other, no angle at all will arise from that meeting, since there is no inclination; rather, the two together will make up one single line. For example, because the straight line AB, when produced, meets the straight line BC, no angle will be produced at B. Likewise, no angle will arise at B from the curved lines AB, BC, because either one, extended along its own bending and oblique course, coincides with the other. They are therefore said to lie in a straight line. And so, for straight lines to make an angle, they must, after meeting, be produced so as to cut one another. Curved lines, however — or lines of which one is curved and the other straight — can genuinely form an angle even without cutting one another; for it is enough that they touch, in such a way that immediately after the contact the one separates from the other, just as they are seen to stand apart before that same point.

The size of any angle consists solely in the inclination, not in the length of the lines; for lines running out farther do not increase their inclination, and therefore do not increase the size of the angle either. There are also two other kinds of angles. The first comprises solid angles, which Euclid [Euclid] discusses in his stereometry and which exist in solid bodies; the second, spherical angles, which are formed on the surface of a sphere by the circumferences of the greatest circles, and which are treated at length in the *Spherics* of Menelaus [Menelaus of Alexandria]. But the explanation of all these we put off to another place, since here the discussion is to concern plane angles alone.

Diagram. Two small specimen figures illustrating lines that "lie in a straight line" (and so form no angle at their meeting-point).

- Upper figure: a horizontal straight line with endpoint **A** (left), **B** (marked at the middle), and **C** (right) — the two straight segments AB and BC continuing as one line, with no angle at B.
- Lower figure: an arc rising from **A** (lower left) to a peak at **B** (top middle) and descending to **C** (lower right) — two curved lines AB and BC joining smoothly at B, with no angle.

IX.

[Euclid, Definition IX] When the lines that contain an angle are straight, that angle is called *rectilinear*.

[Clavius's commentary] Every plane angle is formed either from two straight lines — and this one is called *rectilinear*, the only kind Euclid treats here — or from two curved lines, which may be called *curvilinear*; or from one curved and one straight line, which is not inaptly called *mixed*. Now from these lines, curvilinear angles can be varied in three ways, and mixed angles in two, according to the varying inclination or disposition of the curved lines — namely by the convex or the concave side, as is plainly and clearly seen in the angles set out here. A rectilinear angle, by contrast, cannot be varied by reason of the inclination or disposition of its lines — unless we should choose to call a greater or lesser inclination a "varied disposition," which is absurd; for in this way the rectilinear angle is only increased or diminished, which it has in common with the others, and it is not varied so as to constitute a different kind.

Diagram. A horizontal row of specimen angle-figures — several pointed and hooked forms — illustrating curvilinear angles (both containing lines curved, in convex and concave variants) and mixed angles (one line straight, one curved). The figures carry no lettered points.

X.

[Euclid, Definition X] When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is a *right angle*; and the line that stands on the other is called a *perpendicular* to the line on which it stands.

[Clavius's commentary] The right angle and the perpendicular line are found in very frequent use in geometry, as also the obtuse and the acute angle; and therefore Euclid teaches in this place which rectilinear angle is called "right" among geometers, and which line is "perpendicular." In the two following

definitions he will explain the obtuse and the acute angle — for no rectilinear angle can be given other than the right, the obtuse, and the acute.

So, if a straight line AB, standing on a straight line CD, makes at the point B two angles (which are called by mathematicians "adjacent" [deinceps], precisely because the same line CD, produced, makes them near the same point B) equal to one another — which will happen only when AB inclines no more toward C than toward D, but stands evenly upon CD — then each angle at B will be called a right angle, and AB will be called perpendicular to CD, on which it stands. By the same reasoning, CB will be named perpendicular to AB; for although CB makes only one angle with AB, yet if AB were extended in a straight and continuous line toward the point B, another angle would be produced equal to the first. By what technique a line ought to be drawn making two equal angles with another, Euclid will teach in Propositions 11 and 12 of this first book.

And so, to conclude in geometry that some angle is right, or that the line producing it is perpendicular to another, it is required and sufficient that we prove the angle adjacent to it to be equal to it. By the same reasoning, if some angle is said to be right, or the line forming it perpendicular to another, we may infer that the angle adjacent to it is equal to it as well. For whenever the adjacent angles are equal to one another, each of them is called a right angle, and the line forming them a perpendicular, according to this tenth definition; but whenever they are not equal, neither of them is called right, as will be established from the two following definitions, and therefore the line forming them is not called perpendicular either. I have said this so that you may see what may be inferred from this definition in geometrical matters, and what use the descriptions of terms have among geometers. For without much labor what we have said can be carried over to the other definitions.

Diagram. The perpendicular of Definition X.

- **A** (top): upper end of the vertical line.
- **B** (bottom middle): foot of the perpendicular, where line AB stands on CD.
- **C** (left) and **D** (right): the two ends of the horizontal line CD.
- The vertical line AB stands on CD at B, making the two equal (right) angles ABC and ABD.

XI.

[Euclid, Definition XI] An *obtuse* angle is one that is greater than a right angle.

[Clavius's commentary] When the straight line AB, standing on the straight line CD, does not make the angles at the point B equal, and for that reason neither of them is right, but one is greater than a right angle and the other less, the greater angle is called obtuse — such as is the angle B verging toward the point C, which is contained by the straight lines AB, BC.

Diagram. The obtuse and acute angles of Definitions XI–XII.

- **A** (top left): upper end of the oblique line.
- **B** (middle): the vertex, where AB meets the horizontal line CD.
- **C** (left) and **D** (right): the ends of the horizontal line.
- The oblique line AB meets CBD at B, making the obtuse angle ABC (toward C) and the acute angle ABD (toward D).

XII.

[Euclid, Definition XII] An *acute* angle, in turn, is one that is less than a right angle.

[Clavius's commentary] As in the preceding figure, the lesser angle B, verging toward the point D, contained by the straight lines AB, BD, is called acute. And so the right angle, as is gathered from what has been said, admits of no variation such that one should be greater or less than another, since the perpendicular line producing it must not incline more to one side than to the other. The obtuse and the acute, by contrast, can be increased and diminished

in infinitely many ways, since a straight line can depart in infinitely many ways from that inflexibility of the perpendicular line, as is evident.

Since two lines meet to form any plane angle, and sometimes several angles exist at one point, mathematicians are accustomed — in order to remove confusion — to express any given angle by three letters, the middle of which indicates the point at which the lines form the angle, while the outer two signify the beginnings of the lines that contain the angle. For example, in the figure above they understand the obtuse angle by the angle ABC, and the acute by the angle ABD — which must be carefully noted, so that we may easily distinguish the angles mentioned in the demonstrations.

Now, given some rectilinear angle, if we wish to test whether it is right, obtuse, or acute, we shall do it in this way. Let two straight lines AB, AC contain the angle A. Draw the straight line BC arbitrarily, so as to subtend the angle, and bisect it at D; then from D as center, with radius DA, describe the circumference of a circle. If it

passes through the points B and C, [Margin: a — Elements III.31] the angle A will be right, inasmuch as it stands in the semicircle BAC. But if the same semicircle cuts the straight line BC at E, F, the angle BAC will be obtuse; because, when the straight lines EA, FA are drawn, [Margin: b — Elements III.31] the angle EAF, standing in the semicircle EAF, is right, and it is a part of the angle BAC. And if, finally, the same semicircle cuts the straight line BC *produced* at E, F, the angle BAC will be acute; because, when the straight lines EA, FA are drawn, [Margin: a — Elements III.31] the angle EAF, standing in the semicircle EAF, is right, and it is greater than the angle BAC.

Diagram. Three triangles illustrating the first test (radius DA from the midpoint D of BC), stacked at the right of the page. In each, A is the apex (the angle under test), B the left vertex, C the right vertex, and D the midpoint of the base BC; a dotted circular arc is drawn from D.

- Right-angle case: the arc passes through B, A, C (semicircle BAC).
- Obtuse case: the arc cuts the base at E and F (between B and C); straight lines EA, FA are drawn.
- Acute case: the arc cuts the base *produced* at E and F (beyond B and C); straight lines EA, FA are drawn.

We shall obtain the same result in another way. From the point D — which bisects the straight line BC, the subtense of the given angle A — describe a semicircle with radius DB or DC. If it passes through the point A, [Margin: b — Elements III.31] the given angle will be right, since it stands in a semicircle. But if the same semicircle passes *above* the point A, the given angle will be obtuse. For when the straight line DA is drawn, cutting the circumference at E, let the straight lines EB, EC be joined; [Margin: c — Elements III.31 [?]] then the angle BEC will be right, standing in a semicircle. Since, then, the given angle BAC is [Margin: d — Elements I.21] greater than the angle BEC, the given angle A will be greater than a right angle — that is, obtuse. And if, finally, the same semicircle cuts the straight lines AB, AC, the given angle will be acute. For, taking a point E between the straight lines AB, AC on the circumference, let the straight lines EB, EC be joined; [Margin: e — Elements III.31] then the angle BEC will be right, standing in a semicircle; [Margin: f — Elements I.21] and since it is greater than the given angle A, the given angle A will be less than a right angle — that is, acute.

Let no one think it strange that for the demonstration we assume propositions that are demonstrated later by Euclid, which may seem foreign to the purity of geometrical demonstrations. Let no one, I say, think it strange; for since what we here show is not necessary for the following demonstrations, it can conveniently

be deferred until the required propositions have been demonstrated. It is enough that the practical procedure of this matter be understood here. We shall observe the same in several practical problems; for we shall propose them in their proper places, so far as may be done — as, for instance, we shall teach the division of a rectilinear angle into any number of equal parts in the place where Euclid teaches the division of that same angle into two equal parts, and so on — although the propositions demonstrated later are necessary for the demonstrations of those procedures.

Diagram. Three triangles illustrating the second (alternative) test (semicircle from midpoint D with radius $DB = DC$), stacked at the right of the page.

- Right-angle case: **A** apex, **B** left vertex, **D** midpoint of base, **C** right vertex; the dotted semicircle passes through A.
- Obtuse case: the semicircle passes above A; the line DA, produced, meets the arc at **E** (drawn above A), and EB, EC are joined (angle BEC right). Labels **B, E, A, D, C**.
- Acute case: the semicircle cuts sides AB, AC; a point **E** is taken on the arc between the sides, and EB, EC are joined. Labels **A** (apex), **E** (interior), **B, D, C**.

We shall recognize the same more easily by means of a set-square [normal] accurately made, such as the instrument ABC represents, consisting of two rulers AE, AF joined at a right angle at A. For if the side AB of this set-square is applied to the straight line AB, with the point A falling on the point A: then, if the set-square's side AC also coincides with the straight line AC, the angle A will be right; but if the set-square's side AC falls short of the straight line AC, the angle A will be obtuse; and if, finally, the set-square's side AC falls beyond the straight line AC, the angle will be acute, as is evident.

We shall test the set-square itself — whether it is accurately made or not — as follows. Describe a semicircle BAC of any size from center G, and draw the diameter BC; place the angle A at some point of the circumference, say at A, and let one side of the set-square, namely AB, pass through B, an endpoint of the diameter. Then, if the other side AC passes through the other endpoint C, the set-square ABC is rightly made; [Margin: a — Elements III.31] because then the angle BAC, standing in the semicircle BAC, is right. But if the side AC does not pass through C, the set-square must be corrected, because its angle A is then not right. By the same reasoning we shall test the inner part of the set-square, if we apply the angle D to the circumference and [test whether] its sides DE, DF [pass through] the endpoints B, C, and so on.

Diagram. The set-square (norma) and the semicircle used to test it, at the foot of the page.

- **A** (top-left apex): the outer right-angle vertex of the set-square, where the two rulers meet.
- **E** (lower left) and **F** (lower right): the far ends of the two rulers (AE and AF); the right ruler AF is drawn as a hatched straight-edge.
- **D**: the inner corner of the set-square (just below A), whose inner edges are DE and DF.
- **B** (left) and **C** (right): the endpoints of the diameter BC of the test-semicircle, through which the ruler-sides are to pass.
- **G** (center, on the diameter): the center of the described semicircle BAC.

XIII.

[Euclid, Definition XIII] A *boundary* [terminus] is that which is the extremity of something.

[Clavius's commentary] There are three boundaries according to this definition. For a point is the boundary, or extremity, of a line; a line, of a surface; and a surface, of a body. But a body can bound nothing further, since no other magnitude is found having more than three dimensions. For everything bounded exceeds its own boundary by one dimension, as is evident from the examples adduced.

XIII.

[Euclid, Definition XIV] A *figure* is that which is contained under one or several boundaries.

[Clavius's commentary] Not every magnitude possessing boundaries can be called a figure — lest we be forced to call a finite line a figure. Rather, only those magnitudes that have breadth — namely a bounded surface — and those that have also acquired depth — namely a finite solid — will be called by the name of "figure." For these are properly said to be enclosed by boundaries. For a finite line is not properly said to be enclosed by its extreme points, since points do not surround a line, but rather it is said to be *bounded* by points. And so the boundaries must *surround* the quantity that is called a figure, and not merely bound it. An infinite surface, too, or even a body, since it is enclosed by no boundaries, can in no way be called a figure. Figures enclosed by a single boundary are the circle, ellipse, sphere, spheroid, and the like; but figures

enclosed by several boundaries are the triangle, square, cube, pyramid, and so on. Bounded surfaces are called *plane* figures; but circumscribed solids, *solid* or *corporeal* figures. Moreover, since you will inspect very many forms or types of various figures in what follows — plane figures in the first ten books, solid figures in the last five — no figure seems to need to be drawn in this place.

XV.

[Euclid, Definition XV] A *circle* is a plane figure contained under one line, which is called the *periphery*, to which all the straight lines falling from one of the points placed within the figure are equal to one another.

[Clavius's commentary] Here he defines the circle, the most perfect of plane figures, teaching that that plane figure which is circumscribed by a single line — to which line all the straight lines drawn from one point that lies within the figure are equal — is called a circle. Thus, if a surface, or space, is enclosed by a single line ABC, and has this condition, that from some point taken within it — say from D — all the straight lines falling to the boundary ABC, such as DA, DB, DC, are equal to one another, such a plane figure will be called a circle, and otherwise not. By what method that middle point in the circle ought to be found, Euclid will teach in Proposition 1 of Book III. Euclid also adds that the outermost line of the circle, such as ABC, is called the *periphery*, or, as the Latins render it, the *circumference*.

A circle can also be described in this way: A circle is a plane figure described by a finite straight line carried around, one of its endpoints remaining at rest, when it has been restored again to the same place from which it began to move. This description is very like the one by which the sphere is described by Euclid in Book XI. Thus, if the straight line AD is understood to be moved about the point D, which stays at rest, until it returns to the same place from which it began to move, the straight line itself will describe the whole circular space,... and will trace out the periphery ABC. The resting point D will also be that from which all the lines falling upon the periphery are equal to one another, because the straight line AD, carried around, measures equally all the lines that can be drawn out from D to the periphery. Therefore the ellipse, although it is a plane figure circumscribed by one line, yet because in it there is no point from which all the straight lines to the bounding line itself are equal, cannot be called a circle.

Diagram. The circle of Definition XV.

- **D (center):** the interior point from which the equal radii are drawn.

- **A** (upper left), **B** (upper right), **C** (bottom): three points on the circumference.
- The three radii **DA**, **DB**, **DC** divide the circle into three parts and are equal to one another; **ABC** is the periphery (circumference).

XVI.

[Euclid, Definition XVI] This point, in turn, is called the *center* of the circle.

[Clavius's commentary] He teaches that that point within the circle, from which all the straight lines drawn to the circumference are equal, is called the center of the circle — such as is the point **D** of the preceding figure. From which it is evident that the *pole* of some circle on a sphere — from which all the straight lines falling to the periphery of the circle are equal, as Theodosius [Theodosius of Bithynia] says in the *Spherics* — ought not to be called the center of the circle, since that point which is called the pole lies on the surface of the sphere, and not on the surface of the circle; which, however, is a necessarily required condition for any point to be called a center. Moreover, for some point to be called the center of a circle, it is enough that only three lines falling from it upon the periphery be equal to one another, as Euclid demonstrates in Proposition 9 of Book III. For in this way it comes about that all the others sent out from the same point are equal to one another.

XVII.

[Euclid, Definition XVII] The *diameter* of a circle, in turn, is a certain straight line drawn through the center and terminated on both sides at the periphery of the circle, which bisects the circle.

[Clavius's commentary] If in a circle a straight line **AB** is drawn through the center **C**, so that its endpoints **A** and **B** are terminated at the periphery, it will be called the diameter of the circle. Not, therefore, will every straight line drawn in a circle be called a diameter, but only that one which extends through the center all the way to the periphery on both sides. Hence several diameters can be assigned in a circle, but only one center. As for Euclid's addition that the circle is bisected by the diameter, it can be evident from this: that the diameter is drawn through the middle of the circle, that is, through the center. For hence it comes about that, because of the straight passage of the diameter through the center, equal circumferences are cut off on both sides. This, however, Proclus [Proclus] testifies that Thales of Miletus [Thales] demonstrated in this way.

Let us conceive in our mind the portion ADB to be fitted and adapted to the remaining portion AEB, so that the diameter AB is common to both portions. If, then, the circumference ADB coincides entirely with the circumference AEB, it is manifest that those two portions made by the diameter are equal to one another, since neither exceeds the other. But if the circumference ADB is said not to fall on every side upon the circumference AEB, but either outside it, or partly outside and partly inside, then — a straight line being drawn from the center C, cutting the circumference ADB at D and the circumference AEB at E — the two straight lines CD, CE, drawn from the center to the circumference of the same circle, will be equal (by the definition of the circle), and yet one will be a part of the other, which is absurd. Therefore one circumference will not fall outside another, nor inside, nor partly outside and partly inside, but both will fit to one another, and therefore will be equal — which was what was to be demonstrated.

From this demonstration it is established that the diameter bisects not only the circumference but also the whole area of the circle. For since the semicircumferences coincide with one another, as has been shown, the surfaces themselves — enclosed between the diameter and each circumference — will also coincide, since neither exceeds the other. Therefore they will be equal to one another.

Diagram. The diameter of Definition XVII.

- **A** (left) and **B** (right): the endpoints of the diameter on the periphery.
- **C** (center): the center of the circle, through which the diameter AB passes.
- **E** (top) and **D** [?] (bottom): points on the upper and lower semicircles respectively (the arcs AEB and ADB).

Diagram. The figure for the Thales bisection argument.

- **A** (bottom left) and **B** [?] (bottom right): the shared endpoints of the two circumferences, joined by the base line AB (the common diameter).
- **C** (bottom center): the center, from which the secant line is drawn.
- **D** (on the outer arc, upper): where the secant from C cuts the circumference ADB.
- **E** (on the inner arc): where the secant from C cuts the circumference AEB.
- The figure shows the outer arc ADB and the inner arc(s) AEB not coinciding (the "absurd" case), with the radial line from C crossing them at D and E.

XVIII.

[Euclid, Definition XVIII] A *semicircle*, in turn, is a figure that is contained under the diameter and under that line which is cut off from the periphery of the circle.

[Clavius's commentary] For example, in the circle above, the figure ADB, contained under the diameter AB and the periphery ADB, is called a semicircle, because...

because, as we showed in the preceding definition, it is half the circle. By the same reasoning the figure AEB will be a semicircle. And the same point C, which bisects the diameter, is the center both in the circle and in the semicircle.

Now suppose the straight line BD does not pass through the center E. Then the circle will not be cut by it into two equal halves, but into two unequal portions, BAD and BCD. Of these, the one in which the center of the circle lies — namely the portion BAD — is greater than the other, BCD, outside which the center E lies. That these portions BAD and BCD are unequal can be proved as follows.

Suppose that through the center E a diameter AC is drawn perpendicular to the straight line BD. Now if the said portions were claimed to be equal, and the portion BCD were imagined to be revolved about the straight line BD so as to fall onto the portion BAD, then that portion would coincide with this one, and the straight line FC would coincide with the straight line FA — this because of the right angles at F, which are all equal to one another by definition 10, since they are adjacent to one another. The straight line FC, then — which is now the same as FA — would be greater than EA, a part of that same FA. But since EC is equal to EA (because both are drawn from the center to the circumference), FC would also be greater than EC: the part greater than the whole, which is absurd.

Therefore the portion BCD will not coincide with the portion BAD, but will fall within it — as does the portion BGD, so that the straight line FG, now being the same as FC, could be less than EA or EC. For if it were said instead to fall outside — as if the circle were BCDG, whose center is E, and the portion BCD fell outside BGD (as the portion BAD does) — then FA, now being the same as FC, would again be greater than EG, that is, greater than EC; and so once more the part FC would be greater than the whole EC, which is absurd. From this it is clear that the portion BAD, in which the center E lies, is greater than the remaining portion BCD, since this latter is equal to the portion BGD, which is a part of the portion BAD. For since it has been shown that the portion BCD, revolved about the

straight line BD, can neither coincide with the portion BAD nor fall outside it, it will fall entirely within it — as does BGD.

Diagram. A circle with two mutually perpendicular diameters and an inner arc, illustrating the demonstration that a chord not through the center divides the circle unequally.

- A (top of circle): upper end of the vertical diameter
- C (bottom of circle): lower end of the vertical diameter AC
- B (left of circle): left end of the horizontal chord BD
- D (right of circle): right end of the horizontal chord BD
- E (center): center of the circle, on the vertical diameter
- F (on the vertical diameter, just below E): the point where the chord BD crosses the diameter AC
- G (on the vertical diameter, between E and A): a point of the inner reflected arc BGD

Definition XIX

[Euclid] Rectilinear figures are those contained by straight lines.

[Clavius] After the definition of the circle, Euclid — about to hand down descriptions of various figures — first explains which figures are called rectilinear, since these will be the chief subject of discussion in these books. All plane figures, then, that are enclosed on every side by straight lines are termed rectilinear. From this it is plain that plane figures bounded by curved lines are called curvilinear, while those bounded partly by curves and partly by straight lines are called mixed. Euclid will now describe the various kinds of rectilinear figures.

Definition XX

[Euclid] Trilateral figures are those contained by three [straight lines].

[Clavius] By affirming that those rectilinear figures are called trilateral which are bounded by three straight lines, Euclid plainly hints at how the triangle ought to be defined. For since in rectilinear figures there are as many angles as there are sides (that is, as the straight lines of which they consist), a triangle will be defined as a figure contained by three straight lines — and all its species will be presented shortly.

Definition XXI

[Euclid] Quadrilateral figures are those contained by four [straight lines].

[Clavius] By the same reasoning, a quadrangle will be a figure contained by four straight lines, whose various species will follow shortly.

Definition XXII

[Euclid] Multilateral figures are those contained by more than four straight lines.

[Clavius] Since the species of rectilinear figures are innumerable, owing to the infinite progression of numbers — for three straight lines enclosing a figure produce the first species, under which all triangles are contained; four constitute the second, which embraces all quadrangles; five compose the third species; six the fourth; and so on to infinity — Euclid, so as not to be forced to pursue this endless multitude of figures, calls all the other rectilinear figures that are bounded by more than four straight lines by the general term *multilateral*. He is content with the names of trilateral and quadrilateral figures, perhaps for this reason: that in these earlier books the discussion is chiefly about triangles and quadrangles, and because all the remaining species could easily be defined by anyone on the analogy of these two. For who would not gather from what has been said that a figure contained by five straight lines is called five-sided, one bounded by six lines six-sided, and the rest in the same way? Such figures may likewise be called five-angled, six-angled, seven-angled, and so on.

Definition XXIII

[Euclid] Of trilateral figures, an *equilateral* triangle is one that has its three sides equal.

[Clavius] Euclid now comes down to the individual species of triangles. Since triangles can be divided either with respect to their sides or with respect to their angles, he first sets out the species of the former division, which are just three, because three sides can be related to one another in only three ways. For either all are equal; or only two, the third being either greater or less; or all are unequal. When, therefore, all three sides are equal to one another, the triangle is called *equilateral*. Furthermore, from the equality of all three sides of an equilateral triangle it is inferred that all three of its angles are likewise equal, as we shall demonstrate at the fifth proposition of this book.

Diagram. An equilateral triangle, illustrating Definition XXIII. No labels.

Definition XXIV

[Euclid] An *isosceles* triangle is one that has only two sides equal.

[Clavius] From this equality in turn of the two sides of an isosceles triangle it follows that the two angles upon the remaining side are also equal, as Euclid will demonstrate at proposition 5 of this book. We have set down two isosceles triangles: the first has its third side greater than either of the equal sides, while the second has that side smaller. Thus there are two species of the isosceles triangle: one whose third side is greater than either of the equal sides, and another whose third side is smaller than either of the equal ones.

Diagram. Two isosceles triangles, illustrating Definition XXIV: at left a broad, low triangle (third side greater than either equal side), at right a tall, narrow triangle (third side smaller than either equal side). No labels.

Definition XXV

[Euclid] A *scalene* triangle is one that has its three sides unequal.

[Clavius] Here, finally, from the inequality of all the sides of a scalene triangle the inequality of all its angles is inferred, as will be shown at proposition 18 of this first book. Moreover, it is clear from these considerations that the triangle could have been divided into three species in the same way if the criterion of the equality of the angles had been taken. For since either all three angles are equal to one another; or only two, the third being greater or less; or all three are unequal — every triangle will be either equiangular, having all three angles equal, or having only two angles equal, or having all its angles unequal. Of these, the first corresponds to the equilateral triangle, the second to the isosceles, and the third finally to the scalene. But by what technique the triangles of this classification are to be constructed upon any given finite straight line, we shall show at proposition 1 of this book.

Diagram. A scalene triangle, illustrating Definition XXV. No labels.

Definition XXVI

[Euclid] Further, of trilateral figures, a *right-angled* triangle is one that has a right angle.

[Clavius] He now sets out the species of triangles according to the latter division, taking account of the variety of the angles. Since there are only three different kinds of rectilinear angle — for every rectilinear angle is either right, or obtuse,

or acute, as we said above — it happens that three species of triangles are likewise found under this consideration. For either one angle of the triangle is right, and for that reason the others are acute (as will be established from proposition 17 of book 1); or one is obtuse, and for the same reason the others acute; or finally none is right and none obtuse, but all are acute. When, therefore, some triangle has one right angle, it is called by Euclid and other geometers a *right-angled* triangle. A triangle of this kind can be isosceles or scalene, as the figures indicate, but in no way equilateral. For on account of the equality of the sides, all its angles would also be equal (by what we shall say at proposition 5); and so, since one angle is granted to be right, all three would be right — which conflicts with propositions 17 and 32 of this book.

Diagram. Two right-angled triangles, illustrating Definition XXVI. No labels.

Definition XXVII

[Euclid] An *obtuse-angled* triangle [amblygonium] is one that has an obtuse angle.

[Clavius] An obtuse-angled triangle [amblygonium], or one with an obtuse angle, can likewise be either isosceles or scalene, as is seen in these figures, but not equilateral — for otherwise, by the same reasoning, all three of its angles would be equal (by what we shall show at proposition 5); and so, since one is posited as obtuse, all three would be obtuse, which conflicts even more strongly with propositions 17 and 32 of this book.

Diagram. Two obtuse-angled triangles, illustrating Definition XXVII. No labels.

Definition XXVIII

[Euclid] An *acute-angled* triangle [oxygonium] is one that has three acute angles.

[Clavius] Every acute-angled triangle [oxygonium] can be either equilateral, or isosceles, or scalene, as may be seen in the triangles that we displayed for examination among the species of the earlier division, so that the same figures need not be pointlessly repeated here. From what has been said, then, it is plain that any equilateral triangle is necessarily acute-angled; but that every triangle, whether isosceles or scalene, is either right-angled, obtuse-angled, or acute-angled; and that the isosceles acute-angled triangle is in turn twofold — namely, the isosceles acute-angled triangle having its third side greater than either of the equal sides, and the isosceles acute-angled triangle having its third side smaller than either of the equal sides. Thus there is a single species of the equilateral

triangle, four of the isosceles, and three of the scalene; and in all, eight kinds of triangle: the equilateral, which we said is perpetually acute-angled; the isosceles right-angled; the isosceles obtuse-angled; the isosceles acute-angled having its third side greater than either of the equal sides; the isosceles acute-angled having its third side smaller than either of the equal sides; the scalene right-angled; the scalene obtuse-angled; and the scalene acute-angled. These may also be named with the terms transposed: right-angled isosceles, right-angled scalene, obtuse-angled isosceles, obtuse-angled scalene, acute-angled equilateral, acute-angled isosceles having its third side greater than either of the equal sides, acute-angled isosceles having its third side smaller than either of the equal sides, and acute-angled scalene. Hence it is plain what connection or kinship the triangles of each division have with one another.

That there can be an isosceles acute-angled triangle in which either of the two equal sides is smaller than the third, as Franciscus Barocius [Francesco Barozzi] rightly observed in his *Cosmographia*, we shall show at proposition 15 of book 4. Furthermore, in every triangle, whichever two sides are expressly named, the remaining third side is usually called by mathematicians the *base*, whether it occupies the lowest position in the figure or the highest, and so on. I wanted to advise you of this briefly, so that you would not suppose that some mystery lies hidden in the base of a triangle, and would understand that any side whatever, with every distinction removed, can be called by the name of *base*.

Definition XXIX

[Euclid] Of quadrilateral figures, a *square* [quadratum] is one that is both equilateral and right-angled.

[Clavius] After the species of trilateral figures, Euclid now sets out the quadrilateral figures one by one, reviewing just five of their kinds — of which the first four are regular, but the last, the fifth, is irregular. The first quadrilateral figure is called the *square*, whose four sides are all equal to one another and all whose angles are right. And so a quadrangle that is equilateral but not right-angled, or on the contrary right-angled but not equilateral, will by no means be called a square. Euclid will teach at proposition 46 of this book in what way a square is to be constructed upon a given finite straight line.

Diagram. A square, illustrating Definition XXIX. No labels.

Definition XXX

[Euclid] An *oblong* [altera parte longior] is a figure that is right-angled but not equilateral.

[Clavius] The second quadrilateral figure is called the *oblong* ["longer on one side"], in which the angles are indeed right, but the sides are not all equal to one another — although the two opposite sides are equal to each other. Thus in the oblong ABCD the sides AB and DC are equal to each other, and AD and BC likewise equal to each other; and since ABCD, on account of the rightness of its angles, is a parallelogram, as we shall show at proposition 34 of this book.

Diagram. A rectangle (oblong) labelled ABCD, illustrating Definition XXX.

- A (top-left corner)
- B (top-right corner)
- D (bottom-left corner)
- C (bottom-right corner)

Definition XXXI

[Euclid] A *rhombus* is a figure that is equilateral but not right-angled.

[Clavius] This third figure among the quadrilaterals, which is called the *rhombus*, has conditions altogether opposite to and different from those of the oblong. For it has all its sides equal, but its angles not right and not equal — although the two opposite angles are equal to each other. Thus in the rhombus ABCD the angles A and C are equal to each other, and B and D likewise equal to each other; and since ABCD, on account of the equality of its sides, is a parallelogram, as will be demonstrated at the same proposition 34 of this book.

Diagram. A rhombus labelled ABCD, illustrating Definition XXXI.

- A (top-left corner)
- B (top-right corner)
- D (bottom-left corner)
- C (bottom-right corner)

Definition XXXII

[Euclid] A *rhomboid* is a figure that has its opposite sides and angles equal to one another, but is neither equilateral nor right-angled.

[Clavius] This figure, which is called the *rhomboid*, is in every respect opposite to the square. For neither are all its sides equal, nor is any of its angles right; yet the two opposite sides — such as AB, DC and AD, BC in the rhomboid ABCD — are equal to each other, and likewise the two opposite angles — such as A, C and B, D — are equal to each other. These four quadrilateral figures, then, can be called regular; but all the rest, whatever they may be, are irregular.

Diagram. A rhomboid (oblique parallelogram) labelled ABCD, illustrating Definition XXXII.

- A (top-left corner)
- B (top-right corner)
- D (bottom-left corner)
- C (bottom-right corner)

Definition XXXIII

[Euclid] Besides these, the remaining quadrilateral figures are called *trapezia*.

[Clavius] All the remaining quadrilateral figures, which differ from the four just mentioned — in that they have neither all their sides equal, nor all their angles equal (that is, right), nor the two opposite sides, nor the two opposite angles equal to one another — Euclid names with the general term *trapezia*. Since these can be varied in infinitely many ways, they are rightly called irregular. For two of their angles can be right, or one obtuse and the others acute, or two obtuse and the others acute, and so on. And the same kind of division can be made with respect to the sides: for either some are equal to one another, or none is equal to another, and so on. We shall present certain determinate species of trapezia after the definition of parallel (that is, equidistant) lines and of the parallelogram.

Diagram. An irregular quadrilateral (trapezium), illustrating Definition XXXIII. No labels.

Definition XXXIV

[Euclid] *Parallel* straight lines are those which, being in the same plane and produced to infinity in both directions, do not meet each other on either side.

[Clavius] For two or more straight lines to be called parallel, or equidistant, it is not enough that, produced in any direction even through infinite space, they never come together at a single point; it is also necessary that they lie in one plane surface. For many straight lines that do not lie in the same plane surface,

though produced to infinity, never meet in one point, and yet are not to be called parallel — such as, for example, two straight lines set crosswise in mid-air and not touching each other; for these can never come together.

Two straight lines are said to lie in the same plane surface when some plane surface, fitted to one of them so that it touches all its points, and revolved about it while it stays fixed, can be fitted to the other as well at all its points — even though in actual fact the lines are found in two different surfaces. Thus, given two straight lines AB and CD, if some plane surface is applied to the straight line AB, touching all its points, so that, revolved about it, it also touches all the points of the other straight line CD, then two such straight lines will be said to lie in the same plane surface — otherwise not. If, then, these two straight lines do not meet, even though produced to infinity both toward the parts A, C and toward B, D, they will be called parallel, or equidistant.

But you will understand more clearly and completely, in book 11, in what manner two or even more straight lines are said to lie in the same surface. Let it suffice here to have briefly noted that Euclid rightly laid down both conditions in the definition of parallel lines. For they must lie in the same plane, and, being produced in either direction, never meet in one point, however far this production is continued into infinite space. But if two straight lines, extended through some immense space, are not seen to meet, yet it is established that, drawn out further on one side, they will at length come together at a single point — even though on the other side they always grow more and more distant and separated — they are by no means to be called parallel. Whenever, therefore, two straight lines are said by anyone to be paral-

Diagram. Two parallel horizontal straight lines, illustrating Definition XXXIV.

- A (left end of the upper line) — B (right end of the upper line)
- C (left end of the lower line) — D (right end of the lower line)

Euclid, *Elements*, Book I — end of the Definitions; the Postulates and the Common Notions

(Clavius's commentary on Euclid. The numbered enunciations set in large type are the lemmata; the passages in italic beneath each are Clavius's own commentary and scholia. Definitions XXXV and XXXVI, and Axioms X and XI, are Clavius's own additions, as he notes.)

Definition XXXIV, continued (parallel lines)

[parallel], one must grant that they lie in one and the same surface and can never meet. Likewise, if someone wishes to conclude that two straight lines are parallel, he must first prove that they lie in the same plane and that, produced in neither direction, they cannot be joined. On this point not a few seem to be deceived: they try to show that certain straight lines are parallel merely on the ground that they do not converge in either direction even if produced to infinity, making no mention at all of the other condition — the one that requires these lines to lie in the same plane.

Here Euclid brings the definitions of Book I to a close. But since in this same book mention will be made of the figure called a *parallelogram*, and of the things called the *complements* of a parallelogram, we have judged it necessary to explain, by adding two definitions, what a parallelogram is and what the complements of a parallelogram are, so that the demonstrations may be grasped more easily.

XXXV.

A PARALLELOGRAM is a quadrilateral figure whose two pairs of opposite sides are parallel, or equidistant.

Take a quadrilateral figure ABCD. If side AB is equidistant from (parallel to) side DC, and side AD from side BC, it is called a *parallelogram*.

Diagram. A parallelogram, its four vertices lettered.

- A (top-left vertex)
- B (top-right vertex)
- C (bottom-right vertex)
- D (bottom-left vertex)

There are only four parallelograms: the square [*quadratum*], the figure longer on one side (the oblong) [*figura altera parte longior*], the rhombus, and the rhomboid. The first two are called rectangular, because all their angles are right; the latter two are called non-rectangular, because none of their angles is right. That these four figures are parallelograms we shall show at Proposition 34 of this book (*Elements* I.34).

And so we can — as the ancient geometers did — divide quadrilateral figures into the parallelogram and the trapezium. The parallelogram in turn divides into: the rectangular and equilateral, such as the square; the neither rectangular nor

equilateral, such as the rhomboid; the rectangular but not equilateral, such as the oblong; and the equilateral but not rectangular, such as the rhombus. Of trapezia, likewise, one kind has two opposite sides parallel and the others not, while another kind has no opposite sides parallel. Moreover, the former kind either has the two non-parallel sides equal to one another — and is then called an *isosceles trapezium* — or unequal, in which case it is called a *scalene trapezium*. From all of this, then, seven kinds of quadrilateral figure can be constituted: the square, the oblong, the rhombus, the rhomboid, the isosceles trapezium, the scalene trapezium, and that irregular trapezium in which no sides are parallel.

XXXVI.

When in a parallelogram a diameter [diagonal, *diameter*] has been drawn, and two lines parallel to the sides cut the diameter at one and the same point, so that the parallelogram is divided by these parallels into four parallelograms, then the two through which the diameter does not pass are called the COMPLEMENTS, while the two remaining, through which the diameter runs, are said to lie ABOUT THE DIAMETER.

Let there be a parallelogram ABCD, in which AC is the diameter, and EF a line cutting the diameter at G and parallel to the sides AD and BC. Likewise let HI be a line cutting the diameter at the same point G and parallel to the sides AB and DC. Since this is so, it is evident that the whole parallelogram is divided into four parallelograms. Of these, the two EBIG and GFDH, through which the diameter AC does not pass, are called by geometers the *complements*, or supplements, of the remaining two, AEGH and GICF, which are said to lie about the diameter, since the diameter passes through them — as may be seen in the present figure.

Diagram. A parallelogram ABCD with its diameter AC drawn, and two internal lines EF and HI parallel to the sides, both cutting the diameter at the interior point G, dividing the figure into four smaller parallelograms.

- A (top-left vertex; one end of the diameter)
- E (midpoint of the top side; upper end of line EF)
- B (top-right vertex)
- H (midpoint of the left side; left end of line HI)
- G (interior intersection point, where diameter AC, line EF, and line HI meet)
- I (midpoint of the right side; right end of line HI)
- D (bottom-left vertex)

- F (midpoint of the bottom side; lower end of line EF)
- C (bottom-right vertex; other end of the diameter)
- Sub-parallelograms about the diameter (diameter passes through them): AEGH and GICF.
- Complements (diameter does not pass through them): EBIG and GFDH.

THE PETITIONS, OR POSTULATES [*PETITIONES, SIVE POSTVLATA*]

I.

Let it be granted that from any point to any point a straight line may be drawn.

This first postulate is quite plain, if what we wrote a little earlier about the line is rightly considered. For since a line is a kind of imaginary flow of a point, and a straight line is a flow advancing along an altogether direct path, it follows that, if we conceive some point to be moved directly toward another, a straight line is thereby drawn from point to point. This is just what Euclid postulates in this first petition — as you see here, that a straight line has been drawn from point A to point B; another from the same point to point C; likewise another to point D; and thus innumerable others can be led out from the same point to yet other points.

Diagram. A single point A from which three straight lines fan out to three separate points.

- A (common point at left)
- B (upper endpoint)
- C (middle endpoint)
- D (lower endpoint)

II.

And to produce a finite straight line continuously in a straight line.

If we further conceive that point to be carried along by a direct motion, one free of all deflection, the finite straight line will be produced, and there will never be an end to this production, since we can understand the point to be moved to an infinite distance. Thus the straight line AB is produced, first in a straight line to point C, then to point D, and so on.

Diagram. A straight segment AB continued in a straight line to further collinear points C and D.

- A (left endpoint)
- B (next point along the line)
- C (further point along the same line)
- D (endpoint at right)

III.

Likewise, to describe a circle with any center and radius [*intervallo*].

Now if we have conceived in the mind a finite straight line of any size whatever, applied by one of its ends to any point, and then turned about that fixed point until it returns to the place from which it began to move, a circle will have been described — the very thing the third postulate requires. You have an example in these five lines AB, AC, AD, AE, AF, each of which, revolving about the center A, has described its own circle according to its length, that is, its radius.

Diagram. Five concentric circles sharing the common center A; five radii of increasing length are drawn from A into the upper-right region, their endpoints lettered (innermost to outermost) B, C, D, E, F, each endpoint lying on its own circle.

- A (common center)
- B (endpoint of the shortest radius, on the innermost circle)
- C (endpoint of the next radius)
- D (endpoint of the next radius)
- E (endpoint of the next radius)
- F (endpoint of the longest radius, on the outermost circle)

Besides these three postulates, with which Euclid was content, there are many others

Equally easy, of which I have decided to bring forward only the one that will need to be repeated more often in the course of the whole of geometry; the rest the prudent reader will easily understand by himself.

IV.

Likewise, given any magnitude, that another magnitude can be taken, either greater or lesser.

For every continuous quantity can be increased infinitely by addition, and diminished infinitely by division. Hence there will never be given a continuous quantity so great that a greater cannot be given, nor one so small that a smaller cannot be exhibited. The same holds true of numbers as regards addition: for any number can be increased infinitely by the continual addition of a unit — although in its diminution one arrives at the indivisible unit.

COMMON NOTIONS, OR AXIOMS [*COMMVNES NOTIONES SIVE Axiomata*], which are also usually called Pronouncements [*Pronunciata*], or Dignities [*Dignitates*]

I.

Things equal to the same thing are also equal to one another. And what is greater or less than one of two equals is also greater or less than the other of the equals. And if one of two equals is greater or less than some magnitude, the other of the equals too is greater or less than the same magnitude.

It can in no way happen that two unequal quantities are equal to another single quantity. For if the lesser of them proved equal to the given quantity, the greater of them would necessarily exceed that same quantity; and if the greater were equal to the given quantity, the lesser would be surpassed by it. Hence it is rightly concluded that quantities which are equal to the same quantity are also equal to one another. The remaining parts of the axiom, which we have added, are perfectly clear, since they are in frequent use.

II.

And if equals are added to equals, the wholes are equal.

For if the aggregate, or composite, quantities were unequal, then without doubt more would have been added to the greater than to the lesser — whereas beforehand they were equal. Therefore, from the addition of equal quantities to equal quantities, quantities that are likewise equal will be produced.

III.

And if equals are subtracted from equals, the remainders are equal.

For if the remaining quantities were unequal, more would have been taken from the lesser than from the greater.

IV.

And if equals are added to unequals, the wholes are unequal. And if unequals are added to unequals — greater to greater, and less to less — the wholes are unequal, the former of course greater and the latter less.

Moreover, if unequals are added to equals, the wholes will be unequal: because the greater quantity, added to one of the equals, constitutes a greater quantity than the lesser added to the other of the equals; just as, if equals are added to unequals, the quantity composed from the greater is greater than that composed from the lesser. This second part of the axiom we ourselves have added, on account of its frequent use.

V.

And if equals are subtracted from unequals, the remainders are unequal. And if unequals are subtracted from unequals — the less from the greater, and the greater from the less — the remainders are unequal, the former of course greater and the latter less.

So too, if unequals are subtracted from equals, the remainders will be unequal: because the greater quantity, once taken away, will leave a smaller remaining quantity than the lesser does; just as the remainder of the greater is greater than the remainder of the lesser, if equals are taken from unequals. For the rest, Euclid does not teach what is produced, simply and absolutely, from adding unequal quantities to unequal quantities, nor what is left after the subtraction of unequal quantities from unequal quantities; the reason being that nothing certain can be concluded from it, except when a greater is added to a greater, or a lesser is taken from a greater, as was said in the second part of the axiom, which we added on account of its notable usefulness. For the composite or remaining quantities can be either unequal or equal. Thus, if to 7 and 5 are added 4 and 3, there result 11 and 8, which are unequal. So too, if from 7 and 5 are subtracted 2 and 1, there remain 5 and 4, which are unequal. But if to 7 and 5 are added 4 and 6, there result 11 and 11, which are equal. Likewise, if 3 and 1 are subtracted from 7 and 5, there remain 4 and 4, which also turn out equal.

Furthermore, in all these pronouncements — the first excepted — by the term "equal quantities" one must understand also one and the same quantity common to many. For if the same common thing is added to equals, the wholes will

become equal; and if the same common thing is taken from equals, the remainders will be equal. And if the same common thing is added to unequals, or unequals are added to the same common thing, the wholes will become unequal; and if the same common thing is taken from unequals, or unequals are taken from the same common thing, the remainders will turn out unequal.

VI.

And things that are doubles of the same thing are equal to one another. And what is the double of one of two equals is also the double of the other of the equals.

Likewise, things that are triples, quadruples, quintuples, and so on, of the same thing are equal to one another. For if they were unequal, and the greater of them were double (or triple, etc.) of some quantity, then the lesser would fall short of the double (or triple, etc.); whereas if, on the contrary, the lesser were the double (or triple, etc.) of some quantity, the greater would surely exceed that same double (or triple, etc.). This can also be proved from the second axiom, in this way. If two quantities are equal to some third, [a] and to each of them that third is added, the aggregates will be doubles of that third, [Margin: a — 2. pron. (the 2nd Pronouncement/Axiom)] but also equal to one another, because of the same addition. And if [b],

[Margin: b — 2. pron. (the 2nd Pronouncement/Axiom)] again, to the aggregates that same third is added, the sums will be triples of that same third. Since therefore they are also equal to one another, on account of the same addition, and since the same reasoning holds for the other multiples, the axiom proposed will be evident. The second part of this axiom we have appended, because its use is not rarely required in geometrical matters.

VII.

And things that are halves of the same thing are equal to one another. And conversely, things that are equal are halves of the same thing.

By the same reasoning, things that are the thirds, or fourths, or fifths, and so on, of the same thing are equal to one another. In these two pronouncements, by "the same quantity" one must understand quantities that are also equal. For things that are doubles, or triples, etc., of equals are also equal to one another; likewise things that are halves, or thirds, or fourths, etc., of equals are necessarily equal to one another as well. The second part of this axiom too we have added, because it is used by geometers no less frequently than the first.

VIII.

And things that coincide with one another are equal to one another.

That is: two quantities, one of which laid upon the other, neither exceeds the other, but both coincide with one another, will be equal. Thus two straight lines are said to be equal when, one laid upon the other, the one that is superposed coincides wholly with the other, so as neither to exceed it nor to be exceeded by it. So too two rectilinear angles will be equal when, one laid upon the other, the one superposed neither exceeds the other nor is exceeded by it, but the lines of the one wholly coincide with the lines of the other; for then the inclinations of the lines will be equal, even though the lines themselves may sometimes be unequal to one another.

Conversely: things that are equal to one another will coincide, if the one is laid upon the other. But it must be understood that quantities coinciding with one another are equal only in respect of that in which they coincide. Now length coincides only with length, surface with surface, solid with solid, the inclination of lines with the inclination of lines, and so on.

IX.

And the whole is greater than its part.

Since a part taken from the whole leaves something still remaining — lest the whole itself be taken away — it is evident that every whole is greater than its part.

In the following pronouncements the order of Euclid is interrupted, because we judged that two other axioms, quite necessary, ought to be inserted at this place, since from them Axiom 12 (which is Euclid's tenth) can be demonstrated, as we shall say there. In the margin, however, we have added the numbers corresponding to Euclid's order.

X.

Two straight lines do not have one and the same common segment.

[Margin: o.]

This axiom is not difficult, provided the nature of a straight line is perfectly understood. For since a straight line is produced along an ever-direct path, deflecting in no direction, it can in no way happen that two straight lines have

one part — however small — in common, except the single point at which they mutually intersect. Proclus [Proclus], however, demonstrates this briefly as follows. Let there be, if it be possible, two straight lines AB, AC having a common part AD. From the center D and with the radius DA, [a] let a circle be described cutting the two given lines at the points B and C. [b] Then the two circumferences AB and AC will be equal to one another (for they are the circumferences of equal semicircles, since ADB and ADC are supposed to be diameters) — a part and a whole, which is absurd. Therefore two straight lines do not have one and the same common segment. Which was to be proved.

[Margin: a — 3. petit. (the 3rd Postulate)] [Margin: b — 17. def. (the 17th Definition)]

Diagram. A circle whose center is D; a nearly vertical line runs from A at the top down through D to B and C at the bottom, illustrating the two supposed straight lines AB and AC sharing the common part AD, with the circle (radius DA) cutting them at B and C so that ADB and ADC are diameters of equal semicircles.

- A (top; endpoint above the circle)
- D (center of the circle)
- B (lower endpoint, at the foot of the figure)
- C (lower endpoint, beside B at the foot of the figure)

Yet two straight lines can have a common segment when they constitute one and the same straight line. Thus in this figure, the straight lines AD and BC have the common segment CD, because both together constitute the single straight line AB. But when two straight lines are distinct — as AB and AC were in the earlier example — they cannot possess any common segment, as was rightly demonstrated by Proclus.

Diagram. A single straight line from A to B, with two interior points C and D marked, illustrating that the straight lines AD and BC overlap in the common segment CD while together forming the one line AB.

- A (left endpoint of the line)
- C (interior point, left of center)
- D (interior point, right of center)
- B (right endpoint of the line)

XI.

Two straight lines meeting at one point will, if both are produced, necessarily intersect one another at that point.

[Margin: o.]

This axiom too depends on the nature of the straight line. We shall demonstrate it thus. Let two straight lines AB, CB meet at B. I say that, when produced, they cut one another at B — namely, that CB, when produced, falls at E, above the straight line AB produced. For if CB produced does not fall above AB produced, it will coincide with AB produced, so as to pass through D; and thus the two straight lines ABD, CBD will have the common segment BD, which was shown in the preceding axiom to be impossible. Or else it will certainly fall below AB produced, so that CB produced falls at F, and CBF is a single straight line. Therefore, with center B, let a circle ACFD be described with any radius, cutting the straight lines AB, CB produced at D and F. Since, then, each of the straight lines ABD, CBF is drawn through the center B, both ACD and CF will be a semicircle, by Definition 18; and hence the circumferences ACD and CF will be equal — while, as we demonstrated at Definition 17, they are a whole and a part. Which is absurd.

Diagram. A circle with center B; the straight line ABD runs roughly horizontally through the center from A (left) to D (right), while the line through C (lower left) and B is produced to E (upper right) and to F (lower right). The circle, labelled ACFD, passes through A, C, F, and D and cuts the produced lines at D and F.

- A (left; on the circle, one end of line ABD)
- B (center of the circle; the meeting point of the lines)
- C (lower left; on the circle, one end of the line through B)
- D (right; on the circle, where AB produced meets it)
- E (upper right; where CB produced would fall above AB)
- F (lower right; on the circle, where CB produced falls below AB)

The two axioms just given are not posited by Euclid; but since they are nevertheless necessary for the proofs of the other axioms, we have inserted them here. The three that follow, however, are Euclid's.

Euclid's *Elements*, Book I — Clavius's Commentary (Axioms continued; Prefatory notes to the Propositions)

Axiom XII

[Margin: *x.*]

Likewise, all right angles are equal to one another.

This axiom can be perfectly evident to anyone from Definition 10, by which a right angle is defined — because the inclination of the lines that constitute a right angle can be neither increased nor diminished, but is utterly unchangeable. For a right angle is produced by a perpendicular line, one that stands upon another straight line in such a way as to make the angles on both sides equal, inclining no more to one side than to the other. From this it follows that all right angles are equal to one another, since the inclination is always the same, even though the lines forming them may sometimes be unequal.

Proclus [Proclus], however, attempts to demonstrate this from Definition 10 in the following way. Let there be two right angles ABC and DEF, which I claim are equal to one another. For if it were possible for them to be unequal, let ABC be the greater. If, then, we imagine the point E applied to point B, and the straight line DE laid along the straight line AB, the straight line EF will fall between the straight lines AB and BC — namely along BG — because the angle DEF is supposed smaller than the angle ABC.[Margin: a — 2.pet. (Postulate 2)] Let CB be produced in a straight and continuous line as far as H. Then, since the angle ABC is a right angle,[Margin: b — 10.def. (Definition 10)] the adjacent angle ABH will also be equal to it, and therefore likewise a right angle — and so greater than the angle ABC as well. But when GB is produced in a straight and continuous line as far as I, [Margin: a — 2.pet. (Postulate 2)] the produced portion BI will fall below CB when the latter is produced, as was demonstrated in the preceding axiom. Therefore, since the angle ABG is supposed a right angle,[Margin: b — 10.def. (Definition 10)] the adjacent angle ABI will be made equal to it. Consequently the angle ABH will also be greater than the angle ABI — the part greater than the whole, which is absurd. Therefore the two proposed right angles are not unequal, but equal. Which is what was to be proved; and the same reasoning holds in the other cases.

Pappus [Pappus of Alexandria] rightly points out here that this axiom cannot be converted: for not every angle equal to a right angle is itself a right angle, since a

curvilinear angle too can be equal to a right angle, as we shall demonstrate in Book 5 — and yet such an angle is not called a right angle, because it is not rectilinear. Therefore only a rectilinear angle equal to a right angle will be termed a right angle; and all right angles will be equal to one another, without any exception.

Diagram. The figure for Axiom XII, showing two right angles side by side. On the left, a right angle at B is formed by the vertical line AB (A at top) and the horizontal line through B; the horizontal is extended leftward to H, a ray BG rises from B toward the upper right to G, and a line runs down and to the left from B to I. On the right, a right angle at E is formed by the vertical line DE (D at top) and the horizontal line CEF.

- A (top left): upper end of the vertical AB
- H (left): left end of the horizontal produced through B
- B (center): vertex of the first right angle
- G (upper middle): end of the ray BG rising from B
- I (lower left): end of the line produced downward from B
- C (bottom middle): point on the horizontal, left of E
- D (top right): upper end of the vertical DE
- E (bottom, right of C): vertex of the second right angle
- F (bottom right): right end of the horizontal EF

Axiom XIII

[Margin: xj.]

And if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then those two straight lines, produced to infinity, will meet one another on that side on which the angles are less than two right angles.

Thus, if a straight line EF falling on two straight lines AB and CD makes two interior angles on the same side, BEF and DFE, less than two right angles, Euclid [Euclid] holds that the lines will eventually converge at some single point, toward the side on which the two angles less than two right angles lie, as the appended example shows. The reason for this is clear: when two interior angles on the same side are equal to two right angles, the two straight lines can converge on neither side, but extend forever at an equal distance apart, as will be demonstrated at Proposition 28 of this book. Therefore, if the two interior angles on the same side are made less than two right angles, the space between the said

lines must necessarily be narrowed on that side and widened more and more on the other; and so they will eventually converge at some point.

But because this axiom tends to seem rather obscure to beginners — indeed it is excluded from the number of first principles by Geminus [Geminus] the geometer, by Proclus, and by others, on the ground that it does not readily win everyone's assent, especially since certain other lines are found whose intervening space, though it grows ever narrower (just as happens with the two straight lines AB and CD, as we shall demonstrate at Proposition 28 of this book), nevertheless never meet in a single point even if produced to infinity, as is clear from the conic elements of Apollonius of Perga [Apollonius of Perga] and from the conchoid line of Nicomedes [Nicomedes] — for this reason we defer its fuller explanation to the scholium on Proposition 28 of this book, where we shall demonstrate it geometrically after the view of Proclus, so that it may be assumed, firmly and beyond all doubt as most certain, for Proposition 29 of this book (where its use first begins to appear) and for the demonstrations of the other propositions. This, however, we shall demonstrate in a different way from Proclus — and indeed more geometrically — so that no room for doubt is left.

Diagram. The figure for the parallel axiom (Axiom XIII), showing a straight line A–B cut by a transversal, together with a second line, the interior angles on one side being drawn less than two right angles so that the lines converge. The engraving appears to draw the transversal at two positions; the labels legible in the cut are A (far left), a pair of E's at the top, B (upper right) and D (lower right), C (lower left), and a pair of F's at the bottom. The text of the demonstration works with the lines AB and CD, the transversal EF, and the interior angles BEF and DFE; the right-hand termini are the upper B (line AB) and the lower D (line CD).

- A (far left): end of the upper line
- E, E (top): transversal endpoint(s), drawn at two positions
- B (upper right): endpoint of the upper line AB
- C (lower left): end of the second line
- F, F (bottom): transversal endpoint(s), drawn at two positions
- D (lower right): endpoint of the second line CD

Axiom XIV

[Margin: xij.]

Two straight lines do not enclose a space.

This principle presents no difficulty at all. For if two straight lines meet on one side to form an angle, they must necessarily, when produced, be separated more and more on the other side, as is evident in the example set out. Therefore, in order that any rectilinear surface or space be enclosed on every side, a third line must be joined to the two straight lines. For only so is a triangular space produced — the first of the rectilinear figures.

Proclus, however, demonstrates this principle in the following way. Suppose it were possible for two straight lines ABC and ADC to enclose a surface ABCD, so that those two straight lines meet at two points, A and C. Then, with C taken as center, [Margin: a — 3.pet. (Postulate 3)] let a circle be described with radius CA, [Margin: b — 2.petit. (Postulate 2)] and let the straight lines ABC and ADC be produced in a straight and continuous line to the circumference, namely to the points E and F. Then, since the straight lines ACE and ACF pass through the center C, [Margin: c — 17.def. (Definition 17)] the semicircles AE and AF will be equal to one another, and therefore the arc AE too will be equal to the arc AF — the part equal to the whole, which cannot be. Therefore two straight lines do not enclose a space. Which is what was proposed.

Diagram. Two circles drawn side by side, each illustrating the same construction: a circle with center C and radius CA, in which the two lines from A (through B and through D) pass through the center C and reach the circumference at E and F. Both circles carry the same six labels.

- A (left): starting point, on the circumference at the left
- B (upper center): point on the line ABC, near the center
- C (center): center of the circle
- D (lower center): point on the line ADC, near the center
- F (right): point where line ADC produced meets the circumference
- E (lower right): point where line ABC produced meets the circumference

Now, an opponent might perhaps object that the straight lines ABC and ADC, when produced, meet again at some point of the circumference — say at E, or at F — and that consequently it does not follow that a part is equal to the whole. We shall then demonstrate this same axiom in the following way. Let those two lines meet again, if it be possible, at E. Taking any point F on the straight line ADC whatever, AF will be less than FE, since it is less than AFC — that is, than CHE, which is equal to AFC itself — and therefore much less than FE. A circle, therefore, described from F with radius FA, will cut the straight line FE at H, and likewise CGE at G. Since, then, AFH is a diameter of the circle, AIH will be a

semicircle, as we showed at Definition 17; but the portion AIG, which the straight line ABG cuts off, and in which the center F does not lie, is less than a semicircle, as we demonstrated at Definition 18. The arc AIG is therefore less than AIH — the whole less than the part, which is absurd.

And that the portion AIG is less than a semicircle we shall show as above. For when a perpendicular is drawn from the center F to the straight line ABG, and the portion AIG is revolved about the straight line ABG, the circumference AIG will fall within the circumference AKG, lest a part be greater than the whole, as we demonstrated above.

This axiom is also established from the definition of a straight line. For since a straight line is the shortest extension from one point to another, only a single line can be drawn from one point to another. Therefore, if ABC is a straight line, ADC will not be a straight line. This is also clear from Plato's [Plato] definition. For if ABC is a straight line, its middle points will screen its own extremities from view; but the middle points of the line ADC do not screen its extremities when the sight is directed along the straight line ABC. Therefore ADC is not a straight line.

Diagram. The figure for Proclus's alternative demonstration of Axiom XIV: a large outer circle (with A at the left and K at the bottom) and a second circle passing through I at the top, forming a lens- (vesica-) shaped region across the middle, with the horizontal diameter running from A on the left to the right, and the constructed points marked within. The peripheral labels resolve at native resolution (A, I, K, B, D, F, G, H); only the right vertex (best-read B, semantically E) remains uncertain.

- A (far left): left end of the horizontal, on the outer circle
- B (upper, inside): upper vertex of the lens region
- I (top): top point of the inner circle
- H (center right): constructed intersection near the center
- D (lower left, inside): point on the lower line
- F (lower center): center of the constructing circle
- G (center, below): intersection point
- B / E (far right): right vertex of the lens (best-read B, semantically E) [?]
- K (bottom): lowest point on the outer circle

[Transition — Clavius now adds axioms beyond Euclid's.]

To these axioms laid down by Euclid we shall add certain others, culled from other geometers — no less necessary for the demonstrations to come, both of the

problems and of the theorems, of Euclid as well as of the other mathematicians, than those that Euclid himself handed down to us.

Diagram. The vesica-and-circle figure of Proclus's demonstration of Axiom XIV, repeated at the head of the page (a re-impression of fig-p37-4) so that the continuing demonstration has its figure to hand. The same construction and the same labels appear: the outer circle with A at the left and K at the bottom, the inner circle through I at the top, and the lens-shaped middle region with the interior points B, D, F, G, H (and the right vertex). The labels resolve at native resolution (A, I, K, D, H, F, G); only the right vertex E remains uncertain.

- A (far left): left end of the horizontal, on the outer circle
- I (top): top of the inner circle
- D (upper/left, inside): constructed point
- H (center right): constructed point
- F (lower center): center of the constructing circle
- G (center, below): constructed point
- E (far right): right vertex of the lens [?]
- K (bottom): lowest point on the outer circle

Axiom XV

If unequals are added to equals, the excess of the wholes will be equal to the excess of the things added.

Proclus took this axiom, and the following one, from Pappus. So, to the equal quantities AB and CD let the unequal quantities BE and DF be added, and let BE be greater than DF. From BE let BG be cut off, equal to DF, so that GE is the excess by which the added quantity BE exceeds the added quantity DF. Since, then, to the equals AB and CD equal quantities BG and DF have been added, [Margin: a — 2.pron. (Common Notion 2)] the wholes AG and CF will be equal. Hence it is clear that the whole quantity AE exceeds the whole CF by the same excess GE by which the added magnitude DF is exceeded by the added magnitude BE. Which is what was proposed.

Diagram. The figure shared by Axioms XV and XVI: two parallel line-segments. The upper segment carries the points A, B, G, E (from left to right); the lower segment carries the points C, D, F.

- Upper segment: A (left), B, G, E (right)
- Lower segment: C (left), D, F (right)

Axiom XVI

If equals are joined to unequals, the excess of the wholes will be equal to the excess of those quantities that were unequal at the outset.

In the same figure, to the unequal quantities BE and DF let the equals AB and CD be added. From the greater, BE, let BG be cut off, equal to DF, so that GE is the excess by which the quantity BE exceeds the quantity DF. Since, then, to the equals BG and DF equal quantities AB and CD have been added, [Margin: a — 2.pron. (Common Notion 2)] the wholes AG and CF will be equal. Therefore the whole quantity AE will exceed the whole CF by the same excess GE by which the greater proposed quantity BE exceeds the smaller DF. Which is what was proposed.

Axiom XVII

If unequals are taken from equals, the excess of the remainders will be equal to the excess of the things taken away.

From the equals AB and CD let the unequals BE and DF be taken away. Let EG be the excess by which the quantity BE exceeds the quantity DF, so that BG is equal to DF. Since, then, from the equals AB and CD the equals BG and DF have been taken away, [Margin: b — 3.pron. (Common Notion 3)] the remainders AG and CF will be equal. It is therefore evident that the remainder AE is exceeded by the remainder CF by the same excess EG by which the removed magnitude BE exceeds the removed magnitude DF. Which is what was proposed.

Diagram. The figure for Axiom XVII: two parallel line-segments. The upper segment has endpoints A (left) and B (right), with the interior points E and G marked; the lower segment has endpoints C (left) and D (right), with the interior point F marked.

- Upper segment: A (left), E, G, B (right)
- Lower segment: C (left), F, D (right)

Axiom XVIII

If equals are taken from unequals, the excess of the remainders will be equal to the excess of the wholes.

From the unequals AB and CD let the equals AE and CF be taken away. Let BG be the excess by which the whole quantity AB exceeds the whole quantity CD, so that AG is equal to CD. Since, then, from the equals AG and CD the equals AE and CF

have been taken away, the remainders EG and FD will be equal. Hence the remainder EB will exceed the remainder FD by the same excess BG by which the whole quantity AB exceeds the whole quantity CD. Which is what was proposed.

Diagram. The figure for Axiom XVIII: two parallel line-segments. The upper segment has endpoints A (left) and B (right), with interior points E and G marked; the lower segment has endpoints C (left) and D (right), with interior point F marked.

- Upper segment: A (left), E, G, B (right)
- Lower segment: C (left), F, D (right)

[Clavius's general remark on Axioms XV–XVIII.] In these four axioms just set down, by the term "equal quantities" one may also understand a single quantity common to many. For if unequals are added to the same common quantity, the excess of the wholes will be equal to the excess of the things added. And if the same common quantity is joined to unequals, the excess of the wholes will be equal to the excess of those quantities that were unequal at the outset. And if unequals are taken from the same common quantity, the excess of the remainders will be equal to the excess of the things removed. And if the same common quantity is taken from unequals, the excess of the remainders will be equal to the excess of the wholes.

For with numbers: if to 6 you add 5 and 3, you get 11 and 9, whose excess is 2 — the same as the excess of 5 and 3 themselves. Again, if to 5 and 3 you add 6, you get 11 and 9, whose excess is 2 — the same as that of 5 and 3. Likewise, if from 8 you take 5 and 2, there remain 3 and 6, whose excess is 3 — the same as that of 5 and 2. Finally, if from 10 and 7 you take 3, there remain 7 and 4, whose excess is 3 — the same as that of 10 and 7.

Axiom XIX

Every whole is equal to all its parts taken together at once.

Since all the parts taken together at once constitute the whole of which they are parts, the truth of this axiom is manifest.

Axiom XX

If a whole is the double of a whole, and the part removed the double of the part removed, then the remainder too will be the double of the remainder.

For example: the whole number 20 is the double of the whole number 10; and the removed 6 is the double of the removed 3; therefore the remainder of the former, 14, is likewise the double of the remainder of the latter, 7. In general, however, this will be demonstrated at Proposition 5 of Book 5 — namely: if one magnitude is the same multiple of another as the removed part is of the removed part (as tenfold, or a hundredfold, and so on), then the remainder too will be the same multiple of the remainder as the whole is of the whole.

[Clavius on the distinction between postulates, axioms, problems, and theorems.]

From what has been said, one may gather, with Proclus and Geminus, this distinction between postulates and axioms: although both are self-evident and indemonstrable, the former have the character of problems, in that they demand that something be done, whereas the latter resemble theorems, since they ask that nothing be done but only propose some most evident statement. A postulate differs from a problem in that the construction of a postulate needs no demonstration, whereas no one would grant the construction of a problem without a demonstration, since a problem sets before us something difficult to construct. The same distinction is found between an axiom and a theorem: an axiom ought not to be demonstrated, whereas a theorem must by no means be granted unless it is demonstrated. For no one will require a demonstration, or even a proof, of this proposition: *Things equal to the same thing are also equal to one another*. But of this one someone will at once desire a demonstration: *The three interior angles of every triangle are equal to two right angles*. Hold the same judgment concerning the remaining axioms and theorems, and also the postulates and problems.

It is likewise established that some of the postulates are proper to geometry — such as those three that Euclid set before us — while others are common both to geometry and to arithmetic, of which sort is this: *that a quantity can be increased infinitely*. For both number and magnitude can be increased by addition, so that no end of this increase is ever found. You will say the same of the axioms, or common notions. For the eighth, tenth, eleventh, twelfth, thirteenth, and fourteenth belong to geometry alone, whereas all the rest are applied both to geometric and to arithmetic demonstrations. For just as equal magnitudes taken away from equal magnitudes leave equal magnitudes — whether these magnitudes be lines, or surfaces, or solids — so too equal numbers subtracted from equal numbers leave equal numbers, and so on.

Let this be what we have said about this threefold kind of first principles. Let us now proceed to the demonstrations, from which the nature of all the principles

will be more fully and perfectly perceived. For there are very many principles of the mathematicians of such a kind that they are not clearly understood at all unless their use first appears in demonstrations — as experience itself will sufficiently teach you.

Before we proceed, moreover, to interpreting the propositions of Euclid, a few words must be said about the order and method we have followed in the demonstrations themselves. First, to each proposition we have attached two numbers. One, depicted in the margin, indicates the order that Campanus [Campanus of Novara], following the tradition of the Arabs, observed in Euclid's propositions; the other, written in the series of the propositions itself, reports the arrangement of the propositions from the tradition of Theon [Theon of Alexandria], which we still see observed in the Greek codices. We did this for the following reason: since the propositions of Euclid are cited by some geometers according to the order of Campanus, but by others according to the series of Theon — and since these two interpreters very often differ from each other in the series and order of the propositions, which is especially evident in Books 6, 7, and 10 — we judged it necessary to add the number of each interpreter. For thus it will come about that, whenever a proposition-number cited by some geometer does not correspond to one interpreter, it may at least agree with the other. Then, so that the course of the demonstrations should not be interrupted, we have cited the principles and propositions of Euclid by prefixing in the margin to each citation some little letter of the alphabet, or some other sign, to which a similar little letter or sign corresponds in the demonstration, so that it may be more easily recognized to what place each citation is to be referred. The citations, then, are to be understood in the following way.

Citation	Meaning
1. def.	The first definition; and so of the other numbers, as 4. def., 23. def., etc.
1. pet.	The first petition, or first Postulate.
1. pron.	The first pronouncement, or axiom; and so of the remaining numbers, as before.
1. primi.	The first proposition of the first book.
23. Undec.	The twenty-third proposition of the eleventh book.
6. tertijd.	The sixth of the thirteenth book.
9. sextid.	The ninth of the sixteenth book.
13. duod.	The thirteenth of the twelfth book.

Citation	Meaning
7. quind.	The seventh of the fifteenth book.
5. quartid.	The fifth of the fourteenth book, etc.

From these, the other citations may easily be understood by anyone. For the reasoning is the same in all.

PROPLEMA I. PROPOSITIO I.

Enunciation (Euclid). On a given finite straight line to construct an equilateral triangle.

In every problem there are two things above all to consider: the construction of what is proposed, and the demonstration by which it is shown that the construction has been rightly carried out. For example, since this first problem directs us to construct an equilateral triangle on any given finite straight line, in such a way that the given straight line is one side of the triangle — for a figure is said to be constructed *on* a straight line when that line becomes one side of the figure — we must therefore first construct some triangle out of the granted principles, and then demonstrate that the triangle so constructed is equilateral, that is, that it has all three sides equal to one another. The same may be seen in the other problems. These two elements are also found in nearly every theorem. For very often, in order to demonstrate what is proposed, something must first be constructed and carried out, as will be clear in what follows. Very few indeed are the theorems that require no construction.

Let there then be proposed a finite straight line AB, upon which we are directed to construct an equilateral triangle. With center A and radius the line AB, let a circle CBD be described [Margin: a — Post. 3]; likewise with center B and radius the same line BA, let another circle CAD be described, cutting the first at the points C and D. From either of these, say from C, let two straight lines CA, CB be drawn to the points A and B [Margin: a — Post. 1]; and there will have been constructed on the straight line AB the triangle ABC — that is, a rectilinear figure contained by three straight lines. I say that this triangle, so constructed, is necessarily equilateral. Since the straight lines AB, AC are drawn from the center A to the circumference of the circle CBD, the line AC will be equal to the line AB [Margin: b — Def. 15]; again, because the lines BC, BA are drawn from the center B to the circumference of the circle CAD, the line BC will be equal to the line BA. Both AC and BC, therefore, are equal to the line AB. And so AC and BC will be equal to one another [Margin: c — Axiom 1], and for that reason the triangle ABC

will be equilateral. Therefore on a given finite straight line, and so on. Which was to be done.

Diagram. The construction of Proposition 1. Two equal intersecting circles are drawn on the base line AB; the equilateral triangle ABC stands on that base, with a further point D marking the lower intersection of the circles.

- A (left end of base): center of circle CBD; left vertex of the triangle.
- B (right end of base): center of circle CAD; right vertex of the triangle.
- C (upper intersection of the two circles): apex of the equilateral triangle.
- D (lower intersection of the two circles).

SCHOLIUM

That you may see several demonstrations contained in a single proposition, I have chosen to resolve this first proposition into its own first principles, beginning from the last syllogism of the demonstration. If, then, anyone wishes to prove that the triangle ABC, constructed by the method described above, is equilateral, he will use this demonstrative syllogism:

Every triangle having three equal sides is equilateral [Margin: d — Def. 23]. The triangle ABC has three equal sides. Therefore the triangle ABC is equilateral.

He will confirm the minor premise by this further syllogism:

Things equal to the same thing are also equal to one another [Margin: e — Axiom 1]. The two sides AC, BC are equal to the same side AB. Therefore the two sides AC, BC are also equal to one another. And for that reason all three sides AB, BC, AC turn out to be equal.

The minor premise of this syllogism he will establish in this way:

Straight lines drawn from the center to the circumference of a circle are equal to one another [Margin: f — Def. 15]. The lines AB, AC are drawn from the center A to the circumference CBD. Therefore the lines AB, AC are equal to one another.

By the same reasoning the lines AB, BC will be equal, since they are drawn from the center B to the circumference CAD. And so the minor premise of the preceding syllogism will be wholly confirmed.

In no other way can all the remaining propositions be resolved — not only those of Euclid, but also those of the other mathematicians. Yet mathematicians neglect this resolution in their demonstrations, because they can demonstrate what is

proposed more briefly and easily without it, as may be seen clearly enough from the demonstration above.

But if anyone should wish to construct on a given straight line an isosceles or a scalene triangle as well, he will accomplish it, following Proclus, in this way. Let there be a straight line AB, about which, from the centers A and B, let two circles be described as before [Margin: a — Post. 3]. Then let AB be produced in both directions as far as the circumferences, up to the points C and D. And with center A and radius AD, let a circle EDF be described [Margin: b — Post. 3]; likewise with center B and radius BC, a circle ECF, cutting the first at the points E and F. From either of them, say from E, let two straight lines EA, EB be drawn to the points A and B [Margin: c — Post. 1]. There will have been constructed on the line AB the triangle ABE [Margin: d — Def. 20], which I say is isosceles — namely, that its two sides AE, BE are both equal to one another and greater than the side AB. For since the lines AE, AD are drawn from the center A to the circumference EDF, AE will be equal to the line AD [Margin: e — Def. 15]; likewise, since the lines BE, BC are drawn from the center B to the circumference ECF, BE will be equal to the line BC [Margin: f — Def. 15]. But the lines AD, BC are equal to one another (for each of AC and BD is equal to the line AB, since AB, AC are drawn from the same center A to the circumference, and BA, BD likewise go out from the same center B to the circumference [Margin: g — Axiom 1]. Therefore AC, BD will be equal to one another. And so, when the common line AB is added, the whole AD will be equal to the whole BC [Margin: h — Axiom 2]). Therefore AE, BE will also be equal to one another [Margin: i — Axiom 1]. That each of AE, BE is greater than AB is plain, since AD, shown to be equal to AE, is greater than AB [Margin: a — Axiom 9]; likewise BC, demonstrated to be equal to BE, is also greater than AB [Margin: b — Axiom 9]. There has therefore been constructed on the line AB the isosceles triangle ABE, having its two sides AE, BE equal to one another and greater than the side AB — which was to be done. And this is the demonstration of Proclus and the other interpreters of Euclid.

It seems to me, however, that the triangle ABE can be shown to be isosceles more briefly, in this way. Since AE is equal to the line AD, and the line AD is double the line AB — because BA, BD are equal to one another [Margin: c — Def. 15] — AE too will be double the line AB. Again, since BE is equal to the line BC [Margin: d — Def. 15], and BC is double AB — because AB, AC are equal to one another [Margin: e — Def. 15] — BE too will be double AB. Since, then, each of AE, BE is double the same line AB, AE and BE will be equal to one another [Margin: f — Axiom 6], and greater, on that account, than the line AB. Therefore the triangle ABE is isosceles.

Now if from the point A a straight line AG is drawn to the circumference EGF [Margin: g — Post. 1], one that is not the same as AE or AD, cutting the circumference EHD at the point H, and if from G a line GB is drawn to B [Margin: h — Post. 1], there will have been constructed on the line AB the triangle ABG [Margin: i — Def. 20], which I say is scalene. For AG is greater than AH [Margin: k — Axiom 9]; but AH, AE, drawn from the center A, are equal to one another [Margin: l — Def. 15]; therefore AG will be greater than AE, that is, than BE, which was shown to be equal to AE. And so AG will also be greater than BG [Margin: m — Def. 15], since BG is equal to BE. But BG is also greater than AB, because the whole BC, equal to BG, is greater than AB, its part [Margin: n — Axiom 9]. All three sides of the triangle ABG, therefore, are unequal, and so it is scalene by definition — which was to be done.

We shall also show more briefly that the triangle AGB is scalene, in this way. Since AH, AD, drawn from the center A, are equal, and BG, BC, drawn from the center B, are likewise equal [Margin: o — Def. 15]; and since AD, BC are each double AB, while BD, AC are equal; then AH, BG will also each be double AB, and therefore greater than AB. Since, then, AG is greater than AH, or than BG [Margin: p — Axiom 9], the triangle AGB will be scalene, having AG for its greatest side, BG for its middle side, and AB for its least.

Diagram. The isosceles/scalene construction of the scholium. On the horizontal diameter, the given line AB is produced both ways to C (left) and D (right); circles centered at A (radius AD) and B (radius BC) intersect at E (above) and F (below). The isosceles triangle is ABE; the scalene construction adds the line AG, cutting the inner arc at H. Dotted arcs mark the two constructing circles.

- C (left end of the produced diameter).
- A (left center on the diameter): center of circle EDF.
- B (right center on the diameter): center of circle ECF.
- D (right end of the produced diameter).
- E (upper intersection of the two circles): apex of the isosceles triangle ABE.
- F (lower intersection of the two circles).
- H (upper region, where AG cuts the inner arc EHD).
- G (upper right, endpoint of AG on the circumference EGF).

LIBER I.

(The scholium of Proposition 1 concludes; the marginal citation keys for the passage above continue here on p. 29: g — Axiom 1; h — Axiom 2; i — Axiom 1; a — Axiom 9; b — Axiom 9; c–e — Def. 15; f — Axiom 6; g — Post. 1; h — Post. 1; i — Def. 20; k — Axiom 9; l–m — Def. 15; n — Axiom 9; o — Def. 15; p — Axiom 9.)

PRAXIS

We shall try, in nearly every one of Euclid's problems, to hand down a certain easy and brief practical method by which one may accomplish what Euclid strives to construct with many words and lines. And we shall observe this especially in those cases that are in more frequent use among mathematicians, and in which the practical method seems to bring with it some shortcut.

Thus an equilateral triangle will be constructed easily on the given line AB in this way. From the centers A and B, with radius the given line AB, let two arcs of circles be described, cutting one another at the point C, whether this falls below the line or above it. After this, let two straight lines AC, BC be drawn from the point C to the points A and B; and what is proposed will have been done. The demonstration of this is the same as the one above, provided only the circles were whole and complete — for they would necessarily pass through the points A and B.

Diagram. The practical equilateral construction: two short arcs cross at C above the base AB, forming the triangle ABC.

- A (left end of base).
- B (right end of base).
- C (apex, where the two arcs cross).

An isosceles triangle will be made in this way. From the centers A and B, with a radius greater than AB (if we wish the given line to be the shorter side), or smaller (if we choose the same line to be the longer side), let two arcs be described cutting one another at C. Then let the lines AC, BC be drawn; and an isosceles triangle will have been constructed, since AC, BC will be equal because of the equal radius assumed — greater, that is, or less, than the line AB.

Diagram. Two practical isosceles constructions side by side, each with apex C above a base AB, the crossing arcs shown at the apex. In the left figure the base AB is short relative to the equal sides; in the right figure the base is longer.

- A, B (ends of the base) in each figure.
- C (apex, crossing of the arcs) in each figure.

Finally a scalene triangle will be fashioned in this way on the given line AB. From the center B, with a radius greater than BA, let some arc be described. Likewise from the center A, with a radius still greater than the one assumed before, let another arc be described cutting the former at C. Then let the lines AC, BC be drawn; and a scalene triangle will have been constructed, as is evident from the inequality of the radii that were assumed in the construction.

Diagram. The practical scalene construction: a triangle with apex C above the base AB, the right side (toward B) drawn markedly steeper, the arcs crossing at C.

- A (left end of base).
- B (right end of base).
- C (apex).

For the rest, in what manner a triangle is to be constructed having three sides equal to any three given lines whatsoever, each to each, we shall explain more fully at Proposition 22 of this book.

PROBLEMA 2. PROPOSITIO 2.

Enunciation (Euclid). At a given point to place a straight line equal to a given straight line.

Let there be a given point A, and a given straight line BC, to which another straight line, equal to it, is to be placed at the point A. Taking either end of the line BC, say C, as center, let a circle BE be described with radius the line BC [Margin: a — Post. 3]. And from A to the center C let a straight line AC be drawn [Margin: b — Post. 1] (unless the point A should lie within the line BC: for then AC itself will be taken as the drawn line, as the second figure indicates). On the line AC let an equilateral triangle ACD be constructed [Margin: c — Prop. 1 of Book I], turned upward or downward as one pleases; and let its two sides just constructed, DA and DC, be extended toward the line AC [Margin: d — Post. 2] — DC on the side opposite the given point A, as far as the circumference, at E; and DA on the side opposite the center C, as far as one pleases, at F. Then, with center D and radius the line DE (which passes through the center C), let another circle EG be described [Margin: e — Post. 3], cutting the line DF at G. I say that the line AG, which has been placed at the given point A, is equal to the given line BC. Since DE, DG are drawn from the center D to the circumference EG, they will be equal to

one another [Margin: f — Def. 15]. Therefore, when the equal sides DA, DC of the equilateral triangle ACD are taken away, AG will remain equal to the line CE [Margin: g — Axiom 3]. But the line BC is equal to that same CE [Margin: a — Def. 15] (since both lines CB, CE fall from the center C to the circumference BE). Therefore the lines AG, BC — seeing that each has been shown equal to the line CE — will be equal to one another [Margin: b — Axiom 1]. Therefore at the given point, and so on. Which was to be done.

But if the given point should be at the end of the given line, such as C, the problem will be solved easily. For if, with center C and radius CB, a circle is described [Margin: c — Post. 3], to whose circumference a straight line CE is drawn in any direction [Margin: d — Post. 1], this line, placed at the given point C, will be equal to the given line BC [Margin: e — Def. 15], since both BC and CE go out from the same center C to the circumference BE.

Diagram, first case. The point A lies outside the line BC. A circle is drawn about center C through B; the equilateral triangle ACD is raised on AC; DC is produced to E on that circle and DA to F; the second circle about D (radius DE) cuts DF at G, so that AG equals BC.

- B (left, on the first circle).
- C (center of the first circle, on the base).
- D (top, apex of the equilateral triangle ACD).
- A (given point, near the top center).
- E (lower, where DC meets the first circle's circumference).
- G (right, where the second circle cuts the produced line DF).
- F (below G, endpoint of the produced side DA).

Diagram, second case. The point A lies within the line BC, so AC itself serves as the drawn line; the same lettering is used.

- B (left, on the first circle).
- A (center, the given point within BC).
- C (right, center of the first circle).
- D (top, apex of triangle ACD).
- E (lower right, on the circumference).
- G (bottom, where the second circle cuts the produced line).
- F (bottom, endpoint of the produced side).

EVCLID. GEOM.

SCHOLIUM

This problem can have various cases, as Proclus says. For either the given point is set on the given line itself, or outside it. If on the line, it will be either one of its ends, or it will lie between the two ends. If, however, outside the line, it will be either in a straight line with the given line — so that the line, produced straight and continuous, passes through the point itself — or not in a straight line, so that a line drawn from the point makes an angle with the given line; in which case it will be constructed either above the given line or below, as is plain. But in all these cases the construction and demonstration are always the same. And if in the construction an isosceles triangle ACD is made on the line AC, we shall show in the same way that the line AG is equal to the line BC.

PROBL. 3. PROPOS. 3.

Enunciation (Euclid). From two given unequal straight lines, to cut off from the greater a straight line equal to the less.

Let there be two unequal straight lines: A, the lesser, and BC, the greater; and let it be required to cut off from the greater, BC, a line equal to the lesser, A. At either end of the greater line BC, say at the point B, let some line be placed which is BD, equal to the lesser A [Margin: a — Prop. 2 of Book I]. Then with center B and radius BD let a circle be described [Margin: b — Post. 3], cutting BC at E. I say that BE, so cut off, is equal to A. Since BE is equal to the line BD [Margin: c — Def. 15], and the line A is equal to that same BD by the construction, A and BE will be equal to one another [Margin: d — Axiom 1]. Therefore from two given straight lines, and so on. Which was to be done.

But if the two given lines are joined at one end — as are BD and BC, joined at their common end B — a circle must be described from B, with radius the lesser line BD. For this will cut off BE, equal to BD, as is evident from the definition of a circle.

Diagram. The lesser line A is drawn separately above. A circle is drawn about center B with radius BD (= A); its diameter runs from C on the left through E to B, with D on the upper circumference, so that BE is the segment cut off from BC equal to A.

- A (above the figure): the given lesser line.

- D (upper left, on the circumference).
- C (far left, end of the diameter through the circle).
- E (lower left, where the circle cuts BC).
- B (center of the circle).

SCHOLIUM

That there can also be various cases in this problem, no one is unaware — since the two given unequal lines may either stand apart from one another, so that neither touches the other; or not, but either be joined at one end, or cut one another mutually, or at least the one touch the other with its end only, and so on. On this matter, read Proclus in this place.

THEOREMA 1. PROPOS. 4.

Enunciation (Euclid). If two triangles have two sides equal to two sides respectively, each to each, and have the angle contained by the equal straight lines equal to the angle: then they will also have the base equal to the base; the triangle will be equal to the triangle; and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

Let there be two triangles ABC, DEF, and let each side AB, AC of the one be equal to each side DE, DF of the other — that is, AB to DE, and AC to DF — and let the angle A, contained by the sides AB, AC, be equal to the angle D, contained by the sides DE, DF. I say that the base BC is also equal to the base EF; that the triangle ABC is equal to the triangle DEF; and that each angle B and C is equal to each angle E and F respectively — that is, the angles B and E, which are opposite the equal sides AC, DF, are equal to one another, and the angles C and F, which are opposite the equal sides AB, DE, are also equal to one another. For since the line AB is set equal to the line DE, it follows that, if the one is understood to be laid upon the other, with the point A placed on the point D, they will exactly fit one another [Margin: a — Axiom 8], and the point B will fall on the point E. For no one will be able to say that part of the line AB fits the line DE and part does not, since then two straight lines would have a common segment [Margin: b — Axiom 10], which is impossible. And if anyone should say that, the point A being placed on D, the point B does indeed fall on E, but the line AB falls either to the right or to the left of DE, then two straight lines would enclose a surface [Margin: c — Axiom 14], which cannot happen. Therefore the line AB will fit the line DE, as has been said. Since, then, the angle A is set equal to the angle D, the one will also fit the other [Margin: d — Axiom 8] — that is, the line AC will fit the line DF, and the

point C will fall on the point F, because of the equality of the lines AC, DF. Therefore the base BC will also fit the base EF; for otherwise, if it fell above or below, so that the line EGF or EHF were produced, two lines EF, EGF, or EF, EHF, would enclose a surface (for no one will be able to deny that both EGF and EHF are straight, since each is supposed to be the same as the straight line BC) — which is absurd. For two straight lines cannot enclose a surface. Wherefore the base BC will be equal to the base EF [Margin: f — Axiom 8], since neither exceeds the other; and the triangle ABC will be equal to the triangle DEF; and the angle B to the angle E; and the angle C to the angle F, for the same reason. Therefore, if two triangles have two sides equal to two sides, and so on. Which was to be demonstrated.

(Marginal citation keys for this demonstration: a — Axiom 8; b — Axiom 10; c — Axiom 14; d — Axiom 8; e — Axiom 14; f — Axiom 8.)

Diagram. Two triangles for Proposition 4. The left triangle ABC has apex A over base BC. The right triangle DEF has apex D over base EF; the additional points G and H mark the two impossible positions of a would-be base (EGF above, EHF below) used in the reductio.

- A (apex of the left triangle).
- B (lower left of the left triangle).
- C (lower right of the left triangle).
- D (apex of the right triangle).
- E (lower left of the right triangle).
- F (lower right of the right triangle).
- G (within the right figure, above the base EF).
- H (within the right figure, below the base EF).

SCHOLIUM

Euclid rightly laid down two conditions in the antecedent of this theorem. The first is that the two sides of the one triangle be equal to the two sides of the other, each to each; the second, that the angle of the one contained by those sides be equal to the angle of the other contained by the sides which are equal to them. For if either of these conditions is lacking, neither the bases nor the remaining angles will ever be able to be equal, as is well demonstrated in this place by Proclus. The triangles themselves, however, may indeed be equal when only the latter condition is lacking, as will be clear from the scholium to Proposition 37 of this book — though that happens only very rarely. For let the angles A and D of

the triangles ABC, DEF be equal, namely right angles, and let the sides AB, AC be equal to the sides DE, DF — not indeed each to each, but the two taken together to these two taken together. Let AB be 3 and AC be 4, so that both together make 7; while DE is 2 and DF is 5, so that both together likewise make 7. On these suppositions, the base BC will be 5, and the base EF will be the square root of the number 29, which is indeed greater than 5 but less than 6. Likewise the area of the triangle ABC will be 6, but the area of the triangle DEF will be 5. Finally the angles on the base BC will be unequal to the angles on the base EF. That all this is so we would show here, were it not that its demonstration requires many things not yet established. You see, therefore, that all these quantities are unequal, because it is not the case that each side is equal to each side in the said triangles ABC, DEF.

Diagram. Two right triangles illustrating the numerical counter-example of the scholium. The left triangle ABC has legs 3 and 4, base 5, and area 6; the right triangle DEF has legs 2 and 5, base the square root of 29, and area 5.

- A (apex of the left triangle); "3" on side AB, "4" on side AC, "6" (the area) marked within.
- B (lower left); "5" on the base BC.
- C (lower right).
- D (apex of the right triangle); "2" on side DE, "5" on side DF.
- E (lower left); " $\sqrt{29}$ " (radix quadrata of 29) marked on the base EF.
- F (lower right).

[The page opens in the middle of Clavius's scholium on Proposition 4 (continued from the previous page).]

Again, let triangles ABC and DEF have their sides AB and AC equal to the sides DE and DF, each to each, and let each of them be 5; but let the angles A and D contained by those sides be unequal, with A greater than D. On these assumptions the base BC will be greater than the base EF, as will be shown at proposition 24 of this book [Elements I.24]. Now suppose we take the base BC to be 8 and the base EF to be 4. Then the area of triangle ABC will be 12, while the area of triangle DEF will be the square root of the number 84 — which is indeed greater than 9 but less than 10, a fact perfectly familiar to geometers. Therefore, in order that two triangles may have both their bases and their angles equal, and the triangles themselves equal to one another, it is necessary that each side of the one be equal, respectively, to each side of the other, and that the angles contained by those sides likewise be equal — exactly as Euclid rightly stated.

Diagram. Two triangles drawn side by side, illustrating that equal sides with unequal included angles yield unequal bases and unequal areas.

- Left triangle ABC: **A** (apex); **B** (lower-left base vertex); **C** (lower-right base vertex). The two sides AB and AC are each marked 5 (with tick-marks); the base BC is marked 8; the area **12** is written inside.
- Right triangle DEF: **D** (apex); **E** (lower-left base vertex); **F** (lower-right base vertex). The two sides are each marked 5; the base EF is marked 4; the area is written inside as $\sqrt{84}$ (the square root of 84).

THEOR. 2. PROPOS. 5.

[Margin: v — proposition 5]

[Proposition — Euclid's enunciation] In isosceles triangles the angles at the base are equal to one another; and, if the equal straight lines are produced, the angles under the base will also be equal to one another.

[Clavius's demonstration] Let ABC be an isosceles triangle in which the sides AB and AC are equal to one another. I say that the angles ABC and ACB above the base BC are equal to one another; and likewise, if the equal sides AB and AC are produced as far as one pleases, up to the points D and E, that the angles DBC and ECB below the same base BC are also equal.

For, on the line AE produced indefinitely, let AF be cut off equal to AD [Margin: a — Elements I.3], and let the straight lines BF and CD be drawn [Margin: b — Postulate 1]. Then consider the two triangles ABF and ACD. Since the two sides AB and AF of triangle ABF are equal to the two sides AC and AD of triangle ACD, each to each — namely AB to AC by hypothesis, and AF to AD by construction — and since the angle A contained by the sides AB and AF is equal to the angle A contained by the sides AC and AD (indeed, angle A is common to both triangles), the base BF will be equal to the base CD, the angle F to the angle D, and the angle ABF to the angle ACD [Margin: c — Elements I.4]; for both the former pair and the latter pair stand opposite equal sides in those triangles, as is evident.

Now consider the two triangles BDC and CFB. Since the straight lines AD and AF are equal by construction, it follows that, if the equals AB and AC are taken away from them, the remainders BD and CF are equal [Margin: d — Common Notion 3]. Therefore the two sides BD and DC of triangle BDC are equal to the two sides CF and FB of triangle CFB, each to each — namely BD to CF, and DC to FB, as has been proved. Moreover, the angles D and F contained by those sides are equal, as was also shown. Therefore the angle DBC will be equal to the angle FCB, and the

angle BCD to the angle CBF [Margin: e — Elements I.4]; for both the former pair and the latter stand opposite equal sides, and they lie above the common base BC of each triangle, BDC and CFB.

Now if from the whole equal angles ABF and ACD (which we have already proved equal in the former triangles) the equal angles CBF and BCD are subtracted (which we have likewise just proved equal in the latter triangles), there will remain the angles ABC and ACB above the base BC [Margin: f — Common Notion 3], equal to one another — while the angles DBC and FCB, which lie below the same base BC, are also equal. Therefore both the angles above the base are equal to one another, and the angles below it are equal to one another; and consequently, in isosceles triangles the angles at the base, and so forth. Which was to be demonstrated.

Diagram. The figure for Proposition 5 (the "bridge of asses"). An isosceles triangle ABC with apex A and base vertices B and C; the equal sides are produced below the base to D and E, a point F is taken on AE with $AF = AD$, and the cross-lines BF and CD are drawn.

- A (apex).
- B (left base vertex); C (right base vertex) [glyph of C faint on the leaf, but required by the construction].
- D (lower-left, on AB produced); F (lower-middle, on AE with $AF = AD$); E (lower-right, on AC produced).
- The lines BF and CD cross between the base and the produced ends.

SCHOLIUM.

This proposition is true also of equilateral triangles, since in any equilateral triangle two sides equal to one another can be found — even though Euclid seems to have accommodated it only to isosceles triangles. For, given two sides AB and AC of triangle ABC equal, then — whether the remaining side BC is also equal to them, as happens in an equilateral triangle, or unequal, as occurs in an isosceles one — it necessarily follows both that the angles above the base are equal to one another and that the angles below it are likewise equal, as is clear from the demonstration just given. This theorem is apt to seem rather difficult and somewhat obscure to beginners, on account of the multitude of lines and angles to which they are not yet accustomed. Nevertheless, if the force and demonstration of the preceding theorem are weighed carefully, the present one will be grasped by anyone without much labor — provided only that he remembers that the angles of the triangles were proved equal in the preceding

theorem because they stand opposite equal sides. And the very fact that Campanus [Campanus of Novara] did not supply this is the reason why his demonstration appears confused and rather obscure.

Moreover, the truth of this theorem, in both its parts, can also easily be demonstrated by superposition, just as proposition 4 was demonstrated. For let there again be, in triangle ABC, two equal sides AB and AC, produced as far as one pleases up to D and E. I say both that the angles ABC and ACB above the base BC are equal to one another, and that the angles DBC and ECB below the same base are equal. For if we conceive the triangle ABC in our mind as laid upon the triangle ACB (so that the same triangle serves as though it were two), in such a way that the straight line AB is laid upon AC, the point B will fall on C because of the equality of the sides AB and AC. This being so, the line AC will fall upon the line AB because of the equality — or rather the identity — of the angle A; and the point C will fall on the point B[?] [the printed letter is unclear; the geometry requires B], because of the equality of the sides AC and AB. Therefore the angle ABC will coincide with the angle ACB, and the angle DBC with the angle ECB [Margin: a — Common Notion 8]; and consequently both the former and the latter will be equal to one another.

Diagram. The superposition figure for the alternative proof. A triangle with apex A and base D (lower-left) and E (lower-right); B and C mark the points partway down the two sides (B on AD, C on AE), joined by the segment BC drawn across the figure.

- A (apex); B (on the left side); C (on the right side); D (lower-left); E (lower-right).

COROLLARIUM.

From this fifth proposition it is clear that every equilateral triangle is also equiangular — that is, that the three angles of any equilateral triangle are equal to one another. For let ABC be an equilateral triangle. Since the two sides AB and AC are equal, the two angles B and C will be equal [Margin: a — Elements I.5]. Again, since the two sides AB and BC are equal, the angles C and A will also be equal. Therefore all three angles A, B, and C will be equal. Which was to be shown.

Diagram. An equilateral triangle ABC — A (apex), B (lower-left), C (lower-right) — illustrating that an equilateral triangle is equiangular.

THEOR. 3. PROPOS. 6.

[Margin: vi — proposition 6]

[Proposition — Euclid's enunciation] If two angles of a triangle are equal to one another, then the sides subtended under the equal angles will also be equal to one another.

[Clavius's demonstration] In triangle ABC let the two angles ABC and ACB upon the side BC be equal. I say that the two sides opposite them, AB and AC, are also equal. For if they are not believed to be equal, then — the said angles being nonetheless equal — one will be greater than the other; so let AB, if it be possible, be greater than AC. From AB let there be cut off at D a straight line BD equal to the straight line AC [Margin: b — Elements I.3] (BD being said to be less than AB), and let the straight line CD be drawn. Now consider the two triangles ACB and DBC. In them the two sides AC and CB of triangle ACB are equal to the two sides DB and BC of triangle DBC, each to each — namely AC to DB (for we cut off from AB, by the adversary's own concession, a segment DB equal to AC) and CB to BC, since it is one and the same. Moreover, the angles ACB and DBC contained by those sides are equal, by hypothesis. Therefore the triangles ACB and DBC will be equal [Margin: c — Elements I.4] — the whole to the part, which is impossible. Therefore the sides AB and AC will not be unequal, if the angles B and C upon the side BC are equal — lest we grant that the whole is equal to the part; rather, they will be equal. Therefore, if two angles of a triangle, and so forth. Which was to be demonstrated.

Diagram. The reductio figure for Proposition 6. Triangle ABC with apex A and base vertices B (lower-left) and C (lower-right). On side AB a point D is taken just below A, with $BD = AC$, and the line CD is drawn, forming triangle DBC within triangle ABC.

- A (apex); D (on AB, just below A); B (lower-left); C (lower-right).

SCHOLIVM.

This theorem is the converse of the first part of the preceding one. For there it was demonstrated that if two sides of a triangle are equal to one another, the angles at the base are also equal; here, on the other hand, that if the angles at the base are equal, the sides opposite those angles are also equal. No one ought to find it strange that mathematicians sometimes convert propositions — so that at one time, from some granted antecedent, they collect a consequent by demonstration, while at another time, conversely, from the consequent now

granted they infer that antecedent by another demonstration, as we see Euclid do in these two neighboring propositions. It should not, I say, seem strange, since in mathematical matters antecedent and consequent are not always reciprocal. For in necessary propositions, such as geometrical propositions are, the predicate may sometimes be more universal than the subject, to speak with the logicians; and in that case the proposition cannot be converted. For the proposition "Every man is an animal" is necessary, yet it cannot be converted universally, since not every animal is a man. The same can happen in necessary geometrical propositions, and of this I will now offer just one such example. Euclid demonstrates, at proposition 16 of this book [Elements I.16], that if one side of any triangle is produced, the exterior angle is greater than either of the two interior and opposite angles; and in that proposition antecedent and consequent are in no way reciprocal. For it does not follow that, if in any rectilinear figure one side is produced and the exterior angle is greater than each of the interior opposite angles, that figure is a triangle — since it might also be a quadrilateral, as we shall show at proposition 16 of this book. In the same way many other propositions cannot be converted. For this reason the geometer must first demonstrate that a given proposition can be converted — that is, that its antecedent and consequent are reciprocal — before he collects the antecedent from the granted consequent. Euclid, however, does not convert all the propositions that can be converted, but only those whose conversion is most needed; yet we shall take pains to convert nearly all those that seem to offer any usefulness.

COROLLARIUM.

It follows from this proposition that every equiangular triangle — that is, one all of whose angles are equal — is equilateral. This is indeed the converse of the corollary of the fifth proposition, as is clear. For let the three angles of triangle ABC be equal. I say that it is equilateral. Since the two angles B and C are equal, the sides AB and AC will be equal [Margin: a — Elements I.6]. Again, since the two angles A and B are equal, the sides AC and BC will also be equal; and therefore all three sides AB, BC, AC are equal. Which was to be shown.

Diagram. An equilateral triangle with apex C, base vertices A (lower-left) and B (lower-right), illustrating that an equiangular triangle is equilateral.

EX PROCLO. [*From Proclus.*]

We shall be permitted to convert the second part of the fifth proposition as well, in this way:

[Proposition] If, when the two sides of any triangle are produced, the angles below the base are made equal, then those two sides will also be equal to one another.

[Demonstration] For let the sides AB and AC of triangle ABC be produced to D and E, and let the angles DBC and ECB below the base BC be made equal. I say that the sides AB and AC are also equal to one another. On CE produced as far as one pleases, let CF be cut off equal to BD [Margin: b — Elements I.3], and let the straight lines BF, FD, and DC be drawn. Then consider the triangles DBC and FCB. In them the sides DB and BC are equal to the sides FC and CB, each to each — namely DB to FC by construction, and BC to CB, since it is one and the same; and the angles DBC and FCB contained by those sides are equal by hypothesis. Therefore both the bases CD and BF, and the angles BCD and CBF upon those bases (since they stand opposite the equal sides BD and CF), will be equal [Margin: c — Elements I.4]. Therefore, when these equal angles BCD and CBF are subtracted from the angles FCB and DBC by hypothesis, the remaining angles FCD and DBF will be equal [Margin: a — Common Notion 3]. Now consider again the triangles DBF and FCD. In them, since the sides DB and BF are equal to the sides FC and CD, each to each — namely DB to FC by construction, and BF to CD, as was just shown — and since the angles DBF and FCD contained by those sides are equal, as was also just demonstrated, the angle BDF upon the base DF of triangle DBF will be equal to the angle CFD upon the same base FD of triangle FCD [Margin: b — Elements I.4]; for these stand opposite equal sides. Since, then, in triangle ADF the two angles ADF and AFD are equal, as we have now shown, the sides AD and AF will be equal [Margin: c — Elements I.6]. And if from these the straight lines BD and CF, equal by construction, are taken away, there will remain AB and AC, the sides of triangle ABC, equal [Margin: d — Common Notion 3]. Which was to be shown.

Diagram. The figure from Proclus. Triangle ABC with apex A and base vertices B (left) and C (right), the base BC drawn across; the sides AB and AC are produced below the base to D (below B) and E (below C). On CE a point F is taken with CF = BD, and the lines BF, FD, and DC are drawn, crossing below the base.

- **A** (apex); **B** (left base vertex); **C** (right base vertex) [glyph faint on the leaf — required by the construction]; **D** (lower-left, on AB produced); **F** (on CE); **E** (lower-right, on AC produced).

THEOR. 4. PROPOS. 7.

[Proposition — Euclid's enunciation] On the same straight line, and having the same endpoints as two given straight lines, two other straight lines equal respectively to those two — each to each — cannot be constructed meeting at a different point on the same side as the two lines originally drawn.

[Clavius's demonstration] Upon the straight line AB let two straight lines AC and BC be constructed to some point C. I say that upon the same straight line AB, toward the same side as C, it is impossible to construct at another point, say D, two other straight lines equal to the lines AC and BC, each to each — namely AC to AD, which have the same endpoint A, and BC to BD, which likewise have the same endpoint B. For suppose, if it be possible, that the lines AC and AD are equal to one another, and the lines BC and BD equal to one another as well. Then the point D will either lie on one of the lines AC or BC (so that AD falls on the line AC itself, or BD on BC itself), or inside the triangle ABC, or outside it.

First, let the point D lie on one of the lines AC or BC — say on AC, so that AD is a part of AC. Then, since the lines AC and AD, having the same endpoint A, are said to be equal, the part AD will be equal to the whole AC — which is impossible.

Next, let the point D lie inside triangle ABC; and, the line CD having been drawn, let the lines BC and BD be produced up to E and F. Since in triangle ACD the sides AC and AD are assumed equal, the angles ACD and ADC upon the base CD will be equal [Margin: a — Elements I.5]. But the angle ACD is less than the angle DCE [Margin: b — Common Notion 9], being a part less than the whole; therefore the angle ADC will also be less than that same angle DCE. Hence the angle CDF, being a part of ADC, will be much less than that same angle DCE. Again, since in triangle BCD the sides BC and BD are assumed equal, the angles CDF and DCE below the base CD will be equal [Margin: c — Elements I.5]. But it was shown that the same angle CDF is much less than the angle DCE. Therefore the same angle CDF is both less than the angle DCE and equal to it — which is absurd.

Lastly, let the point D lie outside triangle ABC. Then either it will be in such a position that one line falls upon another, as in the earlier figure — provided that for D you understand C, and for C you understand D — from which again a part will be found equal to the whole, which is absurd. Or it will be in such a position that the two later lines enclose the two earlier ones, as in the latter figure, provided that again for D you understand C, and for C, D. On this supposition we shall fall into the same absurdity, namely that the angle DCF is both less than the angle CDE and equal to it, as is evident. Or finally the point D will lie outside

triangle ABC in such a way that one of the later lines, namely AD, cuts one of the earlier ones, namely BC. Then, the line CD having been drawn, since in triangle ACD the sides AC and AD are assumed equal, the angles ACD and ADC upon the base CD will be equal [Margin: d — Elements I.5]. And accordingly, since the angle ADC is less than the angle BDC [Margin: e — Common Notion 9], a part less than the whole, the angle ACD will also be less than that same angle BDC. Hence the angle BCD, being a part of the angle ACD, will be much less than that same angle BDC. Again, since in triangle BDC the sides BC and BD are assumed equal, the angles BCD and BDC upon the base CD will be equal [Margin: a — Elements I.5]. But it has just been shown that the angle BCD is much less than the angle BDC. Therefore the same angle BCD is both less than the angle BDC and equal to it — which is absurd. Therefore AC and AD are not equal to one another, nor BC and BD to one another. Therefore, upon the same straight line, two other straight lines equal to two given straight lines, and so forth. Which was to be demonstrated.

Diagram. The general setup for Proposition 7. Two pairs of straight lines drawn to the base AB from two different upper points: AC and BC meeting at C, and AD and BD meeting at D, with C and D close together at the top and A (lower-left), B (lower-right) as the common endpoints of the base.

- **C** (upper); **D** (upper, near C); **A** (lower-left); **B** (lower-right).

Diagram. The case where D lies inside the triangle. A small triangle on base AB (A lower-left, B lower-right) with apex C; D is an interior point near C; the lines BC and BD are produced upward to E and F.

- **F** (top-left); **E** (top); **C** (below E); **D** (below C, interior); **A** (lower-left); **B** (lower-right).

Diagram. The case where D lies outside the triangle and one line falls on another. A triangle on base AB (A lower-left, B lower-right) with D at the apex and C just below D on the near side.

- **D** (apex); **C** (below D); **A** (lower-left); **B** (lower-right). [Margin: c — Elements I.5]

Diagram. The case where the line AD cuts the side BC. A triangle on base AB (A lower-left, B lower-right) with D at the top and C below; a line from A crosses BC, meeting it at a point marked E.

- **D** (top); **C** (below D); **A** (lower-left); **B** (lower-right); **E** (intersection on BC) [a second E-like mark appears near the top; reading uncertain]. [Margin: d — Elements I.5; e — Common Notion 9; a — Elements I.5]

SCHOLIVM. [The heading is printed **SHOLIVM** — the C is dropped in the source; read *Scholium*.]

It is possible for the two lines AD and BD to be equal to the two lines AC and BC, each to each — AD to BC, and BD to AC — as the last figure indicates. But in this case those lines that are equal to one another do not proceed from the same point, as is clear: for only AC and AD share the same endpoint A, and likewise only BC and BD share the endpoint B; and Euclid excellently demonstrated that it is impossible for AC and AD to be equal to one another in such a way that BC and BD are also equal to one another. Rightly, therefore, are these words set in the proposition: "having the same endpoints as the two lines originally drawn." Again, the two lines AD and BD taken together may be equal to the two lines AC and BC taken together, as can be seen in the same figure; but Euclid does not show this to be impossible. For he said that the two cannot each be equal to each, and so on.

By the same reasoning, from A and B, below AB, the base of triangle ABC — that is, toward the contrary side — two straight lines AD and BD may be drawn meeting at some point, in such a way that AD, issuing from the point A, is equal to AC, and BD, issuing from B, is equal to BC, as is evident in the figure set beside. Not without reason, therefore, did Euclid add: "on the same side." Finally, two lines AC and AD may be equal to one another, sharing the same endpoint A; but on this supposition it can in no way come about that the remaining two, BC and BD, having the same endpoint B, are also equal to one another, as appears in this figure and was demonstrated by Euclid. Aptly, therefore, was it said in the proposition: "two other straight lines equal to two given straight lines, each to each," and so on. So that the aim Euclid set himself in this proposition may be plainly understood, each single word of the proposition must be weighed carefully.

Diagram. The first scholium figure. A rhombus-like figure with the diagonal AB drawn across: it shows AD and BD equal to AC and BC crosswise.

- **C** (top); **A** (left); **B** (right); **D** (bottom).

Diagram. The second scholium figure. A triangle on base AB (A lower-left, B lower-right) with two upper points C (left) and D (right); the lines AD and BC cross, showing two lines from A and B meeting at points on the same side.

- C (top-left); D (top-right); A (lower-left); B (lower-right).

THEOR. 5. PROPOS. 8.

[Margin: viij.]

[Euclid, Elements I.8 — enunciation] If two triangles have two sides equal to two sides, each to each, and also have the base equal to the base, then they will also have the angle contained by the equal straight lines equal to the angle.

[Clavius's demonstration] Let AB, AC be two sides of the triangle ABC, equal to the two sides DE, DF of the triangle DEF — each to each, namely AB to DE and AC to DF; and let the base BC also be equal to the base EF. I say that the angle A is equal to the angle D — the angles, that is, which are each contained by the said sides. For if the base BC be conceived in the mind as laid upon the base EF, [Margin: a — Book I, common notion 8] neither will exceed the other, but the point B will coincide with the point E, and the point C with the point F, since these bases are posited equal to one another. Next, if the triangle ABC be imagined to fall upon the triangle DEF, the point A will fall either upon the very point D or elsewhere. If the point A falls upon the very point D, the sides of the triangles will mutually coincide, since they are posited equal; [Margin: a — Book I, common notion 8] and therefore the angle A will be equal to the angle D, since neither exceeds the other.

But if the point A be said to fall elsewhere, say at G — in whatever way this may happen, that is, whether onto the side ED, or within the triangle EDF, or outside, as appears in the figures — then EG (which is the same as BA) will always be equal to ED, and FG (which is the same as CA) equal to FD, because the sides of the one triangle are posited equal to the sides of the other. But that this cannot be the case has already been demonstrated, [Margin: b — Book I, Prop. 7] since both the straight lines EG, ED possess the same endpoint E, and the straight lines FG, FD the same limit F. Therefore the point A will not fall anywhere other than upon the point D; and consequently the angle A will be equal to the angle D. Therefore, if two triangles have two sides equal to two sides, etc. Which was to be demonstrated.

Diagram. A horizontal strip of four small triangles illustrating the cases of the superposition argument.

- First triangle: apex A (top), base B (lower-left) to C (lower-right) — the triangle ABC.

- Second, third, and fourth triangles: each is triangle DEF (apex **D**, base **E** lower-left to **F** lower-right), with the additional point **G** marking where the superimposed vertex **A** is supposed to fall — successively on the side **ED**, then inside the triangle, then outside it.
- Labels present: **A**, **B**, **C**, **D**, **E**, **F**, **G**.

[Scholium] [Margin: heading printed "SHOLIVM" — the **C** is omitted]

As you see, this proposition converts the first part of the fourth proposition. For just as there, from the equality of the angles contained by the equal sides, the equality of the bases was inferred, so here Euclid concludes from the equality of the bases the equality of the angles comprehended by the equal sides. We can in the same way, from the first and third part of the conclusion of the fourth proposition, infer the whole antecedent of it, so that the theorem may be stated in this form:

[Restated theorem — enunciation] If two triangles have their bases equal, and the angles set upon the bases equal, each to each, then they will also have the remaining sides equal, each to each — namely those subtended by the equal angles — and the remaining angles included by these sides equal.

[Demonstration] For let the base **BC** be equal to the base **EF**, and the angle **B** to the angle **E**, and the angle **C** to the angle **DFE**. I say that the side **AB** is also equal to the side **DE**, and the side **AC** to the side **DF**, and the angle **A** to the angle **D**. For if the base be laid upon the base, [Margin: a — Book I, common notion 8] their extremities will mutually coincide, and likewise the lines of the equal angles. Therefore all will coincide, and consequently all will be equal to one another. This same theorem, which we have here proposed — which indeed seems to convert the fourth proposition more properly than Euclid's does — Euclid will demonstrate differently in the first part of proposition 26, as we shall note in that place.

[Corollary]

Moreover, from the antecedent of this eighth proposition it can be gathered not only that the angles contained by the equal sides are equal, but also that the remaining angles, which are set at the bases, are equal each to each — as the angle **B** to the angle **E**, and the angle **C** to the angle **F**; indeed that the whole triangle is equal to the whole triangle, as is clear from the same superposition of one triangle upon the other. For both the said angles and the whole triangles will mutually coincide, as is evident. This too can be gathered from the fourth proposition, once it has been demonstrated that the angles comprehended by the

equal sides are equal. For it will follow from this — since the sides too are equal — that both the remaining angles and the whole triangles are equal, as was demonstrated in proposition 4.

FROM PROCLUS.

The followers of Philo [*Philonis familiares* — Philo of Byzantium's circle] demonstrate this same eighth theorem by an affirmative demonstration, in this way. For, the same antecedent being posited, let the base BC be understood to be laid upon the base EF in such a way that the triangle ABC falls on the opposite side, and not upon the triangle DEF — such as is the triangle AEF. Then either the two sides, namely DF and FA, constitute one straight line — which will happen if the two angles C and F are right angles — or they do not. If they do constitute one straight line, say DA, the proposition will be concluded thus. Since in the triangle AED the two sides AE, DE are posited equal (for AE is now the same straight line as AB, which by hypothesis is equal to the straight line DE), [Margin: a — Book I, Prop. 5] the angles A and D upon the base AD will be equal, which was to be shown.

But if neither DF, FA nor DE, EA form a straight line, let a straight line DA be drawn from D to A, which will fall either within the triangles or outside them. Let it first fall within — which happens when the angles at E and F are acute. Since therefore in the triangle AED the two sides AE, DE are posited equal, [Margin: b — Book I, Prop. 5] the two angles EAD, EDA at the base DA will be equal. By the same reasoning, since the two sides AF, DF are equal by hypothesis, the two angles FAD, FDA upon the base DA will be equal. If therefore these equals be added to those equals, [Margin: c — Book I, common notion 2] the whole angles

Diagram. Upper of two figures for Philo's affirmative proof. Triangle ABC (apex **A** upper-left, base **B** lower-left to **C**) set point-to-point at **E** against triangle DEF (apex **D** upper-right, side to **F** at right), with the reflected vertex **A** placed below and the straight line DA drawn — the case where DF, FA lie in one straight line.

- Labels present: A, B, C, D, E, F, and a second A (lower vertex).

Diagram. Lower figure for Philo's proof. Same configuration but with the diagonal DA falling within the triangles, the lines crossing to form the interior construction.

- Labels present: A, B, C, D, E, F, and a second A (lower vertex).

the whole angles EAF, EDF will be equal. Which was to be shown. Next, let the straight line DA fall outside the triangles — which finally happens when the

angles at F are obtuse. Since therefore in the triangle AED the two sides AE, DE are posited equal, [Margin: d — Book I, Prop. 5] the angles EAD, EDA at the base DA will be equal. By the same reasoning, since the two sides AF, DF in the triangle AFD are equal by hypothesis, the angles FAD, FDA upon the base DA will be equal. Therefore, these being subtracted from the former, [Margin: e — Book I, common notion 3] the angles EAF, EDF will remain equal — which was what was proposed to be demonstrated.

Diagram. Upper of two figures continuing Philo's proof — the case where DA falls outside the triangles. Triangle ABC (apex A, base B–C) meeting triangle DEF (apex D, with E, F) at E, and the reflected vertex A below, DA drawn outside.

- Labels present: A, B, C, D, E, F, and a second A.

Diagram. Lower figure for the same case, with the crossing lines of the outside construction.

- Labels present: A, B, C, D, E, F, and a second A.

PROBL. 4. PROPOS. 9.

[Margin: ix.]

[Euclid, Elements I.9 — enunciation] To bisect a given rectilineal angle [*bifariam facere*].

[Clavius's demonstration] Let BAC be the rectilineal angle to be bisected — that is, to be divided into two equal angles. On the straight line AB take any point D, and from AC [Margin: a — Book I, Prop. 3] cut off a straight line AE equal to AD, and draw the straight line DE. Then upon DE [Margin: b — Book I, Prop. 1] construct an equilateral triangle DFE, and draw the straight line AF, dividing the angle BAC into the angles BAF, CAF. I say that these angles are equal to one another. For since the sides DA, AF of the triangle DAF are equal to the sides EA, AF of the triangle EAF, each to each — because DA is equal to EA by construction, and AF is common — and the base DF is also equal to the base EF, because the triangle DFE was constructed equilateral: [Margin: c — Book I, Prop. 8] the angle DAF will be equal to the angle EAF, and therefore the angle BAC is bisected, which was to be done.

Diagram. The construction for bisecting angle BAC. Apex A at top; rays descending to B (lower-left) and C (lower-right). Point D on ray AB and point E on ray AC; below them the equilateral triangle DFE with vertex F at the bottom; the straight line AF drawn through, bisecting the angle.

- Labels present: A, B, C, D, E, F.

[Scholium]

But if in place of the equilateral triangle we construct an isosceles triangle, we shall nonetheless demonstrate the same thing. This can also be done in the next three propositions that follow.

[Practice]

More quickly than can be told, any rectilineal angle, such as BAC, will be bisected in this way. From the centre A, with some compass-opening, cut off equal straight lines AD, AE of any magnitude whatever. Then, without changing the compass (though you could vary it if you wished), from the centres D and E describe two arcs cutting each other at F. The straight line drawn AF will then bisect the angle BAC. For if the straight lines DF, EF were drawn, they would be equal, being the radii of equal circles. From this, as before, it will be demonstrated that the angle DAF is equal to the angle EAF. But we have not drawn the said lines, so that the bare practice might be shown — which we shall observe in the other practices too, as far as may be, lest a multitude of lines cast darkness over us and breed confusion.

Diagram. The practical bisection by arcs. Apex A; on the two rays the points D and E; two arcs struck from D and E crossing at F below; the rays continuing to B and C.

- Labels present: A, B, C, D, E, F.

But if ever a rectilineal angle contained by very short lines, and placed at the edge of some plane, must be bisected, we shall describe from D and E two arcs mutually intersecting at F *above* the angle A — because below the points D and E there is no space in which they could be drawn. For a straight line drawn out from F through A as far as B will bisect the angle A, as before, as appears in the adjoining figure.

Diagram. The edge-of-plane case. Point F at the top (where two arcs cross); the vertex A below it; the points D and E at the bottom; the bisecting line running down through A.

- Labels present: F, A, D, E (with B, the far endpoint of the produced bisector).

[Scholium]

From this it is plainly gathered that any rectilineal angle can be divided also into 4 equal angles, into 8, into 16, into 32, into 64, and so on, always proceeding by doubling. For once any rectilineal angle has been divided into two equal angles, if each of these be again bisected we shall have 4 equal angles; and if these singly be again bisected, we shall obtain 8 equal angles, and so on. Euclid nowhere taught by what method a rectilineal angle can be divided into any given number of equal parts, because up to that day this had been demonstrated by no one. From Pappus of Alexandria, however, we shall teach it, by means of a certain curved or bent line, at the end of Book 6. Meanwhile, if anyone should wish to divide any proposed rectilineal angle into any number of equal parts by rough rule of thumb (as it is said, by "native wit" [*rudī Minerva*]), he will have to use the compass — attempting, as it were, and often repeating the practice itself, until he reaches the desired end — in this manner. Let the rectilineal angle BAC be divided into 5 equal angles. From the centre A describe an arc of a circle BC, at any interval, cutting the straight lines AB, AC at B and C. Then, by means of the compass (widening its legs now more, now less, until they have the due distance), divide the arc into as many equal parts as the proposed angle is to be divided into — as in the example proposed, into five parts, at the points D, E, F, G. For if straight lines be drawn to these points from A, the angle BAC will be divided into five equal angles. For since the intervals BD, DE, etc., taken with the compass are equal, they will all be equal to one another. Therefore the two sides BA, AD of the triangle BAD [Margin: a — Book I, definition 15] will be equal to the two sides EA, AD of the triangle EAD, each to each, since all proceed from the centre out to the circumference. And the base BD is also equal to the base DE, as was said: [Margin: b — Book I, Prop. 8] therefore the angle BAD will be equal to the angle EAD. By the same reasoning it will be demonstrated that the angle EAD is equal to the angle EAF, and so of the rest. But it will be gathered more briefly, from proposition 27 of the third book, that all the angles at A are equal to one another, because the circumferences BD, DE, etc., were all taken equal to one another.

Let no one be surprised that we sometimes exhibit practices whose demonstrations depend on later propositions. For this we do, as we said above at definition 10, with the intent that, so far as may be, each matter be treated in its proper place — the division of any rectilineal angle into any number of equal parts, that is, in the place where Euclid teaches the division of the same angle into two equal parts; and the division of a straight line into any number of equal parts where Euclid divides that same line into two; and so of each. For such practices do not always require the following demonstrations, but only that it be proved that what was required has been rightly effected by them. Therefore he who, not content with the bare practice, requires a demonstration, will be able to go back

to any practice once he has carefully grasped the demonstrations necessary for it. For we shall always cite, in the demonstrations of our practices, those propositions which must be brought to bear on the matter — just as in the preceding practice we cited proposition 27 of the third book.

Diagram. The five-fold division of angle BAC. Apex **A** at top; an arc **B...C** below; on the arc the division points **D, E, F, G**; straight lines fanning from A to each division point.

- Labels present: A, B, C, D, E, F, G.

PROBL. 5. PROPOS. 10.

[Margin: x.]

[Euclid, Elements I.10 — enunciation] To bisect a given finite straight line.

[Clavius's demonstration] Let AB be a given finite straight line, to be bisected — that is, divided into two equal parts. [Margin: a — Book I, Prop. 1] Let an equilateral triangle ABC be described upon AB, and let its angle C be bisected [Margin: b — Book I, Prop. 9] by the straight line CD, and let the straight line CD cut the straight line AB at D. I say that the straight line AB is bisected at D. Since the two sides AC, CD of the triangle ACD are equal to the two sides BC, CD of the triangle BCD, each to each — namely AC to BC, since both are sides of the equilateral triangle, and CD is common — and the angle ACD is also equal to the angle BCD by construction: [Margin: c — Book I, Prop. 4] the base AD will be equal to the base BD. Therefore we have bisected the given straight line AB at D, which was what had to be done.

Diagram. The construction for bisecting the segment AB. Equilateral triangle with apex **C** at the top and base **A** (lower-left) to **B** (lower-right); the point **D** on the base with the line CD drawn from the apex.

- Labels present: A, B, C, D.

[Practice]

From the centre A, with any interval that nevertheless exceeds half the line AB, describe two arcs, one above, one below; and from the centre B, with exactly the same interval, draw two other arcs, which cut the former at C and D. For the straight line drawn CD will bisect the straight line AB at E. For if from A and B four straight lines be drawn to C and D, these will all be equal to one another, since they fall from the centres to the circumferences of equal circles — for the

arcs of the circles were described with the same interval. Since therefore the sides AC, CD are equal to the sides BC, CD, each to each, and the base AD to the base BD, [Margin: d — Book I, Prop. 8] the angle ACD will be equal to the angle BCD. Again, since the sides AC, CE are equal to the sides BC, CE, each to each, and the angle ACE to the angle BCE, as was shown, [Margin: a — Book I, Prop. 4] the base AE will be equal to the base BE.

Diagram. The practical bisection by intersecting arcs. The segment **A–E–B** horizontal (E the midpoint); above it the intersection point **C**, below it the intersection point **D**, where the arcs struck from A and B cross; the line CD drawn vertically through E.

- Labels present: A, B, C, D, E.

Now if the line to be bisected is placed at the edge of some plane, so that below it there is no space in which two intersecting arcs can conveniently be drawn, then, having described above it two arcs intersecting each other at C, we shall describe on the same side two other arcs intersecting each other at D — whether this be done below the point C, as in the adjoining figure, or above C. For the straight line drawn out through C, D will bisect the straight line AB at E.

Diagram. The edge-of-plane bisection (drawn with dotted arcs). Intersection point **C** at the top, a second intersection **D** lower down, and the segment **A–E–B** at the bottom, with the line through C and D meeting AB at the midpoint E.

- Labels present: A, B, C, D, E.

[Scholium]

It is clear that the same straight line AB can be divided in the same way into 4 equal parts, and into 8, 16, 32, etc., just as we said in the preceding proposition about the division of a rectilineal angle. But by what method any proposed straight line is to be divided into any number of equal parts, we shall set forth most fully at proposition 40 of this book. And the same thing we shall afterward accomplish far more easily at proposition 10 of Book 6, where we shall bring forward various and not unpleasant practices. For there seems to be the more proper place for this matter, since practices of this kind are almost all more easily demonstrated by the proportions of lines. Nor indeed, until that place, shall we ever need the division of a line into more than two equal parts.

PROBL. 6. PROPOS. 11.

[Margin: xj.]

[Euclid, Elements I.11 — enunciation] Given a straight line, from a given point on it, to erect a straight line at right angles [i.e., a perpendicular].

[Clavius's demonstration] Let AB be the given straight line, and C the point on it from which we are bidden to erect upon AB a line at right angles, that is, a perpendicular. From the point C [Margin: a — Book I, Prop. 3] take a straight line CD, and cut off CE equal to it. Then upon DE [Margin: b — Book I, Prop. 1] construct an equilateral triangle DEF, and from F draw the straight line FC to C, which I say is perpendicular to AB. Since the sides DC, CF of the triangle DCF are equal to the sides EC, CF of the triangle ECF, each to each — namely DC to EC by construction, and CF common — and the base DF is also equal to the base EF, because of the equilateral triangle: [Margin: c — Book I, Prop. 8] the angles at C contained by the said sides will be equal. [Margin: d — Book I, definition 10] Therefore each will be called a right angle, and so FC is a straight line perpendicular to AB. Given a straight line, therefore, from a given point on it, etc. Which was to be done.

Diagram. The construction of a perpendicular at C. The straight line running **A–D–C–E–B** (C the given point, D and E cut off equal on either side); above C the vertex F (apex of the equilateral triangle DEF), reached by dotted construction arcs; the line FC erected perpendicular.

- Labels present: A, B, C, D, E, F.

[Practice]

From the point C cut off on either side equal lines CD, CE, and from D and E describe two arcs cutting each other at F. For the straight line drawn FC will be the perpendicular. The demonstration is the same as Euclid's, if only the straight lines DF, EF be drawn, which will be equal because of the equal circles described from D and E, which intersect each other at the point F. But if the given point on the straight line be its extremity, the line must be produced straight and continuous, on the side of the given point, so that from it a perpendicular line may be erected according to the given practice. Thus, if the given line be AC and the given point C its extremity, AC must be prolonged to B, and equal lengths CD, CE taken, etc. But if a perpendicular is to be set up on some line, not indeed at an assigned point, but anywhere at all, this will be effected by this method. From two points A and E, whatsoever, of the proposed line, describe both above and below two arcs intersecting each other at C and D. For the straight line drawn CD will be perpendicular to AB — that is, it will make the two angles at E right, or equal. This you will prove no otherwise than as we demonstrated above at the practice by

which we divided a line into two equal parts. For by the 4th proposition the angles at E will be equal, inasmuch as they stand upon equal bases AE, BE, and are set opposite the equal sides AC, BC, which from C would be drawn to the points A and B.

Diagram. The practical perpendicular at a point. The line **A–D–C–E–B** with the given point C; the arcs struck from D and E crossing at **F** above; FC erected.

- Labels present: A, B, C, D, E, F.

Diagram. The perpendicular erected anywhere on a line. Intersection point **C** at the top, **D** at the bottom (arcs struck from two points of the line), the line **A–E–B** across the middle, and CD the perpendicular through E.

- Labels present: A, B, C, D, E.

(Continuing Proclus's additions to Proposition 11 — how to erect a perpendicular when the given point is the very endpoint of the line.)

If the given point on the line happens to be its endpoint, and the line cannot conveniently be extended, we can still draw a perpendicular from the given point without extending the line, by this method. Let AB be the straight line and A the point. From C, any point taken within the line, erect the perpendicular CD, as Euclid taught; and cut off CE equal to AC [Margin: a — *Elements* I.3]. Then bisect the angle C by drawing the straight line CF [Margin: b — *Elements* I.9]. Next, from E again, as Euclid taught, draw EG perpendicular to CD, meeting the straight line CF at G. Then the straight line GA, once drawn, will be perpendicular to AB. For since the sides AC, CG of triangle ACG are equal to the sides EC, CG of triangle ECG, each to each, and the angles contained by these sides are likewise equal, by construction, the angles A and E, which lie opposite the common side CG, will be equal [Margin: c — *Elements* I.4]. But E is a right angle by construction; therefore A too will be a right angle, and so AC is perpendicular to AB [Margin: d — *Elements* I, Def. 10].

Diagram. Construction for erecting a perpendicular from the endpoint of a line (Proclus's method). A rectangular framework stands on the baseline AB (solid horizontal). Dotted lines show the auxiliary construction: a diagonal from the upper-left down toward B, a horizontal line across the top, and a vertical on the right.

- A (bottom-left, on baseline AB): the given endpoint
- B (bottom-right, on baseline AB): far end of the line

- C (foot of the right-hand vertical, on AB near B): the point taken within the line
- D (top-right): upper end of the perpendicular CD
- E (right-hand vertical, below D): point with $CE = AC$
- F (top-left): end of the bisector CF
- G (upper-left): where EG meets CF; GA is the required perpendicular

SCHOLIUM

We shall erect a perpendicular from a given point more briefly — whether that point is an endpoint or not — in this way. Let AB be the given line and A a point on it. From a center C taken outside the line, wherever you please (provided that the straight line AB, when extended, does not pass through C), and with a radius reaching as far as A, describe an arc of a circle cutting AB at D. Then from D through C draw a straight line cutting the arc at E. The straight line EA, once drawn, will be perpendicular to AB. For angle A is a right angle, since it stands in the semicircle DAE, as we shall show in Proposition 31 of Book III.

You will find yet another construction, for when the given point is at the endpoint of the line, in the scholium to Proposition 31 of this Book.

Diagram. A semicircle standing on the diameter DE, with the required perpendicular EA drawn to the baseline AB.

- A (bottom-left, on line AB and on the circle): foot of the perpendicular
- B (right end of the baseline): far end of the given line
- C (center, upper): center of the arc
- D (bottom-right, on AB): where the arc first cuts AB
- E (top): the point where DC produced meets the arc; EA is perpendicular to AB

PROBL. 7. PROPOS. 12.

[Margin: xij.]

Enunciation.—To draw a perpendicular straight line to a given infinite straight line from a given point that is not on it.

Demonstration.—Let AB be a straight line of unlimited extent, and let C be a point outside it, from which a perpendicular is to be drawn to the straight line AB. With center C and any radius, describe a circle cutting AB at D and E — the radius chosen must be just large enough to reach across the straight line AB, otherwise it

would not cut it. Bisect the straight line DE at F [Margin: a — *Elements* I.10], and draw the straight line CF, which I say is perpendicular to AB. For if CD and CE are drawn, the two sides DF, FC of triangle DFC will be equal to the two sides EF, FC of triangle EFC, each to each, by construction; and the base CD is equal to the base CE, since these are drawn from the center C to the circumference. Therefore the angle DFC will be equal to the angle EFC [Margin: b — *Elements* I.8], and so each is a right angle. Thus CF has been drawn perpendicular, which was to be done.

Diagram. A circle with center C, cut by the horizontal line AB at D and E; F is the midpoint of DE, and CF is the perpendicular dropped from the external point.

- A (far left, on line AB): one end of the given line
- B (far right, on line AB): other end
- C (top center): the given external point / center of the circle
- D (left intersection of circle and AB)
- E (right intersection of circle and AB)
- F (center, on AB below C): midpoint of DE; CF is perpendicular to AB

SCHOLIUM

Euclid rightly appended the qualification "infinite." For if the line were finite, a perpendicular could not always be drawn to it from a given point outside it. For instance, if the finite line were BE and the point C, no circle could be described from C cutting BE in two points, and hence no perpendicular could be drawn from C to BE. For this reason, then, Euclid requires the given straight line to be infinite — that is, to have no determinate magnitude — so that a perpendicular can be dropped at least to the line produced. For that is what happens here: if BE is produced until the circle described from C cuts the whole of BA produced at D and E, and so on.

PRAXIS

With C made the center and any radius, describe two arcs cutting the given line at A and B. Then from A and B, with the same radius (or a different one, if you prefer), let two more arcs be described cutting each other at D. For the straight line CD, drawn so as to cut AB at E, will be perpendicular to AB. The demonstration of this operation does not differ from the one given in the praxis of Proposition 10; for the angles at E will be right angles [Margin: a — *Elements* I.4], being equal to one another.

Diagram. A vertical line CD crossing the horizontal AB at E, with the crossing arcs marked at C and D (small "x" intersections above and below).

- A (left, on AB)
- B (right, on AB)
- C (top): upper arc-intersection
- D (bottom): lower arc-intersection
- E (center, on AB): foot of the perpendicular CD

We shall accomplish the same thing in this way. From any point A on the given line, with a radius reaching to C, describe an arc of a circle. Then from any other point B, with a radius reaching to the same C, describe another arc cutting the first at C and D. The straight line CD drawn so as to cut AB at E will be perpendicular to AB. The demonstration is the same as the previous one. It is not necessary, however, that the radius BC be equal to the radius AC, as appears in this figure; yet the operation will be easier and shorter if the same radius is always used.

Diagram. A vesica (lens) formed by two intersecting arcs; the line CD through the two intersection points crosses AB at E.

- A (left, on AB)
- B (right, on AB)
- C (top): upper intersection of the arcs
- D (bottom): lower intersection
- E (center, on AB): foot of the perpendicular

Diagram. A pair of lens-figures drawn side by side, illustrating the case where the two radii (BC and AC) are unequal, so the two arc-pairs intersect at different points. Each half shows the labels C (top), E (center, on the baseline), and D (bottom), with A and B on the baseline. [The paired construction repeats the same five letters A, B, C, D, E in each half.]

When the given point C is too close to the straight line AB, we must proceed thus. With center C and any radius, cut the straight line AB at the two points A and B. From these, with any larger radius — greater than AC or BC — describe two arcs both above and below, intersecting each other at D and E. For the straight line DCE, which when produced necessarily passes through the point E, will be perpendicular to the straight line AB at F. We shall demonstrate this as follows, drawing the straight lines AD, BD, AC, BC. Since the two sides DA, DC of triangle ACD are equal to the two sides DB, DC of triangle BCD, and likewise the bases AC,

BC are equal, the angles at D will be equal [Margin: a — *Elements* I.8]. Therefore, since the two sides DA, DF of triangle ADF are equal to the two sides DB, DF of triangle DBF, and they contain equal angles at D as we have shown, the angles at F will be equal [Margin: b — *Elements* I.4], and so they are right angles, and so on.

Diagram. Two arcs crossing above and below the line AB, with the perpendicular DCF descending through the intersections; dotted lines mark triangle ACD (and its mirror).

- A (left, on AB)
- B (right, on AB)
- C (center, above F): the given point, close to AB
- D (top): upper arc-intersection
- E (bottom): lower arc-intersection (through which DCF passes when produced)
- F (on AB below C): foot of the perpendicular

(Continuing Proposition 12 — further constructions for special positions of the point and line.)

Now if the given point lies next to the edge of some plane, so that the given line cannot be extended, we shall proceed thus. From any point B — which appears to be placed roughly opposite the given point C, that is, nearly at the extremity of the given line AB — describe two arcs above and below the line AB, with radius BC. Then from a point A somewhat removed from the point B already taken (and the farther apart the points A and B are, the more conveniently the arc-intersections will be found), describe two other arcs with radius AC, cutting the earlier ones at C and D. For the straight line CD will be perpendicular to the given straight line AB.

But if, with a point given that is not near the edge of the plane on which the line lies, the line itself lies at the edge of the plane, so that two arcs cannot conveniently intersect below the line — whether the given point C is close to the line AB or not — we shall solve the problem in this way. With radius AC, taking A anywhere, describe from C an arc cutting the straight line AB at D; and from A and D describe two arcs toward the point C, intersecting each other at E. For the straight line drawn from E through C, cutting AB at F, will be perpendicular to AB, as was demonstrated above for the case when the point C was near the line AB.

Diagram. Two figures. The upper shows the line AB with a right-angle bracket: C above and D below at the right (the perpendicular CD). The lower shows a

vertical construction with C (given point) above the baseline, E at top from the intersecting arcs, and the perpendicular meeting AB at F, with arc-marks at the crossings.

- A (left, on AB)
- B (right end)
- C (given point, above AB / upper-right of the first figure)
- D (on AB, where the arc from C cuts it; also lower-right of the first figure)
- E (top, intersection of the arcs from A and D)
- F (on AB): foot of the perpendicular EC produced

As for how we should proceed when the given point lies next to one edge of the plane and the given line runs near the other edge — so that we may neither extend the line nor conveniently make two arcs intersect at D below the given straight line AB — we shall teach this in the scholium to Proposition 31 of this Book.

We shall also accomplish this in the following way. In the figure of the scholium to Proposition 11, let a perpendicular have to be drawn to the straight line AB from the point E, at the extremity of the line, as it were. Draw any straight line ED, and bisect it at C [Margin: a — *Elements* I.10 [?]; the printed digit is ambiguous]. With center C and passing through E, describe an arc of a circle cutting the given straight line at A. (Should it not cut it, the line must be produced until it is cut by the arc.) For the straight line EA will be perpendicular to AB, since the angle A, standing in a semicircle, is a right angle [Margin: b — *Elements* III.31].

Diagram. A semicircle on diameter DE; the perpendicular EA is drawn from E to the baseline AB.

- A (bottom-left, on AB and on the circle)
- B (right end of baseline)
- C (center, upper): center of the arc, midpoint of ED
- D (bottom-right, on AB)
- E (top): the given point from which EA descends

THEOR. 6. PROPOS. 13.

[Margin: xiiij.]

Enunciation.—When a straight line standing on a straight line makes angles, it will make either two right angles or angles equal to two right angles.

Demonstration.—Let the straight line AB, standing on the straight line CD, make two angles ABC, ABD. If AB is perpendicular to CD, the said angles will be two right angles [Margin: c — *Elements* I, Def. 10]. But if AB is not perpendicular, it will make one angle obtuse and the other acute. I say, then, that these two are equal to two right angles. For draw BE from B perpendicular to CD [Margin: d — *Elements* I.11], so that the two angles EBC, EBD are right angles. Now since the right angle EBD is equal to the two angles DBA, ABE [Margin: e — Axiom 19], then, adding the common right angle EBC, the two right angles EBD, EBC are equal to the three angles DBA, ABE, EBC [Margin: f — Axiom 2]. Again, since the angle ABC is equal to the two angles ABE, EBC [Margin: g — Axiom 19], then, adding the common angle ABD, the two angles ABC, ABD are equal to the three angles DBA, ABE, EBC [Margin: a — Axiom 2]. But we have shown that these same three are also equal to the two right angles EBD, EBC; and things equal to the same thing are equal to one another [Margin: b — Axiom 1]. Therefore the two angles ABC, ABD are equal to the two right angles EBD, EBC. So, when a straight line standing on a straight line makes a line, and so on. Which was to be shown.

Diagram. The line AB stands on the base line CD at B; BE is the auxiliary perpendicular to CD.

- A (top): upper end of the standing line AB
- B (center of the base line): foot where AB stands
- C (left end of the base line CD)
- D (right end of the base line CD)
- E (upper-left): end of the perpendicular BE

SCHOLIVM

This proposition seems to rest on a certain common notion of the mind. For by however much the angle ABC exceeds the right angle EBC, by just so much is the remaining angle ABD fallen short of by the right angle EBD. For as the excess there is the angle ABE, so here the deficit is that same angle ABE. Hence the angles ABC, ABD are proved equal to two right angles, since the one gains above a right angle exactly as much as the other loses.

THEOR. 7. PROPOS. 14.

[Margin: xiv.]

Enunciation.—If, with any straight line and at a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another.

Demonstration.—At the point C of the straight line AB, let two straight lines CD, CE be drawn on opposite sides, making with AB the two angles ACD, ACE either right angles or equal to two right angles. I say that CD and CE are set in a straight line, so that DCE is one straight line. For if DCE is not straight, then DC produced straight and continuous through C will fall either above CE, so as to make the straight line DCF, or below CE, so as to make the straight line DCG. If it falls above: since AC stands on the straight line DCF, the two angles ACD, ACF will be equal to two right angles [Margin: a — *Elements* I.13]; but the two angles ACD, ACE are also assumed equal to two right angles, and all right angles are equal to one another [Margin: b — Axiom 12]. Therefore the two angles ACD, ACF will be equal to the two angles ACD, ACE. So, taking away the common angle ACD, the angles ACF, ACE will remain equal to one another [Margin: c — Axiom 3] — the part to the whole, which is absurd. Therefore DC produced will not fall above CE; nor will it fall below, for by the same reasoning the angles ACE, ACG would be proved equal. Therefore DC produced will become the same as CE; and consequently, if with any straight line and at a point on it, and so on. Which was to be demonstrated.

Diagram. A fan of lines meeting at C on the line AB (A above, B below); CD runs left, while CE and the false continuations CF, CG spread to the right.

- A (top): upper part of the line AB through C
- B (just below C): lower part of AB
- C (center): the point on AB
- D (left): end of CD
- E (right): end of CE
- F (upper-right): false continuation DCF (above CE)
- G (lower-right): false continuation DCG (below CE)

SCHOLIVM

This proposition is the converse of the preceding. For in that one it was proved that if DCE is straight, the angles ACD, ACE are equal to two right angles. In this one it is demonstrated that if the said angles are equal to two right angles, the straight lines DC, EC are one straight line.

(Continuing the commentary on Proposition 14.)

Euclid rightly added to this proposition the words "and not on the same side." For, as Porphyry says, it can happen that two lines are drawn to some point of a given line on the same side, making with the given line two angles equal to two right

angles, which nevertheless do not form one straight line, precisely because they are not drawn to opposite sides. For let the point C be given on the line AB. Draw CD perpendicular to AB [Margin: d — *Elements* I.11], and bisect the right angle ACD by the straight line CE [Margin: e — *Elements* I.9 [?]; the printed digit reads like "11"]. Then from D, any point on the straight line CD, draw DE perpendicular [to CD] [Margin: f — *Elements* I.11]. Produce ED toward D and take DF equal to the straight line DE [Margin: g — *Elements* I.3], and draw the straight line FC. Now since the sides ED, DC of triangle EDC are equal to the sides FD, DC of triangle FDC, each to each, and the angles they contain are equal, namely right angles, the base EC will be equal to the base CF, and the angle ECD equal to the angle FCD [Margin: a — *Elements* I.4]. But the angle ECD is half a right angle (for the right angle ACD has been bisected). Therefore FCD too will be half a right angle. Hence CF makes with AC the angle ACF, composed of a right angle and half a right angle; and CE makes with the same AC the angle ACE, also half a right angle. Therefore the two angles ACF, ACE, which the straight lines CF, CE make on the same side with AB, are equal to two right angles — and yet CF, CE are not one straight line, because they are not drawn to opposite sides but to the same side.

Diagram. Porphyry's counter-example: at C on the line AB, the perpendicular CD, its bisector CE, and the auxiliary triangle giving CF, all falling on the same side.

- A (top-left): upper end of the line AB
- B (bottom-left): lower end of AB (AB roughly vertical on the left)
- C (center-left): the point on AB
- D (center-right): foot of the perpendicular direction CD
- E (top-right): end of the bisector CE
- F (bottom-right): end of the line CF (with DF = DE)

THEOR. 8. PROPOS. 15.

[Margin: xv.]

Enunciation.—If two straight lines cut one another, they will make the vertical angles equal to one another.

Demonstration.—Let the two straight lines AB, CD cut one another at the point E, however placed. I say that the angles they make at the vertex E are equal to one another — namely the angle AED to the angle BEC, and the angle AEC to the angle BED. Since the straight line DE stands on the straight line AB, the two angles AED, DEB will be equal to two right angles [Margin: b — *Elements* I.13]. Again, since the straight line BE stands on the straight line CD, by the same reasoning the two

angles CEB, BED will be equal to two right angles. Now since all right angles are equal to one another [Margin: c — Axiom 12], the two angles AED, DEB will be equal to the two angles DEB, BEC. Therefore, taking away the common angle DEB, the angle AED will remain equal to the angle BEC [Margin: d — Axiom 3]. By the same reasoning it will be confirmed that the angles AEC, BED are equal to one another. For the two angles AEC, CEB, which are equal to two right angles [Margin: e — *Elements* I.13], will be equal to the two angles DEB, BEC, which are equal to two right angles. Therefore, taking away the common angle BEC, the angles AEC, BED will remain equal to one another [Margin: f — Axiom 3]. So, if two straight lines cut one another, and so on. Which was to be shown.

Diagram. Two straight lines AB and CD crossing at E, forming the vertical-angle "X."

- A (top-left)
- B (bottom-right) — line AB
- C (bottom-left)
- D (top-right) — line CD
- E (center): the point of intersection

COROLLARIVM I.

From the demonstration of this theorem Euclid infers (following the view of Proclus, since other copies do not contain this corollary) that two straight lines cutting one another make, at the point of section, four angles equal to four right angles. For in the demonstration it was shown that both the two angles AED, DEB and the two angles AEC, CEB are equal to two right angles, by Proposition 13. Therefore all four angles set at E together amount to twice two right angles. Hence they are equal to four right angles.

COROLLARIVM II.

By the same reasoning we shall infer that all the angles set about one and the same point, however many they be, are equal to just four right angles. For if from E any number of other lines are drawn, those four angles set at E will merely be divided into more parts, which all taken together are exactly equal to their wholes [Margin: a — Axiom 19]. Since, then, those four angles are equal to four right angles by the first corollary, all the others taken together will likewise be equal to only four right angles. From this it is clear that the whole space surrounding any point in a plane is equivalent to four right angles, as many authors assert — because all the angles that can be set about that point are equal

to four right angles. In the same way it is established that any number of straight lines cutting one another make, at the point of section, angles equal to four right angles.

EX PROCLO.

Enunciation.—If, with any straight line and at a point on it, two straight lines not taken on the same side make the vertical angles equal, the straight lines will be in a straight line with one another.

Demonstration.—From the point C of the straight line AB, let two straight lines CD, CE go out on opposite sides, making the angles ACE, BCD equal to one another (or else the two angles ACD, BCE). I say that the two lines CD, CE make one straight line. For since the angle ACE is equal to the angle BCD, then, adding the common angle BCE, the two angles ACE, ECB will be equal to the two angles DCB, BCE [Margin: b — Axiom 2]. But the angles ACE, ECB are equal to two right angles [Margin: c — *Elements* I.13]. Therefore the two angles DCB, BCE are also equal to two right angles. Hence CD, CE will be one straight line [Margin: a — *Elements* I.14]. This, as you see, is the converse of Proposition 15.

Diagram. The straight lines meeting at C form the vertical-angle "X"; CD and CE are shown to lie in one straight line.

- A (top-left)
- B (bottom-right)
- C (center): the point on AB
- D (bottom-left)
- E (top-right)

EX PELETARIO.

Enunciation.—If four straight lines going out from one point make the pairs of opposite angles equal to one another, then any two opposite lines will be joined into a straight and continuous line.

Demonstration.—From the point A let four lines AB, AC, AD, AE be drawn, making the two opposite angles BAE, CAD equal to one another, and likewise the two opposite angles BAC, DAE equal to one another. I say that BA, AD make one straight line, and so do CA, AE. Since the angles BAE, CAD are equal, if to them are added the equal angles BAC, DAE, the two angles BAE, BAC will be equal to the two angles CAD, DAE [Margin: b — Axiom 2]. Therefore both this pair and that are half of the four angles set about the point A. But these four are equal to four

right angles, by the second corollary of the preceding proposition. Therefore the two angles BAE, BAC are equal to two right angles; and so CA, AE will make one straight line [Margin: c — *Elements* I.14]. In the same manner it will be shown that BA, AD make one straight line. For by the same reasoning the two angles BAE, EAD will be equal to the two angles DAC, CAB. Hence, as before, the proposition will be concluded. Peletier, however, demonstrates this same thing by an argument leading to what cannot be — a *reductio ad absurdum*. We, nonetheless, have with very good reason set our own direct demonstration before his.

Diagram. Four straight lines radiating from the central point A to B, C, D, E, with BAD forming one straight line and CAE the other.

- A (center): the common point of origin
- B (top-left)
- C (top-right)
- D (bottom): opposite B, so that BAD is straight
- E (bottom-left): opposite C, so that CAE is straight

THEOR. 9. PROPOS. 16.

[Margin: xvj (16)]

(*Enunciation.*) In any triangle, if one of its sides is produced, the exterior angle is greater than either of the interior and opposite angles.

(*Demonstration.*) Let side BA of triangle ABC be produced to D. I say that the exterior angle DAC is greater than the interior and opposite angle ABC, and likewise greater than the interior and opposite angle ACB. Bisect AC at E [Margin: a — I.10], and from B, through E, draw the line BEF so that the segment EF cut off is equal to the straight line EB [Margin: b — I.3]; then draw the line FA. Now the sides CE, EB of triangle CEB are equal to the sides AE, EF of triangle AEF, each to each, by the construction; and the angles at E contained by these sides are equal to one another [Margin: c — I.15], being vertical angles about the point E. Therefore the base CB will be equal to the base AF, and angle ECB to angle EAF [Margin: d — I.4]. But the exterior angle DAC is greater than angle EAF — the whole being greater than its part. Therefore the exterior angle DAC is greater than the interior and opposite angle ACB.

Now if side CA is produced to G, and AB is bisected at H, and the line CHI is drawn so that HI is equal to the straight line HC, and the line IA is drawn, it will be

shown by exactly the same reasoning that the exterior angle GAB is greater than the interior and opposite angle ABC. But angle DAC is equal to angle GAB [Margin: e — I.15], since the lines BD and CG cut one another at A. Therefore angle DAC is also greater than the interior and opposite angle ABC. And this same angle DAC was likewise shown to be greater than the interior and opposite angle ACB. Therefore, in any triangle, when one side is produced, and so on. Which was to be demonstrated.

Diagram. The construction figure for Proposition 16. A triangle ABC lies at the center, with side BA produced upward to D and side CA produced upward-left to G; two auxiliary lines cross the figure.

- G (top left): endpoint of side CA produced
- A (top center): apex of the triangle, the common vertex
- D (top right): endpoint of side BA produced
- I (far left): endpoint of the line CHI (with HI = HC)
- H (center-left): midpoint of AB
- E (center): midpoint of AC
- F (right): endpoint of the line BEF (with EF = EB)
- B (bottom left): base vertex
- C (bottom right): base vertex

SCHOLIUM.

Euclid did not say that the exterior angle DAC is greater than the interior angle BAC adjacent to it, but only that it exceeds in magnitude either of the interior and opposite angles ACB, ABC. For an exterior angle can be equal to its adjacent interior angle — namely when the exterior angle is right, since then the adjacent angle is necessarily right as well. It can also be smaller, namely when it is acute; for on that supposition the adjacent angle will be obtuse. Only when the exterior angle is obtuse, then, will it exceed its adjacent interior angle, since that adjacent angle will then necessarily be acute. All of this is easily gathered from Proposition 13, by which an exterior angle together with its adjacent interior angle is equal to two right angles.

As for what we undertook to demonstrate in the scholium to Proposition 6 of this book — namely that this proposition cannot be converted, since an exterior angle can be greater than any interior and opposite one — we shall now resolve it by the case of a quadrilateral figure, one side of which is produced, in which the exterior angle is greater than any interior and opposite one.

Let ABCD be a quadrilateral figure whose angle BAD is constructed obtuse and whose angle ABC is constructed right, on this condition: that the sides AB, DC, produced toward the parts B and C, meet at a point E, and likewise the sides DA, CB, produced toward the parts A and B, meet at a point F. This will come about if an obtuse-angled triangle ADE is constructed, and, DA being produced toward A, a perpendicular FB is drawn from any point F to AE, cutting side DE at C. I say that, if AD is produced to G, the exterior angle CDG is greater than any of the three interior angles BAD, ABC, BCD opposite to it. For since ADE is a triangle, the exterior angle EDG is greater than the interior opposite angle DAE [Margin: a — I.16]. Again, since DAB is obtuse and greater than the right angle ABC, EDG will also be much greater than ABC. Finally, since in triangle CDF the exterior angle CDG is greater than the interior and opposite angle FCD [Margin: b — I.16], it is manifest that in the quadrilateral ABCD the exterior angle CDG is greater than the interior and opposite angles BAD, ABC, BCD. Therefore this Proposition 16 cannot be converted, since its antecedent and consequent are not reciprocal, as has been demonstrated.

Diagram. The quadrilateral figure of the scholium. Quadrilateral ABCD is embedded in triangle ADE, whose base is the horizontal line FADG; the perpendicular FB rises to meet side DE at C.

- E (top left): the meeting point of AB and DC produced
- C (upper center): the point where the perpendicular FB meets DE
- B (left, below E): the foot on the perpendicular
- F (bottom left): the point from which the perpendicular is dropped
- A (bottom center-left): obtuse-angle vertex of the quadrilateral
- D (bottom center): vertex
- G (bottom right): endpoint of AD produced

EX PROCLO.

It follows from this proposition that from the same point to one and the same straight line no more than two equal straight lines can be drawn. For if it were possible, let three equal straight lines AB, AC, AD be drawn from A to the line BC. Then since the sides AB, AC are equal, the angles ACB and ABC on the base BC will be equal [Margin: a — I.5]. Again, since the sides AB, AD are equal, the angles ADB and ABD on the base BD will be equal [Margin: b — I.5]. Therefore, since each of the angles ACD and ADB is equal to angle ABC [Margin: c — Ax. 1 (pron.)], angle ADB will be equal [to angle ACB] [?] — an exterior angle equal to the interior and opposite one — which is absurd, since by this Proposition 16 the exterior is

greater than the interior. Therefore no more than two equal straight lines, drawn between one another as equals, can be drawn from A to BC. Which was the thing proposed.

Diagram. The tall narrow figure for Proclus's corollary. From apex A three lines descend to the base line BC, meeting it at closely spaced points.

- A (apex): the external point
- B (base, left): base endpoint
- D (base, center-right): point where the third line AD meets the base
- C (base, right): base endpoint

PROBL. 10. PROPOS. 17.

[Margin: xvij (17)]

(Enunciation.) In any triangle, any two of its angles, taken in any combination, are less than two right angles.

(Demonstration.) Let ABC be a triangle. I say that the two angles ABC and ACB are less than two right angles; likewise the two angles CBA and CAB; and likewise the two angles BAC and BCA. For let any two sides, namely CB and CA, be produced to D and E. Since, then, the exterior angle ABD is greater than the interior and opposite angle ACB [Margin: d — I.16], if the common angle ABC is added, the two angles ABD, ABC will be greater than the two angles ABC, ACB [Margin: e — Ax. 4 (pron.)]. But ABD, ABC are equal to two right angles [Margin: f — I.13]. Therefore ABC, ACB are less than two right angles.

By the same reasoning the angles CBA and CAB will be less than two right angles. For since the exterior angle ABD is greater than the interior and opposite angle CAB [Margin: g — I.16], then, adding the common angle ABC, the two angles ABD, ABC will be greater than the two angles CAB, CBA [Margin: h — Ax. 4 (pron.)]. And since those two are equal to two right angles [Margin: i — I.13], these other two will be less than two right angles.

In just the same way we shall show that the two angles BAC, BCA are less than two right angles. For since the exterior angle BAE is greater than the interior and opposite angle BCA [Margin: m — I.16], if the common angle BAC is added, the two angles BAE, BAC will be greater than the two angles BCA, BAC [Margin: n — Ax. 4 (pron.)]; and therefore, since those two are equal to two right angles

[Margin: o — I.13], these two will be less than two right angles. Therefore in any triangle, and so on. Which was to be demonstrated.

Diagram. The figure for Proposition 17. Triangle ABC has side CB produced beyond B to D and side CA produced beyond A to E.

- E (top): endpoint of side CA produced
- A (apex): the upper vertex
- D (bottom left): endpoint of side CB produced
- B (bottom, right of D): base vertex
- C (bottom right): base vertex

EX PROCLO.

From this it is evident that from the same point to the same straight line no more than one perpendicular can be drawn. For if it were possible, let two perpendiculars AB, AC be drawn from A to the straight line BC. Then in triangle ABC the two interior angles B and C will be equal to two right angles, since they are both right — which is absurd. For any two angles in any triangle have been shown to be less than two right angles [Margin: a — I.17]. Therefore no more than one perpendicular can be drawn from A to BC. Which was the thing proposed.

Diagram. A small figure showing the assumed pair of perpendiculars. From apex A two lines descend to the horizontal base, meeting near a point marked C.

- A (apex): the external point
- B (base): one foot (barely distinguished from C on the short base)
- C (base right): the other foot

COROLLARIVM I.

It is also clear from this that in every triangle having one right or obtuse angle, the remaining angles are acute, as we noted at Definition 26 of this book. For since, by this proposition, any two angles are less than two right angles, it is necessary that, if one is right or obtuse, each of the rest be acute — otherwise we should have to admit two right angles in a triangle, or angles greater than two right angles.

COROLLARIVM II.

It also follows from this proposition that, if a straight line makes unequal angles with another straight line — one acute and the other obtuse — the perpendicular

dropped from any point of it to that other line falls on the side of the acute angle. For let the straight line AB make with the straight line CD unequal angles, namely ABD acute and ABC obtuse, and from the point A let a perpendicular AD be dropped to CD at any point. I say that AD falls on the side of the acute angle ABD. For if it does not fall on the side of angle ABD, let it fall, if possible, on the side of the obtuse angle ABC. Then the two angles ABC and ACB — the one obtuse, the other right — in triangle ABC would be greater than two right angles; but they are also less than two right angles [Margin: b — I.17], which is absurd. Therefore the perpendicular drawn from A to CD does not fall on the side of the obtuse angle. Hence it falls on the side of the acute angle.

Diagram. The figure for Corollary II. An obtuse triangle rises from the horizontal line CD, with apex A and the perpendicular from A to the line.

- A (apex): the point from which the perpendicular is dropped
- C (base left): endpoint of the line CD
- B (base center): the point of intersection on CD
- D (base right): the assumed foot of the perpendicular

COROLLARIVM III.

By the same reasoning it becomes manifest from this proposition that all the angles of an equilateral triangle, and the two angles of an isosceles triangle above its base, are acute. For since both any two angles in an equilateral triangle and the two above the base in an isosceles triangle are equal to one another [Margin: c — I.5], and since at the same time both those two and these are less than two right angles [Margin: d — I.17], each of them will be less than a right angle, that is, acute. For if it were right or obtuse, both together would be equal to, or greater than, two right angles.

THEOR. 11. PROPOS. 18.

[Margin: xvij (18)]

(Enunciation.) The greater side of any triangle subtends the greater angle.

(Demonstration.) In triangle ABC let side AC be greater than side AB. I say that angle ABC, subtended by the greater side AC, is greater than angle ACB, which is subtended by the lesser side AB. For from AC let AD be cut off equal to AB [Margin: a — I.3], and let the line BD be drawn. Since, then, the two sides AB, AD are equal by construction, the angles ABD, ADB will be equal [Margin: b — I.5].

But angle ADB is greater than angle ACB [Margin: c — I.16]. Therefore angle ABD is also greater than angle ACB. And since the whole angle ABC is still greater than angle ABD [Margin: d — Ax. 9 (pron.)], angle ABC will be much greater than angle ACB. By the same reasoning, if side AC is supposed greater than side BC, you will show that angle ABC is greater than angle BAC — namely, if a line equal to CB is cut off from CA, and so on. Therefore the greater side of any triangle subtends the greater angle. Which was to be demonstrated.

Diagram. The figure for Proposition 18. Triangle ABC with apex A; the point D lies on side AC, and BD is drawn.

- A (apex): the upper vertex
- D (on side AC): the point where AD (= AB) is cut off
- B (bottom left): base vertex
- C (bottom right): base vertex

COROLLARIVM.

From this it follows that all three angles of a scalene triangle are unequal, as we noted at Definition 15 of this book. For let ABC be a scalene triangle whose greatest side is AC, whose least is BC, and with AB holding the middle place. I say that all its angles are unequal. For since side AC is supposed greater than side AB, by this proposition angle B will be greater than angle C. By the same reasoning angle C will be greater than angle A, since side AB is also supposed greater than side BC. Therefore all three angles are unequal — B the greatest, A the least, and C holding the middle place between the two.

Diagram. A small right-angled scalene triangle.

- A (apex): upper vertex
- B (bottom left): base vertex
- C (bottom right): base vertex

THEOR. 12. PROPOS. 19.

[Margin: xix (19)]

(Enunciation.) In any triangle the greater angle is subtended by the greater side.

(Demonstration.) In triangle ABC let angle B be greater than angle C. I say that side AC, subtending the greater angle B, is greater than side AB, which subtends the lesser angle C. For if side AC is not greater than side AB, it is either equal to it

or less. If AC be said to be equal to AB, then angle B will be equal to angle C [Margin: a — I.5]; but it is also greater, by hypothesis, which is absurd. But if AC be said to be less than side AB, then angle B, subtended by the lesser side AC, will be less than angle C, subtended by the greater side AB; yet it is supposed greater, which is even more absurd. Since, therefore, side AC is neither equal to side AB nor less than it, it is greater. By the same reasoning it will be proved that side AC is greater than side BC, if angle B is granted greater than angle A. Therefore in any triangle the greater angle is subtended by the greater side. Which it was proposed to demonstrate.

Diagram. The plain triangle for Proposition 19. A tall triangle with apex A over base BC. (Faint mirrored letters visible around the figure are show-through from the facing folio.)

- A (apex): the vertex at the greater angle
- B (bottom left): base vertex at the greater angle
- C (bottom right): base vertex at the lesser angle

COROLLARIVM.

It follows from this proposition that, of all the straight lines drawn from any point to a straight line, the one that is perpendicular is the least. For let lines AD, AE, AF, and others be drawn from the point A to the straight line BC, of which AD alone is perpendicular to BC — and no other can be, since from the same point to the same straight line only one perpendicular can be drawn, as we demonstrated after Proposition 16 from Proclus. I say that AD is the least of them all. For in triangle AED, since the two angles ADE, AED are less than two right angles [Margin: b — I.17], and ADE is set right, AED will be acute. Therefore side AE will be greater than side AD [Margin: c — I.19].

Diagram. The figure for the corollary on the least (perpendicular) line. From apex A several lines descend to the base line BC, meeting it at the points F, E, D in order; AD is the perpendicular.

- A (apex): the external point
- B (base left): endpoint of the base
- F (base): foot of line AF
- E (base): foot of line AE
- D (base): foot of the perpendicular AD
- C (base right): endpoint of the base

In the same way we shall show that all the other straight lines are greater than the straight line AD; and hence the perpendicular AD will be the least of all.

EX PROCLO.

We can also demonstrate this same theorem by a direct, affirmative demonstration, without the aid of the preceding one — provided, however, that the following theorem is first demonstrated:

(Enunciation.) If an angle of a triangle is bisected, and the straight line that bisects the angle, drawn to the base, divides it into unequal parts, then the sides containing that angle will be unequal — the greater being the one that coincides with the greater segment of the base, the lesser being the one that coincides with the lesser segment.

(Demonstration.) In triangle ABC let angle BAC be bisected by the straight line AD, which cuts the base BC into unequal parts, and let DC be the greater segment. I say that side AC is greater than side AB. For let AD be produced to E [Margin: a — I.3] so that DE is equal to AD. Then from the greater segment DC let the straight line DF be cut off equal to the lesser segment DB, and through F, from E, let the straight line EFG be produced. Since, then, the sides AD, DB of triangle ADB are equal to the sides ED, DF of triangle EDF, each to each, by construction — and the angles ADB, EDF contained by these sides are equal [Margin: b — I.15] — the bases AB and EF will be equal, and angle FED equal to angle BAD [Margin: c — I.4]. But angle CAD is also equal to angle BAD, by hypothesis. Therefore the angles CAD, GEA of triangle AGE will be equal [Margin: d — Ax. 1 (pron.)]; and hence the sides AG, EG will be equal [Margin: e — I.6]. But the straight line AC is greater than AG; therefore AC is also greater than EG. And since EG is greater than EF, AC will be much greater than EF. Since, then, it has been demonstrated that the straight line EF is equal to the straight line AB, side AC will be greater than side AB. Which was to be shown.

Diagram. The figure for Proclus's theorem. Triangle ABC with apex A; the bisector AD meets base BC at D and is produced downward to E; the line EFG crosses through F.

- A (apex): the bisected-angle vertex
- G (right, on side / on EFG produced): endpoint of the line EFG
- F (center): point on the base region where DF (= DB) ends and EFG passes
- B (bottom left): base vertex
- D (base center): point where the bisector meets the base

- C (base right): base vertex
- E (bottom, below D): endpoint of AD produced ($DE = AD$)

With this theorem established, Proposition 19 will be demonstrated as follows. In triangle ABC let angle ABC be greater than angle C. I say that side AC is greater than side AB. For let the straight line BC (upon which the said unequal angles are set) be bisected at D; from A, through D, let the straight line ADE be produced [Margin: f — I.3] so that DE is equal to AD, and let the line BE be drawn. Since, then, the sides AD, DC of triangle ADC are equal to the sides ED, DB of triangle EDB, each to each, by construction — and the angles ADC, EDB contained by these sides are equal [Margin: g — I.15] — the bases AC and BE will be equal, and angle ACD equal to angle EBD [Margin: a — I.4]. And since angle ACD is supposed less than angle ABC, angle EBD will also be less than that same angle ABC. Hence angle ABE will be divided by the straight line BD into unequal parts. If, then, it is bisected by the straight line BF, BF will fall above ED, because angle ABD is greater than angle EBD [Margin: b — Ax. 9 (pron.)]. And since EF is greater than ED — and ED was set equal to AD — EF will be greater than AD. But AD is still greater than AF [Margin: c — Ax. 9 (pron.)]. Therefore EF will be much greater than AF. And so, since the straight line BF, bisecting angle ABE, cuts the base AE unequally at F — the greater segment being EF and the lesser AF — by the theorem just demonstrated from Proclus, side BE will be greater than side AB. But BE was shown to be equal to side AC. Therefore side AC will also be greater than side AB. Which was to be demonstrated.

Diagram. The figure for the direct demonstration of Proposition 19. Triangle ABC with apex A; base BC is bisected at D and AD produced to E, with BF drawn and F marked on line AE.

- A (apex): the vertex
- F (center): the point where BF meets AE
- B (bottom left): base vertex
- D (base center): midpoint of BC
- C (bottom right): base vertex
- E (bottom, below D): endpoint of AD produced ($DE = AD$)

SCHOLIUM.

This Proposition 19 is the converse of Proposition 18, as is evident. Campanus [Campanus of Novara], however, entirely inverts the order of these two propositions, so that the one that with us is 18 is 19 with him, and vice versa. He

demonstrates both by reduction to the impossible, whereas Euclid confirmed Proposition 18 directly and affirmatively, as is clearly plain from what has been said.

We can also convert the theorem demonstrated from Proclus, in this way:

(Enunciation.) If two sides of a triangle are unequal, the straight line that bisects the angle contained by them will cut the base into unequal parts, and the greater segment will be next to the greater side.

(Demonstration.) Let the two sides AB, AC of triangle ABC be unequal — AC the greater and AB the lesser. And let the straight line AD, bisecting angle BAC, cut the base BC at D. I say that segment DC is greater than segment DB. For if it is not greater, it is either equal or less. If it be said to be equal, produce AD to E [Margin: a — I.3] so that DE is equal to DA, and draw the line EC. Since, then, the sides AD, DB are equal to the sides ED, DC, each to each — namely ED by construction and DC by the adversary's hypothesis — and the angles at D contained by these sides are equal [Margin: b — I.15], the base AB will be equal to the base EC, and angle BAD to angle CED [Margin: c — I.4]. But angle CAD was also set equal to angle BAD. Therefore the angles CED, CAD will be equal; and hence side AC is equal to side EC [Margin: d — I.6]. Since, then, it has been shown that side AB is also equal to side EC, the sides AC, AB will be equal [Margin: e — Ax. 1 (pron.)] — which is absurd, since AC was supposed greater than AB. Therefore segment DC will not be equal to segment DB. But if DC be said to be less, and DB greater, then by Proclus's theorem side AB will be greater than side AC; yet it was supposed less, which is much more absurd. Therefore DC will not be less than DB. Hence it must necessarily be greater. In the same way this theorem can also be demonstrated.

Diagram. The figure for the converse (POTERIMVS) theorem. Triangle ABC with apex A; the bisector AD meets base BC at D and is produced downward to E, with EC drawn.

- A (apex): the bisected-angle vertex
- B (bottom left): base vertex
- D (base center): point where the bisector meets the base
- C (bottom right): base vertex
- E (bottom, below D): endpoint of AD produced ($DE = DA$)

(Enunciation.) If a straight line bisecting an angle of a triangle also bisects the base, the two sides containing the angle will be equal to one another; and if the sides are equal, the straight line that bisects the angle will also bisect the base.

(Demonstration.) First, let the straight line AD, bisecting angle BAC, also bisect the base BC at D. I say that the sides AB, AC are equal to one another. We shall demonstrate this by the same reasoning by which, in the preceding theorem, it was shown that side AC is equal to side AB when segment DC is set equal to segment DB — provided you construct the figure in the same way. For since the sides AD, DB are equal to the sides ED, DC [Margin: a — I.15], and the angles at D contained by these sides are equal [Margin: b — I.4], the bases AB, EC will be equal, and angle CED equal to angle BAD — that is, to angle CAD. Therefore AC will be equal to EC [Margin: c — I.6], that is, to AB.

Next, let the sides AB, AC be equal, and let the straight line AD, cutting the base BC at D, bisect angle BAC. I say that segment DC is equal to segment DB. For since the sides AD, AB are equal to the sides AD, AC, each to each, and the angles at A contained by these sides are also equal by hypothesis, the bases BD, DC will be equal [Margin: d — I.4].

Diagram. The figure for the PRIMVM theorem. Triangle ABC with apex A; AD bisects both the angle and the base BC at D and is produced downward to E.

- A (apex): the bisected-angle vertex
- B (bottom left): base vertex
- D (base center): midpoint of the base
- C (bottom right): base vertex
- E (bottom, below D): endpoint of AD produced

Theorem 13. Proposition 20.

[Margin: xx (proposition index)]

In every triangle two sides, taken in any way whatever, are greater than the remaining side.

Demonstration (Clavius). Let ABC be a triangle. I say that any two of its sides — say AB and AC together — are greater than the remaining side BC. Let one of them, say CA, be produced to D, and let the straight line AD be made equal to the other, unproduced side AB [Margin: c, 3. primi = Elem. I.3]; and let the straight line DB be drawn. Since, then, the two sides AB and AD are equal to one another by construction, the angles ABD and ADB are equal to one another [Margin: f, 5. primi = Elem. I.5]. But the angle CBD is greater than the angle ABD [Margin: g, 9. pron. = Axiom 9, the whole is greater than the part]; therefore the angle CBD is also greater than the angle ADB. So in triangle CBD the side CD, which lies

opposite the greater angle, will be greater than the side BC, which lies opposite the smaller angle CDB [Margin: h, 19. primi = Elem. I.19].

Now the two sides AB and AC together are equal to CD — for if to the equals AB and AD the common AC is added, the wholes become equal [Margin: i, 2. pron. = Axiom 2], namely the line composed of AB and AC and the line composed of AD and AC. Therefore the two sides AB and AC together are also greater than the side BC. In the same way it will be shown that any other two sides are greater than the remaining one. Therefore in every triangle two sides are greater than the remaining side, and so on. Which was to be demonstrated.

Diagram. The main figure for Proposition 20. A triangle with apex D at the top; from D two lines descend to B (bottom-left) and to C (bottom-right). Point A lies on the right-hand descending line (the side CA produced to D), so that CA extended reaches D with AD = AB. The line DB is drawn across the interior.

- D (top apex): endpoint of CA produced, with AD = AB
- A (on the right side, between D and C): the triangle's upper-right vertex
- B (bottom-left vertex)
- C (bottom-right vertex)

From Proclus.

This theorem is demonstrated otherwise by the associates of Hero [Hero of Alexandria] and Porphyry [Porphyry], without producing any side, in the following way. Suppose we must prove that the two sides AB and AC of triangle ABC are greater than the side BC. Let the angle BAC contained by those sides be bisected by the straight line AD [Margin: a, 9. primi = Elem. I.9]. Since in triangle CDA the side CD is produced to B, the exterior angle BDA will be greater than the interior and opposite angle CAD [Margin: b, 16. primi = Elem. I.16]; hence it is also greater than the angle BAD. So in triangle ABD the side AB, opposite the greater angle ADB, will be greater than the side BD, which is opposite the smaller angle BAD [Margin: c, 19. primi = Elem. I.19]. By the same reasoning it will be shown that the side AC is greater than CD, because the angle CDA is greater than the angle BAD — that is, than the angle CAD [Margin: d, 16. primi = Elem. I.16] — and so on. Therefore the two sides AB and AC will be greater than the side BC. And the same reasoning holds for any two sides, provided the angle they contain is bisected.

Diagram. The small figure for the Proclus alternative. A triangle with apex A at the top; its base runs B (left) — D (middle) — C (right). The line AD is the bisector of angle BAC, meeting the base at D.

- A (top apex): vertex of the bisected angle
- B (bottom-left base endpoint)
- D (foot of the bisector, on the base between B and C)
- C (bottom-right base endpoint)

Theorem 14. Proposition 21.

[Margin: xxj (proposition index)]

If on one side of a triangle two straight lines be constructed internally from its extremities, these constructed lines will be less than the remaining two sides of the triangle, but will contain a greater angle.

Demonstration (Clavius). In triangle ABC let two straight lines BD and CD be constructed inside the triangle upon the extremities B and C of the side BC, meeting at the point D. I say that BD and CD together are less than the two sides BA and CA together, but that the angle BDC is greater than the angle BAC. For let one of the interior lines, namely BD, be produced to the point E on the side CA.

Since in triangle BAE the two sides BA and AE are greater than the side BE [Margin: e, 20. primi = Elem. I.20], if the common EC is added, BA and AC will be greater than BE and EC [Margin: a, 4. pron. = Axiom 4]. Again, since in triangle CED the two sides CE and ED are greater than the side CD [Margin: b, 20. primi = Elem. I.20], if the common DB is added, CE and EB will be greater than CD and DB [Margin: c, 4. pron. = Axiom 4]. But it has already been shown that AB and CA are greater than BE and EC. Therefore BA and CA are much greater than BD and CD — which was the first point proposed.

Moreover, since the angle BDC is greater than the angle DEC — an exterior angle greater than the interior one [Margin: d, 16. primi = Elem. I.16] — and the angle DEC is likewise greater than the angle BAC, for the same reason, the angle BDC will be much greater than the angle BAC — which was the second point proposed. So, if on one side of a triangle two lines are constructed from its extremities, and so on. Which was to be shown.

Diagram. The main figure for Proposition 21. A triangle with apex A; B at bottom-left, C at bottom-right. From B and C two interior lines run to the point D inside the triangle; the line BD is produced beyond D to E, which lies on the side AC.

- A (top apex)
- E (on the side AC, below A): where BD produced meets CA
- D (interior point): where BD and CD meet
- B (bottom-left vertex)
- C (bottom-right vertex)

Scholium.

How rightly Euclid said that those two lines constructed inside the triangle must be drawn from the extremities of one side can be clearly understood from what we shall presently demonstrate from Proclus: namely, that in right-angled triangles, or even obtuse-angled ones [amblygonia, obtuse-angled], two lines can be constructed inside the triangle upon one side, about the right or the obtuse angle — one drawn from the extremity of that side, the other from any point near the other extremity of the same side — which are greater than the remaining two sides of the triangle. Likewise, that in scalene triangles two straight lines can in the same way be constructed inside the triangle upon the greatest side, which contain a smaller angle, and so on.

From Proclus.

Let there be a triangle having, for example, the angle ABC obtuse. I say that from the extremity C, and from any point — say D — near the other extremity B of the side BC, two lines can be drawn inside the triangle to some point, which are greater than the two sides BA and AC. For let the straight line DA be drawn. Since in triangle ABD the two angles ABD and ADB are less than two right angles [Margin: e, 17. primi = Elem. I.17 — printed digit at the scan floor], and ABD is assumed obtuse, ADB will be less than a right angle, and therefore less than the angle ABD. Hence the side AD will be greater than the side AB [Margin: a, 19. primi = Elem. I.19]. From DA let the straight line DE be cut off equal to AB [Margin: b, 3. primi = Elem. I.3], and let the remaining line AE be bisected at F [Margin: c, 10. primi = Elem. I.10].

If, then, a straight line CF is drawn from the extremity C to F, the two constructed straight lines CF and DF inside the triangle will be greater than the two sides BA and AC. For in triangle AFC the two sides AF and FC are greater than the side AC [Margin: d, 20. primi = Elem. I.20]; but AF is equal to FE by construction, so CF and FE together are likewise greater than the side CA. So if the equals ED and AB are added, the straight lines CF and FD become greater than the sides CA and AB [Margin: e, 4. pron. = Axiom 4]. Which is what was proposed. But if a straight line

were drawn to F from the extremity B, the two constructed straight lines CF and BF would be less than the two sides CA and AB, as Euclid demonstrated.

Diagram. The first Proclus figure, an obtuse triangle. A at top-left, B at bottom-left (the obtuse angle is at B), C at bottom-right. D is a point on BC near B; DA is drawn; DE (with $DE = AB$) is cut off along DA; the line CF runs from C to F. Dotted compass-arcs near the top mark the bisection.

- A (top-left apex)
- B (bottom-left vertex, obtuse angle)
- C (bottom-right vertex)
- D (on BC, near B)
- E (on DA, between A and D)
- F (bisection point of AE) [?] — indicated by dotted construction arcs; the letter itself is not clearly legible

Diagram. The second Proclus figure, a scalene triangle. A at top; B at bottom-left; the greatest side BC runs to the bottom-right vertex; D lies on BC near B (with $BD = AB$); AD is drawn; E is a point on AD; CE is drawn from C.

- A (top apex)
- B (bottom-left vertex)
- D (on BC, near B)
- E (on AD)
- C (bottom-right vertex) [?] — faint in the scan

But concerning the construction of two lines inside a triangle that are greater than, or equal to, two sides, we shall show more from Pappus [Pappus of Alexandria] at proposition 34 of this book [Margin: 34. huius lib. = Prop. 34 of this book].

Problem 8. Proposition 22.

[Margin: xxij (proposition index)]

From three straight lines equal to three given straight lines, to construct a triangle. But any two of them, taken in every way, must be greater than the remaining one; for in every triangle two sides, taken in any way, are greater than the remaining one.

Demonstration (Clavius). Let three straight lines A, B, and C be given, any two of which are greater than the remaining one (otherwise no triangle could be

constructed from them, as is clear from Proposition 20, in which it was shown that any two sides of a triangle are greater than the remaining one); and let it be required to construct a triangle having its three sides equal to the three given lines. From any assumed straight line DE, of unlimited length, let DF be cut off equal to the line A [Margin: a, 3. primi = Elem. I.3]; from the remainder FE let FG be cut off equal to B; and from the remainder GE let GH be cut off equal to C.

Then, with center F and radius FD, let the circle DIK be described. Likewise, with center G and radius GH, let another circle HIK be described, which will necessarily cut the former at the points I and K. (For since the two lines FD and GH are supposed greater than FG: if from FE a line FL is taken equal to FD, and from GD a line GM equal to GH, the point M will fall between L and D. For if the point M fell at L, then GL and FL — that is, GH and FD — would be equal to the line FG; but if M fell between G and L, then those same two, FL and GM — that is, DF and GH — would be less than FG; and each of these is contrary to the hypothesis. This is evident from the appended figures.)

From either intersection — namely from K — let the straight lines KF and KG be drawn to the points F and G, and the triangle FGK will have been made, whose sides, I say, are equal to the given lines A, B, and C. For since FK is equal to FD [Margin: b, 15. def. = Elem. I, def. 15], and A is by construction equal to that same FD, the side FK will be equal to the line A [Margin: a, 1. pron. = Axiom 1]. Again, since GK is equal to GH, and C to that same GH, the side GK will likewise be equal to the line C. And it was laid down by construction that the remaining side FG is equal to the remaining line B. Therefore all three sides FK, FG, GK are equal to the three given lines A, B, C. We have thus constructed a triangle from three straight lines equal to three given straight lines. Which was to be done.

Diagram. The first case of the construction. Across the top, three reference segments are drawn with end-serifs and labelled A, B, and C (the three given lines). Below, a horizontal baseline carries the points D (left), F, G, and H, with E at the right end. Two circles are drawn: one centered at F through D, the other centered at G through H; they intersect at K (above the baseline) and at I (below it), and the point M is marked near the center of the baseline. The chords KF and KG close the constructed triangle FGK.

- A, B, C (three reference segments at top): the three given lines
- D, F, G, H (points along the baseline, left to right); E (right end of the baseline)
- K (upper intersection of the two circles)
- I (lower intersection of the two circles)

- M (auxiliary point near the center of the baseline); L referenced in the text [?] — not clearly resolved in the scan

Diagram. The second case (the contrary case where point M would coincide with L). Two circles rest on a common baseline carrying the points D, F, G, and H (with E to the right); the circles meet near the center, where the auxiliary points L and M are marked.

- D, F, G, H (baseline points, left to right)
- L, M (auxiliary points near the center where the circles meet) [?] — small and faint in the scan

Diagram. The third case (the contrary case where point M would fall between G and L, so that the circles fail to reach). Two circles on a common baseline with points D, F, G, H and the auxiliary points L, M near the center.

- D, F, G, H (baseline points, left to right)
- L, M (auxiliary points near the center) [?] — small and faint in the scan

Praxis.

Take a straight line DE equal to any one of the given lines, say B, which we now wish to be the base. Then from D, with radius equal to A, describe an arc; likewise from E, with radius equal to C, another arc cutting the former at F. If then the straight lines DF and EF are drawn, a triangle will have been made having its three sides equal to the three given lines. For the side DF will equal the line A, because of the radius A that was used; the side EF will equal C, because of the radius C that was used; and the side DE was taken equal to the line B from the outset.

Diagram. The praxis figure. A triangle stands on the base DE (D at bottom-left, E at bottom-right), with apex F at the top formed by two intersecting arcs struck from D and from E. To the side, the three given lines are shown as reference segments A, B, and C.

- D (bottom-left, base endpoint)
- E (bottom-right, base endpoint)
- F (apex, intersection of the two arcs)
- A, B, C (reference segments for the three given lines)

Scholium.

By this technique we shall construct, for any given triangle, another entirely equal to it — both as to its sides and angles and as to its area. For let ABC be any triangle, to which an equal one in every respect is to be constructed. I regard its sides as three given straight lines AB, BC, CA, any two of which are greater than the remaining one [Margin: b, 20. primi = Elem. I.20]. Then I take a straight line DE equal to one side, namely BC; and from D, with radius equal to the side AB, I describe an arc, and another from E, with radius equal to the remaining side CA, which cuts the former at F, and so on.

Diagram. The scholium figure. On the left stands the given triangle ABC (A at the apex, B at bottom-left, C at bottom-right). On the right, an equal triangle is being constructed on the base DE (D, E), its apex F reached by dotted arcs struck from D and E.

- A, B, C (the given triangle: apex, bottom-left, bottom-right)
- D, E (base of the constructed triangle)
- F (apex of the constructed triangle, at the intersection of the dotted arcs)

Problem 9. Proposition 23.

[Margin: xxiiij (proposition index)]

On a given straight line, at a given point on it, to construct a rectilineal angle equal to a given rectilineal angle.

Demonstration (Clavius). Let AB be the given straight line, C the given point on it, and DEF the given angle. It is required, then, to construct at the point C on the line AB an angle equal to the angle E. On the lines ED and EF let two points G and H be taken at random, and let them be joined by the straight line GH. Then let a triangle CIK be constructed [Margin: a, 22. primi = Prop. 22] having its three sides equal to the three lines EG, GH, HE — so that CI equals EG, CK equals EH, and IK equals GH. (This will easily be done if CI is taken equal to EG, CL equal to EH, and IM equal to GH; then, from the centers C and I, with radii CL and IM, let circles be described cutting one another at K, and so on.)

I say that the angle C is equal to the angle E. For since the two sides CI and CK are equal to the two sides EG and EH respectively, and the base IK equals the base GH, by construction, the angle C will be equal to the angle E [Margin: b, 8. primi = Elem. I.8]. We have thus made at C an angle equal to the angle E, and so on. Which was to be done.

Diagram. The main figure for Proposition 23 (labels small and near the scan floor). On the left is the given angle at vertex E, its sides carrying the points G and H, which are joined by GH (F at the lower end of one side). On the right runs the given line, drawn as a long diagonal from A (top) to B (bottom), with the given point C on it; at C the triangle CIK is constructed (points I and K), together with the auxiliary points L and M of the fuller construction.

- A (top of the given line); B (bottom of the given line)
- C (the given point on AB)
- E (vertex of the given angle); G, H (points on the sides of the given angle, joined by GH); F (lower endpoint of an angle-side)
- I, K (vertices of the constructed triangle at C); M (auxiliary point) [?]; L referenced in the text [?]
- Several peripheral letters are at the scan floor**

Praxis.

The practical construction of this problem does not differ from the one we gave in the preceding problem, since a triangle must be constructed equal to another in order that an angle equal to the given angle may be produced — as is evident. You will, however, carry out the problem more easily by the following technique. Let AB be the given line, C the point on it, and E the given angle. With center E and any radius, describe an arc GH; and with the same radius, from center C, describe an arc IK; and with the compass, take the arc IK equal to the arc GH. For the straight line CK, when drawn, will make at C an angle equal to the angle E. For if the lines IK and GH were drawn, they would be equal, since we took both distances IK and GH with the compass unchanged. Since, then, the two sides IC and CK are equal to the two GE and EH — because of the equal radii with which the arcs were described — the angles ICK and GEH will be equal [Margin: a, 8. primi = Elem. I.8].

For the rest, by what method a straight line can be drawn from a point taken outside a given line, so as to make with the given line an angle equal to a given rectilineal angle, we shall teach at proposition 31 of this book [Margin: 31. huius lib. = Prop. 31 of this book].

Diagram. The praxis figure for Proposition 23 (small labels near the scan floor). Two small angle-figures: on the left, the angle constructed at C (with A above, the arc IK, and points I, K, B, D); on the right, the given angle at E (with A above, the arc GH, and points G, H, F).

- Left figure: C (vertex), A (above), I, K (ends of the arc), B, D
- Right figure: E/A (vertex region), G, H (ends of the arc), F
- Several peripheral letters are at the scan floor**

Theorem 15. Proposition 24.

[Margin: xxiv (proposition index)]

If two triangles have two sides equal to two sides respectively, but one has the angle contained by the equal straight lines greater than the other's, then they will also have the base greater than the base.

Demonstration (Clavius). Let the two sides AB and AC of triangle ABC be equal to the two sides DE and DF respectively — namely AB to DE and AC to DF — but let the angle A be greater than the angle EDF. I say that the base BC is greater than the base EF. At the line DE and at its point D, let the angle EDG be constructed equal to the angle A [Margin: b, 23. primi = Elem. I.23] (and DG will fall outside the triangle DEF, since the angle EDF is assumed smaller than the angle A); and let DG be made equal to DF — that is, to AC. Then, when the straight line EG is drawn [Margin: c, 3. primi = Elem. I.3], it will fall either above the line EF, or on it, or below it.

First case. Let it first fall above EF, and let the straight line FG be drawn. Since the sides AB and AC are equal to the sides DE and DG respectively, and the angle A equals the angle EDG by construction, the base BC will be equal to the base EG [Margin: d, 4. primi = Elem. I.4]. Again, since the two sides DF and DG are equal to one another, the angles DFG and DGF will be equal [Margin: e, 5. primi = Elem. I.5]. But the angle DGF is greater than the angle EGF [Margin: f, 9. pron. = Axiom 9]; therefore the angle DFG is also greater than that same angle EGF, and so the whole angle EFG is much greater than that same angle EGF. So in triangle EFG the side EG will be greater than the side EF [Margin: a, 19. primi = Elem. I.19]. But EG has been shown equal to BC; therefore BC is also greater than EF. Which is what was proposed.

Second case. Next, let EG fall on EF itself. Then, since again, as before, the base EG is equal to the base BC [Margin: b, 4. primi = Elem. I.4], and EG is greater than EF [Margin: c, 9. pron. = Axiom 9], BC will also be greater than EF. Which is what was proposed.

Third case. Third, let EG fall below EF, and let the straight lines DF and DG be produced to H and I, and let the straight line FG be drawn. Then again, as before, the base EG will be equal to the base BC [Margin: d, 4. primi = Elem. I.4]. Then,

since the two sides DF and DG are equal to one another by construction, the angles GFH and FGI below the base FG will be equal [Margin: e, 5. primi = Elem. I.5]. But the angle FGI is greater than the angle FGE [Margin: f, 9. pron. = Axiom 9]; therefore the angle GFH is also greater than that same angle FGE, and so the whole angle EFG is much greater than that same angle FGE. So in triangle EFG the side EG will be greater than the side EF [Margin: g, 19. primi = Elem. I.19]. But EG has been shown equal to BC; therefore the base BC is also greater than the base EF. So, if two triangles have two sides equal to two sides, and so on. Which was to be shown.

Diagram. The figure for the first two cases. On the left stands triangle ABC (A at the apex, B at bottom-left, C at bottom-right). On the right stands triangle DEF (D at the apex, E at bottom-left, F below), with the constructed line DG (G to the lower-right) and EG drawn so as to fall above EF.

- A, B, C (the first triangle: apex, bottom-left, bottom-right)
- D (apex of the second triangle)
- E (bottom-left of the second triangle)
- F (base point of the second triangle)
- G (constructed point, lower-right, with $DG = DF$)

Diagram. The figure for the third case (small labels near the scan floor). On the left, triangle ABC (A apex, B bottom-left, C bottom-right). On the right, triangle DEF with DG, the sides DF and DG produced to H and I, EG falling below EF, and FG drawn.

- A, B, C (the first triangle)
- D (apex of the second triangle); E, F, G (base region); H, I (points where DF and DG are produced) [?] — H and I are small and faint in the scan

Scholium.

Should anyone perhaps ask why, in Proposition 4, Euclid concluded from the fact that two sides of one triangle, each to each, together with the angles contained by those sides, are equal, not only the equality of the bases but also of the triangles and of the remaining angles; whereas here, from the fact that two sides of one triangle are equal to two sides of another, each to each, but the angles contained by those sides are unequal, he gathers only the inequality of the bases, and not of the triangles and remaining angles — to this it must be answered that Euclid, a most skilled geometer, did so of necessity.

For from the premise of this theorem the inequality of the bases always follows, so that the base of the triangle whose angle contained by the assumed sides is greater exceeds the base of the other, whose angle is smaller, as has been demonstrated; but it is not necessary that the former triangle be greater than the latter. For, as we shall demonstrate most clearly from Proclus at proposition 37 of this first book [Margin: 37. huius 1. libri = Prop. 37 of Book I], a triangle having the greater angle is sometimes equal to the triangle having the smaller angle, sometimes less than it, and sometimes greater. It could not therefore be inferred universally, from the fact that one triangle's angle is greater than another's, that the triangle is also greater — since it is sometimes equal, sometimes less, and sometimes greater. The same can be said of the remaining angles.

For in the first figure of this theorem the angle ABC is always less than the angle DEF; since the angle DEG (which is equal to the angle ABC by Proposition 4) is less than that same angle DEF [Margin: a, 9. pron. = Axiom 9], being a part relative to the whole. In the second figure, however, the angle ABC is indeed equal to the angle DEF, by Proposition 4; but the angle ACB is less than the angle DFE, since the angle DFE is greater than the angle DGF — an exterior angle greater than the interior and opposite one [Margin: b, 16. primi = Elem. I.16] — and the angle DGF is equal to the angle ACB. Finally, in the third figure the angle ABC is indeed greater than the angle DEF, because the angle DEG (being equal, by Proposition 4, to the angle...

(Continuation of the scholium on Proposition 24.)

(since angle DFE is greater than angle ABC), it is greater than the same angle DEF — the whole greater than the part [Margin c: Axiom 9 (the whole is greater than its part)]. But angle ACB is less than the same angle DFE. For if the straight line EF is produced so as to cut the straight line DG at K, angle DFE will be greater than angle DKE [Margin d: I.16], the exterior greater than the interior; and angle DKE is in turn greater than angle DGE, again an exterior greater than the interior and opposite. Therefore angle DFE will be much greater than angle DGE, which by Proposition 4 is equal to angle ACB. The same is clear from the second part of Proposition 21 [Margin: cf. *Elements* I.21], for the interior lines DF, EF contain a greater angle than the exterior lines DG, EG contain. Hence nothing certain could be concluded about the inequality of the remaining angles, since one is now greater than the other, now less, and now equal.

Diagram. A composite figure accompanying the discussion above. On the left is a triangle with apex **A** (top), **B** (lower left), and **C** at the base, with **E** just to the right of **C**. On the right is a construction from **D** (top) with the lines running down to

labelled points **F, K, G, H, I** (the points where the produced lines DG, EF, etc. meet).

- A (upper left): apex of the left triangle
- B (lower left): base vertex
- C, E (base of left triangle): base vertices
- D (top right): apex of the right construction
- F, K, G, H, I (right, descending): intersection points of the produced straight lines

Since in triangle DEF the two angles E and F are always less than two right angles [Margin a: I.17], at least one of them will necessarily be acute. Therefore, if on the side of the acute angle (as in the first figure, on the side of angle DFE; in the second and third, on the side of angle DEF) an angle EDG is constructed equal to angle A, and the rest is carried out as stated in the proposition, then the straight line EG, once drawn, will always fall beyond the point F. For since angle DFE is acute, its adjacent angle — formed by producing the side EF — will be obtuse [Margin b: I.13]. Hence, when the straight line DG is produced cutting the side EF extended at H, the straight line DH will be greater than the side DF [Margin c: I.19], since it is opposite the obtuse angle in triangle DFH; and consequently the straight line EG, which cuts off DG equal to DF, will necessarily fall beyond F. For if it passed through F, or fell short of it, it would cut off from DG a line greater than the side DF — namely, either DH or something greater than DH. If this is observed, the first case of the proposition will suffice; and this was perhaps the reason why Euclid, in Theon's [Theon of Alexandria] edition, was content with the first case alone.

Diagram. Two triangles illustrating the acute-angle case. On the left, triangle with apex **A** (top), **B** (lower left), **C** (base). On the right, triangle **D** (top), **E** (lower left), **F** (base), with construction lines drawn down to the right toward the point where DG cuts EF extended.

- A (top left): apex
- B, C (base left triangle)
- D (top right): apex
- E, F (base of right triangle)
- (Additional small construction points at the lower right — not clearly resolved [?])

Problem 16. Proposition 25.

[Margin: XXV]

(Euclid's enunciation.) If two triangles have two sides equal to two sides, each to each, but the base of the one greater than the base of the other, then they will also have the angle contained by the equal straight lines greater in the one than in the other.

(Clavius's demonstration.) Let the two sides AB, AC of triangle ABC be equal to the two sides DE, DF of triangle DEF, each to each — that is, AB to DE and AC to DF — while the base BC is greater than the base EF. I say that angle A is greater than angle D. For if angle A is not greater than angle D, it will be either equal or less. If it is said to be equal, then, since the two sides about A are also equal to the two sides about D, each to each, by hypothesis, the base BC will also be equal to the base EF [Margin a: I.4] — which is absurd, for the base BC was assumed to be greater than the base EF. But if angle A is said to be less than angle D, then, because of the equality of the sides about those angles, the base EF will be greater than the base BC [Margin b: I.24] — which is still more absurd, since EF was assumed to be less than BC. Therefore angle A, since it can be neither equal to angle D nor less, must be greater. So if two triangles have two sides equal to two sides, and so on. Which was to be shown.

Diagram. Two separate triangles. Left triangle: **A** (apex), **B** (lower left), **C** (lower right). Right triangle: **D** (apex), **E** (lower left), **F** (lower right).

- A (apex, left)
- B, C (base, left)
- D (apex, right)
- E, F (base, right)

Scholium

This theorem is the converse of the preceding one. For in that one it was demonstrated, from the greater angle, that the base corresponding to it is greater; whereas in this one it was shown, from the greater base, that the angle corresponding to it is greater. These two theorems — namely, 24 and 25 — differ greatly from those set out in Propositions 18 and 19. For in Proposition 19 it was demonstrated that in one and the same triangle the greater side corresponds to the greater angle; but in Proposition 24 the same was shown for two different triangles, two of whose sides are equal to the two sides of the other, and so forth. You will find the same distinction between Propositions 18 and 25.

Menelaus of Alexandria [Menelaus of Alexandria], as Proclus [Proclus] reports, demonstrates this same theorem directly [ostensively], by the following reasoning. With the same triangles set out, from the greater base BC cut off a straight line BG equal to the smaller base EF [Margin c: I.3]. Construct also an angle GBH equal to angle DEF [Margin d: I.23], and let BH be equal to BA [Margin e: I.3], and therefore to DE. Then, with the straight line AH drawn, let another straight line be drawn through G, from H, cutting AC at I. Now since the two sides BA, BH are equal, the angles BAH, BHA will be equal [Margin f: I.5]. Again, since the sides BG, BH are equal to the sides EF, ED, each to each, and angle GBH is equal to angle DEF by construction, the base HG will be equal to the base DF [Margin a: I.4] — and therefore to AC — and angle GHB equal to angle EDF. And since the straight line HI is greater than HG [Margin b: Axiom 9] (which was shown equal to AC), HI will also be greater than AC; but AC is still greater than AI [Margin c: Axiom 9]. Therefore HI is much greater than AI. Hence angle IAH will be greater than angle IHA [Margin d: I.18]. Therefore, adding the two angles BAH, BHA, which were shown to be equal, the whole angle BAC becomes greater than the whole angle BHG [Margin e: Axiom 4 (if equals are added to unequals, the wholes are unequal)]; but angle BHG was demonstrated to be equal to angle D. Therefore angle BAC will also be greater than angle D — which is what was proposed. And if it should happen that the straight line AH falls outside the triangle, the equal angles BAH, BHA must be subtracted, and so on, so that the remaining angle BAC becomes greater than the remaining angle BHG. But when the straight line AH passes through the point B, nothing need be added or subtracted, as is evident.

Diagram. Menelaus's construction: a kite-shaped figure with apex **A** at the top and vertex **H** at the bottom, the left vertex **B** and the right vertex **C**, with **G** on the base and **I** on side AC near C, and the diagonals/lines AH, BH, and the line through G to I crossing in the interior.

- A (top): apex
- B (left): base vertex
- C (right): base vertex
- G (on base BC)
- I (on AC, near C)
- H (bottom): apex of the lower triangle

Hero [Hero of Alexandria], drawing on the same Proclus, demonstrates it this way. With the same triangles set out, let the smaller base EF be produced to G, so that EG is equal to the greater base BC [Margin f: I.3]. Then, with D as centre and

DF as radius, describe a circle, and let ED be produced to H on the circumference. Now since DH is equal to DF [Margin g: I, Def. 15 (definition of the circle)], DH will also be equal to AC. Therefore, adding the equals DE, AB, the lines AC, AB together will be equal to the whole HE [Margin h: Axiom 2]; but AC, AB together are greater than BC [Margin i: I.20], and therefore greater than EG. So HE will also be greater than EG. Hence the circle described from centre E with radius EG will cut the straight line EH — and thus the circumference of the first circle — at the points I and K; and to K let the straight lines DK, EK be drawn. Now since the two sides AB, AC are equal to the two sides DE, DK, each to each (for DK is equal to DF by the definition of the circle, and DF was assumed equal to the side AC), and the base BC is equal to the base EK (since EK is equal to EG by the definition of the circle, and EG was by con-

Diagram. Hero's construction: at the left a small triangle, and to the right two overlapping circles with several labelled points along a horizontal diameter and around the circumferences. Legible labels include the small triangle's apex [?] and base points, and the circle-points **D** and **E** (the two centres) on the diameter, with **H** at the far right / bottom of a circle, and **F**, **K**, **I** among the upper circumference points. Several peripheral labels are at the scan floor and could not be resolved.

- (small triangle, left): apex [?]; base points A, B [?]
- D (centre of first circle, on the diameter)
- E (centre of second circle, on the diameter)
- F, H (on the circumferences) [?]
- I, K (intersection points of the two circles) [?]

(Continuation of Hero's demonstration, from Proposition 25 scholium.)

struction made equal to the base CB.) Therefore angle BAC will be equal to angle EDK [Margin a: I.8]; but angle EDK is greater than angle EDF [Margin b: Axiom 9]. Therefore angle A will turn out to be greater than angle EDF as well — which is what was proposed. And this demonstration is more general than the earlier one, since it fits all cases.

Theorem 17. Proposition 26.

[Margin: xxvj]

(Euclid's enunciation.) If two triangles have two angles equal to two angles, each to each, and one side equal to one side — either the side that lies between the equal angles, or the side that subtends one of the equal

angles — then they will also have the remaining sides equal to the remaining sides, each to each, and the remaining angle equal to the remaining angle.

(*Clavius's demonstration.*) Let the two angles B and C of triangle ABC be equal to the two angles E and EFD of triangle DEF, each to each — that is, B to E and C to EFD — and let, first, the side BC, which lies between angles B and C, be equal to the side EF, which lies between angles E and EFD. I say that the remaining sides AB, AC are also equal to the remaining sides DE, DF, each to each — that is, AB to DE and AC to DF, namely those that subtend the equal angles — and the remaining angle A equal to the remaining angle D. For if the side AB is not equal to the side DE, let DE be the greater, and from it cut off a straight line EG equal to the straight line AB [Margin c: I.3], and draw the straight line GF. Now since the sides AB, BC are equal to the sides GE, EF, each to each, and the angles B and E are equal by hypothesis, angle C will be equal to angle EFG [Margin d: I.4]. But angle C was assumed equal to angle EFD. Therefore angle EFG will be equal to the same angle EFD — the part to the whole — which is absurd. So the side AB is not unequal to the side DE, but equal. Consequently, since the sides AB, BC are equal to the sides DE, EF, each to each, and the contained angles B and E are equal, the bases AC, DF will also be equal, and the remaining angles A and D equal [Margin a: I.4]. Which is what was proposed.

Diagram. Two large triangles. Left triangle: **A** (apex), **B** (lower left), **C** (lower right). Right triangle: apex **D** (malformed glyph in the scan), **E** (lower left), **F** (lower right), with **G** marked on the side DE and the line GF drawn.

- A (apex, left); B, C (base, left)
- D (apex, right)
- E, F (base, right)
- G (on side DE, near the apex)

Next, let the sides AB, DE, which subtend the equal angles C and EFD, be equal to one another. I say again that the remaining sides BC, CA are equal to the remaining sides EF, FD, each to each — that is, BC to EF and CA to FD — and the remaining angle A equal to the remaining angle D. For if the side BC is not equal to the side EF, let EF be the greater; from it take EG equal to BC [Margin b: I.3], and draw the straight line DG. Now since the sides AB, BC are equal to the sides DE, EG, each to each, and the contained angles B and E are equal by hypothesis, angle C will be equal to angle EGD [Margin c: I.4]. But angle C was assumed equal to angle EFD. Therefore angle EGD will also be equal to the same angle EFD — an exterior angle equal to the interior and opposite — which is absurd [Margin d:

I.16]; for it is in fact greater. So the side BC is not unequal to the side EF. Whereupon, as before, the conclusion will follow from Proposition 4 of this book. Therefore, if two triangles have two angles equal to two angles, and so on. Which was to be demonstrated.

Diagram. Two smaller triangles. Left: **A** (apex), **B** (lower left), **C** (lower right). Right: **D** (apex), **E** (lower left), **F** (lower right), with **G** marked below F on EF (its extension) and the line DG drawn.

- A (apex, left); B, C (base, left)
- D (apex, right); E, F (base, right)
- G (on EF / its extension)

Corollary

It follows from the demonstration of this theorem that the whole triangles are also equal in area. For if the sides AB, BC are equal to the sides DE, EF, as was shown, and by hypothesis they contain equal angles B, E, then the whole triangles will also be equal to one another [Margin e: I.4].

Scholium

The first part of this theorem is the converse of Proposition 4, with respect to that part in which, from the equality of the sides and of the angles contained by them, the equality of the bases and of the angles upon the bases was inferred. For in the first part of this theorem it was demonstrated, from the equality of the bases BC, EF and of the angles upon those bases, that the remaining sides of the one triangle are equal to the remaining sides, and the remaining angle to the remaining angle, and so on. This indeed we have already demonstrated by another method at Proposition 8 of this book, as we noted there.

Diagram. A small pair of triangles for the scholium. Left: **A** (apex), **B** (lower left), **C** (lower right). Right: **D** (apex), **E** (lower left), **F** (lower right), with **G** marked at the upper right near D.

- A (apex, left); B, C (base, left)
- D (apex, right); E, F (base, right)
- G (near apex D)

Here a certain theorem must be demonstrated, one that is thoroughly necessary and highly useful in geometrical matters, namely:

In an equilateral or isosceles triangle, a straight line drawn from the angle contained by the two equal sides, which bisects either that angle or the base, is perpendicular to the base; and if it bisects the angle, it will also bisect the base; but if it bisects the base, it will also bisect the angle. And conversely: a perpendicular line drawn to the base bisects both the base and the angle.

Let there be, in triangle ABC, two equal sides AB, AC, and let the straight line AD first bisect the angle A. I say that AD is perpendicular to BC and bisects the base BC. For since the two sides AB, AD are equal to the two sides AC, AD, and contain equal angles by hypothesis, the bases BD, CD will also be equal, and the angles at D — and therefore right angles [Margin a: I.4].

Next, let the straight line AD bisect the base BC. I say that AD is perpendicular to BC and bisects angle A. For since the two sides BD, DA are equal to the two sides CD, DA, and the base AB is equal to the base AC by hypothesis, the angles at D will also be equal [Margin b: I.8] — and therefore right angles — and consequently, by the corollary to Proposition 8 of this book, the angles at A will be equal.

Now let the straight line AD be perpendicular to BC. I say that it bisects both the base BC and the angle A. For the angles B, C upon the base BC will be equal [Margin c: I.5]. And so, since the two angles ABD [and ADB] are equal to the two angles D, C of triangle ACD, each to each, and the side AD, opposite the equal angles B, C, is common, the remaining sides BD, CD will also be equal, and the remaining angles at A equal [Margin a: I.26]. Which was to be demonstrated.

Diagram. A large isosceles triangle: apex **A** (top), with the base **B** (lower left) — **D** (foot of the line from A, on the base) — **C** (lower right), and the straight line AD drawn from the apex to the base.

- A (apex): vertex of the equal sides
- B (lower left), C (lower right): base vertices
- D (on base BC): foot of AD

But this theorem too is true:

A triangle in which a straight line drawn from one of its angles to the base, and perpendicular to it, bisects either the base or the angle, has the two sides containing that angle equal. And if the base is bisected, the angle will also be bisected; but if the angle is bisected, the base too will be divided in half.

(Continuation of the scholium.)

In the same triangle ABC, let AD, perpendicular to BC, first bisect the base BC. I say that the sides AB, AC are equal, and the angles at A equal. For since the two sides BD, DA are equal to the two sides CD, DA, and contain equal angles — namely right angles — the bases AB, AC will also be equal, and the angles at A [Margin b: I.4]. Which is the first point.

Next, let the perpendicular AD bisect angle A. I say that the sides AB, AC are equal, and so are the straight lines BD, CD. For since the two angles D, A of triangle ABD are equal to the two angles D, A of triangle ACD, and the side AD lying between them is common, the sides AB, AC and the sides BD, CD will also be equal [Margin c: I.26]. Which is the second point.

Theorem 18. Proposition 27.

[Margin: xxvij]

(Euclid's enunciation.) If a straight line falling on two straight lines makes the alternate angles equal to one another, then those straight lines will be parallel to one another.

(Clavius's demonstration.) On the two straight lines AB, CD let the straight line EF fall, making the alternate angles AGH, DHG equal to one another. I say that the lines AB, CD are parallel. For if they are not parallel, they will eventually meet if produced indefinitely; for if they never met, they would be parallel, by the definition of parallels. Let them then meet, toward B and D, at the point I. Now since GIH is a triangle (for AB is continued as a straight line, and likewise CD, as far as the point I), and angle AGH was assumed equal to angle DHG, the exterior angle AGH will be equal to the interior and opposite angle DHG — which is absurd [Margin a: I.16], since the exterior is greater than the interior. But if AB, CD are said to meet toward A and C, at the point K, then again, by the same reasoning, the exterior angle DHG will be equal to the interior and opposite angle AGH — which is absurd. Therefore the lines AB, CD will not meet. Hence they will be parallel. In the same way, if the alternate angles BGH, CHG are assumed equal, it will be demonstrated that the lines AB, CD are parallel. So if a straight line falling on two straight lines, and so on. Which was to be shown.

Diagram. The two straight lines AB and CD crossed by the transversal EF, extended to form an "hourglass" figure. Along the upper line: **A, G** (where EF crosses AB), **B**; along the lower line: **C, H** (where EF crosses CD), **D**; the transversal running **E** (top) to **F** (bottom); with **I** at the right (where AB, CD would meet if produced toward B, D) and **K** at the left (toward A, C).

- E (top of transversal), F (bottom of transversal)
- A, G, B (upper line AB; G on the transversal)
- K, C, H, D (lower line CD; H on the transversal)
- I (right, the hypothetical meeting point toward B, D)

Scholium

It is necessary that the lines said to be parallel lie in the same plane, as is clear from the definition. Therefore it is not enough that two alternate angles be equal to one another for the two lines to be proved parallel, unless it is granted that they lie in one and the same plane. For it can happen that a straight line falling on two straight lines not in the same plane makes the alternate angles equal. Let CD be perpendicular to the straight line AB, which lies in the plane below; and from C, in another plane, let another perpendicular CE be drawn to CD, so that the point E is understood to be aloft. Once this is set up, it is clear that the straight line CD, falling on the straight lines CE, AB, makes the two alternate angles ECD, ADC equal, since they are right angles; and yet CE, AB are not parallel, because they do not lie in the same plane. Euclid, however, did not append this condition — "lying in the same plane" — to the proposition, just as he did not in what follows; because, since in the first six books he deals with plane figures only, as we said above, everything must necessarily be understood to lie in the same plane. But in the eleventh book, and the others that follow it, he will always give notice that certain lines lie in the same plane, or in different planes, because in those books solids are discussed, in which different planes can be considered. The same must be said of points lying outside lines and surfaces, and so on.

Diagram. A right-angle figure illustrating the out-of-plane case. **A—D—B** lie along the horizontal line (in the plane below); **C** stands above D, and **E** is at the upper left, so that CD is perpendicular to AB, and CE is a second perpendicular to CD rising out of the plane.

- A, D, B (the horizontal line AB, with D the foot of the perpendicular)
- C (above D)
- E (upper left, "aloft" — in a second plane)

Theorem 19. Proposition 28.

[Margin: xxvii]

(Euclid's enunciation.) If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same

side, or the interior angles on the same side equal to two right angles, then the two straight lines will be parallel to one another.

(*Clavius's demonstration.*) On the two straight lines AB, CD let the straight line EF fall, making first the exterior angle EGA equal to the interior and opposite angle on the same side, GHC. I say that the straight lines AB, CD are parallel. For since angle GHC is assumed equal to angle EGA, and angle HGB is equal to that same angle EGA [Margin a: I.15], the alternate angles GHC, HGB will be equal [Margin b: Axiom 1 (things equal to the same thing are equal to one another)]. Hence the straight lines AB, CD will be parallel [Margin c: I.27]. The same will be shown if the exterior angle EGB is assumed equal to the interior angle GHD.

Next, let the straight line EF make the interior angles on the same side — namely AGH, CHG — equal to two right angles. I say again that the straight lines AB, CD are parallel. For since the angles AGH, CHG are assumed equal to two right angles, and the angles AGE, AGH are also equal to two right angles [Margin a: I.13], the two angles AGH, CHG will be equal to the two angles AGE, AGH. Taking away the common angle AGH, the angle AGE will remain equal to the exterior angle on the same side. Therefore, as was already shown, the straight lines AB, CD will be parallel. The same will be shown if the two angles BGH, DHG are assumed equal to two right angles. So if a straight line falling on two straight lines makes the exterior angle, and so on. Which was to be demonstrated.

Diagram. Two horizontal straight lines cut by a transversal. Upper line: A—E—G—B; lower line: C—F—D; the transversal runs from E on the upper line down through G and F to the lower line.

- A, E, G, B (upper line AB; E and G lie on it)
- C, F, D (lower line CD; F lies on it)

Scholium

Long ago we rejected the thirteenth "pronunciatum" [the parallel postulate] from the number of the Principles. Since, then, the following Proposition 29 — together with many others — rests upon it so heavily that without its aid it cannot be demonstrated, it will be worthwhile to confirm it here, by a geometrical demonstration, from the theorems and problems demonstrated up to now that in no way depend upon it — as we promised in the exposition of that axiom. In the first place we shall present Proclus's [Proclus] demonstration; then we shall demonstrate the same proposition ourselves more accurately and more evidently. Proclus, then, before demonstrating it, premises two things. The first is:

[This opening continues Clavius's scholium on Euclid's parallel postulate (the thirteenth pronunciatum). The page begins in the middle of an inserted lemma on the unbounded divergence of two rays.]

[Inserted lemma — unbounded ray divergence]

Lemma. If two straight lines making an angle are drawn out from a single point and produced infinitely, their distance will exceed every finite magnitude.

Let two straight lines AB and AC go out from point A, making the angle A. Now points D and E are farther apart from each other than F and G; likewise points B and C are farther apart than D and E; and so on in succession, if the straight lines AB, AC are produced further. It is therefore clear that their outermost points stand at an infinite distance from one another, provided the lines themselves are produced infinitely. For if they were not separated by an infinite distance, their distance could be increased, and hence the lines could be produced still further — which is absurd, since they are assumed to have already been produced infinitely. Therefore, if the said lines AB, AC are produced infinitely, their distance will exceed every finite distance. Aristotle too made use of this pronouncement in Book I of *On the Heavens* [*De caelo* I], where he proved that the world is not infinite.

Diagram. A small angle-figure at the upper right. The vertex is A (at the right), and two straight rays diverge leftward from A.

- A (vertex, right): common origin of the two rays.
- Upper ray, from A outward (right to left): F (nearest A), D (middle), B (farthest).
- Lower ray, from A outward (right to left): G (nearest A), E (middle), C (farthest).
- The farthest point of the upper ray is printed as a malformed glyph resembling "3" [?: read as **B** from the text, which names points B and C as the outermost].

Now, that the straight lines AB, AC diverge more the farther they are drawn out — a point Proclus [Proclus] assumed without demonstration when he said, in the preceding figure, that points D and E are farther apart than F and G, and likewise B and C farther than D and E, and so on — we shall demonstrate as follows.

From points C and D, taken anywhere on the straight line AC, drop the perpendiculars CB and DE to AB. These measure the distances of points C and D from the line AB, since they are the smallest of all the straight lines drawn from C

and D to AB, as we showed in the corollary to Proposition 19. I say that CB is greater than DE, and hence that the line AC is farther from the line AB at the more remote point C than at the nearer point D. For if CB is not greater than DE, it will be either equal or less. Suppose first it is equal. Cut off BF on AE equal to it, so that point F falls either between A and E, or on E, or finally between E and B; and draw the straight line FC. Since, then, the two sides AE, ED of triangle AED are equal to the two sides FB, BC of triangle FBC, each to each, and they contain equal angles — namely right angles — the bases AD, FC and the angles DAE, CFB will also be equal to one another. [Margin: a — 4. pri. (Euclid I.4)] Therefore the exterior angle CFB will be equal to the interior angle DAE, which is absurd. [Margin: b — 16. primi (I.16)] Alternatively, since the exterior angle CFB is equal to the interior DAE, the lines AC, FC will be parallel [Margin: c — 27. primi (I.27)] — which is likewise absurd, since they meet at C.

Next, suppose CB is less than DE, if that be possible. Produce BC and make BG equal to DE, and join the straight line FG. Since, then, the two sides AE, ED of triangle AED are equal to the two sides FB, BG, each to each, and they contain equal angles, the bases AD, FG and the angles EAD, BFG will also be equal to one another. [Margin: d — 4. primi (I.4)] Therefore the exterior angle BFG will be equal to the interior EAD, which is absurd. Or, since the exterior angle BFG is equal to the interior EAD, the lines AC, FG will be parallel to one another [Margin: a — 27. prim. (I.27)] — which is absurd, since they cut one another at H. Consequently BC will be greater than ED, since it can be neither equal nor less, as has been demonstrated.

Diagram. A right triangle standing on the horizontal base A–B, illustrating the divergence argument.

- A (bottom left): vertex of the angle; origin of the two diverging lines.
- Base (left to right): A, F, E, B.
- B (bottom right): foot of the perpendicular BG.
- G (top): apex, top of the perpendicular erected at B.
- C (right, just below G): point on the line AC; foot of perpendicular CB.
- D (middle left, on the rising line): point on the line AC nearer A; foot of perpendicular DE.
- H (middle): intersection point where the lines FG and AC cross (the "cut one another at H").
- The rising line A→G is the ray AC; the perpendiculars DE and CB (and BG) drop to / rise from the base AB.

SECOND [lemma], which Proclus premises, is of this kind.

Lemma. If a certain straight line cuts one of two parallel straight lines, it will also cut the other when it is produced.

Let there be two parallels AB, CD, and let a straight line EF cut AB at G. I say that the straight line EF, if produced, will also cut CD. Since the two straight lines GB, GF make an angle at point G, then, if they are produced infinitely, they will exceed every finite distance; and hence also the distance by which the parallel AB is separated from the parallel CD, since this distance is finite (otherwise the lines would not be parallel). Therefore, once the distance of GB from GF has become greater than that which lies between the parallels, the straight line GF, produced, must have cut the straight line CD. For as long as GF is contained between the two parallels, it will be removed from GB by a smaller distance than CD is from that same GB, as is evident. With these points set out, then, this theorem — which occurs in Euclid as the thirteenth pronouncement — will readily be demonstrated.

Diagram. Two parallel horizontal lines with an oblique transversal.

- A (upper line, left) — B (upper line, right): the parallel AB.
- C (lower line, left) — D (lower line, right): the parallel CD.
- E (upper left, on the transversal): upper end of the transversal.
- G: point where the transversal cuts the upper parallel AB.
- F (on the lower line): point where the transversal EF meets / cuts the lower parallel CD.
- The transversal runs E–G–F.

[Euclid's thirteenth pronunciatum — the parallel postulate]

Pronouncement. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, produced infinitely, will meet one another on that side where the angles are less than two right angles.

[The following is Proclus's proof of the postulate, which Clavius reports.]

Onto the straight lines AB, CD let the straight line EF fall, making the interior angles toward the sides B and D — namely BGH, DHG — less than two right angles. I say that the straight lines AB, CD converge toward these same sides B and D. For since the two angles BGH, DHG are supposed to be less than two right angles, while the two angles DHG, DHF are equal to two right angles [Margin: a —

13. primi (I.13)], the two angles DHG, DHF will be greater than the two angles DHG, BGH. Taking away the common angle DHG, then, the angle DHF will remain greater than the angle BGH. [Margin: b — 5. pron.] If, therefore, at the straight line FG and at the point G an angle KGH is constructed equal to the angle DHF [Margin: c — 23. pri. (I.23)], GK will fall above GB, and, being produced, will cut the straight line AB. [Margin: d — 11. pron.] Now since the falling line EF makes the exterior angle DHF equal to the interior and opposite angle KGH, the straight lines IK, CD will be parallel. [Margin: e — 28. primi (I.28)] But the straight line AB cuts IK at G; therefore, produced, it will also cut CD, as has been demonstrated, toward the sides B and D — namely at point L, which was what was proposed.

Diagram. Two nearly-horizontal straight lines crossed by a transversal, with an auxiliary constructed line.

- A (upper line, far left) with I just below it (left end of the constructed line IK).
- Upper line runs A ... G ... K (K at the right); this is the line IK (through G) / AB.
- G: point on the upper line where the transversal and the constructed line meet.
- E: point on the transversal, above G.
- K (upper right): end of the constructed line, terminus of angle KGH.
- Lower line: C (far left) ... through F/H ... to D and B (at the right); this is CD.
- F, H (lower-middle): F on the transversal, H the point where the transversal cuts the lower line CD.
- Labels read on the figure: A, I, G, E, K (upper); C, F, H, D, B (lower).

By this reasoning, then, Proclus attempts to demonstrate the thirteenth axiom. But since the principle he premised in the first place seems just as doubtful and obscure as that axiom itself, we shall put forward a more careful demonstration — provided we first make clear wherein lies the difficulty, or obscurity, of that principle assumed by Proclus.

In the judgment of Proclus and of other geometers, it must not be conceded without demonstration that two straight lines which continually approach one another do at last, at some point, meet — however true this may be. Such are two straight lines onto which a falling line makes the two interior angles on the same side less than two right angles, one of which is right and the other acute; for these approach one another toward those sides where the two angles are less than two right angles, as we shall shortly demonstrate. This must not be conceded without

proof, because there can be given, in the same plane, two lines — one straight and the other curved (namely either a hyperbola [Hyperbole] or a conchoid line [linea conchoideos, the conchoid]) — continually approaching one another more and more, which nevertheless never meet, even though both are produced to infinity. The former of these was demonstrated by Apollonius of Perga [Apollonius of Perga], the latter by Nicomedes [Nicomedes].

Likewise it does not seem right to admit without demonstration — although it is perfectly true — that two straight lines making an angle exceed every finite magnitude when both are produced to infinity, even though they continually diverge more and more from one another, as we demonstrated above. For there can be exhibited, in the same plane, two lines enclosing an angle — one straight and the other curved, the curve which Nicomedes calls the conchoid — continually diverging from one another more and more, whose distance nevertheless never exceeds or even equals any given straight line, however both may be extended to infinity. For let a straight line AB be given. I say that a straight line and a curve can be described such that one continually recedes from the other, yet their distance

never equals the straight line AB, nor exceeds it, however far both be produced. For let the straight line AB be produced, as far as you please, to C; through B draw BD perpendicular to AC; and with C as pole and AB as interval let the conchoid line AE be described — a curved line, that is, which becomes ever closer to the straight line BD yet never meets it, as is delivered by Nicomedes. Then through A erect the straight line AI perpendicular to AC, which will be parallel to BD [Margin: a — 28. primi (I.28)]. Finally, from any two points H, I on AI, drop perpendiculars HG, ID to BD, which will be parallel [Margin: b — 28. primi], and therefore equal to one another [Margin: c — 34. primi (I.34)], and will make right angles at H, I [Margin: d — 29. primi (I.29)]. For let us now admit — in order to show what is proposed — all the demonstrations of Euclid, as if they did not depend on the thirteenth axiom, or even if they do depend on it, let them nevertheless be granted, just as if that axiom had already been demonstrated before Proposition 29 of this book, where its use first begins to appear — as indeed we shall demonstrate, just before Proposition 29. And so, since FG is greater than ED, as Nicomedes demonstrated, FH (the remainder) will be less than EI (the remainder). Therefore the lines AI, AE are farther apart at points I, E than at points H, F; and thus we shall prove that they continually diverge more and more, the farther they are extended. Nevertheless their distance will always be less than the straight line AB — that is, than the perpendicular dropped from

the line AI to the line BD — since the curved line AE never reaches the line BD, as Nicomedes demonstrated.

Diagram. The conchoid figure: a straight top line, a straight bottom line, and the curved (conchoid) line rising toward the top.

- Top line (left to right): A, H, I — this is the perpendicular AI.
- Bottom line (left to right): B, G, D — this is the line BD.
- C (below B, at bottom left): the pole from which the conchoid is described.
- The curved conchoid line AE runs from A obliquely toward E (upper right); F lies on it.
- F (middle), E (middle right): points on the conchoid.
- Perpendiculars HG and ID drop from H, I on the top line to G, D on the bottom line.

[Margin (left of figure): a — 28. primi; b — 28. primi; c — 34. primi; d — 29. primi]

I know that this principle of Proclus is entirely true for straight lines, and that, if all the demonstrations of Euclid which depend on the thirteenth axiom be conceded, it could easily be demonstrated in this way. Let two straight lines AB, AC contain the angle A, and let a straight line D of any magnitude be given. I say that the distance of the straight lines AB, AC, produced to infinity, exceeds the magnitude D. For from any point E taken on the line AB, drop a perpendicular EF to AC. If this is greater than D, the proposition is established. But if it is not greater, take its multiple that is next greater than D — namely G. Then, taking AB as such a multiple of AE as G is of EF, drop from B the perpendicular BC to AC, which I say is greater than the given straight line D. For since AB is to BC as AE is to EF [Margin: c — 4. sexti (VI.4)] — this by the corollary to Proposition 4 of Book 6 of Euclid, the triangles ABC, AEF being similar because of the parallel lines BC, EF [Margin: f — 28. primi (I.28)] — then, by alternation, as AB is to AE, so BC is to EF. Therefore BC will be the same multiple of EF as AB is of AE, that is, as G is of EF. Since, then, BC and G are equal multiples of EF, they will be equal to one another. But by construction G is greater than D. Therefore BC also — the distance of point B from point C — will be greater than the given straight line D. Which was the proposition.

Diagram. A right triangle with two separate reference segments drawn above it.

- Above the triangle, two horizontal reference segments: G (upper, the longer, marked with a division tick) and D (lower, shorter).
- A (bottom left): vertex of angle A.

- Base A–F–C (left to right); C at the bottom right.
- B (top right): apex; BC is the perpendicular at C.
- Hypotenuse A→B; E lies on it.
- E (on the hypotenuse), with perpendicular EF dropped to F on the base AC.

[Margin (left of figure): c — 4. sexti; f — 28. primi]

But this demonstration has no force without the properties of parallel lines, which rest on that thirteenth axiom; and therefore that principle of Proclus cannot be assumed for demonstrating the thirteenth axiom, lest the principle be begged in the course of demonstrating it. This being so, we have taken diligent pains to demonstrate that very axiom of Euclid solely from those things which are demonstrated before Proposition 29 of the first book. For before Proposition 29 there is no use of that axiom in Euclid. I have been told that this was also done in a certain Arabic Euclid, but I have never been given the opportunity to read that demonstration, though I have insistently demanded it again and again from the man who possesses that Arabic Euclid. For this reason we have devised the demonstration that follows. But first some preliminary matters must also be premised, which, although they are necessarily required for demonstrating what we propose, are nevertheless far more evident and easier than that axiom of Euclid, so that, with all doubt excluded, we may lend them a firm assent. Let the first be of this kind.

I.

Lemma I. A line, all of whose points are equally distant from a straight line lying in the same plane with it, is straight.

For if all the points of the line AB are equally distant from the straight line DC — that is, if all the perpendiculars dropped to DC, such as AD, EH, FG, BC, are equal (for any perpendicular, being the smallest of all the lines drawn from the same point to the straight line DC, measures the distance of the point from which it is drawn, by the corollary to Proposition 19 of this book) — then AB will be a straight line. And this can be clearly established from the definition of a straight line. For if all the points of the line AB are equally distant from the straight line DC, its points will lie evenly between one another — that is, no intermediate point in it will jut up or down from the extremities, nor bend this way or that, and nothing crooked will be found in it, but it will always extend evenly between its points, just like the straight line DC. Otherwise, not all its points would have an equal distance from the straight line DC, which is contrary to the hypothesis. Nor indeed can it be conceived in thought that any line other than a straight one

could have all its points at an equal distance from a straight line lying in the same plane with it. This principle is such that, once it alone is granted, together with those things which have been shown up to Proposition 29 of this book, we shall demonstrate the thirteenth axiom — a principle so clear that it is known by the natural light, and no one of sound mind could deny it. Or at any rate it is beyond all controversy such that anyone assents to it far more readily than to that thirteenth axiom of Euclid.

Diagram. A rectangular figure: a top line AB and a bottom line DC joined by equal perpendiculars.

- Top line (left to right): A, E, F, B.
- Bottom line (left to right): D, H, G, C.
- Perpendiculars: AD, EH, FG, BC — all equal — joining the top points to the bottom points.

The very same thing holds for a circular line. For a curved line surrounding a circular line, all of whose points are equally distant from the circular line — that is, from which all the straight lines falling to the circular line at equal angles are equal — is itself also circular; so that it takes on the nature of the circular line, from which it is always separated by an equal distance, just as a line always equidistant from a straight line takes on the nature of the straight line to which it is always equidistant, and is therefore straight, as we said. That such a curved line about a circular line is also perfectly circular we shall easily show, if we first demonstrate that a straight line drawn from the center of a circle makes two equal angles with the circumference, and conversely, that a straight line which makes equal angles with the circumference passes through the center.

Let there be, then, a circle ABCDEF whose center is G. I say that the straight line GC, drawn from the center, makes both the interior angles GCB, GCD and the exterior angles ICB, ICD equal to one another. For if IG is produced to A, and the semicircle ABC about the diameter AC is imagined to be turned around, it will coincide with the semicircle AEC, since semicircles of the same circle are equal to one another. But if anyone should say that one semicircle cuts the other, a straight line would have to be drawn from the center G cutting both circumferences, and it would follow that a part is equal to the whole — namely a radius drawn to the inner circumference equal to a radius drawn to the outer circumference — which is absurd. Therefore the angles at C, both interior and exterior, will coincide, and hence be equal. Now let the straight line HAK make equal angles at A. I say that it passes through the center. For if it does not pass through, let the straight line GA be drawn from the center G to A, which, by what

was just demonstrated, will make the angles GAB, GAF equal. Then the angles KAB, KAF are not equal — which is contrary to the hypothesis.

Diagram. Two concentric curves: an inner circle ABCDEF and an outer curved (inflexed) line HLIMNO, with connecting radii through the center G.

- G (center): center of the circle; K is a labeled point just to the left of G, near the center.
- Inner circle points (counterclockwise): A (top), B (left), C (bottom), D (lower right), E (right), F (upper right).
- Outer inflexed line, point for point outside the inner circle: H (top, outside A), L (left, outside B), I (bottom, outside C), M (lower right, outside D), N (right, outside E), O (upper right, outside F).
- The vertical diameter runs I – G – A (bottom to top through the center).
- Connecting lines run from each outer point through its inner point toward the center: HA, LB, IC, MD, NE, OF.
- The line HAK is drawn making equal angles at A.
- All labels read on the tile: outer H, L, I, M, N, O; inner A, B, C, D, E, F; center G; and K. (No labels flagged.)

This having been shown, let the inflexed line HLIMNO surround the circular line ABCDEF, and let all its points be equally distant from it — that is, let all the straight lines falling from that line onto the circular line, and making equal angles, be equal to one another, such as are HA, LB, IC, MD, NE, OF. For since, as has been demonstrated, these pass through the center, they will be the smallest of all the lines falling from the points H, L, I, M, N, O onto the convex periphery (for this was demonstrated by Euclid, Proposition 8 of Book 3, which depends solely on the propositions preceding Proposition 29 of this book, so that it may most rightly be transferred here), and thus they will measure the distances of those points from the underlying circular line. I say that the line HLIMNO is circular. For since all those lines, as we just showed, pass through the center G, if to the equal lines HA, OF are added the equal lines AG, FG, the whole lines HG, OG will be equal; and by the same reasoning all the other lines drawn from the inflexed line HLIMNO to G will be equal both to the straight lines HG, OG and to one another. By the definition of a circle, therefore, the inflexed line is circular. Which was to be demonstrated. From this first [lemma], which we premised, the second follows — namely:

II.

Lemma II. If a straight line is moved transversely over another straight line, always keeping right angles with it at its [lower] end, its other end will describe a straight line as well.

For if, in the previous figure, the straight line AD, perpendicular to DC, is moved transversely over the straight line DC, keeping with it at D always right angles — that is, not tottering or wavering, but always advancing evenly — the end A will describe a straight line as well, namely AB. This follows from what we premised in the first place, since all the points of the described line AB are equally distant from the straight line DC, namely by the straight line AD. From this it follows that this line is straight, since, as was said there, its middle parts do not jut up from the extremities but lie evenly between them, seeing that none of them recedes from the straight line CD, up or down, more or less than any other. This is also evident from what we wrote concerning the definition of a straight line. For since the two points D, A are carried by exactly similar motions (in that the straight line AD does not totter or waver in its motion, but the points D, A advance equally fast and always keep an equal distance between themselves), they will describe similar lines, as we taught in that same definition. Since, then, the point D describes the straight line DC, the line AB, which point A describes, will also be straight. Nor can anyone imagine that, if a straight line is moved transversely, uniformly and without any tottering this way or that, its two end-points would describe two lines differing from one another; rather, such a line as the one end describes, such must the other end also describe, on account of that uniform and even motion of both end-points. This principle is as clear and evident as the preceding one.

The very same thing holds entirely for a circular line. For in the same way a straight line flowing transversely over a circular line evenly — keeping, that is, at its end always equal angles with the circular line, without wavering or bending this way and that — describes with its other end a circular line, so that the described line takes on the nature of the underlying circular line, from which it is always equally distant: just as the straight line AD, flowing transversely over the straight line DC evenly — keeping, that is, at its end D always right angles with it, so that it never inclines, bending or tottering this way and that — traces out with its other end A the straight line AB, so that the described line AB takes on the nature of the underlying straight line DC, from which it always recedes equally. That a straight line flowing transversely over a circular line evenly describes with its other end a circular line as well, we shall easily demonstrate. For in the latter

figure let the straight line HA be moved transversely over the periphery ABCDEF evenly, always making at its end A equal angles with it. I say that the line HLIMNO, described by the other end H, is likewise circular. For since the straight line HA, in every position of that imagined motion, makes equal angles with the periphery ABCDEF, it comes about that, produced, it falls upon the center G, as we showed above. Therefore, adding the equal straight lines HA, OF to the equal straight lines AG, FG, the whole straight lines HG, OG, and all the others, will be equal to one another; and consequently, by the definition of a circle, HLIMNO will be the circumference of a circle. Which was to be shown. Hence follows the third [lemma].

III.

Lemma III. If to a straight line two mutually equal perpendicular straight lines are erected, and their end-points are joined by a straight line, then the perpendicular dropped from any point of this [joining] straight line to the first straight line will be equal to either of the prior perpendiculars.

Let there be erected to the straight line AB two mutually equal perpendiculars AC, BD; and, the straight line CD having been drawn, let the perpendicular EF be dropped from any point E of it to AB. I say that EF is equal to each of AC, BD. For if it is not equal, it will be either greater or less. Cut off, then, FG equal [to AC]: if the straight line AC is imagined to be moved transversely along the straight line AB, always making right angles with it, the point C will describe a straight line passing through G, as was shown above — namely CGD. Then the two straight lines CED, CGD will enclose a surface. Which is absurd. Therefore EF is equal to each of AC, BD. Which was to be shown. From this we shall demonstrate the fourth, which follows.

Diagram. A rectangular figure with a lens (vesica) shape formed by two lines CED and CGD between C and D.

- Top line (left to right): C, E, D — with E in the middle.
- G (upper apex, above the line CD): apex of the described curve; label read as G [?: the blackletter apex glyph is somewhat abraded but reads G].
- G (center, below E, on the vertical): point on the perpendicular EF.
- Bottom line (left to right): A, F, B.
- Sides: AC (left) and BD (right) are the equal perpendiculars; EF is the perpendicular dropped from E to F.

- The two lines CED (through E) and CGD (through G) bulge apart and enclose a surface.

IV.

Lemma IV. If to a straight line two mutually equal perpendicular straight lines are erected, and their end-points are joined by a straight line, this [joining] straight line will make a right angle with each perpendicular.

Let there be erected to the straight line AB two mutually equal perpendiculars AC, BD, and let the straight line CD be drawn. I say that each of the angles C, D is a right angle. For let the straight line AB be bisected at E; at E erect the perpendicular EF to AB; and join the straight lines AF, BF. Since, then, the two sides AE, EF of triangle AEF are equal to the two sides BE, EF of triangle BEF, each to each, and they contain equal angles — namely right angles [Margin: a — 4. pri. (I.4)] — the bases AF, BF, and the angles FAE, FBE as well as AFE, BFE, will also be equal to one another. Taking away, then, the equal angles FAE, FBE from the equal right angles CAE, DBE, the remaining angles CAF, DBF...

Diagram. A rectangle with an inscribed isosceles triangle from the midpoint of the base.

- Top line (left to right): C, F, D — with F in the middle.
- Bottom line (left to right): A, E, B — with E in the middle.
- Sides AC (left) and BD (right) are the equal perpendiculars; CD the top joining line.
- EF is the perpendicular erected at the midpoint E, rising to F.
- Diagonals FA and FB are drawn from F to the base ends, forming triangles AEF and BEF.

[Margin: a — 4. pri. (I.4)]

Euclid, *Elements* Book I — Clavius's commentary (continued)

[Conclusion of Lemma IV — that the angles C, D are right]

CAF, DBF. And so, since the two sides CA, AF of the triangle CAF are equal to the two sides DB, BF of the triangle DBF, and they enclose equal angles, as we have shown, [Margin: a — *Elements* I.4] the bases CF, DF will also be equal, and so will

the angles C, D, as well as CFA, DFB, to one another. If, then, to the equal angles CFA, DFB the angles AFE, BFE (shown to be equal) are added, the whole angles CFE, DFE become equal, and hence right. And since the perpendicular EF is equal to each of AC, BD — as we showed in the preceding theorem — the two lines AC, EF, drawn perpendicular to AE, will be equal to each other [?]. Therefore, just as was proved a moment ago concerning the angles C, D, the angles C and EFC will be equal to one another. But EFC has been shown to be right. Therefore C too will be right, and consequently so will the angle D, which has been shown equal to C.

We shall, however, demonstrate that the angles C, D are right by another method, once the following theorem has first been premised.

[Theorem: converging and diverging lines]

If a straight line falling upon two other straight lines makes with one of them an interior right angle, and with the other, on the same side, an acute angle, then those two lines, produced toward the side where the acute angle lies, will always be less distant from each other; but toward the other side they will always be more distant from each other.

Let the straight line AB, falling on the lines AC, BD, make the angle ABD right and the angle BAC acute. I say that the lines AC, BD, produced toward C, D, are always less distant from one another, but more distant if they are produced toward E, F.

Diagram. A figure of two nearly-horizontal straight lines cut by a transversal AB, with a zigzag staircase of alternating perpendiculars between them.

- Upper line (left → right): E, O, M, A, G, I, C
- Lower line (left → right): F, P, N, L, B, H, K, D
- A: point on the upper line where the transversal AB meets it (acute angle BAC)
- B: point on the lower line where the transversal AB meets it (right angle ABD)
- Toward C, D (right side): the perpendiculars BG, GH, HI, IK, KC drop alternately from lower to upper line and back, forming the converging staircase
- Toward E, F (left side): the perpendiculars AL, LM, MN, NO, OP, PE (and EF) rise alternately, forming the diverging staircase

For [Margin: b — *Elements* I.12] let the perpendicular BG be drawn from B to AC. By Corollary 2 to Proposition 17 of this book it will fall toward the side of the

acute angle BAC, and therefore the angle GBD, being part of the right angle ABD, will be acute. So when the perpendicular GH is dropped from G to BD, it too will fall toward the side of the acute angle GBD, by the same corollary; and therefore the angle HGC, being part of the right angle BGC, will be acute. Again, then, when the perpendicular HI is drawn to AC, by the same corollary it will fall toward the side of the acute angle HGC, and so the angle IHD, being part of the right angle GHD, will be acute. And so on to infinity: if from I the perpendicular IK is drawn to BD, and from K the perpendicular KC to AC, and from C the perpendicular CD to BD, and so forth, they will all fall toward the sides of the acute angles, one after another.

Now since in the triangle ABG the angle BAG is acute and AGB is right, [Margin: c — *Elements* I.19] the line AB will be greater than BG. And in the triangle BGH the line BG, being opposite the right angle GHB, is greater than GH; and in the triangle GHI the line GH is greater than HI. And so on continually: the lines nearer to AB will always be greater than those farther off, and there will be no end to this decrease of the perpendiculars. Therefore the line AC is less distant from the line BD at G than at A, and at I less than at G, and at C less than at I, and so on — seeing that the perpendiculars AB, GH, IK, CD, which decrease in order as we have shown, are the least of all the lines falling from the points A, G, I, C onto the line BD (as is clear from the corollary to Proposition 19 of this book), and consequently they measure the distances of those points from the same line BD. It is therefore established that the lines AC, BD, produced toward C, D, become continuously less distant from one another — which was the first point.

Again, let the perpendicular AL be erected from A on AC [Margin: d — *Elements* I.11]. It will divide the obtuse angle BAE into the right angle EAL and the acute angle LAB — that is, it will fall between AB and AE — and the angle ALB will be acute, since both ABL and ALB [Margin: e — *Elements* I.17] are less than two right angles while ABL is right; and consequently the angle ALF will be obtuse. By the same reasoning the perpendicular LM, erected on BF at L, will fall between LA and LF, and will make the angle LMA acute and LME obtuse. Likewise the perpendicular MN, erected on AE at M, will fall between ML and ME, and will make the angle MNB acute and MNF obtuse. And so on to infinity: if from N the perpendicular NO is erected to BF, and from O the perpendicular OP to AE, and from P the perpendicular PE to BF, and from E the perpendicular EF to AE, and so forth, it will divide all the obtuse angles one by one, one after another.

Now since in the triangle ABL the angle ABL is right and ALB [Margin: f — *Elements* I.19] acute, AL will be greater than AB; and in the triangle ALM the line

LM, opposite the right angle LMA, is greater than AL; and in the triangle LMN the line MN, opposite the right angle MNL, is greater than ML; and so on continually: the lines that are farther from AB will always be greater than those nearer to it, to infinity. Hence the line AE is more distant from the line BF at M than at A, and at O more than at M, and at E more than at O, and so on — since the perpendiculars AL, ML, ON, EP [?] (which, as was shown, increase in order) are the least of all the lines falling from the points A, M, O, E onto the line BF. It is therefore clearly established that the lines AC, BD, produced toward E, F, become continuously more distant from one another — which was the second point.

But if some obstinate person should want to maintain that the line AL, which is perpendicular to AC, does not meet DB when produced, that will establish all the more strongly the very thing proposed for demonstration. For the line DB, beyond the point B, will be much more distant from the line CA than at the point B — inasmuch as AL, which is perpendicular to AC, does not meet DB, whereas GB, likewise perpendicular to the same AC, does meet DB at B, and so on.

[Alternative proof that the angles C, D are right]

This being so, we shall demonstrate without difficulty that in the figure above the angles C, D are right. For if the angle C, say, is not said to be right, it will be either acute or obtuse. Suppose first it is acute. Since, then, the line CA, falling on the lines AB, CD, makes the angle CAB right and C acute, the lines CD, AB produced toward D, B will always be less distant from each other — that is, the perpendiculars dropped from CD onto AB will become smaller in order than CA, as was just demonstrated. The perpendicular DB is therefore less than CA, which is absurd, since it is assumed equal to it. So the angle C is not acute; and by the same reasoning neither will the angle D be acute.

Next suppose the angle C is obtuse. Since the line CA, falling on the lines CD, AB, makes the angle CAB right and C obtuse, the lines CD, AB produced toward D, B will always be more distant from each other — that is, the perpendiculars dropped from CD onto AB will increase continually, as we have just shown. The perpendicular DB is therefore greater than CA, which is absurd, since it is assumed equal. So the angle C is not obtuse; and by the same reasoning neither will the angle D be obtuse. But it has not been shown acute either. Each, therefore, is right — which was to be shown.

From what has been demonstrated we shall now easily demonstrate Axiom 13, in this way.

V.

If another straight line, falling on two straight lines, makes the interior angles on the same side less than two right angles, then those two straight lines, produced to infinity, will meet each other on the side where the angles are less than two right angles.

This is Axiom 13 in Euclid, according to our commentaries; in other editions it is the eleventh — the one we have undertaken to demonstrate. Let the straight line AB, falling on the lines AC, BD, make the interior angles ABD, BAC on the same side less than two right angles. I say that the lines AC, BD, produced toward C, D, meet.

Diagram. A "staircase" figure of congruent right triangles built between a vertical transversal AB and an oblique line AC, used to prove Axiom 13 for the case where one interior angle is right.

- Vertical line AB (top → bottom): A, F, G, H, B, I (A at top; I falls below B)
- Oblique line AC (top → bottom): A, E, K, L, C (equal segments $AE = EK$, $KL = AK$, $LC = AL$ — successive doublings)
- Produced points (beside E, K, L respectively): M, N, O (with $EM = EF$, $KN = KG$, $LO = LH$; the joins MK, NL, OC are drawn)
- Horizontal line BD: B (left), D (right)
- The perpendiculars EF, KG, LH, CI are drawn to AB and produced to M, N, O

Suppose first that one of the angles — namely ABD — is right, and the other, BAC, acute. Taking any point E on the line AC [Margin: g — *Elements* I.12], drop from it a perpendicular EF to the line AB; take FG equal to AF, GH equal to AG, and HI equal to AH, so that AG is double AF, AH double AG, and AI double AH.

And since, if along the line AF equal segments are continually cut off, some part will fall beyond the point B — because the finite line AB is at last exhausted by the continual removal of one and the same quantity, seeing that the line AF can be so multiplied (by taking equal parts along AB continuously) that it eventually exceeds the finite line AB. This Euclid frequently assumes in Book 5 and the following books, where, given two unequal magnitudes having a ratio to one another, he usually requires the smaller to be so multiplied until it exceeds the larger. It happens all the more if to AF an equal FG is cut off, and then GH is taken equal to the whole AG (not to AF alone), and HI to the whole AH (not to AF or AG alone), and so on: some part will at last fall beyond the point B. Let the point I,

then — which terminates the third double in the present example — be set beyond the point B.

Take also on the line AC the segment EK equal to AE, KL equal to AK, and LC equal to AL, so that there are as many parts on AC as on AB; and let the lines KG, LH, CI be drawn and produced, together with EF, so that EM, KN, LO become equal to EF, KG, LH respectively; and finally let the lines MK, NL, OC be joined. Now since the two sides KE, EM of the triangle KEM are equal by construction to the two sides AE, EF of the triangle AEF, each to each, and they contain equal vertical angles at E [Margin: I.15], the base MK will be equal to the base AF and the angle M to the angle F [Margin: I.4]. But FG is also equal to AF, and the angle F is right by construction. Therefore MK and FG will be equal to each other [Margin: Common Notion 1], and the angle M too will be right. And so, since at FM two equal perpendiculars FG, MK have been raised, the angle G will be right by the preceding theorem.

Again, since the two sides LK, KN of the triangle LKN are equal by construction to the two sides AK, KG of the triangle AKG, and they contain equal vertical angles at K [Margin: I.15], the base NL will be equal to the base AG and the angle N to the angle G [Margin: I.4]. But GH is also equal to AG by construction, and the angle G is right, as was just shown. Therefore NL and GH will be equal to each other [Margin: Common Notion 1], and the angle N too will be right. Consequently, by the preceding theorem, the angle H will also be right. By the same reasoning we shall show that the angle I is right, and so on, if there were more parts of the line AB. But the angle B is also right by hypothesis; therefore the lines IC, BD [Margin: g — *Elements* I.28] are parallel; and so BD produced will cut the line AC beyond C, and thus at that point of section the lines AC, BD will meet. Which was to be demonstrated.

[Axiom 13 — the case where neither interior angle is right]

Next, let neither of the angles ABD, BAC be right, but ABD acute and BAC either acute or obtuse. Since the two angles ABD, BAC are assumed less than two right angles, while the two angles ABD, DBE [Margin: h — *Elements* I.13] are equal to two right angles, the latter two will be greater than the former two. So when the common angle ABD is removed, the remaining angle DBE will be greater than the remaining angle BAC. [Margin: a — *Elements* I.23] Construct at A, on the line AB, an angle BAF equal to the angle DBE. The line AF will fall above the line AC, since AF makes a greater angle with AB than AC does. Now since the exterior angle DBE is equal to the interior and opposite angle BAF on the same side, [Margin: b — *Elements* I.28] the lines AF, BD will be parallel to each other.

Diagram. A figure showing the transversal AB meeting the lines AC, BD, with the auxiliary line AF (parallel to BD) and the perpendicular AG dropped to BD, used to prove that the lines meet when neither interior angle is right.

- H: upper left, on the line H–A–F drawn across the top
- F: upper right, endpoint of AF (with AF parallel to BD)
- Apex (top center): the point A, where the transversal and the constructed lines meet [label A not clearly legible on the leaf]
- E: lower left, on the lower line (angle DBE)
- G: lower center, foot of the perpendicular dropped from A onto BD
- D: lower right, on the lower line
- B (on the lower line, between E and G) and C (endpoint of AC, toward the lower right) belong to the construction but are not clearly legible at this resolution

Also drop from A to BD [Margin: c — *Elements* I.12] the perpendicular AG, which, by Corollary 2 to Proposition 17 of this book, will fall toward the side of the acute angle ABD; and the angle GAF too will be right. For if it be said to be acute, then the line AG, falling on the lines AF, BD, would make the angle AGD right and GAF acute, and consequently — as we demonstrated just above — the lines AF, BD, produced toward F, D, would at last meet; which is absurd, since they have been shown to be parallel. And if the angle GAF be said to be obtuse, then GAH will be acute; and so, from what was just demonstrated, the lines AF, BD, produced toward H, B, would meet, which is absurd, since they have been shown to be parallel. So the angle GAF is neither acute nor obtuse. It is therefore right, and consequently its part GAC is acute. Now since the line AG, falling on the lines AC, BD, makes the angle G right and GAC acute, the lines AC, BD, produced toward C, D, will meet, as we demonstrated just above. Therefore, if the line AB, falling on the lines AC, BD, makes two angles less than two right angles, neither of the angles ABD, BAC being right, the lines AC, BD, produced toward C, D, will meet. Which was to be shown.

And if it should happen that the line AC lies between AB and the perpendicular AG, it is clearly established that the lines AC, BD meet, since AC would then cut the base BG of the triangle ABG. To grasp this more easily, you may draw the line AC between AB and AG.

[Remark: the postulate is demonstrable]

Although, once our principle is granted, this thirteenth axiom has been excellently demonstrated by us — and by Proclus too, if his principle (harder, indeed, than ours) be admitted — so that it can with full right be reckoned among the theorems and not among the first principles, nevertheless, so as not to alter Euclid's order in any respect, we shall use it in all the propositions whose demonstrations depend on it, as though it were a postulate. But let us now return to the series of Euclid's propositions.

THEOR. 20. PROPOS. 29. [Margin: xxix.]

A straight line falling on parallel straight lines makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite angle on the same side; and the interior angles on the same side equal to two right angles.

Let the straight line EF fall on the parallels AB, CD.

Diagram. The standard figure for Proposition 29: two parallels AB, CD cut by the transversal EF.

- Upper parallel AB: A (left), B (right)
- Lower parallel CD: C (left), D (right)
- Transversal EF: E (upper end), F (lower end)
- G: intersection of the transversal with AB
- H: intersection of the transversal with CD

I say first that the alternate angles AGH, DHG are equal to one another. For if they are not equal, let one — say AGH — be the greater. Then since the angle AGH is greater than the angle DHG, if the common angle BGH is added, [Margin: a — Common Notion 4] the two angles AGH, BGH will be greater than the two DHG, BGH. But the two AGH, BGH [Margin: b — *Elements* I.13] are equal to two right angles. Therefore the two DHG, BGH are less than two right angles. And so, since they are interior and on the same side, B and D, [Margin: c — Common Notion 13 (the postulate)] the lines AB, CD will meet on that side — which is absurd, since they are assumed parallel. So the angle AGH is not greater than the angle DHG; nor is it less, for by the same reasoning it would be shown that the lines meet toward A and C. Therefore the alternate angles AGH, DHG are equal. And the same reasoning holds for the alternate angles BGH, CHG.

I say secondly that the exterior angle AGE is equal to the interior and opposite angle CHG on the same side. For the alternate angle CHG is equal to BGH, as was

shown; and to the same BGH [Margin: d — *Elements* I.15] the angle AGE is equal. Therefore the angles AGE, CHG [Margin: e — Common Notion 1] will also be equal to one another. In the same way it will be demonstrated that the angle BGE is equal to the angle DHG.

I say thirdly that the interior angles on the same side, AGH, CHG, are equal to two right angles. For it was shown that the exterior angle AGE is equal to the interior angle CHE [recte: CHG]; if the common angle AGH is added, [Margin: a — Common Notion 2] the two angles AGE, AGH will be equal to the two CHG, AGH. But the two AGE, AGH [Margin: b — *Elements* I.13] are equal to two right angles. Therefore the two angles CHG, AGH will also be equal to two right angles. In the same way the angles BGH, DHG will be equal to two right angles. Therefore a straight line falling on parallel straight lines makes the alternate angles equal, and so on. Which was to be demonstrated.

SCHOLIUM

This present theorem is the converse of the two preceding theorems, as is evident.

THEOR. 21. PROPOS. 30. [Margin: xxx.]

Straight lines parallel to the same straight line are also parallel to one another.

Let the straight lines AB, CD be parallel to the same straight line EF. I say that AB, CD are also parallel to one another. Since all these lines are assumed to lie in the same plane (for Proposition 9 of the eleventh book will treat of lines in different planes), the straight line GH drawn across will cut them all — namely AB at I, CD at K, and EF at L.

Diagram. Three parallel lines cut by one transversal GH, for Proposition 30.

- Upper parallel AB: A (left), I (crossing), B (right)
- Middle parallel EF: E (left), L (crossing), F (right) [left-hand label of the middle line is faint]
- Lower parallel CD: C (left), K (crossing), D (right)
- Transversal GH: G (upper end), H (lower end)

Now since AB is assumed parallel to EF, [Margin: c — *Elements* I.29] the angle AIL will be equal to the alternate angle FLI. Again, since CD is also assumed parallel to EF, [Margin: d — *Elements* I.29] the angle DKI will be equal to the same angle FLI — namely as interior to exterior, or exterior to interior. Therefore the angles AIL,

DKI [Margin: e — Common Notion 1] will also be equal to one another. And since they are alternate, [Margin: f — *Elements* I.27] the lines AB, CD will be parallel to each other. Therefore straight lines parallel to the same straight line are also parallel to one another. Which was to be demonstrated.

SCHOLIUM

But if someone should say that the two lines AI, BI are parallel to the line CD and yet are not parallel to each other, the reply is that AI, BI are not two lines but only parts of one line. For one must conceive any parallels as produced to infinity; and it is clear that AI produced coincides with BI. For this reason the proposition could have been stated more generally, thus:

Diagram. Three parallel lines cut by a transversal GH, illustrating the scholium to Proposition 30.

- Upper parallel AB: A (left), I (crossing), B (right)
- Middle parallel CD: C (left), K (crossing), D (right)
- Lower parallel EF: E (left), L (crossing), F (right)
- Transversal GH: G (upper end), H (lower end)

Straight lines parallel to the same straight line are also parallel to one another; or else, when they meet each other, they constitute one and the same line.

Diagram. Two lines AB, AC meeting at A, both parallel to DE, with the transversal AF, illustrating that they lie in one straight line.

- B (left) — A (apex, upper right) — C (short segment to the right): the lines AB and AC meeting at A
- Lower line DE: D (left), F (center, where AF meets DE), E (right)
- AF: drawn from A down to F on DE

For let two straight lines AB, AC, meeting at A, be parallel to DE. I say they are set in a straight line. From the point A let the straight line AF be drawn, cutting DE at F, in any manner. Since AB, DE are parallel, [Margin: a — *Elements* I.29] the alternate angles BAF, AFE will be equal. So when the common angle CAF is added, [Margin: b — Common Notion 2] the two angles at A will be equal to the two angles CAF, AFE. But these two [Margin: c — *Elements* I.29] are equal to two right angles, since they are interior between the two parallels AC, DE. Therefore the two angles at A will also be equal to two right angles; and consequently [Margin: d — *Elements* I.14] AB, AC are set in a straight line. Which was the thing proposed.

PROBLEMA 10. PROPOSITIO 31. [Margin: xxxj.]

Through a given point, to draw a straight line parallel to a given straight line.

Through the point A a line is to be drawn parallel to the line BC.

Diagram. The construction for Proposition 31: through A, a parallel FE to the given line BC.

- Upper line FAE (the parallel drawn through A): F (left), A (the given point), E (right)
- Lower line BC (given): B (left), D (center, foot of AD), C (right)
- AD: the auxiliary line from A to D on BC

From A draw a line AD to BC, in any way, making any angle ADB; and at A construct the angle EAD equal to it [Margin: e — *Elements* I.23]. I say that the line EA, extended to F as far as you please, is parallel to BC. For since the alternate angles ADB, DAE are equal by construction, [Margin: f — *Elements* I.27] the lines BC, EF will be parallel. Therefore through a given point we have drawn a straight line parallel to a given straight line, and so on. Which was to be done.

SCHOLIVM

Now the given point must be situated in such a place outside the given line that this line, when produced, does not meet it. This is plainly gathered from the very construction of the problem. For from the given point a line must be drawn making some angle with the given line, which could not happen if the point lay in a straight line with the given line itself. And just as from one and the same point only one perpendicular — not several — can be drawn to the same straight line, as we showed at Proposition 17 from Proclus, so too through the same point only one parallel — not several — can be drawn to a given straight line. For if two were drawn, they would meet [Margin: a — *Elements* I.30] at that same point, which is absurd, since they are parallel to one another — because they are parallel to one and the same line to which they are said to be drawn parallel.

Moreover, from this problem together with the one contained in Proposition 23, we shall easily construct a parallelogram one of whose angles is equal to a given rectilinear angle, and whose sides containing that angle are equal to two given straight lines.

Diagram. The construction of a parallelogram DEGF with a given angle and given sides.

- Parallelogram vertices: D (bottom left), E (bottom right), G (top right), F (top left)
- Given straight lines shown separately at the lower left: A (upper segment), B (lower segment)
- Given rectilinear angle: C (at the left)

For let the straight lines A, B be given, and let it be required to construct a parallelogram having an angle equal to the given rectilinear angle C, with the sides about that angle equal to the given lines A, B. Take a straight line DE equal to the line A; [Margin: b — *Elements* I.23] make the angle EDF equal to the angle C, [Margin: c — *Elements* I.3] and the line DF equal to the line B. Then through E [Margin: d — *Elements* I.31] draw the line EG parallel to DF, and through F the line FG parallel to DE, cutting EG at G. Since, then, both the sides DF, EG and the sides DE, FG are parallel by construction, [Margin: e — Definition 35] DEGF will be a parallelogram. And since by construction it has the angle D equal to the given angle C, and the sides DE, DF about the said angle D equal to the given lines A, B, what was proposed will have been done.

By similar reasoning, from this problem and Proposition 23, we shall draw from a point given outside a given straight line a straight line that makes with the given line an angle equal to a given rectilinear angle — as we promised at Proposition 23.

Diagram. From the point A outside BC, a line AB making with BC an angle equal to the given angle D.

- Given rectilinear angle: D (upper left)
- A: the given point (upper right)
- F: endpoint of EF (to the right)
- Lower line BC: B (left), E (center, the chosen point), C (right)
- AB: drawn through A parallel to EF, meeting BC at B; EF: the angle CEF constructed at E

For let the point A be given outside the line BC, and let the rectilinear angle D be given; and let it be required to draw from A to the line BC a straight line that contains with it an angle equal to the given angle D. Take any point E on the line BC and [Margin: a — *Elements* I.23] construct at it an angle CEF equal to the angle D. If, then, the line EF passes through the given point A, what was ordered will have been done. But if it does not pass through A, [Margin: b — *Elements* I.32] draw through A a line AB parallel to EF, cutting BC at B. I say that the angle ABC is equal to the angle D. For since AB, EF are parallel, [Margin: c — *Elements* I.29] the

angle CEF, as exterior, is equal to the interior and opposite angle ABC on the same side. And since the angle D is also equal to the angle E by construction, [Margin: d — Common Notion 1] the angles ABC and D will be equal to one another — which was proposed.

It does not seem out of place to show here that Euclid, at Proposition 26 — in order to demonstrate the equality of the sides in two triangles from the equality of two angles and one side — rightly prescribed that that side must either be adjacent to the equal angles or else subtend an equal angle. For unless one of these two conditions is present, nothing will be concluded.

Diagram. The counterexample triangle ABC with DE drawn parallel to AC, showing why the side-condition in Proposition 26 is needed.

- A: apex (top)
- B: bottom left; C: bottom right
- D: point on AB
- E: point on BC (near C)
- DE: drawn inside the triangle, parallel to AC

For let there be a triangle ABC, with the sides AB, AC greater than the side BC. Cut off BD equal to BC, [Margin: e — *Elements* I.31] and draw DE parallel to AC; [Margin: f — *Elements* I.29] then the angle BED will be equal to the angle C. And so the two angles B, C of the triangle ABC are equal to the two angles B, E of the triangle DBE, each to each, and the side BC is equal to the side DB — and yet the remaining sides...

(Clavius's commentary — scholium to Proposition 31, continued from the previous pages.)

the remaining sides are not equal to the remaining sides. For BE is less than BC, and so cannot be equal either to side AC or to side AB, since these sides are assumed to be greater than side BC. The reason is this: in triangle ABC the side BC lies adjacent to the said angles, whereas in triangle DBE the side DB is opposite one of those angles—namely the angle DEB, which is equal to angle C.

Again, let there be a triangle FGH, and let the sides FG and FH once more be greater than the side GH, with FH greater than FG. Cut off HI equal to FG [Margin: g — *Elements* I.31], and draw IK parallel to FG [Margin: h — I.29]. Then the angle HKI will be equal to angle G. Therefore the two angles H and G of triangle FGH are equal to the two angles H and K of triangle IKH, each to each, and the side FG is equal to the side IH; and yet the remaining sides are not equal to the remaining

sides. For HK is less than HG, and so cannot be equal either to side HG or to side FH, since FH is assumed to be even greater than GH. The reason is this: in triangle FGH the side FG is opposite angle H, whereas in triangle IKH the side IH is opposite angle K—which is not assumed equal to angle H, but is equal to angle G [Margin: i — I.18], and G is greater than H, because we assumed side FH greater than side FG. We have postponed this demonstration to this place on account of the properties of parallel lines, on which it depends.

Diagram. Two triangles sharing the base region and the right-hand vertex, illustrating the I.26 counter-example. The outer triangle FGH has its apex F at the top, with G at the lower left and H at the lower right. Inside it a second triangle IKH is drawn (with dashed construction lines): I lies on side FH just below F, K lies on the base near G, and IK is drawn parallel to FG.

- F (apex, top): outer-triangle vertex
- I (upper, on FH near F): vertex of inner triangle IKH (HI cut off equal to FG)
- G (lower left): outer-triangle vertex
- K (lower, near G): vertex of inner triangle IKH
- H (lower right): shared vertex of both triangles

Praxis

(Practical constructions for Proposition 31 — drawing a parallel through a point.)

Suppose a parallel to BC is to be drawn through the point A. Draw a straight line AD to BC at any inclination, and from D and A, with any one and the same radius, describe two arcs on opposite sides—one toward B, the other toward C. Then, with the compass, cut off from the second arc an arc GH equal to the arc EF. If now a straight line is drawn from A through H [Margin: k — I.27], it will be parallel to BC; for the alternate angles EDF and HAG are equal, as is established from the construction of Proposition 23, and so on.

Diagram. Construction of a parallel to BC through A by equal arcs. The lower line runs from B (left) to C (right); the transversal AD is drawn obliquely from A down to D on BC. From D an arc is described (endpoints E and F) and from A an equal arc (endpoints G and H); the line AH is the parallel drawn through A.

- A (upper line, left): the given point through which the parallel is drawn
- H (top, arc endpoint): end of the arc centred at A; the parallel AH passes through it
- G (on AD below A): arc endpoint at A (angle HAG)

- E (on BC, left of D): arc endpoint at D (angle EDF)
- D (on BC): foot of the transversal AD
- B (lower line, left) / C (lower line, right): endpoints of the given line
- F: second endpoint of the arc centred at D, on segment AD [?] (label not clearly legible on the leaf)

By another method a parallel to the given line BC will be drawn through the same point A in this way. From center A, with any radius, describe an arc cutting BC at the point D; and with the same radius, from D, mark off a point E on the same straight line BC. Then, with the same radius, from A and E describe two arcs cutting each other at F. For once the straight line AF is drawn, it will be parallel to the straight line BC. Since the same radius was assumed, AF is equal to DE, and AD equal to EF (were these lines drawn); therefore AF, being opposite DE, will be parallel to it, as we shall demonstrate later at Proposition 34.

Diagram. A pair of horizontal lines forming the parallelogram-based construction. The upper line carries A (left) and F (right), where two arcs cross; the lower line runs from B (left) through D and E to C (right). AF is drawn equal and parallel to DE.

- A (upper line, left): the given point
- F (upper line, right): intersection of the two arcs
- B (lower line, left) / C (lower line, right): endpoints of the given line BC
- D (lower line): where the arc from A cuts BC
- E (lower line): point marked off from D at the same radius

But if the point A is close to the straight line BC, the desired parallel is drawn more conveniently by this rule. From A, take a point D on BC at any distance; and from any point of the same line BC—namely E, which should however be somewhat distant from D (for the greater the distance between D and E, the more easily and accurately the parallel is drawn)—describe an arc toward A with the same radius. Then, from A with the radius DE, describe another arc cutting the former at F. For when AF is rightly drawn it will be parallel to BC, as before, because AF is equal to DE by the same radius, and AD equal to EF, were these lines drawn, and so on.

Diagram. The same construction adapted to a point A near the line. Upper line with A (left) and F (right, arc intersection); lower line from B (left) through D and E to C (right).

- A (upper line, left): the given point, close to BC

- F (upper line, right): intersection of the two arcs
- B (lower line, left) / C (lower line, right): endpoints of BC
- D (lower line): point taken on BC
- E (lower line): the more distant point on BC

But we have set out a far easier method of drawing a parallel to a given line through a given point in our *Astrolabe*, Book 1, Lemma 4 (Clavius, *Astrolabium* I, lemma 4).

From these constructions we shall easily draw, from the end point of some line, a line perpendicular to that given line—even if the line cannot be produced—which we promised in the scholium to Proposition 11. Let AB be the straight line from whose end point B a perpendicular is to be raised. Taking any point C, cut off CB equal to CA, and from A and B, with any radius, describe two arcs cutting each other at D; then draw the straight line DC, which will be perpendicular to AB, as we wrote at Proposition 11. Next, through B draw a parallel to CD, according to the construction of this Proposition 31 just explained: from CD cut off any segment CE; from B, with radius CE, describe an arc, which is cut at F by another arc described from E with radius CB; and draw the straight line BF. For this will be parallel to CD, as is clear from the construction of this Proposition 31. Therefore, since the angle ACD is equal to the interior angle CBF [Margin: l — I.29], and since ACD is a right angle, CBF will also be a right angle, and consequently BF will be perpendicular to AB.

Diagram. Raising a perpendicular at the endpoint B of AB. Along the baseline lie A (left), C (middle, with $CA = CB$), and B (right). Above, D is the intersection of the arcs from A and B (so $DC \perp AB$); E is taken on CD, and F is where the arcs of radius CE and CB meet, so that BF is parallel to CD and hence perpendicular to AB.

- D (top): intersection of the arcs from A and B; $DC \perp AB$
- E (middle, on CD): point cut off on CD
- F (middle, right): intersection of the constructing arcs; BF passes through it
- A (baseline, left) / C (baseline, middle) / B (baseline, right): the line AB with C such that $CA = CB$

Similarly, if a straight line AB is given, and a point C outside it—near the edge of the plane in which the line lies—we shall draw from C a perpendicular to AB, without either producing the line or extending the plane below AB, as we promised at Proposition 12, in this way. Take a point D anywhere on AB, and cut off on either side DA and DB equal to each other; from A and B, with any radius,

describe two arcs cutting each other at E; then draw the straight line ED, which by the construction of Proposition 11 will be perpendicular to AB. Next, through C draw a parallel to DE, according to the construction of this Proposition 31: from the given point C, with any radius, describe an arc cutting DE at F; with the same radius, from D toward C, describe another arc, which is intersected at G by another arc drawn from C with radius DF. For the straight line drawn from C through G, cutting AB at H, will be parallel to DE, by the construction of this Proposition 31. Therefore, as we wrote just now, CH will be perpendicular to AB, just as ED is perpendicular to the same.

Diagram. Dropping a perpendicular from an external point C to AB. Along the baseline lie A, D, B, and H (left to right), with $DA = DB$. E lies above D, where the arcs from A and B meet, so $ED \perp AB$. C is the external point (upper right); F is where an arc from C cuts DE, and G is the intersection of the constructing arcs, so that the line CGH is parallel to DE and hence $CH \perp AB$.

- E (top): intersection of the arcs from A and B; $ED \perp AB$
- C (upper right): the given external point
- F (middle, left, on DE): where an arc from C cuts DE
- G (middle, right): intersection of the two constructing arcs; CH passes through it
- A / D / B / H (baseline, left to right): AB with D its chosen point ($DA = DB$) and H the foot of the perpendicular CH

Theorem 22. Proposition 32.

[Margin: xxxij.]

(*Euclid, enunciation.*) In any triangle, when one side is produced, the exterior angle is equal to the two interior and opposite angles. And the three interior angles of a triangle are equal to two right angles.

[p. 56]

(*Clavius's demonstration.*) In triangle ABC, let the side BC be produced to D. I say first that the exterior angle ACD is equal to the two interior and opposite angles A and B together. For draw from C a line CE parallel to the straight line AB [Margin: b — I.31]. Then, since the straight line AC falls upon the parallels AB and CE, the alternate angles A and ACE will be equal [Margin: a — I.29]. Again, since the straight line BD falls upon the same parallels, the exterior angle DCE will be equal to the interior angle B [Margin: b — I.29]. Therefore, adding the equals ACE and A, the whole angle ACD—which is composed of the two angles DCE and ACE—

becomes equal to A and B together [Margin: c — common notion 2 (pronuntiatum 2)]. Which is what was proposed.

I say secondly that the three interior angles of the same triangle—A, B, and ACB—are equal to two right angles. For since the exterior angle ACD, as was shown, is equal to the two interior angles A and B, if the common angle ACB is added, the two angles ACD and ACB will be equal to the three angles A, B, and ACB [Margin: d — common notion 2]. But the two angles ACD and ACB are equal to two right angles [Margin: e — I.13]. Therefore the three interior angles A, B, and ACB are also equal to two right angles. Therefore, in any triangle, when one side is produced, and so on. Which was to be demonstrated.

Diagram. The triangle for Proposition 32. Triangle ABC has its apex A at the top; the base runs from B (lower left) to C, and BC is produced beyond C to D (lower right). From C a line CE is drawn upward to the right to E, parallel to AB.

- A (apex, top): vertex whose interior angle is under consideration [?] (label partly obscured by an arrowhead on the tile)
- B (lower left): base vertex
- C (base, right of B): vertex at which the side is produced
- D (lower right): the point to which BC is produced
- E (upper right): endpoint of CE, drawn parallel to AB

Scholium

Since it was demonstrated in Proposition 16 that the exterior angle of any triangle is greater than either of the interior and opposite angles, whereas here the same exterior angle is shown to be equal to those two interior angles together, it is clear that either one of the interior and opposite angles is exceeded by the exterior angle by the amount of the other interior and opposite angle. Thus, in the triangle proposed, the interior angle A is exceeded by the exterior angle ACD by the interior angle B; and the interior angle B is exceeded by the same exterior angle ACD by the interior angle A—since the angle ACD has been shown in this place to be equal to A and B together. Again, since it was demonstrated in Proposition 17 that any two angles of a triangle, taken in any way, are less than two right angles, while here all three are equal to two right angles, it is clear that the two fall short of two right angles by the remaining angle of the triangle. Thus, in the same triangle, the two angles A and B fall short of two right angles by the angle ACB, to which that deficit is equal.

Moreover, that every triangle has its three angles equal to two right angles was, as Eudemus [Eudemus of Rhodes] reports, first of all demonstrated by the Pythagoreans in this way. Let there be a triangle ABC, and through the point A [Margin: a — I.31] draw a line DE parallel to BC. Then, since the alternate angles DAB and ABC are equal [Margin: b — I.29], if the equal angles EAC and ACB are added (for these too are alternate), the two angles DAB and EAC will be equal to the two angles ABC and ACB [Margin: c — common notion 2]. Therefore, adding the common angle BAC [Margin: d — I.2 (pronuntiatum)], the three angles DAB, BAC, CAE will be equal to the three angles ABC, BAC, ACB. But the angles DAB, BAC, CAE are equal to two right angles, as is established from Proposition 13. Therefore in triangle ABC also the angles ABC, BAC, ACB are equal to two right angles—which is what was proposed.

From this, moreover, we shall easily conclude that the exterior angle ACF, if the side BC is produced, is equal to the two interior and opposite angles ABC and BAC. For since the angles ABC, BAC, ACB are equal to two right angles, as was shown [Margin: e — I.13], and the angles ACF and ACB are also equal to two right angles, the angles ABC, BAC, ACB will be equal to the angles ACF and ACB. Therefore, removing the common angle ACB, the angle ACF remains equal to the two angles ABC and BAC [Margin: f — common notion 3 (pronuntiatum 3)].

Diagram. The Pythagorean demonstration. Triangle ABC has apex A; through A the line DAE is drawn parallel to the base BC (D on the left, E on the right). The base runs from B (lower left) through C, with BC produced beyond C to F (lower right), forming the exterior angle ACF.

- D (top left): endpoint of the parallel through A (angle DAB)
- A (apex): vertex of the triangle
- E (top right): endpoint of the parallel through A (angle EAC)
- B (lower left): base vertex
- C (base): base vertex
- F (lower right): point to which BC is produced (exterior angle ACF)

The first part of Euclid's proposition can also easily be converted; that is: if from one angle of a triangle a straight line is drawn so that the exterior angle equals the two interior and opposite angles, then that line is set in a straight line with the side. For from C draw the straight line CD, and let the angle ACD be equal to the two angles ABC and BAC. I say that BC and CD lie in a straight line. Since the angle ACD is equal to the angles ABC and BAC, if the common angle ACB is added, the angles ACD and ACB will be equal to the angles ABC, BAC, ACB [Margin: g —

common notion 2]. But ABC, BAC, ACB are equal to two right angles [Margin: h — I.32]. Therefore the angles ACD and ACB will also be equal to two right angles. Therefore BC and CD constitute one straight line [Margin: i — I.14].

How Many Right Angles All the Interior Angles of Any Rectilineal Figure Are Equivalent To

(An additional proposition by Clavius.) By two methods, from this Proposition 32, we shall gather how many right angles the interior angles of any rectilineal figure are equivalent to; the first of these is as follows.

(Enunciation.) All the angles of any rectilineal figure are equal to twice as many right angles as the number giving the figure's rank among the rectilineal figures.

That is: all the angles of the first rectilineal figure are equal to twice one right angle, that is, to two right angles; the angles of the second rectilineal figure are equal to twice two right angles, namely four right angles; the angles of the third rectilineal figure are equal to twice three right angles, namely six right angles; and so with the rest. The rank that any rectilineal figure holds among the rectilineal figures is indicated by the number of its sides or angles, less two—since two straight lines do not enclose a surface, and therefore do not form a figure, but at least three straight lines are required to constitute a figure. From this it follows that the triangle, since it has three sides and as many angles, is the first among the rectilineal figures; for when two is subtracted from three, one remains. Thus a figure having 20 sides or angles will be the eighteenth among rectilineal figures, since two subtracted from 20 leaves 18. The same judgment is to be made about the other figures. Accordingly, a figure contained by 20 sides, being the eighteenth, will have 20 angles equivalent to 36 right angles—namely twice 18 right angles, as was said. Likewise all 10 angles of a figure contained by 10 sides will be equivalent to 16 right angles, since such a figure is the eighth among the rectilineal figures. This will be demonstrated in the following way. Every rectilineal figure is divided into as many triangles as its rank among the figures—that is, as many as it has angles or sides, less two. For from any one of its angles straight lines can be drawn to all the opposite angles; only to the two neighboring angles can they not be drawn. Therefore it will be divided into as many triangles as it has angles, less those two. Thus you see that a triangle cannot be divided into other triangles; a quadrilateral can be cut into two; a pentagon into three; a hexagon into four, and so on. Since, then, the angles of these triangles constitute all the angles of the proposed rectilineal figure, and all the angles of any triangle are equal to two right angles, it is clear that all the angles of

any rectilineal figure are equal to twice as many right angles as the number of triangles into which it is divided—that is, as its rank among the rectilineal figures. This is indeed clearly seen in the figures shown.

Diagram. A row of four unlabelled rectilineal figures illustrating division into triangles by diagonals drawn from a single vertex: a triangle (undivided), a quadrilateral (split into 2 triangles by one diagonal), a pentagon (split into 3), and a hexagon (split into 4). No point-labels are printed.

[p. 57]

all the angles of any rectilineal figure are equal to twice as many right angles as the number of triangles into which it is divided—that is, as its rank among the rectilineal figures. This is indeed clearly seen in the figures shown.

The second method, by which the value of the angles of any rectilineal figure is found, is this.

(Enunciation.) All the angles of any rectilineal figure are equal to twice as many right angles, less four, as it contains sides or angles.

That is: the angles of any triangle are equal to twice three right angles less four—namely two right angles. Likewise, the angles of a figure containing 20 sides will be equivalent to twice 20 right angles less four—namely 36 right angles—and so on. The demonstration is as follows. If from any point taken inside the figure straight lines are drawn to all the angles, as many triangles are produced as the figure contains sides or angles. Then, since the angles of any triangle are equal to two right angles [Margin: a — I.32], all the angles of those triangles will be equal to twice as many right angles as there are sides bounding the figure. But the angles of those triangles that meet at the point taken inside the figure do not belong to the angles of the proposed rectilineal figure, as is evident. Therefore, if these are taken away, the remaining triangle-angles—those that constitute the angles of the proposed figure—will likewise be equal to twice as many right angles, less those formed at the assumed point, as the figure contains sides or angles. Now all those angles about the said point, however many they are, are equal to just four right angles, as we gathered from Proposition 15. For this reason the angles of every figure are equal to twice as many right angles, less four, as the figure contains angles or sides—which is what was proposed.

Diagram. A row of unlabelled rectilineal figures illustrating the second method—division into triangles from a single interior point: a figure divided into triangles

about an interior point, then a quadrilateral (4 triangles), a pentagon (5 triangles), and a hexagon (6 triangles). No point-labels are printed.

Furthermore, from this second method it is clear that, if the several sides of any rectilineal figure are produced in order toward the same side, all the exterior angles together are equal to four right angles. For any exterior angle, together with the interior angle next to it, equals two right angles [Margin: b — I.13]; and so all the exterior angles together with all the interior angles will be equal to twice as many right angles as the figure contains sides or angles. But the interior angles alone are equal to twice as many right angles less four, as we demonstrated. Therefore, if the interior angles are taken away, the exterior angles will remain equal to just four right angles—the four, namely, by which the interior angles fall short, so that the interior and exterior together make up twice as many right angles as there are sides bounding the figure. For example: in any triangle the interior and exterior angles together are equal to six right angles. Since, then, the interior angles are equal to two right angles [Margin: c — I.32], the exterior alone will be equal to just four right angles. In a quadrilateral, the exterior and interior angles together are equal to eight right angles. Since, then, the interior angles alone are equal to four right angles, as we showed, the exterior alone will also be equal to four right angles. In a pentagon, or five-angled figure, the interior and exterior angles are equal to 10 right angles. But since the interior angles amount to six right angles, as we demonstrated, the exterior angles will remain equal to just four right angles. All of this may be seen in the figures shown; and the reasoning is the same in all other figures.

Diagram. A small schematic strip beside the text, showing rectilineal figures (a triangle and further polygons) with their sides produced in order, so that the exterior angles are marked at each vertex—illustrating that the exterior angles together make four right angles. The figures are drawn schematically; no point-labels are clearly legible on the leaf.

From Campanus

(A demonstration drawn from Campanus of Novara.) If the individual sides of a pentagon are produced in both directions, so that any two meet outside the pentagon, five angles will be formed by the meeting sides, equal to just two right angles.

In the pentagon ABCDE, let the sides, produced in both directions, meet at the points F, G, H, I, K. I say that the five angles F, G, H, I, K are equal to just two right angles. For in triangle BHK, since the side HB is produced to F, the exterior angle

FBK will be equal to the two interior and opposite angles H and K [Margin: d — I.32]. By the same reasoning, in triangle AIG the exterior angle FAG will be equal to the two interior and opposite angles I and G. Therefore the two angles FBA and FAB are equal to the four angles G, H, I, K. Adding, then, the common angle F [Margin: e — common notion 2], the three angles A, B, F of triangle ABF will be equal to the five angles F, G, H, I, K. But the angles A, B, F of triangle ABF are equal to two right angles [Margin: f — I.32]. Therefore the five angles F, G, H, I, K are also equal to two right angles. Which is what was proposed.

Diagram. A five-pointed star (pentagram). The inner pentagon has vertices A, B, C, D, E; each side, produced in both directions, meets a neighbouring produced side at the outer star-points F (top), G (right), H (lower right), I (lower left), and K (left). The construction triangles ABF, BHK, and AIG referred to in the text are formed by these produced sides.

- F (top apex): star-point (vertex of triangle ABF)
- A (upper left, inner): pentagon vertex
- B (upper centre, inner): pentagon vertex
- G (right): star-point
- C (centre-right, inner): pentagon vertex
- D (lower centre, inner): pentagon vertex [?] (printed glyph ambiguous; position corresponds to the fifth pentagon vertex D)
- E (left-centre, inner): pentagon vertex
- K (far left): star-point [?] (printed glyph ambiguous between K and R; text names it K)
- I (lower left): star-point
- H (lower right): star-point

Corollary I

From this Proposition 32 it follows that the three angles of any triangle taken together are equal to the three angles of any other triangle taken together, since both those three and these are equal to two right angles [Margin: g — I.32]. Hence, if two angles of one triangle are equal to two angles of another triangle, the remaining angle of the one will also be equal to the remaining angle of the other, and the triangles themselves will be equiangular.

Corollary II

It is also established that, in every isosceles triangle whose angle contained by the equal sides is a right angle, each of the remaining angles is half a right angle. For

the remaining two together make one right angle, since all three are equal to two right angles [Margin: h — I.32], and that third one is assumed to be a right angle. Therefore, since the two remaining angles are equal to each other [Margin: i — I.5], each of them will be half a right angle. But if the angle contained by the equal sides is obtuse, each of the others is less than half a right angle, for the remaining two together will be less than one right angle, and so on. Finally, if the said angle is acute, each of the remaining two is greater than half a right angle, since the remaining two together will be greater than one right angle, and so on.

Corollary III

It is also clear that any angle of an equilateral triangle is two-thirds of one right angle, or a third of two right angles. For the two right angles to which the three angles of an equilateral triangle are equal [Margin: k — I.32], divided into three angles, make two-thirds of one right angle each.

Euclid, *Elements*, Book I — Clavius's Commentary (continued)

COROLLARY IV

It is also clear that if from one angle of an equilateral triangle a perpendicular is dropped to the opposite side, two scalene triangles are formed. Each of them has one right angle, at the foot of the perpendicular; a second angle equal to two-thirds of a right angle — namely the angle that is also an angle of the equilateral triangle; and, finally, a remaining angle equal to one-third of a right angle.

SCHOLIUM

Moreover, from the third corollary a method can be derived for dividing a right angle into three equal angles. Let ABC be a right angle. On the straight line AB construct an equilateral triangle ABD [b: *Elements* I.1]. Since, by the third corollary, the angle ABD makes two-thirds of the right angle ABC, the angle CBD will be one-third of that same right angle. Now bisect the angle ABD by the straight line BE [c: *Elements* I.9]; then each of the angles ABE and EBD will likewise be one-third of a right angle. The right angle ABC is thus divided into three equal angles — which is what was proposed.

Diagram. A small figure illustrating the trisection of a right angle. The right angle ABC stands on the base, with an equilateral triangle ABD raised on AB and the bisector BE drawn.

- A (bottom-left): endpoint of the base line
- B (bottom, right of A): vertex of the right angle ABC
- C (upper right): endpoint of the vertical arm of the right angle
- D (top): apex of the equilateral triangle ABD
- E (left, interior): point on the bisector BE of angle ABD

THEOREM 23. PROPOSITION 33

[Margin: xxxij.]

Straight lines that join equal and parallel lines toward the same parts are themselves also equal and parallel.

Let AB and CD be equal and parallel straight lines, and let the straight lines AC and BD join them toward the same parts. I say that AC and BD too are equal and parallel. Draw the straight line AD. Since AD falls upon the parallels AB and CD, the alternate angles BAD and CDA are equal [d: *Elements* I.29]. Therefore the two sides BA and AD of triangle BAD equal the two sides CD and DA of triangle CDA, each to each, and the angles contained by these sides are likewise equal; so the bases BD and AC are equal, and the angle ADB equals the angle DAC [a: *Elements* I.4]. Since, then, these are alternate angles between the straight lines AC and BD, those lines are parallel [b: *Elements* I.27]. And it has already been proved that they are equal. Therefore straight lines that join equal and parallel lines, and so forth. Which was to be demonstrated.

Diagram. A parallelogram-like figure with the diagonal AD drawn. The equal, parallel lines AB and CD are joined by AC and BD.

- A (top-left)
- B (top-right)
- C (bottom-left)
- D (bottom-right)

The diagonal runs from A to D.

SCHOLIUM

Euclid said that equal and parallel lines must be joined *toward the same parts*, so that the joining lines are themselves both equal and parallel. For if they were joined toward opposite parts — say at A and D, and again at B and C — the joining lines would never be parallel, but would always cut one another; nor would they be equal, except very rarely, as will be clear from the following proposition.

THEOREM 24. PROPOSITION 34

[Margin: xxxiv.]

In parallelogrammic areas the opposite sides and the opposite angles are equal to one another; and the diameter bisects them.

Let ABCD be a parallelogram, of the kind we defined in Definition 35. I say that the opposite sides AB and DC are equal to one another, and likewise the opposite sides AD and BC; that the opposite angles B and D are equal to one another, and likewise the opposite angles DAB and DCB; and finally that, when the diameter AC is drawn, the parallelogram itself is bisected.

Since AB and DC are parallel, the alternate angles BAC and DCA are equal [c: *Elements* I.29]. Again, since AD and BC are parallel, the alternate angles BCA and DAC are also equal [d: *Elements* I.29]. And so, since the angles BAC and BCA of triangle ABC equal the two angles DCA and DAC of triangle ADC, each to each, and the side AC, which lies between these angles, is common to both triangles, the straight line AB equals the opposite straight line DC, and the straight line BC equals the opposite straight line AD [e: *Elements* I.26] — which is the first point. For the same reason the angle B equals the angle D. And since, if to the equal angles BAC and DCA the equal angles DAC and BCA are added, the whole angles BAD and BCD are likewise made equal [a: Common Notion 2], the second point is established — namely, that the opposite angles are equal. Finally, since the two sides AB and BC of triangle ABC equal the two sides CD and DA of triangle CDA, each to each, and the angle B equals the angle D, as we have just shown, the triangles ABC and CDA are equal [b: *Elements* I.4]; and therefore the parallelogram ABCD is bisected by the diameter AC, which was the third point proposed. Therefore, in parallelogrammic areas, the opposite sides and angles are equal to one another, and so forth. Which was to be demonstrated.

Diagram. A parallelogram with the diameter AC drawn.

- A (top-left)

- B (top-right)
- D (bottom-left)
- C (bottom-right)

The diameter runs from A to C.

SCHOLIUM

Euclid aptly said that only the *parallelograms* are bisected by the diameter, and not the *angles* as well. For only in the square and the rhombus are the angles too bisected by the diameter; but in the oblong [Latin: *altera parte longius* — "longer on one side"] and in the rhomboid they are divided into unequal parts. All of this will be clear if we first show that these four figures — the square, the oblong, the rhombus, and the rhomboid — are parallelograms. This we shall demonstrate by the three following theorems, of which the first is this:

Every quadrilateral having its opposite sides equal is a parallelogram.

In quadrilateral ABCD let the opposite sides AB and DC be equal, and likewise the opposite sides AD and BC. I say that ABCD is a parallelogram — that is, that the lines AB and DC are parallel, and likewise the lines AD and BC. When the diameter AC is drawn, the two sides AB and BC of triangle ABC will equal the two sides CD and DA of triangle CDA, each to each, the base AC being common. Therefore the angle B will equal the angle D [c: *Elements* I.8]. Again, since the sides AB and BC equal the sides CD and DA, each to each, and the angles B and D have been shown equal, the angle

Diagram. A quadrilateral with the diameter AC drawn.

- B (top-left)
- A (top-right)
- C (bottom-left)
- D (bottom-right)

The diameter runs from A to C.

BAC will equal the alternate angle DCA, and the angle BCA the alternate angle DAC. Therefore AB and DC are parallel, and likewise AD and BC [a: *Elements* I.27] — which is what was proposed.

From this it is established that the rhombus and the rhomboid are parallelograms, since their opposite sides are equal to one another, as is clear from their definitions. By the same reasoning the square will be a parallelogram,

since it has its opposite sides equal — indeed all four of its sides are equal to one another, by its definition. This theorem, moreover, converts the first part of Proposition 34, as is evident.

The second theorem is this:

Every quadrilateral having its opposite angles equal is a parallelogram.

In quadrilateral ABCD let the opposite angles A and C be equal, and likewise the opposite angles B and D. I say that ABCD is a parallelogram — that is, that the lines AB and DC are parallel, and likewise the lines AD and BC. For if to the equal angles A and C the equal angles B and D are added [b: Common Notion 2], the two angles A and B will equal the two angles D and C, and therefore the angles A and B will make half of the four angles A, B, C, and D. Since these four angles equal four right angles, as we demonstrated at Proposition 32, the two angles A and B equal two right angles. Therefore AD and BC are parallel [c: *Elements* I.28]. For the two angles A and D will likewise equal the two angles B and C. From this too it is clear that the rhomboid is a parallelogram, since its opposite angles are equal, by definition. Similarly the square and the oblong; for their opposite angles too are equal, being right angles, from their definitions.

Diagram. A small parallelogram (drawn as an oblong).

- A (top-left)
- B (top-right)
- D (bottom-left)
- C (bottom-right)

This theorem converts the second part of Proposition 34, as is plain. The third part, however, cannot be converted. For a trapezium too can be bisected by a diameter and yet not be a parallelogram. Let ABCD be an oblong or a rhomboid, which has been shown to be a parallelogram; in it, when the diameter AC is drawn, construct on AC a triangle AEC equal to triangle ABC but in reverse order, so that the side CE equals the side AB, and AE equals CB, as we taught in the scholium to Proposition 22; and let the trapezium AECD be formed. Now since the triangle ABC equals the triangle ADC — because the diameter AC bisects the parallelogram [d: *Elements* I.34] — the triangle AEC will also equal the triangle ADC; and accordingly the trapezium AECD is bisected by the diameter AC.

Diagram. Two figures (the oblong case and the rhomboid case), each showing the parallelogram ABCD with the diameter AC and the added triangle AEC forming the trapezium AECD.

- A (top-left)
- B [?] (top-right)
- C (bottom-right)
- D (bottom-left)
- E (right side, between B and C): apex of the constructed triangle AEC

But if any quadrilateral is bisected by *each* of its two diameters, it will be a parallelogram, as we shall show at Proposition 39. This indeed cannot happen in any trapezium.

The third theorem is of this kind:

Every quadrilateral having all its angles right is a parallelogram.

In quadrilateral ABCD let all four angles be right. I say that it is a parallelogram — that is, that the lines AB and DC are parallel, and likewise AD and BC. Since the two angles A and B equal two right angles (being themselves two right angles), AD and BC are parallel [c: *Elements* I.28]. In the same way AB and DC are parallel; and thus ABCD is a parallelogram, which is what was proposed.

Diagram. A square.

- D (top-left)
- C (top-right)
- A (bottom-left)
- B (bottom-right)

From this it is again established that the square and the oblong are parallelograms, since all their angles are right, as is clear from their definitions.

Having in this way demonstrated that the square, the oblong, the rhombus, and the rhomboid are parallelograms, we shall easily show that the angles of the square and of the rhombus are bisected by the diameter, but that the angles of the oblong and of the rhomboid are not bisected, as we noted a little earlier. Let ABCD be a square or a rhombus, with the diameter AC. Since the two sides BA and AC of triangle BAC equal the two sides DA and AC of triangle DAC, each to each, and the base BC equals the base DC (for these figures are equilateral), the angles BAC and DAC are equal [f: *Elements* I.8]. Therefore the angle BAD is bisected. In the same way we shall demonstrate that the remaining angles are bisected by the diameter.

Diagram. Two small figures (the square and the rhombus), each with a diagonal drawn.

- A (top-left)
- B (top-right)
- D (bottom-left)
- C (bottom-right)

Again, let ABCD be an oblong or a rhomboid, with the diameter AC, and let AB be the greater side. Since in triangle ABC the side AB is greater than the side BC, the angle BCA is greater than the angle BAC [g: *Elements* I.18]. But the angle BCA equals the alternate angle DAC [h: *Elements* I.29], because BC and AC are parallel (for ABCD has been shown to be a parallelogram). Therefore the angle DAC too will be greater than the angle BAC. And for this reason the angle BAD is divided unequally by the diameter AC. The same reasoning holds for the other angles. Hence Euclid aptly said, in the third part of this proposition, that only the parallelograms are bisected by the diameter, and not the angles as well.

Diagram. Two small figures (the oblong and the rhomboid), each with a diagonal drawn.

- A (top-left)
- B (top-right)
- D (bottom-left)
- C (bottom-right)

In nearly the same way we shall show that the two diameters in a square and in an oblong are equal, but that in a rhombus and a rhomboid they are unequal — the greater being the one that divides the acute angles, the lesser the one that divides the obtuse angles. Let ABCD be a square or an oblong, with the diameters AC and BD, which I say are equal. Since the two sides AB and BC of triangle ABC equal the two sides AB and AD of triangle BAD, each to each, and the angle ABC equals the angle BAD (each being right), the base AC equals the base BD [i: *Elements* I.4]. And accordingly the diameters in a square and in an oblong are equal.

Diagram. Two figures (the square and the oblong), each with both diameters drawn, meeting at the centre E.

- A (top-left)
- B (top-right)

- D (bottom-left)
- C (bottom-right)
- E (centre): intersection of the diameters AC and BD

Diagram. Two figures (the rhombus and the rhomboid), each with both diameters drawn, meeting at E.

- A (top-left)
- B (top-right)
- E (centre): intersection of the diameters AC and BD
- C (bottom, left figure) / (bottom-right)
- D (bottom)

Again, let ABCD be a rhombus or a rhomboid, with the diameters AC and BD, and let the angle BAD be the greater and ABC the lesser. For they are not equal; otherwise each would be right, since both together equal two right angles [a: *Elements* I.29] — which is absurd, and contrary to the definitions of the rhombus and rhomboid. I say that the diameter BD is greater than the diameter AC. Since the two sides AB and AD of triangle BAD equal the two sides AB and BC of triangle ABC, each to each, and the angle BAD is greater than the angle ABC, the base BD will be greater than the base AC [b: *Elements* I.24] — which is what was proposed. From this it is clear why, in Proposition 33, Euclid asserted that only those lines are equal which join equal parallels toward the same parts, as we noted there. For in the rhombus and rhomboid the straight lines AC and BD are unequal, even though they join the equal parallels AB and DC, because they do not join them toward the same parts, as is evident.

In every parallelogram, nevertheless, the diameters bisect one another. For the two angles EAD and EDA of triangle AED equal the alternate angles ECB and EBC of triangle BEC, each to each [c: *Elements* I.29]; and the side AD equals the side BC opposite to it in the parallelogram ABCD [d: *Elements* I.34], each of which lies between equal angles. Therefore AE will equal CE, and DE will equal BE [e: *Elements* I.26]. And so each diameter is bisected at the point E.

We shall also demonstrate the converse of what we have just said, namely:

Every quadrilateral in which the diameters bisect one another is a parallelogram.

In quadrilateral ABCD let the diameters AC and ED [recte: BD] bisect one another at E. I say that ABCD is a parallelogram. Since the sides AE and EB of triangle AEB equal the sides CE and ED of triangle CED, and the angles contained at the vertex

E are also equal [f: *Elements* I.15], the bases AB and CD are equal, and the angle ABE equals the alternate angle CDE [g: *Elements* I.4]. Therefore the straight lines AB and CD are parallel [h: *Elements* I.27]. By the same reasoning AD and CB are shown to be parallel. Therefore ABCD is a parallelogram.

Diagram. A small parallelogram with both diameters drawn, meeting at E.

- A (top-left)
- B (top-right)
- E (centre): intersection of the diameters
- D (bottom-left)
- C (bottom-right)

And so, if one wishes to construct a parallelogram readily, let him draw two lines AC and BD intersecting one another at E in any manner whatever. For if he cuts off EA and EC equal to one another, and likewise EB and ED equal to one another, the lines AB, BC, CD, and DA, when drawn, will form a parallelogram, as has here been demonstrated. But if the angles at E are right, and all four lines EA, EC, EB, and ED are equal to one another, the parallelogram so formed will be a square, because the eight angles at A, B, C, and D are each half a right angle, as will easily be demonstrated by Propositions 5 and 32 of this book.

This theorem too may be referred to here:

A straight line that bisects the diameter of a parallelogram in any way whatever also bisects the parallelogram; and a straight line that bisects the parallelogram in any way whatever also bisects the diameter.

In parallelogram ABCD let the straight line EF bisect the diameter AC at the point G. I say that the parallelogram is divided in half. Since the angle EAG equals the alternate angle FCG [i: *Elements* I.29], and the angle EGA equals the angle FGC [k: *Elements* I.15]; and since the side AG equals the side CG by hypothesis — each of which lies between equal angles — the sides EG and FG are also equal [l: *Elements* I.26]. Therefore, since the sides AG and GE equal the sides CG and GF, and the contained angles are also equal, the triangles AGE and CGF are equal [m: *Elements* I.4]. So, when the common area BCGE is added, the triangle ABC equals the trapezium BCFE [n: Common Notion 2]. But the triangle ABC is half of the parallelogram ABCD [o: *Elements* I.34]. Therefore the trapezium too is half of the same parallelogram; and accordingly the straight line EF bisects the parallelogram.

Diagram. A large parallelogram ABCD with the diameter AC and the bisecting line EF, together with an auxiliary line for the converse.

- A (top-left)
- E (top, between A and B)
- B (top-right)
- G (interior, upper): point where EF meets the diameter AC
- H (interior, lower): auxiliary point used in the converse
- D (bottom-left)
- F (bottom, left of centre)
- I (bottom, centre-right)
- C (bottom-right)

Now let EF bisect the parallelogram; I say that the diameter AC too is bisected at G. For if the diameter AC is *not* bisected at G [p: *Elements* I.10], let it be bisected at some other point, say H, through which let the straight line EHI be drawn. Then, as we have just demonstrated, EICB will be the trapezium that is half of the parallelogram ABCD, and thus equal to the trapezium EFCB, which is also supposed to be half of the same parallelogram — a part equal to the whole, which is absurd. Therefore AC is bisected at G and at no other point, which was what was proposed.

From this it is easily gathered how, if a point is marked on some side of any parallelogram — or even inside the parallelogram, or outside it, provided it does not lie on the diameter unless it bisects the diameter — a line ought to be drawn from that point so as to bisect the parallelogram. For if the diameter is drawn, and from the given point a straight line is drawn through the midpoint of the diameter, what is proposed will be accomplished. For instance, if the point is E, on the side AB, one must draw the straight line EF through G, the point at which the diameter AC is bisected; and so with other points.

Peletier [Jacques Peletier] also demonstrates here a not-unpleasing problem, namely:

Between two infinite straight lines that form an angle, to place a straight line equal to a given line, which shall make with one of them an angle equal to any given angle. But this given angle, together with the angle contained by the given lines, must be less than two right angles.

Let the two infinite straight lines AB and AC contain the angle BAC; let D be any given finite straight line, and E a given angle, on this condition: that the two

angles E and BAC are less than two right angles. It is required, then, to place between the straight lines AB and AC a straight line equal to the given line D, making with one of them — namely with AC — an angle equal to the given angle E. Make the angle CAF equal to the angle E [q: *Elements* I.23]; and, F A having been produced to G, let AG be equal to the straight line D [r: *Elements* I.3]. Through G draw GB parallel to AC, cutting AB at B [s: *Elements* I.31]. Then through B draw BC parallel to AG, cutting AC at C. I say that the straight line BC, placed between the straight lines AB and AC, is equal to the straight line D, and that the angle BCA equals the angle E. For since it is a parallelogram, by con-

Diagram. Peletier's construction. At the left stand the given angle E and the given finite line D; at the right the two infinite lines AB and AC meet at A, with the constructed parallelogram AGBC.

- E (upper left): the given angle
- D (left): the given finite straight line
- G (top, left of B): point on the line FA produced, with AG equal to D
- B (top-right): point where GB (parallel to AC) meets AB
- A (centre-left): vertex where the infinite lines AB and AC meet
- C (bottom-right): point where BC (parallel to AG) meets AC
- F (bottom, below A): point on line AC through which the angle CAF equal to E is made

(Continuation of the demonstration begun on the previous page.)

by construction, ACBG [is a parallelogram], and so the line BC will be equal to the line GA.[Margin a: *Elements* I.34] But GA is equal, by construction, to the line D. Therefore BC too will be equal to the line D. Again, since the angle BCA is equal to the alternate angle CAF,[Margin b: *Elements* I.29] and the angle E is also equal, by construction, to that same angle CAF, the angles E and BCA will be equal — which is what was proposed. Moreover, anyone can see from the construction why the two given angles must be less than two right angles; for otherwise triangle ABC would not be formed, if the angles BAC and BCA were equal to, or greater than, two right angles, as is clear from Proposition 17 or 32.

SCHOLIUM II.

It will not be out of keeping with our plan to add here some remarks about lines drawn inside a triangle, which Pappus of Alexandria [Pappus] demonstrates in Book 3 of his *Mathematical Collections* — as we undertook to do at Proposition 21. First, then, in every triangle that is not equilateral, and is not an isosceles triangle

whose base is smaller than either of its [equal] sides, one can construct from the two endpoints of the base not only two straight lines within the triangle whose sum is greater than the sum of the two sides — as Proclus [Proclus] showed for right-angled or obtuse-angled triangles at Proposition 21 — but even two lines whose sum is equal to it.

For let ABC be the first triangle, with side AB greater than side AC, and let BD be half of the two sides AB and AC together. Take any point E between A and D, and draw EF parallel to BC; from any point G draw GH parallel to AB, and join GC.

Diagram. A scalene triangle with apex at the top and a circular arc swung across its lower right. Labelled points:

- A (apex, top)
- D (near the apex, upper left of A)
- E (just below, between D and F)
- F (near the apex, upper right of A)
- B (lower-left vertex)
- H (on the base, immediately right of B)
- K (on the base, toward the centre — where the arc cuts CH)
- G (on the right-hand slant, below the apex)
- C (right-hand vertex)
- I (below and right, the far end of the drawn arc)

Now since EA and AF together are greater than EF, [Margin c: Elements I.20] and likewise CF and FG are greater than GC, the lines EA, AC, FG together will be greater than EF and GC together; and when the common FG is removed, EA and AC together will be greater than EG and GC together, and therefore much greater than GC alone. Let GC then be produced, and let GI be made equal to EA and AC together. And since BE is greater than EA and AC together — because BE exceeds BD, the half of BA and AC together, whereas EA and AC together fall short of the half of those same lines, namely by DA and AC together — BE will also be greater than GI, which is equal to EA and AC together. Since, then, GH is equal to BE [Margin d: Elements I.34] in the parallelogram BEGH, GH too will be greater than GI. If, therefore, a circle is described with centre G and radius GI, it will cut the line GH, and consequently will first cut CH itself at K. Join GK. I say that GH and GK together are equal to AB and AC together; for GH is equal to BE, and GI — that is, GK — is equal to EA and AC together. And this can be done in infinitely many ways, according as the point E is taken farther from D, and the point G

farther from the point E; that is, according as both AD and EF can be divided at infinitely many points.

Next, let ABC be an isosceles triangle whose base BC is greater than either side. With centre B, through A, describe the arc AD cutting the line BC at D; draw any line BF cutting the arc at E and the side AC at F; through any point G of EF draw GH parallel to BC; through any point I of GH draw IK parallel to BF; join the line IC; and cut off AL equal to EG.

Diagram. An isosceles triangle with apex at the top and a large circular arc swept across the lower part of the figure. Labelled points:

- A (apex, top; L marked just beside it on side AB)
- L (on AB, close below A)
- E, G, F, H (a cluster along EF / GH in the upper middle, left to right)
- I (on the segment GH)
- B (lower-left vertex)
- K, N (on the base, right of B)
- D (on the base, toward the centre)
- C (right-hand vertex)
- M (far right, where IC is produced to M)

Then BG will be equal to AB and AL together, while LC is less than BE (or AB), and for that reason much less than BG. And since IK is equal to BG[Margin e: Elements I.34] in the parallelogram BGIK, IK too will be equal to AB and AL together, and greater than LC. Now since GF and FH are greater than GH,[Margin f: Elements I.20] and HC and HI greater than IC, the lines GF, FC, HI together will be greater than GH and IC; and when the common HI is removed, the remaining GF and FC together will be greater than GI and IC together; and therefore, when the common BG is added, BF and FC together will become greater than BG, GI, IC together. But BA and AC together are greater than BF and FC.[Margin g: Elements I.20] Therefore BA and AC together will be much greater than BG, GI, IC together. Since, then, BG is equal to BA and AL together, the remaining LC will be greater than the remaining GI and IC together, and therefore much greater than IC alone. Let IM (with IC produced) be made equal to LC. And since IK has been shown to be greater than LC — that is, than IM — if with centre I, through M, an arc of a circle is described, it will cut the line IK, and consequently will first cut CK itself at N. Join IN. I say that IK and IN together are equal to AB and AC together; for IK is equal to AB and AL together, and IN — that is, IM — is equal to LC. And this too

can be done in infinitely many ways, since BF can be drawn in infinitely many ways, and both EF and GH can be divided in infinitely many ways.

It is clear, moreover, that if within the same triangle ABC two straight lines are drawn enclosing the lines HG, GK in the first triangle, or the lines KI, IN in the second,[Margin h: Elements I.21] their sum will be greater than sides AB and AC together — since they would be greater than the lines HG, GK in the first triangle, or the lines KI, IN in the second.

But in an equilateral triangle, or in an isosceles triangle whose base is smaller than either side, it can be shown as follows that no two lines can be constructed inside whose sum is greater than, or equal to, the two sides together — rather, whatever interior lines are drawn are smaller. For in triangle ABC, whose two sides AB, AC are equal, and whose base BC is not greater but either equal or smaller — so that angle A is not greater than angle B or C,[Margin i: Elements I.5 or 18] but either equal or smaller — let two lines DE, EF be constructed at random, which I say are together smaller than the two sides AB and AC together.

Diagram. A tall triangle with apex A and base along the bottom, with several lines (cevians) drawn from near the apex to points on the base. Labelled points:

- A (apex, top)
- H (below A, on side AC)
- G (below H)
- E (below G — the meeting-point of the interior lines)
- B (base, left end)
- D (base)
- F (base)
- C (base, right end)

For let DE be produced, cutting side AC at G, and join the line GB. Since, then, the two angles ABC and C are equal,[Margin k: Elements I.5] and ABC is greater than angle GBC, C too will be greater than angle GBC. But the angle GDB is also greater than angle C,[Margin l: Elements I.16] being exterior to the interior [angle].

Therefore angle GDB will be much greater than angle GBC;[Margin m: Elements I.19] and consequently side GB will be greater than side GD. And since angle BGA is greater than angle C,[Margin n: Elements I.16] exterior to interior, and angle C is either equal to angle A or greater, as [we shall] show—

—it will follow that angle BGA is also greater than angle A. Therefore side AB will be greater than side GB.[Margin a: Elements I.19] But the line GB has been shown

to be greater than GD, and consequently much greater than DE. Therefore side AB will be much greater still than DE. By the same reasoning, if FE is produced, cutting AC at H, and the line HB is joined, we shall show that side AB — and consequently AC too — is greater than FE. Therefore sides AB and AC together are greater than the two lines DE and EF together. But if FE, produced, were to cut side AB, we should demonstrate in the same way that side AC is greater than the line EF.

Now, if it seems marvellous to those unacquainted with geometry that, in the triangles we have described, interior lines can be drawn whose sum is greater than, or equal to, the two sides together, it will surely seem still more marvellous that within those same triangles two lines can be constructed, each of which singly is greater than, or equal to, each single side. Let there first be constructed, in triangle ABC, individual lines each greater than the individual sides — the triangle whose side AB is not less than side AC, while side BC is greater than either side AB or AC. With centre B describe the arc of a circle AD; take any point E between the arc AD and the side AC, and draw the line BE [Margin b: Elements I.21] cutting the arc at F, such that BE is greater than BF — that is, than BA — and join EC.

Diagram. A triangle with apex A and a circular arc (centre B) swept across the lower right. Labelled points:

- A (apex, top)
- F (near the apex, on line BE where it meets the arc)
- E (just below F, between the arc AD and side AC)
- B (lower-left vertex)
- H (on the base, near B)
- G (on the base, near H — where the arc from E cuts CB)
- D (on the base, toward the centre — foot of arc AD)
- C (right-hand vertex)
- K (far right, on the produced line EC / arc)
- I [?] (a small point on line EH where it meets the arc; not clearly resolvable on the woodcut)

Now since BE and EC together are less than AB and AC together, [Margin: Elements I.21] and BE is greater than AB, EC will be much less than AC. Let EC, then, be produced, and let EK be made equal to AC, and hence less than EB; and from E, through K, describe an arc of a circle, which will cut EB, and consequently will first cut CB itself at G. Finally, taking any point H between B

and G, join the line EH cutting the arc at I; and what was proposed will be done. For BE is greater than BA, and EH is greater than EI — that is, than EK, or AC.

Next, let there be constructed, within the same triangle, individual lines each equal to the individual sides. Let the same construction be made, omitting the line EH, but let the arc KG cut the line EB at H. Suppose side BA greater than side AG; then BE will be a line greater than AB, since it is greater than BF, which is equal to AB. Then, taking EI equal to AB, the point I will fall between B and H — since EB is greater than AB, and EH is less than AB.

Diagram. A triangle with apex A and a large circular arc across the lower part of the figure. Labelled points:

- A (apex, top)
- H, F, E (a cluster near the apex, along line BE)
- B (lower-left vertex)
- L (on the base, close to B, where the arc from E cuts BC)
- G (on the base, near L)
- D (on the base, toward the centre)
- C (right-hand vertex)
- K (far right, on the produced line / arc)
- I [?] (small point on EB, taken so that $EI = AB$; not clearly resolvable on the woodcut)

From E, through I, describe an arc of a circle cutting BC at L, and join the lines EL, EG; and what was proposed will be done. For EL is equal to EI — that is, to AB — and EG is equal to EK — that is, to AC.

Besides these, Pappus demonstrates that within the same triangles we have described two lines can be constructed that are greater than the two sides and that have a given ratio to those sides — which someone might well find incredible. But this cannot be demonstrated here, since it depends on ratios.

FROM PROCLUS.

Proposition (from Proclus). In every rectilinear figure having an even number of sides, if it is equilateral and equiangular, any two opposite sides will be parallel to one another.

Clavius's demonstration. "Opposite sides" are called those two which have an equal number of sides on either hand: thus in the hexagon ABCDEF the opposite sides will be AB and ED, since toward the parts A and E there are two sides, just

as toward the parts B and D. In the octagon ABCDEFGH the opposite sides will be AB and FE, since toward the parts A and F there are three sides, just as toward the parts B and E. And so in the other equilateral figures with an even number of sides: on either hand of the opposite sides there will be as many sides as there are in half the number of sides, less one. Thus in the quadrangle there will be one, in the hexagon two, in the octagon three, in the decagon four, in the twelve-sided figure five, and so on. I say, then, that any opposite sides are parallel — AB, namely, to ED in the hexagon; AB to FE in the octagon; and likewise for the rest.

Diagram. A regular hexagon. Labelled vertices:

- A (top left)
- B (top right)
- C (right)
- D (bottom right)
- E (bottom left)
- F (left)

Diagram. A regular octagon, illustrating the opposite sides AB and FE. The woodcut labels are small and faint; on it I can read the vertices A and B (across the top edge) and F, G, E toward the lower left, while the remaining vertices C, D, H are too indistinct to resolve with confidence. Per the text the vertices are lettered A B C D E F G H in order.

- A [?] (top left)
- B (top right)
- C [?]
- D [?]
- E (lower left region)
- F (lower left region)
- G (left)
- H [?]

For let the two extremities of the opposite sides be joined, toward the same parts, by a straight line — such as BD in the hexagon and BE in the octagon. And since, as we demonstrated at Proposition 32,[Margin c: Elements I.32] the six angles of a hexagon are equal to eight right angles, the three angles B, C, D of that hexagon will be equal to four right angles, because all the angles are supposed equal. But the angles BCD, CBD, CDB of triangle BCD are equal to two right angles. Therefore the remaining angles ABD, EDB will be equal to two right angles. Hence AB and

ED will be parallel.[Margin d: Elements I.28] Again, since the eight angles of an octagon are equal to twelve right angles, its four angles B, C, D, E will be equal to six right angles. But the four angles of the quadrilateral BCDE are equal to four right angles. Therefore the two remaining angles ABE, FEB will be equal to two right angles,[Margin e: Elements I.28] and so AB and FE will be parallel. In the same way it will be demonstrated, in all other figures of this kind, that the two angles standing at the straight line that joins the extremities of the opposite sides are equal to two right angles. For in the decagon that line cuts off a pentagon, whose angles are equal to six right angles; but five angles of a decagon are equal to eight right angles. Removing the six, then, two right angles remain. In the equilateral and equiangular figure of twelve sides the same line will cut off a hexagon, whose angles are equal to eight right angles; but six angles of the whole figure are equal to ten right angles. Removing the eight, then, two right angles remain, and so on.

Although every equiangular figure with an even number of sides has its opposite sides parallel, as we have shown, nevertheless only a quadrilateral figure that has its opposite sides parallel is customarily called a *parallelogram* by Euclid [Euclid] and the other geometers; and for that reason, in the Definitions, we said that a parallelogram is a quadrilateral figure, and so forth.

THEOREM 25. PROPOSITION 35. [Margin: XXXV.]

Euclid's enunciation. Parallelograms constructed on the same base and between the same parallels are equal to one another.

Clavius's demonstration. Between the two parallels AB, CD, on the base CD, let there stand two parallelograms CDEA, CDBF. (Parallelograms are said to be between the same parallels when two opposite sides are parts of the parallels, as is seen in the example proposed.) I say that these parallelograms are equal to one another — not as to their angles and sides, but as to their area, or capacity. For, first, the point F will fall between A and E.

Diagram. Two overlapping parallelograms on the common base CD, between the parallels AB and CD. Labelled points, along the upper parallel from left to right: A, F, E, B; along the lower parallel: C (left), D (right).

- A (upper line, left)
- F (upper line, between A and E)
- E (upper line)
- B (upper line, right)

- C (lower line, left)
- D (lower line, right)

Since, then, in parallelogram CDEA the line AE is equal to the opposite line CD, [Margin a: Elements I.34] and FB is equal to that same CD as its opposite in parallelogram CDBF, AE and FB will be equal to one another. [Margin b: Common Notion 1] Removing, then, the common FE, AF will remain equal to EB. [Margin c: Common Notion 3] But AC is also equal to ED, its opposite in parallelogram CDEA; [Margin d: Elements I.34] and the angle BED is equal to the angle FAC, exterior to interior. [Margin e: Elements I.29] Therefore triangle FAC will be equal to triangle BED. [Margin f: Elements I.4] Adding, then, the common trapezium CDEF, [Margin g: Common Notion 2] the whole parallelogram CDEA will become equal to the whole parallelogram CDBF. Which is what was proposed.

Let the point F fall, secondly, on the point E. I say again that the parallelograms CDEA, CDBE are equal. For, as before, the lines AE, EB will be equal, and likewise the angles BED, EAC; and hence the triangles EAC, BED [will be] equal. [Margin h: Elements I.4] Adding, then, the common triangle CDE, [Margin i: Common Notion 2] the parallelograms CDEA, CDBE will become equal.

Diagram. The second case: a figure on base CD in which the two upper vertices coincide, with crossed lines forming two triangles. Labelled points: E and B along the top, C and D along the bottom. (The top-left vertex A, required by parallelogram CDEA, is not clearly resolvable on the woodcut.)

- A [?] (top left)
- E (top, centre-left)
- B (top right)
- C (bottom left)
- D (bottom, centre)

Let the point F fall, thirdly, beyond E, so that the line CF cuts the line DE at G. Since, then, as before, the lines AEFB are equal, if the common EF is added, [Margin k: Common Notion 2] the whole AF will be equal to the whole EB, and likewise the angles BED, FAC will be equal; and hence triangle FAC [will be] equal to triangle BED. [Margin l: Elements I.4] Removing, then, the common triangle EGF, [Margin m: Common Notion 3] the trapezium AEGC will remain equal to the trapezium FGDB. And so, adding the common triangle CDG, the whole parallelogram CDEA will become equal to the whole parallelogram CDBF. Therefore parallelograms constructed on the same base and between the same parallels are equal to one another. Which was to be demonstrated.

Diagram. The third case: two parallelograms on base CD whose upper sides overlap, the line CF crossing DE at G. Labelled points along the top, left to right: E, F, B; the crossing-point G in the middle; C and D along the bottom. (The top-left vertex A is not clearly resolvable on the woodcut.)

- A [?] (top left)
- E (top left)
- F (top, centre)
- B (top right)
- G (interior, where CF meets DE)
- C (bottom left)
- D (bottom, centre)

SCHOLIUM.

We shall easily convert this proposition, in this way: *Equal parallelograms constructed on the same base and toward the same parts will be between the same parallels.*

Let there be two equal parallelograms ABCD, CDEF on the same base CD and toward the same parts. I say that the line AB, produced, lies in a straight line with EF, and that therefore the parallelograms are between the same parallels. For otherwise AB, when produced, will fall either below EF or above it. Let it fall first below, as AH does. Then the parallelogram CDGH will be equal to the parallelogram ABCD; [Margin n: Elements I.35] but the parallelogram CDEF is supposed equal to that same parallelogram ABCD. Therefore the parallelograms CDEF, CDHG would be equal — the whole and the part — which is absurd. Therefore AB will not fall below EF.

Diagram. Two parallelograms sharing the base region, with diagonals drawn, illustrating the case in which AB produced falls below EF. Labelled points: along the upper line A, B (left) and E, F (right); the interior points G, H; along the lower line D (left) and C (right).

- A (upper line, left)
- B (upper line, right of A)
- E (upper line, right)
- F (upper line, far right)
- G (interior, left of centre)
- H (interior, right of centre)

- D (lower line, left)
- C (lower line, right)

Let AB, secondly, when produced, fall above EF. Then FE, extended, will fall below AB. Therefore, as before, the parallelograms ABCD, CDHG will be equal — the whole and the part — which is absurd. The same absurdity would follow if CF, DE were produced as far as AB extended. And the same demonstration will fit all the cases that can occur — that is, whether the point F lies beyond B or not, as is evident. Therefore AB will not fall above EF; but neither below, as was demonstrated. Therefore, when produced, it lies in a straight line with EF; and consequently the parallelograms ABCD, CDEF are between the same parallels.

Diagram. A small inset figure for the second case of the scholium (AB produced above EF), with diagonals drawn. Labelled points: A, B across the top; H, G, E, F across the middle; D, C along the bottom.

- A (top left)
- B (top, right of A)
- H (left)
- G (middle)
- E (middle right)
- F (far right)
- D (bottom left)
- C (bottom, right of D)

THEOREM 26. PROPOSITION 36. [Margin: XXXVI.]

Euclid's enunciation. Parallelograms constructed on equal bases and between the same parallels are equal to one another.

Clavius's demonstration. Let there be two parallelograms ACEF, GHDB on the equal bases CE, HD, between the same parallels AB, CD. I say that they are equal. For let the extremities of the lines CE, GB be joined, toward the same parts, by the straight lines CG, EB.

Diagram. Two parallelograms on equal bases CE and HD, between the parallels AB and CD, with the connecting lines CG, EB drawn. Labelled points: along the upper line, left to right, A, F, G, B; along the lower line, left to right, C, E, H, D.

- A (upper line, left)
- F (upper line)

- G (upper line)
- B (upper line, right)
- C (lower line, left)
- E (lower line)
- H (lower line)
- D (lower line, right)

Since, then, the line CE is supposed equal to the line HD, and GB is equal to that same HD[Margin o: Elements I.34] as its opposite in parallelogram GHDB, CE and GB will be equal to one another;[Margin p: Common Notion 1] and they are also parallel, by hypothesis. Therefore CG, EB, which join them, will also be parallel[Margin q: Elements I.33] and equal, and so CEBG will be a parallelogram. Thus, since the parallelograms ACEF, GCEB are between the same parallels and on the same base CE,[Margin r: Elements I.35] the parallelogram ACEF will be equal to the parallelogram GCEB. Again, since the parallelograms GCEB, GHDB are between the same parallels and on the same base GB,[Margin s: Elements I.35] the parallelogram GHDB will also be equal to that same parallelogram GCEB. Therefore the parallelograms ACEF, GHDB will be equal to one another.[Margin t: Common Notion 1] Therefore parallelograms constructed on equal bases and between the same parallels [are equal to one another], and so forth. Which was to be shown.

(Marginal citations are keyed by the letters a, b, c... that stand at the corresponding step in the text. In them "primi" means "of Book I" — i.e. Elements I.n — and "pron." means pronunciatum, a common notion / axiom. They are rendered below verbatim inside [Margin ...].)

SCHOLIVM [to Prop. 36]

The converse of this theorem is twofold, in this manner.

(Converse, enunciation — Clavius.) Parallelograms that are equal, stand on equal bases, and are set up toward the same side lie between the same parallels. And parallelograms that are equal and lie between the same parallels, if they do not have the same base, stand on equal bases.

(Demonstration.) First, let there be two equal parallelograms ABCD and EFGH, on the equal bases BC and FG, and set up toward the same side. I say that they lie between the same parallels — that is, that AD, when produced, meets EH in one straight line. For otherwise it would fall either below EH or above it; and on that supposition it would follow that a whole and a part are equal, exactly as was said

in the converse of the preceding proposition, and as the figure readily shows. The equal bases are to be understood as given on the same straight line BG.

Diagram. Two adjoining/overlapping parallelograms on a common base-line, illustrating the converse of Prop. 36 (equal parallelograms on equal bases).

Readable labels:

- Along the base (line BG): B (far left), C, F, G (far right)
- Along the top: A, D, and E, H
- K (upper region, above the base near the construction) and I, E (interior/base construction points)

[Several of the interior and upper labels sit very small at this resolution — the construction points I and K in particular are not fully resolved.]

(Second part.) Secondly, let the same equal parallelograms lie between the same parallels AH, BG. I say that the bases BC, FG are equal. For if one of them, say BC, be claimed the greater, cut off BI equal to the straight line FG [Margin a: 3.primi], and draw IK parallel to CD [Margin b: 31.primi]. Then the parallelogram ABIK will be equal to the parallelogram EFGH [Margin c: 36.primi], and therefore to the parallelogram ABCD — a part equal to the whole, which is absurd. So BC is not greater than FG; and by the same reasoning it will not be smaller either. Therefore the bases BC, FG are equal.

The following theorem too is easily demonstrated from this.

(Second converse, enunciation — Clavius.) If two parallelograms between the same parallels have unequal bases, the one whose base is greater will be the greater. And conversely, if two parallelograms between the same parallels are unequal, the base of the greater will be the greater.

(Demonstration.) For in the latter figure let there be parallelograms BD, FH between the parallels AH, BG, and let base BC be greater than base FG. I say that parallelogram BD is greater than parallelogram FH [Margin d: 3.primi]. Cut off the straight line BI equal to FG [Margin e: 31.primi], and draw IK parallel to AB [Margin f: 36.primi]. Then the parallelograms BK, FH, on the equal bases BI, FG, will be equal. Since therefore BD is greater than BK [Margin g: 9.pron.], the same BD will be greater than FH.

Again, let the parallelograms BD, FH be unequal, and let BD be the greater. I say that base BC is greater than base FG. For if it were equal, the parallelograms would be equal [Margin h: 36.primi], which is absurd, since BD is supposed the greater. And if it were smaller, the parallelogram FH would be greater than BD,

as we have just shown — which is far more absurd, since BD is supposed greater than FH. Therefore the base BC, being neither equal to FG nor smaller, will be greater than FG, which is what was proposed.

Diagram. The large "latter figure": two tall parallelograms between the parallels AH (top) and BG (base), for the comparison of unequal bases. Readable labels:

- Top line (AH): A (far left), D, E, H (far right)
- Base line (BG): B (far left), C, F, G (far right)
- Construction points I (on the base near C) and K — small and only partly resolved at this resolution.

THEOR. 27. PROPOS. 37.

[Margin: XXXVII.]

(*Euclid — enunciation.*) Triangles set up on the same base and between the same parallels are equal to one another.

(*Demonstration — Clavius.*) Between the parallels AB, CD, and on the base CD, let two triangles ACD, BCD be set up. (A triangle is said to be set up between two parallels when its base is part of one of them and the opposite vertex touches the other.) I say that these triangles are equal. Through D draw DE parallel to the straight line AC [Margin i: 31.primi], and DF parallel to BC. Then the parallelograms ACDE, BCDF will stand on the same base CD and between the same parallels [Margin k: 35.prim.]. But the triangles ACD, BCD are the halves of these, since the diagonals AD, BD bisect the parallelograms ACDE, BCDF [Margin l: 34.primi]. Therefore the triangles ACD, BCD will be equal [Margin m: 7.pron.]. Therefore triangles on the same base, etc. Which was to be demonstrated.

Diagram. The Prop. 37 figure: two triangles on the common base CD between parallels. Labels (top line, left to right): E, A, B, F; (base): C, D. The diagonals cross, forming the two parallelograms ACDE and BCDF.

SCHOLIVM [to Prop. 37]

The converse of this proposition will be demonstrated by Euclid at prop. 39. Furthermore, from this proposition we shall easily demonstrate, along with Proclus [Proclus], the following: that triangles whose two sides are equal to the two sides of another, each to each, while the angle contained by those sides in the one is greater than the corresponding angle in the other, are sometimes equal and sometimes unequal — which is what we promised at prop. 24. For let there be two triangles ABC, DEF, with sides AB, AC equal to the sides DE, DF, and angle

A greater than angle EDF; and let these two angles first be equal to two right angles. I say the triangles are equal. Produce ED to H and FD to I [Margin n: 23.prim.]; and make angle EDG equal to angle A [Margin o: 17.prim.], toward the acute angle DFE (for one of the angles DEF, DFE will always be acute, since both together are less than two right angles) [Margin p: 3.prim.]; and let DG equal the straight line DF, that is AC; and draw the straight lines EG, GF. EG will always fall beyond the acute angle DFE, as we showed at the end of the scholium to prop. 24. Now since the angles A and EDF are supposed equal to two right angles, and angle EDG has been made equal to angle A, the angles EDG, EDF will also be equal to two right angles [Margin q: 13.prim.]; but the angles EDG, GDH are likewise equal to two right angles [Margin r: 3.pron.]. Therefore the angles EDG, EDF are equal to the angles EDG, GDH. So, taking away the common angle EDG, the angle EDF will remain equal to angle GDH; and angle EDF is equal to angle GDH. But it is also equal to angle GDH [Margin s: 15.prim.]... [the two remaining angles EDF and GDH being equal]; and so angle GDH is the half of the whole angle GDI. Again, since the sides DG, DF are equal, the angles DFG, DGF are equal; and these, being equal to the exterior angle GDI, each of them — namely DGF — is the half of angle GDI. But it has been shown that angle GDH is the half of the same GDI [Margin t: 5.prim.]. Therefore the angles GDH, DGF will be equal [Margin u: 32.prim.]; and since they are alternate between EH, FG, EH and FG will be parallel [Margin x: 7.pron.]. And [triangles DEG, DEF] will be equal, since they have the same base DE and lie between the same parallels [Margin y: 27.prim.]. And since triangle DEG is equal to triangle ABC — because the sides DE, DG are equal to the sides AB, AC, and angle A is equal to angle EDG [Margin z: 37.prim.] — triangle ABC will also be equal to triangle DEF [Margin a: 4.prim.], which is what was proposed.

Diagram. The scholium's first-case figure (angles equal to two right angles): a plain triangle ABC at the left (A apex; B, C base), and at the right the construction on triangle DEF with ED produced to H and FD produced to I, the added lines DG, EG, GF, and the intersection point K. Readable labels — left: A, B, C; right: D, E, F, G, H, I, and K (near F). At this resolution the crowded right-hand cluster (H, I, G, K in particular) is only partly resolved.

(Case 2.) Next, let the angles A and EDF be greater than two right angles. I say that triangle ABC, which has the greater angle, is less than triangle DEF. Produce ED to H and FD to I [Margin a: 23.prim.]; make angle EDG equal to angle A [Margin b: 3.prim.], and let DG equal DF, that is AC; and draw the straight lines EG, GF. Now since the angles A and EDF are supposed greater than two right angles, the angles EDG, EDF will also be greater than two right angles; but the angles EDG, GDH are

equal to two right angles [Margin c: 13.primi]. Therefore the angles EDG, EDF are greater than the angles EDG, GDH. So, taking away the common EDG, angle EDF remains greater than angle GDH. And since angle EDF is equal to angle HDI [Margin d: 15.primi], HDI too will be greater than GDH; and so GDH is less than the half of angle GDI. Again, since the sides DG, DF are equal, the angles DFG, DGF are equal [Margin e: 5.primi]; and since they are equal to the exterior angle GDI [Margin f: 32.primi], each of them — namely DGF — is the half of angle GDI. But it has been shown that angle GDH is less than the half of the same GDI. Therefore DGF will be greater than GDH. Cut off from angle DGF the angle DGK equal to the alternate angle GDH [Margin g: 23.primi]. Then GK will be parallel to DE [Margin h: 27.prim.] and will cut the straight line EF. From D to K, where GK cuts the straight line EF, draw the straight line DK. Then triangle DEG will be equal to triangle DEK [Margin i: 37.primi]. And since triangle DEG is equal to triangle ABC — because the sides DE, DG are equal to the sides AB, AC, and angle A is equal to angle EDG [Margin k: 4.prim.] — triangle ABC will also be equal to triangle DEK. Since therefore DEK is less than triangle DEF [Margin l: 9.pron.], the triangle ABC too will be less than triangle DEF. Which is what was proposed.

Diagram. The scholium's second-case figure (angles greater than two right angles): a small triangle ABC at the left (A apex; B, C base), and a right-hand construction on triangle DEF with the produced lines to H and I, the added line DG, and the point K on EF. Readable labels — left: A, B, C; right: D, E, F, H, I, K, G. The right cluster is crowded and small; H, I, K are not all fully resolved.

(Case 3.) Thirdly, let the angles A and EDF be less than two right angles. I say that triangle ABC, which has the greater angle, is greater than triangle DEF. Produce ED to H and FD to I [Margin m: 23.prim.]; make angle EDG equal to angle A [Margin n: 3.primi], and let DG equal DF, that is AC; and draw the straight lines EG, GF. Now since the angles A and EDF are supposed less than two right angles, the angles EDG, EDF will also be less than two right angles. But the angles EDG, GDH are equal to two right angles [Margin o: 13.primi]. Therefore EDG, EDF are less than EDG, GDH; and, the common EDG being taken away, EDF remains less than GDH. But EDF is equal to HDI [Margin p: 15.primi]. So HDI too will be less than GDH; and thus GDH is greater than the half of angle GDI. And since DGF is the half of the same angle GDI, as was shown above, GDH will be greater than DGF. Make the angle DGK equal to angle GDH [Margin q: 23.primi], drawing the straight line GK, which will cut the straight line EF produced at K; and draw the straight line DK. Then, as before, GK will be parallel to DE [Margin r: 27.prim.], and triangle DEG will be equal to triangle DEK [Margin s: 37.primi]. But again DEG is equal to ABC [Margin t: 4.primi]. Therefore ABC too is equal to DEK

[Margin u: 9.pron.]. And so, since DEK is greater than DEF, ABC too will be greater than DEF. Which was to be demonstrated.

Diagram. The scholium's third-case figure (angles less than two right angles): a large triangle ABC at the left (A apex; B, C base), and to the right the construction on triangle DEF with ED, FD produced to H and I, the added lines DG, EG, GF, and the intersection K on EF. Readable labels — left: A, B, C; right: D, E, F, G, H, I, K. The upper labels H, I and the point K are small and only partly resolved.

From all this it is clear why Euclid, in prop. 24, concluded only the inequality of the bases and not of the triangles, as we noted there.

THEOR. 28. PROPOS. 38.

[Margin: XXXVIII.]

(*Euclid — enunciation.*) Triangles set up on equal bases and between the same parallels are equal to one another.

(*Demonstration — Clavius.*) Between the parallels AB, CD, and on the equal bases CE, FD, let the triangles ACE, BFD be set up. I say they are equal. Draw EG parallel to AC, and DH parallel to BF [Margin x: 31.primi]. Then the parallelograms ACEG, BFDH will be equal [Margin y: 36.primi]. Since therefore the triangles ACE, BFD are the halves of these [Margin z: 34.prim.], they will be equal to one another [Margin a: 7.pron.]. Therefore triangles on equal bases, etc. Which was to be shown.

Diagram. The Prop. 38 figure. Top line (left to right): A, G, B, H. Base line (left to right): C, E, F, D. The zig-zag internal lines form the two parallelograms ACEG and BFDH and the two triangles ACE, BFD on the equal bases CE, FD.

SHOLIVM [to Prop. 38]

[Printed "SHOLIVM" — the initial C is dropped; see anomalies.]

The converse of this will be shown by Euclid at prop. 40.

Moreover, from this proposition it follows that, if from any vertex of a given triangle a straight line is drawn bisecting the opposite side, the triangle too is bisected. For in triangle ABC let the straight line AD be drawn from vertex A, bisecting side BC at D. I say that triangle ABC is also bisected. For if a parallel to BC is drawn through A, the two triangles ABD, ADC will lie between the same parallels and on equal bases [Margin b: 38.primi]; therefore they will be equal.

EX PELETARIO [From Peletier]

(Problem — enunciation.) From any given point on one side of a proposed triangle, to draw a straight line that bisects the given triangle.

(Solution.) Let ABC be the triangle, and D the given point on side BC. We must, then, draw a straight line from D that bisects the triangle. Now if the point D bisects the side BC, then the straight line DA, drawn to A, will bisect the triangle, as was shown in this scholium. But if D does not bisect BC, let BC be bisected at E [Margin c: 10.primi]. Then from D to the opposite vertex A draw the straight line DA, and through E draw EF parallel to DA [Margin d: 31.primi], cutting AC at F. If now the straight line DF is drawn, the triangle will be bisected by the line DF. For, drawing the straight line EA, the triangles EFA, EFD will be equal [Margin e: 38.primi], since they stand on the same base EF and between the same parallels EF, AD. Adding therefore the common triangle CFE, the whole triangles AEC, CDF will be equal [Margin f: 2.pron.]. But AEC is the half of the whole ABC, as was shown. Therefore CDF too is the half of the same triangle ABC, which was to be proved.

Diagram. The "Colligitur" scholium figure: a triangle with apex A and base B–D–C, D being the midpoint; a dotted horizontal line through A marks the parallel to BC. Labels: A, B, C, D.

Diagram. The Peletier figure: triangle with apex A; base line B, D, E, C (left to right); F on side AC; internal lines DA and EF (parallel to DA). Labels: A, F, B, D, E, C.

But if the point D falls in the other half EC, we shall carry out the problem in the same way; only then the triangle will be cut off toward the B side, and the trapezium toward the C side, as the present figure sufficiently shows. The demonstration is the same, provided the letter B in it is changed to C and C to B. We shall, however, propose this problem in a much more general form at the end of Book VI.

Diagram. The Peletier figure for the alternative case (point D in the other half): triangle with apex A; base line B, E, D, C (left to right); F on side AC; the internal construction lines. Labels: A, F, B, E, D, C.

THEOR. 29. PROPOS. 39.

[Margin: XXXIX.]

(*Euclid — enunciation.*) Equal triangles set up on the same base and toward the same side are also between the same parallels.

(*Demonstration — Clavius.*) Let there be two equal triangles ABC, DBC on the same base BC and toward the same side. I say they are set up between the same parallels — that is, that the straight line AD, when drawn, is parallel to BC. For if it is not, draw through A a parallel to BC [Margin a: 31.primi], which will fall either above AD or below it. Let it first fall above, as AE, meeting BD produced at E; and draw the straight line EC. Now since AE, BC are parallel, triangle EBC will be equal to triangle ABC [Margin b: 37.primi]; but by hypothesis triangle DBC too is equal to the same triangle ABC. Therefore the triangles DBC, EBC will be equal [Margin c: 1.pron.] — a part and a whole, which is absurd. And if the parallel drawn through A falls below AD, as AF, then, drawing the straight line FC, by the same reasoning the triangles BFC, BDC will be equal — a part and a whole — which is absurd. Therefore AD will be parallel to BC. Therefore equal triangles on the same base, etc. Which was to be shown.

Diagram. The Prop. 39 figure: base BC with the two equal triangles ABC (apex A) and DBC (apex D) toward the same side; AE the parallel meeting BD produced at E; AF the parallel falling below. Readable labels: A, E, B (upper area), B, C (base), D, F. Several of the small internal labels (D, F and the direction of E) are only partly resolved at this resolution.

SHOLIVM [to Prop. 39]

[Printed "SHOLIVM" — the initial C is dropped; see anomalies.]

From these Campanus [Campanus of Novara] infers the following theorem.

(*Theorem — enunciation.*) A straight line bisecting two sides of a triangle will be parallel to the remaining side.

(*Demonstration.*) Let the line DE bisect the sides AB, AC of triangle ABC at D and E. I say DE is parallel to the side BC. For since the triangles ADE, BDE stand on the equal bases AD, DB and between the same parallels — if through E a parallel to AB were drawn — triangle BDE will be equal to triangle ADE [Margin d: 38.primi]; and by the same reasoning triangle CED will be equal to the same triangle ADE. This is also established from the scholium of the preceding proposition: for the straight line ED will bisect triangle AEB, and the same line DE will likewise bisect triangle ADC, since by hypothesis the bases AB, AC are bisected by the line DE. Therefore the triangles DBE, ECD will be equal [Margin e: 1.pron.]; and they have the same base DE and are set up toward the same side.

Therefore they will lie between the same parallels [Margin f: 39.primi], and hence DE, BC will be parallel. Which is what was proposed.

Diagram. The Campanus scholium figure: triangle ABC (apex A; B, C base) with D, E the midpoints of AB, AC; the line DE and the crossing internal lines to B and C. Labels: A, D, E, B, C.

That which we promised at the end of the second theorem in the scholium to prop. 34 we shall now easily demonstrate from this proposition. Namely:

(Theorem — enunciation.) Every quadrilateral that is bisected by both of its diagonals is a parallelogram.

(Demonstration.) For let the quadrilateral ABCD be bisected by both diagonals AC, BD. I say it is a parallelogram. Since the triangles ADC, BDC are halves of the same quadrilateral ABCD [Margin g: 7.pron.], they will be equal to one another. And so, since they have the same base DC and are toward the same side, they will lie between the same parallels [Margin h: 39.primi]; and therefore the straight lines AB, DC are parallel. In no other way we shall show that AD, BC are parallel. Therefore ABCD is a parallelogram. Which is what was proposed.

Diagram. The "OMNE quadrilaterum" figure: a quadrilateral (parallelogram) with the two diagonals AC, BD drawn. Labels: A (top-left), B (top-right), C (bottom-right), D (bottom-left).

THEOR. 30. PROPOS. 40.

[Margin: XXXX.]

(Euclid — enunciation.) Equal triangles set up on equal bases and toward the same side are also between the same parallels.

(Demonstration — Clavius.) Let there be two equal triangles ABC, DEF on the equal bases BC, EF (placed on the same straight line) and set up toward the same side. I say they lie between the same parallels — that is, that the straight line drawn from A to D is parallel to the straight line BF. For if it is not, the parallel to BF drawn through A will fall either above AD or below it. Let it first fall above, meeting ED produced at G; and draw the straight line GF. Now since AG, BF are parallel, triangle EFG will be equal to triangle ABC [Margin i: 38.primi]; but triangle DEF too is supposed equal to the same triangle ABC. Therefore the triangles DEF, GEF will be equal [Margin k: 1.pron.] — a part and a whole — which is absurd. And if the parallel drawn through A falls below AD, as AH, then, drawing the straight line HF, by the same argument the triangles HEF, DEF will

be equal — a part and a whole — which is absurd. Therefore AD is parallel to BF. Therefore equal triangles on equal bases, etc. Which was to be demonstrated.

Diagram. The Prop. 40 figure: base line B, C, E, F (the collinear equal bases BC, EF); triangle ABC (apex A) and triangle DEF (apex D); AG the parallel meeting ED produced at G; AH the parallel below (H interior near D). Readable labels — top: A, D, G; base: B, C, E, F; and H. The top cluster D, G and the interior H are small; not all fully resolved.

SCHOLIVM [to Prop. 40]

[The initial S–C–H–O is set with a decorative dotted ring for the O.]

This proposition is the converse of prop. 38 in one way. But it can be converted in another way, just as we converted prop. 36 in its scholium — namely:

Clavius, *Opera Mathematica*, Tomus I — Commentary on Euclid, Book I (folios 67–69)

(This batch continues Clavius's scholium on Euclid I.37–40, extending the results on parallelograms and triangles to trapezia, then a theorem on bisecting the two parallels in a triangle, and finally the problem of dividing a line into any number of equal parts. Throughout, the theorem-enunciations set in capitals are Clavius's own within the scholium; the italic passages are his demonstrations. Marginal notes are Clavius's citations: "primi" (also "pri."/"prim.") = a proposition of Euclid's Elements Book I; "pron." = one of the pronuntiata, i.e. the common notions/axioms. The reference numerals are given as printed.)

Theorem. Equal triangles between the same parallels, if they do not stand on the same base, will be constructed on equal bases.

Demonstration. Let ABC and DEF be equal triangles between the parallels AD and BF, standing on the bases BC and EF, which I say are equal. For if they are not equal, let BC be the greater. Cut off, then, the straight line CG equal to EF [Margin: a — Elements I.3], and draw the straight line GA; then triangle AGC will be equal to triangle DEF [Margin: b — Elements I.38]. But triangle ABC is also assumed equal to that same triangle DEF; therefore triangles AGC and ABC will be equal [Margin: c — Common Notion 1] — a part and the whole, which is absurd. Hence the bases BC and EF are not unequal, but equal. Which is what was proposed.

The following theorem, too, will be proved with little trouble.

Theorem. If two triangles between the same parallels have unequal bases, the one whose base is greater will be the greater. And conversely, if two triangles between the same parallels are unequal, the base of the greater will be greater.

Diagram. Two triangles standing between two parallel lines. The upper parallel runs through A (left apex) and D (right); the lower parallel carries, from left to right, B, G, C, E, F. Triangle ABC and triangle AGC share the apex A; triangle DEF has apex D. The straight line CG is cut off equal to EF, and GA is drawn.

- A (upper parallel, left): apex of triangles ABC and AGC
- D (upper parallel, right): apex of triangle DEF
- B (lower parallel, far left): base endpoint
- G (lower parallel): endpoint of the cut-off segment CG
- C (lower parallel): shared base endpoint
- E (lower parallel): base endpoint of triangle DEF
- F (lower parallel, right): base endpoint of triangle DEF

Demonstration. For let there be, in the adjoining figure, between the parallels AD and BF, the triangles ABC and DEF, with base BC greater than base EF. I say that triangle ABC is greater than triangle DEF. For when the straight line CG has been cut off equal to EF [Margin: d — Elements I.3] and the straight line AG drawn, the triangles AGC and DEF, standing on the equal bases GC and EF, will be equal [Margin: e — Elements I.38]. Since, then, triangle ABC is greater than triangle AGC [Margin: f — Common Notion 9], that same triangle ABC will be greater than DEF.

Again, let the triangles ABC and DEF be unequal, with ABC the greater. I say that base BC is greater than base EF. For if it be called equal, triangle ABC will be equal to triangle DEF [Margin: g — Elements I.38], which is absurd, since ABC is assumed the greater. But if it be thought smaller, then triangle DEF will be greater than triangle ABC, as we have just shown — which is far more absurd, since ABC is assumed the greater. Therefore the straight line BC will be greater than EF, since it has been shown neither equal nor smaller. Which is what was proposed.

Now the results demonstrated so far concerning parallelograms and triangles set between the same parallels can also easily be demonstrated of trapezia described between the same parallels, in nearly the same order, as follows.

I.

Theorem. Trapezia in the same parallels and on the same base, whose opposite bases are equal to one another, are equal to one another; and trapezia that are

equal, in the same parallels and on the same base, have their opposite bases equal.

(Definition.) Trapezia are said to be in the same parallels when their two opposite sides are parallel and are parts of the same parallels.

Diagram. Two trapezia sharing the base CD, set between the parallels AB (upper) and CD (lower). Along the upper parallel, from left to right: A, E, F, B. Along the lower parallel: C (left), D (right). The diagonals EC and FD are drawn, crossing between the parallels, so that the interior resembles a "W" within the outer trapezium. Trapezium ACDE and trapezium FCDB are formed.

- A (upper parallel, far left): vertex of trapezium ACDE
- E (upper parallel): opposite-base endpoint (AE)
- F (upper parallel): opposite-base endpoint (FB)
- B (upper parallel, far right): vertex of trapezium FCDB
- C (lower parallel, left): shared base endpoint
- D (lower parallel, right): shared base endpoint

Demonstration. Between the parallels AB and CD, then, on the same base CD, let two trapezia ACDE and FCDB be set up, whose opposite bases AE and FB are equal. I say that they are equal to one another. For when the straight lines EC and FD have been drawn, the triangles ECD and FCD, on the same base CD and between the same parallels, will be equal to one another [Margin: h — Elements I.37], as will the triangles ACE and FDB, on the equal bases AE and FB and between the same parallels [Margin: i — Elements I.38]. If, then, to the equal triangles ECD and FCD there be added the equal triangles ACE and FDB, the whole trapezia ACDE and FCDB will be equal to one another [Margin: k — Common Notion 2].

Now let the trapezia ACDE and FCDB be equal. I say that the opposite bases AE and FB are also equal. For again the triangles ECD and FCD will be equal [Margin: l — Elements I.37]. If, then, from the equal trapezia these be taken away, the remaining triangles ACE and FDB will be equal [Margin: m — Common Notion 3]. Hence, as we show in this scholium, the bases AE and FB will be equal.

II.

Theorem. Trapezia in the same parallels and on the same base, whose opposite bases are unequal, are themselves unequal, and the greater is the one whose base is greater; and unequal trapezia in the same parallels and on the same base have

their opposite bases unequal, and the greater base belongs to the greater trapezium.

Demonstration. As in the same figure: if base AE be said to be greater than base FB, I say that trapezium ACDE is greater than trapezium FCDB. For again the triangles ECD and FCD are equal [Margin: n — Elements I.37]; but triangle ACE is greater than triangle FDB, as we demonstrated in this scholium [Margin: o — Common Notion 4], because base AE is assumed greater than base FB. Therefore the whole trapezium ACDE will be greater than the whole trapezium FCDB.

Again, if trapezium ACDE be said to be greater than trapezium FCDB, I say that base AE is greater than base FB. For again the triangles ECD and FCD are equal [Margin: p — Elements I.37]. Hence the remaining triangle ACE is greater than the remaining triangle FDB [Margin: q — Common Notion 5]; and therefore, as was shown in this scholium, base AE will be greater than base FB. Which was to be shown.

III.

Theorem. Trapezia in the same parallels and on equal bases, whose opposite bases are equal, are equal; and equal trapezia in the same parallels and on equal bases have their opposite bases equal.

Diagram. Two trapezia between the parallels AB (upper) and CD (lower), standing on the equal bases CE and GD. Along the upper parallel, from left to right: A, F, H, B. Along the lower parallel, from left to right: C, E, G, D. The straight lines FC and HD are drawn, so that two triangles (FCE and HGD) point upward. Trapezium ACEF and trapezium HGDB are formed.

- A (upper parallel, far left): vertex of trapezium ACEF
- F (upper parallel): apex/opposite-base endpoint (AF)
- H (upper parallel): apex/opposite-base endpoint (HB)
- B (upper parallel, far right): vertex of trapezium HGDB
- C (lower parallel, far left): base endpoint
- E (lower parallel): division point (base CE)
- G (lower parallel): division point (base GD)
- D (lower parallel, far right): base endpoint

Demonstration. Between the parallels AB and CD, on the equal bases CE and GD, let two trapezia ACEF and HGDB be set up, whose opposite bases AF and HB are likewise equal. I say that the trapezia are equal. For when the straight lines FC and HD have been drawn, the triangles FCE and HGD, on the equal bases CE and

GD, will be equal [Margin: r — Elements I.38], as will the triangles ACF and BDH, on the equal bases AF and BH and between the same parallels, equal to one another. Therefore the whole trapezia will be equal [Margin: s — Common Notion 2].

Now let the trapezia ACEF and HGDB be equal, on the equal bases CE and GD. I say that the opposite bases AF and BH are also equal. For again the triangles FCE and HGD, on the equal bases CF [recte: CE] and GD, are equal [Margin: t — Elements I.38]; and when these are taken away from the equal trapezia, the remaining triangles ACF and BDH will be equal [Margin: u — Common Notion 3]; and therefore, as we showed a little earlier in this scholium, the bases AF and BH will be equal.

IV.

Theorem. Trapezia in the same parallels and on equal bases, whose opposite bases are unequal, are unequal, and the greater is the one whose base is greater; and unequal trapezia in the same parallels and on equal bases have their opposite bases unequal, and the greater base belongs to the greater trapezium.

Demonstration. As in the same figure: if base AF be said to be greater than base HB, then trapezium ACEF will be greater than trapezium HGDB. For again the triangles FCE and HGD, on the equal bases CE and GD, are equal [Margin: a — Elements I.38]; but triangle ACF is greater than triangle BDH, as shown above [Margin: b — Common Notion 4], because base AF is assumed greater than base BH. Therefore the whole trapezium ACEF will be greater than the whole trapezium HGDB.

Again, if trapezium ACEF be assumed greater than trapezium HGDB, then base AF will be greater than base BH. For again the triangles FCE and HGD, on the equal bases CE and GD, are equal [Margin: c — Elements I.38]; and when these are taken away from the unequal trapezia, the remaining triangle ACF will be greater than the remaining triangle BDH [Margin: d — Common Notion 5]; and therefore, as was shown in this scholium, base AF will be greater than base BH. Which was to be demonstrated.

Moreover, the following theorem does not seem one to be omitted here, since by it we shall with little trouble divide any straight line into any number of equal parts — a task we undertook to carry out at this place in the scholium to Proposition 10 of this book. For although the same could be demonstrated, and indeed more easily, by the proportions of lines, as we shall show in Book 6, it is nonetheless pleasant to understand that the same can be accomplished, with no

labor, by the propositions demonstrated so far, without the aid of proportions. The theorem, then, is as follows.

Theorem. If in a triangle a straight line be drawn parallel to one of the sides, then a straight line drawn from the opposite angle and bisecting either of the parallels also bisects the other.

Diagram. The first case: triangle ABC with apex A at the top. The base BC carries B (left), F (middle), C (right); the parallel DE, drawn across higher up, carries D (left), G (middle), E (right). The cevian AF runs from A through G on DE to F on BC, meeting both at right angles.

- A (apex): the opposite angle
- D (left end of parallel DE): on side AB
- G (middle of DE): where AF meets DE
- E (right end of parallel DE): on side AC
- B (base, left): base endpoint
- F (middle of base): where AF meets BC
- C (base, right): base endpoint

Diagram. The second case (angles at F not right): triangle ABC, apex A. Base B (left), F (middle), C (right); parallel DE with D (left), G (middle), E (right). To the right, the constructed point H, with the straight lines AH, HF, FD, FE drawn, and angle AGH constructed equal to angle AGD.

- A (apex): the opposite angle
- D, G, E (on parallel DE, left to middle to right)
- H (right, beyond E): constructed point, where angle AGH = angle AGD and GH = GD
- B, F, C (base, left to middle to right)

Demonstration. In triangle ABC let DE be parallel to BC, and let the straight line AF cut either BC or DE into two equal parts, at F or at G. I say that it bisects the other as well.

First, let the angles at F be right. [Margin: e — Elements I.29] Granting this, the angles at G will also be right. If, then, BC is bisected, angle A will also be bisected, by what we demonstrated in the scholium to Proposition 26 of this book; and therefore the straight line AG, being perpendicular to the base DE of triangle ADE and bisecting angle A, will also bisect the base DE, as we showed in the same place — which is what was proposed. But if DE be assumed bisected, then by the

same scholium to Proposition 26 of this book angle A is also bisected; and therefore the straight line AF, being perpendicular to the base BC and bisecting angle A, will also cut the base BC into two equal parts, which is what was proposed.

Next, let the angles at F not be right, but AFC obtuse and AFB acute. [Margin: f — Elements I.29] Granting this, AGE will also be obtuse and AGD acute, since these are equal to those, as external to internal. If, then, BC is bisected, let there be constructed on the acute angle AGD an angle AGH equal to it [Margin: g — Elements I.23], with the straight line GH equal to GD [Margin: h — Elements I.3], and let the straight lines AH, HF, FD, FE be drawn. Now since both the whole triangles ABF and ACF are equal, by the scholium to Proposition 38 of this book [Margin: i — Elements I.38], and likewise the triangles cut off, DBF and ECF, on account of the equal bases BF and CF, the triangles ADF and AEF will also be equal [Margin: k — Common Notion 3]. Again, since the two sides AG, GD of triangle AGD are equal to the two sides AG, GH, and they contain the equal angles AGD, AGH by construction, the triangles AGD and AGH will also be equal [Margin: l — Elements I.4]. By like reasoning, since the two sides DG, GF are equal to the two sides HG, GF by construction, and they contain the equal angles DGF, HGF — for since by construction the angles AGD, AGH are equal, the remainders of the two right angles, DGF and HGF, will be equal [Margin: m — Elements I.13]; for both AGD together with DGF, and AGH together with HGF, are equal to two right angles — the triangles DGF and HGF will also be equal [Margin: n — Elements I.4]; and therefore the whole triangles ADF and AHF will be equal [Margin: o — Common Notion 2]. But triangle AEF has been shown equal to that same triangle ADF. Therefore the triangles AHF and AEF will also be equal to one another [Margin: p — Common Notion 1]; and on that account they will lie between the same two parallels — that is, when the straight line EH is drawn, it will be parallel to the common base AF [Margin: q — Elements I.39]. Therefore the triangles HGF and EGF, between the same parallels and on the same base FG, will also be equal to one another [Margin: r — Elements I.37]. But triangle HGF has been shown equal to triangle DGF. Therefore the triangles DGF and EGF will also be equal to one another; and on that account the base DG too will be equal to the base GE, as was demonstrated in this scholium — which is what was proposed.

Diagram. The case in which DE is assumed bisected: triangle ABC, apex A. Parallel DE with D (left), G (middle), E (right), and the constructed point H at the far right; base B (left), F (middle), C (right). The straight lines AH, HG, GB, GC are drawn.

- A (apex)
- D, G, E (on parallel DE, left to right)
- H (far right): constructed point
- B, F, C (base, left to right)

Demonstration. But if DE be assumed bisected, let there be constructed an equal angle [Margin: s — Elements I.23], and let the straight lines AH, HG, GB, GC be drawn [Margin: t — Elements I.3]. Now since both the triangles [AGB, AGH?] and the triangles DGB, EGC, on account of the equal bases DG, EG [Margin: u — Elements I.38], [are respectively equal], the whole triangles AGB and AGC will also be equal [Margin: x — Common Notion 2]. Again, since the two sides AF, FB are equal to the two sides AF, FH, and they contain the equal angles AFB, AFH by construction, the triangles AFB and AFH will also be equal [Margin: y — Elements I.4]. By like reasoning, since the two sides GF, FB are equal to the two sides GF, FH, and they contain the equal angles GFB, GFH by construction, the triangles GFB and GFH will also be equal [Margin: z — Elements I.4]; and when these are taken away from the equal triangles AFB, AFH, the remaining triangles AGB and AGH will be equal [Margin: a — Common Notion 3]. But triangle AGC has been shown equal to that same triangle AGB. Therefore the triangles AGC and AGH will also be equal to one another [Margin: b — Common Notion 1]. Hence they will also lie between the same parallels — that is, when the straight line CH is drawn, it will be parallel to AF [Margin: c — Elements I.39]; and on that account the triangles GHF and GFC will be equal to one another [Margin: d — Elements I.37]. But triangle GHF has been shown equal to triangle GBF. Therefore the triangles GFC and GFB will also be equal to one another [Margin: e — Common Notion 1]; and so, as we demonstrated in this scholium, the bases FC and FB will also be equal. Which is what was proposed.

Diagram. The "Aliter" (alternative) case: triangle ABC, apex A. Parallel DE with D (left), G (middle), E (right); the base carries B (left), H, F, C (right), so that the reductio-point H lies on BC between B and F. The straight lines FD, FE are drawn.

- A (apex)
- D, G, E (on parallel DE, left to right)
- B (base, far left)
- H (base): the assumed bisection point of BC in the reductio
- F (base, middle)
- C (base, far right)

Another way. First let BC be bisected at F. I say that DE too is bisected at G. For if DG and GE are not equal, let DG be the greater, and draw the straight lines FD, FE. Then, by what we demonstrated in this scholium, both triangle ADG will be greater than triangle AEG, and triangle FDG greater than triangle FEG. Therefore the whole triangle ADF will be greater than the whole triangle AEF [Margin: f — Common Notion 4]; and if to these be added the triangles DBF, ECF, ECF — which are equal because of the equal bases BF, CF [Margin: a — Elements I.38] — the whole triangle ABF will be made greater than the whole triangle ACF [Margin: b — Common Notion 4]; and so it will be the greater: but it is also assumed equal. Which is absurd. Therefore DE is bisected at G. Which is what was proposed.

Now let DE be bisected at G. I say that BC too is bisected at F. If not, let BC be bisected at H [Margin: c — Elements I.10], and draw the straight line AH, cutting DE at I. Now since AH cuts BC into two equal parts at H, it will also cut DE itself into two equal parts at I, as we have just shown. Which is absurd, since DE is assumed bisected at G; for it would follow that a part is greater than the whole. For if DI is equal to IE, then since IE is greater than GE, DI will also be greater than GE — that is, greater than DG, which is assumed equal to GE. Therefore BC is bisected at F. Which was to be demonstrated.

This theorem being shown, let us now come to the division of a straight line into any number of equal parts.

Problem. To cut a given finite straight line into any number of equal parts you please.

Diagram. [The page is heavily foxed; the labels below were confirmed at high magnification.] A large triangle with apex C at the top, from which two sides descend, extended slightly above C by a ruled line. Three horizontal transversals cross the figure: an upper one carrying, left to right, A, N, M, L, K, B; a middle one carrying E, I, H, G, F, D; and the base carrying A, N, M, L, K, B. Straight lines run from C through the division points F, G, H, I of the parallel DE down to the points K, L, M, N on AB.

- C (apex): the meeting-point of AC and BD
- A (left end of AB): extremity of the given line
- B (right end of AB): extremity of the given line
- K, L, M, N (on AB, between B and A): the four division points giving parts BK, KL, LM, MN, NA
- D (right end of the parallel DE): the point taken on BC
- E (left end of the parallel DE): where AC meets DE

- F, G, H, I (on DE, between D and E): the equal-part marks DF, FG, GH, HI, IE

Demonstration. Let AB be the given straight line, to be cut into five equal parts. Through the extremity B draw any straight line BC, and, taking any point D on BC — whether below B or above it — draw through D a line DE parallel to AB [Margin: d — Elements I.31], and on it cut off [Margin: e — Elements I.3] five parts equal to one another, DF, FG, GH, HI, IE, yet under this rule: that, when the point D is below B, the straight line DE, composed of the five equal parts, be greater than the given line AB, but smaller when the point D is above B — so that, of course, the straight line AC, drawn through the other extremity A and through E, can meet BD at some point, as at C. And if from C straight lines be drawn through the points F, G, H, I, the given straight line AB will be cut into the five equal parts BK, KL, LM, MN, NA. For since in triangle CBL the straight line DG is parallel to BL — or, in triangle CDG, the straight line BL is parallel to DG — and DG is bisected at F, then BL too will be bisected at K, as we demonstrated in the preceding theorem. And by the same reasoning KM will be bisected at L, just as FH is bisected at G. Therefore the three parts BK, KL, LM are equal to one another, just as the three DF, FG, GH; and so on for the rest.

Another way. At the extremity A of the line AB, which is to be cut into five equal parts, construct any rectilinear angle A, and from the straight line AC cut off [Margin: f — Elements I.3] five parts of any size, equal to one another, AD, DE, EF, FG, GC; and when the straight line CB has been drawn [Margin: g — Elements I.31], draw parallels to it, GL, FK, EI, DH. I say that the straight line AB is cut into five equal parts.

Diagram. [Foxed; labels confirmed at high magnification.] A right-triangle "staircase." The base runs, left to right, A, H, I, K, L, B; the slanting side rises from A through D, E, F, G to the apex C at the top right. From the division points D, E, F, G of the slanting side, lines DH, EI, FK, GL are drawn parallel to the closing side CB, meeting the base at H, I, K, L. The interior points M and N (on the parallels GM, FN drawn to AB) are marked near G and F.

- A (bottom left): the vertex where the angle is constructed
- C (top right): apex
- D, E, F, G (on the slant AC, ascending): equal divisions AD, DE, EF, FG, GC
- H, I, K, L (on the base AB, ascending): division points AH, HI, IK, KL, LB
- B (bottom right): extremity of the given line
- M, N (interior): points on the parallels GM, FN drawn to AB

For when through G and F there have been drawn parallels GM, FN to AB [Margin: h — Elements I.31] — which will also be parallel to one another [Margin: i — Elements I.30] and to BL, parallelograms GB, FL being formed [Margin: k — Elements I.34] — the angles FGN and GCM, external and internal in the parallels GL, CB, will be equal to one another, as will the angles CGM and GFN, external and internal in the parallels GM, FN [Margin: l — Elements I.29]. Now since the two angles C, G of triangle CGM are equal to the two angles G, F of triangle GFN, each to each, and the sides adjacent to them, CG and GF, are equal by construction, the sides GM and FN will also be equal [Margin: m — Elements I.26]; and since these have been shown equal to the straight lines BL, LK, then BL and LK too will be equal to one another. And by the same reasoning we shall show that KI, IH, and HA are equal; and therefore the straight line AB will be divided into five equal parts. Which is what was proposed.

Another way. At the two extremities A, B of the line AB, which is to be divided into five equal parts, construct two equal angles on opposite sides, ABC and BAD. And on each of the lines BC, AD — which, on account of the alternate equal angles A, B, are parallel to one another [Margin: n — Elements I.27] — take four parts wholly equal to one another (namely one fewer than the number of parts into which the line is to be cut), of the sort BE, EF, FG, GC and AH, HI, IK, KD [Margin: o — Elements I.3], and let the straight lines CH, GI, FK, ED be joined, which will be parallel to one another, since they join the extremities of equal parallels [Margin: p — Elements I.31]. I say that the straight line AB is cut into five equal parts.

Diagram. [Foxed; labels confirmed at high magnification.] A rhomboid/parallelogram-like figure formed by the two equal, oppositely-directed lines BC and AD crossing the given line AB. The upper region carries, left to right, C, G, Q, F, P, E, then B at the right; the central line AB carries A (left), then M, N, O; the lower region carries H, I, K, D. Transversals CH, GI, FK, ED are drawn, together with the parallels EP, FQ to AB.

- A (left of central line): extremity of the given line
- B (right): extremity of the given line
- C (upper left): apex of triangle ABC
- D (lower right): apex of triangle BAD
- E, F, G (on BC, from B toward C): equal divisions BE, EF, FG, GC
- H, I, K (on AD, from A toward D): equal divisions AH, HI, IK, KD
- M, N, O (on AB): division points giving AL/LM/MN/NO/OB [with L]
- P, Q (interior): points on the parallels EP, FQ drawn to AB

For when through E and F there have been drawn parallels EP, FQ to AB [Margin: q — Elements I.31] — which will also be parallel to one another [Margin: r — Elements I.30], and equal to NO, MN, parallelograms EN, FM being formed [Margin: s — Elements I.34] — the angles EFP and FGQ, external and internal in the parallels FK, GI, will be equal to the angles GFQ and FEP, external and internal in the parallels QF, PE [Margin: t — Elements I.39]. And so, since the two angles G, F of triangle FGQ are equal to the two angles F, E of triangle EFP, each to each, and the sides adjacent to them, FG and EF, are equal by construction, the sides FQ and EP will also be equal to one another [Margin: u — Elements I.26]; and since these have been shown equal to the straight lines MN, NO, then MN and NO too will be equal to one another. And by the same reasoning OB, NO, MN, LM, AL will all be equal to one another; and therefore the straight line AB will be divided into five equal parts. Which is what was proposed.

This practical construction will be demonstrated more briefly thus. Since the straight lines ED, FK, GI, GH [recte: CH] are parallel — because they join the extremities of equal and parallel lines [Margin: x — Elements I.33] — the straight line BL will be cut into four equal parts, and likewise the straight line AO, as we showed in the preceding practical construction. Therefore all five parts AL, LM, MN, NO, OB will be equal.

Another way. Let an instrument be prepared, suited to the division of lines into equal parts, in this manner. When two parallels CD, EF have been drawn a fairly large distance apart, let there be taken on each any number of parts wholly equal to one another [Margin: y — Elements I.3] — as many on the one as on the other — and let them be joined, at a modest interval, by straight lines, which will be parallel to one another, since they join the extremities of equal parallels [Margin: z — Elements I.33]. If, then, by means of a compass the straight line AB, which is to be divided into five equal parts, be transferred from some point G as far as a point H, so that five intervals of the parallels are enclosed between G and H, the line GH thus drawn will be divided by those parallels into five equal parts [Margin: a — Elements I.3]; and if from these parts equal ones be taken on the given line AB, then AB too will be divided into five equal parts [Margin: b — Elements I.30]. That GH is cut into five equal parts will be demonstrated thus. When through C and N there have been drawn parallels CI, NM to GH [Margin: c — Elements I.34] — which will also be parallel to one another and equal to GK, KL, parallelograms GI, KM being formed [Margin: d — Elements I.29] — the angles CNI, NOM [recte: NKO?], external and internal in the parallels NK, OL, [and the angles] ONM, NCI, external and internal in the parallels NM, CI, will be equal to one another. Therefore, since the two angles N, C of triangle CNI are equal to

the two angles O, N of triangle NOM, each to each, and the sides adjacent to them, CN, NO, are equal by construction [Margin: e — Elements I.26], the sides CI, NM will also be equal to one another; and since these have been shown equal to the straight lines GK, KL, then GK and KL too will be equal to one another. And by the same reasoning all the parts of the straight line GH will be shown equal; and therefore the straight line GH will be divided into five equal parts.

Diagram. [Foxed; labels confirmed at high magnification.] The dividing instrument: a four-sided frame between the two parallels CD (upper, running C...E) and DF (lower, running D...F), ruled with several equally-spaced horizontal lines between them, and crossed by oblique transversals. Upper edge carries C (top left), G, Q, then E (top right); the interior carries N, K on the first interior line, I lower, O and M on the left, L in the middle, H and P lower down; the base carries D (bottom left) and F (bottom right).

- C (top left): corner
- E (top right): corner
- D (bottom left): corner
- F (bottom right): corner
- G, Q (on the upper edge): the transferred segment's endpoints / division marks
- N, K, I, L, M, O, H, P (interior): intersection points of the transferred line GH with the ruled parallels

Diagram. A short horizontal straight line AB marked with four ticks dividing it into five equal segments — the line AB to be divided, shown for transfer by the compass onto the instrument.

- A (left end): extremity
- B (right end): extremity

This practical construction can also be shown more briefly, as follows. Mark off five intervals of the line EF, from E all the way to P, and with the compass transfer the line AB from P to some point of the line CE, say to Q. In this way the line PQ, once drawn, will be divided by the parallels into five equal parts, as was demonstrated in the second construction. Therefore, if the parts of the line PQ—which by construction equals the given line AB—are transferred onto the given line AB, then AB too will be divided into five equal parts. Which is what was proposed.

THEOR. 31. PROPOS. 41.

[Margin: xlj (41)]

Enunciation. If a parallelogram has the same base as a triangle and lies in the same parallels, the parallelogram will be double the triangle.

Demonstration. Between the parallels AB and CD, and on the base CD, let a parallelogram ACDE and a triangle BCD be set up. I say that the parallelogram is double the triangle BCD. For when the diagonal AD is drawn in the parallelogram, the triangles ACD and BCD will be equal [Margin: (a) Elements I.37]; but the parallelogram ACDE is double the triangle ACD, since the triangles ACD and ADE are also equal to one another [Margin: (b) Elements I.34]. Therefore the same parallelogram ACDE will also be double the triangle BCD [Margin: (c) common notion 6 [pronunciatum]]. And so, if a parallelogram has the same base as a triangle, and so on. Which was to be demonstrated.

Diagram. A parallelogram ACDE with the triangle BCD standing on the same base CD in the same parallels AB, CD. The diagonal AD divides the parallelogram; lines from A and from the apex complete the triangle BCD.

- A (top left): upper-left vertex of the parallelogram, on the upper parallel.
- E (top, above D): upper vertex of the parallelogram, on the upper parallel.
- B (top right): apex of the triangle, on the upper parallel.
- C (bottom left): left end of the base.
- D (bottom, middle): right end of the base CD.

SCHOLIUM

Hence it follows that if a triangle has double the base and lies in the same parallels as a parallelogram, the triangle will be equal to the parallelogram. For if the base CD is produced to F so that DF equals CD, and the line FB is drawn, the triangle BCF will be double the triangle BCD [Margin: (d) Elements I.38], since the triangles BCD and BDF are equal. But the parallelogram ACDE is also double the same triangle BCD [Margin: (e) Elements I.41]. Therefore the triangle BCF and the parallelogram ACDE will be equal [Margin: (f) common notion 6].

Diagram. A parallelogram ACDE and a triangle BCF, where the base CF is double the base CD of the parallelogram (D being the midpoint of CF).

- A (top left): upper-left vertex of the parallelogram.
- E (top, above D): upper vertex of the parallelogram.
- B (top right): apex of the triangle BCF.

- C (bottom left): left end of the doubled base.
- D (bottom, middle): midpoint, with $DF = CD$.
- F (bottom right): right end of the base CF.

This same theorem of Euclid's can be demonstrated in the same way if the parallelogram and the triangle have equal bases—though not the same base—and lie in the same parallels, as you see in the parallelogram ACDE and the triangle BFG, whose bases CD and FG are equal. For when the diagonal AD is drawn in the parallelogram, the triangles ACD and BFG will be equal [Margin: (g) Elements I.38]. Since, then, the parallelogram ACDE is double the triangle ACD—because the diagonal AD bisects the parallelogram ACDE [Margin: (h) Elements I.34]—it will also be double the triangle BFG [Margin: (i) common notion 6]. By the same reasoning, if the base FG were doubled and a line drawn to B, the triangle would come out equal to the parallelogram, since this triangle would then be double the triangle BFG as well, and so on.

Diagram. A parallelogram ACDE and a triangle BFG standing in the same parallels on equal but distinct bases CD and FG.

- A (top left): upper-left vertex of the parallelogram.
- E (top, above D): upper vertex of the parallelogram.
- B (top right): apex of the triangle BFG.
- C (bottom left): left end of base CD.
- D (bottom): right end of base CD.
- F (bottom): left end of base FG.
- G (bottom right): right end of base FG.

The converse

The converse of this theorem is twofold, as follows.

Enunciation. If a parallelogram is double a triangle, and they have the same base, or equal bases, and are set up toward the same side, they will lie in the same parallels. And if a parallelogram is double a triangle and lies in the same parallels, the bases will be equal—unless it is the same base.

Demonstration. Let the parallelogram ABCD be double the triangle EBC, whether they have the same base or equal bases. I say that the parallel AD, when produced, falls on the point E. For otherwise it would fall either above E or below it. Hence, as will be shown after the manner of proposition 39 or 40, a part would equal the whole, as the figure also indicates. For the parallelogram ABCD would then also be double the triangle BFC or BGC [Margin: (k) Elements I.41]. Therefore

the triangles EBC and FBC, or the triangles EBC and GBC, would be equal—a part and the whole [Margin: (l) common notion 6]. Which is absurd.

Diagram. A crowded figure for the first converse: a parallelogram ABCD with a triangle EBC on the base, and the alternative apex points F (above E) and G (below E) illustrating the "falls above or below" argument; a second, similar configuration appears to the right. The apex labels are printed close together and overlap.

- A (top left): upper-left vertex of the parallelogram.
- D (top): upper vertex of the parallelogram.
- E (top, on the produced parallel): apex of the triangle EBC.
- F (raised above E): alternative apex position (falls above E).
- G (below E): alternative apex position (falls below E).
- B (bottom left): left end of the base.
- C (bottom): right end of the base.

Next, let the parallelogram ABCD be double the triangle EFG, in the same parallels. I say that the bases BC and FG are equal. For if one of them—say BC—were greater, then, cutting off CH equal to the other and drawing HI parallel to AB [Margin: (m) Elements I.31], we shall prove that the parallelograms ABCD and IHCD are equal, the whole and the part (since each is double the triangle EFG—the one by hypothesis, the other by proposition 41). Which is absurd. We shall show the same if the base FG is said to be the greater. For if FH equal to BC be cut off and the line HE drawn, the triangles EFH and EFG will be equal, the part and the whole (since each is half of the parallelogram ABCD—the one by proposition 41, the other by hypothesis). Which is absurd.

Diagram. A small figure at the lower left for the second converse: the parallelograms ABCD and IHCD and the triangle EFG. The figure is small and faint, and several labels are hard to resolve.

- A (top left): upper-left vertex.
- I (top): upper vertex of the inner parallelogram IHCD. [?]
- D (top): upper vertex.
- E (top right): apex of the triangle EFG.
- B (bottom left): left end of base BC.
- H (bottom): point on the base, with the cut-off segment.
- C (bottom): shared base point.
- F (bottom): left end of base FG.

- G (bottom right): right end of base FG.

The following additional theorems, inserted after proposition 41, are drawn from Proclus.

Enunciation. If a triangle and a trapezium stand on the same base and in the same parallels, and the greater parallel line of the trapezium is the base of the triangle, the trapezium will be less than double the triangle. But if the lesser parallel line of the trapezium is the base of the triangle, the trapezium will be greater than double the triangle.

Demonstration. Between the parallel lines AE and BC, let a trapezium ABCD and a triangle EBC be set up on the same base BC—which base, however, is greater than the other line AD, the parallel side in the given trapezium. I say that the trapezium ABCD is less than double the triangle EBC. For since AD is assumed to be less than BC, take AF equal to BC and draw the line CF, which will be parallel to AB [Margin: (a) Elements I.33]; and so ABCF will be a parallelogram, which is double the triangle EBC [Margin: (b) Elements I.41]. Therefore the trapezium ABCD, being a part of the parallelogram, will be less than double the same triangle EBC. Which is what was proposed.

Diagram. A trapezium ABCD and a triangle EBC on the base BC, with AF taken equal to BC to complete the parallelogram ABCF; here the base BC exceeds the parallel side AD.

- A (top left): left end of the upper parallel.
- D (top): upper vertex of the trapezium.
- F (top): point with AF = BC, completing parallelogram ABCF.
- E (top right): apex of the triangle EBC.
- B (bottom left): left end of the base BC.
- C (bottom): right end of the base BC.

Again, let there be a trapezium and a triangle as before, but let the base BC be less than the other parallel line AD in the given trapezium. I say that the trapezium ABCD is greater than double the triangle EBC. For since AD is greater than BC, cut off DF equal to BC and draw the line BF, which will be parallel to CD [Margin: (c) Elements I.33]; and so BCDF will be a parallelogram, which is double the triangle EBC [Margin: (d) Elements I.41]. Therefore the whole trapezium ABCD, which exceeds the parallelogram BCDF, will be greater than double the same triangle EBC. Which is what was proposed.

Diagram. A trapezium ABCD and a triangle EBC, with DF cut off equal to BC to form the parallelogram BCDF; here the parallel side AD exceeds the base BC.

- A (top left): left end of the upper parallel.
- F (top): point with $DF = BC$.
- D (top): upper vertex of the trapezium.
- E (top right): apex of the triangle EBC.
- B (bottom left): left end of the base BC.
- C (bottom): right end of the base BC.

The same will be concluded if the trapezium and the triangle are set up on equal bases, in such a way that now the base of the trapezium is greater than its opposite parallel side, and now less.

Enunciation. A trapezium having two opposite sides parallel is double the triangle whose base is the one side of the trapezium joining the two parallels, and whose vertex is at the midpoint of the opposite side.

Demonstration. Let ABCD be a trapezium whose two opposite sides AB and DC are parallel, and on the base BC let a triangle EBC be set up, having its vertex E at the midpoint E of the side AD. I say that the trapezium ABCD is double the triangle EBC. For let one side of the triangle running to the vertex, namely BE, be produced until it meets CD, extended to F. Since AB and CF are parallel, the alternate angles BAE and FDE will be equal [Margin: (e) Elements I.29]; and the angles AEB and DEF are also equal, being vertical angles at E [Margin: (f) Elements I.15]; and the side AE of triangle ABE equals the side DE of triangle DEF, by hypothesis. Therefore the remaining sides AB and BE will be equal to the remaining sides DF and FE, each to each, and the remaining angles ABE and DFE will be equal; and hence the triangles ABE and DFE will be equal, by the corollary of proposition 26 of this book [Margin: (g) Elements I.26]. So, adding the common triangle CDE, the triangles ABE and CDE together will be equal to the triangle CEF [Margin: (h) common notion 1]. But the triangle BCE is also equal to the same triangle CEF, since the bases BE and EF have been shown to be equal and the triangles themselves lie between the same parallels—if through C a parallel to BF were drawn [Margin: (i) Elements I.38]. Therefore the triangle CBE will be equal to the triangles ABE and CDE [Margin: (k) common notion 1]; and consequently the triangle CBE will be half of the trapezium ABCD. Which is what was proposed.

Diagram. A trapezium ABCD with AB parallel to DC, and a triangle EBC whose vertex E is the midpoint of the side AD; the side BE is produced to meet CD extended at F.

- A (left): left vertex of the trapezium.
- B (bottom left): lower-left vertex, base end.
- C (bottom right): base end.
- D (right): right vertex of the trapezium.
- E (middle): midpoint of AD, vertex of the triangle.
- F (top right): where BE produced meets CD extended.

PROBLEMA 11. PROPOSITIO 42.

[Margin: xlij (42)]

Enunciation. To construct, in a given rectilineal angle, a parallelogram equal to a given triangle.

Demonstration. Let the given triangle be ABC and the given rectilineal angle D. We must therefore construct a parallelogram equal to the triangle ABC, having an angle equal to angle D. Bisect one side of the triangle, namely BC, at E [Margin: (a) Elements I.10], and make the angle CEF equal to angle D [Margin: (b) Elements I.23], as you please—that is, whether the angle CEF inclines toward C or toward B, whichever seems more convenient. Likewise draw through A a line AF parallel to BC [Margin: (c) Elements I.31], to cut EF at F. Again, through C or B, draw CG parallel to EF, meeting AF produced at G. Then in the angle CEF—which has been made equal to the given rectilineal angle D—there will be constructed the parallelogram CEFG, which I say is equal to the triangle ABC. For when the line EA is drawn: the parallelogram CEFG is double the triangle AEC [Margin: (d) Elements I.41], and the triangle ABC is likewise double the same triangle AEC [Margin: (e) Elements I.38], because the triangles AEC and ABE stand on equal bases EC and BE and in the same parallels, and so are equal. Therefore the parallelogram CEFG and the triangle ABC will be equal to each other [Margin: (f) common notion 6]. And since the angle CEF has been made equal to angle D, the proposition stands. And so, for a given triangle, we have constructed a parallelogram equal to it in a given rectilineal angle. Which was to be done.

Diagram. The triangle ABC with its side BC bisected at E, and the parallelogram CEFG built in the angle CEF; the given angle D is drawn separately to the right.

- A (top left): apex of the triangle.
- F (top): upper vertex of the parallelogram CEFG.
- G (top right): upper vertex of the parallelogram.
- B (bottom left): left end of the base.
- E (bottom, middle): midpoint of BC.

- C (bottom): right end of the base.
- D (right, separate): the given rectilineal angle.

SCHOLIUM

The practical construction of this problem is very easy from the construction itself.

Here Peletarius [Jacques Peletier] adds the following problem, which is the converse of this proposition.

Enunciation. To construct, in a given rectilineal angle, a triangle equal to a given parallelogram.

Demonstration. Let the given parallelogram be ABCD and the given angle G. Make the angle CBE equal to angle G, and let the line BE cut AD, produced to E. Extend BC also to F so that CF equals the line BC, and join EF. I say that the triangle BEF, having the angle EBF equal to the given angle G, is equal to the parallelogram ABCD. For when the line CE is drawn, the parallelogram ABCD is double the triangle BCE [Margin: (g) Elements I.41]; likewise the triangle BEF is double the same triangle BCE [Margin: (h) Elements I.38], since the triangles EBC and ECF are equal. Therefore the parallelogram ABCD and the triangle BEF will be equal to each other [Margin: (i) common notion 6]. Which is what was proposed.

Diagram. The parallelogram ABCD and the triangle BEF, where BE cuts AD produced at E and BC is extended to F; the given angle G is drawn separately to the right.

- A (top left): upper-left vertex of the parallelogram.
- D (top): upper-right vertex of the parallelogram.
- E (top right): where BE meets AD produced; apex of the triangle.
- B (bottom left): common vertex.
- C (bottom): lower-right vertex of the parallelogram.
- F (bottom right): point on BC produced, with CF = BC.
- G (right, separate): the given angle.

The practical construction of this problem of Peletier's is likewise very easy from the construction itself.

THEOR. 32. PROPOS. 43.

[Margin: xliij (43)]

Enunciation. In every parallelogram, the complements of the parallelograms about the diameter are equal to one another.

Demonstration. In the parallelogram ABCD, let there be, about the diameter AC, the parallelograms AEGH and CFGI, and the complements DFGH and EBIG, as we stated in definition 36. I say that these complements are equal to one another. For since the triangles ABC and CDA are equal [Margin: (a) Elements I.34], and likewise the triangles AEG and GHA, then if the latter are subtracted from the former, the trapezia CBEG and CDHG will remain equal [Margin: (b) common notion 1]. But the triangles CGI and CGF are also equal [Margin: (c) Elements I.34]. Therefore, if these are taken away from the trapezia, the complements DFGH and EBIG will remain equal [Margin: (d) common notion 3]. In every parallelogram, then, the complements, and so on. Which was to be shown.

Diagram. A parallelogram ABCD with the diameter AC drawn; about the diameter stand the parallelograms AEGH and CFGI, meeting at G, with the two complements DFGH and EBIG.

- A (top left): upper-left vertex.
- E (top): point on AB.
- B (top right): upper-right vertex.
- H (middle left): point on AD.
- G (center): point on the diameter AC where the inner parallelograms meet.
- I (middle right): point on BC.
- D (bottom left): lower-left vertex.
- F (bottom): point on DC.
- C (bottom right): lower-right vertex.

SCHOLIUM

This theorem is demonstrated in the same way by Proclus even if the two parallelograms about the diameter do not meet at the point G, but either one is set apart from the other, or the two intersect each other. Let one first be at a distance from the other, so that the complements are five-sided figures. Thus, in the parallelogram ABCD, about the diameter AC, let the parallelograms AEFG and CHIK stand. I say that the complements DEFIH and BKIFG are equal. For since the triangles ABC and CDA are equal to one another [Margin: (e) Elements I.34], and likewise the triangles AEF and CHI [recte: CHI; source prints GHI] are equal to the triangles AGF and CKI, the remaining complements DEFIH and BKIFG will be equal [Margin: (f) common notion 3]. Which is what was proposed.

Diagram. A parallelogram ABCD with diameter AC; two smaller parallelograms AEFG and CHIK stand about the diameter but are separated, so that the two complements are pentagons (DEFIH and BKIFG). Several inner labels are crowded and faint.

- A (top left): upper-left vertex.
- B (top right): upper-right vertex.
- E, F (upper area): vertices of the parallelogram AEFG.
- G (mid): vertex of parallelogram AEFG.
- H, I, K (lower area): vertices of parallelogram CHIK.
- D (bottom left): lower-left vertex.
- C (bottom right): lower-right vertex.

Now let the parallelograms AEFG and CHIK, standing about the diameter, cut each other, so that they have the common part ILMF. I say that here too the complements DELH and BGMK are equal. For since the triangles ABC and CDA are equal [Margin: (g) Elements I.34], and likewise the triangles AFG and AFE, the remaining quadrilaterals BCFG and DCFE will be equal [Margin: (h) common notion 3]; and again the triangles IFM and IFL are equal [Margin: (i) Elements I.34]. Therefore, if these are added to the said quadrilaterals, the figures BCIMG and DCILE will be equal [Margin: (k) common notion 2]. Since, then, the triangles CIK and CIH are also equal [Margin: (l) Elements I.34], the remaining complements BGMK and DELH will likewise be equal [Margin: (m) common notion 3]. Which is what was proposed.

Diagram. A parallelogram ABCD with two inner parallelograms AEFG and CHIK that intersect, sharing the common part ILMF; the complements are DELH and BGMK. The figure is small with several overlapping labels.

- A (top left): upper-left vertex.
- G (top): point on the upper edge.
- B (top right): upper-right vertex.
- E (mid left): point on the left edge.
- L, M (center): vertices of the shared inner region ILMF.
- F (center): vertex of the shared inner region.
- I (center): intersection point.
- K (right): point on the right edge.
- D (bottom left): lower-left vertex.
- H (bottom): point on the lower edge.

- C (bottom right): lower-right vertex.

We shall also demonstrate the converse of this theorem, following Peletarius, in this way.

Enunciation. If a parallelogram is divided into four parallelograms such that two opposite ones of them are equal, the remaining two will lie about the diameter.

Demonstration. When two lines EF and GH, parallel to the lines BC and CD and cutting each other at I, are drawn, let the parallelogram ABCD be divided into four parallelograms, of which the two opposite ones BEIH and DFIG are equal. I say that the remaining two, AEIG and CFIH, lie about the diameter—that is, that the diameter drawn from the point C to the point A passes through the point I. For if it does not pass through it, let the diameter CKA cut the line GH at K (if that be possible), and through K [Margin: (n) Elements I.31] let LM be drawn parallel to BC. Then the complements BHKL and DGKM will be equal [Margin: (o) Elements I.43]; but DGKM is greater than DGIF [Margin: (p) common notion 9]. Therefore BHKL will also be greater than DGIF. Since, then, DGIF is assumed equal to BEIH, BHKL will also be greater than BEIH—the part greater than the whole [Margin: (q) common notion 1]. Which is absurd. Therefore the diameter AC does not cut the line GH at K, but passes through the point I. Which is what was proposed.

Diagram. A large leaning parallelogram ABCD divided into four parallelograms by the lines EF and GH crossing at I; the diagonal is drawn, and the point K (on GH) with the line LM illustrates the reductio. The labels are crowded and several overlap near the center.

- A (top left): upper-left vertex.
- E (top): point on the upper edge AB.
- B (top right): upper-right vertex. [?]
- G (left): point on the left edge AD.
- I (center): intersection of EF and GH.
- K (center, near I): where the diameter cuts GH (in the reductio).
- L, M (center/lower): endpoints of LM parallel to BC. [?]
- H (right): point on the right edge BC.
- D (bottom left): lower-left vertex.
- F (bottom): point on the lower edge DC.
- C (bottom right): lower-right vertex.

PROBLEMA 12. PROPOSITIO 44.

[Margin: xliv (44)]

Enunciation. To apply to a given straight line, in a given rectilineal angle, a parallelogram equal to a given triangle.

Demonstration. Let the given straight line be A, the given triangle B, and the given rectilineal angle C. We must therefore construct a parallelogram equal to the triangle B, having an angle equal to angle C and one side equal to the line A. Construct a parallelogram DEFG equal to the triangle B, having the angle EFG equal to angle C [Margin: (a) Elements I.42], and produce GF to H so that FH equals the line A; and through H [Margin: (b) Elements I.31] draw HI parallel to FE, meeting DE produced at I. Then extend from I, through F, the diameter IF, meeting DG produced at K; and through K [Margin: (c) Elements I.31] draw KL parallel to GH, cutting IH produced at L, and produce EF to M. I say that the parallelogram LMFH is the one sought. For it has the side FH equal to the given line A, and the angle HFM equal to the given angle C [Margin: (d) Elements I.15]—since the angle HFM equals the angle EFG, which was made equal to angle C. And finally the parallelogram LMFH is equal to the triangle B [Margin: (e) Elements I.43], since it is equal to the complement DEFG, which was made equal to the triangle B. And so, to a given straight line, for a given triangle, and so on. Which was to be done.

Diagram. The given line A, the given triangle B, and the given angle C at the left; then the construction: the parallelogram DEFG equal to triangle B, extended into the applied parallelogram LMFH along the diameter IF.

- A (top left, segment): the given straight line.
- B (left, triangle): the given triangle.
- C (left, angle): the given rectilineal angle.
- D (top): upper vertex of parallelogram DEFG.
- E (top): upper vertex.
- I (top right): where HI meets DE produced.
- G (middle left): vertex of DEFG.
- F (center): common vertex on the diameter IF.
- H (middle right): point with FH = A.
- K (bottom left): where the diameter meets DG produced.
- M (bottom): point on EF produced.
- L (bottom right): where KL meets IH produced.

SCHOLIUM

But if one should wish the given line A itself to be one side of the parallelogram, it will not be difficult to transfer the parallelogram FMLH onto the line A, from what we taught in the scholium of proposition 31 of this book. For if at N, the extremity of the line A, there be made...

Euclid, *Elements* Book I — Propositions 45–47 (with Clavius's commentary and scholia)

(continuation of the scholium to Proposition 44)

at A, let an angle be made equal to the angle MFH, and let the straight line NO be taken equal to the straight line FM. Then, when the parallelogram is completed, as was taught in the said scholium, what is sought will have been accomplished.

But perhaps we shall carry out this problem more in keeping with Euclid's own intention if we construct the parallelogram upon the given straight line itself, and not upon one equal to it — such as the straight line FH was — and so on. So let the straight line AB again be given, together with the given triangle CDE and the given angle F; and let it be required to construct upon the straight line AB a parallelogram equal to triangle CDE, in the given angle F. We shall do it as follows. One side of the triangle — say DE — being bisected at G, and the straight line CG being drawn (which, by the scholium to Proposition 38 of this book, bisects triangle CDE), let the straight line AB be produced toward the part where the parallelogram must have its angle equal to the given angle F — here, toward A; and let AH be made equal to EG, half of the side DE. Next, at A, [Margin: (f) I.23] let the angle HAI be constructed equal to the angle E — downward, if the parallelogram is to be constructed toward the upper part, but upward if toward the lower. With the straight line AI set equal to EC, let the straight line HI be joined. And since the two sides HA, AI are equal to the two sides GE, EC, each to each, and contain equal angles, [Margin: (g) I.4] the triangles AHI, EGC will be equal. After this, [Margin: (h) I.31] let IK be drawn through I parallel to HB, [Margin: (i) I.23] and at A, toward this parallel, let the angle HAL be constructed equal to the given angle F, and let AL cut the straight line IK at L. [Margin: (k) I.31] Let HM also be drawn through H parallel to AL, cutting IK at M; [Margin: (l) I.41] then the parallelogram AHML will be double the triangle AHI. But triangle CDE is likewise double triangle EGC, since triangles EGC, DGC have been shown to be equal. Therefore, since triangles AHI, EGC have been proved equal, [Margin:

(m) Ax. 7] the parallelogram AHML and the triangle CDE will be equal. Now, with the straight line LK taken equal to the given straight line AB, let KA be drawn and produced until it meets MH (produced) at N; through N let NO be drawn parallel to HB until it meets KB (produced) at O; and finally let LA be produced as far as P, on the straight line NO. I say that the parallelogram ABOP is the one sought. For it has been constructed upon the given straight line AB, and it has the angle BAP equal to the given angle F, [Margin: (n) I.15] since it is equal to the angle HAL, which was made equal to angle F. It is moreover equal to the given triangle CDE, [Margin: (o) I.43] since it is equal to the parallelogram AHML, which was shown to be equal to triangle CDE.

Praxis. The practical construction of this problem does not differ from the construction itself, except that the straight line HI need not be drawn, nor the straight line CG; for these were drawn only for the sake of the demonstration.

Diagram. The large construction figure for the Proposition 44 scholium (parallelogram AHML transformed into the required parallelogram ABOP), a dense network of intersecting lines. Labels read on the leaf:

- N, P (top edge)
- O, E, G, D (upper-right region)
- H (left)
- B (center-right)
- C (lower right); F (lowest right)
- I (lower left), M, L, K (along the lower edge)
- A [?] (a central label near the diagonal intersection is not cleanly resolvable)

Diagram. A small separate parallelogram at the top right of the page (the completed result parallelogram). Legible corner labels: N (top left), A (top), O (lower left). The two remaining corner labels are not legible on the scan, even under magnification — 2 labels unread.

Here Peletier [Jacques Peletier] adds another problem, in this way.

[Peletier — problem] *On a given straight line, to construct a triangle equal to a given parallelogram, in a given rectilineal angle.*

Let the given straight line be AB, the given parallelogram CDEF, and the given angle L. Let CD be produced to G so that DG is equal to CD, and let the straight line GE be joined. Then triangle CEG will be equal to the parallelogram CDEF, as we demonstrated in the scholium to Proposition 41. [Margin: (P) cf. schol. I.41] Now

upon the given straight line AB let a parallelogram ABHI be made equal to triangle CEG — that is, to the parallelogram CDEF — having the angle A equal to angle L; and let AI be produced to K so that IK is equal to AI, and let the straight line BK be joined. I say that the triangle ABK, constructed upon the given straight line AB and having the angle A equal to the given angle L, is equal to the given parallelogram CDEF. For since triangle ABK is equal to the parallelogram ABHI (by the scholium to Proposition 41), which was constructed equal to the parallelogram CDEF, the proposition is established.

Praxis. The practical construction of this problem does not differ from the theoretical construction, and it is not difficult, provided one uses the preceding practical construction, by which the parallelogram ABHI is set up on the given straight line AB, in the given angle A, equal to the triangle ECG.

Diagram. Two adjacent figures for Peletier's problem: at the left, triangle ABK with the parallelogram ABHI (and I produced to K); at the right, the parallelogram CDEF with triangle CEG; the given angle L is shown separately. Labels read on the leaf:

- Left figure: B (apex), H [?] (top left of parallelogram — reads ambiguously), A, I, K (lower edge)
- Right figure: E, F (top), C, D, G (lower edge)
- L (given angle, shown at the side)

PROBLEMA 13. PROPOS. 45.

[Margin: xlv.]

[Euclid, Prop. 45 — enunciation] *On a given straight line, to construct a parallelogram equal to a given rectilineal figure, in a given rectilineal angle.*

Although Euclid states this problem in absolute form, without tying us to any determinate given straight line — as he had done in the preceding Proposition 44 — nevertheless, because in what follows it is frequently applied to a given straight line, it seemed good to state it together with a given straight line. So let the given straight line be EF, the rectilineal figure ABC, and the given angle D. It is required, then, to construct upon the given straight line EF a parallelogram equal to the rectilineal figure ABC and having an angle equal to angle D. Let the rectilineal figure be resolved into the triangles A, B, and C. Then [Margin: (a) I.44] upon triangle A let a parallelogram EFGH be constructed, on the straight line EF, having the angle F equal to angle D. Likewise, upon the straight line GH, a parallelogram GHIK equal to triangle B, having the angle G equal to angle D.

Likewise, upon the straight line IK, a parallelogram IKLM equal to triangle C, having the angle K equal to angle D. And let one proceed in the same way if there are more triangles in the given rectilineal figure; and what is required will be done. For the three parallelo-

Diagram. The Proposition 45 figure: at the left, the rectilineal figure ABC resolved into triangles (interior regions marked A, B, C); at the right, the three parallelograms EFGH, GHIK, IKLM joined into a single oblique strip; the given angle D is shown separately. Labels read on the leaf:

- Rectilineal figure: A [?], B, C (interior regions); D (angle)
- Parallelogram strip: E, H, I, M (upper edge); F, G, K, L (lower edge)

(Proposition 45, commentary continued — p. 74)

grams constructed, which are indeed equal to the given rectilineal figure ABC, make up one whole parallelogram — which is proved as follows. The two angles EFG, HGK [Margin: (b) Ax. 1] are equal to one another, since each is equal to angle D. Adding, therefore, the common angle FGH, the two angles EFG, FGH — [Margin: (c) Ax. 29] which are equivalent to two right angles — [Margin: (d) Ax. 2] will be equal to the two angles HGK, FGH; and hence these angles too will be equal to two right angles. [Margin: (e) I.14] Therefore FG, GK will make one straight line. By the same reasoning we shall show that EH, HI make one straight line, because the two angles EHG, [Margin: (f) I.34] HIK are equal to one another — since they are equal to the equal opposite angles EFG, HGK — [Margin: (g) Ax. 29] and the two angles HIK, IHG are equal to two right angles, and so on. Since, then, EI and FK are parallel, [Margin: (h) I.30] and likewise EF and IK (because each is parallel to the straight line HG), the figure EFKI will be a parallelogram. In the same way it will be shown that the parallelogram IKLM, joined to the parallelogram EFKI, makes up one whole parallelogram EFLM. Therefore upon the given straight line EF we have constructed, equal to the given rectilineal figure ABC, a parallelogram EFLM having the angle F equal to the given angle D. Which was to be done.

Scholium

In the same way, given any number of rectilineal figures, we shall construct a parallelogram equal to them, if they are all resolved into triangles for which equal parallelograms are exhibited, one for each, by Proposition 44 — just as was done in this problem. For since all these parallelograms make up one parallelogram, as was demonstrated here, a parallelogram will have been

constructed equal to the given rectilinear figures. For example, suppose someone considers two given rectilinear figures AB and C; suppose AB is resolved into the triangles A and B; and suppose that for the individual triangles A, B, C, individual parallelograms EG, GI, IL are constructed, upon the straight lines EF, HG, IK, according to the method of this problem, equal to them, by Proposition 44. Then the whole parallelogram EFLM will have been constructed, equal to the two rectilinear figures AB and C. And so for more.

The practical construction of this problem must be sought from the practical construction of the preceding proposition, repeated as often as needed.

To this place may be referred a most useful problem from Peletier, which we shall, however, demonstrate by another and shorter method, in this way.

[Peletier — problem] *Given two unequal rectilinear figures, to find the excess of the greater over the less.*

Let the given rectilinear figures be A and B, and let A be the greater. It is required, then, to find out by what magnitude the rectilinear figure A exceeds the rectilinear figure B. [Margin: (i) I.45] Let a parallelogram CDEF be made, in any angle D, equal to the greater rectilinear figure A. And upon the straight line CD let a parallelogram CDGH be made, in the same angle D, equal to the lesser rectilinear figure B. Since, then, the parallelogram CDEF exceeds the parallelogram CDGH by the parallelogram EFHG, the figure A will also exceed the figure B by the same parallelogram EFHG. Which was the thing proposed.

Diagram. The "excess" figure: at the left, two small oblique parallelograms marked A and B (the two given figures rendered as parallelograms); at the right, a rectangle divided by a vertical line into the parallelogram CDGH and the excess parallelogram EFHG. Labels read on the leaf:

- A, B (the two parallelograms at the left)
- C, H, F (upper edge of the rectangle); D, G, E (lower edge)

PROBLEMA 14. PROPOSITIO 46.

[Margin: xlv.]

[Euclid, Prop. 46 — enunciation] *On a given straight line, to describe a square.*

Let AB be the given straight line, upon which a square must be described.

[Margin: (a) I.11] From A and B let AD, BC be drawn perpendicular to AB, and let them be equal to AB itself, and let the straight line CD be joined. I say that ABCD is

a square. For since the angles A and B are right, [Margin: (b) I.28] AD and BC will be parallel; and they are also equal, since each is equal to AB. Therefore [Margin: (c) I.33] AB and DC are also parallel and equal; and so ABCD is a parallelogram, in which — since AD, DC, CB are equal to AB — all four sides are equal. And all four angles are also right, since C and D [Margin: (d) I.34] are equal to the opposite right angles A and B. Therefore ABCD is a square, by definition; and consequently we have described a square on a given straight line. Which was to be done.

Diagram. The square ABCD erected on AB. Labels read on the leaf: D, C (top edge); A, B (bottom edge).

From Proclus [Ex Proclo]

[Proclus] *The squares of equal straight lines are equal; and the straight lines of equal squares are equal.*

First let the straight lines AB, CD be equal. I say that their squares ABEF, CDGH are equal also. For when the diagonals BF, DH are drawn, the two sides BA, AF of triangle BAF will be equal to the two sides DC, CH of triangle DCH, [Margin: (e) I.4] each to each — since, by the definition of a square, the straight lines AF, CH are equal to the straight lines AB, CD. And the angles A and C are also equal, namely right. [Margin: (f) I.34] Therefore triangles BAF, DCH will be equal. And since these are halves of the squares, the whole squares will be equal too. Which was the thing proposed.

Diagram. Two squares side by side, each with a diagonal drawn: the square ABEF (diagonal BF) and the square CDGH (diagonal DH). Labels read on the leaf: F, E, H, G (upper edge, left to right); A, B, C, D (lower edge, left to right).

Next let the squares ABDE, BCFG be equal. I say that their straight lines AB, BC are equal too. For let the squares be joined at the angle B, so that the straight lines AB, BC are set in a straight line. And since the angles ABG, ABD are right, [Margin: (g) I.14] the straight lines GB, BD will also be set in a straight line. Let the diagonals AD, CG be drawn, and let the straight lines AG, CD be joined. Since, then, the squares ABDE, BCFG are equal, the triangles ABD, BCG, their halves, will be equal too. Adding, therefore, the common triangle BCD, the whole triangle ACD will become equal to the whole triangle GDC. Therefore triangles ACD, GDC, since they have the same base CD and are on the same side of it, [Margin: (h) I.39] are within the same parallels; and so AG, CD are parallel. And since — as we showed in the scholium to Proposition 34 — the diagonal in a square bisects the angles of the square, the angles DAC, GCA will be alternate half-right angles, and therefore equal. For this reason [Margin: (i) I.27] AD, CG are parallel too. Therefore ADCG is

a parallelogram; and consequently [Margin: (k) I.34] the straight lines AD, CG are equal. Since, then, in the triangles ABD, BCG the sides AD, CG are equal, and the angles at which those sides lie are also equal to one another (being half-right, as was shown in the scholium to Proposition 34), [Margin: (l) I.26] the remaining sides will be equal, namely AB to BC, and so on. Which was the thing proposed.

Diagram. The two squares ABDE and BCFG joined at the vertex B, with the diagonals AD, CG and the joining lines AG, CD, drawn tilted. Labels read on the leaf:

- A (top); E (upper right)
- G (left); B (center); D (right)
- F (lower left); C (lower center)

(Scholium to Proposition 46, continued — p. 75)

Scholium

All this could be proved much more briefly by the superposition of one square upon another. For if the straight lines are equal, and one is placed upon the other, they will coincide with one another. Since, then, the angles too are equal — namely right — they will likewise coincide with one another; and so the whole square will coincide with the whole square. But if the squares are equal, they coincide with one another on account of the equality of their angles; therefore the straight lines too are equal — otherwise one square would be greater than the other.

The practical construction of this problem is very easy. For if, at a given straight line AB, at one of its ends — say at A — a perpendicular AD is erected equal to the given straight line AB, and from B and D, at the interval of that same AB, two arcs are described cutting one another at C, and the straight lines BC, DC are joined, a square will have been constructed. For ABCD, since by construction it is a figure of equal sides — and moreover has its opposite sides equal — will be a parallelogram, as we demonstrated at the beginning of the scholium to Proposition 34. Since, then, the angle A is right, [Margin: (m) I.29] B and D will also be right, as well as [Margin: (n) I.34] the opposite angle C, and so on.

THEOR. 33. PROPOS. 47.

[Margin: xlvj.]

[Euclid, Prop. 47 — enunciation (the Pythagorean theorem)] *In right-angled triangles, the square described on the side subtending the right angle is equal to the squares described on the sides containing the right angle.*

In the triangle ABC let the angle BAC be right, [Margin: (a) I.46] and let there be described, on AB, AC, BC, the squares ABFG, ACHI, BCDE. I say that the square BCDE, described on the side BC which is opposite the right angle, is equal to the two squares ABFG, ACHI, which are described on the other two sides — whether these two sides be equal or unequal. For let the straight line AK be drawn [Margin: (b) I.31] parallel to BE, or to CD, cutting BC at L, and let the straight lines AD, AE, CF, BH be joined. And since the two angles BAC, BAG are right, [Margin: (c) I.14] the straight lines GA, AC will be one straight line; and in the same way IA, AB will be one straight line. Again, since the angles ABF, CBE are equal (being right), if the common angle ABC is added, [Margin: (d) Ax. 2] the whole angle CBF becomes equal to the whole angle ABE; and similarly the whole angle BCH equal to the whole angle ACD. Since, then, the two sides AB, BE of triangle ABE are equal to the two sides FB, BC of triangle FBC, each to each — as is clear from the definition of a square — and the angles ABE, FBC contained by these sides are equal, as we have shown, [Margin: (e) I.4] the triangles ABE, FBC will be equal. But the square (or parallelogram) ABFG [Margin: (f) I.41] is double the triangle FBC, since they are between the parallels BF, CG and on the same base BF; and the parallelogram BEKL is double the triangle ABE, since they are between the parallels BE, AK and on the same base BE. Therefore [Margin: (g) Ax. 6] the square ABFG and the parallelogram BEKL will be equal. By the same reasoning it will be shown that the square ACHI and the parallelogram CDKL are equal. [Margin: (h) I.4] For again the triangles ACD, HCB will be equal, and therefore their doubles — namely the parallelogram CDKL and the square ACHI — will be equal. Wherefore the whole square BCDE, which is composed of the two parallelograms BEKL, CDKL, is equal to the two squares ABFG, ACHI. Therefore in right-angled triangles the square, and so on. Which was to be demonstrated.

Diagram. The classic Euclid I.47 "windmill" figure, drawn twice (for the two cases named in the text — legs equal, and legs unequal): triangle ABC with the right angle at A, squares ABFG, ACHI, BCDE on the three sides, the line AK dropped through L on BC to K on DE, and the joining lines AD, AE, CF, BH. Labels read on the leaf (page heavily stained at the head):

- Left figure: G, I (top); F (left); A (right-angle vertex); H (right); B, L, C (middle row); E, K, D (lower edge)

- Right figure (second case): G [?], I (top); F (left); A; H (right); B, L, C (middle); E, K, D (lower edge)

[— duplicated figure with dense labelling on a foxed page; a top-left label on the second figure is not cleanly resolvable]

Scholium

Anyone will easily understand from this theorem that in an obtuse-angled triangle the square of the side opposite the obtuse angle is greater than the two squares together of the other two sides; while in any triangle the square of the side opposite one of the acute angles is less than the two squares together of the other two sides. For if the angle were narrowed until it became right — the sides containing it remaining the same — the opposite side would turn out less; but if the acute angle were widened until it became right — the same sides containing it remaining — the opposite side would become greater, as is evident. Since, then, the square of the side opposite the right angle has here been shown equal to the two squares together of the other two sides, it is clear that the square of the side opposite the obtuse angle is greater than the two squares together of the other two sides, while the square of the side opposite the acute angle is less than the two squares together of the remaining two sides. But by how much the former is greater, and by how much the latter is less, Euclid will demonstrate in Book 2, Propositions 12 and 13.

Since this theorem is most beautiful and has outstanding uses, I judged it worth the effort to try whether it could be demonstrated by other ways, with the construction somewhat varied — which Peletier denied could be done without proportions. So let there again be a triangle ABC whose angle BAC is right, and let the sides BA, CA be produced toward the parts of the right angle, and let AD, AE be made equal to AC, AB; and through C, D, B, E let parallels be drawn to AB, AC, meeting at F, G. And since in the parallelograms CD, BE [Margin: (i) I.34] both the sides DF, FC are equal to the opposite sides AC, AD, and the sides EG, GB to the opposite sides AB, AE, each parallelogram will be equilateral; and moreover all their angles are right. [Margin: (k) I.34] For F, G, being equal to the opposite right angles DAC, EAB, are themselves right; and the angles C, D, B, E are right, because each of C and D together with the right angle DAC, and each of B and E together with the right angle EAB, [Margin: (l) I.29] is equal to two right angles. Therefore CD, BE are the squares of the sides AC, AB. The sides FD, GE being also produced until they meet at H, let there be erected at B and C, perpendicular to BC, the straight lines BI, CK, cutting GH, FH at I, K, and let the straight line IK be joined. And since, if from the right angles IBC, ABG the common angle ABI is taken away,

the remaining angles ABC, GBI are equal — and the right angles BAC, BGI are also equal — the two angles A, B of triangle ABC will be equal to the two angles G, B of triangle GBI, each to each; and the sides AB, GB adjacent to them are equal, on account of the square BE. [Margin: (m) I.26] Therefore both the sides BC, BI and AC, GI will be equal, and the remaining angles C, I equal. In just the same way, in the triangles ABC, FKC, both the sides BC, CK and AB, FK will be equal, and the angles B, K — because the two angles A, C of triangle ABC are equal to the two angles F, C of triangle FKC (since A, F are right, and the two angles C are the remainder of the two right angles once the common angle ACK is removed), and the sides AC, CF adjacent to them are equal, on account of the square CD. Therefore the straight lines BI, CK [Margin: (n) Ax. 1] are also equal to one another; [Margin: (o) I.28] and since they are also parallel, [Margin: (P) I.33] BC, IK will also be equal and parallel. Therefore BCKI is the square of the side BC, since its four sides are equal and all its angles right — for the angles I, K are equal to the opposite right angles C, B. I say, then, that the square CI is equal to the two squares BE, CD,

Diagram. The large figure for the alternative (Peletier-inspired) demonstration of the Pythagorean theorem: the right triangle ABC with its legs produced, the squares CD and BE built outward, the point H where FD, GE meet, the perpendiculars BI, CK, and the joined line IK, together with numerous interior division and intersection points (some on dotted construction lines). Labels read on the leaf (page heavily stained/foxed):

- H (top apex)
- E (upper left); I, T, K (upper row)
- G (far left); R, S, P (middle-upper row)
- D (right)
- A (center)
- M, N [?], O [?] (lower dotted line)
- B, L, C (bottom edge); F (lower right)

[— ~18 labels on a foxed page; the lower-row points on the dotted line are not all cleanly resolvable]

(Continuation of Clavius's commentary on Proposition 47 of Book I — the Pythagorean theorem. The alternative demonstrations begun on the preceding page conclude here; all of the following is Clavius's commentary, not Euclid's own text.)

Diagram. The large combined construction figure for the alternative demonstrations of the Pythagorean theorem. A right triangle sits at the center

(right angle at A), with the three squares erected outward and an outer square **BCKI** built on the hypotenuse; numerous auxiliary lines (several dotted) subdivide the outer square into the two parallelograms discussed in the text. The sheet is foxed, but at native resolution the labels read as follows.

- H (top apex): summit of the upper construction
- E (upper, below H): point on the left slope
- I (left, upper): vertex on the left square
- T (upper middle): division point on the top edge
- K (upper right): outer vertex on the right
- G (left edge): outer vertex on the left
- R (left middle): interior division point
- S (upper center): interior division point
- P (right center): interior division point
- A (center): the right-angle vertex of the triangle
- D (right edge, middle): vertex of the right-hand square
- M (lower left): interior/foot point
- N (lower middle): interior/foot point
- [?] (lower right, between N and C): a small mark; the text calls for point **O** here, but the glyph is not clearly resolvable
- F (right edge, lower): outer point on the right
- B (bottom-left corner): base vertex
- L (bottom middle): base division point
- C (bottom-right corner): base vertex

[Clavius, alternative demonstration — conclusion of the first alternative]

that the squares BE and CD are equal. For when the straight line HTAL is drawn from H through A, it will be parallel to BI and CK. Since GE is equal to BA [Margin: (q) Elements I.34], and EH is equal to AD — that is, to GI, which was shown equal to AC (or AD) — then, adding the common segment EI, the whole HI will be equal to the whole GE, that is, to AB. And since HI and AB are also parallel, BI and AH will likewise be parallel and equal [Margin: (r) Elements I.33]. In the same way AH will be shown parallel and equal to CK. Now the square BE is equal to the parallelogram BH on the same base AB [Margin: (s) Elements I.35], and the parallelogram BT is equal to that same parallelogram BH on the same base BI; therefore the square BE is also equal to the parallelogram BT [Margin: (t) pron. 1 (Common Notion 1)]. Likewise, since the square CD is equal to the parallelogram CH on the same base AC [Margin: (u) Elements I.35], and the parallelogram CT is

equal to that same parallelogram CH on the same base CK, the square CD is also equal to the parallelogram CT [Margin: (x) pron. 1 (Common Notion 1)]. Therefore the whole square BCKI, composed of the two parallelograms BT and CT, is equal to the two squares BE and CD. Which was to be demonstrated.

[Clavius, second alternative demonstration]

Otherwise. In place of the straight line HTAL, let the two straight lines AI and AK be joined, and through A let TAL be drawn parallel to BI and CK. Then, since the square BE is double the triangle ABI on the same base AB, while the parallelogram BT is likewise double that same triangle ABI on the same base BI [Margin: (y) Elements I.41] — for we shall show, as before, that BK, BE, and CD are the squares on the sides BC, AB, and AC — the square BE will be equal to the parallelogram BT [Margin: (z) pron. 6 (Common Notion 6)]. By the same reasoning, since the square CD is double the triangle ACK on the same base AC, while the parallelogram CT is double that same triangle ACK on the same base CK [Margin: (a) Elements I.41], the square CD will also be equal to the parallelogram CT [Margin: (b) pron. 6]. Wherefore, as before, the whole square BCKI, made up of the two parallelograms BT and CT, is equal to the two squares BE and CD. Which was to be shown.

[Clavius, third alternative demonstration]

Otherwise. In place of the straight lines HTAL, AI, and AK, let there be drawn through F and G the lines FM and GP, parallel to BC and IK, cutting AB and CK at N and O, and BI and CE at R and S. Then BR will be equal to KO. For since the angles O and R are right [Margin: (c) Elements I.29], and the angles OKF and RBG are equal — because each was shown above to be equal to the angle ABC — the two angles O and K of the triangle FKO will be equal to the two angles R and B of the triangle GBR, each to each. Moreover the sides FK and BG adjacent to them are equal, because each was shown above to be equal to BC. Therefore the sides KO and BR will also be equal [Margin: (d) Elements I.26]. And so, since the parallelograms BCFN and BCOM on the same base BC are equal [Margin: (e) Elements I.33], and the parallelogram BCPR is equal to the parallelogram IKOM (on account of the straight lines BR and KO shown to be equal — for from this it follows that if one is superimposed on the other they exactly coincide, because of the equality of their sides and angles), both parallelograms BCFN and BCSG will be equal to the whole square BCKI, which is composed of the two parallelograms BCOM and IKOM [Margin: (f) Elements I.35]. Since therefore the square CD is equal to the parallelogram BCFN on the same base CF, and the square BE is equal

to the parallelogram BCSG on the same base BG, both squares CD and BE will also be equal to that same square BCKI. Which was to be demonstrated.

**

[Clavius, scholium on the theorem]

And in this way still other demonstrations can be devised. I consider Euclid's own demonstration the simpler and more direct; yet it is not without pleasure to see that one and the same truth can be confirmed by a variety of demonstrations.

The discovery of this admirable and most beautiful theorem is ascribed to Pythagoras, who — as Vitruvius writes in Book 9 — sacrificed victims to the Muses because they had aided him to so splendid a discovery. Some think he sacrificed a hundred oxen; but if Proclus is to be believed, he offered only one. Perhaps, however — as some maintain — Pythagoras took his occasion from numbers in order to investigate this theorem. For when he had carefully contemplated these three numbers, 3, 4, and 5, and had seen that the square of the greatest is equal to the squares of the other two, he constructed a scalene triangle whose greatest side was divided into 5 equal parts, the least into 3 of the same magnitude, and the remaining side into 4. This done, he considered the angle contained by these two sides and found it to be right. He observed the same in very many other numbers, as in 6, 8, 10, and in 9, 12, 15, and so on. He therefore judged it worth inquiring whether in every right triangle the square of the side opposite the right angle is equal to the squares of the remaining sides, seeing that all triangles whose sides had magnitude according to the said numbers contained one right angle. And so at last, with the greatest delight of mind, he lit upon this marvelous theorem and demonstrated it by a firm argument. Euclid, however, amplified it in a wonderful manner in Book 6, Proposition 31, where he demonstrated not only that the square of the side opposite the right angle is equal to the squares of the other two sides, but also that any rectilinear figure whatever constructed upon the side opposite the right angle — whether it be a triangle, a quadrangle, or the like — is equal to the two figures described upon the remaining sides, provided they are similar to the first and similarly described, as we shall show in that place.

Moreover, since we have made mention of three numbers the square of whose greatest is equal to the squares of the others, it will not be beside the point to explain briefly by what method such numbers are found. Given, then, these three numbers 3, 4, 5: if they are doubled, another three are obtained, 6, 8, 10; if the same are tripled, there arise another three, 9, 12, 15; and if they are quadrupled,

these three are found, 12, 16, 20. In this way as many others as one pleases will be found, if those first three are multiplied by any number whatever. Two rules, however, are handed down by Proclus by which the said numbers are found without any regard to those first three. The first is ascribed to Pythagoras and is of this kind. Take for the least any odd number you please, say 5, from which you will find the others thus. From the square of the number taken — here, from 25 — reject unity. For half of the number remaining, namely 12, will be the second number; and if unity is added to this, the third number, 13, arises. The square of this, therefore, is equal to the squares of the others. And if the odd number taken had been 3, the other two found by this rule would be 4 and 5. The second rule is attributed to Plato and is as follows. Take any even number, say 6. From the square of its half, namely 9, subtract one, and to the same add one, and you will have the remaining two numbers, 8 and 10; the first being 6, that is, the even number taken. If by this rule the even number 10 is taken, the other two will be found to be 24 and 26.

[Clavius, transition to the appended propositions]

From this most celebrated discovery of Pythagoras many things not unpleasant to know may be gathered, theorems as well as problems, of which I have thought it best to bring forward only those that are believed likely to bring great utility to geometrical matters, taking my beginning here.

I.

If in any square a diagonal is drawn, the square described on the diagonal will be double the said square.

In the square ABCD let the diagonal AC be drawn. I say that the square on the diagonal AC is double the square ABCD. For since in the triangle ABC the angle B is right [Margin: (g) Elements I.47], the square on the side AC will be equal to the two squares on the sides AB and BC. And since the squares on the lines AB and BC are equal — because the lines AB and BC are equal — the square on AC will be double either of them, that is, double the square on the line AB; so the square on the diagonal is double the square ABCD. Which is what was proposed.

Diagram. Proposition I: a square with a diagonal.

- A (top-left) and B (top-right): upper vertices
- D (bottom-left) and C (bottom-right): lower vertices
- Diagonal drawn from A to C

II.

The square on the diagonal of an oblong (a figure longer on one side) is equal to the two squares on the unequal sides.

In the oblong ABCD let the diagonal AC be drawn; and since in the triangle ABC the angle B is right [Margin: (h) Elements I.47], the square on the side AC will be equal to the two squares on the unequal sides AB and BC. Which is what was proposed.

Diagram. Proposition II: an oblong (rectangle) with a diagonal.

- A (top-left) and B (top-right): upper vertices
- D (bottom-left) and C (bottom-right): lower vertices
- Diagonal drawn from A to C

III.

If there are two right triangles whose sides opposite the right angles are equal, then the two squares on the remaining two sides of the one triangle will be equal to the two squares on the remaining two sides of the other.

Let the angles A and D of the triangles ABC and DEF be right, and let the opposite sides BC and EF be equal. I say that the two squares on the sides AB and AC taken together are equal to the two squares on the sides DE and DF taken together. For the squares on the lines BC and EF are equal to one another, since the lines themselves are assumed equal. But the squares on the lines AB and AC are equal to the square on the line BC [Margin: (i) Elements I.47], and the squares on the lines DE and DF are equal to the square on the line EF. Therefore the squares on the straight lines AB and AC are equal to the squares on the straight lines DE and DF [Margin: (k) pron. 1 (Common Notion 1)]. Which is what was proposed.

Diagram. Proposition III: two right triangles side by side.

- A (apex of the first triangle, right angle here); B (bottom-left) and C (bottom-right)
- D (apex of the second triangle, right angle here); E (bottom-left) and F (bottom-right)

IV.

Given two unequal squares, to find another two squares which are both equal to each other and, taken together, equal to the two unequal squares proposed taken together.

Let A and B be the sides of two unequal squares. Make a right angle DCE, and let DC be a straight line equal to B, and CE a straight line equal to A. Then draw the straight line DE joining the two points D and E, and upon it construct two half-right angles DEF and EDF [Margin: (l) pron. 13 (Common Notion / axiom 13)], and let the straight lines DF and EF meet at F. Now since in the triangle FDE the angles FDE and FED are equal, the sides DF and EF will also be equal [Margin: (m) Elements I.6], and therefore the squares on those sides equal as well. I now say that these same squares on the lines DF and EF are equal to the squares on the lines A and B — that is, to the squares on the lines CE and CD. For since in the triangle DEF the angles FDE and FED make one right angle, the remaining angle F will be right [Margin: (n) Elements I.32]. Wherefore the squares on the lines DF and EF will be equal to the square on the line DE [Margin: (o) Elements I.47]. But to that same square on the line DE the squares on the lines CD and CE are also equal [Margin: (p) Elements I.47]. Therefore the squares on the lines DF and EF are equal to the squares on the lines DC and EC [Margin: (q) pron. 1]. Which is what was proposed.

Diagram. Proposition IV: the construction on the right angle DCE.

- A and B (two short reference segments at the top): the sides of the two given squares
- C: vertex of the right angle DCE
- D and E: endpoints of the two legs (CD, CE)
- F: point where DF and EF (drawn at half-right angles from D and E) meet

V.

Given two unequal lines, to find that by which the greater is more powerful than the lesser.

The power [potentia] of a straight line is called its square; for a straight line is said to be capable of just as much as its square amounts to. Let there be, then, two unequal lines A and B, and let it be required to know by how much the square of the greater line A exceeds that of the lesser B. On any straight line CD let CE be taken equal to A, and EF equal to B. Then, with center E and radius EC, let a

semicircle CGD be described, and from F let FG be drawn perpendicular to CD. I say that the square on the line A — that is, on the line CE, which is equal to it — is greater than the square on the line B — that is, on the line EF, which is equal to it — by the square on the line FG. For when the line EG is drawn, its square will be equal to the squares on the lines EF and FG [Margin: (r) Elements I.47]; that is, the square on the line EC, being equal to it, will exceed the square on the line EF by the square on the line FG. Which is what was proposed.

Diagram. Proposition V: a semicircle on the diameter, with a perpendicular.

- A and B (two reference segments at the top): the two unequal given lines
- C, E, F, D (along the base line, left to right): C the left end of the diameter, E the center, F between E and D, D the right end
- G (on the arc above F): top of the perpendicular FG drawn up from F

VI.

Given any number of squares whatever, whether equal or unequal, to find a square equal to all of them.

Let the sides of five squares be A, B, C, D, E, and let it be required to find a square equal to all five of them. Make a right angle FGH, and let the line FG be equal to A, and the line GH to B. Then, when the line HF has been drawn, make a right angle FHI, and let HI be equal to C. Again, drawing the line IF, make a right angle FIK, and let IK be equal to D. Finally, drawing the line KF, make a right angle FKL, and let KL be equal to E, and let the line FL be drawn. I say that the square on the line FL is equal to the five squares proposed. For the square on FH is equal to the squares on FG and GH [Margin: (s) Elements I.47], that is, to the squares on A and B. Again, the square on FI is equal to the squares on FH and HI [Margin: (t) Elements I.47], and therefore to the squares on A, B, and C. Likewise the square on FK is equal to the squares on FI and IK [Margin: (u) Elements I.47], and thus to the squares on A, B, C, and D. Finally the square on FL is equal to the squares on FK and KL [Margin: (x) Elements I.47], and therefore to the squares on A, B, C, D, and E. Which is what was proposed.

Diagram. Proposition VI: a spiral of right triangles about the common apex F, with five reference segments A–E at the right.

- F (bottom center): common apex where all the drawn lines meet
- G (bottom-left): vertex of the first right angle (FGH)
- H (left, above G): vertex of the second right angle (FHI)

- I (upper left): vertex of the third right angle (FIK)
- K (top-left) and L (top-right): outer vertices; FL is the final hypotenuse
- A, B, C, D, E (five vertical segments at the right): the sides of the five given squares

VII.

Given any two squares whatever, to adjoin to one of them a figure equal to the other square, in such a way that the whole composite figure is also square.

Let ABCD and EFGH be two squares, and let it be proposed to adjoin to the square ABCD a figure equal to the square EFGH, and so on. Take the straight line BI equal to the line FG, a side of the square EFGH. Then, having drawn the straight line AI and produced the line BA toward A, take BK equal to the line AI, and complete the square BKLM. I say that the figure ADCMLK, adjoined to the square ABCD, is equal to the square EFGH. For since the square on the line AI — that is, the square BKLM — is equal to the squares on the lines AB and BI [Margin: (y) Elements I.47], that is, to the squares ABCD and EFGH, then, if the common square ABCD is removed, there will remain the figure ADCMLK, equal to the square EFGH [Margin: (z) pron. 3 (Common Notion 3)]. Which is what was proposed.

Diagram. Proposition VII: the large square BKLM containing the small square ABCD in its lower-left corner, with the separate square EFGH beside it.

- K (top-left), L (top-right), M (bottom-right), B (bottom-left): the large completed square BKLM
- A (on the left side, above B), D (interior, right of A), C (on the base, right of B): with B, the small square ABCD in the corner
- I (on the base line, to the right of C): endpoint of the diagonal AI drawn from A
- The gnomon remaining when ABCD is removed is the figure ADCMLK
- E (top-left), H (top-right), F (bottom-left), G (bottom-right): the separate given square EFGH

VIII.

Given any two sides whatever of a right triangle, to arrive at the knowledge of the remaining side.

Let the angle A be right in the triangle ABC, and first let the sides AB and AC about the right angle be known, of which let AB be set at 6 palms and AC at 8.

Since, then, the squares on the lines AB and AC — namely the squares of 36 and 64 palms — are equal to the square on the line BC [Margin: (a) Elements I.47], if they are joined together the result is 100 square palms. Therefore the side BC will contain 10 palms; for that is the side, or square root, of 100 palms, as is plain among the arithmeticians. Next, let the sides AB and BC be known, and let AB be 6 palms and BC 10. Since, then, the squares on the lines AB and AC are equal to the square on the line BC [Margin: (b) Elements I.47], if the square on the line AB, which contains 36 palms, is subtracted from the square on the line BC, which is 100 palms, there will remain the square on the line AC, 64 palms. Therefore the side AC will contain 8 palms; for that is the square root of 64 palms. Which is what was proposed. For the rest, rational numbers will not always be found by this art, since not every number has a side, or square root, as is known among the arithmeticians. Hence the side found can very often not be expressed except by what they call a surd root. But of these matters elsewhere.

Diagram. Proposition VIII: a right triangle with its sides marked off in units.

- A (apex, right angle here)
- B (bottom-left) and C (bottom-right)
- Side AB marked 6, side AC marked 8, base BC marked 10 (each side ruled into unit tick-marks)

IX.

If from the angle contained by two unequal sides of a triangle a perpendicular is drawn to the base, falling within the triangle, the base will be cut into unequal parts, and the greater part will be on the side of the greater side. And conversely, if the base is cut by the perpendicular not into two equal parts, the two sides will be unequal, and the greater will be the one adjacent to the greater segment of the base.

Let the perpendicular AD fall first within the triangle ABC, whose side AB is greater than the side AC, dropped to BC — which happens only when each of the angles B and C is acute, as is clear from Corollary 2 of Proposition 17. I say that the segment BD is greater than the segment CD. For since the square on AB together with the squares on AD and BD is equal to the square on AC together with the squares on AD and CD [Margin: (c) Elements I.47], and the square on AB is greater than the square on AC — because the side AB is assumed greater than the side AC — the two squares on AD and BD will also be greater than the two squares on AD and CD. And when the common square on the line AD is taken away, the remaining square on BD will be greater than the remaining square on

CD. Therefore the straight line BD will be greater than the straight line CD; which was the first point.

Next, let the perpendicular AD make the segment BD greater than the segment CD. I say that the side AB is greater than the side AC. For the square on BD will be greater than the square on CD; and when the common square on AD is added, the two squares on BD and AD will be greater than the two squares on CD and AD. Since, then, the square on AB together with the squares on BD and AD is equal to the square on AC together with the squares on CD and AD [Margin: (d) Elements I.47], the square on AB will also be greater than the square on AC, and consequently the side AB will be greater than the side AC. Which is what was proposed.

Diagram. Proposition IX: a triangle with a perpendicular from the apex to the base.

- A (apex)
- B (bottom-left) and C (bottom-right): the base endpoints
- D (foot of the perpendicular AD on the base, between B and C)

[Clavius, closing scholium and transition to the general (Pappus) theorem]

And in this way very many other things could be gathered from this discovery of Pythagoras, which, deliberately — so as not to be tiresome to the reader — we have judged should here be omitted and deferred to another place.

Far more universal than this Pythagorean theorem is the one demonstrated by Pappus [Pappus of Alexandria] for every triangle, whether it be right-angled or not, and concerning any parallelograms whatever constructed upon the sides of the triangle, whether right-angled or not, even if they are not equiangular with one another. Reducing this into the form of a theorem, we have proposed it more clearly in this manner, and — in my judgment — still more generally than Pappus:

In every triangle, any parallelograms whatever described upon two of its sides are equal to a parallelogram set upon the remaining side, one of whose sides is equal and parallel to the straight line drawn from the angle which the two sides contain to the point where the sides of the parallelograms opposite to the sides of the triangle meet, when they are produced toward the parts of that angle.

(The demonstration of this general theorem begins on the following page. Book I of the Elements has not yet ended: the running heads remain "EVCLID. GEOM." /

"LIBER I." throughout, and Proposition 48 — the converse of the Pythagorean theorem — does not appear here.)

(Continuation of Clavius's generalized area theorem — the "Pappus theorem": parallelograms erected on two sides of a triangle are together equal to a parallelogram on the base. The demonstration begins here.)

[Clavius — demonstration]

Let ABC be any triangle, and on the sides AB and AC let there be erected any parallelograms whatever, ABDE and ACFG, whose sides DE and FG — the sides that, within the parallelograms, are opposite to the sides AB and AC taken in the triangle — when produced toward the side of the angle A meet at H; and let the straight line AH be drawn. I say that the parallelograms AD and AF together are equal to a parallelogram described on the base BC, one of whose other sides is equal and parallel to the straight line AH.

For let HA be produced so as to cut BC at L, and through B and C let BI and CK be drawn parallel to AH; and join the straight line IK. Now since BIHA and CKHA are parallelograms, [Margin: a. 34. primi (Elements I.34)] each of BI and CK is equal to AH; and hence [Margin: b. 30. primi (Elements I.30)] BI and CK are equal to one another. And since they are also parallel — because each is parallel to the same line AH — [Margin: c. 33. primi (Elements I.33)] BC and IK are likewise parallel and equal. Therefore BCKI is a parallelogram on the base BC, having its other side BI equal and parallel to AH; and it is to this that the parallelograms AD and AF must be shown equal. [Margin: d. 35. primi (Elements I.35)] Now the parallelograms AD and ABIH are equal, since they have the same base AB and lie between the same parallels AB and HD; [Margin: e. 35. primi (Elements I.35)] and ABIH is equal to the parallelogram IL, since these two likewise have the same base BI and lie between the same parallels BI and LH. Therefore AD too is equal to IL. In the same way we shall show that AF is equal to KL. Therefore the parallelograms AD and AF together are equal to the parallelogram BK. Which was the thing proposed.

[Clavius — Pappus's alternative construction]

Pappus [Pappus of Alexandria] constructs the figure differently. For he takes the straight lines AC, AE and AB, AG set in a straight line, so arranged that the parallelograms AD and AF are equiangular, having their angles AED and AGF [Margin: f. 29. primi (Elements I.29)] equal, as interior angles, to the exterior angle BAC — as this figure of his indicates. But we have set out the matter more generally, as is evident.

Diagram. The principal figure for the generalized area theorem. A triangle ABC stands with its base BC at the bottom (B left, C right) and vertex A at the centre; above it the apex H is formed where the produced sides of the two parallelograms meet. Parallelograms ABDE (on side AB) and ACFG (on side AC) are erected toward H; the line AH is produced downward to L on BC. Lines BI and CK are drawn parallel to AH, and IK is joined.

- H (top apex): meeting point of the produced parallelogram sides
- E (upper left): outer vertex of parallelogram ABDE
- G (upper right): outer vertex of parallelogram ACFG
- A (centre): vertex of the triangle
- I (mid-left): foot of BI $\parallel$ AH
- K (mid-right): foot of CK $\parallel$ AH
- D (lower left): vertex of parallelogram ABDE, near B
- B (bottom left): base vertex
- L (bottom centre): where AH produced meets BC
- C (bottom right): base vertex
- F (far bottom right): vertex of parallelogram ACFG, beyond C

Diagram. Pappus's alternative figure for the same theorem. Same lettering as the principal figure, but with the sides so placed (AC in line with AE, and AB in line with AG) that the two parallelograms are equiangular.

- H (top apex): meeting point of the produced sides
- E (upper left): parallelogram vertex
- G (upper right): parallelogram vertex
- A (centre): vertex of the triangle
- I (mid-left): foot of BI $\parallel$ AH
- K (mid-right): foot of CK $\parallel$ AH
- B (bottom left): base vertex
- L (bottom centre): foot of AH on BC
- C (bottom centre-right): base vertex
- F (bottom right): parallelogram vertex beyond C

THEOR. 34. PROPOS. 48.

[Margin: xxxviii (= 48, in the outer margin beside the heading)]

[Euclid — enunciation]

If the square that is described on one of the sides of a triangle is equal to the squares that are described on the two remaining sides of the triangle, the angle contained by those two remaining sides is a right angle.

[Clavius — demonstration]

Let there be given a triangle ABC, and let the square on the side AC be equal to the squares on the remaining sides BA and BC. I say that the angle ABC is a right angle. For draw BD perpendicular to BA and equal to the straight line BC, and join the straight line AD. Now since, in the triangle ABD, the angle ABD is a right angle, [Margin: g. 47. primi (Elements I.47)] the square on AD will be equal to the squares on BA and BD. But the square on BD is equal to the square on BC, because the lines are equal. Therefore the square on AD will be equal to the squares on BA and BC. And since the square on AC is assumed equal to the squares on BA and BC, [Margin: h. 1. pronunc. (Common Notion 1)] the squares on AD and AC are equal to one another, and consequently the straight lines AD and AC are themselves equal. Now since the sides BA, BD of triangle ABD are equal to the sides BA, BC of triangle ABC, and the base AD has been shown equal to the base AC, [Margin: i. 8. primi (Elements I.8)] the angles ABD and ABC are equal. But the angle ABD is by construction a right angle. Therefore the angle ABC too will be a right angle. If, then, the square described on one of the sides of a triangle [is equal to the squares on the remaining sides], and so on. Which was to be demonstrated.

[Clavius — SCHOLIUM]

This theorem is the converse of the preceding Pythagorean theorem, as is plainly evident.

Diagram. Figure for Proposition 48 (the converse of the Pythagorean theorem). Triangle ABC with A at the lower left, C at the lower right, and B on the upper right; D is above, with BD drawn perpendicular to BA and equal to BC, and AD joined.

- D (top): endpoint of $BD \perp BA$, with $BD = BC$
- B (upper right of the base region): vertex whose angle is proved right
- A (lower left): vertex
- C (lower right): vertex

TRIANGULORUM COMPARATIONES

[Comparisons of Triangles]

[Clavius — scholium]

Euclid [Euclid] has compared two triangles with one another in nine ways in this book. **First** [Margin: k. 4. primi (Elements I.4)], when two sides are equal to two sides, each to each, and they contain an angle equal to an angle, he gathered the equality of the bases, of the remaining angles, and hence of the whole triangles.

Next [Margin: l. 8. primi (Elements I.8)], when two sides are equal to two sides, each to each, and the base equal to the base, he inferred the equality of the angles contained by those sides. Here we concluded also the equality of the remaining angles, and proved the whole triangles to be equal.

Third [Margin: m. 24. primi (Elements I.24)], when two sides are equal to two sides, each to each, but they contain unequal angles, he showed that the base opposite the greater angle is greater than the base opposite the lesser angle.

Fourth [Margin: n. 25. primi (Elements I.25)], when two sides are equal to two sides, each to each, but the bases are unequal, he demonstrated that the angle opposite the greater base is greater than the angle opposite the lesser base.

Fifth [Margin: o. 26. primi (Elements I.26)], when two angles are equal to two angles, each to each, and one side is equal to one side — whether the side that lies adjacent to the equal angles, or the side that is opposite one of the equal angles — he proved that the remaining sides of the one are equal to the remaining sides of the other,

and the remaining angle to the remaining angle. Here we taught that it follows also that the whole triangles are equal.

Sixth [Margin: a. 37. primi (Elements I.37)], he demonstrated that two triangles set on the same base and between the same parallels are equal.

Seventh [Margin: b. 38. primi (Elements I.38)], he showed that two triangles set on equal bases and between the same parallels are equal.

Eighth [Margin: c. 39. primi (Elements I.39)], he taught that two equal triangles set on the same base and toward the same part lie between the same parallels.

Ninth, finally [Margin: d. 40. primi (Elements I.40)], he proved that two equal triangles set on equal bases in the same straight line, and toward the same part, lie between the same parallels.

And upon these modes of arguing about two given triangles nearly the whole of geometry seems to rest, so that they ought to be committed diligently to memory. But just as, beginning from proposition 26, we excluded in the scholium to proposition 31 two comparisons — namely, when two angles are equal to two angles and one side to one side, yet not the side that lies adjacent to the equal angles nor the side that is opposite one of the equal angles in both triangles, but rather the side that in the one lies adjacent to the equal angles while in the other is opposite one of the equal angles; or the side that in the one is opposite one of the equal angles while in the other is opposite the other of the equal angles — just as, I say, we excluded these two comparisons because from them the equality of the remaining sides does not follow, as we demonstrated in the scholium to proposition 31: so too, in the case of proposition 4, three comparisons must be excluded, from which the equality of the bases can in no way be inferred.

First, then, when two sides are equal to two sides, each to each, and one angle to one angle — the angle that is opposite one of the equal sides — the bases cannot be equal, unless the angle in each triangle that is opposite the corresponding equal side is either less than a right angle or not less than a right angle.

Diagram. Figure for the first excluded comparison. Two triangles share a common baseline. On the left, triangle ABC with apex A over the base points B and C; on the right, the isosceles triangle DFG with apex D over the base points E, F, G. The baseline is lettered, from left to right, B — C — E — F — G.

- A (upper left apex): vertex of triangle ABC
- D (upper right apex): vertex of the isosceles triangle DFG ($DF = DG$)
- B (baseline, far left): base vertex of triangle ABC
- C (baseline): base vertex of triangle ABC
- E (baseline): point to which base GF is produced
- F (baseline): base vertex of the isosceles triangle
- G (baseline, far right): base vertex of the isosceles triangle

[Clavius — demonstration of the first excluded case]

For let DFG be an isosceles triangle having the sides DF and DG equal; let the base GF be produced as far as E, let the straight line ED be joined, and upon triangle DEG let there be constructed an equal triangle ABC, whose sides AB, AC are equal

to DE, DG, and whose base BC is equal to the base EG. Then two sides AB, AC are equal to two sides DE, DF, each to each, and the angles B, E, which are opposite the equal sides AC, DF, are equal; and yet the base BC is greater than the base EF, since BC is equal to EG. The reason is that the angle C is acute, whereas DFE is obtuse. For since [Margin: e. 5. primi (Elements I.5)] the angles DFG and G are equal, and [Margin: f. 17. primi (Elements I.17)] taken together are less than two right angles, each will be less than a right angle, and therefore DFE is greater than a right angle — seeing that [Margin: g. 13. primi (Elements I.13)] the two angles at F are equal to two right angles. But if two sides AB, AC are equal to two sides DE, DG, the angle B equal to the angle E, and each of the angles C, G either less than a right angle or not less, then and only then does it follow that the bases BC, EG are equal, and so on. For if the base BC were less than EG: were AB laid upon DE, then BC would indeed coincide with EG, on account of the equality of the angles B, E, but the point C would fall short of G, say at F. Since, then, the sides DF, DG are equal — because DF is the same as AC, which is equal to DG — [Margin: h. 5. primi (Elements I.5)] the angles DFG and G will be equal, and therefore each less than a right angle, since [Margin: i. 17. primi (Elements I.17)] both together are less than two right angles. Since, then, [Margin: k. 13. primi (Elements I.13)] the two angles at F are equal to two right angles, DFE — that is, C, which does not differ from DFE — will be greater than a right angle. Therefore it is not the case that each of the angles C and G is less than a right angle; rather C is acute and G obtuse, which is contrary to the hypothesis. In the same way, if the base BC be said to be greater than the base EF (the sides AB, AC being set equal to DE, DF, the angle B equal to E, and each of the angles C and F either less than a right angle or not less), the point C will fall beyond F, say at G; and again the angle C — that is, G — will be acute, and DFE obtuse, which is contrary to the hypothesis.

[Clavius — the second excluded case]

Next, when two sides are equal to two sides, each to each, and one angle to one angle, so arranged that the one is opposite one of the equal sides and the other opposite the other, nothing can be concluded either. For let ABC be a triangle whose side AB is greater than the side AC. Then [Margin: l. 18. primi (Elements I.18)] the angle ACB will be greater than the angle ABC. Let the angle ACD be made equal to the angle ABC: then CD will fall above CB. Let there also be described, from A through B, an arc of a circle cutting CD at D, and let the straight line AD be joined. Then two sides AB, AC are equal to two sides AD, AC, each to each, and the angle ABC, opposite the side AC, is equal to the angle ACD, opposite the side AD; and yet the bases BC, DC are not equal. For if they were equal, then

both the two straight lines AB, AD and the two CB, CD would be equal, [Margin: m. 7. primi (Elements I.7)] which is absurd; or else the angles BAC, DAC would be equal, [Margin: n. 8. primi (Elements I.8)] a part equal to the whole, which is absurd.

Diagram. Figure for the second excluded comparison. Triangle ABC with A at the upper right, B at the lower centre, and D at the left; the angle ACD is made equal to angle ABC, and an arc of a circle described from A through B cuts CD at D. Lines AD, AB, AC, DB and DC are drawn.

- A (upper right apex): vertex
- D (left): point where the arc cuts CD
- B (lower centre): vertex
- C (lower right): vertex

[Clavius — the third excluded case]

Lastly, when two sides are equal to two sides, each to each, and the angle contained in the one triangle by those sides is equal to the angle in the other triangle that is opposite one of those sides, nothing can be inferred either. For let ABC be a triangle each of whose sides AC, CB is greater than the side AB, and let BC be greater than AC. Then [Margin: a. 18. primi (Elements I.18)] the angle BAC will be greater than the angle ABC. If, then, the angle ABD be made equal to the angle BAC, BD will fall below BC. Let there also be described an arc of a circle from A through C, which will fall below BC and will cut BD at D. When the straight line AD is joined, then two sides AB, AC will be equal to two sides AB, AD, each to each, and the angle BAC, contained by those sides AB, AC, will be equal to the angle ABD, which is opposite the side AD; and yet the bases BC, BD are not equal. Otherwise both the straight lines AC, AD and BC, BD would be equal, [Margin: b. 7. primi (Elements I.7)] which is absurd; or else the angles BAC, BAD would be equal, [Margin: c. 8. primi (Elements I.8)] the whole and a part, which is absurd.

Diagram. Figure for the third excluded comparison. Quadrilateral-like figure with A at the top, B at the left, C at the right, and D at the bottom; the diagonals cross, an arc of a circle is described from A through C, and the line AC is drawn dashed.

- A (top): vertex from which the arc is described
- B (left): vertex
- C (right): vertex through which the arc passes

- D (bottom): point where the arc cuts BD

[Clavius — conclusion of the scholium]

Rightly, then, did Euclid in proposition 4 prescribe that the angle contained by those sides in the one triangle must be equal to the angle of the other triangle that is likewise contained by those sides; for without this condition nothing can be concluded, as we have demonstrated.

FINIS ELEMENTI PRIMI.

[End of the First Element — Book I]

(Book I ends here. Book II does not begin on these pages: the catchword "EVCLI-" at the foot of the last page points forward to the opening of Book II — presumably its title and Definitions — on the following leaf, beyond this range.)

EUCLID'S ELEMENTS, BOOK TWO

[EVCLIDIS ELEMENTVM SECVNDVM]

DEFINITIONS

I.

[Euclid, Definition] Every rectangular parallelogram is said to be *contained* by the two straight lines that contain the right angle.

[Clavius's commentary] In this second book Euclid treats the powers of straight lines [potentiae linearum rectarum], inquiring how large are the squares of the parts of any divided straight line, and the rectangular parallelograms [contained] under the parts of that same divided line — both among themselves and compared with the square of the whole line, and so on. To carry this out conveniently, he first sets out, in two definitions, the two things most necessary for a right understanding of what is to be demonstrated.

By the first definition he explains under which straight lines any rectangular parallelogram is said to be contained, and what it is for a parallelogram to be contained under two straight lines. To understand this, one must first explain that a parallelogram is called *rectangular* when all its angles are right. Of these there are only two kinds: the square, and the oblong [altera parte longius, "longer

on one side"]. In both, all the angles are right, as is clear from their definitions. Furthermore, in any parallelogram, if only one angle is given to be right, the remaining three will necessarily be right as well. For let angle A in parallelogram ABCD be right. I say that the remaining three angles B, C, D are right too. Since AD and BC are parallel, [Margin: a. Elements I.29] the interior angles A and B are equal to two right angles; but angle A is right, by hypothesis; therefore B will be right too. And since [Margin: b. Elements I.34] each angle is equal to its opposite — angle A to angle C, and angle B to angle D — the angles C and D will be right as well.

Diagram. A tall upright rectangle illustrating Definition I, parallelogram ABCD.

- A (top-left corner)
- B (top-right corner)
- C (bottom-right corner)
- D (bottom-left corner)

Euclid therefore says that any rectangular parallelogram is contained under the two straight lines that contain one of its right angles. Thus the rectangular parallelogram ABCD is said to be contained under the two straight lines AB, AD; or under AD, DC; or under DC, CB; or finally under AB, BC — because any such pair of lines expresses the whole magnitude of the parallelogram: one of them, such as AB or DC, its length, and the other, such as AD or BC, its breadth. Hence, once the two lines containing the right angle of a rectangular parallelogram are given, its whole quantity is at once conceived and understood — namely its length and its breadth. There is this too: such a parallelogram is produced by the imagined motion of one line along the other. For if one conceives in the mind the straight line AB moving downward transversely along the straight line AD, always making a right angle with AD, until point A reaches point D and point B reaches point C, the whole parallelogram ABCD will be traced out. The same will happen if AD is supposed to move transversely along the straight line AB, and so on. For this reason a rectangular parallelogram is with full justice said to be contained under two such straight lines.

Therefore a rectangular parallelogram said to be contained under two straight lines will be the one whose two sides about a single right angle are equal to those two straight lines, each to each. Thus the rectangular parallelogram contained under the straight lines E and F will be the same as parallelogram ABCD, since side AB is equal to line E, and side AD to line F.

Diagram. Two separate horizontal line segments, drawn one above the other, labelled at their left ends. They represent the two given straight lines of the preceding paragraph.

- E (upper segment)
- F (lower segment)

It is clear from what has been said that a rectangular parallelogram contained under two *equal* lines is a square. For since each of those equal lines is equal to the line opposite it, all four sides of the rectangular parallelogram will be equal; and therefore, by the definition of a square, it will be a square.

Likewise it is evident that if two straight lines are equal to two other straight lines, each to each, the rectangular parallelogram contained under the first two is equal to the one contained under the latter two — a rectangular

parallelogram — because both the angles and the sides of the one are equal to the angles and sides of the other. This, however, can also easily be demonstrated in the following way. Let straight lines AB, BC be equal to straight lines DE, EF, each to each. I say that the rectangular parallelogram ABCG, contained under AB, BC, is equal to the rectangular parallelogram DEFH, contained under DE, EF. For when the diagonals AC, DF have been drawn: since the sides AB, BC of triangle ABC are equal to the sides DE, EF of triangle DEF, and the angles B and E are equal — namely right — [Margin: c. Elements I.4] the triangles ABC, DEF will be equal. By the same reasoning triangles AGC, DHF will be equal. Therefore the whole parallelograms ABCG, DEFH will be equal.

Diagram. Two rectangles, one above the other, each divided by a diagonal into two triangles, showing the two equal rectangular parallelograms ABCG and DEFH.

Upper rectangle (ABCG), with diagonal AC:

- A (top-left corner)
- G (top-right corner)
- B (bottom-left corner)
- C (bottom-right corner)

Lower rectangle (DEFH), with diagonal DF:

- D (top-left corner)
- H (top-right corner)
- E (bottom-left corner)

- F (bottom-right corner)

This containing of a rectangular parallelogram under the two straight lines that contain the right angle has a great affinity with the multiplication of one number by another. For just as the number 12 is produced from multiplying 3 by 4, and is arranged into the form of a parallelogram — whence it too is said to be contained under 3 and 4 — so also the parallelogram ABCD, contained under the two straight lines AB, BC (of which the former is 3 palms and the latter 4), consists of 12 square palms. These are produced by drawing line AB, of 3 palms, into line BC, of 4 palms, as the figure shows. This is well known to arithmeticians and geometers, and is demonstrated by Johannes Regiomontanus [Regiomontanus], *On Triangles* [De triangulis] I.16. Hence some say that a rectangular parallelogram is generated by drawing one of two lines about the right angle into the other —

Diagram. A lattice of small open circles (nodes) arranged in a rectangular array, illustrating the product $3 \times 4 = 12$ as an arrangement of units.

- 3 (label at left, the number of one dimension)
- 12 (label at the centre of the array, the product)
- 4 (label at the foot, the number of the other dimension)

Diagram. A rectangle ABCD ruled into a grid of unit squares (3 by 4), illustrating the same product $3 \times 4 = 12$ as an area.

- A (top-left corner)
- D (top-right corner)
- B (bottom-left corner)
- C (bottom-right corner)
- 3 (label at the left side, the length AB)
- 12 (label at the centre, the total area in square palms)

— just as the immediately preceding parallelogram is produced by drawing line AB into line BC, or (what is the same) by drawing line BC into line AB. For the same parallelogram is produced whether the smaller line is drawn into the larger or the larger into the smaller — just as the same number is produced whether the smaller number is multiplied by the larger or the larger by the smaller, as Euclid demonstrates in Book 7, Prop. 16. For this number 12 is produced from multiplying 3 by 4 just as much as from multiplying 4 by 3.

In passing, it also seems right that I warn the reader that Euclid, in this second book and in the others that follow, calls a rectangular parallelogram simply a

"rectangle" — a usage the other geometers observe as well, so that by the name "rectangle" one must always understand a rectangular parallelogram. Again, to avoid repeating the same letters so often, geometers customarily denote a parallelogram — whether rectangular or not — by just two letters, those that lie opposite each other across the diagonal. Thus they call the adjacent parallelogram AC, or BD.

Diagram. A single upright rectangle (drawn with dotted sides), illustrating the practice of naming a parallelogram by the two opposite corner-letters, AC or BD.

- A (top-left corner)
- B (top-right corner)
- D (bottom-left corner)
- C (bottom-right corner)

II.

[Euclid, Definition] In every parallelogrammic space, let each one of the parallelograms that lie about its diagonal, taken together with the two complements, be called a *Gnomon*.

[Clavius's commentary] In parallelogram ABCD — whether it be rectangular or not — let the diagonal AC be drawn; and from any point G on it let the straight lines EF, HI be drawn parallel to the sides of the parallelogram, so that the parallelogram is divided into four parallelograms, of which two, EH and IF, are said to be *about the diagonal*, while the other two, BG and GD, are the *complements*, as is clear from the last definition of Book 1. Thus the figure composed of either parallelogram about the diagonal — say IF — together with the two complements BG, GD, such as the figure EBCDHG, which the arc KLM encloses, is called a Gnomon. By the same reasoning the figure FDABIG, composed of the parallelogram EH about the diagonal together with the two complements BG, GD, will be called a Gnomon.

Diagram. The left of two figures for Definition II — a rectangle ABCD with the diagonal AC drawn, divided by the lines EF and HI (crossing at point G on the diagonal) into four parallelograms; an arc KLM marks off the gnomon.

- A (top-left corner)
- D (top-right corner)
- B (bottom-left corner)
- C (bottom-right corner)

- H (on the top edge; upper end of vertical line HI)
- I (on the bottom edge; lower end of HI)
- E (on the left edge; left end of horizontal line EF)
- F (on the right edge; right end of EF)
- G (interior point on diagonal AC, where EF and HI cross)
- K (on HI, above G)
- L (on HI / near diagonal, below G)
- M (on EF, left of G)
- KLM (the arc enclosing the gnomon)

Diagram. The right of the two figures for Definition II — the same construction shown in an oblique (non-rectangular) parallelogram ABCD, with diagonal AC, the lines EF and HI meeting at G, and the arc KLM marking the gnomon.

- A (top-left corner)
- D (top-right corner)
- B (bottom-left corner)
- C (bottom-right corner)
- H (on the top edge; upper end of HI)
- I (on the bottom edge; lower end of HI)
- E (on the left edge; left end of EF)
- F (on the right edge; right end of EF)
- G (interior point on diagonal AC, where EF and HI cross)
- K (on HI, above G)
- L (below G, near the diagonal)
- M (on EF, left of G)
- KLM (the arc enclosing the gnomon)

But let us now come to the propositions of this second book. In them it will surely be worthwhile to spend much labor understanding them precisely, because of their manifold use both in geometry and in human commerce. For from several of these propositions are demonstrated those admirable rules of Algebra, than which I scarcely believe anything more excellent can be found in the human disciplines: they draw out certain miracles of numbers (so to speak) so abstruse and hidden that the art seems to surpass every human grasp — and yet do so with such ease and delight that nothing seems easier.

From other propositions of this book, in turn, are drawn the demonstrations by which surd numbers (as they are called) are added, subtracted, multiplied, and

divided among themselves — that is, numbers that can in no way be expressed, such as the roots of non-square or non-cubic numbers, which not even by Divine power can be exhibited among numbers, since the thing implies a contradiction (as the philosophers and theologians put it). What could be more wonderful than this? For who would believe that by demonstration one can know what results from the square root of the number 8 added to the square root of the number 18 — when each root is unknown and can by no means be expressed (the former being a little less than 3, the latter a little more than 4) — and yet the sum made from the two is gathered, by force of Proposition 4 of this book, to be the square root of the number 50, which is a little more than 7?

Moreover, from Propositions 12 and 13 of this same book the area and quantity of any triangle is known most exactly; and from that knowledge, in turn, flows and derives the measurement of all magnitudes. Finally, the last proposition of this book reduces every rectilinear figure — even an irregular one, or several of them — to an equal square, with wonderful ease. So that this book truly deserves to be called golden: though very small in bulk, it contains uses that are almost infinite.

EUCLID, GEOMETRY

THEOREM 1. PROPOSITION 1.

[Euclid, enunciation] If there be two straight lines, and one of them is cut into any number of segments: the rectangle contained under those two straight lines is equal to the rectangles contained under the uncut line and each of the segments.

[Clavius's demonstration] Let there be two straight lines, A and BC, of which BC is cut in any manner into any number of segments BD, DE, EC. I say that the rectangle contained under A and BC is equal to all the rectangles taken together that are contained under the undivided line A and each segment — namely the rectangle under A and BD; likewise under A and DE; likewise under A and EC.

For let the rectangle BF be contained under A and BC — that is, let GB be equal to A. This will be so if at BC two perpendiculars BG, CF are erected equal to line A, and line FG is drawn. For BG, CF will be [Margin: a. Elements I.28] parallel, on account of the right angles B, C; and they are also [Margin: b. Axiom 1 (Common Notion 1)] equal to each other, since each is set equal to A. [Margin: c. Elements I.33] Therefore FG, BC will also be parallel and equal to each other; and consequently BF will be a rectangle, contained under A (that is, GB) and BC, by Definition 1 of this book. Next, from D and E draw the lines DH, EI parallel to BG

or CF. So DH, EI, since they are parallel to BG, will [Margin: d. Elements I.30] also be parallel to each other. Again, these same lines, since by construction BH and BI are parallelograms, will [Margin: e. Elements I.34] be equal to BG, and therefore to A.

Diagram. A rectangle BF (with top edge GF and bottom edge BC), divided by the two interior verticals DH and EI into three sub-rectangles (BH, DI, EF). Below and to the left, the separate straight line A is drawn.

- G (top-left corner)
- H (on top edge, above D)
- I (on top edge, above E)
- F (top-right corner)
- B (bottom-left corner)
- D (on bottom edge, first cut)
- E (on bottom edge, second cut)
- C (bottom-right corner)
- A (the separate straight line, drawn below the rectangle at the left)

Since, therefore, line BG is equal to line A, the rectangle BH will be contained under the cut line A and segment BD. By the same reasoning DI will be contained under A and segment DE; likewise EF under A and segment EC. Therefore, since the rectangles BH, DI, EF together are equal to the whole rectangle BF, it is clear that the rectangle contained under A and BC is equal to all the rectangles contained under A and the segments BD, DE, EC. If, then, there be two straight lines, and one of them be cut, and so on. Which was to be demonstrated.

SCHOLIUM.

Since at Book 9, Proposition 14 we shall also demonstrate in numbers the first ten theorems of this second book (which Euclid adapts to lines), provided the numbers are divided as the lines are, it will not be out of place to apply briefly to numbers what is here demonstrated at greater length concerning lines — especially since the multiplication of one number by another corresponds to the drawing of one line into another, as we said above. So, given any two numbers, say 6 and 10, let the latter be divided into three parts, 5, 3, and 2. I say that 60, the number produced from 6 times 10, is equal to the three numbers 30, 18, and 12, which are generated from the multiplication of 6 by 5, by 3, and by 2 — which is clear.

At this point Federico Commandino [Federicus Commandinus] demonstrates two other theorems, which now follow.

[THEOREM — first addition of Commandino]

[Euclid/Commandino, enunciation] If there be two straight lines, and both are cut into any number of segments: the rectangle contained under those two straight lines is equal to the rectangles contained under each segment of the one and each of the segments of the other.

[Clavius's demonstration] Let there be two straight lines AB, AC containing the right angle A, cut into the parts AD, DE, EF, FB [on AB] and AG, GH, HC [on AC]. I say that the rectangle contained under AB, AC is equal to the rectangles contained under AD, AG; AD, GH; AD, HC; DE, AG; DE, GH; DE, HC; EF, AG; EF, GH; EF, HC; FB, AG; FB, GH; FB, HC. Let the rectangle AI be completed, and draw DK, EL, FM parallel to AC (or BI); likewise HN, GO parallel to AB (or CI), which cut the former lines at P, Q, R, S, T, V. Since, therefore, the rectangle AS is contained under AD, AG; and GP under AD, GH; and HK under AD, HC — [Margin: f. Elements I.34] because the lines GS, HP are equal to AD — and likewise the rectangles DT, SQ, PL are contained under DE, AG; DE, GH; DE, HC — [Margin: g. Elements I.34] because DS, SP, PK are equal to AG, GH, HC, and ST, PQ equal to DE — and by the same reasoning the rectangles EV, TR, QM are contained under EF, AG; EF, GH; EF, HC; and also the rectangles FO, VN, RI under FB, AG; FB, GH; FB, HC — it is clear that the rectangle under AB, AC is equal to the rectangles contained under each of the parts AD, DE, EF, FB and each of the segments AG, GH, HC. Which was proposed.

Diagram. The rectangle AI (with sides AB along the bottom and AC up the left), ruled by the interior verticals DK, EL, FM and horizontals HN, GO into a grid of cells (4 columns by 3 rows). Interior cells are lettered P, Q, R (upper) and S, T, V (lower).

Corners:

- A (bottom-left)
- B (bottom-right)
- I (top-right)
- C (top-left)

Bottom edge AB (left to right): A, D, E, F, B Left edge AC (bottom to top): A, G, H, C
Top edge CI (left to right): C, K, L, M, I Right edge IB (top to bottom): I, N, O, B
Interior points/cells: P, Q, R, S, T, V

[THEOREM — second addition of Commandino]

[Euclid/Commandino, enunciation] If there be two straight lines, and both are cut in any way: the rectangle contained under those two straight lines, together with the rectangle contained under one part of the one and one part of the other, is equal to the rectangles contained mutually under the whole lines and the said parts, together with the rectangle contained under the remaining parts.

[Clavius's demonstration] Let there be two straight lines AB, AC containing the right angle A, cut in any way at D and E. I say that the rectangle contained under AB, AC, together with the rectangle contained under the parts AD, EC, is equal to the rectangles contained under AB, EC and under AC, AD, together with the rectangle contained under DB, AE. For let the rectangle AF be completed; through D draw the line DG parallel to AC (or BF); and through E draw the line EH parallel to AB (or CF), cutting DG at I. Since, therefore, the rectangle AF is equal to the rectangles EF, DH, AI: if the common rectangle EG is added, the rectangles AF, EG (namely those contained under the wholes AB, AC and the parts AD, EC) will be equal to the rectangles EF, AG, DH — contained under AB, EC; AC, AD; and DB, AE. Which was proposed.

Diagram. The rectangle AF, with D on the bottom side AB and E on the left side AC; the vertical DG and the horizontal EH cross at the interior point I, dividing the rectangle into four sub-rectangles.

- C (top-left corner)
- F (top-right corner)
- A (bottom-left corner)
- B (bottom-right corner)
- D (on bottom edge AB)
- E (on left edge AC)
- G (on top edge CF, above D)
- H (on right edge BF, level with E)
- I (interior intersection of DG and EH)

(Continuation of the numerical scholium to Proposition 1, from the preceding page.)

The same things are also clear in numbers. For suppose these two numbers 10 and 8 are proposed, of which the first is divided into three parts, 2, 3, 5, and the second into two, 3, 5: it is evident that 80, the number produced from 10 times 8, is equal to the six numbers 6, 10, 9, 15, 15, 25, which are produced from each of the parts 2, 3, 5 multiplied into any of the parts 3, 5, and which added together

make 80 [?], as Federico Commandino's [Federico Commandino] first theorem required.

Again, if the same numbers are cut in any other way — the first, say, into 3, 7, the second into 2, 6 — it is likewise plain that 80, the number produced from 10 times 8, together with 18, the number produced from 3 times 6, is equal to the three numbers 60, 24, 14, which are produced from 10 times 6, from 8 times 3, and from the remaining part 7 into the remaining part 2, since on both sides the total comes to 98, as Commandino's second theorem required.

THEOR. 2. — PROPOS. 2. [Euclid, *Elements* II.2] [Margin: ij.]

Enunciation (Euclid). If a straight line is cut at random, the rectangles contained by the whole and each of the segments are together equal to the square described on the whole.

Demonstration (Clavius). Let the straight line AB be divided at random at C into two parts. I say that the two rectangles contained by the whole AB and the segments AC, CB, taken together, are equal to the square on the whole line AB. For let AD, the square on the line AB, be described, and from C let CF be drawn parallel to AE or BD; [Margin: a — Euclid I.34] CF will be equal to AE — that is, to AB, to which AE is equal by the definition of a square. Since, then, AE is equal to AB, the rectangle AF will be the one contained by the whole AB and the segment AC. Similarly the rectangle CD will be the one contained by the whole AB and the segment CB. Therefore, since the rectangles AF, CD together are equal to the square AD, it is evident that the rectangles contained by AB and the segments AC, CB are equal to the square on the line AB. Therefore, if a straight line is cut at random, and so on. Which was to be demonstrated.

Diagram. The square on AB (the square AD), for Proposition 2. The base runs A (bottom-left) — C (bottom-middle) — B (bottom-right); the top runs E (top-left) — F (top-middle) — D (top-right). The vertical CF divides the square into two rectangles: AF (the left rectangle, on the segment AC) and CD (the right rectangle, on the segment CB).

- A (bottom-left corner)
- C (bottom edge, division point)
- B (bottom-right corner)
- E (top-left corner)

- F (top edge, foot of CF at top)
- D (top-right corner)

Otherwise (ALITER). Let the line D be taken equal to AB. Since AB is divided at C, the rectangle contained by the undivided line D and the line AB — that is, the square on AB — is [Margin: b — Euclid II.1] equal to the two rectangles contained by D and the divided line, that is, by AB and each of the segments AC, CB. Which is what was proposed.

Diagram. Two horizontal reference segments for the alternative proof: the line D (undivided) and the line AB cut at C. Legibly labelled points: D (on the one segment); B and C (on the other, divided, segment). The far endpoint of the divided segment (A) is drawn with an arrowhead terminal and is faint.

- D (undivided segment)
- B (endpoint of divided segment)
- C (division point of divided segment)
- A [?] (opposite endpoint, faint/arrowhead terminal)

SCHOLIUM [printed: SHOLIVM]

Although Euclid states this theorem for a straight line divided into just two parts in any way, the same result is proved by the same means if the line is divided into any number of parts, as these figures make clear. In numbers the same thing is seen as follows. Let the number 10 be divided into two parts, 7 and 3. I say that the numbers 70 and 30, produced by multiplying 10 by 7 and by 3, are together equal to 100, the square of the number 10 itself, as is evident. In the same way, if 10 is divided into several parts — 3, 2, 4, and 1 — the numbers 30, 20, 40, and 10, generated from 10 times 3, 2, 4, and 1, will be equal to 100, the square of the number 10. Federico Commandino demonstrates this theorem here in the same way.

Diagram. A figure illustrating the theorem for a line divided into several parts: a rectangle divided by vertical lines into a row of strips, accompanied by reference line-segments. Legibly along the top edge: G, H, I, K (with B at the top-right corner); along the base: A, C, D, then a further division point, and B at the right. Below the rectangle are two reference segments — one plain, marked L at its left end, and one divided segment carrying marks at C, D, E with an arrowhead terminal and B at the far end. Several interior division-labels are degraded by a diagonal offset-smudge across this region.

- G, H, I, K (top edge, division points of the strips)

- B (top-right and bottom-right corner)
- A (bottom-left corner)
- C, D (base, division points)
- [?] (a further base division point, obscured)
- L (left end of the plain reference segment)
- C, D, E (marks on the lower divided reference segment)

Federico Commandino demonstrates this theorem here in the same way.

[Added theorem — generalization of Proposition 2]

Enunciation. If a straight line is cut into any number of segments, the square described on the whole is equal to the rectangles contained by each of the segments taken with every one of the segments.

Demonstration. Let AB be divided into any number of parts AC, CD, DB. I say that the square described on AB is equal to the rectangles contained by each of the parts AC, CD, DB and every one of the segments AC, CD, DB — that is, to the rectangles contained by AC and AC; AC and CD; AC and DB; CD and AC; CD and CD; CD and DB; DB and AC; DB and CD; DB and DB. For let a line EF be taken equal to AB, and divided into parts EG, GH, HF equal to the parts AC, CD, DB. Then, by what we demonstrated at Proposition 1 of this book, the rectangle contained by AB and EF — that is, the square on AB — will be equal to the rectangles contained by AC and EG; AC and GH; AC and HF; CD and EG; CD and GH; CD and HF; DB and EG; DB and GH; DB and HF. Since, then, the parts AC, CD, DB are equal to the parts EG, GH, HF, the square on AB will also be equal to the rectangles contained by AC and AC; AC and CD; AC and DB; CD and AC; CD and CD; CD and DB; DB and AC; DB and CD; DB and DB. Which is what was proposed.

In numbers the same is evident. For if the number 10 is divided into 2, 3, 5, then 100, the square of the whole, will be equal to these nine numbers — 4, 6, 10, 6, 9, 15, 10, 15, 25 — which are produced from each of the parts 2, 3, 5 multiplied by every one of the parts 2, 3, 5, as is clear.

Diagram. The square on AB for the generalized theorem, AB being divided into three parts. The square is divided into a 3×3 grid of nine rectangles. Along the top edge: A, C, D, B (A at the left, C and D the interior divisions, B at the right). Below is the auxiliary line EF, equal to AB and divided at G, H: E, G, H, F.

- A, C, D, B (top edge of the square — the divisions of AB)
- E, G, H, F (the auxiliary line EF, divided at G and H)

THEOR. 3. — PROPOS. 3. [Euclid, *Elements* II.3] [Margin: iij.]

Enunciation (Euclid). If a straight line is cut at random, the rectangle contained by the whole and one of the segments is equal both to the rectangle contained by the segments and to the square described on the aforesaid segment.

[Book II continues]

Demonstration (Clavius). Let the straight line AB be divided at random at the point C. I say that the rectangle contained by the whole AB and either segment — say AC (whether this segment be the greater or the less) — is equal to the rectangle contained by the segments AC, CB together with the square on the first segment taken, AC. For let the square on the said segment AC be constructed, namely AD; and from B let BF be drawn parallel to AE, until it meets ED, produced, at F. Since, then, AE is equal to the line AC by the definition of a square, the rectangle AF will be the one contained by the whole AB and the segment AC. Again, since CD is by the same reasoning equal to AC, the rectangle CF will be the one contained by the segments AC and CB. Since, then, the rectangle AF is equal to the square AD together with the rectangle CF, it plainly follows that the rectangle contained by the whole AB and the segment AC is equal to the rectangle contained by the segments AC, CB together with the square on the aforesaid segment AC. And so, if a straight line is cut at random, and so on. Which was to be shown.

Diagram. The figure for Proposition 3. A rectangle standing on AB: the base runs A (bottom-left) — C (bottom-middle) — B (bottom-right); the top runs E (top-left) — D (top-middle) — F (top-right). The left portion AD is the square on the segment AC; BF, drawn from B up to F, completes the rectangle AF on the whole AB, while CF (through D) marks off the rectangle CF on the segment CB.

- A (bottom-left corner)
- C (bottom edge, division point)
- B (bottom-right corner)
- E (top-left corner)
- D (top edge, above C)
- F (top-right corner)

Otherwise (ALITER). Let the line D be taken equal to the segment AC. Since AB is divided at C, the rectangle contained by D and AB — that is, by AB and AC — is [Margin: a — Euclid II.1] equal to the rectangle contained by D and CB — that is, by AC, CB — together with the rectangle contained by D and AC, that is, the square on the segment AC. Which is what was proposed.

Diagram. Two horizontal reference segments for the alternative proof: the line D (equal to AC) and the line AB cut at C. Legibly labelled: D (the short upper segment); C and B on the lower, divided segment (its far endpoint drawn with an arrowhead terminal). [Margin: a — Euclid II.1]

- D (upper segment, equal to AC)
- C (division point of lower segment)
- B (endpoint of lower segment)

SCHOLIUM

To fit this theorem to numbers, let 10 be divided into 7 and 3. I say that the number 70, produced from 10 times 7, is equal to the number 21, produced from 7 times 3, together with 49, the square of the first part, 7 — as the thing itself shows. By the same reasoning, the number 30, generated from 10 times 3, will be equal to the number 21, produced from 3 times 7, together with 9, the square of the aforesaid number 3.

THEOR. 4. — PROPOS. 4. [Euclid, *Elements* II.4] [Margin: iv.]

Enunciation (Euclid). If a straight line is cut at random, the square described on the whole is equal both to the squares described on the segments and to twice the rectangle contained by the segments.

Demonstration (Clavius). Let the straight line AB be divided at random at C. I say that the square on the whole line AB is equal to the squares on the segments AC, CB together with the rectangle contained twice by the segments AC, CB. For let the square AD be described on AB, and let the diagonal BE be drawn. Then from C let CF be drawn parallel to BD, cutting the diagonal at the point G; and through G let HI in turn be drawn parallel to AB. The square AD will thus be divided into four parallelograms. Now, since in the triangle ABE the two sides AB, AE are equal, [Margin: a — Euclid I.5] the two angles ABE, AEB will be equal. But [Margin: b — Euclid I.32] the three angles ABE, AEB, BAE of the triangle ABE are equal to two

right angles, and BAE is a right angle; the remaining two, ABE and AEB, will therefore be half-right angles. By the same reasoning you will show that the angles DBE, DEB are half-right. This also follows from what we demonstrated at Proposition 34 of Book 1; for, as was shown there, the diagonal BE bisects the right angles ABD, AED. Since, then, the three angles of the triangle EFG are likewise [Margin: c — Euclid I.32] equal to two right angles, and the angle EFG is a right angle [Margin: d — Euclid I.29] (being equal, as external to internal, to the right angle D), while FEG has been shown to be half-right, the remaining angle EGF will also be half-right, and therefore equal to the angle FEG. [Margin: e — Euclid I.6] The sides EF, FG will therefore be equal; and since these are [Margin: f — Euclid I.34] equal to the opposite sides GH, HE, the parallelogram FH will be a square, all its sides being equal and all its angles right — for, one angle (namely FEH, or F) being right in the parallelogram FH, all four are right, as we demonstrated at Definition 1 of this book. By the same reasoning CI will be a square. Hence CI and FH are the squares on the segments AC, CB, since the side HG is [Margin: g — Euclid I.34] equal to AC. The rectangles AG, DG likewise will be contained by the segments AC, CB, because CG, GI are equal to CB (on account of the square CI), and FG is equal to GH (on account of the square FH, that is, [Margin: h — Euclid I.34] to AC). Consequently, since the square AD is equal to the squares CI, FH together with the rectangles AG, DG, it follows that the square AD on the whole line AB is equal to the squares on the segments AC, CB together with the rectangle contained by those same segments AC, CB taken twice. Therefore, if a straight line is cut at random, the square described on the whole, and so on. Which was to be demonstrated.

Diagram. The figure for Proposition 4. The square AD on AB, with the diagonal BE drawn from B (bottom-right) to E (top-left), and divided into four parallelograms by CF (vertical, through the base-point C) and HI (horizontal). The base runs A (bottom-left) — C (bottom-middle) — B (bottom-right); the top runs E (top-left) — F (top-middle) — D (top-right); the horizontal midline runs H (left) — G (centre, on the diagonal) — I (right). The parallelograms CI and FH (about the diagonal) are the squares on the segments; AG and DG are the two equal rectangles.

- A (bottom-left corner)
- C (bottom edge, division point)
- B (bottom-right corner)
- E (top-left corner)
- F (top edge, above C)

- D (top-right corner)
- H (left edge, midpoint)
- G (centre, on the diagonal BE)
- I (right edge, midpoint)

Otherwise (ALITER). Since AB is divided at C, the square on the whole AB is [Margin: i — Euclid II.2] equal to the rectangles contained by the whole AB and the segments AC, CB. But the rectangle contained by AB and AC is [Margin: k — Euclid II.3] equal to the rectangle contained by AC, CB together with the square on the segment AC; likewise the rectangle contained by AB and CB is equal to the rectangle contained by CB, AC together with the square on the segment CB. Therefore the square on AB is also equal to the squares on the segments AC, CB together with the rectangles contained by AC, CB and by CB, AC. Which is what was proposed.

COROLLARIUM I

Hence it is evident that the parallelograms about the diagonal of a square are squares. This follows from the earlier demonstration of this theorem, in which it was shown that the rectangles CI, FH, which lie about the diagonal BE, are squares; for in all other squares the demonstration will be the same. Yet this corollary is to be understood of those parallelograms about the diagonal of a square which have some angle in common with the whole square — of which kind are the said parallelograms CI, FH. For CI has the angle ABD in common with the square, and FH the angle AED.

The same is nonetheless true of any parallelograms about the diagonal of a square, even one produced beyond it, provided they have some angle in common with the square and their sides are parallel to its sides. For about the diagonal AC of the square BD let there stand a parallelogram, whether inside the square or outside, provided only that it has its sides parallel to the sides of the square BD. I say that FH is a square. For since AB, EF are parallel, [Margin: l — Euclid I.29] the angles BAC, FEG will be equal, internal and external; and by the same reasoning the angles BCA, FGE will be equal. But the angles BAC, BCA are half-right, as was already shown. Therefore the angles FEG, FGE will also be half-right; [Margin: m — Euclid I.6] and consequently the sides EF, FG opposite them are equal, [Margin: n — Euclid I.32] and the angle F is right. Therefore, since the sides EF, FG are [Margin: o — Euclid I.34] equal to the opposite sides GH, HE, the parallelogram FH will be equilateral. But it is also right-angled, from what we have shown; for

the single angle F has been demonstrated to be right. Therefore FH will be a square. Which was proposed.

Diagram. The figure for Corollary I: a square with its diagonal, carrying a smaller similar parallelogram (FH) set about that same diagonal. The large square is BD (with the diagonal drawn between corners A and C); the smaller parallelogram about the diagonal is EFGH. Legibly labelled points include E, F, G, H (the smaller parallelogram) and A, B, C, D (the large square). The relative positions of several corner-labels of the two nested figures could not all be resolved with certainty against the offset show-through. [Margin: l — Euclid I.29; m — Euclid I.6; n — Euclid I.32; o — Euclid I.34]

- E, F, G, H (the parallelogram FH about the diagonal)
- A, B, C, D (the large square, diagonal drawn A–C)

COROLLARIUM II

It likewise follows from the demonstration of this Proposition 4 that the diagonal of any square bisects its angles. For it was proved that the angles AEB, DEB are half-right, and so on — which we also demonstrated at Proposition 34 of Book 1.

SCHOLIUM [printed: SHOLIVM]

In numbers this fourth theorem will be exercised thus. Let the number 10 be divided at random into 7 and 3. You see, then, that 100, the square of the whole number, is equal to 49 and 9, the squares of the parts 7 and 3, together with the number 21, produced from 7 times 3, taken twice. For 49, 9, 21, and 21 make 100.

It will be very easy, moreover, to demonstrate from this fourth theorem the following further theorem.

[Added theorem — the doubling of a line and its square]

Enunciation. If one straight line is double another, the square described on the former is quadruple the square described on the latter; and if a square is quadruple another square, the side of the former is double the side of the latter.

Demonstration. Let AB be, first, double the line K. I say that the square on AB is quadruple the square on K. For if AB is bisected at C, and the construction of this fourth theorem is carried out, there will be four parallelograms AG, CI, IF, FH, which are squares and equal to one another, since all their sides are equal and all their angles right, as can easily be demonstrated from Proposition 34 of Book 1. Therefore, since the square AD is equal to the four squares AG, CI, IF, FH, the

square on the line AB will be quadruple the square on the line AC — that is, on the line K, which is equal to AC; for each of them is half of the line AB.

More briefly (BREVIVS). [Margin: p — Euclid II.4] The square on AB is equal to the squares on the lines AC, CB together with the rectangle contained twice by the lines AC, CB. Since the squares on the equal lines AC, CB are equal, and the rectangle contained by the equal lines AC, CB is itself also a square, equal to the square on AC, it follows that the square on AB is quadruple the square on AC, being equal to four squares each equal to the square on AC.

Next, let the square on AB be quadruple the square on K. I say that the side AB is double the side K. For if AB is bisected at C, so that AB is double AC, then, as has already been demonstrated, the square on AB will be quadruple the square on AC. But it was assumed also to be quadruple the square on K. Therefore the squares on the lines K and AC are equal, and so the lines K, AC are equal; but AB, by its construction, is double AC. It will therefore also be double K. All this, however, we shall demonstrate otherwise at Proposition 20 of Book 6.

Diagram. The figure for the doubling theorem: the square AD on AB, with the diagonal EB and the internal divisions CF and HI making the four equal squares AG, CI, IF, FH. The base runs A (bottom-left) — C (bottom-middle) — B (bottom-right); the top runs E (top-left) — F (top-middle) — D (top-right); the horizontal midline runs H (left) — G (centre) — I (right). At the left stands the separate reference segment K, equal to AC. [Margin: p — Euclid II.4]

- A (bottom-left corner)
- C (bottom edge, division point)
- B (bottom-right corner)
- E (top-left corner)
- F (top edge, above C)
- D (top-right corner)
- H (left edge, midpoint)
- G (centre, on the diagonal EB)
- I (right edge, midpoint)
- K (separate reference segment, equal to AC)

Furthermore, from this Proposition 4 a method can be gathered for finding a number which, together with any given number, makes a square number as well. For if 1 is subtracted from the given number, the square produced by multiplying half of the remainder by itself is the number sought. Thus, if the given number is

15, and a square number is sought which, together with the given 15, makes another square number, we subtract 1 from 15 and take half of the remaining 14, namely 7. For the square number 49, produced by multiplying that half by itself, is the number which, together with the given 15, makes up 64, a square number whose side is 8 — that is, a number greater by one unit than 7, the side of the square found. The reason for this is as follows. Since the square AD on the line AB is [Margin: q — Euclid II.4] equal to the two squares FH, CI on the parts AC, CB, together with the rectangle contained twice by AC, CB: if we set the gnomon HBF — which, together with the square FH, makes up the whole square AD — equal to 15 (that being the given number), and set the square CI equal to 1, so that its side CB is also 1, then each of the rectangles AG, DG will be 7, and hence the segment AC will be 7 (so that indeed AC times CB, which is 1, makes 7). The square FH will therefore be 49; and the whole side AB will be 8, greater by one unit than AC, the side of the square FH. This, then, is why, from the given number — that is, from the gnomon HBF — we subtract 1 and set half of the remainder as the side of the square FH that is sought.

But so that fractional numbers may be avoided, the given number should be odd, so that, once 1 is removed, the remaining number can be halved. The rule, however, is true whatever the given number may be.

We shall achieve the same thing if from the given number we take away any square number and divide half of the remainder by the side of the square taken away; the quotient will be the side of the square sought. Thus, if the number 45 is given for the gnomon HBF, we take away the square number 9 for the square CI, whose side is 3. Half of the remainder 36 is 18, for AG; this, divided by 3 — that is, by CG, the side of the square taken away, CI — gives the quotient 6, for the side GH, whose square FH, namely 36, added to the gnomon HBF (that is, to 45), makes the square number 81. But so that fractions may be avoided, the number must be so given that, when some square number is taken away, the remaining number has a half which the side of the square taken away measures exactly.

Conversely, we shall find a square number from which any given number, subtracted, leaves another square number. For if 1 is taken from the given number, and 1 is added to half of the remainder, the side of the square sought will be produced. Thus, if [again] ...

Clavius, *Opera Mathematica*, Tomus I — Commentary on Euclid's *Elements*, Book II

(continuation of the numerical scholium to Proposition 4)

again, let the given number be 15. Take away 1, and the half of the remaining number 14 will be 7; add 1 to this, and 8 will be produced, the side of the square sought. For 49 is a square number whose side is 7, which is always one unit less than the side of the square found. The ground of this rule, too, is easily drawn from the same figure of Proposition 4 of this [book]. For if we set the gnomon HBF — which, subtracted from the square AD, leaves the square FH — equal to 15 (which is of course the given number), and set the square CI equal to 1, and accordingly its side CB equal to 1 as well, then each of the rectangles AG and DG will be 7, and consequently the segment AC will be 7, as was said above, and the whole line AB will be 8, and so on.

[Margin: v.]

THEOR. 5. PROPOS. 5.

[Euclid, *Elements* II.5 — enunciation.] If a straight line is cut into equal and into unequal parts, the rectangle contained by the unequal segments of the whole, together with the square on the [segment] lying between the points of section, is equal to the square described on the half.

[Demonstration.] Let the straight line AB be divided in two at C, and unequally at D, so that the [line] between the sections is the straight line CD — the amount, that is, by which the half CB exceeds the smaller segment DB, or by which the greater segment AD exceeds the half AC. I say that the rectangle contained by the unequal segments AD and DB, together with the square on the straight line CD (which lies between the two sections), is equal to the square on the half CB.

Describe CF, the square on the half CB; and, the diagonal [diámetro] BE having been drawn, from D draw the straight line DG parallel to BF, cutting the diagonal BE at the point H, through which draw IK parallel to BC. Likewise from A draw AL parallel to CE, cutting IK produced at L. Then, by Corollary 1 of the preceding proposition, DI and KG will be squares, and therefore DH will be equal to the straight line DB; [Margin (a): *Elements* I.34] and KH is also equal to CD. Hence the rectangle AH will be contained by AD and DB, and KG will be the square on the straight line CD. What must be proved, therefore, is that the rectangle AH together with the square KG is equal to the square CF.

Now since [Margin (b): *Elements* I.43] the complements CH and FH are equal, if the common square DI is added, the parallelogram DF will be equal to the parallelogram CI. [Margin (c): *Elements* I.36] But AK is also equal to that same CI, since the bases AC and CB are equal. Therefore DF and AK will also be equal to one another; and if the common CH is added to them, the gnomon MNO will be equal to the rectangle AH. Consequently, since the gnomon MNO and the square KG are equal to the rectangle AH, the rectangle AH together with the square KG is equal to that same square CF. Therefore, if a straight line is cut into equal and unequal parts, and so on. Which was to be shown.

Diagram. The figure for Proposition 5. A large rectangle standing on the base line A–C–D–B (read left to right), with the square on the half CB built up at the right and a rectangle adjoined to its left; a semicircular arc is drawn about the central point H, and a diagonal runs from the top-left corner E to the bottom-right corner B. Labelled points:

- E (top-left corner): upper end of the diagonal EB
- G (top edge, middle): division of the top edge
- F (top-right corner)
- M (interior, upper-middle): corner-letter of the gnomon MNO
- L (left end of the middle horizontal)
- K (middle horizontal, left of centre)
- O (middle horizontal, near centre): corner-letter of the gnomon MNO
- H (centre): meeting of the diagonal, the horizontal IK, and the vertical through D
- I (right end of the middle horizontal)
- A (bottom-left corner)
- C (bottom edge, midpoint of AB)
- D (bottom edge, point of unequal section)
- B (bottom-right corner)

[The gnomon is named MNO in the text; M and O are legible on the figure, but a distinct point-letter N is not separately visible.]

SCHOLIUM.

[Scholium to Proposition 5.] Otherwise. We shall demonstrate this theorem together with Francesco Maurolico [Francesco Maurolico], in the manner in which the same result is demonstrated in numbers by the monk Barlaam [Barlaam of Seminara]; and this we shall do, together with the same Maurolico, in

the following propositions up to the eleventh exclusive. You will find these demonstrations adapted to numbers at Proposition 14 of Book 9 [*Elements* IX.14]. We shall therefore carry out our purpose as follows.

Since the square on CB [Margin (d): *Elements* II.4] is equal to the squares on CD and DB together with twice the rectangle contained by DB and CD; and since the rectangle contained by DB and CD, together with the square on DB, [Margin (e): *Elements* II.3] is equal to the rectangle contained by DB and CB — the square on CB will be equal to the remaining square on CD, together with the remaining rectangle contained by DB and CD, and the rectangle contained by DB and CB (that is, by DB and AC). But the rectangle contained by DB and AC, together with that contained by DB and CD, [Margin (f): *Elements* II.1] is equal to the rectangle contained by DB and the whole AD. Therefore the square on CB will be equal to the rectangle contained by AD and DB; that is, the rectangle contained by AD and DB, together with the square on CD, will be equal to the square on CB. Which was to be demonstrated.

Diagram. A small marginal line-diagram accompanying the scholium : a single horizontal straight line marked with points, illustrating AB bisected at C and cut unequally at D. Legible labels along the line: A (left end), C (middle), D (right of C). [The label at the right end (B) is not clearly legible: B?]

Now the same is manifest in numbers. Let the number 10 be divided equally into 5 and 5, and again unequally into 7 and 3, so that the middle number between the sections is 2 — that, namely, by which the half-number 5 exceeds the smaller part 3, and so on. You see, then, that the number 21, produced from 7 times 3, together with 4, the square of the intermediate number 2, is equal to 25, the square of the half-number 5.

From this Proposition 5, too, we shall find, in another way, a square number with which any given number will likewise make up a square number. For if 1 is added to the given number, and 1 is subtracted from the half of the composite number, the remainder will be the side of the square sought. Thus, if the given number is 15, adding 1 makes 16; and if 1 is taken from its half, 8, there remains 7, the side of the square 49 — with which the given number 15 makes up the square number 64, whose side is 8, which is always one unit greater than 7, the side of the square found. The ground of this rule is not difficult, from the demonstration of Proposition 5. For since the rectangle AH, together with the square KG, [Margin (g): *Elements* II.5] is equal to the square CF: if the rectangle AH is set equal to 15, as the given number, while the square DI is set equal to 1, and accordingly its side DB equal to 1 as well, so that 15 is produced from AD into

DB — namely the rectangle AH — then, 1 being added, the whole line AB will be 16; and if 1 is taken from its half, 8, there will remain CD, 7, which is of course the side of the square KG.

Here too, if fractional numbers are to be avoided, the given number must be odd, so that, 1 being added, it may be divided in two.

Conversely, we shall find a square number from which, if the given number be subtracted, the remainder will likewise be a square number. For if 1 is added to the given number, half the composite number will be the side of the square sought. Thus, if the given number is 15, adding 1 makes the half of the composite number 8, the side of the square 64 sought. For if you subtract the given number 15, the remaining number 49 will be a square, whose side, 7, is always one unit less than the side of the square found. The ground of this too is clear from the same figure of Proposition 5. For if the square KG (which, taken from the square CF, leaves the rectangle AH) is set equal to 15, as the given number, and the square DI set equal to 1, and accordingly its side DB equal to 1, then the whole line AB will be 16, and its half CB will be 8, and so on.

[Margin: vj.]

THEOR. 6. PROPOS. 6.

[Euclid, *Elements* II.6 — enunciation.] If a straight line is bisected and some straight line is added to it in a straight line, the rectangle contained by the whole together with the added line and by the added line, together with the square on the half, is equal to the square on the line that is composed both of the half and of the added line, as though described on a single line.

[Demonstration of Proposition 6.] Let the straight line AB be bisected at C, and let BD be added to it in a straight line. I say that the rectangle contained by the whole composite line AD and the added line DB, together with the square on the half CB, is equal to the square on the line CD, which is composed of the half CB and the added line BD.

Describe CE, the square on CD; and, the diagonal DF having been drawn, from B draw the straight line BG parallel to DE, cutting the diagonal DF at the point H, through which draw IK parallel to CD. Likewise from A draw AL parallel to CF, cutting IK produced at L. Then, by Corollary 1 of Proposition 4 of this book, BI and KG will be squares, and therefore the straight line DI will be equal to the straight line DB; [Margin (a): *Elements* I.34] and KH is also equal to the straight line CB. Hence the rectangle AI will be contained by the straight lines AD and DB, and KG

will be the square on the straight line CB. What must be proved, therefore, is that the rectangle AI together with the square KG is equal to the square CE.

Now since the parallelogram AK [Margin (b): *Elements* I.36] is equal to the parallelogram CH — the bases AC and CB being equal — [Margin (c): *Elements* I.43] and since the parallelogram HE is likewise equal to that same CH (complement to complement), AK and HE will be equal to one another. Therefore, the common CI being added, the rectangle AI will be equal to the gnomon MNO. Consequently, since the gnomon MNO and the square KG are equal to the square CE, the rectangle AI together with that same square KG will also be equal to that same square CE. And so, if a straight line is bisected and some straight line is added to it in a straight line, and so on. Which was to be demonstrated.

Diagram. The figure for Proposition 6. A large square standing on the base line A–C–B–D (read left to right), where AB is bisected at C and BD is added beyond B; the square is built on CD, with subdivisions marked; a semicircular arc is drawn about the central point H, and the diagonal runs from the top-left corner F down to the bottom-right corner D. Labelled points:

- F (top-left corner): upper end of the diagonal FD
- G (top edge, middle)
- E (top-right corner): [glyph slightly abraded, but required as the corner opposite C of the square CE]
- M (interior, upper-middle): corner-letter of the gnomon MNO
- L (left end of the middle horizontal)
- K (middle horizontal, left of centre)
- O (middle horizontal, near centre): corner-letter of the gnomon MNO
- H (centre): meeting of the diagonal, the horizontal IK, and the vertical through B
- I (right end of the middle horizontal)
- A (bottom-left corner)
- C (bottom edge, midpoint of AB)
- B (bottom edge, point of addition)
- D (bottom-right corner)

[As in the previous figure, the gnomon is named MNO; M and O are legible, but a distinct point N is not separately visible.]

SCHOLIUM.

[Scholium to Proposition 6.] Otherwise. Since the square on CD [Margin (d): *Elements* II.4] is equal to the squares on CB and BD together with twice the rectangle contained by DB and CB — that is, to the squares on CB and BD together with the rectangles contained by DB and CB and by DB and AC (taking AC once for CB, to which it is set equal) — and since the rectangles contained by DB and AC, by DB and CB, and by DB and DB (which is the square on DB) [Margin (e): *Elements* II.1] are equal to the rectangle contained by DB and the whole AD: therefore the square on CD is equal to the remaining square on CB, together with the rectangle contained by DB and AD. That is, the rectangle contained by AD and DB, together with the square on CB, is equal to the square on CD. Which was to be demonstrated.

Diagram. A small marginal line-diagram accompanying this scholium : a single horizontal straight line marked with points, illustrating the line AB with BD added. The legible labels along the line (reading as printed) are D, B, C, A. [Point-order reading uncertain owing to the small type: D?, B?, C?, A?]

Furthermore, if we are willing to consider the matter more attentively, we shall find that this Proposition 6 can most easily be demonstrated by means of the preceding Proposition 5 — a thing which that learned man, most practised in every kind of learning, Maurice Bressius [Maurice Bressius] of Grenoble, once Royal Professor of the mathematical disciplines in the University of Paris, also observed. For let the straight line AB be bisected at C, and let a straight line BD of any length be added to it. I say that the rectangle contained by the whole composite line AD and the added line BD, together with the square on the half CB, is equal to the square on the straight line CD, composed of the half and the added line.

For, added onto AD from the other side, let there be the straight line AE, equal to the added line BD; then the whole ED will be cut equally at C and unequally at B, since EA and AC are equal to DB and BC. The straight line EB will also be equal to the straight line AD, since EB is equal to DB and AB is common to both. [Margin (f): *Elements* II.5] Therefore the rectangle contained by the unequal parts EB and BD, together with the square on the intermediate section CB — that is, the rectangle contained by AD and BD, together with the square on the half CB — will be equal to the square on the half CD; that is, to the square on the line composed of the half CB of the given line AB and of the added line BD. Which is what was proposed.

Diagram. A small marginal line-diagram for the Bressius construction : a single horizontal straight line divided by tick-marks, illustrating ED cut equally at C and unequally at B, with EA added at the left. Legible labels along the line (left to right): E, A, C, B, D.

Now let the number 10 be bisected (so that we may adapt this theorem too to numbers) into 5 and 5, and let the number 2 be added to it. You see, then, that the number 24, which is produced from the whole composite number 12 into the added number 2, together with 25, the square of the half-number, is equal to 49, the square of that number 7 which is composed of the half 5 and the added 2.

Moreover, from this theorem there is gathered a remarkable property of arithmetical proportionality, which consists in the always-equal excess of proportional quantities. For since AD exceeds CD by the magnitude AC (that is, CB), and CD exceeds BD by the same magnitude BC, the three lines AD, CD, BD will have arithmetical proportionality; inasmuch as the first, AD, exceeds the second, CD, by the same excess AC (or CB) by which the second, CD, exceeds the third, BD. Hence, since it has been shown that the rectangle contained by the extremes AD and BD, together with the square on the excess CB, is equal to the square on the mean line CD, it is clearly gathered that in every arithmetical proportionality of three lines, the rectangle contained by the extremes, together with the square on the excess, is equal to the square on the mean line.

For three lines that are arithmetically proportional can always be so joined together as to make up a single line bisected in two (a line, that is, equal to the excess between the first and the third) together with the third. This is plain if three straight lines AD, CD, BD are given. For if from the first, AD, there be cut off DC, equal to the mean, and from this mean DC there be taken away the third DB, then AB (the excess between the first and the third) will be cut in two at C, with the third BD added to it. For since BC, the excess between the mean CD and the third BD, must be equal (on account of the arithmetical proportionality) to AC, namely the excess between the first AD and the mean CD, AB will necessarily be cut in two at C. And the same reasoning holds in the other cases. The same is seen in numbers that have the same excess. For in the numbers 4, 7, 10, which have the same excess 3, the number 40, produced from the extremes 4 and 10, together with 9, the square of the excess 3, is equal to the square number 49, which arises from the mean number 7.

From this Proposition 6, too, we shall find — as from Proposition 4 — a square number with which any given number will make up a square number as well. For if 1 is taken from the given number and the remaining number is bisected,

the square number of this half will be the one that is sought. Thus, if the given number is 15: 1 taken away leaves 14, whose half, 7, will give the square number 49 sought. For 15 with 49 makes the square number 64, whose side, 8, is one unit greater than 7, the side of the square found. The ground of this is manifest from this Proposition 6. For since the rectangle AI, together with the square GK, [Margin (g): *Elements* II.6] is equal to the square CE: if the rectangle AI is set equal to 15, and the square BI equal to 1, and accordingly its side BD equal to 1, then the whole line AD will be 15, since, multiplied into BD (1), it produces the rectangle AI, 15. So, 1 (BD) being taken away, there remains AB, 14, whose half, 7, will give CB, the side of the square KG sought.

It is necessary here too that the given number be odd, so that fractional numbers may be avoided — that is, so that, 1 being taken away, it may be divided in two.

Conversely, we shall find a square number from which, when the given number is subtracted, a square number likewise remains. For if 1 is taken from the given number, and 1 is again added to the half of the remaining number, the side of the square sought will be produced. Thus, if the given number is 15: 1 taken away leaves 14, the half of the remaining number 7; and, 1 being added, the side of the square sought, 8, is produced. For if from its square, 64, you subtract the given number 15, the remainder will be the square number 49, whose side, 7, is always one unit less than the side of the square found...

[printed p. 60 (recte 90)]

(continuation of the scholium to Proposition 6)

found, 8. The reason is that, if the rectangle AI (which, subtracted from the square CE, leaves the square KG) is set equal to 15, and the square BI [equal to 1], and accordingly its side BD equal to 1, then the whole line AD will be 15, as was said. So, 1 (BD) being taken away, AB will be 14; and if 1 is again added to its half CB, which is 7, the side CD, 8, will be produced, which is what is sought, and so on.

[Margin: vij.]

THEOR. 7. PROPOS. 7.

[**Euclid, *Elements* II.7 — enunciation.**] If a straight line is cut in any way, the squares on the whole and on one of the segments — whether the greater or the smaller — both together, are equal both to twice the rectangle contained by the whole and the said segment, and to the square made on the remaining segment.

[Demonstration.] Let the straight line AB be cut in any way at C. I say that the square on the whole AB, and the square on the segment (whether the greater or the smaller) AC, are equal to twice the rectangle contained by the whole AB and the said segment AC, together with the square on the remaining segment CB.

Describe upon AB the square AD; and, the diagonal BE having been drawn, from C draw the straight line CF parallel to AE, cutting the diagonal at the point G, through which draw HI parallel to AB. Then, by Corollary 1 of Proposition 4 of this book, CI and HF will be squares; and since the straight line GH [Margin (a): *Elements* I.34] is equal to the straight line AC, HF will be the square on the segment AC. Again, since AE is equal to AB, the rectangle AF will be contained by the whole AB and the segment AC. By the same reasoning the rectangle HD will be contained by the same straight lines AB and AC, since the straight lines DE and EH are equal to AB and AC, on account of the squares AD and FH. Now since the rectangles AF and FI, together with the square CI — that is, the gnomon KLM together with the square CI — are equal to the square AD, then, if the common square HF is added, the squares AD and HF will be equal to the rectangles AF and DH (each of which is contained by the whole AB and the segment AC), together with CI, the square on the remaining segment CB. Therefore, if a straight line is cut in any way, and so on. Which was to be demonstrated.

Diagram. The first (left) figure for Proposition 7. A square on the base A–C–B, subdivided by a vertical through C and a horizontal HI; the diagonal runs from the top-left corner E to the bottom-right corner B, and a semicircular arc is drawn about the central point G. Labelled points:

- E (top-left corner): upper end of the diagonal EB
- F (top edge, middle)
- D (top-right corner)
- H (left end of the middle horizontal HI)
- G (centre): meeting of the diagonal and the horizontal
- K (middle horizontal, right of centre): corner-letter of the gnomon KLM
- I (right end of the middle horizontal)
- M (interior, lower-middle): corner-letter of the gnomon KLM
- A (bottom-left corner)
- C (bottom edge, point of section)
- B (bottom-right corner)

[The gnomon is named KLM; K and M are legible, but a distinct point L is not separately visible.]

Diagram. The second (right) figure for Proposition 7, of the same construction with the section point taken so that the other segment is the greater. A square on the base A–C–B, with the diagonal from the top-left corner E to the bottom-right corner B and a semicircular arc about the central point G. Labelled points:

- E (top-left corner)
- F (top edge, middle)
- D (top-right corner)
- H (left end of the middle horizontal)
- G (centre)
- K (middle horizontal, right of centre)
- M (interior, lower-middle)
- A (bottom-left corner)
- C (bottom edge, point of section)
- B (bottom-right corner)

[A right-end horizontal label (I) is expected but not clearly legible in this second figure: I?]

SCHOLIVM.

[Scholium to Proposition 7.] Otherwise. Since the square on AB [Margin (b): *Elements* II.4] is equal to the squares on AC and CB together with twice the rectangle contained by AC and CB: if the common square on AC is added, the squares on AB and AC will be equal to twice the square on AC and the square on CB, together with twice the rectangle contained by AC and CB. But the rectangle contained by AC and CB, together with the square on AC, [Margin (c): *Elements* II.3] is equal to the rectangle contained by AB and AC; and therefore twice [that rectangle] is equal to twice the rectangle contained by AB and AC. Therefore the squares on AB and AC are equal to the remaining square on CB, together with twice the rectangle contained by AB and AC. Which was to be demonstrated.

Diagram. A small marginal line-diagram accompanying this scholium : a single horizontal straight line divided by a tick-mark, illustrating AB cut at C. The point-letters are too small to resolve individually: [labels illegible — A?, C?, B?].

In numbers, however: let the number 10 be divided in any way into 6 and 4. Then 100, the square number of the whole number 10, and 36, the square number of the part 6, are equal to the number 120 (which is made twice from the whole 10 into the part 6) together with 16, the square number of the other part 4, as is evident. So likewise 100, the square number of the whole number 10, and 16, the

square number of the part 4, are equal to the number 80 (which is made twice from the whole 10 into the part 4) together with 36, the square number of the other part 6.

EX FEDERICO COMMANDINO.

(From Federico Commandino [Federico Commandino].)

[Enunciation.] If a straight line is cut into unequal parts, the squares on those parts are equal to the rectangle contained twice by the said parts, together with the square on the line by which the greater part exceeds the smaller.

[Demonstration.] Let the straight line AB be cut into unequal parts AC and CB, and let AC be the greater part; and let a line AD be set equal to the smaller part CB, so that DC is the excess by which the part AC exceeds the part CB. I say that the squares on the parts AC and CB are equal to the rectangle contained twice by AC and CB, together with the square on the line DC.

For let the squares AE and CH be set up, and let DI be drawn parallel to CE, and let HG be produced to K. Now since AD is equal to CB, then, the common DC being added, the whole AC — that is, CE — will be equal to the whole DB. But CG is also equal to CB, since CH is a square. Therefore the remainder GE will also be equal to the remainder DC. And accordingly, since IE is also equal to DC, GE and IE will be equal; and therefore IG will be a square described on the excess DC. Now since the rectangles AI and DH are contained by the parts AC and CB (for AC is equal to each of the lines AF and DB, and CB to each of AD and BH), it is manifest that the squares AE and CH, on the parts AC and CB, are equal to the rectangles AI and DH — which are contained by the parts AC and CB — together with the square IG on the excess DC. Which is what was proposed.

Diagram. The figure for Commandino's theorem. A rectangle on the base A–D–C–B (read left to right), subdivided by verticals and a horizontal so as to display the two squares on the parts and the square on the excess. Labelled points:

- F (top-left corner)
- I (top edge, middle)
- E (top-right corner)
- K (middle horizontal, left)
- G (centre): meeting of the horizontal and the vertical through C
- H (middle horizontal, right)
- A (bottom-left corner)

- D (bottom edge, left division; AD equal to the smaller part CB)
- C (bottom edge, point of section)
- B (bottom-right corner)

SCHOLIUM.

[Scholium to Commandino's theorem.] In numbers. Let the number 10 be cut unequally into 4 and 6, so that the greater part exceeds the smaller by the number 2. You see, then, that the numbers 16 and 36, the squares of the parts — which make 52 — are equal to the number 24 (which is made from 4 into 6), taken twice, together with 4, the square of the excess 2, as the theorem required.

[Margin: viij.]

THEOR. 8. PROPOS. 8.

[Euclid, *Elements* II.8 — enunciation.] If a straight line is cut in any way, four times the rectangle contained by the whole and one of the segments, together with the square made on the remaining segment, is equal to the square described on the whole together with the said segment, as on a single line.

Book II — Proposition 8 (Theorem 8), demonstration

[Clavius's demonstration of the enunciation set out at the close of the preceding page.]

Let the straight line AB be divided at C in any way whatever. I say that the rectangle contained four times by AB and the segment CB — whether the larger or the smaller — together with the square on the remaining segment AC, is equal to the square on the line composed of the straight line AB and the said segment CB.

For let AB be produced toward the said segment CB, as far as D, and let BD be a straight line equal to the segment CB; and on the whole of AD let the square AE be described. Now, with the diagonal DF drawn, let BG and CI be drawn parallel to DE, cutting the diagonal at the points H and K; and through these points let LM and OP be drawn parallel to AD, cutting the earlier parallels at N and Q. Therefore, by the first corollary of Proposition 4 of this book, OI, NQ, BM, LG, and CP are squares about the diagonal DF.

And since OK is equal to the straight line AC [Margin (a): *Elements* I.34], OI will be the square on the segment AC. Again, since NH is equal to the straight line CB [Margin (b): *Elements* I.34], NQ will be the square on the segment CB, and

therefore equal to the square BM, since the straight lines CB and BD are equal. Hence the straight lines BH and HQ are equal to the segment CB; and so the two rectangles AH and LQ will be contained by AB and the segment CB, since LH is equal to the straight line AB [Margin (c): Elements I.34]. By the same reasoning the two rectangles NG and HE will be contained by AB and CB, since the straight lines NH and HM are equal to the straight lines CB and BD [Margin (d): Elements I.34], and the straight lines GH and EM are equal to FL — that is, to LH, that is, to AB.

And since the squares NQ and BM are equal, if the common square KG be added, BM and KG together will be equal to the rectangle NG. Therefore the five rectangles AH, LQ, HE, BM, and KG, which make up the gnomon RST, are equal to the rectangle contained four times by the straight line AB and the segment CB. Since, then, the gnomon RST together with the square OI is equal to the square AE, the rectangle contained four times by the given straight line AB and the segment CB, together with the square on the remaining segment AC, will be equal to the square on the line AD, which is composed of AB and the said segment CB. Therefore, if a straight line be divided in any way whatever, and so on. Which was to be demonstrated.

Diagram. A large square standing on the base AD, with the diagonal drawn from the top-left corner F to the bottom-right corner D, an inscribed circle, and an internal grid of parallels producing the squares and rectangles of the proof.

- A (bottom-left corner): endpoint of the divided line
- C (base, between A and B): the point of division of AB
- B (base, between C and D): the point where the smaller segment begins
- D (bottom-right corner): the endpoint to which AB is produced
- E (top-right corner): opposite corner of the square on AD
- F (top-left corner): corner from which the diagonal FD runs
- G (top edge, above B): where BG meets the top side
- O (left edge, upper): endpoint of parallel OP
- L (left edge, lower): endpoint of parallel LM
- P (right edge): endpoint of OP
- M (right edge, lower): endpoint of LM
- H, K (on the diagonal): intersections of BG, CI with the diagonal DF
- N, Q (interior): intersections of the horizontal parallels with the earlier verticals
- R, S, T (interior): the vertices marking the gnomon RST

- I (interior, right region): corner of square OI / CP [?]

Diagram. A second square on the base AD, drawn for the alternative case, again with a diagonal from the top-left corner F to the base, an inscribed circle, and the same lettering of internal parallels.

- A (bottom-left corner), C (base), B (base), D (bottom-right corner): the base as before
- F (top-left corner), I (top edge), G (top edge), E (top-right corner): the top side, left to right
- O (left edge), L (left edge): endpoints of the horizontal parallels
- P (right edge), M (right edge): the opposite endpoints
- R, T, K, Q (upper interior, along and about the diagonal): intersection points
- S (lower interior): vertex of the gnomon

Scholium

Alternatively: since the square on AD is equal to the squares on AB and BD together with twice the rectangle contained by AB and BD [Margin (e): Elements II.4] — that is, to the squares on AB and BC together with twice the rectangle contained by AB and BC — and since the squares on AB and BC are equal to twice the rectangle contained by AB and BC together with the square on AC [Margin (f): Elements II.7], the square on AD will be equal to the rectangle contained four times by AB and BC together with the square on AC. Which was to be demonstrated.

Now let the number 10 be divided in any way whatever, say into 6 and 4. Then the number 240, which is four times the product of the whole, 10, into the part, 6, together with 16 — the square of the other part, 4 — that is, the number 256, is equal to the square of the number 16, which is composed of the given number 10 and the said part 6, as is clear. In the same way, the number 160, which is four times the product of 10, the whole, into the part 4, together with 36 — the square of the other part, 6 — that is, the number 196, is equal to the square of the number 14, which is composed of 10 and 4, as is evident.

This proposition 8 can also be stated in the following way.

[Alternative enunciation:] If a straight line be cut in any way whatever, and to it there be added in a straight line another straight line equal to one of the segments, then the square on the whole composed line is equal to the rectangle contained four times by the given line and the added line — that is, the said segment — together with the square on the other segment.

For the straight line AB has been cut at C in any way whatever, and to it has been added BD equal to the segment CB; and it has been demonstrated that the square on the whole line AD is equal to the rectangle contained four times by the given line AB and the added line BD (that is, the said segment CB), together with the square on the other segment AC.

Or thus.

[A further enunciation:] If a straight line be bisected, and to it there be added in a straight line another straight line of any length whatever, then the square on the whole composed line is equal to the rectangle contained four times by the line composed of the half and the added part, and by the half, together with the square on the added part.

For the straight line DC has been bisected at B, and to it has been added CA; and it has been proved that the square on the whole AD is equal to the rectangle contained four times by AB (which is composed of BC, the half, and the added AC) and by the half CB, together with the square on the added part AC.

Diagram. A short vertical straight line marked with four points, illustrating the two alternative enunciations above.

- A (top): one end of the line
- C (below A): point of division / the half-point
- B (below C): the second point of division
- D (bottom): the far end

Moreover, from this Proposition 8 also we shall find, by another method, a square number such that any given number combined with it produces likewise a square number. For if from the fourth part of the given number 1 be subtracted, the remainder will be the side of the square sought. Thus, if the given number be 32, let 1 be taken from its fourth part, 8. The remaining number, 7, will give the square 49, with which the given number 32 makes the square 81, whose side, 9, is always greater by one unit than the fourth part of the given number. The reason for this is easy. For since the rectangle AH taken four times, together with the square OI, is equal to the square AE [Margin (g): Elements II.8], if the rectangle AH be set at 8 — namely the fourth part of the given number — while the square CH, and therefore CB its side, is 1, then the line AB will be 8, so that, multiplied into CB (=1), it makes 8. When 1 is therefore taken away, the remainder will be AC, the side of the square OI, equal to 7, which is what was sought.

But so that fractions may be avoided, it is necessary that the given number can be divided into four equal parts, so that it may be counted off by 4.

Conversely, we shall also find a square number from which the given number, subtracted, leaves a number that is likewise square. For if 1 be added to the fourth part of the given number, the side of the square sought will be produced. Thus, if the given number again be 32, and if 1 be added to its fourth part, 8, the side becomes 9; and if from the square of this, 81, the given number 32 be subtracted, there will remain the square number 49, whose side, 7, is always one unit less than the fourth part of the given number. The reason is that if the fourth part of the gnomon RST — which, taken from the square AE, leaves the square OI — that is, if the rectangle AH, which we demonstrated to be the fourth part of the gnomon RST, be set at 8, the fourth part of the given number 32, while the square BM, and therefore its side BD, is 1, then the line AB will be 8, as has been said, and so the whole side AD will be 9. From the square of this, AE, which is 81, if the gnomon RST be removed, which we assumed to be 32, the remainder will be the square OI, 49, whose side AC is 7 — inasmuch as AB was 8, and CB, that is BD, equal to it, is 1. And this side AC is perpetually one unit less than the fourth part of the given number 32, as has been said.

THEOR. 9. PROPOS. 9. [Margin: prop. viii.]

[Euclid's enunciation:] If a straight line be cut into equal and unequal parts, the squares on the unequal segments of the whole are together double both of the square on the half and of the square on the line lying between the points of section.

Book II — Proposition 9 (Theorem 9), demonstration

[Clavius's demonstration:] Let the straight line AB be bisected at C, and cut unequally at D. I say that the squares on the unequal segments AD and DB are together double of the squares that arise from the half AC and from CD, the line between the points of section. For from C let the perpendicular CE be erected to AB, equal to the half AC, or to CB, and let the straight lines EA and EB be joined. Then from D let the perpendicular DF likewise be drawn to AB, cutting EB at the point F; through F let FG be drawn parallel to AB, cutting CE at G; and finally let AF be joined.

Since, then, in the triangle ACE the sides CA and CE are equal, the angles CAE and CEA will be equal [Margin (a): Elements I.5]. But the angle ACE is a right angle; therefore the remaining angles [Margin (b): Elements I.32] will make up another

right angle, and so AEC will be half a right angle. By the same reasoning the angle BEC will be half a right angle; and on that account the whole angle AEB is a right angle. Again, since in the triangle FGE the angle EGF is equal to the right angle ECB — the exterior to the interior [Margin (c): Elements I.29] — the remaining two angles will be equal to one right angle [Margin (d): Elements I.32]. But it has been shown that the angle FEG is half a right angle. Therefore EFG too will be half a right angle, and consequently the angles EFG and FEG will be equal; and hence also the sides EG and GF are equal to one another [Margin (e): Elements I.6]. In the same way it will be shown that in the triangle BDF the sides BD and DF are equal. For the angle FDB is a right angle, and B half a right angle, and so on.

And so, since in the triangle ACE the angle C is a right angle, the square on the side AE will be equal to the two squares on the sides AC and CE [Margin (f): Elements I.47]. But these two squares are equal to one another, since the lines AC and CE are equal. Therefore the square on the side AE will be double the square on the side AC. Again, since in the triangle EGF the angle G is a right angle, the square on the side EF will be equal to the two squares on the sides EG and GF [Margin (g): Elements I.47]. But these two squares are equal to one another, because of the equality of the lines EG and GF. Therefore the square on the side EF will be double the square on the side FG — that is, the square on the line CD. For CD is a straight line equal to FG [Margin (h): Elements I.34].

Hence the two squares on the straight lines AE and EF are double the two squares on the straight lines AC and CD. But the two squares on AE and EF are equal to the square on AF [Margin (i): Elements I.47]; and the square on AF is equal to the two squares on AD and DF; therefore the two squares on AD and DF are double the two squares on AC and CD. But the square on DF is equal to the square on DB, for it has been shown that DF and DB are equal. Therefore the two squares on AD and DB, the unequal segments, are likewise double the squares on AC and CD — the half of the line and the line between the points of section. If, then, a straight line be cut into equal and unequal parts, and so on. Which was to be demonstrated.

Diagram. A right isosceles triangle standing on the base AB (extended to D), with apex E, the perpendicular CE, the perpendicular DF meeting EB at F, FG parallel to the base, and AF joined.

- A (bottom-left): endpoint of the line
- C (base, midpoint): point of bisection, foot of the perpendicular CE
- D (base, between C and B): point of unequal section, foot of the perpendicular DF

- B (base, right): the other endpoint
- E (apex, top): summit of the perpendicular CE
- F (interior, below and right of E): where DF meets EB
- G (interior, on CE): where FG meets the perpendicular CE

Scholium

Alternatively. Since the square described on the straight line AD is equal to the squares described on AC and CD, together with the rectangular parallelogram contained twice by the straight lines AC and CD [Margin (k): Elements II.4], if the common square on DB be added, the two squares on AD and DB will be equal to the three squares on AC, CD, DB, together with the rectangle taken twice under AC, CD (or under BC, CD). But the square on DB, together with twice the rectangle under BC, CD, is equal to the squares on BC (or AC) and on CD [Margin (l): Elements II.7]. Therefore the squares on AD and DB are equal to twice the squares on AC and CD; and consequently the squares on AD and DB are double the squares on AC and CD. Which was to be shown.

Alternatively, after Federico Commandino [Federico Commandino]. Since AC is equal to CB, and CB exceeds CD by the straight line DB, AC too will exceed CD by that same straight line DB [?]. Hence, as we showed at Proposition 7 of this book, the squares on the straight lines AC and CD are equal to twice the rectangle under AC, CD together with the square on the straight line DB; and consequently the squares on AC and CD, and the rectangle under AC, CD taken twice, together with the square on DB, are double the squares on AC and CD. But to the squares on AC and CD, together with the rectangle under AC, CD taken twice, the square on the straight line AD is equal [Margin (m): Elements II.4]. Therefore the squares on the straight lines AD and DB too are double the squares on AC and CD.

Now again let the number 10 be divided equally into 5 and 5, and also unequally into 7 and 3, so that the intermediate section is the number 2, as was said at Proposition 5. Then the squares of the numbers, 49 and 9, of the unequal parts 7 and 3, are double the squares 25 and 4 — of the half-number 5 and of the number 2 between the two sections — as is manifest.

And if it be pleasing, we shall also demonstrate this Proposition 9 from the construction itself, like the preceding ones, in the following way. Let the square AE be described on the whole line AB, and, with the diagonal BF drawn, let CG and DH be drawn through C and D parallel to the sides AF and BE, cutting the diagonal BF at Q and P. Since, then, by the first corollary of Proposition 4 of this book, all the rectangles about the diagonal are squares [Margin (n): Elements

I.34], and since LI is equal to AC, and QK to CD, and NK to AD, the figures NH and DO will be the squares on the unequal parts AD and DB, while LG and QP will be the squares on the bisected line AC and on CD, the intermediate section. I say that the former are double the latter.

For since the squares NH and DO exceed the squares LG and QP by the square DO and by the rectangles NI and IH — and since the square DO and the rectangles NI and IH are equal to the squares LG and QP, as will presently be demonstrated — it is clearly established that the squares NH and DO are double the squares LG and QP, since the latter are contained twice in the former. That the three rectangles DO, NI, and IH are equal to the squares LG and QP will be shown thus. The rectangle NI is equal to the rectangle QM [Margin (o): Elements I.36], because the straight lines NQ and QO are equal to the equal straight lines AC and CB [Margin (p): Elements I.34]. Therefore NI comprehends the square QP, and moreover KM. If, then, KM, DO, and IH be equal to the square LG, then DO, NI, and IH will necessarily be equal to the two squares LG and QP. That KM, DO, and IH are equal to the square LG we shall demonstrate thus. The rectangles KM and DO together are equal to PE, because the straight line BM [is equal to] CB, on account of the square CM [Margin (r): Elements I.34], and ME is equal to FL — that is, to LI, on account of the square LG, that is, to AC. When, therefore, the common IH is added, KM, DO, and IH will be equal to IE — that is, to the square LG [Margin (t): Elements I.36] — because IE and LG are equal, on account of the straight lines LI and IM being equal to the equal straight lines AC and CB [Margin (u): Elements I.34]. Which is what was proposed.

Diagram. The square AE on AB, with the diagonal from F to B, and the internal parallels CG, DH, LM, NO producing the squares LG, QP, NH, DO and the rectangles of the proof.

- F (top-left corner), G (top edge), H (top edge), E (top-right corner): the top side, left to right
- A (bottom-left corner), C (base), D (base), B (bottom-right corner): the base, left to right
- L (left edge), N (left edge): endpoints of the horizontal parallels on the left
- M (right edge), O (right edge): the opposite endpoints on the right
- I (interior, centre): intersection of the central parallels
- P (interior, upper, on the diagonal): where DH meets the diagonal
- Q (interior, lower, on the diagonal): where CG meets the diagonal
- K (interior, lower right): corner of the square QK

- [Illegible: a small mark resembling "2" appears in the lower-left interior cell — possibly a stray or rotated letter [?]]

THEOR. 10. PROPOS. 10. [Margin: prop. x.]

[Euclid's enunciation:] If a straight line be bisected, and there be added to it in a straight line any straight line whatever, then the square on the whole with the added line, and the square on the added line, both together, are double both of the square on the half and of the square on the line composed of the half and the added line, described as on a single line.

Book II — Proposition 10 (Theorem 10), demonstration

[Clavius's demonstration:] Let the straight line AB be bisected at C, and let BD be added to it in a straight line. I say that the two squares on the straight lines AD and BD together are double of the squares described on the straight lines AC and CD. For on AB, from C, let the perpendicular CE be erected, equal to the half AC or CB, and let the straight lines AE and EB be joined. Then through D let DF be drawn parallel to CE, meeting the straight line EB, produced, at G; and through E let a line EF be drawn parallel to CD, cutting DF at F; and let the straight line AG be joined.

It will now be shown that the angle AEB is a right angle, as in the preceding proposition, and that CEB is half a right angle; and hence that its alternate angle EGF is likewise half a right angle [Margin (a): Elements I.29]. But the angle F is a right angle [Margin (b): Elements I.34], since in the parallelogram CF it is opposite the right angle C. Therefore the remaining angle FGE too [Margin (c): Elements I.32] will be half a right angle, and on that account equal to EGF. Hence the straight lines EF and FG, subtending the equal angles FEG and EGF, will likewise be equal [Margin (d): Elements I.6]. By the same technique you will show that the straight lines BD and DG are equal, because the angle BDG is a right angle and BGD half a right angle, and so on.

Since, then, the square on the straight line AE is equal to the equal squares on the equal straight lines AC and CE [Margin (e): Elements I.47], the square on the straight line AE will be double the square on the straight line AC. Again, since the square on the straight line EG is equal to the equal squares on the equal straight lines EF and FG [Margin (f): Elements I.47], the square on EG too will be double the square on EF — that is, on CD, since CD is a straight line equal to EF [Margin (g): Elements I.34]. Therefore the two squares on AE and EG are double the squares described on AC and CD. But to the two squares on AE and EG the square

on the straight line AG is equal [Margin (h): Elements I.47]; and to the square on AG are equal the two squares described on the two straight lines AD and DG. Therefore the squares on AD and DG are double the squares described on AC and CD. Since, then, the square on the straight line DG is equal to the square on the straight line BD, the squares on the straight lines AD and DB too will be double the squares described on AC and CD. And so, if a straight line be bisected, and so on. Which was to be shown.

Diagram. The bisected line AB extended to D, with the perpendicular CE, the parallel DF, the line EF, and AG joined, forming the figure of the proof.

- A (bottom-left): the far endpoint
- C (base, midpoint): the point of bisection, foot of the perpendicular CE
- B (base, right): the near endpoint
- D (base, right of B): the end of the added line BD
- E (upper-left): summit of the perpendicular CE
- F (upper-right): where EF meets DF
- G (lower-right, below D): where DF, produced, meets EB produced

Scholium

Alternatively. Since the square on AD is equal to the squares on AC and CD, together with twice the rectangle under AC, CD (or under BC, CD) [Margin (i): Elements II.4], if the common square on BD be added, the two squares on AD and BD will be equal to the three squares on AC, CD, BD, together with twice the rectangle under BC, CD. But the square on BD, together with twice the rectangle under BC, CD, is equal to the squares on CD and BC — that is, the squares on AC and CD [Margin (k): Elements II.7]. Therefore the squares on AD and BD are equal to twice the squares on AC and CD; and consequently the squares on AD and BD are double the squares on AC and CD. Which was to be demonstrated.

Alternatively, after Federico Commandino. Since AC is equal to CB, and [CB] exceeds CD [?], CD too will exceed AC by the same straight line BD. Hence, as we showed at Proposition 7 of this book, the squares on the straight lines AC and CD are equal to twice the rectangle under AC, CD together with the square on the straight line BD; and consequently the squares on AC and CD, and the rectangle under AC, CD taken twice, together with the square on BD, are double the squares on the straight lines AC and CD. But to the squares on the straight lines AC and CD, together with the rectangle under AC, CD taken twice, the square on the straight line AD is equal [Margin (l): Elements II.4]. Therefore the squares on AD and BD too are double the squares on AC and CD.

Let the number 10 be bisected into 5 and 5, and to it let any number, say 3, be added, so that the whole composed number is 13. Then the squares of the numbers 169 and 9, of these numbers 13 and 3, are double the square numbers 25 and 64, which are generated from the two numbers 5 and 8, as is evident.

Diagram. A short vertical straight line marked with four points, accompanying the alternative demonstrations above.

- A (top): one end
- C (below A): the point of bisection
- B (below C): the near endpoint
- D (bottom): the end of the added line

But this Proposition 10 too will easily be demonstrated from the preceding Proposition 9, just as above Proposition 6 was demonstrated from Proposition 5. For let the straight line AB be bisected at C, and let a straight line BD of any length be added to it. I say that the two squares on the straight lines AD and BD together are double of the two squares on the straight lines AC and CD together. For if to AB, on the side of A, there be added the straight line AE, which is equal to the added BD, then the whole ED will be cut at C into two equal parts, and at B into unequal parts, because EA and AC are equal to DB and BC. The straight line EB too will be equal to AD, and AB is common to both. Therefore the squares on the straight lines EB and BD — that is, on AD and BD — will be double the squares on the straight lines EC and CB [Margin (m): Elements II.9], that is, on CD and AC. Which is what was proposed.

Diagram. A horizontal straight line marked with five points, illustrating the reduction of Proposition 10 to Proposition 9.

- E (leftmost): the endpoint of the added segment
- A (next): the original left endpoint
- C (middle): the point of bisection
- B (next): the original right endpoint
- D (rightmost): the end of the added line BD

And if it be pleasing to demonstrate this same Proposition 10 from the construction itself, as we showed the preceding Proposition 9, it will be done in the following way. Let AD be produced to E, so that DE is equal to the added line BD; and, with the square AF described on the whole line AE, let CH, BI, DK be drawn through C, B, D parallel to the sides AG, EF, cutting the drawn diagonal EG at the points L, M, N; and through these let OP, QR, ST be drawn parallel to the

sides AE, GF, cutting the earlier lines at V, X, Y, Z, a, b. Since, then, by Proposition 4 of this book, all the rectangles about the diagonal EG are squares [Margin (n): Elements I.34], and since SN is equal to AD, and OL to AC, and aN to CD, then SK will be the square on the straight line AD, and DT the square on the straight line DE (or BD), and OH and aX the squares on the straight lines AC and CD. I say that the squares SK and DT together are double of the squares OH and aX together.

For since the squares SK and DT exceed the squares OH and aX by the square DT and by the rectangles SL and LK — and since the square DT and the rectangles SL and LK are equal to the squares OH and aX, as will presently be demonstrated — it clearly follows that the squares SK and DT are double of the squares OH and aX, since the latter are contained twice in the former. That the three rectangles DT, SL, and LK are equal to the squares OH and aX will be made plain thus. The rectangle QL is equal to the square OH [Margin (o): Elements I.36], because the lines QO and OG are equal — inasmuch as QO is equal to YL, and YL to OG, on account of the equal rectangles YV and OH, whose sides OL and LV are equal [Margin (p): Elements I.34]. Likewise SY is equal to aM, and LI to YV, and VK to VZ, and DT to bZ [Margin (q): Elements I.34]. Therefore the five rectangles QL, SY, LI, VK, DT — that is, the three rectangles DT, SL, LK — are equal to the five rectangles OH, aM, YV, VZ, bZ, that is, to the two squares OH and aX. Which is what was proposed.

Diagram. The large square AF on AE, with the diagonal from G to E and a four-column grid of internal parallels producing the squares SK, DT, OH, aX and the rectangles of the proof.

- G (top-left corner), H (top edge), I (top edge), K (top edge), F (top-right corner): the top side, left to right
- A (bottom-left corner), C (base), B (base), D (base), E (bottom-right corner): the base, left to right
- O (left edge), Q (left edge), S (left edge): the division points down the left side
- P (right edge), R (right edge), T (right edge): the division points down the right side
- L, V, X (upper interior row): intersections along the parallels
- Y, M, Z (middle interior row): intersections, M on the diagonal
- a, b (lower interior row): intersections [?]
- N (lower interior, on the diagonal): where ST meets the diagonal

PROBLEMA 1. PROPOSITIO 11. [Margin: prop. xj.]

[*Euclid's enunciation:*] To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square that is made from the remaining segment.

Euclid, *Elements*, Book II — Clavius's commentary (continued)

Proposition 11 (Problem 1) — demonstration concluded

Let AB be the given straight line, which we must cut into two parts in such a way that the rectangle contained by the whole AB and one of its segments — namely the smaller — is equal to the square on the remaining segment, that is, on the larger. On AB describe a square, and, having bisected at E the side AD (which makes a right angle with the given line AB), join the straight line EB. Along EA produced take EF equal to EB, and from the straight line AB cut off AG equal to AF.

For AB is greater than AF. [Margin: a — I.20] Since EB is equal to EF by construction, and the sides AE, AB are together greater than the side EB, the straight lines EA and AB will likewise be greater than the straight line EF; and so, when the common part AE is taken away, the remainder AB will be greater than the remainder AF.

I say that AB has been cut at G in such a way that the rectangle contained by AB and BG is equal to the square on the straight line AG — so that AG is the larger segment and BG the smaller. For through G draw the straight line HI parallel to DF, cutting CD at I; and through F draw FH parallel to AG, cutting HI at H. Then the parallelogram AH will be the square on the segment AG, since all four of its sides are equal — inasmuch as [Margin: b — I.34] FH and GH are equal to their opposites AG and AF, which are equal — and all its angles are right on account of the right angle at A, as we showed at Definition 1 of this book. The rectangle CG, moreover, will be contained by AB and the segment BG, since AB is equal to BC. It must therefore be proved that the rectangle CG and the square AH are equal.

Now since the straight line DA has been bisected at E and AF added to it in a straight line, [Margin: c — II.6] the rectangle DH (since FH is equal to FA), together with the square on the half AE, is equal to the square on the straight line EF — that is, to the square on the straight line EB, which is equal to EF. But the

square on EB is [Margin: d — I.47] equal to the squares on the straight lines AE and AB. Therefore the rectangle DH, together with the square on the straight line AE, is likewise equal to the squares on the straight lines AE and AB. When the common square on AE is removed, the rectangle DH will remain equal to the square on the straight line AB — that is, to the square AC. Therefore, when in turn the common rectangle AI is taken away, the rectangle CG and the square AH will remain equal to one another. This was what was proposed. We have therefore cut the given straight line AB, and so on. Which was to be done.

Diagram. The square described on AB, drawn with its interior division for the golden-section construction. All nine point-labels were read on the leaf.

- C (top-left corner of the square)
- B (top-right corner of the square)
- I (on the left side CD, at mid-height)
- G (interior, at the crossing of the horizontal line HI with the vertical rising from A; the point of section)
- H (on the right side BF, at mid-height)
- D (bottom-left corner)
- E (on the bottom side, left of centre; midpoint of the side DA)
- A (on the bottom side, right of centre)
- F (bottom-right corner)

A horizontal line I–G–H divides the square at mid-height; a vertical line rises from A to G; a diagonal runs from E up to B. Per the demonstration, the small square A–F–H–G (lower right) is the square on the greater segment AG, and the rectangle CG (upper region) is the rectangle contained by AB and the lesser segment BG.

Scholium **

This problem cannot in any way be fitted to numbers. For no number can be divided into two numbers such that the number produced from the whole into one of the parts is equal to the square of the other part — as we shall demonstrate at Proposition 14 of Book 9, where we shall also demonstrate in numbers the ten theorems that precede this one in the present book. And more clearly, at Proposition 29 of the same Book 9, we shall show that this Problem 11 can in no way be accommodated to numbers.

The practical execution of this problem, however, is not difficult. For at the extreme point A, where the greater segment must terminate, erect on the given line AB a perpendicular AD equal to the given line AB; bisect it at E; and if, at the

interval EB, EA be cut back and produced to F, then AF will be equal to the greater segment AG, as has been demonstrated.

Theorem 11. Proposition 12 **

[Margin: xij.]

Enunciation. In obtuse-angled triangles, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by one of the sides about the obtuse angle — namely the one onto which, when produced, the perpendicular falls — and the line taken outside, cut off by the perpendicular near the obtuse angle.

Demonstration. Let the triangle ABC have the angle ABC obtuse, and from A let the perpendicular AD fall onto the side CB produced toward the part of the obtuse angle. I say that the square on the side AC, which is opposite the obtuse angle, is greater than the squares on the sides AB, BC by twice the rectangle contained by CB and BD — that is, that the square on the side AC is equal to the two squares on the sides AB, BC together with twice the rectangle contained by CB and BD.

For since the straight line CD has been divided at B in any manner whatever, [Margin: a — II.4] the square on the straight line CD will be equal to the two squares on the straight lines CB, BD together with twice the rectangle contained by CB, BD. Therefore, when the common square on the straight line AD is added, the two squares on the straight lines CD, DA will be equal to the three squares on the straight lines CB, BD, DA together with twice the rectangle contained by CB, BD. But [Margin: b — I.47] the square on the straight line AC is equal to the squares on the straight lines CD, DA. Therefore the square on the straight line AC will also be equal to the three squares on the straight lines CB, BD, DA together with twice the rectangle contained by CB, BD. And since [Margin: c — I.47] the square on the straight line AB is equal to the squares on the straight lines BD, DA, the square on the straight line AC will be equal to the squares on the straight lines CB, BA together with twice the rectangle contained by CB, BD. In obtuse-angled triangles, therefore, the square that is formed, and so on. Which was to be shown.

Diagram. The obtuse triangle for Proposition 12, with the perpendicular dropped onto the base produced. Four point-labels read on the leaf.

- A (apex, upper right)
- C (lower-left vertex)
- B (on the base, right of C; the obtuse angle ABC)

- D (foot of the perpendicular from A, on CB produced beyond B, at lower right)

The perpendicular AD is drawn from A to D; AC subtends the obtuse angle at B.

Scholium **

Here Euclid has demonstrated by how much, in an obtuse-angled triangle, the square on the side opposite the obtuse angle is greater than the squares on the other two sides. In the following Proposition 13 he will show by how much the square on the side opposite an acute angle is less than the squares on the two remaining sides — as we noted at the beginning of the scholium to Proposition 47 of Book 1.

Now since Euclid assumed that the perpendicular drawn from A falls onto the side CB produced toward the part of the obtuse angle, we shall demonstrate this point briefly. Let the angle ABC be obtuse in the triangle ABC, and let the side CB be produced toward B. I say that the perpendicular drawn down from A falls outside the triangle, onto the side CB produced — of which sort is the straight line AD. For if it were to fall inside the triangle — of which sort is the straight line AE — the two angles ABE and AEB would be greater than two right angles, since the former is obtuse and the latter right. And this is contrary to Proposition 17 of Book 1. But if it were to fall outside the triangle onto the side BC produced toward the part of C — of which sort is the straight line AF — then again, in the triangle ABF, the two angles ABF and AFB would be greater than two right angles, since the former is obtuse and the latter right. Which is absurd.

Diagram. The obtuse triangle illustrating that the perpendicular from A falls outside the triangle. Apex and five base-labels read on the leaf.

- A (apex, upper right)
- F (leftmost point of the base, beyond C)
- C (base vertex, left)
- E (on the base, between C and B; the hypothetical interior foot)
- B (base vertex; the obtuse angle)
- D (rightmost, on CB produced beyond B; the true foot of the perpendicular)

Lines are drawn from A to F, C, E, B, and D (variously dotted/dashed).

But this theorem too is true:

Enunciation (converse of Proposition 12). If in a triangle the square on one side is greater than the two squares on the two remaining sides, then the angle opposite that side will be obtuse.

Demonstration. In the triangle ABC let the square on the side AC be greater than the squares on the sides AB, BC. I say that the angle B, which the side AC subtends, is obtuse. For from B, to AB, draw the perpendicular BD equal to the line BC, and join the straight line AD. Since then the square on AD is [Margin: d — I.47] equal to the squares on AB, BD — that is, on AB, BC — while the square on AC is assumed greater than the squares on AB, BC, the square on AD will be less than the square on AC, and therefore the straight line AD is less than the straight line AC. And so, since the sides AB, BC of the triangle ABC are equal to the sides AB, BD of the triangle ABD, each to each, while the base AC is greater than the base AD, [Margin: e — I.25] the angle ABC will be greater than the angle ABD. But ABD is a right angle. Therefore ABC is greater than a right angle, and will be obtuse. This is what was proposed.

Diagram. The triangle for the converse of Proposition 12, with the auxiliary perpendicular BD.

- A (apex, upper right)
- C (lower-left vertex)
- B (on the base, between C and D — the obtuse angle; the printed glyph reads ambiguously as "E" [?], but the construction requires B here)
- D (lower right, foot of the perpendicular BD dropped from B; below A)

AD is joined; BD is the perpendicular to AB, set equal to BC.

Moreover, we shall demonstrate the converse of this Proposition 12 — namely:

Enunciation (second converse). If in a triangle the square on one side is greater than the squares on the two remaining sides by twice the rectangle contained by one of these sides and by the exterior line which the straight line dropped from the opposite angle cuts off from that side when produced: then this dropped line will be perpendicular to the produced side, and the angle of the given triangle opposite that former side will be obtuse.

Demonstration. In the triangle ABC let a straight line AD be dropped from the opposite angle A onto the side CB produced, and let the square on the side AC be greater than the squares on the sides AB, BC by twice the rectangle contained by CB, BD. I say that AD is perpendicular to CD, and that the angle ABC is obtuse. For

if AD is not the perpendicular, let a perpendicular be drawn from A to CB, which I say must of necessity fall onto CB produced toward the part of B.

First, let it fall, if it can, at B, so that AB is the perpendicular to CB. Then [Margin: f — I.47] the square on AC will be equal to the squares on AB, BC — which is absurd, since it is assumed greater. The perpendicular dropped from A onto CB therefore does not fall at the point B.

Second, let the perpendicular dropped from A fall, if it can, onto the side BC — of which sort is AF. Then [Margin: g — I.47] the square on AC will be equal to the squares on AF, FC. But the square on AC is assumed greater than the squares on AB, BC; therefore the squares on AF, FC will also be greater than the squares on AB, BC. Yet the squares on AB, BC are in fact greater than the squares on AF, FC, since the straight line AB, being opposite the right angle AFB, is [Margin: h — I.19] greater than AF, and BC as a whole is greater than its part FC. The perpendicular dropped from A therefore does not fall onto CB.

Third, let the perpendicular dropped from A onto the side BC fall, if it can, at C, so that AC is the perpendicular. Then [Margin: i — I.47] the square on AB will be equal to the squares on AC, CB, and so the square on AB will be less than the squares on AB, BC — which is absurd, since it is assumed greater. The perpendicular dropped from A onto BC therefore does not fall at C.

Fourth, let the perpendicular from A fall, if it can, onto BC produced toward the part of C — of which sort is AG. Since then [Margin: k — I.17] the two angles AGC, ACG are less than two right angles, and AGC is right, ACG will be less than a right angle, and consequently ACB obtuse. [Margin: l — I.19] The straight line AB will therefore be greater than AC, and on that account the square on AG will be less than the square on AB, and so very much less than the squares on AB, BC; yet it is assumed greater — which is absurd. Since therefore the perpendicular dropped from A onto CB falls neither at B, nor between C and B, nor at C, nor beyond C, it will fall beyond B — of which sort is AD, as will be demonstrated. [Margin: m — I.17] Hence the angle ABE will be acute and ABC obtuse. Which was the thing proposed in the second place, to be demonstrated.

That the angle ABC is obtuse, if the square on the straight line AC is greater than the squares on the straight lines AB, BC, we have demonstrated more easily in the preceding theorem of this scholium. From this it follows (as we showed at the beginning of this scholium) that the perpendicular dropped from A onto CB falls outside the triangle ABC, toward the part of the obtuse angle.

That AD is perpendicular to CB, we shall demonstrate thus. If it is not, let AE be drawn perpendicular to CB, falling beyond B, as we showed. Since then the angle ABC has been shown obtuse, the square on the straight line AC will be greater than the squares on the straight lines AB, BC by twice the rectangle contained by CB, BE. But the square on the same straight line AC is also assumed greater than the squares on the same straight lines AB, BC by twice the rectangle contained by CB, BD. Therefore the squares on AB, BC together with the rectangle contained by CB, BD will be equal to the squares on AB, BC together with twice the rectangle contained by CB, BE. And when the common squares on the straight lines AB, BC are taken away, twice the rectangle contained by CB, BE will be equal to twice the rectangle contained by CB, BD; and consequently the rectangle contained once by CB, BE will be equal to the rectangle contained once by CB, BD, and the straight line BE equal to the straight line BD — a part to the whole, or the whole to a part. Which is absurd. AD, therefore, is perpendicular to CD, and no other line is. This is what was proposed.

Diagram. The tall triangle enumerating the possible positions of the foot of the perpendicular from A. Apex and the crowded base-labels read on the leaf.

- A (apex, at top)
- G (leftmost of the base, on BC produced beyond C)
- C (base vertex, next to G)
- F (on the base, between C and B)
- B (base vertex; the obtuse angle)
- E (on CB produced beyond B)
- D (on CB produced beyond B; the true foot)
- (a further label at the extreme right of the base reads as "E" [?] — the base is very crowded at this resolution)

Lines run from A to each base-point (variously solid, dashed, and dotted).

Now, since neither this twelfth theorem nor the following thirteenth can, in every case, be explained through numbers — because, one side of a triangle being posited as any number of equal parts you please, the other sides, and the parts of them made by the perpendicular lines, for the most part cannot be expressed in numbers, but are lines incommensurable with that side (which did not happen in the preceding propositions, since, a divided straight line being posited as any number of equal parts, its parts can be set at will commensurable with it, as was done in the numerical examples employed) — we shall prescribe certain rules by which geometrical

obtuse-angled and acute-angled triangles (as many as one wishes) may be constructed of such a kind that all their sides, and the parts of them made by the perpendicular lines, are commensurable lines — so that the truth of each theorem may appear in numbers as well. Here, however, we shall treat of obtuse-angled triangles, and in the scholium to the following proposition, of acute-angled ones.

Now since an obtuse-angled triangle is either isosceles — in which the third side, which is opposite the obtuse angle, is always [Margin: m — I.9] the greater (for it cannot be equilateral, as we said at Definition 27 of Book 1) — or scalene; and since in the scalene case the perpendicular line falls either onto the smallest side produced, or onto the middle side: we shall set out three rules. In the first we shall exhibit an isosceles obtuse-angled triangle with commensurable sides, in which the segment of either of the equal sides produced, lying between the perpendicular and the obtuse angle, is also commensurable with those same sides. In the second we shall construct a scalene obtuse-angled triangle, likewise with commensurable sides, and in which the segment of the smallest side produced, lying between the perpendicular and the obtuse angle, is commensurable with those same sides. In the third, finally, we shall propose a scalene obtuse-angled triangle with commensurable sides, and in which the segment of the middle side produced, lying between the perpendicular line and the obtuse angle, is commensurable.

Rule I **

To construct an isosceles obtuse-angled triangle with commensurable sides, in which the exterior segment of one of the equal sides produced, lying between the perpendicular line and the obtuse angle, is likewise commensurable with those same sides: let the exterior segment be set as a number of equal parts such that their number is counted by 7 — as 7, or 14, or 21, or 28, and so on. Then let each of the equal sides be taken as two and four-sevenths ($2\frac{4}{7}$) of the said segment, and the greatest side, opposite the obtuse angle, as four and two-sevenths ($4\frac{2}{7}$) of the same segment — as, in the triangle ABC, let CD be drawn to BA, the side produced.

If, then, AD is set at 7 parts, and each of the sides AB, AC at 18 parts (which number you get if you double 7 and add four-sevenths of it), and the side BC at 30 parts (which number you will have if you quadruple 7 and add to it two-sevenths of it), the triangle ABC will be the one sought. For in this triangle the square on the side BC is 900, to which are equal the squares on the sides AB, AC — namely 324 and 324 — together with twice the rectangle contained by AB, AD, that is,

with 126 and 126. For these also make up the sum 900. Hence, as we have shown in this scholium, CD drawn to BD will be perpendicular, and the angle BAC obtuse.

And if you multiply each of the numbers of this triangle by any number, you will produce other lines of the triangle commensurable, yet proportional to the former ones. Thus if you double the numbers found, you will make AD 14, both AB and AC 36, and BC 60. You will also find the required triangle by setting the exterior segment AD at any number of parts, even if they are not counted from 7 — but then the sides will be whole numbers with fractions. Finally, you will hit upon the triangle by setting whatever side you wish to begin from at any number of parts, provided that the greatest is made four and two-sevenths of the segment AD, and each of the equal sides two and four-sevenths of the same segment.

Diagram. The isosceles obtuse triangle for Rule I, with numerical labels read on the leaf.

- D (apex region, top; BA produced meets the perpendicular from C at D)
- A (just below D; vertex of the obtuse angle BAC)
- B (bottom-left vertex)
- C (bottom-right vertex)
- Side AB (left): 18
- Side AC (right): 18
- Base BC: 30
- Exterior segment AD: 7

Rule II **

To produce a scalene obtuse-angled triangle with commensurable sides, in which the perpendicular falling onto the smallest side produced makes an exterior segment commensurable with the sides also: let the exterior segment be set as any number of equal parts you please, counted from 5 — as 5, or 10, or 15, or 20, and so on. To these if you add three-fifths, you will have the smallest side. But if you triple the same parts of the said segment and add one-fifth — or if you double the parts of the smallest side already found — you will make the middle side. And if, finally, you quadruple the same parts of the said segment, you will find the greatest side.

Thus, if in the triangle set before us the segment AD is set at 10, then AB will be 16, AC 32, and BC 40. Here too you see that to the square on the side BC, which is 1600, are equal the squares on the sides AB, AC — namely 256 and 1024 —

together with twice the rectangle contained by AB, AD, that is, with 160 and 160. From this it follows that CD is perpendicular to BD, and the angle BAC obtuse, as we showed above in this scholium.

Now if you multiply each of the numbers found by any number, other numbers proportional to them will be generated, which will do the same. Thus if you double them, the side AB will be 32, AC 64, and BC 80 — which numbers will indeed be found by the same art if you set the exterior segment AD at 20 parts, which have the double proportion to the former parts, 10. So too, if you triple the same numbers, you will make the exterior segment AD 30 parts, the side AB 48, AC 96, BC 120; and so on in succession.

Diagram. The scalene obtuse triangle for Rule II, with numerical labels read on the leaf.

- D (top; the smallest side produced meets the perpendicular at D)
- A (below D; vertex of the obtuse angle)
- B (bottom-left vertex)
- C (bottom-right vertex)
- Exterior segment AD: 10
- Side AB: 16
- Side AC: 32
- Base BC: 40

Rule III **

For a scalene obtuse-angled triangle with commensurable sides, in which the perpendicular line falling onto the middle side produced likewise makes a segment commensurable with the sides of the triangle: take again the exterior segment as any number of parts you please, counted from 5, as in the preceding rule. If you triple these and add one-fifth, you will produce the smallest side. But if you multiply the same by 6 and add two-fifths, you will have the middle side. And if, finally, you multiply the same by 8, you will produce the greatest side.

Thus, if in the triangle proposed the segment AD is made 5 parts, then AB will be 16, AC 32, and BC 40. And so, to the square on the side BC — namely 1600 — will be equal the squares on the sides AB, AC — namely 256 and 1024 — together with twice the rectangle contained by AC, AD, that is, with 160 and 160. And consequently BD will be perpendicular to CD, and the angle BAC obtuse, as we demonstrated above in this scholium.

And if you multiply the numbers found by any number, you will find other numbers of the sides proportional to them, in which this same Proposition 12 will be verified. Thus if you quadruple them, you will make the exterior segment AD 20 parts, the side AB 64, AC 128, and BC 160. And if you multiply those same former numbers by 10, the exterior segment AD will be 50, the side AB 160, AC 320, and BC 400. And so on to infinity.

Diagram. The elongated scalene obtuse triangle for Rule III, with numerical labels read on the leaf.

- D (upper left; the middle side produced meets the perpendicular at D)
- A (next to D; vertex of the obtuse angle)
- B (bottom-left vertex)
- C (bottom-right vertex, far to the right)
- Exterior segment AD: 5
- Side AB: 16
- Side AC: 32
- Base BC: 40

Beware, however, of supposing that, one side of an obtuse-angled triangle (or the exterior segment) being posited as any number of equal parts, the other sides must necessarily keep with it those proportions which we have set out in the foregoing rules: so that, one being known, the remaining

(Continuation of the scholium to Proposition 12, from the foot of the preceding page.)

may also be known. For this is false, since those [values] can be varied in a thousand ways and can bear one proportion after another. And so, from the three prescribed rules, the only thing to be gathered will be this: that from straight lines which preserve the said proportions an obtuse-angled triangle [triangulum amblygonium] can be constructed, together with the exterior segment, in which the truth of proposition 12 can be examined.

THEOR. 12. PROPOS. 13.

[Margin: xiiij.]

Enunciation (Euclid, Elements II.13).

In acute-angled triangles [oxygoniis triangulis], the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle — namely the one on which the perpendicular falls — and the straight line cut off within by the perpendicular toward the acute angle.

Demonstration (Clavius).

Let all the angles of triangle ABC be acute, and from A let the perpendicular AD, when dropped, fall on the side BC. I say that the square on the side AB, which is opposite the acute angle ACB, is less than the squares on the sides AC, CB about the said acute angle by twice the rectangle contained by BC, CD — that is, that the square on the side AB, together with twice the rectangle contained by BC, CD, is equal to the two squares on the sides AC, CB.

For since the straight line BC has been divided at D, in whatever manner, the squares on BC, CD [Margin: a. Elem. II.7] are equal to twice the rectangle contained by BC, CD together with the square on BD. Adding, then, the common square on DA, the three squares on BC, CD, DA will be equal to twice the rectangle contained by BC, CD together with the two squares on BD, DA. But the two squares on CD, DA are equal to [Margin: b. Elem. I.47] the square on CA. Therefore the two squares on BC, CA are equal to twice the rectangle contained by BC, CD together with the two squares on BD, DA. Since, then, the two squares on BD, DA are equal to [Margin: c. Elem. I.47] the square on AB, the two squares on BC, CA will be equal to twice the rectangle contained by BC, CD together with the square on AB — which is what was proposed.

In the same way it will be shown that the squares on AB, BC are equal to twice the rectangle contained by CB, BD together with the square on AC — that is, that the square on the side AC is less than the squares on the sides AB, BC by twice the rectangle contained by CB, BD. Therefore, in acute-angled triangles, the square on the side [subtending the acute angle], and so on. Which was to be demonstrated.

Diagram. Three acute-angled triangles set side by side, illustrating the different positions the foot of the perpendicular may take. In each, A is the apex, BC the base, and AD the perpendicular dropped from A to the base (foot D).

- A (apex of each triangle): vertex from which the perpendicular is dropped
- B (left end of base): base vertex
- C (right end of base): base vertex (the acute angle in question, ACB)

- D (on the base, foot of the perpendicular AD): in the first triangle D lies close to C; in the second, roughly midway; in the third, close to B — showing the perpendicular meeting BC at varying points

SCHOLIUM.

And so, from three propositions — namely I.47, and II.12, and II.13 of this book — we know how great the square on any side of a triangle is, compared with the squares on the other two sides: whether, that is, it is equal to them, or greater, or less, and by how much it is greater or less, according as the angle opposite the assumed side is right, or obtuse, or acute.

Although Euclid proposes this theorem only for acute-angled triangles — those, namely, that have all their angles acute — the same is nevertheless true also for right-angled and obtuse-angled triangles [in the three triangles set out in the figure]. For in these triangles the two remaining angles are necessarily acute, as can be clearly gathered from I.17 or I.32. Only this must be observed: in right-angled and obtuse-angled triangles the perpendicular must be drawn from the right or obtuse angle, but in acute-angled triangles from any angle whatever; for in this way the perpendicular will always fall inside the triangle, as Euclid assumed in the demonstration. This is easily demonstrated as follows. Let there be in triangle ABC two acute angles ABC, ACB, while the angle BAC is right, or obtuse, or acute. I say that the perpendicular dropped from A falls inside the triangle. For if it fell outside, on CB produced toward B — such as the straight line AE — then in triangle ABE the exterior angle ABC, being acute, would be [Margin: d. Elem. I.16] greater than the interior and opposite right angle AEB, which is absurd. And if it fell outside BC, on the part produced toward C, such as the straight line AF, we should fall into the same absurdity, as is manifest.

Diagram. A single triangle with apex A and a base line carrying several marked points, showing that the perpendicular from A falls within the base.

- A (apex): vertex from which lines are drawn to the base
- E (leftmost point on the base): foot of the line AE, an assumed perpendicular falling outside toward B
- B (base, right of E): base vertex
- D (base, between B and C): foot of the true perpendicular AD, falling inside
- C (base, right of D): base vertex
- F (rightmost point on the base): foot of the line AF, an assumed perpendicular falling outside toward C

Federico Commandino [Federico Commandino] demonstrates this same theorem for right-angled and obtuse-angled triangles as well, even if the perpendicular AD does not fall on the side BC but is either the same as the side AB (as happens in right-angled triangles) or falls outside the triangle (as happens in obtuse-angled ones) — just as we demonstrated in the scholium of the preceding proposition. And that comes about precisely when the perpendicular is dropped not from the right or obtuse angle, but from one of the acute angles.

The right-angled case.

Let there be a right-angled triangle ABC whose angle B is right; and from the acute angle A let the perpendicular AD be drawn to BC — which will be the same as the side AB, because of the right angle B. I say that the square on the side AB, subtending the acute angle C, is less than the squares on the sides AC, CB by twice the rectangle contained by the side CB (on which the perpendicular falls) and by the line CD, which lies between the perpendicular AD and the acute angle C. For since the squares on AB, CB [Margin: e. Elem. I.47] are equal to the square on AC, then, adding the common square on CB, there will be three squares — namely that on AB and twice that on CB — equal to the two squares on AC, CB. But the square on CB is the same as the rectangle contained by CB, CD. Therefore the square on AB too, together with twice the rectangle contained by CB, CD, is equal to the squares on AC, CB. And accordingly the square on AB is less than the squares on AC, CB by twice the rectangle contained by CB, CD. Which is what was proposed.

The obtuse-angled case.

Again, let there be an obtuse-angled triangle ABC whose angle B is obtuse; and from the acute angle A let the perpendicular AD be drawn to BC, falling outside the triangle. I say that the square on the side AB, subtending the acute angle C, is less than the squares on the sides AC, CB by twice the rectangle contained by CB and CD. For since the squares on AD, CD

Diagram. An obtuse-angled triangle drawn with its apex A at upper right; the base runs C–B–D from left to right, and the perpendicular AD is shown as a dotted line dropping from A to D, which lies beyond B (the perpendicular falls outside the triangle).

- A (upper right, apex): vertex from which the perpendicular is dropped
- C (left end of base): base vertex, the acute angle ACB
- B (base, middle): the obtuse-angle vertex

- D (base, rightmost, directly below A): foot of the perpendicular AD, outside segment CB

(Continuation of the obtuse-angled case.)

are equal to [Margin: f. Elem. I.47] the square on AC, then, adding the common square on CB, the three squares on AD, CD, CB will be equal to the two squares on AC, CB. But the square on CD is equal to [Margin: g. Elem. II.4] the squares on CB, BD together with twice the rectangle contained by CB, BD. Therefore the two squares on AD, CB, together with the squares on CB, BD and twice the rectangle contained by CB, BD, are equal to the squares on AC, CB. But the squares on AD, BD are equal to [Margin: h. Elem. I.47] the square on AB. Therefore the square on AB too, and twice the square on GB [recte: CB], together with twice the rectangle contained by CB, BD, are equal to the squares on AC, CB. But the square on CB, together with the rectangle contained by CB, BD, is equal to [Margin: i. Elem. II.3] the rectangle contained by CD, CB. And therefore twice the square on CB, together with twice the rectangle contained by CB, BD, is equal to twice the rectangle contained by CD, CB. Therefore the square on AB too, together with twice the rectangle contained by CB, CD, is equal to the squares on AC, CB. And accordingly the square on AB is less than the squares on AC, CB by twice the rectangle contained by CB, CD.

Another proof, and shorter.

Since the square on AC is [Margin: k. Elem. II.12] greater than the squares on AB, BC by twice the rectangle contained by CB, BD — that is, the square on AC is equal to the squares on AB, BC together with twice the rectangle contained by CB, BD — the square on AB will be less than the square on AC by the square on BC and twice the rectangle contained by CB, BD. And accordingly, if to the square on AC the square on BC be added, the square on AB will be less than the squares on AC, BC by twice the square on BC and twice the rectangle contained by CB, BD. Since, then, the square on BC together with the rectangle contained by CB, BD is equal to [Margin: l. Elem. II.3] the rectangle contained by CB, CD, the square on AB too will be less than the squares on AC, CB by twice the rectangle contained by CB, CD. Which is what was proposed.

In a certain ancient scholium the following theorem is demonstrated, after the manner of the one that we demonstrated in the scholium of the preceding proposition in the second place. Namely:

Converse theorem.

If in a triangle the square on one side is less than the two squares on the two remaining sides, the angle opposite that side will be acute.

Demonstration.

In triangle ABC, let the square on the side AB be less than the squares on the sides AC, CB. I say that the angle C, which the said side subtends, is acute. For let there be drawn from C, perpendicular to AC, the line CD equal to the line CB, and let the straight line AD be joined. Since, then, the square on AD is equal to [Margin: m. Elem. I.47] the squares on AC, CD — that is, on AC, CB — while the square on AB is assumed less than the squares on AC, CB, the square on AD will be greater than the square on AB; and therefore the straight line AD is greater than the straight line AB. And so, since the two sides AC, CD of triangle ACD are equal to the two sides AC, CB of triangle ACB, each to each, and the base AD is greater than the base AB, the angle ACD will be [Margin: n. Elem. I.25] greater than the angle ACB. But ACD is right, by construction. Therefore ACB is less than a right angle, and will be acute.

Diagram. Three small triangles set side by side, each showing the construction for the converse theorem: triangle ACB (base C on the left, B on the right, apex A) with the right-angled construction line CD erected at C equal to CB.

- A (apex): vertex opposite the base CB
- B (right end of base): base vertex
- C (left end of base): the angle under test (ACB)
- D (near the apex region, top): endpoint of the perpendicular CD ($CD \perp CA$, and $CD = CB$); its position relative to A differs across the three copies

Converse theorem (second).

But we shall also demonstrate the converse of this proposition 13, namely:

If in a triangle the square on one side is less than the squares on the remaining two sides by twice the rectangle contained by one of these two sides and by the straight line lying between the angle opposite the former side and the straight line dropped from that angle onto the other of those sides — then this dropped line will be perpendicular to that other side, and the angle of the proposed triangle opposite the former side will be acute.

Demonstration.

In triangle ABC, onto the side BC let there be dropped, from the opposite angle A, the straight line AD; and let the square on the side AB be less than the squares on

the sides AC, CB by twice the rectangle contained by BC, CD. I say that AD is perpendicular to BC, and that the angle ACB is acute. For if AD is not perpendicular, let a perpendicular be drawn from A to BC — which, I say, must necessarily fall short of the point C, toward B; that is, either on the side BC, or at the point B, or on the side BC produced toward B. For let it fall, if it can, at C, so that AC is perpendicular to BC. Then [Margin: o. Elem. I.47] the square on AB will be equal to the squares on AC, CB. Which is absurd, since it is assumed less. Therefore the perpendicular dropped from A to BC does not fall at the point C.

Next, let the perpendicular from A fall, if it can, beyond C, at E — such as AE. Then [Margin: p. Elem. I.47] the square

Diagram. Three tall triangles stacked one above the other, each with apex A and a horizontal base crowded with marked points; solid and dotted lines run from A to these points, illustrating the successive cases of the reductio (the perpendicular assumed to fall at various feet). The base points are densely packed and their exact left-to-right order in each figure is hard to fix at this resolution.

- A (apex of each figure): vertex from which the lines descend
- D (on the base): foot of the true perpendicular AD
- E (on the base, toward the right end): foot of the assumed perpendicular AE falling beyond C
- F (repeated at several base positions): feet of the trial perpendiculars AF in the various cases
- B, C (base vertices of the triangle)
- Base-point sequence, first figure (crowded): F, B, F, D, F, C, E [?]; second figure: F, B, F, C, E [?]; third figure: F, D, F, B, F, C, E [?] — exact ordering uncertain

(Continuation of the second converse demonstration.)

on AB will be equal to the squares on AE, EB. But the square on AB is assumed less than the squares on AC, CB. Therefore the squares on AE, EB too will be less than the squares on AC, CB. Since, then, the square on AC is equal to [Margin: q. Elem. I.47] the squares on AE, EC, the squares on AE, EB will also be less than the squares on AE, EC, CB. And, the common square on AE being taken away, the remaining square on EB will be less than the squares on EC, CB. Which is absurd. For since the square on EB is equal to [Margin: r. Elem. II.4] the squares on EC, CB together with twice the rectangle contained by EC, CB, the square on EB will be greater than the squares on EC, CB. Therefore the perpendicular does not fall beyond C. But neither does it fall at C, as we have shown. Therefore it will fall

short of the point C — such as AD, as will be demonstrated; and on that account [Margin: s. Elem. I.17] the angle ACB will be acute. Which is what was proposed to be demonstrated in the second place.

For the rest — that the angle ACB is acute, if the square on AB is less than the squares on AC, CB — we demonstrated more easily in the immediately preceding theorem of this scholium. From which it follows (as was shown in Corollary 2 of I.17) that the perpendicular dropped from A to BC falls short of the acute angle ACB, toward B.

That AD, moreover, is perpendicular to BC, we shall show thus. If it is not, let AF be the perpendicular to BC, falling short of the point C (as has been proved), wherever this may happen — whether inside the triangle, or at the point B, or outside the triangle. (Only in the second triangle will the adversary not claim that the perpendicular falls at B, since it would then be the same as AD, which he denies.) Since, therefore, the angle ACB has been shown acute, [Margin: t. Elem. II.13] the square on AB will be less than the squares on AC, CB by twice the rectangle contained by BC, CF — that is, the square on AB, together with twice the rectangle contained by BC, CF, is equal to the squares on AC, CB. But the same square on AB is assumed to be less than the squares on AC, CB also by twice the rectangle contained by BC, CD — that is, the same square on AB, together with twice the rectangle contained by BC, CD, is assumed equal to the squares on AC, CB. Therefore the square on AB, together with twice the rectangle contained by BC, CD, will be equal to the same square on AB together with twice the rectangle contained by BC, CF. And, the common square on AB being taken away, twice the rectangle contained by BC, CD will be equal to twice the rectangle contained by BC, CF; and accordingly the rectangle contained once by BC, CD will be equal to the rectangle contained once by BC, CF, and therefore the lines CD, CF will be equal — the part and the whole. Which is absurd. Therefore AD is perpendicular to BC, and no other line. Which is what was proposed.

Now, just as in the scholium of the earlier proposition we constructed obtuse-angled triangles of commensurable sides and lines, in which the truth of proposition 12 could be examined by numbers, so in this place we shall set up acute-angled triangles of commensurable sides and lines, in which the truth of this proposition 13 may be explained. And since an acute-angled triangle is either equilateral, or isosceles, or scalene, we shall embrace this whole doctrine in seven rules. The first will treat of the acute-angled equilateral triangle. The second, of the isosceles in which the perpendicular falls on the unequal third side, whether that side be greater or less than either of the equal sides. The third,

of the isosceles whose third side is greater, and the perpendicular falls on one or other of the equal sides. The fourth, of the isosceles whose third side is less, and the perpendicular again falls on one or other of the equal sides. The fifth, of the scalene in which the perpendicular falls on the smallest side. The sixth, of the scalene in which the perpendicular is dropped on the middle side. The seventh, finally, of the scalene in which the perpendicular is drawn to the greatest side. I am speaking here also of those triangles in which the perpendicular line falls inside the triangle — and accordingly the two angles at the base are acute — whether the opposite angle is acute as well or not. For the proposition is to be understood of these too, as we have said. But how triangles in which the perpendicular line falls either outside them, or coincides with one of the sides, may likewise be fitted to numbers, we shall teach at the end of the rules.

REGVLA I.

If any side of an equilateral triangle be set at whatever number of equal parts — an even number, however, so that fractions be avoided — all the sides will be commensurable both with one another and with the segments made by the perpendicular. Thus in the equilateral triangle ABC the perpendicular AD divides the opposite side BC into two equal parts, as we demonstrated in the scholium of I.26. Wherefore, any side being set at 8, the segments BD, CD will be 4 and 4. Here you see that the square on the side AB, opposite the acute angle C (for all the angles in an equilateral triangle are acute, by Corollary 3 of I.17) — that is, 64 — together with twice the rectangle contained by BC, CD, that is, with 32 and 32, makes 128: which is precisely what the two squares on the sides AC, CB make, namely 64 and 64.

Diagram. An equilateral triangle ABC with apex A, base B–D–C, and the perpendicular AD dropped to the base, annotated with the numerical values used in the rule.

- A (apex): vertex from which the perpendicular AD is dropped
- B (left base vertex)
- D (foot of the perpendicular, midpoint of the base)
- C (right base vertex, the acute angle C)
- Numerical labels: "8" on side AB (left), "8" on side AC (right); "4" and "4" on the two base segments BD and DC; "8" beneath the base BC

REGVLA II.

If each of the equal sides of an isosceles triangle be set at any number of equal parts, and the third side at any even number of parts (so that fractions be avoided) — whether fewer parts be placed in this side than in either of the equal ones, or more, provided only they be not as many as in both together, or more; for so a triangle could not be made, since the two equal sides would not be greater than the third side, but either equal to it or less, which contradicts I.20 — if, I say, the sides be expressed in numbers in this way, all the lines of the triangle will be commensurable with one another. Thus, in the first of these isosceles triangles, let each of the sides AB, AC be set at 7, and BC at 4; in the second, let each of AB, AC be 5, and BC 6. Since, then, the perpendicular AD divides the base BC into two equal parts, by the scholium of I.26, the segments BD, CD will be, in the first triangle, 2 and 2, and in the second, 3 and 3. Here too it is clear that the square on the side AB, opposite the acute angle C (for each of the angles B, C is acute, by Corollary 3 of I.17), together with twice the rectangle contained by BC, CD, is equal to the squares on the sides AC, CB together, and so forth.

Diagram. Two isosceles triangles side by side, each with apex A, base B–D–C, and the perpendicular AD, annotated with numerical values.

- First (left) triangle: A (apex), B, D (foot of perpendicular), C; equal sides labelled "7" and "7"; base segments "2" and "2"; base BC labelled "4"
- Second (right) triangle: A (apex), B, D (foot of perpendicular), C; equal sides labelled "5" and "5"; base segments "3" and "3"; base BC labelled "6"

REGVLA III.

In an isosceles triangle in which the third side is greater, and the perpendicular falls on one of the equal sides, the segment near the greater third side will be the greater, as we demonstrated at I.47. If, therefore, you set the smaller segment at any number of parts and multiply them by 8, you will have the greater segment; but if you multiply the same by 12, you will produce the third side; and finally the two segments added together will make up each of the equal sides — which you will also produce from the multiplication of the smaller

Euclid's *Elements*, Book II — Clavius's commentary (continued)

Scholium to Proposition 13 — numeric rules (continued)

Rule III (conclusion)

the segment by 9.

For instance, if in the isosceles triangle ABC the smaller segment BD is set at 4, the larger segment CD will be 32, the side AC will be 48, and each of the two sides AB and BC will be 36. Now the square of side AB, opposite the acute angle C — namely 1296 — together with twice the product of BC and CD, that is 1152 and 1152, makes 3600; and this is exactly what the two squares on AC and CB produce, namely 2304 and 1296. In the same way the square on AC, namely 2304, together with twice the product of CB and BD, that is 144 and 144, makes 2592; and this number is likewise produced by the squares of the sides AB and BC, namely 1296 and 1296. In a triangle of this kind, therefore, the unequal segments stand in the ratio of eight to one; either of the equal sides to the smaller segment stands in the ratio of nine to one; and finally the third, greater side to that same smaller segment stands in the ratio of twelve to one.

Diagram. Isosceles triangle for Rule III, apex A above, base BC running left to right with D the foot of the perpendicular AD (drawn as a dashed line from A).

- A (apex): vertex
- B (base, left): vertex
- D (on BC): foot of the perpendicular from A
- C (base, right): vertex
- Segment BD: "4"
- Segment DC: "32"
- Side AC (right edge): "48"
- Side AB (left edge): "36" [reads as stacked "3 / 6"]
- Base BC: "36" [a second numeral near the base foot is faint — possibly a repeated "36"; the value 36 is confirmed by the text]

Rule IV

[Margin: (rule number IV)]

In an isosceles triangle whose third side is the smaller and onto which the perpendicular again falls on one of the equal sides, the segment nearer to the smaller side will be the third [part], smaller, as we have demonstrated at Prop. 47 of Book I. If you set this segment at any even number of parts, multiply them by 3 and add to the product half of the same (or multiply the same parts by 7), you will produce the greater segment. But if you multiply the same parts of the smaller segment by 3, you will make the third side, the smaller. Finally, each of the equal sides is composed of the two segments, which is also had by multiplying half of the smaller segment by 9. Thus, if in the isosceles triangle ABC the smaller segment CD is set at 6, the larger BD will be 21, and each side — both AB and BC — will be 27. The square of side AB, therefore, opposite the acute angle C, namely [729], together with twice the product of BC and CD, that is 162 and 162, makes 1053; and this number is also composed of the squares of the sides [AC and AB], namely 324 and 729.

Diagram. Isosceles triangle for Rule IV, apex A, base BC with D the foot of the perpendicular AD.

- A (apex): vertex
- B (base, left): vertex
- D (on BC): foot of the perpendicular
- C (base, right): vertex
- Segment BD: "21"
- Segment DC: "6"
- Side AB (left edge): "27"
- Base BC: "27"
- Side AC (right edge): "18" [glyph small; value confirmed by the arithmetic $324 = 18^2$]

Rule V

[Margin: (rule number V)]

In a scalene triangle in which the perpendicular is let fall onto the smallest side, the segment [nearer that side] will be the smaller, as was demonstrated at Prop. 47 of Book I. If this segment is set at some number of equal parts counted by 70 — that is, 70, or 140, or 210, and so on — and there are added to them $\frac{29}{70}$ of them, the greater segment will be produced; and from the addition of the parts of these segments the [smallest] side [is produced]. But if the parts of the smaller segment are doubled and there is added to the product $\frac{44}{70}$ of the parts, the middle side

will be generated. And finally, if $\frac{55}{70}$ — that is $\frac{11}{14}$ — of those same parts of the smaller segment are added, the greatest side will be put together. Thus in the scalene triangle ABC, if the smaller segment BD is set at 70, the greater CD will be 99, and the whole smallest side BC will be 169. The middle side AB will be 182, and the greatest AC 195. The square of side AB, therefore, opposite the acute angle C — that is, 33124 — together with the number produced by twice the product of BC and CD, that is 16731 and 16731, makes the number [66586], equal to the two squares taken together, 38025 and 28561, of the sides AC and CB, and so on.

Diagram. Scalene triangle for Rule V, apex A, base BC with D the foot of the perpendicular AD.

- A (apex): vertex
- B (base, left): vertex
- D (on BC): foot of the perpendicular
- C (base, right): vertex
- Side AB (left edge): "182"
- Side AC (right edge): "195"
- Segment BD: "70"
- Segment DC: "99"
- Base BC: "169"

Rule VI

[Margin: (rule number VI)]

In a scalene triangle, when the perpendicular falls onto the middle side, the smaller segment will be the one adjacent to the smallest side — as we showed from Prop. 47 of Book I. If this segment is set at some number of equal parts counted by 5 — as 5, or 10, or 15, and so on — and there are added $\frac{4}{5}$ [?], the greater segment will be made; and the parts of these segments, gathered into one sum, will compose the middle side. But if to the doubled parts of the smaller segment there are added $\frac{2}{5}$ [?] of the same, the smallest side will be produced. Finally, the triple of those same parts of the smaller segment will give the greatest side. Thus in the scalene triangle ABC, if the smaller segment BD is made 5, the greater segment CD will be 9, and the whole side BC — the middle — [14]. The smallest side AB will be 13, and the greatest AC 15, and so on.

Diagram. Scalene triangle for Rule VI, apex A, base BC with D the foot of the perpendicular AD (dashed vertical).

- A (apex): vertex
- B (base, left): vertex
- D (on BC): foot of the perpendicular
- C (base, right): vertex
- Side AB (left edge): "13"
- Perpendicular AD: "12"
- Side AC (right edge): "15"
- Segment BD: "5"
- Segment DC: "9" [glyph reads as "3" on the leaf; value 9 confirmed by the text]
- Base BC: "14"

Rule VII

[Margin: (rule number VII)]

In a scalene triangle, finally, where the perpendicular is drawn to the greatest side, the segment adjacent to the smallest side will be the one we wrote of at Prop. 47 of Book I. If, then, one takes for the smaller segment 33, or any other number counted by 33 — as 66 or 99 — and adds $\frac{2}{33}$, that is $\frac{3}{11}$, the greater segment will be made; and the two segments composed together will yield the greatest side. But if to the doubled number of the smaller segment there are added $\frac{4}{33}$, the middle side will be produced. And finally, if to the same smaller segment there are joined $\frac{32}{33}$, the smallest side will be exhibited. Thus, if in the scalene triangle ABC the smaller segment CD is 33, the greater segment BD will be 42, the whole greatest side BC 75, the middle side AB 70, and the smallest AC 65, and so on.

Diagram. Large scalene triangle for Rule VII, apex A, base BC with D the foot of the perpendicular AD (dashed).

- A (apex): vertex
- B (base, left): vertex
- D (on BC): foot of the perpendicular
- C (base, right): vertex
- Side AB (left edge): "70"
- Side AC (right edge): "65"
- Segment BD: "42"
- Segment DC: "33"

- A dimension mark below D reads "5" [?] — unclear; the base total $BC = 75$ is confirmed by the text

Concluding remarks of the scholium

If you multiply the several numbers of these rules by one and the same number, whatever it may be, you will generate other numbers to be assigned to the sides of triangles. The same numbers will also be found if the smaller segment, or the greater, or [any other] be [multiplied], provided the proportions are kept that we have set out in the rules given above.

Here, too, one must understand that the prescribed proportions are not to be found among the sides in every acute-angled triangle, since their proportions can be varied in a thousand ways. Nor indeed do those rules teach this; their use consists in this alone: that, keeping the proportions we have explained, acute-angled triangles can be formed in which Proposition 13 of this Book II is fitted to numbers.

Now if the perpendicular coincides with that side whose square is to be proved less than the two squares of the other two sides, and so on — so that the proposed triangle is right-angled — three numbers must be sought for the sides, from those we wrote of in the scholium to Prop. 47 of Book I. Thus, in the figure of Rule 6, if in the triangle ABD the smallest side BD is set at 5, AD will be 12 and AB 13. Here you see that the square of side AD, opposite the acute angle B, namely 144, together with twice the product of BD and

BD, that is 25 and 25, makes 194 — which is exactly what the two squares on AB and BD together produce, namely 169 and 25. By the same reasoning, if the smallest side BD is set at 6 parts, AD will be 8 and AB 10. Here too you see that the square of side AD, subtended by the acute angle B, namely 64, together with twice the product of BD and BD, that is 36 and 36, makes 136 — a number that the two square numbers of the sides AB and BD, namely 100 and 36, also produce. So likewise you see that the square number of side BD, opposite the acute angle A, that is 36, together with twice the product of AD and AD, that is 64 and 64, makes 164 — as much as the two square numbers of the sides AB and AD together produce, namely 100 and 64.

Finally, if in another triangle ACD the smallest side CD is set at 9, you will find from what we wrote at Prop. 47 of Book I that AD is 40 and AC is 41. Here it is manifest that the square number of side AD, opposite the acute angle [C], namely 1600, together with the number produced by twice the product of CD and CD, that is 81 and 81, makes 1762 — which number is equal to the square numbers of the

two sides AC and CD, that is, to the two numbers 1681 and 81; for these too make up the sum 1762. And if you make the smallest side CD 7, you will find AD to be 24 and AC 25, in which numbers you will experience the same thing. For the square number of side CD, subtended by the acute angle A, that is 49, together with the number twice produced by AD and AD, that is 576 and 576, makes 1201 — as much as the two square numbers of the sides AC and AD produce, that is, 625 and 576.

But if the perpendicular falls outside the triangle, so that the triangle is obtuse-angled, one must seek the exterior segment together with the sides, as we taught in the three rules of the scholium to the preceding proposition. For the exterior segment, together with the produced side, will give the whole segment lying between the assumed acute angle and the perpendicular. Thus in the figure of Rule 3 of the scholium to the preceding proposition, the whole segment CD, between the acute angle C and the perpendicular BD falling outside the triangle, will be 37. Here it is clear that the square of side AB, opposite the acute angle C — that is, 256 — together with twice the rectangle contained under AC and CD, that is 1184 and 1184, composes the number 2624; which is equal to the two [squares taken together], namely to the aggregate of the square numbers 1600 and 1024, and so on.

Diagram. Obtuse triangle ABC with the perpendicular BD dropped from B onto the side CA produced, its foot D lying beyond A (outside the triangle). B is at the lower left, C at the lower right, and D–A across the top left.

- A (upper left): vertex; the obtuse angle
- B (lower left): vertex
- C (lower right): vertex
- D (top, beyond A): foot of the perpendicular from B
- Segment AD: "5"
- Side AB: "16"
- Side AC: "32"
- Base BC: "40"

Added theorem (from Pappus of Alexandria)

Nor did I think it out of place here to demonstrate the following theorem as well, taken from Pappus of Alexandria [Pappus]:

If in a triangle a straight line is drawn from any angle, bisecting the opposite side, then the two squares of the sides containing that angle together are

double of two squares taken together — one of which is described on the drawn line, the other on the half of the [bisected] side.

In the triangle ABC, let the straight line AD cut the side BC in half at D. I say that the squares on AB and AC are double the squares on AD and BD.

For let a perpendicular be dropped from A to BC. First let it coincide with the straight line AD, so that the triangle ABC is equilateral or isosceles, as we demonstrated at Prop. 26 of Book I. [Margin: (u) Elements I.47] Now since both the square on AB is equal to the squares on AD and DB, and the square on AC is equal to the squares on AD and DC, the two squares on AB and AC will be equal to the square on AD taken twice, together with the squares on DB and DC. And since the squares on DB and DC are equal, if the two squares on AD and DC are taken away, there will have been removed half of the four squares — namely, of the square on AD taken twice and the square on DC. Therefore the two squares on AB and AC are double the two squares on AD and DB. This is what was proposed.

The same is shown more clearly thus. [Margin: (x) Elements I.47] Since the square on AB is equal to the two squares on AD and DB, and since the two squares on AB and AC are double the square on AB (because of the equality of the lines AB and AC), the squares on AB and AC will also be double the squares on AD and DB. This is what had to be demonstrated.

Diagram. First case: isosceles (or equilateral) triangle in which the perpendicular from A coincides with the median AD. Apex A, base BC with D at its midpoint.

- A (apex): vertex
- B (base, left): vertex
- D (midpoint of BC): foot of the perpendicular = bisection point
- C (base, right): vertex

Next let the perpendicular AC coincide with the side AC [the right angle falling at C]. Since the square [on BC] is fourfold both the square on BD and the square on DC — as we showed in the scholium to Prop. 4 of this Book — the same square on BC will be double of the two squares on DB and DC. But the two squares on AC and CD, taken twice, are also double the square on AD, [Margin: (y) Elements I.47] because the squares on AC and CD taken once are equal to the square on AD. Therefore the square on BC, together with the squares on AC and CD taken twice, will be double the squares on DB and DC. And since the square on CD taken twice is double the square on CD, [Margin: (z) pronuntiatum 20] the remaining square on BC, together with the square on AC taken twice, will likewise be double of the remaining squares on DB and [AD]. But [Margin: (a) Elements I.47] the squares on

BC and AC are equal to the square on AB. Therefore the squares on AB and AC (which are equal to the square on BC together with the square on AC taken twice) are likewise double the squares on AD and DB. This is what was proposed.

Diagram. Second case: right triangle with the right angle at C, A above C. Base BC with D its midpoint; AD is the median.

- A (upper right, above C): vertex
- B (base, left): vertex
- D (midpoint of BC): bisection point
- C (base, right): vertex

Thirdly, let the perpendicular AE fall inside the triangle, between the points D and C. [Margin: (b) Elements II.9] Since the squares on BE and EC are double the squares on BD and DE, and likewise the squares on AE and DE [taken twice] are double the square on AD — [Margin: (c) Elements I.47] because the squares on AE and ED taken once are equal to the square on AD — the squares on BE and EC, together with the squares on AE and DE taken twice, will be double the squares on BD, DE, and AD. And since the square on DE taken twice is double the square on DE, [Margin: (d) pronuntiatum 20] the remaining squares on BE and EC, together with the square on AE taken twice, will likewise be double of the remaining squares on DB and AD. Therefore, since the square on AB is equal to the squares on BE and AE, [Margin: (e) Elements I.47] and the square on AC to the squares on EC and AE, the squares on AB and AC will also be double the squares on AD and DB. This is what was proposed.

Diagram. Third case: acute triangle with apex A, base marked B, D, E, C from left to right (D the midpoint of BC, E the foot of the perpendicular AE falling inside, between D and C).

- A (apex): vertex
- B (base, far left): vertex
- D (on base): midpoint of BC
- E (on base): foot of the perpendicular AE
- C (base, far right): vertex

Lastly, let the perpendicular AE fall onto the side BC produced. [Margin: (f) Elements II.10] Since the squares on BE and CE are double the squares on BD and DE, and likewise the squares on AE and DE, taken twice, are double the square on AD — [Margin: (g) Elements I.47] because the squares on AE and DE taken once [are equal] to the square on AD — the squares on BE and CE, together with the

squares on AE and DE taken twice, will also be double the squares on BD, DE, and AD. And since the square on DE taken twice is double the square on DE, [Margin: (h) pronuntiatum 20] the remaining squares on BE and CE, together with the square on AE taken twice, will likewise be double of the remaining squares on BD and AD. Therefore, since the square on AB is equal to the squares on BE and AE, [Margin: (i) Elements I.47] and the square on AC to the squares on CE and AE, the squares on AB and AC will also be double the squares on AD and DB. This is what had to be shown.

Diagram. Fourth case: triangle with apex A, base marked B, D, C, E from left to right (D the midpoint of BC, E the foot of the perpendicular AE falling on BC produced, beyond C).

- A (apex): vertex
- B (base, far left): vertex
- D (on base): midpoint of BC
- C (on base): vertex
- E (on base produced, beyond C): foot of the perpendicular AE

Do not be disturbed, however, that for the demonstration of this theorem we have employed the pronounce-

-ment 20, which is universally demonstrated by Euclid for every kind of multiples at Prop. 5 of Book V. For in the double ratio it can easily be granted without demonstration, as we said in the exposition of its principle; or else the theorem itself can be demonstrated after Book V, so that no circularity at all is committed in the demonstration, even if that principle is demonstrated in Book V — since Proposition 5 of Book V does not depend on this theorem.

Scholium (continued): finding the area of a triangle whose sides are known

From what has been demonstrated in the two preceding theorems and at Prop. 47 of Book I, we shall find the area of any triangle whose sides are known — as Campanus [Campanus of Novara] rightly advises in this place, and as Federico Commandino [Federico Commandino] demonstrates, by roughly this method.

Let there be, first, a right triangle ABC whose sides are known, namely AB 10, AC 8, BC 6. When the side BC is bisected at D, so that CD and BD are each 3 palms, let the rectangle ACDE be completed, which is equal to the triangle ABC, as we showed in the scholium to Prop. 41 of Book I. Since from drawing CD, 3 palms,

into CA, 8 palms, there is produced the area of the rectangle CE, 24 square palms — as we taught at the beginning of this Book — the triangle ABC, equal to the rectangle CE, will contain just as many square palms. This is what was proposed.

Diagram. Right triangle ABC with the completed rectangle ACDE. The right angle is at C; A above C, B at the lower left. D is the midpoint of the base BC; E the fourth corner of the rectangle above D.

- A (top right): vertex
- E (top, above D): corner of the rectangle
- B (lower left): vertex
- D (midpoint of BC): bisection point
- C (lower right): vertex; the right angle
- Hypotenuse AB: "10"
- Side AC: "8"
- Base BC: "6"
- Segments BD, DC: "3" and "3"
- Area of rectangle CE (inside): "24"

Let there be, next, an obtuse triangle ABC whose sides are known: AB 20 palms, AC 13, BC 11. First, the length of the perpendicular AD, dropped from A onto the side BC produced, must be found, in this way. Since the square of side AB [Margin: (k) Elements II.12] is greater than the squares of the sides AC and BC by twice the rectangle contained under BC and CD: if the squares of the sides AC and BC, namely 169 and 121, which make 290, are subtracted from 400, the square of side AB, there will remain 110 for twice the rectangle contained under BC and CD; and half of this number, 55, will give the rectangle under BC and CD. If, then, 55 — the rectangle under BC and CD — is divided by the known side BC, 11, the remaining side CD, 5 palms, will come out, as Johannes Regiomontanus [Regiomontanus] demonstrates at Prop. 17 of Book I of his *On Triangles*, and as we have demonstrated in our book *On the Measurement of All Magnitudes*. And since the squares of the sides AD and CD [Margin: (l) Elements I.47] are equal to the square of side AC: if the square 25 palms — namely of side CD, 5 palms, just found — is taken away from 169, the square of side AC, there will remain 144 for the square of side AD; and AD will be 12 palms, since the square root of this number 144 is 12. Now, the side BC being bisected at E, let there be drawn out from C and E, perpendicular to BD, the perpendiculars CF and EG, meeting the straight line AG (which is drawn through A parallel to BD) at the points F and G. This done, if EC, $5\frac{1}{2}$ palms, is drawn into CF, 12 palms — [Margin: (m) Elements

I.34] for CF is equal to AD — the area of the rectangle CG, 66 palms, will arise; and since this is equal to the triangle ABC by the scholium to Prop. 41 of Book I (for the rectangle CG and the triangle ABC are between the same parallels, and so on), the area of the triangle ABC will likewise be 66 square palms. This is what was proposed.

Diagram. Obtuse triangle ABC (obtuse at C), with the perpendicular AD dropped from A onto BC produced (D beyond C). E is the midpoint of BC; the perpendiculars CF and EG rise from C and E to the line AG drawn through A parallel to BD, meeting it at F and G. The rectangle CG (i.e. CEGF) is shaded and labelled with its area. Points G, F, A run across the top; B, E, C, D along the bottom.

- A (top): vertex
- G, F (top): corners of the rectangle on the line through A parallel to BD
- B (bottom left): vertex
- E (on BC): midpoint of BC
- C (bottom, before D): vertex
- D (bottom, beyond C): foot of the perpendicular AD
- Side AC: "13"
- Perpendicular AD (= CF): "12"
- Segment CD: "5"
- Segments BE, EC: "5½" each [a second fractional numeral reads "5¼" on the tile — likely "5½" repeated; flagged]
- Base produced BD: "16"
- Area of rectangle CG (inside): "66"
- An additional interior construction point/mark near the left of the rectangle is present but not clearly labelled [?]

Let there be, lastly, an acute triangle ABC having its sides known: AB 13 palms, AC 15, BC 14. Here too, first, the length of the perpendicular AD, dropped from A onto BC, must be found, by this reasoning. Since the square of side AB [Margin: (n) Elements II.13] is less than the squares of the sides AC and BC by the rectangle contained twice under BC and CD: if the square of side AB, namely 169, is subtracted from the squares of the sides AC and BC, that is from 225 and 196, which make 421, there will remain 252 for the rectangle contained twice under BC and CD; and half of this number, 126, will give the rectangle under BC and CD. If, then, 126 — the rectangle under BC and CD — is divided by the known side BC, that is by 14, the remaining side CD, 9 palms, will come out, as is established from Johannes Regiomontanus, *On Triangles*, Book I, Prop. 17, and as we have

demonstrated in the book *On the Measurement of All Magnitudes*. And since the squares on AD and CD [Margin: (o) Elements I.47] are equal to the square on AC: if the square 81 palms — namely of side CD, 9 palms, just found — is taken away from 225, the square of side AC, there will remain 144 palms for the square of side AD. Therefore, since the square root of this number 144 is 12, the side AD will be 12 palms. Now, the side BC being bisected at E, let there be drawn out from C and E, perpendicular to BC, the perpendiculars CF and EG, meeting the straight line AF (which is drawn through A parallel to BC) at the points G and F. This done, if CE, 7 palms, is drawn into EG, 12 palms — [Margin: (p) Elements I.34] for EG is equal to the straight line AD — the area of the rectangle EF, 84 palms, will arise. And since this is equal to the triangle ABC, by the scholium to Prop. 41 of Book I (for the rectangle EF and the triangle ABC are between the same parallels, and so on), the area of the triangle ABC will likewise be 84 palms. This is what was proposed.

Diagram. Acute triangle ABC with apex A, base marked B, D, E, C from left to right (D the foot of the perpendicular AD, E the midpoint of BC). The perpendiculars CF and EG rise to the line AF through A parallel to BC; the rectangle EF is shaded and labelled with its area. Points A, G, F run across the top.

- A (apex): vertex
- G (top, next to A): rectangle corner [lowercase "a" on the tile]
- F (top right): rectangle corner
- B (bottom left): vertex
- D (on base): foot of the perpendicular AD
- E (on base): midpoint of BC
- C (bottom right): vertex
- Side AB: "13"
- Perpendicular AD (= EG): "12"
- Side AC: "15"
- Segment BD: "5"
- Segment DC: "9"
- Segment EC: "7"
- Base BC: "14"
- Area of rectangle EF (inside): "84"

And so, universally, the area of any triangle is produced from half of the base into the perpendicular that is dropped from the vertex to the base, as is manifest in the examples given. But more about this matter in the book *On the Measurement*

of All Magnitudes, which — God willing — we shall shortly bring to light. As for how, when two sides together with one angle are known, or two angles with one side, the remaining angles and sides may be found — this we have demonstrated in our *Plane Triangles*.

Problem 2. Proposition 14

[Margin: *xiiij.*]

To construct a square equal to a given rectilineal figure.

Let A be the given rectilineal figure, to which a square is to be constructed equal. [Margin: (a) Elements I.45] Let a parallelogram BCDE be constructed, equal to the rectilineal figure A and having a right angle, one side of which — say DC — is produced to F, and let CF be a straight line equal to the straight line BC. Let DF also be bisected at the point G, which will fall either on the point C or not. If it falls on the point C, then BC (since it is set equal to CF) will be equal to CD. Therefore the rectangle BD will be a square, since the sides DE and EB [Margin: (b) Elements I.34] are equal to the opposite sides BC and CD; and so a square will have been constructed equal to the rectilineal figure A.

But if the point G does not fall on C, then with G as centre let a semicircle FHD be described with radius GD (or GF), and let BC be produced until it cuts the circumference at H. I say, then, that the square of the straight line CH is equal to the rectilineal figure A. For let GH be drawn. Since the straight line DF is bisected at G and not bisected at C, the rectangle contained under DC and CF — that is, the rectangle BD — together with the square of the straight line GC, [Margin: (c) Elements II.5] is equal to the square of the straight line GF, that is, to the square of GH (since GF and GH are equal). But the square of GH [Margin: (d) Elements I.47] is equal to the squares of the straight lines GC and CH. Therefore the rectangle BD, together with the square of GC, will also be equal to the squares of GC and CH. And so, the common square of GC being taken away, there will remain the rectangle BD — that is, the rectilineal figure A — equal to the square of CH. Therefore we have constructed a square equal to a given rectilineal figure. Which was to be done.

Diagram. Three figures side by side. At the left, the given rectilineal figure A (an irregular quadrilateral). In the middle, the right-angled parallelogram BCDE (a square, in the case where G falls on C): E and D across the top, B and C across the bottom, with G marked. At the right, the general construction: the rectangle BCDE

with DC produced to F below, the semicircle FHD drawn on centre G, and BC produced to H on the circumference.

- A: the given rectilineal figure (label inside the quadrilateral)
- B (lower left of rectangle): vertex
- C (lower, on the base line): vertex
- D (upper, above C): vertex
- E (upper left): vertex
- F (below C, on DC produced): endpoint, $CF = BC$
- G (bisection point of DF; centre of the semicircle): point
- H (on BC produced, on the circumference): point; CH is the side of the required square

Scholium [to Proposition 14]

(Clavius's scholium, continued.)

Constructing, on a given straight line, a rectangle equal to a given rectilineal figure of many angles is laborious — and this is something we often need to do. By Proposition 44 of Book I we can build a rectangle equal to a triangle; so it may be easier to describe a square equal to a given rectilineal figure by first resolving the given figure into triangles, then producing a square equal to each triangle — that is, by finding the side of a square equal to each single triangle. This is very easily done, as I shall show shortly. For if, by the method set out in the scholium to Proposition 47 of Book I, we then find the side of a further square equal to the squares on all the sides already found, we shall have accomplished what is proposed.

Thus, suppose the given rectilineal figure is ABCDEF. We resolve it into the triangles ABF, FBC, CFD, DEF, and we find the sides G, H, I, K whose squares are respectively equal to those triangles. Next we set up a right angle at M, and lines LM and MN equal to the sides G and H. Then, once NL has been drawn, we erect on it the perpendicular NO equal to the side I. Similarly, once OL has been drawn, we raise on it the perpendicular OP equal to the side K, and we join the straight line PL. We proceed in this way until a perpendicular equal to the last side has been taken — as here the perpendicular OP was equal to the last side K. For the square on the straight line PL, drawn last, will be equal to all the squares on the sides G, H, I, K, as we demonstrated at Proposition 47 of Book I, and therefore equal to all the triangles ABF, FBC, CFD, DEF together — that is, to the given rectilineal figure ABCDEF.

Diagram. The construction for resolving a rectilinear figure into triangles and combining their equal-squares. On the left, a six-sided figure ABCDEF triangulated from the common vertex F; to its right, four short vertical line-segments representing the sides G, H, I, K; below, the perpendicular "staircase" construction on points L, M, N, O, P.

- A (top of the polygon)
- B (upper-left vertex)
- C (left vertex)
- D (lower-left vertex)
- E (lower-right vertex)
- F (right vertex; apex to which all the diagonals FB, FC, FD run, forming triangles ABF, FBC, CFD, DEF)
- G, H, I, K (four vertical line-segments, the sides whose squares equal the several triangles)
- M (bottom-left; the initial right angle)
- N (left)
- O (top; foot of the perpendicular NO)
- P (right; foot of the perpendicular OP)
- L (bottom-centre; PL joined last)

Now the practical method [praxis] for finding a square equal to any triangle is straightforward. Suppose the square to be found is one equal to the fourth triangle, DEF, of the given rectilinear figure. Bisect any one side, say DE, at A, and extend it as far as you like. Then, through the opposite vertex F, draw FB parallel to the side DE; through E draw HEB perpendicular to DE, cutting FB at B; and let EH be equal to AE, that is, to half of DE. Finally, once the whole line BH has been bisected at G, describe with centre G and radius GH (or GB) an arc cutting the side DE, extended, at I. I say that the square on the side EI is equal to the triangle DEF. For if the rectangle AEC is completed and the semicircle on BH is drawn, and the straight line GI is joined, we shall show — as in this Proposition 14 — that the square on EI is equal to the rectangle AB. Since, then, the rectangle AB is equal to the triangle DEF (by the scholium to Proposition 41 of Book I, because the base DE is double the base AE), the proposition is established.

Diagram. The construction for a square equal to triangle DEF, using the semicircle on BH.

- C, B, F (top row: the parallel FB drawn through the vertex; C at left, B at centre, F at right)

- G (centre; midpoint of BH, centre of the semicircle)
- D (mid-left, an endpoint of the bisected side)
- A (bottom; midpoint of DE)
- E (bottom, beside A; foot of the perpendicular)
- H (bottom; the point with $EH = AE$)
- I (right; where the arc cuts DE extended)
- (a dotted semicircle is drawn on BH)

It is not, however, necessary to construct the rectangle AB. For if the right angle IEB is made, and on BH is taken EH equal to half the side DE, and the straight line EB is made equal to the perpendicular dropped from F onto the side DE, and BH is bisected at G, then the circumference described with centre G on BH will cut off the straight line EI, whose square is equal to the rectangle contained by BE and EH — that is, equal to the triangle DEF.

Now since, in this second book, Euclid has discussed many things about squares and about rectangular parallelograms, two problems concerning the square and the rectangular parallelogram may fitly be inserted here — neither unpleasant nor unprofitable. The second of them was solved most ingeniously by the learned and excellent mathematician Marino Ghetaldi [Marino Ghetaldi], a patrician of Ragusa and a close friend of mine, and solved in a way far different from, and much briefer than, the way in which it was once solved by someone else. To append that other solution would be superfluous, since I have set down here Ghetaldi's own, as the briefer and easier one.

I.

Problem. Given the excess of the diameter of some square over its side, to find the side of that square.

Demonstration. Let the diameter of some square exceed its side by the straight line AB, and let the side of that square be sought. On the straight line AB describe the square AC, and let its diameter BD be extended to E so that DE is equal to the straight line AD. I say that EB is the side of the square whose diameter exceeds that same side BE by the given excess AB. For draw EF perpendicular to BE, meeting BA extended at F. Now since, in the triangle BEF, the angle FEB is right and the angle EBF is half a right angle [semirectus], by Corollary 2 of Proposition 4 of this book [Margin: c) Bk. 1, Prop. 32], the angle BFE will also be half a right angle. [Margin: f) Bk. 1, Prop. 6] Therefore the straight lines BE and FE are equal. If, then, FG is drawn from F parallel to BE, and from B the straight line BG is drawn parallel to EF, meeting the first at G, the square on the line BE will have

been constructed. And if the straight line AE is drawn [Margin: g) Bk. 1, Prop. 5], the angles AED and EAD, opposite the equal lines DE and DA, will be equal. So if these are subtracted from the right angles DEF and DAF, the remaining angles AEF and EAF will be equal [Margin: h) Bk. 1, Prop. 6], and therefore the straight lines EF and AF will be equal. But since the diameter BF exceeds the straight line AF by the given straight line AB, that same diameter BF will exceed the side of the square EF by the same straight line AB — which is what was proposed.

Diagram. The figure for Problem I: the square AC on AB, its diameter BD extended to E, with EB the side sought and the auxiliary lines EF, AE, FG, BG.

- E (top; the extension of the diameter, $DE = AD$)
- D (upper-centre; corner of the square, on the diameter BE)
- C (upper-right; corner of the square AC)
- F (left; where EF meets BA extended)
- A (centre-lower)
- B (lower centre-right)
- G (bottom; the far corner of the constructed square on BE)

Corollary

Thus the straight line composed of the diameter of the square described on the given excess and of the side of that same square exhibits the side of the square that is sought. And the same diameter, together with twice the excess, constitutes the diameter of the square sought.

II.

Problem. Given the two excesses of the diameter of some rectangular parallelogram over each of its sides, to find the sides of that rectangle.

Demonstration. Let the diameter exceed the lesser side by the straight line AB, and the greater side by the straight line BC. Extend AC in both directions; take AD equal to AB; and, having described a semicircle on CD, erect BE perpendicular to AC, cutting the circumference at E, and take CF equal to BE. Now since, as is clear from the demonstration of Proposition 14 of this book, the square on the line BE is equal to the rectangle contained by DB and BC, BE will be greater than BC, the lesser side of the said rectangle. Therefore CF too will be greater than BC. So, taking FG equal to BC, the point G will fall between C and FE; and with GH — perpendicular to AF — made equal to BF, I say that AG is the greater side, and GH the lesser, so that the rectangle

Diagram. The figure for Problem II: the semicircle on CD, with BE perpendicular to the base and the auxiliary construction for the rectangle's sides.

- I (left; on the base line, extended)
- D, A, B, C (along the horizontal base line, left to right)
- F (right; on the base line)
- E (centre; where the perpendicular BE meets the circumference)
- K (upper-left arc of the semicircle)
- H (upper-right)
- G (right, below H)
- M (bottom of the figure)

(Problem II, demonstration continued.)

is that which was sought, GK. That AG is greater than GH will be proved thus. Since BC and FG are equal, and the common segment CG is added, BG and CF will also be equal. Adding, then, the unequal segments — AB to BG, and BC to CF — AG will be greater than BF, that is, than GH: since AB, once added, gives more than BC once added. That AG and GH are the sides sought will be demonstrated as follows. With centre A and radius AG describe a circle; draw through A and H a straight line cutting the circumference at L and M; and extend AD, cutting the circumference at I. Now since the rectangle contained by BC and BD [Margin: i) Bk. 2, Prop. 1] is equal to the rectangles contained by CB and BA, and by CB and AD — that is, to twice the rectangle on the excesses AB, BC — and since the square on BE is equal to the rectangle contained by BC and DB, as we showed, the square on BE will also be double the rectangle contained by AB and BC; that is, the square on BE will be equal to twice the rectangle contained by AB and BC.

Adding, then, the common square on BC, together with twice the rectangle contained by BG and BC — that is, by CF and BC — the squares on BE (or CF) and on BC, together with twice the rectangle contained by CF and BC, will be equal to twice the rectangle contained by AB and BC, together with the square on BC and twice the rectangle contained by CF and BC. But the squares on CF and BC, together with twice the rectangle contained by CF and BC [Margin: k) Bk. 2, Prop. 4], are equal to the square on BF. Therefore the square on BF too will be equal to twice the rectangle contained by AB and BC, together with the square on BC and twice the rectangle contained by CF and BC (or by BG and BC). Since, then, twice the rectangles contained by AB and BC and by BG and BC is equal to the rectangle contained by IG (because IG is double AB, BG taken together) and by BC (or FG), the square on BF will also be equal to the rectangle contained by IG and FG,

together with the square on BC (or FG). But the rectangle contained by IG and GE, together with the square on FG [Margin: l) Bk. 2, Prop. 3], is likewise equal to the rectangle contained by IF and FG. Adding, therefore, the common square on AG, the squares on GH and AG will be equal to the rectangle contained by IF and FG, together with the square on AG. But the square on AH is equal to the squares on GH and AG [Margin: m) Bk. 1, Prop. 47], and the square on AF is equal to the rectangle contained by IF and FG, together with the square on AG. [Margin: n) Bk. 2, Prop. 6] Therefore the square on AH is equal to the square on AF, and the straight line AH is equal to the straight line AF; and when the equals AL and AG are removed, the remainders LH and GF will be equal. The diameter AH of the rectangle GK — that is, the straight line AF — therefore exceeds the greater side AG, namely by GF, by the given excess BC; and it exceeds the lesser side GH itself, namely the straight line BF (equal to GH), by the given excess AB. Which is what was proposed.

Corollary

Thus the straight line BE (or CF), whose square is double the rectangle contained by the given excesses AB and BC, when added to the lesser excess BC, produces the lesser side sought, BF — that is, GH. And the same line BE — that is, BG — added to the greater excess AB, makes up the greater side sought, AG. And finally the same line BE (or CF), taken with both excesses AB and BC, constitutes the diameter AF (or AH).

FINIS ELEMENTI SECVNDI. (*End of the Second Element.*)

EUCLID'S THIRD ELEMENT

(*EVCLIDIS ELEMENTVM TERTIVM*)

DEFINITIONS

I.

Definition (Euclid). Circles are equal whose diameters are equal, or whose straight lines drawn from their centres are equal.

Commentary (Clavius). Since Euclid in this third book demonstrates various properties of the circle, he first explains certain terms that will be frequently used in it. He teaches first, then, that those circles are equal whose diameters, or whose semidiameters, are equal. For since a circle is described by the revolution of a semidiameter about one fixed and immovable extremity — as we said in

Book I — it is evident that those circles are equal whose semidiameters, that is, whose straight lines drawn from the centres, are equal; or likewise those whose whole diameters are equal. Thus if the diameters AB and BC, or the straight lines DF and EG drawn from the centres D and E, are equal, the circles AFB and BGC will be equal. So too, conversely, if the circles are equal, their diameters, or the straight lines drawn from their centres, will be equal. From this it is clear that circles whose diameters, or whose lines drawn from the centres, are unequal are themselves unequal — and indeed that the one whose diameter or semidiameter is greater is the greater. And conversely, that the diameters or semidiameters of unequal circles are unequal, the greater belonging to the greater circle and the lesser to the lesser.

Diagram. Four circles illustrating equal and unequal circles: the upper pair equal, the lower pair unequal (one large, one small). Each circle has a horizontal diameter and a vertical radius drawn to the circumference.

- Upper-left circle (AFB): A (left), D (centre), B (right); F (bottom, on the circumference)
- Upper-right circle (BGC): left endpoint [B?] (label not clearly legible [?]), E (centre), C (right); G (bottom, on the circumference)
- Lower-left circle (AFB, larger): A (left), D (centre), B (right); F (bottom)
- Lower-right circle (BGC, smaller): left endpoint [B?] (label not clearly legible [?]), E (centre), C (right); G (bottom)

II.

Definition (Euclid). A straight line is said to touch a circle when, on touching the circle, it does not — if produced — cut the circle.

Commentary (Clavius). Thus the straight line AB, if it touches the circle BFD at B in such a way that, when produced to C, it in no way cuts the circle but lies wholly outside it, will be said to touch the circle. But the straight line EF, because it so touches the same circle at F that, when produced to G, it cuts the circle and falls within it, will not be said to touch the circle, but to cut it.

Diagram. A circle BFD with a tangent line and a secant line, illustrating the distinction between touching and cutting.

- A, B, C (the tangent straight line across the top; B is the point of contact at the top of the circle)
- E, F, G (the secant line: E outside/upper-left, F where it meets the circumference, G where it passes within)

- D (bottom of the circle)

III.

Definition (Euclid). Circles are said to touch one another when, touching one another, they do not cut one another.

Book III — Definitions (continued)

Definition III (continued) — circles touching one another

Diagram. A group of illustrative figures at the head of the page, showing the ways in which two circles can meet. In the upper row are three small figures; in the lower row two larger pairs of circles.

- Upper-left figure: two circles touching, one lying within the other; labelled **C** (top of the inner circle), **B** (middle), **A** (at the bottom, near the point of contact) — internal tangency.
- Upper-middle figure: a single circle, labelled **A** (inside, upper part).
- Upper-right figure: a circle labelled **B** (top) and **C** (within, upper-left).
- Lower-left pair: two large circles overlapping and cutting one another (mutual intersection); no interior labels legible.
- Lower-right pair: a larger and a smaller circle in contact (external/internal tangency); no interior labels legible.

(Clavius's commentary on Definition III): In the same way, two circles AC and BC are said to touch one another at C, if they meet at C in such a way that neither cuts the other. This contact of circles is of two kinds. Either the circles touch one another externally — as when one is placed outside the other — or internally, when one is set inside the other. But if two circles touch one another in such a way that the one also cuts the other, then those circles are said to cut one another mutually, and not to touch.

IV.

(Euclid's definition): In a circle, straight lines are said to be equally distant from the centre when the perpendiculars drawn to them from the centre are equal. And a line is said to be farther off onto which the greater perpendicular falls.

(Clavius's commentary): Among all the straight lines drawn from any point to a given straight line, the shortest is the perpendicular, and it is always one and the

same, whereas the others can vary in infinitely many ways. The distance of a point from a straight line is therefore rightly reckoned by the perpendicular line.

Diagram. A triangle-like figure with apex above a horizontal base line, from which several lines descend.

- **A** (apex, above): the point from which the lines are drawn.
- Base line **BC** (horizontal), with the feet of the drawn lines marked, in order from the left: **B**, **F**, **E**, **D** (and the line runs on toward C).
- Lines **AD** (perpendicular, foot at D), **AE** and **AF** (oblique).

Thus the distance of the point A from the straight line BC is said to be the perpendicular AD, and not AE, or AF, or any other that is not perpendicular; for AD is shorter than all of them, by the corollary of Proposition 19 of Book 1 [Margin: a — *Elements* I.19]. Indeed, not only are AE and AF greater than AD, but they are also unequal among themselves. For AF is greater than AE, since the angle AEF is obtuse and AFE acute; and so with the other non-perpendicular lines. That AFE is acute is clear from this: in the triangle ADF the two angles ADF and AFD are less than two right angles [Margin: b — *Elements* I.17]; hence, since ADF is a right angle, the angle AFD must be less than a right angle. By the same reasoning the angle AED will be shown to be acute, because in the triangle ADE the two angles ADE and AED are less than two right angles [Margin: c — *Elements* I.17], and ADE is a right angle. Consequently, since AED and AEF together are equal to two right angles [Margin: d — *Elements* I.13], AEF will be obtuse.

This is why Euclid has defined the equal distance of straight lines in a circle from its centre by means of equal perpendiculars, and unequal distance by unequal ones. Thus two straight lines AB and CD in the circle ABCD are said to be equally distant from the centre E if the perpendiculars EF and EG are equal. But the line CD is said to be farther from the centre E than the line HI, if the perpendicular EG is greater than the perpendicular EK.

Diagram. A circle with centre E and several chords, with perpendiculars dropped from the centre.

- **A** (upper-left) and **B** (upper-right): endpoints of the chord AB.
- **C** (left) and **D** (lower-left): endpoints of the chord CD.
- **E** (centre).
- **F**: foot of the perpendicular EF on chord AB.
- **G**: foot of the perpendicular EG on chord CD.
- **H** (bottom) and **I** (right): endpoints of the chord HI.

- **K**: foot of the perpendicular EK on chord HI.

V.

(Euclid's definition): A segment of a circle is a figure contained by a straight line and a circumference of the circle.

(Clavius's commentary): Thus, if in the circle ABCD any straight line BD is drawn, then both the figure BAD — contained by the arc BAD and the straight line BD — and the figure BCD — contained by the straight line BD and the arc BCD — will be called a segment of the circle. From this it follows that there are three kinds of circular segment. There is the semicircle, when the straight line BD passes through the centre E; the segment greater than a semicircle, when the straight line BD does not pass through the centre, yet the centre lies within it, such as the segment BAD; and the segment less than a semicircle, outside of which the centre of the circle falls, such as the segment BCD. This we demonstrated in connection with Definition 18 of Book 1. By most geometers the straight line BD is called the *chord*, and the arc BAD, or BCD, the *arc*.

Diagram. A circle divided by a horizontal chord into an upper and a lower segment.

- **A** (top): a point on the upper arc.
- **E** (centre).
- **B** (lower-left) and **D** (lower-right): endpoints of the chord BD.
- **C** (bottom): a point on the lower arc.
- Upper segment BAD; lower segment BCD.

VI.

(Euclid's definition): An angle of a segment is one contained by a straight line and a circumference of the circle.

(Clavius's commentary): Euclid now defines the three kinds of angle that are considered in circles. In the first place, the angle of a segment: he says that the mixed angle BAC (or BCA), contained by the straight line AC and the arc ABC, is called the angle of a segment. If the segment of the circle is a semicircle, it will be called the angle of a semicircle. If the segment is greater than a semicircle, it will be called the angle of the greater segment; and finally, if the segment is less than a semicircle, it will be named the angle of the lesser segment.

VII.

(*Euclid's definition*): An angle in a segment is formed when some point is taken on the arc of the segment, and from it straight lines are joined to the ends of the straight line that is the base of the segment: that angle, I say, is the one contained by those joined lines.

(*Clavius's commentary*): Let ABC be any segment of a circle, whose base is the straight line AC. From any point B taken on the arc, let straight lines BA and BC be drawn to the points A and C, the ends of the base. The rectilineal angle ABC is then said to exist in the segment ABC.

Diagram. A segment of a circle with an inscribed angle.

- **B** (top, on the arc): the vertex of the inscribed angle.
- **A** (lower-left) and **C** (lower-right): the ends of the base AC.
- Lines BA and BC drawn; the angle ABC stands in the segment.

VIII.

(*Euclid's definition*): When the straight lines containing an angle cut off some arc, the angle is said to stand upon that arc.

(*Clavius's commentary*): From any point A taken on the circumference of the circle ABCD, let two straight lines AB and AD be drawn to the two ends B and D of the arc BCD, which those two straight lines AB and AD cut off. The rectilineal angle BAD is then said to stand upon the arc BCD. It is clear, however, that this angle does not differ from the preceding one except in name. For the very same rectilineal angle BAD is, by the preceding definition, said to be *in* the segment BAD, if the straight line BD is drawn as base; while by the present definition it is said to *stand upon* the arc BCD.

Diagram. A circle with an inscribed angle at A.

- **A** (top): the vertex of the angle.
- **D** (lower-left) and **B** (lower-right): the ends of the arc, to which the lines AD and AB are drawn.
- **C** (bottom): a point on the arc BCD.

Nevertheless, an angle in some segment must not be confused with an angle that stands upon the arc, even though it is one and the same angle; for they are referred to different things. An angle in a segment regards the segment in which it lies, whereas an angle standing upon an arc regards the arc that is the base of

the angle. Hence, if one takes some segment BCD of the circle in the circle ABCD, the angle lying in this segment and the angle standing upon its arc will not be the same. For the angle lying in it will be BCD; but the angle standing upon its arc BCD will be BAD, which differs greatly from it. On this point Orontius [Oronce Finé], Peletier [Jacques Peletier], and several other commentators have gone remarkably astray.

That the angle in a segment and the angle standing upon some arc are referred to different arcs will become clearer than daylight from the last proposition of Book 6, which can apply only to the arcs of circles upon which the angles stand, and not to those in which they lie — as we shall show in that place. The same is also readily established from the Greek word βεβηκέναι [Greek: bebēkenai, "to have stepped up," i.e. "to have ascended"], which means "to have ascended." For the angle DAB ascends above the arc BCD.

(*Clavius, further*): Besides the three angles just mentioned, geometers also consider the angle of contact [angulus contingentiae], which is contained by a straight line touching a circle and the arc of the circle; or else by two arcs touching one another, whether this happens externally or internally. For example: if the straight line AB touches the circle CDE at C, the mixed angle ACD (or BCE) will be called the angle of contact, or of tangency. Again, if the circle CED touches the circle EFG externally at E, and likewise the circle HFI touches the circle EFG internally at F, then the curvilinear angle CEF, as well as EFH (or GFI), will be called an angle of contact, or of tangency. And so, as you see, there are three angles of contact — one mixed, and the other two curvilinear.

Diagram. A circle with an inscribed triangle.

- A (top).
- B (left) and D (right).
- C (bottom).
- Chords drawn forming the inscribed triangle (ABD) and the diagonals, illustrating the angle standing on the arc.

Diagram. Two circles side by side, touching one another, with a straight line tangent above — illustrating the angle of contact.

- Along the top (tangent line): A, C, B, F (C and F marking points of contact).
- Left circle (CDE): D (far left), E (interior).
- Right circle (EFG / HFI): H (interior), G (far right).
- The point I is not legibly labelled on the tile [?].

IX.

(Euclid's definition): A sector of a circle is formed when an angle is constructed at the centre of the circle — the figure, namely, contained both by the straight lines containing the angle and by the arc cut off by them.

(Clavius's commentary): If in the circle ABCD, whose centre is E, the straight lines AE and CE form the angle AEC at the centre E, then the figure AECD — contained by the straight lines AE and EC and by the arc ADC, which those lines cut off — will be called a sector of the circle. From this it is also clearly gathered that the angle explained in Definition 8 is referred to the arc that is its base, and not to the one in which it lies, as many commentators have supposed. For just as in this definition Euclid understands the arc ADC — which is the base of the angle constructed at the centre — when he makes mention of the arc cut off by the straight lines AE and CE, so too in that other definition he must have understood, by the name "arc," the one that is the base of the angle constructed at the circumference and cut off by the straight lines; since in both definitions he used the same Greek word ἀπολαμβάνειν [Greek: apolambanein, "to cut off," i.e. to subtend an arc].

Diagram. A circle with two radii drawn from the centre, forming a sector.

- **B** (top) and **C** (right): points on the arc.
- **E** (centre).
- **A** (left) and **D** (bottom): further points on the circumference.
- Radii EA and EC drawn, cutting off the arc ADC (the sector AECD).

X.

(Euclid's definition): Similar segments of a circle are those that admit equal angles, or in which the angles are equal to one another.

(Clavius's commentary): That is to say, the segments ABDF and DCAE of the same circle ABCDEF, which admit these two equal angles ABF and DCE — or, what is the same thing, in which these same angles are equal — are, by Definition 7, called similar; and their arcs ABDF and DCAE are likewise called similar.

Diagram. A circle ABCDEF with inscribed chords forming two segments and their angles.

- **B** (top-left) and **C** (top-right).
- **A** (left) and **D** (right).

- F (lower-left) and E (lower-right).
- Chords drawn making the angles ABF and DCE, marking the similar segments ABDF and DCAE.

**

Definition X (commentary continued)

(Clavius's commentary): In the same way, segments of different circles — whether equal or unequal — are called similar by geometers when they either admit equal angles, or when equal angles exist in them. Thus, if in the circles ABCK, DEFL, GHIM the angles ABC, DEF, GHI are equal, then the segments (or arcs) ABC, DEF, GHI, which admit those angles — or in which the said angles exist — will be called similar.

Now this similarity of segments (or of arcs) consists in this: that whatever part one arc is of its own whole circumference, such a part also is the other arc, which is called similar to it, of its own whole circumference; so that, just as, and as great as, the part which the arc ABC is of the whole circumference ABCK, such and so great a part also is the arc DEF of the whole circumference DEFL; and likewise such and so great is the arc GHI of the whole circumference GHIM. Or rather, the similarity of the segments consists in this: that the similar segments (or arcs) have the same ratio to their whole circumferences. That segments which either admit equal angles, or in which equal angles exist, are indeed of this sort, we shall demonstrate in the last proposition of Book 6. For now let it suffice that such segments of circles — or also arcs, or circumferences — are called similar.

Diagram. Three circles side by side, each with an inscribed triangle, illustrating similar segments.

- First circle (ABCK): B (top), A (lower-left), C (lower-right), K (on the lower arc); inscribed triangle ABC.
- Second circle (DEFL): E (top), D (lower-left), F (lower-right), L (on the lower arc); inscribed triangle DEF.
- Third circle (GHIM): H (top), G (lower-left), I (lower-right), M (on the lower arc); inscribed triangle GHI.

(Clavius, further): By the same reasoning, arcs or circumferences are called similar when equal angles stand upon them, according to Definition 8. Thus, if in the same circles the angles ABC, DEF, GHI are equal, then the arcs (or circumferences) AKC, DLF, GMI upon which they stand will be called similar. Indeed, if the angles at the centres standing upon the arcs AKC, DLF, GMI are

equal, the arcs themselves will still be similar — which we shall demonstrate in the scholium to Proposition 22 of this Book.

PROBLEM 1. — PROPOSITION 1.

To find the centre of a given circle.

(*Clavius's demonstration*): Let ABCD be the given circle, whose centre is to be found. Draw in it any line AC, and let it be bisected at E [Margin: a — *Elements* I.10]; and through E let the perpendicular BD be drawn to AC, terminated on both sides at the circumference in the points B and D. Then, this line BD having been bisected at F, I say that F is the centre of the proposed circle.

For on the straight line BD no point other than F will be the centre, since every other point divides it unequally, whereas at F it was divided equally. If, then, F is not the centre, let some point G, off the straight line BD, be the centre, and from it let the lines GA, GE, GC be drawn. Now since the sides AE and EG of the triangle AEG are equal to the sides CE and EG of the triangle CEG, and the base AG to the base CG (for they are said to be drawn from the centre), the angles AEG and CEG will be equal [Margin: b — *Elements* I.8], and therefore right angles. But the angle AEF was also a right angle, by construction. Therefore the right angles AEF and AEG are equal — the part and the whole — which is absurd. Point G, therefore, is not the centre; and the same reasoning holds for every other point. Hence F will be the centre. And so we have found the centre of the given circle. Which was to be done.

Diagram. A circle with a horizontal chord AC and a vertical chord BD crossing it, with the false centre G marked above.

- **B** (top) and **D** (bottom): ends of the perpendicular chord BD.
- **A** (left) and **C** (right): ends of the chord AC.
- **E** (where BD meets AC; midpoint of AC).
- **F** (on BD; midpoint of BD — the true centre).
- **G** (above, off BD — the supposed false centre), with lines GA, GE, GC drawn.

COROLLARY.

(*Clavius*): From this it is manifest that, if in a circle some straight line cuts another straight line into two equal parts and at right angles, the centre of the circle lies on the cutting line. For from the fact that the straight line BD cuts the straight line AC into two equal parts at E and at right angles, it was shown that its midpoint F is necessarily the centre of the circle.

THEOREM 1. — PROPOSITION 2.

If two points whatever are taken on the circumference of a circle, the straight line joined to those points will fall within the circle.

(*Clavius's demonstration*): In the circle ABC let any two points A and C be taken on its circumference. I say that the straight line drawn from A to C falls within the circle, so that it cuts the circle. For if it does not fall within, let it fall outside, as does the line ADC, as the adversary would have it. Then, the centre E having been found [Margin: a — *Elements* III.1], let there be drawn from it, to the points A and C that were taken, and also to any point D on the line ADC, the straight lines EA, EC, ED; and let ED cut the circumference at B.

Now since the two sides EA and EC of the triangle whose base is the line ADC are equal (for they are drawn from the centre), the angles EAD and ECD will be equal [Margin: b — *Elements* I.5]. But the angle EDA is greater than the angle ECD [Margin: c — *Elements* I.16], being the exterior angle opposite the interior, since the side CD in the triangle ECD is produced to A. Therefore the same angle EDA will also be greater than the angle EAD. Hence the straight line EA, subtending the greater angle ADE — that is, the line EB, equal to it — will be greater than the line ED [Margin: d — *Elements* I.19], which subtends the lesser angle DAE: the part greater than the whole. Which is absurd. Therefore the straight line drawn from A to C will not fall outside the circle, but within.

For it will be demonstrated in the same way that the line drawn from A to C cannot fall upon the arc ABC, so as to coincide with the circumference ABC. For then the straight line EA would be greater than the straight line EB. This is also clear from the definition of a straight line, since ABC, being an arc, is a curved line and not a straight one. And so, if two points whatever are taken on the circumference of a circle, and so on. Which was to be proved.

Diagram. A circle with centre E; a supposed line ADC bulging outside above, with radii drawn to the centre.

- **D** (top): the point on the falsely-assumed exterior line ADC.
- **B** (just below D): where the radius ED cuts the circumference.
- **A** (left) and **C** (right): the two given points on the circumference.
- **E** (centre), with lines EA, EC, ED drawn.

SCHOLIUM

(*Clavius, continuing the scholium to Proposition 2*)

This same theorem can also be proved affirmatively, as follows. Let the straight line AB join two points A and B on the circumference of the circle AB whose centre is C. I claim that AB falls within the circle, so that all its intermediate points lie inside the circle. For take any one of its intermediate points, D, and draw from the centre the straight lines CA, CB, and CD. Since the two sides CA and CB of the triangle CAB are equal, the angles CAB and CBA are equal (d). But the angle CDA is greater than the angle CBA, being an exterior angle greater than the interior one (e). Therefore the same angle CDA is also greater than the angle CAD, and for that reason the side CA will be greater than the side CD (f). Hence, since CA is drawn from the centre all the way to the circumference, the line CD will not reach the circumference, and so the point D falls within the circle. The same can be shown for any other point taken. Therefore the whole straight line AB falls within the circle — which was to be proved.

[Margin: (d) Elements I.5] [Margin: (e) Elements I.16] [Margin: (f) Elements I.19]

Diagram. A circle with a chord across its upper portion and a triangle inscribed to the centre below.

- A (upper left, on circumference): endpoint of the chord
- B (upper right, on circumference): other endpoint of the chord
- D (on the chord AB, between A and B): the arbitrary intermediate point
- C (centre, lower middle): centre of the circle; radii CA, CB, CD are drawn from it

COROLLARIUM

From this it is clear that a straight line which is tangent to a circle, in such a way that it does not cut it, touches it at one point only. For if it were to touch it at two points, the part of the straight line lying between those two points would fall within the circle (g). It would therefore cut the circle — which is contrary to the hypothesis.

[Margin: (g) Elements III.2]

THEOR. 2. PROPOS. 3.

[Margin: iij.]

(Euclid's enunciation)

If in a circle a straight line drawn through the centre bisects another straight line not drawn through the centre, it will also cut it at right angles. And if it cuts it at right angles, it will also bisect it.

(Clavius's demonstration)

Let the straight line CE, extended through the centre A of the circle BCD, divide the straight line BD (which is not drawn through the centre) into two equal parts at F. I claim that AF is at right angles to BD. For, when the straight lines AB and AD are drawn, the two sides AF and FB of the triangle AFB will be equal to the two sides AF and FD of the triangle AFD, and the bases AB and AD will be equal. Therefore the angles AFB and AFD will be equal (a) — that is, right angles. This is what was proposed in the first place.

[Margin: (a) Elements I.8]

Now let AF be at right angles to BD. I claim that BD is bisected at F by the line CE. For when the straight lines AB and AD are again drawn, since the sides AB and AD of the triangle ABD are equal, the angles ABD and ADB will be equal (b). Since, then, the two angles AFB and ABF of the triangle ABF are equal to the two angles AFD and ADF of the triangle ADF, and the sides AB and AD, which lie opposite the equal right angles, are also equal, the sides FB and FD will be equal (c). This is what was proposed in the second place. Therefore, if in a circle a straight line drawn through the centre, and so on — which was to be demonstrated.

[Margin: (b) Elements I.5] [Margin: (c) Elements I.26]

This second part, which is indeed the converse of the first, could also easily be proved by the following reasoning. If AF is perpendicular to BD, then the square on AB will be equal to the squares on AF and FB (d), just as the square on AD will be equal to the squares on AF and FD. Since, then, the square on AB is equal to the square on AD, the squares on AF and FB will also be equal to the squares on AF and FD. Hence, when the common square on AF is taken away, the remaining squares on FB and FD will be equal, and therefore the lines FB and FD will also be equal.

[Margin: (d) Elements I.47]

Diagram. A circle with a vertical diameter and a horizontal chord crossing it, with an inscribed triangle.

- C (top, on circumference): upper endpoint of the diameter CE
- A (centre): centre of the circle

- F (on the chord BD, at its crossing with the diameter): point of bisection
- B (left, on circumference): left endpoint of the chord
- D (right, on circumference): right endpoint of the chord
- E (bottom, on circumference): lower endpoint of the diameter CE
- Straight lines AB and AD are drawn, forming the triangle ABD

COROLLARIUM

From this demonstration we shall easily infer that, in any triangle having two equal sides — whether it is equilateral or isosceles — the line that bisects the base is perpendicular to the base; and, conversely, that the line perpendicular to the base bisects the base. For in the triangle ABD, whose two sides AB and AD are equal (so that a circle can be described from the centre A through B and D): from the fact that the line AF bisects the base BD it has been shown that the angles at F are right angles; and from the fact that the angles at F are right angles it has been demonstrated that the base BD is bisected by the line AF. But we also demonstrated this at Proposition 26 of Book 1, and indeed more generally.

SCHOLIUM

We judge that this is also the place to prove the following theorem, which is not without use for what follows — namely:

(theorem)

If two circles are described from the same centre, and from some point of the outer circumference a straight line is drawn cutting the inner circumference, then the segments of it lying between the two circumferences will be equal to each other.

(demonstration)

Let two circles ABC and DEFG have the same centre H, and from the point A let there first be drawn, through the centre H, the straight line AC, cutting the inner circumference at D and G. I claim that the lines AD and CG are equal. For if from HA and HC, which are equal (e), we take away HD and HG, which are equal for the same reason, then the lines AD and CG will be equal (f).

[Margin: (e) Elements I, Definition 15] [Margin: (f) Common Notion 3]

Next, from the same point A let the straight line AB be drawn, not through the centre, cutting the inner circumference at E and F. I claim, again, that the lines AE and BF are equal. For when the perpendicular HI is drawn from the centre H to

AB (g), it will bisect both the line AB in the circle ABC and the line EF in the circle DEFG (h). Therefore, when the equal parts IE and IF are taken away from the equal parts IA and IB, the remainders AE and BF will be equal (i) — which is what was proposed.

[Margin: (g) Elements I.12] [Margin: (h) Elements III.3] [Margin: (i) Common Notion 3]

Diagram. Two concentric circles about a common centre H, with a diameter and an oblique secant drawn from the top point A.

- A (top, on outer circumference): the point from which the lines are drawn
- D (upper, on inner circumference): where the diameter meets the inner circle
- E (upper, on inner circumference): where the oblique line AB meets the inner circle
- I (interior, on line AB): foot of the perpendicular HI from the centre
- H (centre): the common centre of both circles
- F (lower left, on inner circumference): where the oblique line meets the inner circle again
- G (lower, on inner circumference): where the diameter meets the inner circle again
- B (lower left, on outer circumference): lower end of the oblique line
- C (bottom, on outer circumference): lower end of the diameter AC

[Margin: iv.]

THEOR. 3. PROPOS. 4.

(Euclid's enunciation)

If in a circle two straight lines cut each other, neither being drawn through the centre, they will not bisect each other.

(Clavius's demonstration)

Let two straight lines AB and CD cut each other at E in the circle ACBD, neither drawn through the centre. I claim that it is impossible for them to bisect each other. For if one of them passes through the centre, it is certain that it is not bisected by the other: only at the centre, through which the other is supposed not to pass, is it divided into two equal parts. But if neither is drawn through the centre, then although one of them may sometimes be bisected by the other, still

the other will by no means be bisected. So let both AB and CD, if it be possible, be bisected at E. Then, once the centre F of the circle has been found (a), let the straight line FE be drawn from it to E. Since, then, FE is supposed to cut the line AB into two equal parts at E, it will cut it at right angles (b). By the same reasoning CD will be cut at right angles, since it is supposed to be bisected at E. Hence the right angle FED is equal to the right angle FEB — a part equal to the whole, which is absurd. Therefore, if in a circle two straight lines cut each other, and so on — which was to be demonstrated.

[Margin: (a) Elements III.1] [Margin: (b) Elements III.3]

Diagram. A circle ACBD with two chords crossing inside it and a line drawn from the centre to their crossing point.

- F (centre, upper middle): centre of the circle
- A (left, on circumference): endpoint of the chord AB
- B (right, on circumference): other endpoint of the chord AB
- C (lower left, on circumference): endpoint of the chord CD
- D (right, on circumference): other endpoint of the chord CD
- E (interior): the point where AB and CD cross; the line FE is drawn to it

THEOR. 4. PROPOS. 5.

[Margin: vj.]

(Euclid's enunciation)

If two circles cut each other, they will not have the same centre.

(Clavius's demonstration)

Let two circles ABD and EBD cut each other at B and D. I claim that they do not have the same centre. For let there be, if it be possible, the same centre C for both, from which two straight lines are drawn: CB to the point of section B, and CA cutting both circumferences at A and E. Since, then, C is supposed to be the centre of the circle EBD, the line EC will be equal to the line CB. Again, since C is supposed to be also the centre of the circle ABD, the line AC will likewise be equal to the same line BC. Hence the lines EC and AC will be equal to each other (a) — a part and the whole, which is absurd. Therefore, if two circles cut each other, and so on — which was to be shown.

[Margin: (a) Common Notion 1]

Diagram. Two circles intersecting at two points, with a supposed common centre C and two lines drawn from it.

- B (top): one of the two points where the circles cut each other
- C (centre region): the assumed common centre
- A (left, on the outer circumference): where the line CA meets one circle
- E (between A and C, on the other circumference): where the line CA meets the other circle
- D (bottom): the other point where the circles cut each other

THEOR. 5. PROPOS. 6.

[Margin: vij.]

(Euclid's enunciation)

If two circles touch each other internally, they will not have the same centre.

(Clavius's demonstration)

Let two circles AB and BC touch each other internally at B. I claim that they do not have the same centre. For let them have, if it be possible, the same centre D, from which two straight lines are drawn: DB to the point of contact B, and DC cutting both circumferences at A and C. Since, then, D is supposed to be the centre of the circle AB, the line AD will be equal to the line BD. Again, since D is supposed to be the centre of the circle BC, the line CD will be equal to the same line BD. Hence the lines AD and CD will be equal to each other (a) — a part and the whole, which is absurd. Therefore, if two circles touch each other internally, and so on — which was to be demonstrated.

[Margin: (a) Common Notion 1]

Diagram. Two circles touching internally, with a supposed common centre D and two lines drawn from it.

- B (top): the point of internal contact of the two circles
- D (centre): the assumed common centre
- C (lower left, on the outer circumference): where the line DC meets the outer circle
- A (near C, on the inner circumference): where the line DC meets the inner circle

SCHOLIUM

Euclid proposed this theorem only for circles that touch each other internally, since for circles touching each other externally — one being outside the other — it is manifest that they cannot have the same centre.

THEOR. 6. PROPOS. 7.

(Euclid's enunciation)

If in the diameter of a circle some point is taken which is not the centre of the circle, and from that point certain straight lines fall onto the circle, then: the greatest will be the one on which the centre lies, and the least will be the remaining part [of the diameter]; and of the others, the one nearer to the line drawn through the centre is always greater than one more remote; and only two equal straight lines fall from the same point onto the circle, one on either side of the least or of the greatest.

(Clavius's demonstration)

In the diameter AB of the circle ACDEB, whose centre is F, let any point G be taken other than the centre, and from G let any number of lines GC, GD, GE fall onto the circle. I claim that, of all the lines drawn from G to the circumference, the greatest is GA, on which the centre lies, and the least is the remaining part GB, which completes the diameter. Next, that the line GC, which is nearer to the line GA drawn through the centre, is greater than the line GD, which is farther from that same GA; and by the same reasoning GD is greater than GE; and so on with the other lines, if they were to be drawn, to infinity. Finally, that from G, on either side of the least line GB or the greatest line GA, only two lines equal to each other can be drawn. Let the straight lines FC, FD, FE be drawn from the centre F to C, D, and E. Since, then, the two sides GF and FC of the triangle GFC are greater than the side GC (a); but the lines GF and FC are equal to the lines GF and FA, that is, to the whole line GA; GA will also be greater than GC. By the same reasoning the line GA will be greater than GD, and than GE. Hence GA is the greatest of all the lines that fall from G onto the circle.

[Margin: (a) Elements I.20]

Next, since in the triangle EFG the side EF is less than the two sides FG and GE (b), and EF is equal to FB, FB will also be less than the two lines FG and GE. Therefore, when the common line FG is taken away, GB will still remain less than GE. By the

same reasoning GB will be less than GD and than GC. Hence GB is the least of all the lines that fall from G onto the circumference of the circle.

[Margin: (b) Elements I.20]

Diagram. A circle with a vertical diameter AB, a centre F, an interior point G on the diameter, and several lines fanning out from G to the circumference.

- A (top, on circumference): upper end of the diameter; the greatest line GA runs to it
- C (upper left, on circumference): endpoint of line GC
- D (left, on circumference): endpoint of line GD
- F (centre): centre of the circle; lines FC, FD, FE drawn from it
- G (below F, on the diameter): the assumed point from which the lines fan out
- E (lower left, on circumference): endpoint of line GE
- B (bottom, on circumference): lower end of the diameter; the least line GB runs to it
- H (lower right, on circumference): endpoint of the line GH (introduced later in the demonstration on the facing page)

(continuation of the demonstration of Prop. 7)

Again, since the two sides GF and FC of the triangle GFC are equal to the two sides GF and FD of the triangle GFD, while the whole angle GFC is greater than the angle GFD, the base GC will be greater than the base GD (c). By the same reasoning GC will be greater than GE; likewise GD will be greater than GE. Hence the line nearer to the one drawn through the centre is greater than the one more remote.

[Margin: (c) Elements I.24]

Now let there be made, on the other side, an angle BFH equal to the angle BFE, and let the straight line GH be drawn. Since, then, the sides EF and FG of the triangle EFG are equal to the sides HF and FG of the triangle HFG, and the angles contained by these sides, EFG and HFG, are equal, the lines GE and GH — one on each side of the least line GB or the greatest line GA — will be equal to each other (d). And it is settled that no other line can be equal to these two. For if from G another is drawn which falls above the point H, it will, being nearer to the line drawn through the centre, be greater than GH; but if it falls below H, it will, being more remote from that same GA drawn through the centre, be less than GH, as was shown. Therefore only two equal straight lines fall on either side of the least

line GB or the greatest line GA. And so, if in the diameter of a circle some point is taken, and so on — which was to be demonstrated.

[Margin: (d) Elements I.4]

SCHOLIUM

[recte: SCHOLIUM]

Although Euclid demonstrated only that the line nearer to the line drawn through the centre is greater than the more remote one when both lines lie on the same side of the diameter AB, nevertheless the same is also true if they are drawn on different sides. For example, if someone says that the line GD is nearer to the line GA than the line GH is, I claim that GD is greater than GH. For if, on the other side, a line GE is drawn equal to GH, as was said — namely, if the angle GFE is made equal to the angle GFH, and so on — the point E will fall between D and B, because GH and GE are equally distant from GA on account of the equality of the angles GFH and GFE, whereas GD is supposed to be less distant than GH, and therefore less distant than GE. Since, then, GD is greater than GE (c), the same GD will also be greater than GH.

[Margin: (c) Elements III.7]

Diagram. A circle with a vertical diameter AB, centre F, an interior point G, and lines fanning out from G to points on the circumference (the same figure that serves Proposition 7).

- A (top, on circumference): upper end of the diameter
- C (upper left, on circumference): endpoint of line GC
- F (centre): centre of the circle
- D (left, on circumference): endpoint of line GD
- G (below F, interior on the diameter): the point from which the lines fan out
- E (lower left, on circumference): endpoint of line GE
- H (lower right, on circumference): endpoint of line GH
- B (bottom, on circumference): lower end of the diameter

Furthermore, some convert this proposition as follows.

(the converse, as stated by some)

If within a circle a point is taken, and of the straight lines falling from that point onto the circle one is the greatest and one the least, while of the rest some are

unequal and some equal, then: the greatest will pass through the centre; the least will be the remaining part of the diameter; and of the others, the greater ones will be nearer to the greatest, while the equal ones will be equally distant from that greatest (or from the least).

(demonstration)

In the circle ABC let a point D be taken, from which any number of lines DA, DC, DE, DF, DB fall onto the circumference, of all of which let DA be the greatest and DC the least; and of DE and DF let DE be the greater; and finally let DE and DB be equal. I claim that DA passes through the centre; that DC is the remaining part of the diameter — that is, that DC is in a straight line with DA; likewise that DE, which is supposed greater than DF, is nearer to the greatest line DA than DF is; and finally that DE and DB are equally distant from that same DA (or DC). For, first, if DA does not pass through the centre, then when some straight line is drawn from D through the centre, it will be the greatest of all the lines falling from D (f) — which is absurd, since DA is supposed to be the greatest. Therefore DA passes through the centre.

[Margin: (f) Elements III.7]

Next, if DC is not in a straight line with DA, then when AD is produced in a straight line, there will be another line, distinct from DC, falling from D — namely the produced part of AD — which is the least of all (g); which is absurd, since DC is supposed to be the least. Therefore DC is the remaining part of the diameter.

[Margin: (g) Elements III.7]

Again, if DE is not nearer to the greatest line DA than DF is, then either they will be equally distant from it, or DE will be farther from it. If they are said to be equally distant from DA on both sides, they will be equal (h) — which is absurd, for DE is supposed the greater. But if they are said to be equally distant from DA on the same side, then DE and DF will be one and the same line, and so DE will not be greater than DF, which is contrary to the hypothesis. And if DF is said to be nearer to the greatest line DA, then it will be greater than DE (i) — which is even more absurd.

[Margin: (h) Elements III.7] [Margin: (i) Elements III.7]

Finally, if DE and DB are not equally distant from DA (or DC), then the one that is farther from DA will be the smaller (k) — which is absurd, since DE and DB are supposed equal.

[Margin: (k) Elements III.7]

Diagram. A circle with an interior point D near the bottom, from which several lines are drawn to the circumference.

- A (top, on circumference): endpoint of the greatest line DA (through the centre)
- B (upper right, on circumference): endpoint of line DB
- E (upper left, on circumference): endpoint of line DE
- F (left, on circumference): endpoint of line DF
- D (interior, lower middle): the point from which the lines are drawn
- C (bottom, on circumference): endpoint of the least line DC (in a straight line with DA)

THEOR. 7. PROPOS. 8.

[Margin: viij.]

(Euclid's enunciation)

If some point is taken outside a circle, and from that point certain straight lines are drawn to the circle, of which one is extended through the centre while the rest are drawn as one pleases, then: of those falling on the concave arc, the greatest is the one drawn through the centre, while of the others the one nearer to the line drawn through the centre is always greater than the more remote. But of those falling on the convex arc, the least is the one that lies between the point and the diameter, while of the others the one nearer to the least is always less than the more remote. And only two equal straight lines fall from that point onto the circle itself, one on either side of the least or of the greatest.

(Clavius's demonstration)

From the point A, outside the circle BCDE whose centre is K, let lines cutting the circle be drawn, of which AI passes through the centre, while the others, AH, AG, AF, are drawn as one pleases. I claim that of all of them the greatest is AI, which runs through the centre. Next, that the line AH, which is nearer to the line AI drawn through the centre, is greater than the line AG, which is more remote from that same AI; and by the same reasoning that AG is greater than AF. On the contrary, that the line AB is the least of all those [parts] lying outside the circle. Next, that the line AC, which is nearer to the least line AB, ...

[The demonstration of Proposition 8 continues.]

Euclid's *Elements*, Book III — Clavius's Commentary (continued)

THEOR. 7. PROPOS. 8 (continued) — lines drawn from a point outside the circle

[Clavius's demonstration, continued from the previous page.]

that the least line AB is less than the more remote AD; and by the same reasoning that AD itself is less than AE. Finally, that from A, toward either side of the least line AB or of the greatest line AI, only two straight lines equal to one another can be drawn.

From the center K draw to the points C, D, E, F, G, H the straight lines KC, KD, KE, KF, KG, KH. Now the two sides AK, KH of triangle AKH are greater than the straight line AH [Margin: a — *Elements* I.20]; but the straight lines AK, KH together are equal to AK, KI, that is, to the whole line AI. Therefore AI too is greater than AH. By the same reasoning AI is greater than AG, and than AF. Hence AI is the greatest of all the lines that fall from A into the circle.

Next, since the sides AK, KH of triangle AKH are equal to the sides AK, KG of triangle AKG, and the whole angle AKH is greater than the angle AKG, the base AH will be greater than the base AG [Margin: b — *Elements* I.24]. By the same reasoning AH will be greater than AF, and AG greater than AF. Therefore the line nearer the one drawn through the center is greater than the line farther from it.

Again, since in triangle ACK the straight line AK is less than the two lines AC, CK [Margin: c — *Elements* I.20], if the equals BK, CK are subtracted, AB will still remain less than AC. By similar reasoning AB will be less than AD, and than AE. Therefore AB is the least of all the lines drawn from A outside the circle.

Again, since within triangle ADK the two straight lines AC, CK fall from the extremities of the side AK, the lines AC, CK will be less than AD, DK [Margin: d — *Elements* I.21]. When therefore the equals CK, DK are taken away, AC will still remain less than AD. By parity of reasoning AC is less than AE, and likewise AD is less than AE. Therefore the line nearer the least line AB is less than the line more remote from it.

Finally, let the angle AKL be made equal to the angle AKC, and let the straight line AL be drawn. Since the sides AK, KC of triangle AKC are equal to the sides AK, KL of triangle AKL, and the angles AKC, AKL contained by those sides are equal, the

straight lines AC, AL — one on either side of the least line AB or of the greatest AI — will be equal to one another [Margin: e — *Elements* I.4]. And it is evident that no other line can be equal to these. For if a line be drawn from A falling beyond L, it, being more remote from the least, will be greater than AL. But if it fall between B and L, it, being nearer the least, will be less than AL, as has been shown. Therefore only two equal straight lines fall on either side of the least, or of the greatest. If, then, a point be taken outside a circle, and from that point certain straight lines be drawn to the circle, of which one is extended through the center, and so on. Which was to be demonstrated.

SHOLIVM. [transcribed verbatim; the C is dropped — read "SCHOLIVM."]

[*Clavius's scholium.*]

By the same reasoning, from A only two equal lines will fall on the concave circumference, one on either side of the greatest line AI. For let the angle AKM be made equal to the angle AKH, and let the straight line AM be joined. Since then the sides of triangle AKH are equal to the sides AK, KM, and the angles AKH, AKM are also equal, the bases AH, AM will be equal [Margin: f — *Elements* I.4]. Nor indeed can any other line be shown equal to these two. For if from A a line be drawn toward the side of H, it will be either greater or less than AH, according as it is drawn short of AH or beyond it, as is manifest from the demonstration of the theorem.

This proposition is true also when one of the lines falling from A touches the circle. For since such a line lies farther from the line drawn through the center, it will be less than all the others falling on the concave circumference — as is the straight line AM in the earlier figure. For when the straight line KM is drawn from the center K, the two sides AK, KF of triangle AKF will be equal to the two sides AK, KM, while the angle AKF is greater than the angle AKM. Therefore the base AF will be greater than the base AM [Margin: g — *Elements* I.24]. By the same reasoning any other line falling on the concave circumference will be greater than the straight line AM. And by the same reasoning the tangent line AM will be greater than all the others falling on the convex circumference. For since the two sides AE, EK are less than the two sides AM, MK [Margin: h — *Elements* I.21], if the equals KE, KM be subtracted, the remainder AE will be less than the remainder AM. And the like reasoning holds for the rest.

Although Euclid demonstrated only that the line nearer the one drawn through the center is greater than the more remote, and likewise that the line nearer the

least is less than the more remote — provided both lines lie on the same side, that of the greatest or of the least — nevertheless the same is true also when the lines are drawn toward opposite sides. This will be demonstrated in no other way than the same was demonstrated in the scholium of the preceding proposition concerning two lines drawn toward opposite sides.

Diagram. The figure for Proposition 8: an external point A stands above a circle whose center is K, with straight lines drawn from A to the several points of the circle.

- A (apex, above the circle): the external point from which the lines are drawn
- L, B, C, D (in a row along the upper / near circumference, left to right): points where the drawn lines meet the near (convex) arc
- E (upper right, on the circumference)
- M (to the right, outside the circle)
- K (center of the circle)
- F (right, on the circumference)
- G (lower right, on the circumference)
- H (lower right, on the circumference)
- I (bottom, on the circumference — the far endpoint of the line through the center; AI is the greatest line)

Diagram. The figure accompanying the scholium: the external point A above a circle with center K, illustrating the equal lines falling on the concave circumference.

- A (apex, above the circle)
- K (center)
- H (lower left, on the circumference)
- M (lower right, on the circumference)
- I (bottom, on the circumference — far endpoint of the line AKI through the center)

THEOR. 8. PROPOS. 9 — a point from which more than two equal lines fall is the center [Margin: ix.]

[Euclid's enunciation.] If in a circle some point be taken, and from that point more than two equal straight lines fall to the circle, the point taken is the center of the circle.

[Clavius's demonstration.] Let there fall from the point A, taken in the circle BCD, more straight lines than two — AB, AC, AD — equal to one another. I say that A is the center of the circle. For let the points B, C, D be joined by the straight lines BC, CD; and these being bisected at E and F, let the straight lines AE, AF be drawn. Since then the sides AE, EB of triangle AEB are equal to the sides AE, EC of triangle AEC, and the bases AB, AC are also assumed equal, the angles AEB, AEC will be equal, and therefore right angles. In the same way we shall show that the angles at F are right angles. Therefore, since the straight lines AE, AF divide the straight lines BC, CD in half and at right angles, each of them, when produced, will pass through the center, by the corollary of Proposition 1 of this book. The point A, then, in which they mutually cut one another, will be the center of the circle. For if some other point were the center, both lines would not pass through the center. If, then, a point be taken in a circle, and so on. Which was to be demonstrated.

ALITER (another demonstration)

If the point A is not the center of the circle, let the center found be E [Margin: b — *Elements* III.1], and through E and A let the diameter FG be drawn. Since then in the diameter FG, besides the accepted center, lies the point A, from which the straight lines AD, AC fall to the circumference, the straight line AD — which is nearer to the line AG drawn through the center E — will be greater than the straight line AC, which is more remote from AG [Margin: c — *Elements* III.7]; which is absurd, for the lines AD, AC were assumed equal. The same absurdity will follow if any other point besides A be set as center.

But if it should happen that the straight line drawn through the center E and the point A coincides with one of the three given equal lines — as, for instance, if the three equals be called AB, AF, AC, where EA coincides with AF — then AF will be the least of all the lines falling from the point A [Margin: d — *Elements* III.7], and so less than AB and AC; which is absurd, for it is assumed equal to each of them.

THEOR. 9. PROPOS. 10 — a circle does not cut a circle in more than two points [Margin: x.]

[Euclid's enunciation.] A circle does not cut a circle in more than two points.

[Clavius's demonstration.] For let the circle ABCDEF cut the circle AGDBHE, if it be possible, in more than two points A, B, and D; let these be joined by the straight lines AB, BD; and these being bisected at I and K [Margin: a — *Elements* I.10], let the perpendiculars IL, KL be raised from I and K to AB and BD [Margin: b —

Elements I.11]. Since then the straight lines IL, KL cut the straight lines AB, BD in the circle AGBDHE in half and at right angles, each of them will pass through the center of that circle, by the corollary of Proposition 1 of this book. Therefore the point L, in which they divide one another, will be the center of the said circle. In the same way we shall demonstrate that the point L is the center of the circle ABCDEF. Two circles, then, cutting one another mutually, would possess the same center [Margin: c — *Elements* III.5]; which is absurd. Therefore a circle does not cut a circle in more than two points. Which was to be demonstrated.

ALITER (another demonstration)

Let the same two circles cut one another, if it be possible, in three points A, B, and D. Let I be the center found of the circle AGBDHE [Margin: d — *Elements* III.1], and from it let the straight lines IA, IB, ID be drawn to the three said points; these will be equal to one another by the definition of the circle. Since then within the circle ABCDEF the point I has been assumed, from which more than two equal straight lines fall to the circumference, I will be the center of the circle ABCDEF [Margin: e — *Elements* III.9]. But this same point I was the center of the circle AGBDHE. Therefore two circles cutting one another mutually would have the same center [Margin: f — *Elements* III.5]. Which is absurd.

THEOR. 10. PROPOS. 11 — internally tangent circles: the line of centers passes through the point of contact [Margin: xj.]

[*Euclid's enunciation.*] If two circles touch one another internally, and their centers be taken, then the straight line joined to their centers and produced will fall on the point of contact of the circles.

[*Clavius's demonstration.*] Let the circle ABC touch the circle ADE internally at A; let F be the center of the circle ABC, and G the center of the circle ADE — which will necessarily be distinct from F, since two circles touching internally cannot have the same center [Margin: a — *Elements* III.6]. I say that the straight line extended through G and F falls on the contact A. For if it does not so fall, let it cut each circle in the points D, B, C, E, and from the contact A let the straight lines AF, AG be drawn to the centers F, G. Since then in triangle AFG the two sides GF, FA are greater than the side GA [Margin: b — *Elements* I.20]; and GA is equal to the straight line GD (because G is set as the center of the circle ADE); the straight lines GF, FA together will be greater than GD. When therefore the common GF is subtracted, FA will remain greater than FD. And since FA is equal to FB (because F was set as the center of the circle ABC), FB too will be greater than FD — the part greater than the whole; which is absurd.

But if anyone should wish to maintain that F is the center of the circle ADE, and G the center of the circle ABC, the argument will be framed in this way. In triangle AFG the two sides FG, GA are greater than the side FA [Margin: c — *Elements* I.20]. But the straight line FA is equal to FE (since F is set as the center of the circle ADE). Therefore the straight lines FG, GA are greater than FE. When the common FG is subtracted, GA will remain greater than GE. And since GA is equal to GC (because G is set as the center of the circle ABC), GC too will be greater than GE — the part greater than the whole; which is absurd. The same absurdity will follow if the center of the greater circle be placed outside the smaller. Therefore the straight line FG, produced, will not cut both circles, but will fall on the contact A. Hence if two circles touch one another internally, and so on. Which was to be demonstrated.

THEOR. 11 PROPOS. 12 — externally tangent circles: the line of centers passes through the contact [Margin: xij.]

[*Euclid's enunciation.*] If two circles touch one another externally, the straight line joined to their centers will pass through the point of contact.

THEOR. 11. PROPOS. 12 (continued)

[*Clavius's demonstration.*] Let the two circles ABC, DBE touch one another externally at B; let the center of the circle ABC be F, and the center of the circle DBE be G. I say that the straight line extended through F and G passes through the contact B. For if it does not so pass, let it cut the circumferences in C and E, and from the centers F, G let the straight lines FB, GB be drawn to the contact B. Since then in triangle FBG the two sides BF, BG are greater than the side FG [Margin: a — *Elements* I.20]; and the straight line BF is equal to FC (because F is set as the center of the circle ABC), and the straight line GB is equal to GE (because G is set as the center of the circle DBE); the straight lines FC, GE together will be greater than the straight line FG — the part greater than the whole, since FG contains, besides FC and GE, the further straight line CE — which is absurd. If, then, two circles touch one another externally, and so on. Which was to be demonstrated.

Diagram. The figure for Proposition 12: two circles touching externally at B, with a dotted straight line running through the two centers and the point of contact.

- A (top of the left circle)
- D (top of the right circle)
- B (middle, the point of external contact)
- F (center of the left circle ABC)

- G (center of the right circle DBE)
- C (lower center, on the left circle near the contact)
- E (lower center, on the right circle near the contact)

THEOR. 12. PROPOS. 13 — a circle does not touch a circle in more than one point [Margin: xiiij.]

[*Euclid's enunciation.*] A circle does not touch a circle in more points than one, whether it touch internally or externally.

[*Clavius's demonstration.*] Let the circles ABCD, AECF touch one another internally, if it be possible, in more points than one, namely A and C. Let the centers of these circles, G and H, be taken; these will be distinct [Margin: a — *Elements* III.6]. Through them let the straight line GH be extended in both directions; and this must fall on the contacts A and C [Margin: b — *Elements* III.11]. Therefore, since G is the center and the straight line AGHC is a diameter, AGHC will be bisected at the point G. By similar reasoning the same line AC will be bisected at H; which is absurd. For one straight line is bisected at one point only. For if GC is the half of the whole AC, then HC will necessarily be less than the half, since it is a part of the half, GC.

But if anyone should say that the straight line GH, extended toward the side of G, falls indeed on the contact A, but toward the side of H does not reach the contact C, but rather cuts both circles at I and K, as is plain in the second figure — (for one might say that in the preceding proposition it was shown that a straight line drawn through the two centers of circles touching internally falls on only one contact, and not on the other; which, however, no one could rightly affirm, since the demonstration of the preceding proposition suits both contacts; but whatever anyone may say, we shall show that this is absurd) — then from the centers G, H there must be drawn to the contact C the straight lines GC, HC. Let G, then, first be set as the center of the circle ABCD, and H as the center of the circle AECF. And since in triangle GHC the two sides GH, HC are greater than the side GC [Margin: c — *Elements* I.20]; and the straight lines GH, HC are equal to GK (because HC, HK, drawn from the center H, are equal, and GH is common), and the straight line GC is equal to GI (since these are drawn from the center G); the straight line GK too will be greater than GI — the part greater than the whole; which is absurd.

Let G, secondly, be set as the center of the circle AECF, and H as the center of the circle ABCD. Since then the straight lines HG, GC are greater than the straight line HC [Margin: d — *Elements* I.20]; and HC is equal to HA (since both are drawn from the center H); the straight lines HG, GC together will be greater than GA.

When therefore the common HG is subtracted, GC will remain greater than GA; which is absurd, since both are drawn from the center G. Therefore the circles will not touch internally in more points than one.

[*The external case.*] Now let the circles AB, CB touch one another externally in more points than one, near F. From D, the center of the circle AB, let the straight line DE be drawn to E, the center of the circle CB; this will necessarily pass through the contact F [Margin: e — *Elements* III.12]. If, then, they also touch in some point other than F, let them touch at B. When the straight lines DB, EB are drawn, the straight lines DB, EB will be equal to the straight lines DF, EF, that is, to DE itself; but they are also greater [Margin: f — *Elements* I.20], which is absurd. Therefore the circles will not touch externally in more points than one.

ALITER (another demonstration)

If the circles AB, CB touch externally in two points B and F, then the straight line BF drawn will fall within one of the circles, by Proposition 2 of this third book, and therefore outside the other, which is contrary to the same proposition. Therefore a circle does not touch a circle in more than one point, and so on. Which was to be demonstrated.

SCHOLIVM.

[*Clavius's scholium.*]

If Euclid's demonstration be rightly considered — the one by which he proved that a circle cannot be touched internally by a circle in more points than one — it appears to conclude principally that a circle cannot be touched by a circle in two or more points that lie far apart, but not that one and the same contact cannot have more than one point; although this very thing could readily be demonstrated by the same argument. Therefore, so that the matter may be left confirmed on every side — that a circle cannot touch a circle in more points than one — we shall show briefly that in one and the same contact there cannot be more points than one.

For let the circle ABC touch the circle ABD near A, and through their centers E and F — which are distinct [Margin: g — *Elements* III.6] — let the straight line EF be drawn; this will necessarily reach the contact, namely the point A [Margin: h — *Elements* III.12]. I say, then, that these circles touch one another only at the point A. For if they touch also at some other point, say B, then, drawing the straight lines BE, BF from B to the centers E and F, the straight lines EF, FB will be greater than the straight line EB [Margin: i — *Elements* I.20]. But EB is equal to EA,

since both are drawn from the center E. Therefore the straight lines EF, FB together will be greater than EA. When the common EF is subtracted, FB will remain greater than FA; which is absurd, since FB, FA are drawn from the center F to the circumference, and so by the definition of the circle are equal. Therefore the circles ABC, ABD touch one another only at the point A, and at no other.

There seems, moreover, to be a further theorem that should not be omitted in this place, namely:

If in the produced semidiameter of a circle a point be taken beyond the center, the circle from that point,

Diagram. The figure for the alternative demonstration of Proposition 9: a circle with a horizontal diameter FG; near the center lie the assumed point A and the found center E (marked with a dot), with lines drawn from A to points on the circumference.

- F (left endpoint of the diameter, on the circumference)
- G (right endpoint of the diameter, on the circumference)
- A (near the center — the assumed point)
- E (near the center, marked with a dot — the found center)
- B (upper left, on the circumference)
- C (lower left, on the circumference)
- D (bottom, on the circumference)

Diagram. The first figure for Proposition 10: two circles overlapping in a lens-shaped intersection, with the chords AB and BD and the perpendicular bisectors IL, KL meeting at L.

- G (top)
- A (upper left) and B (upper right): two of the assumed intersection points
- I (just below G, on chord AB): midpoint of AB
- F (far left)
- L (center — where the perpendiculars meet)
- K (below and right of L, on chord BD): midpoint of BD
- C (far right)
- E (lower left)
- D (lower right): the third assumed intersection point
- H (bottom)

Diagram. The second figure for Proposition 10 (the ALITER): two overlapping circles with straight lines drawn to the assumed intersection points.

- G (top)
- A (upper left) and B (upper right)
- F (far left)
- C (far right)
- E (lower left)
- D (lower right)
- H (bottom)
- [The internal center point I named in the text is not clearly labelled on the figure, even at high magnification — [?].]

Diagram. The figure for Proposition 11: a larger circle with a smaller circle tangent to it internally at the top point A; a horizontal diameter runs from E through the centers to D, with lines drawn from A to the centers (partly dotted).

- A (top, the point of internal contact)
- E (left endpoint of the diameter)
- C (left of center, on the diameter)
- G (near the center, on the diameter)
- F (center, on the diameter)
- B (right of center, on the diameter)
- D (right endpoint of the diameter)

Diagram. The first of the paired figures for Proposition 13 (internal contact): a circle with a smaller circle tangent internally at A, with the diameter AC and lines drawn toward the alleged second contact C.

- A (left endpoint of the diameter, the point of contact)
- B (top)
- E (below B, on the inner circle)
- G and H (near the center, on the diameter)
- C (right, the alleged second contact)
- F (lower left)
- D (bottom)

Diagram. The second of the paired figures for Proposition 13, showing the case where the line of centers, produced, cuts both circles at I and K rather than reaching the contact C.

- A (left endpoint of the diameter)
- B (top)
- E (below B)
- G and H (near the center, on the diameter)
- C (right)
- F (lower)
- D (bottom)
- I and K (on the diameter, where the produced line cuts the circles — set close between the two figures, hard to resolve even at high magnification) [?]

Diagram. The figure for the external case of Proposition 13: two circles touching externally at F, with the line of centers DE and the alleged second contact B.

- A (top left)
- C (top right)
- B (center, near the contact — the alleged second contact)
- D (center of the left circle AB)
- E (center of the right circle CB)
- F (the point of external contact)

Diagram. The figure for the scholium: two circles touching internally near the top, with centers E and F and lines drawn from the contact and from the alleged second point B to the centers.

- A (top, the point of contact)
- B (top, near A — the alleged second point)
- F (center, upper)
- E (center, below F)
- D (lower)
- C (bottom)

[The demonstration of the preceding proposition concludes here. Its enunciation, whose final clause opens this page, states that a circle described with that point as its center, drawn through the endpoint of the semidiameter, will touch the earlier circle at the said endpoint of the semidiameter, and will fall entirely outside it.]

[Demonstration.] Let there be a circle ABC with center D, and on the semidiameter AD, produced, let any point E be taken. With E as center and EA as radius, let the circle AF be described. I say that it touches the circle ABC at the

single point A. For the point E is either inside the circle ABC or outside it. If it is inside: since AE is greater than AD — that is, than DC, and hence, all the more, than EC — then EF too, which is equal to AE, will be greater than EC. The point F therefore lies outside the circle ABC, and so the circle AF lies outside the same circle ABC near the point F. All the more will the point F fall outside the circle ABC if E lies outside that circle. Now suppose the circle AF does not fall entirely outside the circle ABC so as to touch it at the single point A; then, if it be possible, let the circle AF cut or touch the circle ABC at some other point B, and let the straight line DB be drawn. Since, then, the point D has been taken in the diameter of the circle AF, other than its center E, DA will be the least of all the straight lines falling from D [Margin k: Elements III.7]. Therefore DA is less than DB — which is absurd, since DA and DB are equal, being drawn from the center D to the circumference of the same circle ABC. The circle AF, therefore, does not cut or touch the circle ABC at any point other than A, but falls entirely outside it. This is what was proposed.

But if a point is taken in the semidiameter *not* produced, on the near side of the center, then the circle described from that point as center, through the endpoint of the semidiameter, will likewise touch the earlier circle at the said endpoint of the semidiameter, and will fall entirely inside it. Thus, if in the semidiameter AE of the circle AF the point D is taken, and with D as center and DA as radius the circle ABC is described, this circle will fall entirely inside the other and will touch it at the single point A. For since it has just been shown that the circle AF falls entirely outside the circle ABC, conversely this circle ABC will fall entirely inside AF, so that the two touch one another at the single point A.

Diagram. A large circle with a smaller circle internally tangent to it at the top point A; a vertical diameter runs down through the point of tangency.

- A (top): point of tangency of the two circles
- B (upper left): second intersection point assumed in the reductio
- D (upper middle, on the vertical line): center of circle ABC
- E (middle, below D): center of circle AF
- G (lower middle): point on the vertical line inside the inner circle
- F (bottom): endpoint of the diameter of circle AF

THEOR. 13. PROPOS. 14.

[Dual numbering: Theorem 13 = Proposition 14 (Euclid, Elements III.14).]

[Margin: xiiij.]

[Euclid's enunciation.] In a circle, equal straight lines are equally distant from the center; and those that are equally distant from the center are equal to one another.

[Demonstration.] Let there be, in a circle ABCD with center E, two equal straight lines AB and CD. I say that they are equally distant from the center E. From the center E let two perpendiculars EF, EG be drawn to the lines AB, CD, and let the straight lines EA, ED be joined. The lines EF, EG will bisect the lines AB, CD [Margin a: Elements III.3]. Therefore, since the whole lines AB, CD are assumed equal, their halves too — namely the lines AF, DG — will be equal. Now since the squares of the equal lines EA, ED are equal to one another; and the square of the line EA is equal to the squares of the lines AF, FE [Margin b: Elements I.47], while the square of the line ED is equal to the squares of the lines DG, GE; the squares of the lines AF, FE will also be equal to the squares of the lines DG, GE. Taking away, then, the equal squares of the equal lines AF, DG, the remaining squares of the lines FE, GE will be equal, and therefore the lines EF, EG will be equal as well. Hence the lines AB, CD are equally distant from the center E, by the 4th definition of this book.

Again, let the lines AB, CD be equally distant from the center E. I say that they are equal to one another. For let the perpendiculars EF, EG be drawn once more from the center E to AB, CD, which by the 4th definition of this book are equal; they will bisect the lines AB, CD [Margin c: Elements III.3]. When, then, the lines EA, ED have been drawn, their squares will be equal. But the square of the line EA is equal to the squares of the lines AF, FE, and the square of the line ED to the squares of the lines DG, GE [Margin d: Elements I.47]. Therefore the squares of the lines AF, FE are equal to the squares of the lines DG, GE. And since, of these, the equal squares of the equal lines EF, EG are taken away, the remaining squares of the lines AF, DG will be equal, and consequently their doubles AB, CD will be equal too. Thus in a circle equal straight lines are equally distant from the center, and so on. Which was to be demonstrated.

Diagram. A circle with center E containing two chords, AB and CD, each with a perpendicular dropped to it from the center and joined to the center by radii.

- A (top left) and B (bottom left): endpoints of one chord AB
- C (bottom right) and D (top right): endpoints of the other chord CD
- E (center): center of the circle
- F (middle left): foot of the perpendicular EF on AB
- G (middle right): foot of the perpendicular EG on CD

THEOR. 14. PROPOS. 15.

[Theorem 14 = Proposition 15 (Euclid, Elements III.15).]

[Margin: xv.]

[Euclid's enunciation.] In a circle the diameter is the greatest line; and of the others, the one nearer the center is always greater than the one farther off.

[Demonstration.] In a circle ABCDEF with center G, let the diameter be AF; let the line HI be nearer the center, and CD farther off. I say that AF is the greatest of all, and that HI is greater than CD. From the center G let the straight lines GK, GL be drawn perpendicular to CD, HI. And since CD is farther from the center than HI, GK will be greater than GL, by the 4th definition of this book. From GK cut off a line GM equal to GL, and through M draw BME perpendicular to GK, and join GB, GC, GD, GE. Since, then, the perpendicular lines GM, GL are equal, the lines BE, HI will be equally distant from the center, by the 4th definition of this book [Margin a: Elements III.14], and therefore they will be equal to one another. Again, since the lines GB, GE are indeed greater than BE [Margin b: Elements I.20], but are equal to the diameter AF, the diameter AF too will be greater than BE. By the same reasoning AF will be shown to be greater than all the other lines. Next, since the sides GB, GE of the triangle BGE are equal to the sides GC, GD of the triangle CGD, and the angle BGE is greater than the angle CGD, the line BE will be greater than CD [Margin c: Elements I.24]; and hence HI, which was shown equal to BE, will also be greater than CD. In a circle, therefore, the diameter is indeed the greatest line, and so on. Which was to be demonstrated.

Diagram. A circle with center G and horizontal diameter AF; three further chords are drawn parallel to one another, each met by a perpendicular from the center.

- A (far left) and F (far right): endpoints of the diameter AF
- G (center): center of the circle
- C (upper left) and D (upper right): endpoints of the chord CD (the farther chord)
- K (top): foot of the perpendicular GK on CD
- B (left) and E (right): endpoints of the constructed chord BE
- M (center, above G): foot of the perpendicular GM on BE
- H (lower left) and I (lower right): endpoints of the chord HI (the nearer chord)
- L (bottom): foot of the perpendicular GL on HI

SCHOLIUM.

The theorem will be demonstrated in nearly the same way if several lines fall from one and the same point of the circumference.

For let several lines AB, AC, AD fall from a point A, of which AD passes through the center E. I say that AD is the greatest of all, and that AC is greater than the more remote AB. For when the lines BE, CE have been drawn, then, since in the triangle AEC the sides AE, EC are greater than the side AC [Margin d: Elements I.20], and since the lines AE, EC are equal to the lines AE, ED — that is, to the line AD — AD will be greater than AC.

Next, since the two sides AE, EC of the triangle AEC are equal to the two sides AE, EB of the triangle AEB, and the angle AEC is wholly greater than the angle AEB [Margin e: Elements I.24], the base AC will be greater than the base AB. By the same argument, AC will be greater than any other line more remote from the center.

Moreover, here too only two equal lines can be drawn from the point A to either side of the greatest line AD. For if an angle AEF is made equal to the angle AEC, and the straight line AF is joined, then — since the sides AE, EC are equal to the sides AE, EF, and the contained angles AEC, AEF are also equal — the bases AC, AF will be equal [Margin f: Elements I.4]. And no other line equal to these two can be produced; for any line drawn from A above AC will be less than AC, while any drawn below AC will be greater, as has already been demonstrated.

But if AB, AF are drawn to opposite sides of the diameter AD, and AF is said to be nearer the center than AB, it will be shown, as in the scholium to Proposition 7 of this book, that AF is greater than AB [Margin g: Elements III.14]. For if, as was just shown, a line AC equal to AF is drawn on the other side — that is, if an angle AEC is made equal to the angle AEF, and so on — the point C will fall between B and D. This is because AF and AC are equally distant from AD, on account of their equality, or at any rate on account of the equal angles AEF, AEC. Hence it follows that, since AF is assumed nearer the center than AB, AC too, being equal to AF, is nearer the center than AB, and consequently the point C lies between B and D. Since, then, AC is greater than AB — as has been demonstrated in this scholium — AF too will be greater than AB.

THEOR. 15. PROPOS. 16.

[Theorem 15 = Proposition 16 (Euclid, Elements III.16) — the famous horn-angle proposition.]

[Margin: xvj.]

[Euclid's enunciation.] The line drawn at right angles to the diameter of any circle, from the extremity of the diameter, will fall outside that circle; and into the space enclosed between that straight line and the circumference no other straight line can fall; and the angle of the semicircle is greater than any acute rectilinear angle, while the remaining angle is less.

[Demonstration.] In a circle ABC with center D, let the diameter be AC, and to it, from A, its extreme point, let a perpendicular be drawn. I say that this perpendicular line necessarily falls outside the circle. For if it falls inside the circle — say, like AB — then, when DB has been drawn, the two angles DAB, DBA will be equal [Margin a: Elements I.5]; but DAB is a right angle by construction, so DBA too will be a right angle, which is absurd. For the two angles in a triangle are less than two right angles [Margin b: Elements I.17]. The perpendicular, therefore, does not fall inside the circle; nor, for the same reason, does it fall on the circumference itself, but outside — say, like EF.

I say further that no other straight line can fall from A into the space between the straight line AE and the circumference AB. For let one fall, if it be possible — the straight line AG — and from D let a perpendicular DH be drawn to it, cutting the circumference at I, which will necessarily lie toward the side of the acute angle DAG, by the 2nd corollary to Proposition 17 of Book I. Since, then, in the triangle DAH the two angles DHA, DAH are less than two right angles [Margin c: Elements I.17], and DHA is a right angle by construction, the angle DAH will be less than a right angle; and therefore the line DA — that is, the line DI, equal to it — will be greater than DH [Margin d: Elements I.19], the whole greater than the part, which is absurd. No straight line, therefore, will be intercepted between AE and the circumference AB; rather, whatever line is drawn from A below AE will cut the circle.

Finally, I say that the angle of the semicircle, contained by the diameter AC and the circumference AB, is greater than any acute rectilinear angle; while the remaining angle — the angle of contact, contained by the straight line AE and the circumference AB — is less than any acute rectilinear angle. For since it has been shown that every straight line drawn from A below the perpendicular AE falls inside the circle, such a line will necessarily make with AC a rectilinear acute angle less than the angle of the semicircle; but with AE it will make a rectilinear acute angle greater than the angle of contact, since the former is a part of the angle of the semicircle, whereas the latter is something whole in comparison with the angle of contact. This is plainly evident once the straight line AB has been

drawn in any way below AE. For since this line AB falls inside the circle, as has been demonstrated, the rectilinear acute angle CAB will be less than the angle of the semicircle contained by the diameter AC and the circumference ABC, since the former is a part of the latter; while the angle of contact contained by the tangent line AE and the circumference ABC will be less than the rectilinear acute angle BAE, since the former is a part of the latter. And the same reasoning holds for all the other acute rectilinear angles, since they are all contained by the diameter AC, or the tangent AE, and by lines drawn from A below AE, all of which fall inside the circle, as we have demonstrated. The angle of the semicircle, therefore, is greater than any acute rectilinear angle, while the remaining angle of contact is less. And so, the line drawn at right angles to the diameter of any circle from its extremity, and so on. Which was to be shown.

Diagram. A circle with center D and vertical diameter AC; at A the perpendicular is drawn (falling outside), with the various auxiliary lines of the reductio near the top.

- A (top): extremity of the diameter, from which the perpendicular is raised
- C (bottom): other extremity of the diameter
- D (center): center of the circle
- B (just below A, inside): endpoint of the line AB assumed (wrongly) to fall inside
- E (upper right): endpoint of the perpendicular line AE
- F (upper left): endpoint of the perpendicular line (EF) [?]
- G (upper right, near A): endpoint of the line AG assumed in the horn-space
- H (upper right): foot of the perpendicular DH on AG
- I (near A, inside): point where DH meets the circumference

COROLLARIUM.

From this it is manifest that the line drawn at right angles from the extremity of the diameter of a circle touches that circle. For it has been shown to fall outside the circle. Hence it meets the circle only at that extreme point of the diameter.

Therefore, if we are required to draw, through a given point A on the circumference of a circle AB, a straight line that touches the circle at A, we shall draw from A to the center C the straight line AC, and to it we shall raise the perpendicular DAE. For this line will touch the circle at A, as has been demonstrated.

Diagram. A circle with the tangent line drawn horizontally across its top through the point of contact.

- D (left) and E (right): endpoints of the tangent line DAE
- A (top center): point of contact
- C (center): center of the circle
- B (lower right): a point on the circumference

EX ORONTIO.

[From Oronce Finé — an alternative demonstration.]

This same theorem can be demonstrated by direct proof, in the following way. Let AC be the diameter in a circle ABC whose center is D. From A, at right angles to AC, let the perpendicular AE be drawn, which I say falls outside the circle. For let any point G be taken on it, and let the straight line GD be joined. Since, then, in the triangle ADG the two angles DAG, DGA are less than two right angles [Margin e: Elements I.17], and DAG has been made a right angle, DGA will be less than a right angle. Therefore the line DG will be greater than DA [Margin f: Elements I.19], and consequently the point G will lie outside the circle. And the same reasoning holds for all the other points of the line AE. The whole line AE will therefore fall outside the circle. Now from A, below AE, let the straight line AH be drawn, which I say necessarily cuts the circle. For let an angle ADI be made equal to the angle EAH. When, then, the common angle DAH is added, the two angles ADI, DAH will together be equal to the whole right angle DAE, and therefore less than two right angles. Hence the lines AH, DI will meet at some point, say at I [Margin g: Elements I.13 [?] — reference glyph uncertain]. I say, then, that the point I lies inside the circle. For since the three angles in the triangle DAI are equal to two right angles [Margin h: Elements I.32], and the two angles DAI, ADI have been shown equal to the right angle DAE, the remaining angle AID will be a right angle, and therefore greater than the acute angle DAI. Hence the line DA is greater than DI [Margin i: Elements I.19]. Therefore DI will not reach the circumference; and for that reason the point I will lie inside the circle; and consequently the line AH will cut the circle. The remaining parts of the theorem are shown as before.

Diagram. The figure for Oronce Finé's demonstration (the same construction, reprinted at the head of this page): a circle with center D and vertical diameter AC, the perpendicular AE across the top, and the auxiliary line AH cutting the circle.

- A (top center): extremity of the diameter, where the perpendicular is raised
- C (bottom): other extremity of the diameter
- D (center): center of the circle
- E (upper right): endpoint of the perpendicular AE
- F (upper left): endpoint of the perpendicular line on the left
- G (top, between A and E): point taken on the perpendicular AE
- H (right side, lower): endpoint of the line AH
- I (near center, above D): meeting point of AH and DI
- B (left): a point on the circumference

SCHOLIUM.

Peletier [Jacques Peletier] holds that the angle of contact, which Euclid here proved to be less than any acute rectilinear angle, is nothing at all — that indeed no angle whatever is produced by the contact of a straight line and a circumference. To make his view clearer, we shall set out the digression he undertakes at this point. He speaks thus:

When I considered rather attentively the last part of this theorem, it did indeed occur to me, at first sight, that geometry is not sufficiently consistent with itself — that it even admits a contradiction within itself.

For, in the first place, it is beyond understanding that among quantities there could be a smallest — such as the angle here, which they call the angle of contact, or more properly of contact, which we have set down as less than every acute angle. No more does it accord with reason that there should be a greatest quantity — such as this angle of the semicircle, which is set down as greater than every acute rectilinear angle. For a quantity is called a quantity precisely because it consists of parts, and because with respect to it a thing is said to be equal and unequal. And a continuous quantity is divisible to infinity. And so, when I had come upon the first proposition of Book X, I began all the more anxiously to weigh by what means so open a contradiction, as it appeared, could be reconciled. For the first proposition of Book X runs as follows:

[Euclid, Elements X.1:] If from the greater of two quantities there be taken away more than the half; and again from the remainder more than the half; and this be done continually; there will at last be left a magnitude less than the lesser magnitude set out.

For example, let there be two angles: A, a rectilinear one; and BCD, an angle of contact (if it is indeed an angle). The first proposition of Book X requires that, if more than the half be taken away from the angle A; and again more than the half from the remaining part; and thus continually more than the half from the residual parts; there be left at last an angle less than BCD. I do not here supply the demonstration, since it depends on what follows. Yet there is no proposition in all of geometry that is (so to speak) more naturally true. This becomes clearer than daylight from numbers, in which are the images of all things. For who does not see that, given the two numbers 8 and 2, when you take from the eight more than its half — say five — and then from the remaining three more than its half — say two — there is left a unit less than the two set out? Nor is it to the purpose that Campanus [Campanus of Novara] there makes an exception, holding that the sense of the proposition is to be understood of quantities of the same kind. For that reconciliation is worthless; indeed it is contrary to the mind of Euclid, as we shall make plain when we come to that place. Nay, Campanus even contradicts himself, since in demonstrating the second proposition of Book XII, and in several other propositions concerning solids, he takes a straight thing away from a curved one.

We, then, shall settle this difficulty in the following way: by saying that a straight line which touches a circle makes no angle with the circumference — that is to say, that BCD ought in no way to be called an angle. For every angle consists in a section, not in a contact. And where the section ceases, the form of the angle ceases too. And, to put it in a word, all the kinds of angles are completed in a crossing (I take "crossing" and "section" here without distinction). For let two lines AB and CD, cutting one another at right angles at the point E, be conceived to be moved around — namely, about the fixed point E — so that from CD there comes FG. From the right angle AEC there will then arise the obtuse angle AEF; and from the right angle BEC the acute angle BEF. And when HK has been made, on the one side the angle HEA will become more obtuse, and on the other the angle BEH more acute; and so on continually, until the moving line reaches AB and is confined within the same bounds together with it. For then, the line CD being (so to speak) merged into the line AB, the angle vanishes.

Nor is the case any different with a curved line. For let there be, in a circle ABCA with center D, a line DE passing beyond the circumference and cutting it at A, a fixed point, about which let DE itself be carried around, through the points F, G, H. Then angles will be made continually varying with the circumference, at the

very point A; until, the crossing ceasing, the line ED becomes EK and touches the circle. And then the line DE is understood to be no longer inclined, but merged into the line BAC, so far as the angle is concerned — no otherwise than if BAH were a straight line. Nor does it tell against this that lines are drawn out and make the space CAK. For this is done solely by the line AC, which shrinks from the straight line, yet embraces it at the point A. Since, then, every angle consists in no more than a single point, it comes about that the point A is as unfit for constituting an angle as, a moment ago, was the point of section E of the straight lines. Perhaps you will say that the point A of the straight line remains in its straightness, and the point A of the circumference in its roundness; that the two are not the same point; but that the lines only, as it were, graze one another, because the one shrinks from the other utterly and at every point — so that contraries, set side by side, may be made more manifest. But this the senses do not admit. For two circles touching one another externally would leave an intervening straight line untouched: that is, if we conceived a circle touching the circle ABC externally at the point A — which the nature of lines does not allow. But grant that it could be so, so that nothing falls into thought that geometry does not somewhere represent — even so it will not press the point in the least. On the contrary, all the less will the contact of lines be an angle. For their meeting will gape open on either side. But let us confirm these things by geometric reasonings, through theorems.

Diagram. Two figures side by side illustrating Peletier's example: at the left a narrow rectilinear angle; at the right a circle with a tangent line, showing the angle of contact.

- A (apex of the narrow wedge at left): the rectilinear angle A
- C (upper left of the circle): the point of contact
- B (upper right of the circle): endpoint of the tangent line
- D (right of the circle): point on the circumference marking the arc of the contact angle BCD

Diagram. A star-like figure of straight lines crossing at a common point, illustrating the "crossing" (decussation): two lines AB and CD cross at right angles at E, with further lines FG and HK shown in rotated positions.

- E (center): the point of crossing
- A (right) and B (left): endpoints of the line AB
- C (top) and D (bottom): endpoints of the line CD
- F (upper left) and G (lower right): endpoints of the rotated line FG

- H (left) and K (lower right): endpoints of the rotated line HK

Diagram. A circle with the line DE swept about the fixed point A on the circumference; the moving line is shown in several successive positions (its outer endpoint marked E in each), passing through F, G, H, until it becomes the tangent.

- A (top, on the circumference): the fixed point about which the line turns
- D (lower left, inside) and center of the circle
- E (several positions across the top): successive positions of the outer endpoint of the swept line
- F (lower center): one intermediate position
- G (lower right): a further position
- H (right): a further position, toward the tangent
- B (left) and C (lower right): points on the circumference

[This opening continues Clavius's scholium (digression) on the angle of contact, appended to Book III, Proposition 16. Clavius first lays out the demonstrations that Jacques Peletier had proposed, then refutes them. The display-capital lines below are Peletier's theorem-statements as Clavius quotes them.]

The interior contact of two circles is not a quantity.

Let there be a circle AFBA whose centre is C and whose diameter is AB; and through the endpoint A of that diameter let a line DE be drawn at right angles. From the corollary to this sixteenth proposition [III.16] it is established that the line DE touches the circle AFBA, and therefore that the angle DAF is less than any acute angle — this on Euclid's own authority, namely by the last part of this sixteenth proposition. Now between the points C and B let a centre G be taken on the diameter AB, and with the radius GA let a second, larger circle AHKA be described. I say that FAH is a quantity. For it is established that the circle AHKA passes between the straight line DA and the curve AF, since the radius GA is greater than the radius CA. It is likewise clear, from the same corollary to this sixteenth proposition, that the line DE touches the circle AHKA, and therefore that DAH is less than any acute angle. Let a third circle AMNA be described according to the still greater radius LA. By the same corollary DAM too will be less than any acute angle. In this way, on to infinity, all the contacts that the line DE makes with the circles drawn through the point A — whose centres lie on the line AB — will be less than any rectilinear [acute angle]; and thus all equal, if only equality can be predicated of things that are not quantities. Therefore the contact DAM will be equal to the contact DAF; and so it comes about that MAF, the

interior contact of the circles, neither increases nor diminishes the contact DAM. Therefore MAF is not a quantity. Which was to be demonstrated.

We shall also prove that the interior contact of the circles is not a quantity in the following way. Since all circles are similar, the semicircles too will be similar. Therefore the angles made by a diameter and the circumference are equal in all circles, by the converse of the definition of similar segments. (For the mixed angles are not excluded from this equality of angles.) Hence the angle BAF will be equal to each of the angles KAH and NAM; and therefore the contact FAM adds nothing to the angle BAF. So FAM is not a quantity. Which was to be demonstrated. From this the next result follows.

The contact of a straight line with a circle is not a quantity.

For, the same construction remaining, if DAF is a quantity, it will certainly be divisible either by a straight line or by an oblique one. Not by a straight line, since the last part of this sixteenth proposition forbids it; nor by an oblique line, such as the line AM. For then FAM would be a part of DAF; but FAM is not a quantity, as we have just proved. Therefore FAM is not a part of DAF. So DAF can be divided neither by a straight line nor by an oblique one. Therefore DAF is not a quantity. Which was to be proved. From this the third result arises.

The exterior contact of two circles is not a quantity.

In the same construction, let the diameter BA be extended to a point P. Then with centre P and radius PA let a circle AQRA be described, touching the circle ABFA externally at the point A. I say that the contact FAQ is not a quantity. This indeed is clear from the later demonstration. For it is not divided by an oblique line, since FAM is not a quantity, by the first of these; nor by a straight one, since DAF is not a quantity, by the second of them; nor is DAQ a quantity, by the same. Therefore, since FAQ has no parts, it will not be a quantity. Which was to be proved.

From these results this proposition emerges, which no one in geometry has hitherto thought admissible.

The angles made by a diameter and the circumference, whether inside or outside the circle, are right angles, and equal to rectilinear right angles.

As in the later figure, the angle BAF is equal to the angle BAD, since nothing accrues to it on account of the contact DAF, which is not a quantity; and for the same reason QAP too is a right angle, and equal to DAP, since DAQ adds nothing. Which was to be proved.

This is what Peletier says here. But in the letter he writes to Cardano [Girolamo Cardano] he brings forward another demonstration, which he himself says is fuller. I have therefore decided to append it here. First of all he sets out this theorem.

In circles, the angles made by a diameter and the circumference are equal.

Let there be, upon the centre A, two circles BCDB and EFGE, whose diameters are BD and EG; and let EG cut both circles at the points E, B, D, and G. I assert that the two angles CBD and FED are equal. For if FED were greater than CBD — the reverse cannot occur, that CBD should in any way be greater than FED — then let several circles be described upon the same centre A, of which one, HKLH, will suffice here. It will finally come about, through continual increase, that an angle made by a diameter and the circumference — for example the angle KHL — is greater than a right angle: which is contrary to Euclid's own view, for he holds all these angles to be less than a right angle. Therefore the interior angles at B and E are equal to one another. Which was to be shown. The same verdict holds for the exterior angles. Nor is there any paralogism in this manner of demonstrating. For although there is no comparison between the angles they call angles of contact and rectilinear angles, nevertheless there will be some comparison of the angles made by the cutting of a straight line and the circumference to those same rectilinear angles. For angles of this kind, which are called mixed, plainly become greater and less.

Diagram. A set of concentric circles about a common centre A, with a fourth circle set within them, illustrating Peletier's theorem that the angles a diameter makes with the circumference are equal across circles.

- A (centre): common centre of the three concentric circles
- M (just below A): centre of the inner circle BFNB
- H (top), K (left), L (bottom): points of the outermost circle HKLH (diameter HL)
- E (upper, on vertical), F (left), G (lower, on vertical): points of the middle circle EFGE (diameter EG)
- B (on vertical, below E), C (left), D (lower, on vertical): points of the circle BCDB (diameter BD)
- N (bottom): point of the inner circle BFNB (centre M)

Having demonstrated these things in this way, I assert that the interior contact of the circles is not a quantity. Upon a centre M placed on the same line HL, let a circle BFNB be described with a radius as great as the circle EFGE — that is, with

MB set equal to the radius AE — a circle that touches the circle BCDB at the point B. It is clear that the angle FBD is equal to the angle FED, on account of the equality of the circumferences and diameters. Therefore this same angle FBD will also be equal to the angle CBD. So the contact CBF adds nothing to the angle CBD. Hence CBF is not a quantity. Which was to be proved. And from this proceeds the demonstration of those theorems which we have adduced in Book III.

Diagram. The horn-angle construction: a straight line DE touching, at the point A, a family of circles whose centres lie along the diameter, together with an exterior tangent circle AQRA above the line.

- A (centre of the line DE): common point of tangency
- D (left end) and E (right end): the tangent line DE drawn through A at right angles to the diameter
- P (top), Q (upper left), R (upper right): the exterior tangent circle AQRA, centre on the extension of the diameter above A
- C, G, L (stacked below A on the diameter): the successive centres of the tangent circles
- B (below): point of the circle AFBA (centre C)
- K (below B): point of the circle AHKA (centre G)
- N (bottom): point of the circle AMNA (centre L)
- F, H, M (on the left, inner to outer): the points where the circumferences of the three circles lie against the line

This, then, is Peletier's view of the angle of contact, which is wholly contrary to Euclid. For if Euclid had held

that the angle of contact is nothing at all, and that the angle of the semicircle is equal to a rectilinear right angle, why, I ask, would he have laboured so hard to demonstrate that the angle of contact is less than every acute rectilinear angle, but the angle of the semicircle greater? For what could be clearer than that nothing — which is what the angle of contact is, on Peletier's view — is less than any angle whatever? And what more evident than that a right angle, which is what Peletier makes the angle of the semicircle to be, is greater than any acute angle? Therefore we affirm that the angle of contact is truly an angle and a quantity; and further, that the angle of the semicircle is less than a rectilinear right angle. Hence we must dissolve all the sophisms Peletier has brought forward in this digression to confirm that the angle of contact is nothing.

First, then, we admit that it is beyond understanding that there should be a least among quantities; but we deny that we assert the angle of contact to be a least

quantity. On the contrary, we maintain that any angle of contact, even though Euclid showed it to be less than every acute rectilinear angle, can be divided into infinitely many parts — not indeed by a straight line, as Euclid demonstrated, and excellently, but by a circular line. Thus, given the angle of contact CAB, if through A a circle AEH is described larger than the circle ABG, touching the straight line CD at A, the angle of contact CAE will be made less than the angle of contact CAB. And if a still larger circle AFI is described, the angle of contact CAF will be much less than the same angle of contact CAB. And so, without end, we shall always make a smaller angle of contact. Therefore an inequality is found among the angles of contact, even though each of them is less than any acute rectilinear angle whatever — just as any acute angle is less than a right angle, and yet a smaller one can be given than any acute angle assigned, since it can be divided infinitely. Thus the acute angle BAC is less than the right angle BAF; and yet, if a straight line AD is drawn between AC and AB, the acute angle BAD becomes smaller. And if another straight line AE is drawn between AB and AD, the acute angle BAE will be much smaller, and there will never be an end to this diminution.

Diagram. A straight tangent line CD meeting, at the point A, a nested family of circles ABG, AEH, AFI, illustrating that angles of contact can be made ever smaller.

- C (left) and D (right): the ends of the tangent line CD, drawn through A
- A (top): the point of tangency
- B, E, F (on the left, inner to outer): points where the three circles lie against the line
- G, H, I (on the vertical, inner to outer): the lower points of circles ABG, AEH, AFI respectively

By parallel reasoning we assert that the angle of contact can be increased infinitely, so that, given any angle of contact, others greater than it may be assigned without number. Thus in the upper circular figure the angle of contact CAF is less than the angle of contact CAE, and much less than the angle of contact CAB; and so on in turn: if smaller and smaller circles than ABG are described, touching the straight line CD at A, the angle of contact will be perpetually increased, yet it will always remain less than any rectilinear acute angle. In the same way, given any acute angle, other acute angles beyond number can be given that are greater. Thus in the next figure the acute angle BAE is less than the acute angle BAD, and this much less than the acute angle BAC. And if another straight line is drawn between the perpendicular AF and the straight line AC, a still

greater acute angle will be made, and there will never be an end to this increase. Therefore, just as in this kind of increase we never arrive at any acute angle equal to a right angle, or greater than a right angle, unless the acute angle is changed into a right or an obtuse angle — an acute angle always being less than a right or an obtuse one — so too, in the growth of the angle of contact, we never come to any angle of contact equal to a given acute rectilinear angle, or greater than an acute angle, unless the angle of contact is changed into another mixed angle (which is indeed made by two lines cutting one another, such as the angle of a segment). But the angle of contact is always less than an acute angle.

Diagram. A fan of straight lines from a single vertex A, illustrating that ever-greater acute angles can be formed short of a right angle.

- A (vertex, upper left): the common vertex
- B (bottom, below A): the endpoint of the base line AB
- E, D, C (on the right, in ascending order): endpoints of the intermediate rays AE, AD, AC
- F (upper right): the endpoint of the perpendicular AF

In the same way we deny that we posit the angle of the semicircle as a greatest quantity. On the contrary, we assert that, given any angle of the semicircle — even though Euclid showed it to be greater than every acute rectilinear angle — infinitely many greater ones can be given. Thus in the same upper circular figure the angle of the semicircle FAI is greater than the angle of the semicircle BAG; and so on in turn: if larger and larger circles than AFI are described, touching the straight line CD at A, the angle of the semicircle will be perpetually increased, yet it will always remain less than the right angle CAI. Conversely, the angle of the semicircle EAH is less than the angle of the semicircle FAI, and the angle of the semicircle BAG much less still; and there will never be an end to this diminution, and yet each remains greater than any acute rectilinear angle, however infinitely it is diminished, as Euclid demonstrated. Just so, given any acute angle, innumerable others are found that are greater, and others that are smaller.

That the angles of contact are unequal among themselves, and not all equal, as Peletier will have it — and likewise the angles of the semicircles — is clear from this: that every angle consists in a single point and in the inclination of the lines, which do not lie in a straight line, as is established from the definition of a plane angle. For hence it comes about that the equality of angles of the same kind requires the same inclination of the lines, so that the lines of the one coincide entirely with the lines of the other, if one is superimposed on the other; for those things are equal which mutually coincide, according to the eighth common

notion [postulate 8]. Since, then, in angles of contact, and likewise in angles of semicircles, the same inclination is by no means always found — for when one is superimposed on the other, their lines do not correspond but altogether diverge from one another, as is evident from the figures above — not all angles of this kind will be equal among themselves. On the contrary, any angle of contact can be increased and divided infinitely by a curved line, even though it cannot be cut by a straight one, as Euclid rightly shows. And for this fact the following cause too can be given. If a line touching a circle is conceived to be moved about the fixed point of contact, it will continuously cut the circle until it touches it again: for only then will it cease to cut the circle. Hence, if it is understood to be inclined even in the least degree about that fixed point of contact, it will cut the circle, since a straight line can touch a circle at one point only, as we gathered from Proposition 2 of this book.

Therefore only those angles of contact, and likewise only those angles of the semicircles, will be equal among themselves which are made by equal circumferences: for in these alone do the lines mutually coincide. But angles of contact made by smaller circumferences will be greater, and those made by larger circumferences smaller. And angles of the larger semicircles will finally be greater, of the smaller ones smaller, as can be understood without difficulty from the figures above. For the lines of such angles do not mutually coincide.

It is established, then, that any angle of contact has parts; and that one can be exhibited equal to another, and again unequal — namely greater or less — just as we see happen in all other angles. This can also be said of any angle of the semicircle. Peletier therefore did not correctly understand the angle of contact to be a least quantity, and the angle of the semicircle a greatest.

Next, there is no reason why the first proposition of the tenth book should trouble Peletier. For it is to be understood of quantities both of the same kind and of different kinds, provided that either of them, when multiplied, can exceed the other; and of such a kind are not the angle of contact

and the acute rectilinear angle. For any angle of contact, whether it be taken twice, or three times, or four times, or finally as many times as one pleases, is always less than an acute rectilinear angle. For if any number of angles of contact — say a hundred — equal to one another were to exceed an acute rectilinear angle, or to equal it, then one of them too would be greater than, or equal to, that rectilinear acute angle which is the hundredth part of the proposed acute rectilinear angle; which is absurd, since any angle of contact is less than any acute rectilinear angle whatever, as was demonstrated by Euclid. Hence it will in

no way be permissible to conclude from that proposition that if from an acute angle more than the half is taken away, and again from the remainder more than the half, and this is done continually, there is at last left an acute angle smaller than the proposed angle of contact; because the angle of contact, as has been said, however many times it is multiplied, or however much it is increased, always remains less than the acute angle, and can never exceed it — which nonetheless is necessarily required for the demonstration of that proposition. For to demonstrate it, the smaller proposed quantity must be multiplied so many times that it exceeds the greater, as may be seen in all the commentators on Euclid. Hence Campanus [Campanus of Novara] — or rather Euclid himself through Campanus — when in the stereometry he takes a straight quantity from a curved one, or the reverse, as the first proposition of Book X will have it, always assumes such magnitudes of which either, when multiplied, can exceed the other.

Now by no means shall we grant to Peletier that an angle is made only by two lines cutting one another. For it suffices, in order that an angle be made, that two lines in a plane be inclined toward one another, provided that they do not lie in a straight line, as is established from the description of a plane angle handed down by Euclid — even though they do not cut one another mutually if they are produced; of which kind are circumferences that touch one another. Therefore they truly make an angle, as we said before.

Furthermore, Peletier's demonstration by which he attempts to show that the interior contact is not a quantity is of no weight. For although all the angles of contact made by a tangent line and the circumference are less than any acute angle, it does not on that account follow that they must all be equal among themselves; rather one can be greater than another, and less, as we have said — just as all acute angles are less than a right angle, and yet they are not all equal among themselves. So too all ants (to take an example from natural things as well) are less than a man or a mountain, and yet they are quite unequal among themselves. Therefore Peletier does not rightly conclude that the interior contact is nothing.

Next, we deny that the angles of similar segments are equal, as he assumes in the following demonstration. For Euclid did not signify this in the last definition of this third book; rather he taught that those segments are similar in which the rectilinear angles are equal, or which admit equal rectilinear angles — not those whose angles are equal. On the contrary, there is a great difference between the angles in the segments and the angles of the segments, as will be clearly established from Proposition 31 of this book.

From these considerations the remaining demonstrations of Peletier will be easily rejected. For the angle of contact made by a tangent line and the circumference is divided by a circular line, so that the interior contact is truly a part of it, since he has by no means shown the contrary. In the same way the angle of contact which two circles touching one another externally constitute is divided both by a circular line and by the straight line that touches them both. By the same reasoning that theorem is refuted by which he asserted that the angle of the semicircle is equal to a rectilinear right angle. For the right angle exceeds the angle of the semicircle by that angle of contact which the tangent line and the circumference make. Finally, Peletier was also deceived in that demonstration which he sent to Cardano. For although the angle of the semicircle in a larger circle is greater than the angle of the semicircle in a smaller circle, it does not on that account follow that any angle of the semicircle is greater than a right angle. For a right angle will always surpass any angle of the semicircle by that angle of contact which the circumference and the tangent line make. Just as, given any acute angle, others greater than it can be given, yet no acute angle will ever exceed a right angle, as may be seen in the preceding figure. It is established, then, from these considerations, that the angle of contact truly is an angle, but that Peletier's demonstrations are sophistical, of no weight, and utterly unworthy of a man who is a geometer.

Moreover, this dispute of mine about the angle of contact so moved and disturbed Peletier that he supposed his reputation to have been altogether wounded and offended. And so, about five years after the publication of my commentary on Euclid — that is, in the year 1579 — in order to defend himself, he wrote an Apologia against me. In it, however, he neither dared to overthrow or attack the solutions I have adduced here, nor to confirm his own opinion — that novel and unheard-of one — with new arguments (for he had none), but attempts to defend himself only with words and abuse, as those who read it through will easily judge. To this Apologia I would long since have replied — not so much to clear myself as to defend the truth — had not the weighty authority of most learned men, who judged it altogether unworthy of a reply, called me back from that resolve. But now, since this place seems to demand of me that in this second edition I should again take up the defence of the truth, I have thought it not out of place to repel briefly, and with as much moderation as I can, what that Apologia charges me with. For although I already did this in the year 1586, in my *Spherical Triangles*, where mention necessarily had to be made of the matter itself and of his Apologia — yet, since perhaps not everyone has seen that book of mine, I judged it worth my while, if I insert that whole reply here, where its proper place is, so that the fair-minded reader may understand that Peletier, without cause,

blazed up against me with as much distress of mind and anger as he displays, and falsely imputed many things to me; whereas I, on the contrary, poured out no injurious word or abuse against him — which he complains of in vain in the dedicatory letter, where he openly testifies that I do not refrain from abuse. And in that *Apologia* nothing so offends me as that Peletier charges me with having acted not sincerely but with animosity, and even out of envy, in either censuring or praising him in my commentaries — faults which I have always held to belong to a petty mind, and utterly foreign to a man liberally and in a Christian manner educated. But how far these are from the discipline of our Society, and from my own habit, no one who knows us and our ways is ignorant. For those who do not know our ways, this very book will bear witness that everything is said sincerely, nothing out of envy, nothing out of animosity — plainly, so that it may appear that the truth was sought, and no one's authority was despised. Nor do I so far derogate from myself (though I claim nothing) as to think that I alone am forbidden, if something in another's writings should seem false and the occasion offers, to show why it is less approved by me — provided (as the custom of prudent and well-mannered men requires) I do this without abuse or mockery. And this very book, which... that though you should read the whole book through, you would not find even a single word that has any appearance of slander or abuse. For in this scholium on the angle of contact I could swear plainly that nothing was further from my thought than to attack Peletier in a spirit of disparagement; but that I had this one purpose (if only I could attain it): that the truth should have its place. And this I did the more freely because I judged that I should be doing it not only with Peletier himself unwilling, but even willing,

— even on his own authority. For he did not hesitate to dissent not only from Cardano [Gerolamo Cardano] and Campanus [Campanus of Novara], but also from Proclus [Proclus], Theon [Theon of Alexandria], Apollonius [Apollonius of Perga], Eratosthenes [Eratosthenes], Pappus [Pappus of Alexandria], Ptolemy [Ptolemy], and Hippocrates of Chios [Hippocrates of Chios] — the lights of Geometry — and finally from Euclid himself, the master of them all. No wonder: for, as he himself truly said in the *Apologia*, in every discipline, and especially in Geometry, it is not authority that must be regarded, but truth. And yet I do not see how a man can be a friend of truth in whose eyes truth breeds hatred — unless perhaps he either thinks himself incapable of being deceived (he who now and then proclaims that those pillars of Geometry have erred at times), or loves and seeks truth in other men's affairs rather than in his own. For my part, if anyone should show that I have erred in some matter — which, given the weakness of human intellect, I see can happen — truly I shall be most grateful to him who leads me, wandering, back into the way of truth.

But Peletier [Jacques Peletier] proves his zeal for truth: he cannot bear insults. What insults, pray? you ask. He calls my demonstrations sophisms. Now at last I understand what he means by insults; for I am quite certain there are no others in this book of mine, and from these — if they are insults — I confess I do not abstain. But I, a simple man and ignorant of words, never considered them insults, since they were spoken truly. I had no other word by which to call demonstrations that are plainly false and spurious; nor did the customary usage of philosophers and mathematicians supply me a more fitting one. Moreover, since I saw that Peletier — a man most careful in his speech — called the genuine demonstrations of Campanus and Cardano "paralogisms," I supposed that in refuting his false demonstrations I might with more impunity use a very similar word. But if "sophism" is a more insulting word than "paralogism" in France, let him pardon one ignorant of that custom, and let him suppose that I meant to say "paralogisms."

And so that Peletier may plainly understand that I wrote those things not out of a desire to contradict, let him consider with me how much material for refuting him he gave me, had it been my intention to refute the man rather than to explain the matter then in hand — although I often let slip an opportunity for reproaching him that had fallen into my lap, lest I should seem to have singled him out as the one I most wished to attack. For how splendid an opportunity there was in Props. 4 and 8 of Book 1, and in Prop. 24 of Book 3 (which he makes his 23rd)! In all these he rejects the most ancient demonstrations of Euclid as non-geometric, since in them one must conceive in the mind one figure being superimposed on another — which he thinks foreign to the dignity of Geometry, led solely by this reasoning: that he supposes such superposition [superpositio] to be something mechanical, and that nearly all propositions can, as he says, be demonstrated in this way, even problems, in which something is set forth to be constructed. And as an example of this he adduces Props. 2 and 3 of Book 1, which are problems. Here indeed I could rightly have censured Peletier, had it been my purpose (as he falsely charges) — especially in this, that he supposed superposition would be equally useful in demonstrating problems and theorems. For he seems not to have sufficiently understood in what way geometers employ superposition. They do not wish the superposition to be actually carried out in fact (that would be something mechanical), but only in thought and mind, which is the work of reason and intellect. And so in theorems this kind of argument will have its place, but not in problems. For in theorems, on account of the equality of the magnitudes, anyone's intellect grasps without any hesitation that the one is or is not superposed so as not to exceed the other, even though that superposition is not actually made — as was done in Prop. 4 of Book 1. But in problems, in which

one is bidden to construct a magnitude equal to another, even if in his mind he thinks of the given magnitude being transferred to another place, he will nonetheless accomplish nothing thereby, since in fact no transfer has been made. So it is astonishing that Peletier could persuade himself that Props. 2 and 3 of Book 1, and nearly all the others, could be demonstrated by superposition, or by the transfer of lines and figures, if this manner of arguing were permitted in Geometry.

And certainly in this matter Peletier calls into accusation not only Euclid, but also Archimedes [Archimedes], than whom, by everyone's judgment, no one was keener and subtler in demonstration, together with his most weighty commentator, that most learned Eutocius of Ascalon [Eutocius of Ascalon], who use the same kind of argument in the books *On Plane Equilibria* [*De aequaeponderantibus*] — indeed he thereby refutes all geometers, who not rarely employ this kind of argument. But let us see to what our excellent Geometer here at last descended, when he criticized all other geometers. Peletier had seen (for he could not fail to see a thing so obvious) that, if he removed this manner of arguing, he would utterly overturn the whole of Geometry, since very many — and those the chief — propositions in Geometry are demonstrated from Props. 4 and 8 of Book 1 and from Prop. 24 of Book 3, which indeed cannot be demonstrated in any other way than by that superposition of figures — not actually existing, but grasped only in thought, as I said. Where, then, was he to turn? What was he to do? He devised something even more foreign to Geometry than that superposition of figures. For he was forced to assert that Prop. 4 of Book 1 is a definition of equal angles (and who ever heard such a thing?), and hence must be conceded without demonstration; and that Prop. 8 of the same book is a principle known through itself. And to make this more credible, he writes thus on Prop. 4 of Book 1: ("For by no more evident mark is the equality of figures recognized than from the equality of their sides.") And he confirms and, as it were, repeats the same on Prop. 8 of the same book, when he speaks thus: ("For who would deny that two surfaces are equal, whose sides are equal in quantity and number?")

These things Peletier contrived so that the said propositions of Euclid might be admitted without demonstration — things which are entirely false, so that it is greatly to be wondered at that he could put forward propositions so foreign to Geometry so inconsiderately. Of course it is true, as the philosophers assert: given one absurdity, the rest follow. For Peletier had assumed Props. 4 and 8 of Book 1 as principles — which is indeed false and absurd. Hence he necessarily fell into those absurdities which even those who have barely yet touched the first

principles of Geometry could easily avoid. For who does not see that a rhombus and a square, even if they have sides equal in quantity and number, can nevertheless be very unequal to one another? The same can be seen in equilateral pentagons too, and in other figures of many equal sides — which it is not the place here to explain at greater length. Since, therefore, in all many-sided figures inequality is found even when the sides are equal in quantity and number, it was necessary for Euclid to demonstrate that the equality of triangles is gathered from the equality of their sides — seeing that in other figures it is not so gathered. Therefore neither will Prop. 4 be a definition, nor Prop. 8 a principle; and consequently all the propositions that rest on them — which are well-nigh innumerable — must necessarily collapse, unless Euclid's demonstrations are taken up again, since they cannot be demonstrated in any other way. For the new demonstration of Prop. 4 which Peletier contrived is nothing else than (to speak with the logicians) a begging of the question [*petitio principii*]. This will be evident to anyone who considers it carefully. For in it only one triangle is constructed equal to the latter from the two given — nay, the very same triangle — and this indeed most ineptly, since to accomplish it Peletier describes circles which, however, he does not use in the demonstration — a thing wholly faulty in Geometry. Then he infers that this constructed triangle, which does not differ from the latter set out from the two, is equal to the first, without any demonstration; yet it is certain that this was from the beginning what was set forth to be demonstrated. Hence he manifestly begs the question, since with the same ease he could have concluded straightway at the outset that the given triangles are equal, even without employing any construction — inasmuch as that construction does not bear on the matter. I say the same of the demonstration of Prop. 24 of Book 3, which he likewise newly contrived: what their judgment is, into whose hands his commentaries have come, I leave to them. I pass over countless other places in which he misuses Euclid's propositions in demonstrating, so that what he most often does —

— namely, clumsily takes the second proposition of Book 1 for the third, and so on. For it is not now my intention to examine his commentaries, but only to drive off from myself the slanders which he frequently employed in his *Apologia*. Since this is so, what he falsely said of me I could truly say of him: that (to use his own words) I blush that he fell to being an interpreter of Euclid who does not now emend Theon or Campanus, but faults Euclid himself without cause — whereas I, as is fitting, defend Euclid from his slanders, and uncover and refute all the snares and fallacies he had set against him. It is clear, then, that I was not of a mind to refute Peletier, since I passed over so many and such great errors in silence — errors which even now I would not have brought to light, did I not wish

everyone both to understand how much benefit Peletier received from me (of whom he complains so bitterly), and to gain some fruit and profit from this brief disputation.

Now, so that it may appear with how unequal a spirit he treated me, I will briefly set out his slanders, and so refute and dissolve them that all may see with their own eyes that they are slanders. Yet in this I will use such moderation that I in no way forget the modesty that befits a religious man. For I will not answer him as he provoked me.

First, then, Peletier objects that, in citing his demonstrations, I conducted myself in such a way as to show that, if I could in any way have suppressed his name, I would gladly have done so. How false this is, those who have read my commentaries will easily judge; since everywhere I name him honorably, and ascribe very many demonstrations to him as his own — which, however, I demonstrate otherwise than he does, and much more briefly, and sometimes even (what is more) more universally, as is clearly established from what I demonstrated after Peletier both at Prop. 38 and at Prop. 45 of Book 1, to say nothing meanwhile of others which I could not unjustly have claimed for myself. So that I marvel what came into his mind, to reckon that done by me insincerely, and even enviously, which I feared someone might rather judge to have been done too ambitiously in his favor. As for the fact that at Prop. 16 of Book 3, and at the first two definitions of Book 5, I set out — "boldly," as he objects; "freely," as I confess — what I thought of his demonstrations, this I did, as I said before, not for the sake of injuring anyone, but of seeking the truth. For such is the nature and condition of those who devote themselves to the liberal arts, that even if one now and then attacks another's opinion, they do not on that account seem to contend among themselves with hatreds rather than with intellects.

What others' feeling may be I do not know; for my part, as I said above, I am so disposed that, if anyone were to warn me of some error committed in demonstration, I would be most grateful to him. And so that they might do this more freely, I earnestly asked not a few, and now again I wish the same men, and others too, to be entreated as friends. For I know how easily anyone may be deceived in his own discoveries; I see (as Peletier himself also wisely asserted in the *Apologia*) that it is common to all men to err. Next, that in the additions to Prop. 47 of Book 1 I made no mention of him is nothing he should take ill, since those propositions were not discovered by him. For some of them were demonstrated long before him, either by Campanus, or by Proclus, or by Theon; and some indeed I myself demonstrated before I had seen his demonstrations.

They are so manifest and easy that they need no proof, but are, as it were, corollaries of Prop. 47. So that no praise or glory at all would seem to accrue to him, even if I had most emphatically proclaimed them to have been discovered by him (which, however, is not true); since anyone who has tasted mathematical studies even with the tip of his lips can gather them from Prop. 47 with no trouble. So that I do not see why he should, after all, call those propositions of such great weight, when they are, by everyone's judgment, most trifling — so much so that in most of them not even Peletier himself adduces any demonstration, on account of their evidentness, but confesses that they need no proof.

Finally, there is no reason for him to be so angry with me on the ground that I did not attribute to him the construction of an equilateral and equiangular pentagon upon a given finite straight line: for in that construction I borrowed nothing at all from him — which I leave to be judged by those who have compared my construction with his. For mine is entirely different, and he, in his own (to say nothing of other faults), makes wonderful misuse of Prop. 9 of Book 3, since from it he proves that a certain point is the center of a circle which has not yet been described. A true geometer, surely, would have said that that point is of such a kind that a circle described from it, at the interval of any one of those three straight lines which are there shown to be equal, passes through the extremities of the two remaining equal lines. For Prop. 9 of Book 3 does nothing to the purpose in that place, since the proposition in hand can be concluded from the construction itself and from what has already been demonstrated (as I just said), even if that proposition were not true, or had nowhere been demonstrated. Peletier commits the same fault in all the propositions of Book 4, in which a circle is to be described either within or about a rectilineal figure. But if I am to be blamed on the ground that I did not ascribe to him one proposition which I demonstrated in a very different and briefer way, I do not see how he himself does not fall into the same fault, since he claims for himself this problem ("given two unequal lines, to find the power of the greater over the lesser") which was demonstrated by Theon many centuries before him — for this reason only, as I judge: that he demonstrated it in another way, though a longer one.

I say nothing of this other problem ("given a rectilineal angle, to construct a curvilinear angle equal to it") which in the *Apologia* he calls his own, and boasts to have been wanting until now — although I myself demonstrated that very thing, taken from Proclus (who flourished many centuries before him), at Def. 5 of Book 5, much more briefly and clearly. For, as I showed in that place, if the straight lines containing a given rectilineal angle are made equal, and about them

two semicircles (which will be equal) are described toward the same parts, at once a curvilinear angle will be constructed equal to the given rectilinear angle; nor is there need to use as many roundabout steps as Peletier employs to demonstrate that thing — although the force of his demonstration is the same as that of mine. And what is more astonishing, Peletier admits that he saw my demonstration, and nonetheless dares to claim it for himself as his own. See, then, why Peletier cries out that I tried, not very honorably, to filch not a few demonstrations in order to claim them for myself! But who does not see that he blames in another what he counts glorious in himself? And so I can condemn him of that same charge much more truly and justly than he can condemn me, since I never called myself the discoverer of those propositions, as he does, but only kept his name silent, for the reasons I have set out.

Next, he tries with many words to show, against me, that the angle of contact and the acute rectilinear angle are of the same kind. But somehow he strays, as they say, from the mark. For he only proves that both angles are contained under the same kind of quantity — that is, that both are plane angles; and that an acute rectilinear angle, or even a right one, can be composed of an angle of contact and another mixed angle: which neither I, nor any geometer ever, denied. I denied that those angles are of the same kind for this reason alone: that an angle of contact, however much multiplied, cannot exceed an acute rectilinear angle, as I clearly showed in the scholium to this Prop. 16. For hence it comes about that the one has no ratio to the other, and so they are in a certain way of different kinds — just as, for the same reason, a finite and an infinite straight line are not reckoned to be of the same kind, since the one has no ratio to the other, although they are comprehended under the same kind of magnitude, namely under "straight line." This, then, he may strike so as to seem to have taken aim; but though he do everything, he will never hit the mark; so far is he from it that —

For that magnitudes of which the one, multiplied, cannot exceed the other, are not reckoned of the same kind (as far as ratio is concerned), even though they be placed under the same kind of quantity — that is, under length, or breadth, or depth, or number — is clearly established from Def. 5 of Book 5, where Euclid explains plainly enough what sort of magnitudes those of the same kind must be, among which a ratio is found. Therefore let others consider why Peletier, a thoughtful man, called me — thinking about this very thing — an unthinking man, as though I had not correctly understood which magnitudes are of the same kind and which are not. For I never said (a thing he fastened upon me in order to carp) that of two magnitudes placed under different kinds of quantity — such as line, surface, solid, and number — either one could be so multiplied as to exceed

the other: in which, with no one gainsaying it, he wearies himself in vain to teach that this cannot be done. But I spoke only of those magnitudes which, though they lie in the same kind of quantity, may nonetheless be reckoned of a different kind: such as a rectilineal surface and a curvilineal or mixed one; and likewise a straight line and a curved one. For these seem to differ among themselves in such a way —

— that Aristotle [Aristotle] plainly affirmed that the one cannot be equal to the other. This, however (with Aristotle's leave be it said), is not everywhere true, since Archimedes in his book *On Spiral Lines* [*De lineis spiralibus*] demonstrated what straight line can be equal to the circumference of any given circle, and we ourselves have shown the same in the quadrature of the circle. Therefore Peletier will not be able to deny that in Def. 5 of Book 5 Euclid excludes certain quantities from the definition of ratio, and that they are on that account in a certain way of a different kind, as far as ratio is concerned, even though they be placed in the same kind of magnitude: such as the angle of contact and the rectilineal angle, and likewise a finite and an infinite straight line. Nor will he be able to deny that many magnitudes which most people excluded are comprehended in that same definition of ratio — such as curvilineal and rectilineal surfaces, as also the circular and the straight line, as we said a little before and will explain more fully at Def. 5 of Book 5.

But Peletier, lest he be compelled to abandon that opinion of his about the angle of contact which he had once imbibed, was unwilling to accept this exposition of Def. 5 of Book 5; indeed, to attack it, he seems in the *Apologia* to have strained every nerve — forgetful of himself, who had formerly expounded that definition in Book 5 in almost the same way. Except that he does not rightly infer from it that the angle of contact is not a quantity, on the ground that, when multiplied, it cannot (as he says) exceed any magnitude. For this (with his leave I say it) is false. For although the angle of contact, multiplied, cannot exceed a rectilineal angle, it will nonetheless exceed another angle of contact. Hence from that definition only this is rightly gathered: that the angle of contact has no ratio at all to a rectilineal angle, but that it does have some ratio to any other angle of contact. But whether he understood that definition in this way (as can be gathered from his commentaries on Book 5), or otherwise (as he seems to indicate in the *Apologia*), I do not much labor over. Certainly, that it is to be understood as I have expounded, no one who weighs Euclid's words carefully will deny. But truly, if Peletier will not put faith in me, he will surely — unless he wishes to be thought arrogant — put faith either in Proclus, that most weighty writer, who in Book 2 of his commentary on Book 1 of Euclid, at the definition of a plane angle,

understood that definition in the same way; or in Pedro Nunes the Portuguese [Pedro Nunes], whom he esteems so highly (and rightly so, for he was a man of the keenest intellect, inferior to none in Mathematics in our age) that he cites him alone as a witness in place of many thousands, and as an approver of his demonstrations — who, in the plainest words, both in his book *On the Errors of Oronce Finé* [Oronce Finé] and in his *Algebra*, explains that definition just as it has been set forth by me. Indeed, in the same place he asserts that from that definition it follows that the angle of contact has no ratio to a rectilineal angle, and a finite line none to an infinite: so that Pedro Nunes, whom Peletier had produced as a witness for himself, now speaks testimony for me.

And from these two passages of Pedro Nunes anyone will easily understand how without reason, and with how great a zeal for contradiction, Peletier insults me, when once and again he odiously demands whence I could have carried off an infinite line to him. For into the same charge (if it is a crime to name an infinite line by way of example) he draws even Pedro Nunes, his own witness — indeed all the philosophers, whose saying is unheard by no one except Peletier: that a finite line has no ratio to an infinite. Let him therefore cease to inquire of me whence I carried off an infinite line: for I shall answer, "from where Pedro Nunes, from where all the philosophers carried it off." What of it? Is not that sophism of Peletier's — I always slip up here; I meant to say that demonstration, and indeed his crowning one — by which he tries to show that Prop. 1 of Book 10, together with Prop. 16 of Book 3, cannot stand if the angle of contact is granted to be a quantity, refuted by Pedro Nunes, Peletier's own authority, on exactly the same reasoning by which it is refuted by me myself?

And if it is a genuine demonstration, I wonder what the reason is that Nunes — most learned in Geometry, and Peletier's own approver — approved it the less; and why the demonstrations of the same Peletier, by which he makes plain (as he thinks) that the angle of contact is not a quantity, did not move that same Nunes in the least. For this (unless I am mistaken) is what those words of Nunes declare: ("If anyone should wish to embrace Peletier's opinion about the angle of contact.") For if he had considered them demonstrations, he would surely have said that Peletier's teaching must be retained on that point; for geometrical demonstrations are of such a kind that they wring out assent and exclude all doubt, and in no way allow anyone to be pulled apart by a wavering opinion, so as now to assent, if he wishes, and now, if he does not wish, to dissent. Behold why Peletier despises others' judgments on the strength of Nunes's testimony — behold the splendid testimony that Pedro Nunes gave to his demonstrations, so

that he may bear with a more even mind that they are held to be demonstrations no more by me than by that approver of his!

Thirdly, he cries out that it is false that I said he holds the angle of tangency [angulus contingentiae] to be nothing: for, he says, he nowhere said it is nothing, but that it is not a quantity. Is that so, indeed? But in the category of Quantity — a thing which is neither a point (for who would say that that inclination is a point?), nor a quantity — by what other name should it be called than Nothing? But however he put it, it is truly not only astonishing but monstrous, something like a portent in Geometry, to suppose that the angle of contact is not a quantity, when, afterward added to other angles, it makes a curvilinear angle equal to a rectilinear one. For what geometer ever joined to a magnitude that which is not a quantity, so as to make it equal to another? I pass over the fact that a three-sided curvilinear figure enclosed within three mutually tangent circles would, on Peletier's view, have no angle — because those three tangencies are not angles — even though it is contained by three distinct lines; which is altogether new and unheard-of among geometers. Likewise, if four or more circles were mutually tangent, so as to make a curvilinear quadrilateral, or a figure of more sides, it would have no angle. And again, if two straight lines containing an angle were tangent to one and the same circle, that three-sided figure, having its third side curved, would have but a single angle. If all these things are absurd, Peletier's opinion is not consistent with reason.

But perhaps I have said too much toward overturning that "Nothing" of Peletier's, for the defense of which he brings nothing. But since — lest he seem to do nothing on behalf of his "Nothing," when the matter itself cannot be defended — he carps at my words, I will defend those words, which indeed he distorts by I know not what sleights of hand, so that I seem to say that Nothing is less than any angle whatever; and (to entangle me, I suppose, as a simple man) he asks me what kind of speech that is. Let him see to it, from whose workshop it came. For I do not acknowledge any such statement; I nowhere said that Nothing is less than any angle whatever, except as following from Peletier's own view. But of course, being a vehement man, in order to avenge that "Nothing" of his, he fabricated another "Nothing" and attributed it to me, so that he might fight it with impunity. But how like a wrestler he fights! How pleased he is with himself here, while with marvelous elegance he turns against me the very argument by which he had been attacked! For he argues thus: ("The angle of contact is nothing; the angle of contact is greater than another angle of contact; therefore an angle is greater than nothing. But Clavius sets the same as less than nothing. Therefore the angle of contact is greater than nothing, and the same is less than nothing.") Then, as

though he had throttled that "Nothing" fashioned by himself, he exclaims: ("Behold Clavius's arguments — which, in the end, are they? Peletier's sophisms, or rather Clavius's figments, since it is he who says his angle of contact is nothing, not I?")

But so that this man of straw and shadow may not go on demanding anything to no purpose, I will point out a man with whom, if he wishes, he could contend with credit. In order to show that the angle of contact, on Euclid's view, is truly an angle, and that the angle of a semicircle [text uncertain at the center gutter of these two lines; here the reading joins with] is less than a right rectilineal angle of the circle [?], I argued as follows. If Euclid had held the angle of contact to be utterly nothing (that is, as Peletier understands it, that it is not an angle, or not a quantity) and had held the angle of a semicircle equal to a right rectilineal angle, why, I ask, would he have sweated so hard to demonstrate that the angle of contact is less than every acute rectilineal angle, but the angle of a semicircle greater? For what is clearer than nothing — of what sort the angle of contact is, on Peletier's view, that is, than —

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than that what is not a quantity should be less than any angle whatever? And again, what is more evident than that a right angle — the very sort of angle that Peletier makes the angle of a semicircle to be — is greater than any acute angle? Let Peletier therefore acknowledge that we did not take that "Nothing" of his in a bad sense, and let him take his revenge in this way: by refuting this argument of mine. For if I said that the angle of contact is nothing — rather than asserting, following Peletier's own view, that it is nothing — I shall gladly give in. It seems that Peletier either did not understand my argument or was unwilling to understand it, unless one were to say that he deliberately chose to cavil at my words — which he also seems to do in a great many other places. For I never said that the angle of contact is less, or greater, than nothing. I affirmed only that any angle of contact whatever is less, or greater, than some other angle of contact; and it was not I who said this to be nothing, but Peletier, while Euclid duly demonstrated the same angle to be less than any acute rectilineal angle.

But so that Peletier may understand that I have learned dialectic — which he denies — let us examine for a moment that neat and subtle syllogism of his, by which he has so charmingly constructed that "Nothing" he fabricated for himself; so that it may appear with what right he calls others ignorant of dialectic. For to me at least that syllogism seems badly turned, and fit to be sent back to the anvil. Since it operates in the third figure, in it the major extreme (as the dialecticians

put it), which is "Nothing," ought to be predicated in the direct case of the minor extreme, which is "greater than the angle of contact," in this way:

The angle of contact is nothing. The angle of contact is greater than an angle of contact. Therefore something that is greater than an angle of contact is nothing.

This conclusion does indeed follow correctly from the premises — of which the first is Peletier's, not mine, while the latter is mine, and Proclus's, indeed Euclid's too. But that conclusion of Peletier's — "Therefore an angle is greater than nothing" — can in no way be inferred from the premises. For if, when he says "angle," Peletier means the angle of contact, then the middle term is taken up [into the conclusion] — the term that is the subject in both premises; and Aristotle in the *Prior Analytics* [*Analytica priora*], and all the dialecticians, cry out that this is illicit. But if he means some other angle, then a term is taken into the conclusion of which no mention at all was made in the premises; and there is no one so leaden in dialectic as not to know that this is no more permissible. Nor let Peletier contend that mention was made of the angle in the minor extreme, where it was said that the angle of contact is greater than an angle of contact. For the angle placed in the minor extreme stands in the oblique case, whereas in the conclusion it is made the subject in the direct case — which, as all the logicians teach on Aristotle's authority, cannot be done without fault.

To make this plain, let us use the very exercise-ground we learned from him. Suppose someone argued thus:

An angle in a semicircle is a right angle. An angle in a semicircle is greater than an acute angle. Therefore an acute angle is greater than a right angle.

Who, if he has any grounding in dialectic, would approve such reasoning, when the premises are true but the conclusion false? Such is that syllogism of Peletier's, who among the untrained crowd wished to seem a second Chrysippus. The conclusion that would correctly follow from the premises would be this: "Therefore some angle that is greater than an acute angle is a right angle." Yet I think he should be pardoned, since he professes to be a geometer by profession — and of geometry I frankly confess myself ignorant in the way I am ignorant of his dialectic, if it recoils from Aristotle's.

Next, Peletier admits that he does not understand how I can say both that no least rectilinear angle can be given, and yet that the angle of contact is less than any acute rectilinear angle — less than the least acute rectilinear angle — even though

that word "least" ... [Peletier claims] the angle of contact is less than every rectilinear acute angle, so that I may oblige the man without reluctance, although from the scholium on this proposition 16 he could [have seen] that I could say this, in the same way that we say an obtuse rectilinear angle is greater than every rectilinear obtuse angle, and that a rectilinear acute angle is less than every obtuse rectilinear angle. Likewise, just as it is one thing that a rectilinear acute angle is less than ... it is less than every obtuse rectilinear, than that a rectilinear acute angle is the least of obtuse rectilineals: because that acute angle is not obtuse, just as the angle of contact is not rectilinear, [as I noted] in book 2, on the definition of a right, obtuse, and acute angle. These matters belong rather to the grammarians than to the geometers.

Then, so as [not to make my book] too thick, he complains that I recited his own words; he reproaches me for the very thing in which I [wished readers] to see plainly that I reported his opinion sincerely and faithfully, [in a way] he himself would not have brought his mind to do in reciting others' words. For it was of much greater [importance] — what reason had I to make my labor, my book, thicker with Peletier's words? — that one who embraces in his volume neither rare nor empty commentaries on all the books of Euclid, [comprising the demonstrations only] of those books, has judged [it thick] enough. For in his *Apologia*, lest it should seem empty of matter, he inserts three demonstrations that have nothing whatever to do with the point, as I said above; the earlier of which he wrongly makes his own, as I noted above, [claiming] to have made it clearer, whereas I did so far more briefly and lucidly (unless love of my own things deceives me), as will most clearly appear from my *Practical Geometry* [*Geometria Practica*]. If it was permitted to him to attach companions and attendants in a new fashion, why should it not be permitted me [to weave others' opinions] into my writings? Either, then, let him reproach everyone — and above all [Oronce Finé, in exposing] Finé's paralogisms — or let him admit that he wrongly charged me with this as a fault. Peletier accuses me of having wrongly interpreted his opinion — [as if,] had I brought his view forward [in other words]. Indeed anyone will easily persuade himself...

Fourthly — to pass over these lighter matters — he thinks this his trump card: that he shows me laboring to prove that angles of contact are unequal to one another; because I wrote that equality of angles of the same kind requires the same inclination of the lines, so that the lines of the one coincide entirely with the lines of the other, if the one is superposed on the other, according to the eighth postulate. In this he says I err doubly. First, because I say that equality of angles of the same kind requires the same inclination of the lines — whereas a

rectilineal angle has been shown by me to be equal to a curvilinear one, and so of the same kind as it, even though the inclination of the lines is not the same in both. Secondly, because I hold that angles of contact are unequal to one another on the ground that they do not mutually coincide.

Now, if I understood that I had erred at all in this, and acknowledged the error — as befits a man of honorable and liberal upbringing — I would give thanks to Peletier as my corrector and emender (with whatever intent he did it). But as it is, since, having weighed the whole matter again and again, I see no fault at all in it, I shall so resolve the objections raised as still to be grateful to Peletier for giving me occasion to explain this passage more carefully. In that passage, then, I understood angles of the same kind to be those that have one straight line and one circular line — such as angles of contact and of semicircles, which we were then discussing. Therefore, since the straight line of the one coincides with the straight line of the other, but the circular line does not likewise coincide with the circular (unless the circles be assumed equal), it follows that those angles are unequal to one another, since the one exceeds the other. By the same reasoning, if two curvilinear angles of equal circles are given equal, the lines of the one must coincide with the lines of the other, if the one is superposed on the other. And if Peletier attacks this doctrine, let him know that he is now making war not on me but on Proclus, who in book 3 on the first book of Euclid, at [proposition 4, teaches exactly the same as I. For he says (Of angles,

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however, we shall take the equality according to the correspondence of the sides in rectilineal figures; but in all the rest that are of the same species — as in lunes, in systroids, and in figures convex on both sides, etc.) And further on: (Things given equal coincide with one another. This, however, is not true in all cases, but in those that are similar in species. By similar in species I mean, for example, a straight line to a straight line, and a circumference to a circumference of the same circle, and angles contained by similar lines similarly situated. Of these, I say, that whichever are given equal coincide with one another.)

Is it not clearer than daylight, from these words, that Proclus grants those angles alone to be equal whose straight line and whose curve mutually coincide? Rashly, then, does Peletier throw up against me the rectilineal and the curvilinear, the triangle and the square, and other such things, of which there was no talk in that passage — since they are not of the same species, and so maintain their equality even though the one does not coincide with the other. So that I now begin to fear that our Peletier is more eager for contention than for truth.

Lastly, so as to leave nothing untouched, he calls me ignorant not only of geometry but of logic too: because in book 5 I said that Proportion is not rightly divided by some into proportion of equality and of inequality, since many proportions of inequality are also irrational. Now I — though I am not one to claim anything for myself in any field — have nevertheless always maintained that in these studies, in which I am moderately versed, I am plainly no novice. How little I can do in either subject, let others judge, who are free from love and hatred; but I think I have thus far answered Peletier himself in such a way that I may now perhaps seem less ignorant of geometry and dialectic than he had supposed.

Now, so that he may see that I am neither obstinate nor at odds with the precepts of the dialecticians — which he stirs up — I gladly grant him that the division he charges me with censuring is sound, provided that in each member of the division the thing divided [*Divisum*] be understood. Nor indeed did I ever deny this, since elsewhere I use similar divisions. I contended only, in that passage, that Proportion is, in my judgment, more rightly divided universally by a twofold division: the first into rational and irrational proportion, the second into proportion of equality and of inequality — which I hold to be most true, and think no one will deny who weighs the matter more carefully — since both pairs of dividing members exhaust the whole thing divided, as the logicians say; whereas if the first member of the first division, that is, rational proportion, be cut into proportion of equality and of inequality, these dividing members extend more widely than the thing divided, unless the thing divided is understood in them.

And that twofold division commends itself to me the more because there are those who divide Proportion first into proportion of equality and of inequality, and then divide the latter into rational and irrational proportion — that is, in the opposite manner to the former group. So, in order to settle this controversy and to remove the doubt as to which do it more rightly, the former or the latter, I resolved that Proportion should be cut by two divisions, each of which is most absolute and most perfect. In no other way, I judge, will all be the more convinced, if a body be cut by a twofold division — first into living and non-living, and then into what is white, black, or of mixed color — than if a living body be divided into white, black, and mixed-colored, for the reason already given; even though this subdivision is sound, if the thing divided be always understood. I could adduce six hundred divisions of this kind; but it is enough that I have set out to the prudent reader my plan in dividing Proportion, and why I preferred that twofold division to the subdivision of others.

But if we were willing to be such sharp and severe judges of every single word or impropriety that now and then slips out through inadvertence, why, no writer would be free of some fault — not even Peletier himself, as is partly clear from what has been said, and as can partly appear from his other demonstrations too: which, if one cared to test against that fixed norm of the dialecticians, there would surely be no lack of material for censure. But it is no part of my plan to scrutinize others' faults out of a zeal for refutation; only, where occasion offers itself, to set out my own opinion — such as it is — about the views of others.

I ask only this of him (since he boasts himself so keen a dialectician that he seems to despise others — though from the syllogism above, which he turned against me, it is plainly evident how skilled he is in dialectic): from what logic did he draw this argument — "All angles of contact are less than any acute rectilineal angle: therefore they are all equal to one another"? And likewise this: "Angles of semicircles, the greater the circles from which they are made, are the greater: therefore we shall at last arrive at one that is greater than a right rectilineal angle"? — on which point he writes to Cardano [Cardanus] that there is no paralogism. I am utterly amazed that he could fall into hallucinations so open, and altogether unworthy of a geometer. But arguments of this kind have been more than sufficiently refuted by me in the scholium on this proposition 16, with the most evident counter-instances brought against them.

Then, that he presses me as if I understood too little what rational and what irrational proportion is, gives me little trouble. For it is plain that he said this out of a desire to disparage me — whereas, since I have everywhere defined these proportions after the view of the weightiest authors, he will not be able to show that I committed any fault in the matter. Certainly my commentary on Euclid's book 10 abundantly declares whether I understood them or not. Finally, his charge that in the definitions of book 5 I confound the name of "proportion" with the name of "ratio" is in no way true. For in plain words I taught, in definition 4 of book 5, that in my commentary I would, with many geometers, call the comparison of two quantities "Proportion," and the relation of proportions "Proportionality"; although in the text, following the translator, I call the former "Ratio" and the latter "Proportion." For I did not wish to change anything in the text of Euclid. And so there can be no ambiguity in my words.

From what we have said, it appears sufficiently, I think, that those most learned men whom I mentioned at the outset judged Peletier's *Apologia* empty, and unworthy of a reply, not without cause. Yet I, lest I should seem to scorn a man I have always thought worthy of praise, was moved by the occasion and thought it

best to reply in a friendly spirit. Let him hold that the angle of contact is not a quantity, and hence that the angle of a semicircle is equal to a right rectilineal angle: I for my part shall defend the contrary view, until something else is demonstrated to me by someone. For Peletier's reasonings are fallacious and contain nothing of probability in them, as I showed in the scholium on this proposition 16, where I dissolved them all; and to my own solutions he replied not so much as a single word (except for those points I brought forward above from book 10) — which yet was what most concerned the *Apologia*. So that not without cause have very many judged that he wrote that *Apologia* not out of zeal for truth, but so as not to appear to have yielded to the truth.

Nor let anyone suppose that I alone hold the angle of contact to be truly an angle, and the angle of a semicircle to be less than a right rectilineal angle. For I can cite many authors of that view, and those of great weight: Theon, Campanus [Campano], Pedro Nunes [Petrus Nonius], and (as Nunes reports) Archimedes, and Jordanus [Jordanus Nemorarius]; indeed also — what I value most — Euclid himself, and his most celebrated commentator Proclus; to say nothing of not a few most excellent and learned men from among the French, chief of whom is François de Foix-Candale [Franciscus Candalla], sprung from the most illustrious family of Foix, who published a notable volume on the geometric Elements of Euclid, where, on this very proposition 16, he teaches most openly that angles of contact are truly angles, from the definition of a plane angle, and that some are greater than, equal to, and less than others. Those who think otherwise (he doubtless means Peletier — for down to this day no one else has written on the matter) he says conclude many absurdities from false...

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false suppositions. To this let there be added Henri de Monantheuil [Henricus Monantholius], royal professor of the mathematical arts, who, when he had seen Peletier's *Apologia* written against me, published a learned little work against Peletier on the angle of contact.

That the studious reader may see what Proclus thinks in this matter, I shall bring forward a few things from his commentaries on Euclid's book 1, which I noted in passing, and from which it will clearly stand established that his view is utterly contrary to Peletier's contrivance. First, then, he writes thus in book 2 on the first book of Euclid, on the definition of a plane angle:

(For two circumferences, by cutting one another or by touching one another, make angles. Moreover, an angle is contained by a straight line and a convex circumference, as the horn-like [Cornicular] angle.)

Now by the name "Cornicular angle" he understands the mixed angle of contact. For a little before he had said that the Cornicular angle is less than every rectilineal — which is the property of the angle of contact alone. Then, in the same book, on the definition of a right, obtuse, and acute angle, he has this:

(For the Cornicular angle is less than every right angle, since [it is less] than an acute angle too, yet is not itself acute: the semicircular angle likewise is less than every right angle, but is not acute.)

What is clearer than that Proclus here asserts the angle of a semicircle to be less than a right angle? Again, in book 3, on proposition 4 of Euclid's book 1, he writes thus:

(For we shall learn that the Cornicular angle is always unequal to an acute angle, and never equal; and the semicircular angle likewise, and that the passage from greater to less does not take place altogether through equality.)

See how openly he teaches that the angle of a semicircle cannot be equal to a rectilineal angle, and that the passage from greater to less therefore does not occur through equality — both of which Peletier denies, and he dares to call the latter a paralogism. Finally, in the same book 3, on proposition 23, these words are found:

(Since no mixed angle can be equal to a rectilineal, etc.)

And yet Peletier does not hesitate to make the angle of a semicircle, which is mixed, equal to a right rectilineal angle, against Proclus's view. From these things I judge it clear — to say nothing of the rest — that Proclus holds the same about the angle of contact and of the semicircle as I wrote against Peletier. And who would deny that Proclus's authority is greater, and his arguments better, than Peletier's?

In passing, I also think the reader should be warned here that what I taught about the angle of contact that occurs in a circle, following the view of Euclid and Proclus, is true likewise of the angle of contact that is produced in the conic sections — namely in the parabola, hyperbola, and ellipse. For, as Apollonius of Perga [Apollonius Pergaeus] demonstrates in book 1, proposition 32, into the space that lies between a conic section and the straight line tangent to it, no other

straight line falls; and so that angle of contact is less than every acute rectilineal angle, while the remaining angle out of the right angle (namely, if a perpendicular be raised from the point of contact to the tangent line) is greater than every acute rectilineal angle. If, therefore, as Peletier supposes, the angle of contact is not a quantity (for the reasoning here is the same as in the circle), then all angles of contact will be equal to one another — that is, as he would have it, not unequal — and each of the remaining angles will be equal to a right rectilineal angle. Here indeed a greater absurdity appears, to the senses, in an ellipse that has a very small breadth, and in a hyperbola that seems almost a straight line. For the angles set at the vertex of that ellipse and hyperbola are seen to be very unequal; so that it is altogether incredible — unless demonstrated by firm reasoning — that those angles of contact set at the vertices of the sections are equal to one another, and the remaining angles out of the right angles are also equal to one another; because in that ellipse the tangent line is seen to recede more from the circumference of the ellipse than in a circle, while in that hyperbola it recedes less. But let us leave these matters to be examined at another time: now let us return to the interrupted exposition of Euclid.

EX CARDANO. [From Cardano]

A certain quantity can be continually and infinitely increased, and another infinitely diminished; and yet the increase of the former, however great it be, will always be less than the decrease of the latter.

Diagram. A composite figure illustrating Cardano's claim, in two parts. On the left: a horizontal straight line **E–A–F** with **A** as the point of contact at its midpoint; below **A**, two nested circles are drawn tangent to the line at **A** (the larger with diameter toward **B** at the bottom, the smaller inner circle passing through the interior points), the angle of contact **BAE** being formed between line **EA** and the largest circle. On the right: a rectilineal acute angle with its vertex **G** at the top and rays descending to the base points, so that angle **HGI** is the outermost acute angle and the intermediate rays **GK**, **GL** fall within it.

- **E** (top left, on tangent line): left endpoint of the tangent
- **A** (top center): point of contact; vertex of the angle of contact
- **F** (top right, on tangent line): right endpoint of the tangent
- **B** (bottom of largest circle): marks the largest circle forming angle **BAE**
- **C** (interior, on middle circle): a smaller tangent circle
- **D** (interior, on innermost circle): the smallest tangent circle
- **G** (apex of right-hand figure): vertex of the acute rectilineal angle

- H (base, leftmost): outer ray endpoint (angle HGI)
- K (base): intermediate ray endpoint
- L (base): intermediate ray endpoint
- I (base, rightmost): outer ray endpoint (angle HGI)

The accompanying explanation reads: Let there be proposed the angle of contact **BAE** and the acute angle **HGI**. If, then, other smaller circles **AC**, **AD** be described, tangent to the straight line **EF** at **A**, the angle of contact will be continually increased, as was said. If again, between the straight lines **GH**, **GI**, other straight lines **GK**, **GL** fall, the acute angle will be continually diminished. And yet the angle of contact, however much it be increased, is always less than the acute angle, however much this be diminished.

EX CAMPANO. [From Campanus]

Moreover, from this proposition 16 it is evident that the argument is faulty which Brison [Bryson] used in squaring the circle, as Aristotle reports. Namely:

There is a passage from less to greater, or the reverse, and through all the means; therefore through the equal. Or: it happens that one finds this greater and that less than the same; therefore it happens that one finds an equal.

Diagram. A circle **ABC** with vertical diameter **AC** (**A** at top, **C** at bottom) and **B** on the circumference at the left. From the fixed point **A** at the top, three lines radiate to the right to the points **F** (uppermost), **E** (middle), and **D** (lower), representing successive positions of the far end of the rotating diameter as it turns about **A** until it becomes tangent at **A**.

- A (top): the fixed extremity of the diameter, about which it rotates
- B (left, on circumference): a point of the circle
- C (bottom): the other extremity of the diameter AC
- F (upper right): a swept position (through point F)
- E (middle right): a swept position (through point E)
- D (lower right): a swept position (through point D)

The accompanying explanation reads: Let a circle **ABC** be described, whose diameter **AC** is understood to be moved about the fixed extreme point **A**, through the points **D**, **E**, **F**, until it becomes tangent to the circle at **A**. This being granted, it is manifest that so long as the straight line **AC** cuts the circle, an acute angle is made less than the angle of a semicircle; but as soon as it ceases to cut it, a right

angle is made greater than the same angle of a semicircle. Since, therefore, the passage has been made through all the intermediate rectilineal angles, the flaw of the earlier argument is plain. Similarly, because no rectilineal angle is found equal to the angle of a semicircle — (for a right or an obtuse angle is greater; and an acute one, less) — it is established likewise that the latter consequence is faulty.

Book III

Problem 2. Proposition 17. [Euclid, *Elements* III.17]

[Margin: xvj.]

Proposition (Euclid). From a given point to draw a straight line that touches a given circle.

Demonstration (Clavius). From the point A a line is to be drawn that touches the circle BC, whose centre is D. Draw the straight line AD, cutting the circle BC at B. Then, with centre D and radius DA, describe the circle AE; and from B draw BE perpendicular to AD, cutting the circle AE at E. Finally, drawing the straight line ED — which cuts the circle BC at C — join the straight line AC; I say that this touches the circle BC at C. For the two sides DE, DB of the triangle BDE are equal to the two sides DA, DC of the triangle CDA, each to each, as is clear from the definition of a circle; and the angle D, contained by those sides, is common to both. Therefore [Margin: a — I.4] the bases BE, CA and the angles DBE, DCA standing upon them will be equal. But DBE is a right angle by construction; therefore DCA will also be a right angle. And so CA, being drawn perpendicular at C, the extremity of the radius CD, will touch the circle by the corollary of the preceding proposition. From the given point A, therefore, the straight line AC has been drawn touching the circle BC at C — which was to be done.

Diagram. Two concentric circles about the centre D: the outer circle AE and the inner circle BC. The straight line AD is drawn from A through B to the centre D; the radius DE reaches the top point E; BE is drawn perpendicular to AD; ED cuts the inner circle at C; and AC is the constructed tangent.

- A (upper left, on outer circle): the given external point
- E (top, on outer circle): foot of the perpendicular BE on the outer circle
- C (upper right, on inner circle): point of tangency
- B (left of centre, on inner circle): where AD meets the inner circle

- D (centre): centre of the given circle BC and of the auxiliary circle AE

Scholium

The practical construction of this problem is easily drawn from the demonstration itself. You will, however, find an easier construction at proposition 31 of this book.

But the following problem we shall solve together with Cardano [Gerolamo Cardano].

[Added problem: a common tangent to two circles]

Proposition. Given two circles, neither of which includes the other, to draw a straight line that touches both circles.

First case — equal circles. Let there first be two given equal circles AB and CD, whose centres E and F are joined by the straight line EF. To EF let the perpendiculars EA and FC be drawn, cutting the circumferences at the points A and C. I say that the straight line drawn through A and C touches both circles. For since EA and FC, the radii of equal circles, are equal, and [Margin: b — I.28] also parallel — because the angles E and F are right angles — [Margin: c — I.33] EF and AC will likewise be equal and parallel; and therefore [Margin: d — I.20 (?)] (printed "20"; sense requires I.29) the angles A and C are right angles. Hence, by the corollary of proposition 16 of this book, the straight line AC will touch both circles, since it makes right angles at the extremities of the radii.

Diagram. Two equal circles AB and CD standing side by side, their centres E and F joined by EF; the radii EA and FC are drawn perpendicular to EF up to the tangent line AC, so that AEFC forms a rectangle. The page prints this equal-circles construction in several near-identical variants (external and overlapping positions).

- A (top left, tangency point on circle AB)
- C (top right, tangency point on circle CD)
- E (centre of circle AB)
- F (centre of circle CD)
- B (bottom, on circle AB)
- D (bottom, on circle CD)

Second case — unequal circles. Next let two unequal circles AB and CD be given, whose centres E and F are again joined by the straight line EF. With centre E, the

centre of the larger circle, and with EF as radius, describe the circle FH (provided the larger circle does not pass through the centre of the smaller). Then, having drawn EA perpendicular to EF, cut off from it the portion AG equal to the radius of the smaller circle CD; and from G draw GH perpendicular to EA, as far as the circumference of the circle FH just described. Draw the straight line HE, and make the angle FEI equal to the angle HEA; and through F draw the straight line FC parallel to EI. I say that the straight line drawn through the points I and C touches both circles AB and CD. For let there be cut off from IE the straight line IK equal to AG, that is, to the radius FC of the smaller circle,

*(text continues *

Diagram. The construction for the unequal-circles common tangent, printed large at the foot of the page (and repeated at the head A large circle of centre E carries an inner auxiliary circle; a smaller circle of centre F stands to the right. The tangent line runs across the top through A (and the neighbouring point I) to C; the auxiliary radii and the equal triangles HEG, FEK are drawn to the interior.

- A (top left, on the large circle): tangency point on the larger circle
- I (top, just right of A): point through which the tangent IC passes
- C (upper right, on the smaller circle): tangency point on the smaller circle
- G (interior left): foot of the cut-off AG
- K (interior, right of G): where $IK = AG$ is cut off on IE
- H (interior right): endpoint of GH on the auxiliary circle
- E (centre): centre of the larger circle
- F (right of centre): centre of the smaller circle
- B (bottom): opposite point of the larger circle
- D (lower centre-right): opposite point of the smaller circle

equal to the radius of the smaller circle, so that the remainders EG and EK are also equal; and draw the straight line KF. Since, then, the sides HE, EG of the triangle HEG are equal to the sides FE, EK of the triangle FEK, and the angles contained by them are equal, by construction, [Margin: e — I.4] the angles HGE and FKE will be equal. And accordingly, since HGE is a right angle by construction, FKE will also be a right angle. Again, since CF and IK are equal and parallel by construction, [Margin: f — I.33] IC and KF will likewise be equal and parallel; and therefore the angle EIC, [Margin: g — I.29] being equal to the exterior angle FKE, will be a right angle; [Margin: h — I.29] and so ICF too will be a right angle. Wherefore, by the corollary of proposition 16 of this book, the

straight line IC will touch both circles, since it makes right angles at the extremities of the radii — which was proposed.

Diagram. The full unequal-circles construction: a large circle of centre E with an inner auxiliary circle, and a smaller circle of centre F overlapping it at the right. The tangent IC runs across the top; the equal right triangles HEG and FEK are drawn to the interior, together with KF.

- A (top): tangency point on the larger circle
- I (top, right of A): point through which the tangent passes
- C (upper right): tangency point on the smaller circle
- G (interior left) and K (interior centre): feet of the equal cut-offs
- H (interior right): endpoint of the perpendicular GH
- E (centre) and F (right of E): centres of the larger and smaller circles
- B (bottom): opposite point of the larger circle
- D (lower right): opposite point of the smaller circle

Diagram. A row of four small figures repeating the same two-circle construction in the successive positions (the "modes" in which the circles can stand). Each carries the same lettering — A, I, C at the top; G, K, H in the interior; E, F at the centres; B, D at the foot — but at this size several labels do not resolve. [Several labels illegible]

On the five and seven modes. It is sufficiently clear that, if the given circles are equal, this can happen in five ways: either one falls outside the other; or they touch each other; or they cut one another through their centres; or they do not, yet so that either the centres lie in their common segment, or else outside it.

Likewise, if the given circles are unequal, it can happen in seven ways. For either the smaller falls wholly outside the larger; or it touches it; or it cuts it, so that its centre lies on the circumference of the larger, or within the larger circle — yet on the condition that the circumference of the smaller passes short of the centre of the larger; or the centre of the smaller lies outside the larger circle; or it falls on the centre of the larger; or, finally, so far within that the circumference of the smaller includes the centre of the larger.

Diagram. Two circles, the smaller lying largely within the larger and overlapping it, illustrating the construction begun from the smaller circle. The interior carries the auxiliary lines and the points G, K, H.

- I (top left) and C (top right): the tangent passes through I and C

- A (below I): tangency point
- E (left of centre) and F (right): the two centres
- G (below E), K (centre), H (right of K): interior construction points
- D (bottom): opposite point
- [One further foot-point at lower left illegible — [?]]

A further problem: to describe circles tangent to given circles. Since, from what has been demonstrated, we can draw a straight line that touches a circle both through a point set on the circumference of the circle and through a point lying outside the circle — through a point on the circumference by the corollary of proposition 16, and through a point outside the circle by this proposition 17 — and since it has also been shown in this scholium that a straight line can be drawn to touch two circles, provided neither of them is wholly included in the other, it will not be out of place to demonstrate here how, through a given point, a circle is to be described touching a given circle, whether internally or externally; and likewise by what method a circle is to be described touching two other circles, one of which is not wholly within the other.

First. Let there be, in the circle AB whose centre is C, a given point A on the circumference, through which a circle is to be described touching the circle AB at A. Draw from A through the centre C the straight line AC, and take on it any point D; with radius DA describe the circle AE, which will touch the circle AB at A. And if the point D lies beyond the centre C, the circle AE will fall wholly outside the circle AB; but within it, if the point D lies on the radius AC — as we demonstrated in the scholium to proposition 13 of this book.

Then. Outside the same circle AB let a point G be given, through which a circle is to be described touching the circle AB. Draw from G through the centre C the straight line GCA, and, bisecting it at D, describe from D through A the circle AE, which will pass through the given point G and will touch the circle AB at A, by what we demonstrated in the scholium to proposition 13 of this book.

Otherwise. Let the point E again be given, outside the circle AB. Draw the straight line EA at random from the given point E, so that it cuts the circle and does not pass through the centre; then draw from A through the centre C the straight line AC. The angle EAC will be acute, since it is part of the angle of a semicircle, which is less than a right angle. At E let the angle AED be made equal to the angle EAC; then the straight line ED will meet AC at D, because the two angles DAE and DEA are less than two right angles. Since, then, the angles DAE and DEA are equal, [Margin: i — I.6] the sides DA and DE will also be equal.

Therefore the circle described from D through A will pass through the given point E and will touch the circle AB at A, by what we demonstrated in the aforesaid scholium to proposition 13 of this book.

Diagram. A single circle with an inscribed construction: from the top point A lines are drawn to points around and below the circle, the tangent circle being built on the diameter through the centre.

- A (top): the point of tangency / given point on the circumference
- B (right): a point on the circle
- C (centre) and D (just below C): centre of the circle and the taken point D
- E (lower left): the given external point
- F (bottom centre) and G (foot): further construction points

Third. Let a point F be given within the circle AE. Draw from F through the centre D the straight line FA, and, bisecting it at C, describe from C through A the circle AB, which will pass through the given point F and, by the said scholium, will touch the circle AE at A.

Otherwise. Let the point B again be given, within the circle AE whose centre is D. Draw any straight line AD through the centre D, not passing through the given point B; joined with the straight line AB, it will make the acute angle BAD — acute inasmuch as it is smaller than the angle of a semicircle, which is less than a right angle. At B, then, let the angle ABC be made equal to the angle BAD; the straight line BC will meet AD at C, because the two angles BAC and ABC are less than two right angles. Since, then, the angles BAC and ABC are equal, [Margin: k — I.6] the sides CA and CB will also be equal. Therefore the circle described from C through A will pass through the given point B and will touch the circle AE at A, by what we demonstrated in the scholium to proposition 13 of this book.

And if through a point N, given outside the circle QAR, a circle is to be described touching the circle QAR externally, so that neither falls within the other: draw from N, the given point, to the centre P of the circle QAR the straight line NP, cutting the circumference of the circle at A. Then, bisecting the straight line NA at L, describe from L through A a circle, which will pass through the given point N and will touch the circle QAR at A, because both circles touch, at A, the straight line DE drawn through A perpendicular to NP — by the corollary of proposition 16 of this book.

Diagram. A stacked figure: at the top a small circle QAR of centre P, with the horizontal line DAE drawn through A; below A, larger concentric circles share the

vertical diameter through C, G, L, B, K down to N, and the line from M through H and F sweeps to the left.

- Q (top left), P (top, centre of the upper circle), R (top right): the circle QAR
- A (junction of the circles): the point of tangency
- D (left) and E (right): the tangent DAE perpendicular to NP
- M (left), H, F (left of centre): points on the lower circles
- C, G, L (on the vertical diameter, descending): centres / division points
- B (lower), K (below B), N (bottom): the given external point N on the vertical

In the same way, if a point A is given on the circumference of the circle QAR, we shall describe a circle that touches the circle QAR externally at A. For, drawing from the centre P the straight line PAC through the given point A, if from any point of this line a circle be described through A — say AMN, with centre L — this will touch the circle QAR, as was said, at A.

Diagram. The same stacked construction repeated for the continuing text: the upper circle QAR of centre P with the horizontal DAE through A, and below it the larger circles on the vertical diameter through C, G, L, B, K, N, with M, H, F to the left.

- Q (top left), P (top centre), R (top right): the circle QAR
- A (junction): point of tangency
- D (left), E (right): the perpendicular tangent DAE
- M (left), H, F (left of centre): points on the lower circles
- C, G, L (descending on the vertical): division points
- B, K, N (lower vertical): with N the external point

[Margin: l — III.12]

From this it is clear that through the same given point several circles can be described touching a given circle — namely if, from the given points E and B in the earlier figure, other straight lines be drawn differing from the lines EA and BA; likewise if, from the given points F and G, other lines be sent out not passing through the centres C and D but cutting the circles; and at length, on the straight line AG of the same figure, or on PAC of the latter figure (when the point A is given on the circumference), other centres different from the centres C, D, L be assumed — whereas, through a given point outside the circle, only one single line toward the same side can be drawn touching the circle. But by what method a circle touching two circles is to be described, when one of them is not wholly

within the other, we shall demonstrate on this principle together with Giovanni Battista Benedetti [Ioan. Bapt. Benedicto].

First. Let there first be two unequal circles touching or cutting each other — AB the smaller and CDE the larger — whose centres are F and G; through these let the straight line FG be drawn, meeting the circumferences at A and C. If the circles touch at E, this line will pass through the point of contact E. Cut off the straight line EH from the radius of the larger circle, equal to the radius of the smaller; and from F, the centre of the smaller, with radius HC — which is greater than the radius of the smaller — describe an arc cutting the circumference of the larger at I; and from I draw through the centres F, G the straight lines IF, IG, meeting the circumferences at B, D. Since CH is equal to IF, and HE to FB, by construction, the wholes CE and IB will be equal. Since, then, the diameter ID is also equal to the diameter CE, IB too will be equal to ID. Therefore the circle described from I through B will pass through D; and since the same falls wholly outside both circles, by the scholium to proposition 13 of this book, it will touch both circles at B, D — which was proposed.

Diagram. Two overlapping circles — the smaller AB and the larger CDE — with their centres F, G on the horizontal line AC; the constructed touching circle passes through B and D, meeting at I below.

- B (top left, on the smaller circle) and D (top right, on the larger): the two points of contact
- A (far left), F (centre of smaller), E (contact), H (cut-off point), G (centre of larger), C (far right): points along the central line
- I (bottom): the centre of the touching circle

Then. Next let there be two circles touching or cutting each other, equal, AB and CD, through whose centres E, F the straight line AC is drawn. Then bisect the straight line EF, between the two centres, at G. (When the circles touch each other, the straight line EF is bisected at the point of contact, because of the equal radii of equal circles.) At G erect GH perpendicular to EF; and, taking any point H on it, draw from it through the centres E, F straight lines meeting the circumferences at B, D. Since, then, the two sides GE, GH are equal to the two sides GF, GH, and contain equal angles — namely right angles — [Margin: m — I.4] the bases HE, HF will also be equal; and, the equal radii EB, FD being added, the whole lines HB, HD will be equal. Therefore the circle described from H through B will pass through D, and will fall outside both circles, by the scholium to proposition 13 of this book. Hence it will touch both at B, D — which was proposed.

Diagram. Two equal overlapping circles AB and CD, centres E and F on the horizontal line AC, with the midpoint G, the perpendicular GH, and the touching circle through B, D meeting at H below.

- B (top left) and D (top right): the two points of contact
- A (left), E (centre), G (midpoint of EF), F (centre), C (right): points on the central line
- H (bottom): apex of the perpendicular and centre of the touching circle

Third. Third, let there be two circles, whether equal or unequal, AB and CD, one of which lies wholly outside the other. Through their centres E, F draw the straight line AC. And if the circles are unequal, describe from E, the centre of the smaller, with radius equal to the diameter of the larger CH, an arc; let it be cut at G by another arc described from F, the centre of the larger, with radius FI, compounded of the radius of the larger FH and of HI, equal to the radius of the smaller. Let us now suppose that these arcs intersect one another; when they do not cut, we shall say presently what is to be done. From G let straight lines be drawn through the centres E, F, cutting the circumferences at B, D. Since FG is equal to FI, the equals FK, FH being taken away, the remainders KG, HI will be equal. But HI was taken equal to the radius of the smaller, EB. Therefore KG too will be equal to EB. Since, then, EG is also equal to the diameter of the larger KD, the whole straight lines GB, GD will be equal. Wherefore the circle described from G through B will pass through D, and will fall outside both circles, by the scholium to proposition 13 of this book; and accordingly it will touch both at B, D — which was proposed.

Diagram. Two circles standing wholly apart, with the centres E, F on the horizontal line AC; the two auxiliary arcs meet below at G, and the touching circle is built through B, D.

- A (far left), E (centre of one circle), I, H (centre points), F (centre of the other), C (far right): points along the central line
- K (below the central line): where FK is cut off
- G (bottom): intersection of the two arcs and centre of the touching circle
- [The contact points B, D at the top of the two circles are not clearly resolved at this size — [?]]

But because those two arcs described from E and F do not always intersect — which happens on account of too great a distance of one circle from the other — we shall solve this problem in another way, which is general, whether the circles touch or not, even if they do not cut, and whether they are equal or unequal.

Draw the straight line AC through the centres of the circles, and take equal straight lines AG, CH, such that each exceeds the half of AC itself; and from centre E, through G, describe the arc GI; let another arc HI, described from centre F through H, cut it at I; and finally from I, through the centres E, F, let straight lines be sent out meeting the circumferences at B, D. And since EG, EI are equal, if the equals AE, BE be added, the wholes AG, BI will be made equal. By the same reasoning CH, DI will be equal. Since, then, AG, CH were laid down equal, BI, DI too will be equal. Therefore the circle described from I through B will pass through D, and, falling wholly outside both circles by the scholium to proposition 13 of this book, will accordingly touch both at B, D — which was proposed.

Diagram. The general construction: two circles on the horizontal line AC, with the auxiliary arcs from E and F meeting at I below, and the touching circle passing through B and D.

- B (top left) and D (top right): the two points of contact
- A (left), E (centre), H, G (centre points), F (centre), C (right): points along the central line
- I (bottom): intersection of the arcs and centre of the touching circle

(Continued from the preceding page.) By the same technique we shall now describe a circle of any given size that touches the two given circles, provided the radius of the circle to be described is greater than half the line AC drawn through the centres of the circles as far as their circumferences. For if from that line we cut off AG and CH equal to the radius of the given circle, we shall carry out the rest as was just explained.

Moreover it is also clear that, if from the midpoint of the line AC — drawn through the centres of the circles out to their circumferences — a circle is described through A and C, that circle touches both circles AB and CD, at A and at C, by the scholium to Proposition 13 of this book.

FROM PELETIER [Jacques Peletier du Mans]

Problem. *Parallel to a straight line that cuts a circle, to draw a line that touches the same circle.*

Diagram. A circle with centre D. A horizontal tangent line FAG runs across the top, touching the circle at A. A chord BC lies just below, parallel to FG; the line DA is perpendicular to BC, meeting it at E and extended up to A.

- F (top left): left end of the tangent line

- A (top, on circumference): point of tangency
- G (top right): right end of the tangent line
- B (left end of chord)
- E (on chord BC, foot of the perpendicular from the centre)
- C (right end of chord)
- D (centre of the circle)

Let the straight line BC cut the circle ABC, whose centre is D, and let a line parallel to BC be required, touching the circle ABC. From the centre D draw DE perpendicular to BC, and extend it to the point A on the circumference; and from A draw FG perpendicular to AD [Margin: n — El. I.28]. Then FG will be parallel to BC, and it will touch the circle at A, by the corollary to Proposition 16 of this book. Which was to be done.

THEOR. 16. PROPOS. 18.

[Margin: xvij.]

Enunciation (Euclid III.18). **If a straight line touches a circle, and from the centre a straight line is joined to the point of contact, the joining line will be perpendicular to the tangent.**

Diagram. A circle with centre E. A tangent line runs across the top through the points A, C, F, B (left to right), touching the circle at C. From the centre E two lines run up to C and to F, forming the narrow triangle CEF; D lies on the circumference at the left.

- A, C, F, B (top tangent line, left to right; C is the point of contact)
- E (centre)
- D (on the circumference, left)

Let the straight line AB touch the circle CD, whose centre is E, at C; and from E to C draw EC. I say that EC is perpendicular to AB. For if it is not, let EF be drawn perpendicular to AB, cutting the circumference at D. Now in the triangle CEF the two angles ECF, EFC are less than two right angles [Margin: a — El. I.17]; and EFC is a right angle by construction; therefore ECF is less than a right angle. Hence [Margin: b — El. I.19] the side EC — that is, ED — will be greater than EF, the part greater than the whole, which is absurd. Therefore EC is perpendicular to AB. And so, if a straight line touches a circle, etc. Which was to be demonstrated.

ANOTHER WAY. If EC is not perpendicular to AB, then one of the angles at C will be obtuse and the other acute. Let ECB be the acute one; and since it is greater

than the angle of the semicircle ECD, the angle of the semicircle would be less than some acute angle — which is absurd, since every angle of a semicircle is greater than every acute angle [Margin: c — El. III.16].

SCHOLIUM

Here it will be permissible to demonstrate the following theorem, which is not without use for what follows; namely:

When two circles are described from the same centre, all the straight lines that touch the inner circle and are extended as far as the circumference of the outer circle will be equal to one another, and will each be bisected at the points of contact.

Diagram. Two concentric circles with common centre G. The outer circle carries the points A and B (top), C (right) and D (bottom left); the chords AC and BD of the outer circle each touch the inner circle, AC at E and BD at F.

- A (top left), B (top right): on the outer circle
- E, F (inner circle): points where AC and BD touch the inner circle
- G (common centre)
- D (lower left), C (lower right): on the outer circle

Let two circles ABCD and EF be described from the same centre G, and let the straight lines AC, BD touch [the inner circle] at the points E, F. I say that the lines AC, BD are equal, and are bisected at E, F. For from the centre G to the points of contact E, F draw the straight lines GE, GF, which will be perpendicular to AC, BD [Margin: d — El. III.18]; and therefore, by Definition 4 of this book, the lines AC, BD are equally distant from the centre G in the circle ABCD, since the perpendiculars GE, GF are equal. Therefore the lines AC, BD are equal [Margin: e — El. III.14]. And they are also bisected at E, F by the perpendiculars GE, GF [Margin: f — El. III.3]. Which establishes what was proposed.

THEOR. 17 PROPOS. 19.

[Margin: xvii.]

Enunciation (Euclid III.19). If a straight line touches a circle, and from the point of contact a straight line is erected at right angles to the tangent, the centre of the circle will lie on the line so erected.

Diagram. A circle touched at its top by the tangent line ACB (A left, C the point of contact, B right). The perpendicular CE runs down from C through the interior to E at the bottom; F is an interior point (the falsely assumed centre), joined to C.

- A (top left), B (top right): ends of the tangent
- C (top, point of contact)
- F (interior, the assumed false centre)
- E (bottom, other end of the perpendicular CE)

Let the straight line AB touch the circle CDE at C; and from C draw CE perpendicular to AB. I say that the centre of the circle lies on CE. For if it lies outside CE, let F be the centre; and from F to C draw the straight line FC, which will be perpendicular to AB [Margin: a — El. III.18]. Then the right angle FCB will be equal to the right angle ECB, the part to the whole, which is absurd. Therefore the centre of the circle will not lie outside CE. And so, if a straight line touches a circle, etc. Which was to be demonstrated.

THEOR. 18. PROPOS. 20.

[Margin: xix.]

Enunciation (Euclid III.20). **In a circle, the angle at the centre is double the angle at the circumference, when the same arc is the base of both angles.**

Diagram. A circle with centre D. A is at the top, B at the lower left, C at the lower right. The lines DA, DB, DC run from the centre; the diameter AE is extended from A through the centre D down to E at the bottom, and AB, AC are the chords.

- A (top): vertex of the inscribed angle
- B (lower left), C (lower right): ends of the base
- D (centre)
- E (bottom, second end of the diameter AE)

In the circle ABC, whose centre is D, let the angle BDC be set up at the centre on the base BC, and on the same base the angle BAC at the circumference. I say that the angle BDC is double the angle BAC. First, let the two lines AB, AC enclose the two lines DB, DC; and through the centre D let the straight line AE be extended. Now since DA, DB are equal, the angles DAB, DBA will be equal [Margin: a — El. I.5]; and the exterior angle BDE is equal to the two interior angles DAB, DBA [Margin: b — El. I.32]; therefore BDE is double one of them, namely the angle DAB. In the same way the angle CDE will be shown to be double the angle DAC. Therefore the whole angle BDC is double the whole angle BAC. For when two

magnitudes are each the double of two others, one of each, then their sum is likewise double the sum of the others. Which establishes the proposition.

Diagram. A circle with centre D. A is at the top right, B at the lower left, C at the lower right; the line BA is a diameter passing through the centre D, and DC, AC are drawn.

- A (top right): vertex
- B (lower left): the diameter BA runs through D to A
- C (lower right): third point
- D (centre)

Next, let the lines AB, AC not enclose the lines DB, DC, but let AB be extended through the centre. Then since the exterior angle BDC is equal to the two interior angles DAC, DCA [Margin: c — El. I.32] — and these two are equal to one another, because the sides DA, DC are equal [Margin: d — El. I.5] — the angle BDC will be double one of them, namely the angle BAC. Which is the proposition.

Diagram. A circle with centre D. A is on the right, E on the left, B at the lower left, C at the lower right. The line AE passes through the centre D; DB, DC, AB and AC are drawn, so that the chord AB cuts DC.

- A (right): vertex
- E (left, second end of the diameter AE through the centre)
- B (lower left), C (lower right): ends of the base
- D (centre)

Thirdly, let the line AB cut the line DC, and through the centre D let the line AE be extended. Then since the angle EDC at the centre and the angle EAC at the circumference have the same base EC, and AE is drawn through the centre, the angle EDC will be double the angle EAC, as was shown in the second part. In the same way the angle EDB will be double the angle EAB, since these angles have the same base EB. Therefore the remaining angle BDC will be double the remaining angle BAC [Margin: e — Ax. 20]. For when the whole is double the whole, and the part taken away double the part taken away, the remainder too is double the remainder. Therefore, in a circle, the angle at the centre is double, etc. Which was to be demonstrated.

SCHOLIUM

To demonstrate the first part of this proposition, this principle was assumed: (*If two magnitudes are each the double of two magnitudes, one of each, then their sum*

is also double the sum of the others). And in the demonstration of the third part this other principle was used: (*If the whole is double the whole, and the part taken away double the part taken away, then the remainder too is double the remainder*). Both of these are demonstrated universally by Euclid, for every kind of multiple and for any number of magnitudes, in Book 5, Propositions 1 and 5. Yet the demonstration of this proposition is not on that account to be judged faulty, as though it assumed things not yet demonstrated: for those two principles are so evident by the natural light in the case of double magnitudes that anyone readily grants them without any demonstration; and both of those propositions of Book 5 can be demonstrated before this third book, since they do not depend on it — so that those two principles may with full right be used here as though demonstrated. Nor indeed does Euclid commit a circular argument in this matter, since the two principles by which Proposition 20 of this book is demonstrated do not rest on this same proposition, nor on any other propositions that depend on this twentieth.

But, if you please, we shall demonstrate the first part of this proposition without the former principle, and the third part without the latter (for the second part needs neither of them), in the following way. Let the first figure be repeated. Since [Margin: f — El. I.5] the angles DAB, DBA are equal to one another, and likewise the angles DAC, DCA, the whole angle BAC will be equal to the two angles B, C taken together; and therefore the three angles BAC, B, and C taken together will be double the angle BAC. But since the angle BDC is equal to those same three angles BAC, B, and C — for since [Margin: g — El. I.32] BDE equals the two interior angles B and DAB, and CDE equals the two interior angles C and DAC, the whole BDC will be equal to all four, that is, to the three BAC, B, and C — the angle BDC too will be double the angle BAC, which is the proposition.

ANOTHER WAY. Since [Margin: h — El. I.32] the six angles of the triangles ABD, ACD are equal to four right angles, and the three angles BDC, CDA, ADB are likewise equal to four right angles, by the second corollary of Proposition 15 of Book 1, these three will be equal to those six; and when the common angles ADB, ADC are removed, the remaining angle BDC will be equal to the remaining angles BAC, B, and C [Margin: i — Ax. 3]. Therefore, as before, BDC will be double BAC.

Let the third figure also be repeated, where the line DC cuts the line AB, as at E.

Diagram. A circle with centre D at the top. A is on the right, B at the lower left, C at the bottom; E is a central point where DC cuts AB, and F an interior point on AE. Lines DA, DB, DC, AB, AC and CF are drawn.

- D (centre, upper part)
- A (right): vertex
- F (interior point on AE, where CF cuts AE)
- E (central crossing point of DC and AB)
- B (lower left), C (bottom): ends of the base

Since, when the line AD has been drawn, the two angles DAC, DCA are equal [Margin: k — Ax. 5], and the angle DAC is greater than the angle EAC, the angle DCA too will be greater than the same angle EAC. If, then, the angle ACF is made equal to the angle EAC, CF will cut the line AE, as at the point F; and the angle EFC will be equal to the interior angles FAC, FCA [Margin: l — El. I.32], and therefore double the angle BAC. But since [Margin: m — El. I.5] the angles DAC, DCA are equal, and the angles FAC, FCA taken away are also equal, the remaining angles DAB, ECF will be equal. Then since [Margin: n — El. I.5] DAB is equal to DBA, ECF too will be equal to that same angle DBE; and the angles DEB, FEC at the vertex E are equal [Margin: o — El. I.15]. Therefore the remaining angles BDE, EFC will also be equal, by the first corollary of Proposition 32 of Book 1. And since the angle EFC has been shown to be double the angle BAC, the angle BDC too will be double the angle BAC, which is the proposition.

Diagram, left of the upper pair. A circle with centre D. B is at the top left, A at the top right, C on the right; E is at the bottom (label faint, matching its twin figure at right). Lines DB, DC and the chords AB, AC cross at the centre, illustrating the case where BD, CD do not form the angle on the base side.

- B (top left), A (top right), C (right)
- D (centre)
- E (bottom, second end of the diameter AE) [?]

Diagram, right of the upper pair. A circle with centre D. A is at the top, B on the left, C on the right, E at the bottom; the diameter AE passes through D, and the lines DB, DC and chords are drawn.

- A (top): vertex
- B (left), C (right)
- D (centre)
- E (bottom, second end of the diameter AE)

But if the lines BD, CD do not form the angle at the centre on the side of the base BC — which happens only when the segment BAC is either a semicircle or a smaller segment — nonetheless that space at the centre will be double the angle

at the circumference that has the same base as the space. For when the line AE is drawn through the centre, the angle BDE at the centre will be double the angle BAE at the circumference, and the angle CDE at the centre double the angle CAE at the circumference, as was shown. Therefore the space at the centre D, having the base BEC and made up of the two angles BDE, CDE, is double the whole angle BAC. Which is the proposition.

THEOR. 19 PROPOS. 21.

[Margin: xx.]

Enunciation (Euclid III.21). **In a circle, the angles that are in the same segment are equal to one another.**

Diagram. A circle with centre E. A is at the top left, B at the top right, D at the lower left, C at the lower right. The inscribed angles A and B both stand on the base DC (lines AD, AC, BD, BC), and DE, CE are drawn to the centre.

- A (top left), B (top right): the two inscribed angles in segment DABC
- D (lower left), C (lower right): ends of the common base
- E (centre)

In the circle ABCD, whose centre is E, let there be angles A and B in the segment DABC. I say that they are equal. First, let the segment DABC be greater than a semicircle; and let the straight lines DE, CE be drawn to the centre E. Then since the angle DEC at the centre is double both the angle DAC and the angle DBC at the circumference [Margin: a — El. III.20] — since all have the same base DC — the angles A and B will be halves of the angle E. Therefore they will be equal to one another [Margin: b — Ax. 7]. And by the same reasoning all the other angles in the segment DABC will be shown to be equal.

Next, let the segment DA, BC be either a semicircle or less than a semicircle.

Diagram, left of the middle pair. A circle with centre E. A is at the upper left, B at the upper right, D at the left, C at the right; G is at the lower left and F at the lower right, with the diameters AF, BG through the centre and lines DE, CE joined.

- A (upper left), B (upper right)
- D (left), C (right)
- E (centre)
- G (lower left, second end of diameter BG), F (lower right, second end of diameter AF)

Diagram, right of the middle pair. A companion circle with centre E, showing the same construction. A at the upper left, B at the upper right, D at the left, E the centre, G at the lower left, F at the lower right.

- A (upper left), B (upper right)
- D (left)
- E (centre)
- G (lower left), F (lower right)

Through the centre E let the straight lines AF, BG be drawn, and in the smaller segment let the lines DE, CE be joined. Then since the angle DEF at the centre is double the angle DAF at the circumference [Margin: c — El. III.20], and likewise the angle CEF double the angle CAF, the two angles DEF, CEF together are double the two angles DAF, CAF together — that is, of the whole angle DAC. But the angles DEG, GEF together are equal to the angle DEF; therefore the three angles DEG, GEF, FEC together are also double the angle DAC. By the same reasoning those same three angles are double the angle DBC. Therefore the angles DAC, DBC are equal [Margin: d — Ax. 7].

ANOTHER WAY. As we demonstrated in the scholium to the preceding proposition, the space at the centre E, whose base is DGFC, is double each of the angles DAC, DBC at the circumference; therefore the angles DAC, DBC themselves are equal to one another [Margin: e — Ax. 7].

Diagram, left of the lower-middle pair. A circle with centre E. A at the upper left, B at the upper right, D at the left, C at the right; the chords AC, BD cross at F, and AB is joined; E is the centre below F.

- A (upper left), B (upper right)
- D (left), C (right)
- F (crossing point of AC and BD)
- E (centre)

Diagram, right of the lower-middle pair. A companion circle with centre E showing the same crossing construction. A at the upper left, B at the upper right, D at the left, C at the right, F the interior crossing point, E the centre.

- A (upper left), B (upper right)
- D (left), C (right)
- F (interior crossing point)
- E (centre)

ANOTHER WAY. Let the lines AC, BD cut one another at F, and let the line AB be joined. Then since the three angles of the triangle AFD are equal to the three angles of the triangle BFC — for both sets are equal to two right angles [Margin: f — El. I.32] — if the equal angles AFD, BFC are taken away [Margin: g — El. I.15], the remaining angles ADF, DAF will be equal to the remaining angles BCF, CBF. But the angles ADF, BCF have been shown to be equal, being in the greater segment ADCB. Therefore the remaining angles DAC, DBC are also equal.

Diagram, left of the lowest pair. A circle with centre E. A at the left, B at the top, D toward the left, C at the right, F at the bottom; many chords are drawn (DF, CF, AF, BF and the sides of segment DABCF).

- A (left), B (top)
- D (left of centre), C (right)
- E (centre)
- F (bottom, the arbitrary point on the circumference)

Diagram, right of the lowest pair. A companion circle with centre E showing the same construction. A at the upper left, B at the top, D at the left, C at the right, F at the bottom.

- A (upper left), B (top)
- D (left), C (right)
- E (centre)
- F (bottom)

ANOTHER WAY. From any point F on the circumference, let the lines DF, CF be drawn enclosing the centre E, in such a way that both DABCF and FDABC are the greater segment; and let the lines AF, BF also be joined. Then since the angles DAF, DBF, being in the same greater segment DABCF, are equal, and likewise the angles FAC, FBC, being in the greater segment FDABC — if these are added to those, the whole angle DAC becomes equal to the whole angle DBC. And so, in a circle, the angles that are in the same segment, etc. Which was to be shown.

SCHOLIUM

We shall also easily convert this theorem, in this way: **If from two points four straight lines are drawn, two from each, which toward the same side contain two equal angles, then the circle described through those two points and through one of those angles will also pass through the other angle.**

Diagram. A small circle. B and C are at the base (B toward the right, C at the lower left), A and D above (A at the upper left, D at the upper right); E is the point on AB where the circle would cut it in the false case. Lines BA, CA, BD, CD and CE are drawn.

- A (upper left), D (upper right): the two equal angles
- B (base, right), C (base, lower left): the two points from which the lines issue
- E (on AB, where the displaced circle cuts it)

For from the two points B, C let four straight lines BA, CA, BD, CD be drawn, two from each, forming toward the same side two equal angles A, D. I say that the circle described through the points B, C and the angle D also passes through the angle A. For if it does not pass through A, it will pass either beyond A or short of it, cutting the line AB at E. Then when the line CE is drawn, the angles E and D will be equal [Margin: h — El. III.21]. But since the angle A is also supposed equal to the angle D, the angles CAB, CEB will be equal — an exterior and an interior angle — which is absurd; for the exterior angle is greater than the interior [Margin: i — El. I.16]. Therefore the circle passes through A, which is the proposition.

Clavius, *Opera Mathematica*, Tomus I — Commentary on Euclid, Book III

THEOR. 20. PROPOS. 22.

(Euclid's enunciation.)

The opposite angles of quadrilaterals inscribed in circles are equal to two right angles.

(Demonstration.) Let a quadrilateral ABCD be inscribed in a circle whose center is E. I say that the two opposite angles ABC and CDA — and likewise BCD and DAB — are equal to two right angles. Draw the two diagonals of the quadrilateral, AC and BD. [Margin: a — 21. tertij (*Elements* III.21; "tertij" = Book III)] Then the two angles ABD and ACD, being in the same segment ABCD, are equal. Similarly the two angles CBD and CAD, being in the same segment CBAD, are equal. Therefore the two angles ABD and CBD — that is, the whole angle ABC — are equal to the two angles ACD and CAD. Adding the common angle CDA, the two angles ABC and CDA are equal to the three angles ACD, CAD, and CDA. [Margin: b — 32. prim

(*Elements* I.32; "prim" = Book I)] But these three are equal to two right angles. Therefore the two angles ABC and CDA are also equal to two right angles.

In the same way we shall show that the angles BCD and DAB are equal to two right angles. For again the two angles ABD and ACD are equal; likewise the two BCA and BDA; and therefore the whole angle BCD is equal to the two angles ABD and BDA. Adding, then, the common angle BAD, the two angles BCD and BAD are equal to the three angles ABD, BDA, and DAB. [Margin: c — 32. prim (*Elements* I.32)] But these three are equal to two right angles. Therefore the two angles BCD and DAB will also be equal to two right angles. Therefore the opposite angles of quadrilaterals inscribed in circles, etc. Which was to be demonstrated.

Diagram. Two circles side by side, each carrying a cyclic quadrilateral for Proposition 22.

- Left circle — center E (marked by a dot at the middle); inscribed quadrilateral with vertices A (far left), B (upper left), C (upper right), D (right); both diagonals drawn.
- Right circle — center E (dot at middle); inscribed quadrilateral with vertices A (lower left), B (upper left), C (right), D (lower right); both diagonals AC and BD drawn.

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The converse of this theorem can also be demonstrated, in this way.

If in a quadrilateral the opposite angles are equal to two right angles, then a circle described through any three of its angles will also pass through the remaining fourth angle; and so a circle can be described about the quadrilateral itself.

For in the quadrilateral ABCD let the opposite angles B and D be equal to two right angles, and let a circle be described through the angles A, B, C. (How this can be done will be shown at Proposition 25 of this book and at Proposition 5 of Book IV.) I say that it passes through D as well. For if not, it will pass either beyond D or short of D. So draw the straight lines CE and AE to the circumference, in such a way that they do not cut the lines CD and AD. This done, [Margin: d — 32. prim (*Elements* I.32)] the angles B and E will be equal to two right angles. But the angles B and D were also equal to two right angles. Therefore the two angles B and E are equal to the two angles B and D; and, removing the common angle B, the remaining angles D and E are equal — which is absurd. For, when the straight line AC has been drawn, [Margin: e — 21. primi (*Elements* I.21)] angle D will be

greater than angle E, or conversely angle E greater than angle D. Therefore the circle passes through the point D. Which is what was proposed.

It is enough, moreover, to take only two opposite angles equal to two right angles for this theorem to hold; because if two opposite angles are equal to two right angles, the remaining two opposite ones will necessarily also be equal to two right angles, since all four together are equal to four right angles, as we demonstrated at Proposition 32 of Book I.

From this proposition we shall also easily demonstrate the following theorem, which mathematicians frequently use as a principle, namely:

(Additional theorem.)

If similar segments are removed from semicircles or from circles, the remaining segments will also be similar.

First, let there be in the circles ABCDEF and GHIKLM the similar arcs CDE and IKL. I say that the remaining arcs EFABC and LMGHI are also similar. For, taking points D, B, K, H anywhere on the said arcs, join the straight lines DC, DE, BC, BE, KI, KL, HI, HL. Now since the arcs CDE and IKL are similar, the angles CDE and IKL will be equal, by the definition of similar segments. Since, then, the two angles CDE and CBE are equal to the two angles IKL and IHL, [Margin: f — 22. tertij (*Elements* III.22)] because both these and those are equal to two right angles, [Margin: g — 3. pron (Common Notion 3; "pron." = pronuntiatum, a common notion)] the remaining angles CBE and IHL will also be equal; and therefore, by definition 10 of this book, the arcs EBC and LHI will be similar.

Next, in the semicircles AEC and GLI let the arcs EC and LI be similar. I say that the remaining arcs AE and GL are also similar. For, taking points D, F, K, M anywhere on those arcs, join the straight lines CD, DE, CE, EF, FA, AE; IK, KL, IL, LM, MG, GL. Then, since the arcs EC and LI are similar, the angles CDE and IKL are equal, by the definition of the angle. Since, then, the two angles CDE and EAC are equal to the two angles IKL and LGI, [Margin: h — 22. tertij (*Elements* III.22)] because both these and those are equal to two right angles, the remaining angles EAC and LGI will also be equal. But the angles AEC and GLI — being in semicircles, that is, in the similar segments AEC and GLI — are also equal, by the definition of similar segments. Therefore in the triangles AEC and GLI the remaining angles ACE and GIL will also be equal, by the corollary of Proposition 32 of Book I. Since, then, the two angles ACE and EFA are equal to the two angles GIL and LMG, [Margin: i — 22. tertij (*Elements* III.22)] because both these and those are equal to two right angles, the remaining angles EFA and LMG will also

be equal; and accordingly, by the definition of similar segments, the remaining arcs AE and GL in the semicircles will be similar. Which was to be shown.

Diagram. A circle carrying the construction for the converse scholium: a circle drawn through three vertices of a quadrilateral, with the disputed fourth vertex.

- D (top): vertex drawn just outside the circle through A, B, C.
- E (upper arc, below D): the point constructed on the circumference where CE and AE meet it.
- A (right): vertex on the circle.
- C (lower left) and B (lower right): the other two vertices.
- A further D-like mark near the center [?] (possibly the true position of the vertex).

Diagram. Two circles for the theorem on removing similar segments, each with an inscribed polygonal construction (and the diameter marking off a semicircle).

- Left circle: A (top), B (right), C (bottom), D (lower left), E (left), F (upper left); diameter AC.
- Right circle: G (top), H (right), I (bottom — serified capital, reads I though close to a T [?]), K (lower left), L (left), M (upper left); diameter GI.

Since the discussion here has fallen upon similar segments, three theorems should not be omitted at this point — theorems highly necessary both for geometry and especially for astronomy. Let the first of them be this, which we undertook to demonstrate at definition 10 of this book, namely:

(Additional theorem — first of three.)

Angles standing on similar arcs of circles — whether at the centers or at the circumferences — are equal to one another. And conversely, arcs on which equal angles stand, whether at the centers or at the circumferences, are similar.

(Demonstration of the theorem just enunciated.) In the circles ABCD and EFGH, whose centers are I and K, let the arcs BCD and FGH first be similar, on which stand the angles I and K at the centers, and the angles A and E at the circumferences. I say that both the former and the latter are equal to one another. For let angles be constructed at C and G on those arcs, which by the definition of similar segments will be equal. [Margin: k — 22. tertij (*Elements* III.22)] Now the two angles C and A, and likewise the two G and E, are equal to two right angles. So, removing the equal angles C and G, the remaining angles A

and E will also be equal. And since the angles I and K [Margin: l — 20. tertij (*Elements* III.20)] are the doubles of these, [Margin: m — 6. pron (Common Notion 6)] they too will be equal. Which is what was proposed.

Next, let both the angles I and K, and the angles A and E, standing on the arcs BCD and FGH, be equal to one another. I say that the arcs BCD and FGH are similar. For if the two angles A and C, and the two E and G, are equal to two right angles, [Margin: n — 22. tertij (*Elements* III.22)] the remaining angles C and G will also be equal; and therefore, by the definition of similar segments, the arcs BCD and FGH — on which the equal angles A and E stand at the circumferences — are similar. But if the angles I and K at the centers are equal, [Margin: o — 7. pron (Common Notion 7)] their halves will also be equal, that is, the angles A and E. Hence, as before, the arcs BCD and FGH are similar. Which was to be shown.

Diagram. Two circles, each with an inscribed angle at the circumference and the corresponding central angle standing on the same arc.

- Left circle — center I (dot at middle); A (top), B (lower left), C (bottom), D (right); lines from A and from I to the arc BCD.
- Right circle — center K (dot at middle); E (top), F (lower left), G (bottom), H (right); lines from E and from K to the arc FGH.

The second theorem, which we also demonstrated in our work on the sphere at the end of the first chapter, let it be of this kind:

(Additional theorem — second of three.)

If two or more circles are described from the same center, and from the center two or more straight lines are drawn, then the arcs intercepted between any two of the lines will be similar.

Let two circles ABC and DEF be described from the same center G, and from the center G let two straight lines GB and GC be drawn. I say that the arcs EF and BC are similar. For, producing BG to A, join the straight lines AC and DF; likewise, taking points H and I anywhere, draw the straight lines BH, CH, EI, FI. Now since the angle G, standing at the center on the arcs EF and BC, [Margin: p — 20. tertij (*Elements* III.20)] is the double both of the angle EDF and of the angle BAC, [Margin: q — 7. pron (Common Notion 7)] the angles EDF and EIF will be equal to the two angles BAC and BHC, because [Margin: r — 22. tertij (*Elements* III.22)] both these and those are equal to two right angles. So, removing the equal angles EDF and BAC, [Margin: s — 3. pron (Common Notion 3)] the remaining angles EIF

and BHC will be equal; and accordingly the segments EF and BC, in which they lie, will be similar, by the definition of similar segments. Which was to be shown.

More briefly, thus. [Margin: t — 7. pron (Common Notion 7)] Since the angles A and D are equal — because the angle G at the center [Margin: u — 20. tertij (*Elements* III.20)] is the double of each — then by the preceding theorem the arcs BC and EF, on which they stand at the circumferences, are similar. Which is what was proposed.

Or still more briefly. Since equal angles at the center stand on the arcs EF and BC — indeed one and the same angle BGC — then by the preceding theorem the arcs EF and BC are similar. And in this demonstration no construction at all is needed, as is evident.

Diagram. Two concentric circles about a common center G.

- G (center).
- Outer circle: A (upper right), B (lower left), C (right).
- Inner circle: D (below A), E (inner left), F (inner right).
- I (bottom) and H (bottom, toward the right): points where the chords BH, CH, EI, FI meet the circumferences.

Finally, let the third — which we have also shown by another method in our *Astrolabe* — be this:

(Additional theorem — third of three.)

If two or more circles touch one another internally at a single point, from which two or more straight lines are drawn, then both the arcs intercepted between any two of the lines, and the arcs intercepted between any one line and the point of contact, will be similar.

Let two circles ABC and ADE touch one another internally at the point A, from which two straight lines AB and AC are drawn. I say that the arcs DE and BC, as well as AD and AB, and AED and ACB, are similar. For, taking two points F and G anywhere, join the straight lines BF, CF, DG, EG. Now since the two angles DAE and DGE are equal to the two angles BAC and BFC — because [Margin: x — 22. tertij (*Elements* III.22)] both these and those are equal to two right angles — then, removing the common angle BAC, [Margin: y — 3. pron (Common Notion 3)] the remaining angles DGE and BFC will be equal; and accordingly the arcs DE and BC will be similar, by the definition of similar segments.

More briefly, thus. Since the angle at A, common to both the arcs DE and BC, stands at the circumferences, then by what was demonstrated in the immediately preceding theorem the arcs DE and BC, on which it stands, are similar. Which is what was proposed.

And since, if a straight line AC is imagined drawn through the centers of the circles, [Margin: z — 11. tertij (*Elements* III.11)] this falls on the point of contact A, and in the same way the arcs DE and BC are similar, it follows that — these similar arcs being removed from similar semicircles — the remaining arcs AD and AB are also similar; and, these being removed from the whole similar circumferences, the remaining arcs AED and ACB are likewise similar, as we demonstrated a little earlier. Which is what was proposed.

Diagram. A larger circle and a smaller circle touching it internally at the top point A, with two lines from A.

- A (top): point of contact.
- B (left) and C (bottom): points on the larger circle, joined to A.
- D (upper, on the smaller circle) and E (center, on the smaller circle): where the lines meet the smaller circle.
- F (lower left) and G (interior): the points taken on the arcs, with chords BF, CF, DG, EG.

The following theorem can also be referred to this place.

If one side of a quadrilateral inscribed in a circle is produced, the exterior angle will be equal to the angle that stands opposite to the interior angle adjacent to it.

In the circle ABCD let any quadrilateral ABCD be inscribed, whose side AD is produced to E. I say that the angle EDC is equal to the angle B. For since the two angles at D [Margin: a — 13. prim (*Elements* I.13)] are equal to two right angles, and likewise [Margin: b — 22. tertij (*Elements* III.22)] the two angles B and D in the quadrilateral, the former two are equal to these latter two. Therefore, removing the common angle ADC, the remaining exterior angle EDC will be equal to the remaining angle B. Which is what was proposed.

Diagram. A circle with an inscribed quadrilateral, one side produced.

- A (upper left), B (lower left), C (lower right), D (top): the four vertices, inscribed in the circle.
- E (upper right, outside the circle): the point to which side AD is produced.

THEOR. 21. PROPOS. 23.

(Euclid's enunciation.)

Upon the same straight line, two similar and unequal segments of circles cannot be constructed on the same side.

(Demonstration.) For if it were possible, let two similar and unequal segments ACB and ADB be constructed on the same side of the straight line AB. Now it is clear that they intersect one another only at the points A and B, [Margin: a — 10. tertij (*Elements* III.10)] since a circle does not cut a circle in more than two points. Hence the whole circumference of one segment will lie outside the circumference of the other. So draw the straight line AD, cutting the circumferences at C and D, and join the straight lines CB and DB. Since the segments are supposed to be similar, then by definition 10 of this book the angle ACB will be equal to the angle ADB — the exterior to the interior — [Margin: b — 16. primi (*Elements* I.16)] which is absurd. Therefore the segments are not similar. And so, upon the same straight line, etc. Which was to be demonstrated.

Diagram. Two arcs raised on the same base AB, on the same side, one enclosing the other.

- A (lower left) and B (lower right): the endpoints of the base.
- C (on the inner arc) and D (top, on the outer arc): the points where the straight line AD cuts the two circumferences; chords CB and DB drawn.

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By the same reasoning, neither will two similar and unequal segments of circles be constructed on opposite sides of the same straight line. For if one of them is imagined to be rotated about the line AB, so that both now lie on the same side, we shall fall into the same absurdity — as the figure shows — since the one will not coincide with the other, because of their inequality.

Diagram. A lens-shaped figure showing two unequal arcs on the base AB, brought to the same side.

- A (left) and B (right): endpoints of the base.
- C (upper) and D (top): points on the two arcs. A D-like label also appears at the foot of the figure [?].

THEOR. 22. PROPOS. 24.

(Euclid's enunciation.)

Upon equal straight lines, similar segments of circles are equal to one another.

(Demonstration.) Upon the equal straight lines AB and CD let similar segments AEB and CFD be constructed. I say that they are equal to one another. For since the lines AB and CD are equal, they will coincide with each other if one is placed upon the other. I say, then, that the segment AEB also coincides with the segment CFD. For if it does not coincide, it will fall either outside, or inside, or partly outside and partly inside. If it falls outside or inside, then upon the same straight line CD two similar and unequal segments AEB and AFB will be constructed, one of which falls wholly outside the other — which is absurd, [Margin: a — 23. tertij (*Elements* III.23)] for the contrary has been demonstrated. But if it falls partly outside and partly inside, they will cut each other in more than two points, namely at A, B, and G — which is absurd, [Margin: b — 10. tertij (*Elements* III.10)] since circles do not cut one another in more than two points. Therefore the segment AEB will coincide with the segment CFD, and so they will be equal to one another. And so, upon equal straight lines, etc. Which was to be demonstrated.

Diagram. Two similar segments raised on equal bases, together with several small nested figures illustrating the impossible cases.

- Main left segment: A (bottom left), E (top), B (bottom right).
- Main right segment: C (bottom left), F (top), D (bottom right).
- To the right, a row of small superposed figures (segments falling outside/inside one another and cutting at a third point), carrying the labels E, F, G over the base points A, B and A, C, D, B [some small labels indistinct].

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Not only is it shown in this proposition that the similar segments AEB and CFD are equal upon the equal bases AB and CD, but also that their circumferences are equal, because, as has been demonstrated, they coincide with one another.

The converse of this proposition and of the preceding one will also easily be demonstrated: namely, that segments of circles which are equal — whether constructed upon equal lines or upon the same line — are similar. For on account of their equality, the one will coincide with the other; and so they will be similar, since by this reasoning all the angles constructed in them are equal. And if

anyone should say that the segments do not coincide with one another, then necessarily one circumference will cut the other; for that one does not fall outside the other is contrary to the hypothesis. So let the segments AFB and AGB be equal upon the same straight line AB, and let them cut each other at G. Then the circles AFB and AGB cut each other at the points A, G, B — more than two — which is absurd, [Margin: c — 10. tertij (*Elements* III.10)] since circles do not intersect one another in more than two points. The matter will be demonstrated in the same way if equal segments are constructed upon equal straight lines: namely, if the one is placed upon the other, etc.

PROBL. 3. PROPOS. 25.

(*Euclid's enunciation.*)

Given a segment of a circle, to describe the circle of which it is a segment.

(*Demonstration.*) Let ABC be the segment of a circle that must be completed. Subtend the straight line AC, bisect it at the point D, through D draw the perpendicular DB, and join AB. Then the angle DBA is either greater than the angle DAB, or equal to it, or less.

Let it first be greater — which will happen when the segment ABC is less than a semicircle. For then, since BD passes through the center (by the corollary of Proposition 1 of this book) — the part of it that lies outside the segment, since the segment is supposed to be less than a semicircle — DA will be greater than DB, because DB, which completes the diameter, [Margin: a — 7. tertij (*Elements* III.7)] is the least of all the lines that fall from the point D to the circumference.

Therefore the angle DBA [Margin: b — 18. prim (*Elements* I.18)] will be greater than the angle DAB. Then make the angle BAE equal to the angle DBA, and let the straight line AE cut the straight line BD, produced, at E. I say that E is the center of the circle of which ABC is a segment. For, when the straight line EC has been drawn, the sides AD and DE of the triangle ADE will be equal to the sides CD and DE of the triangle CDE, and the contained angles are right. [Margin: c — 4. prim (*Elements* I.4)] Therefore the bases EA and EC will be equal; [Margin: d — 6. primi (*Elements* I.6)] and EA is also equal to EB, since the angles EAB and EBA are equal. Therefore the three lines EA, EB, EC will be equal, [Margin: e — 9. tertij (*Elements* III.9)] and consequently E will be the center of the circle ABC, since from E more than two equal straight lines fall onto the circumference.

Next, let the angle DBA be equal to the angle DAB (which will finally happen when the segment ABC is a semicircle; for then AC will be the diameter, and D the

center, and so the lines DA and DB are equal, whence [Margin: f — 5. primi (*Elements* I.5)] the angles DAB and DBA are also equal). [Margin: g — 6. primi (*Elements* I.6)] Therefore the lines DA and DB will be equal; and DC was also equal to DA, since the line AC was bisected. Therefore, since the three lines DA, DB, DC fall from D onto the circumference, [Margin: h — 9. tertij (*Elements* III.9)] D will be the center.

(The third case — where the angle DBA is less than DAB, that is, where the segment is greater than a semicircle — continues on the following page.)

Diagram. The construction for the first case of Problem 25 (segment less than a semicircle).

- A (left), B (top), C (right): the three given points of the segment.
- D (on the chord AC, at its midpoint): foot of the perpendicular BD.
- E (below D, on BD produced): the constructed center; lines AE, EC, AB drawn.

Diagram. The construction for the second case (segment a semicircle).

- A (left), B (top), C (right): the three given points, forming a semicircle on the diameter AC.
- D (center, on AC): the center, with the perpendicular BD and the chord AB drawn. Faint additional construction lines appear within the figure [?].

[Euclid, *Elements* Book III — continuation of PROPOS. 25 / THEOR. 23: to complete a circle upon a given segment]

[Clavius's demonstration, continued — third case]

Let it be, thirdly, that the angle DBA is less than the angle DAB. (This indeed happens when the segment ABC turns out to be greater than a semicircle. For then, since BD passes through the center — by the corollary of Prop. 1 of this book [Margin: i — Elem. III.7] — and since BD lies within the segment, the segment being assumed the greater, DB will be the greatest of all the lines that fall from D to the circumference; it will therefore be greater than DA, and consequently the angle DAB greater than the angle DBA. [Margin: k — Elem. I.18]) Let the angle BAE be made equal to the angle DBA, and let the line AE cut the line BD at the point E, which will be shown to be the center in the same way we showed it when the angle DBA was greater than the angle DAB, as is evident if the line EC is drawn. Therefore upon the given segment of a circle we have described the circle of which it is the segment. Which was to be done.

Diagram. A circular segment (more than a semicircle) with its completed circle. The base AC is horizontal at the bottom; B is the apex above; the constructed center E lies inside, on the line BD.

- A (bottom left): left end of the base chord AC.
- B (top): apex of the segment; lines run from B to A, to C, and down through E to D.
- C (bottom right): right end of the base.
- D (bottom, on AC): foot of the perpendicular from B (midpoint of AC).
- E (center): the located center of the circle, where AE meets BD.

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But perhaps this demonstration of Euclid — which proceeds through all three kinds of segment of a circle — may be set up more easily as follows. Bisect the line AC at D, raise the perpendicular DB, and draw AB; then DB is either less than DA, or equal to it, or greater.

If it is less, the angle DBA will be greater than the angle BAD. [Margin: l — Elem. I.18] Therefore, having constructed the angle BAE equal to the angle ABD, we shall show, as before, that E is the center. And this happens when the segment is less than a semicircle.

But if DB is equal to DA, then all three lines DA, DB, DC will be equal, [Margin: m — Elem. III.9] and therefore D is the center. This case applies in the semicircle.

If, finally, DB is greater than DA, the angle DBA will be less than the angle DAB. [Margin: n — Elem. I.18] Therefore, having constructed the angle BAE equal to the angle ABE, E will be shown to be the center, as before. And this occurs with the greater segment.

Another way. The same problem will be solved in this manner. With the same things laid down as before, let the angle EAB be made equal to the angle DBA. And if the line AE falls below AC, the segment will be less than a semicircle, as in the first figure. But if AE falls above AC, the segment will be greater than a semicircle, as in the third figure; and in every case E will be the center, as has been demonstrated. And if, finally, the line making at A with AB an angle equal to the angle ABD coincides with AC, the segment will be a semicircle and the point D will be the center, as in the second figure.

A shorter way, however, will find the center of any given segment — whether it be less than a semicircle, or equal, or greater — by this method. Take on the arc

of the segment any three points whatever, A, B, and C; join them by the two straight lines AB and BC; and let these be bisected at D and at E. Then from D and E raise the perpendiculars DF, EF to AB and BC. Now since, by the corollary of Prop. 1 of this book, both DF and EF pass through the center of the circle of which ABC is the segment, the two will meet at the center, as at F. Therefore the center has been found. Which is what was proposed.

Diagram. A circle in which two chords AB and BC are drawn from B, with their perpendicular bisectors DF and EF meeting at the center F.

- A (lower left): one end of chord AB, on the circle.
- B (top): common endpoint of the two chords.
- C (right): end of chord BC, on the circle.
- D (left): midpoint of AB; foot of perpendicular DF.
- E (upper right): midpoint of BC; foot of perpendicular EF.
- F (center): intersection of the two perpendicular bisectors — the center of the circle.

Another way, as the practitioners [mechanics] do. Take on the circumference any two points, A and B; about them describe two arcs with any equal radius, which intersect one another at E and at F. Then from two other points, C and D, describe two more arcs cutting one another at G and H, with any radius — whether the same as before or different. If then the straight lines EF and GH are drawn, both will pass through the center. Therefore the point I, where they meet, will be the center.

That the lines EF, GH pass through the center is demonstrated thus. Draw the lines AE, AF, BE, BF, which will be equal to one another because of the equality of the circles [radii]. Since, then, the sides AE, EF of the triangle AEF are equal to the sides BE, EF of the triangle BEF, and the bases AF, BF are likewise equal, the angles AEF, BEF will be equal. [Margin: o — Elem. I.8] Again, draw the straight line AB cutting EF at K. Since the sides AE, EK of the triangle AEK are equal to the sides BE, EK of the triangle BEK, and the angles AEK, BEK have also been shown equal, the bases AK, BK and the angles AKE, BKE will be equal, [Margin: p — Elem. I.4] and therefore right angles. Thus, since EF divides the line AB within the circle into two equal parts and at right angles, it will pass through the center, by the corollary of Prop. 1 of this book. In the same way it will be shown that GH passes through the center. But the four points A, B, C, D must be so placed that the lines EF, GH do not meet in a straight line, but cut one another. For should it happen that EF, GH are set in a straight line, the line comprised within the

circumference must itself be bisected; for the point of division will then be the center of the circle, since that line is then a diameter of the circle, passing through the center, by the corollary of Prop. 1 of this book.

Diagram. The practitioners' construction: two pairs of intersecting arcs about the chords, with the lines EF and GH meeting at the center I.

- A (left): first base point.
- B (upper right): second base point (paired with A).
- C (upper right, further): third base point.
- D (lower right): fourth base point (paired with C).
- E (top left): intersection of the arcs about A and B (above).
- F (center left): the other intersection of the arcs about A and B (below).
- G (right): intersection of the arcs about C and D.
- H (middle right): the other intersection of the arcs about C and D.
- I (lower center): meeting point of EF and GH — the center.

THEOR. 23. — PROPOS. 26. [Euclid, *Elements* III.26] [Margin: xxv.]

[Enunciation] In equal circles, equal angles stand on equal arcs, whether they are constructed at the centers or at the circumferences.

[Clavius's demonstration] In equal circles ABC, DEF, whose centers are G, H, let there be constructed, first, at the centers, the equal angles AGC, DHF. I say that the arcs AC, DF on which they stand — that is, over which they were raised — are equal. For take on the arcs ABC, DEF two points B, E, to which draw the lines AB, CB, DE, FE, and join the lines AC, DF. Now since the angles G and H are equal, [Margin: a — Elem. III.20] they will themselves also be equal to one another. Therefore the segments ABC, DEF will be similar. And since the sides AG, GC of triangle

AGC are equal to the sides DH, HF of triangle DHF because of the equality of the circles, and the angles they contain, G and H, are equal by hypothesis, [Margin: b — Elem. I.4] the bases AC, DF will be equal. Since, then, the similar segments ABC, DEF stand upon the equal lines AC, DF, they will themselves be equal to one another. [Margin: c — Elem. III.24] Therefore the arcs AC, DF will also be equal. [Margin: continues — see below]

[Second part] Now let there be constructed, at the circumferences, two equal angles B and E; I say again that the arcs AC, DF over which they were raised are

equal. For, as before, the segments ABC, DEF will be similar. Since, then, they stand upon equal lines AC, DF — for since the angles G, H are equal, being double of the equal angles B and E, [Margin: d — Elem. III.20] the lines AC, DF will, as before, be equal — the segments will be equal to one another. [Margin: e — Elem. III.24] If then they are taken away from equal circles, the remaining arcs AC, DF will also be equal. And so, in equal circles, equal angles [stand on equal arcs], etc. Which was to be demonstrated.

SCHOLIUM [Clavius]

This second part will be demonstrated more briefly thus. Since the angles G, H are double of the equal angles B, E, they will be equal to one another. Therefore, as was shown before, the arcs AC, DF over which they were raised will be equal.

But now, if it should happen that the lines AG, CG; DH, FH do not form the angles at the centers upon the bases AC, DF, but upon the bases ABC, DEF — as the figures set below show — then the second part of this proposition must be demonstrated thus. Since the spaces [sectors] at the centers G, H, standing on the arcs AC, DF, are equal to one another — being, by the scholium of Prop. 20 of this book, double of the angles B, E at the circumferences — the angles AGC, DHF, which together with the space about G, and likewise the space about H, make up four right angles, will also be equal to one another, by the corollary 2 of Prop. 15 of Book 1. Therefore, as we shall show as before, the bases AC, DF of the triangles AGC, DHF are equal, and hence the similar segments ABC, DEF, on account of the equality of the angles B, E, standing upon the equal bases AC, DF, are equal, and so on.

Diagram. Two equal circles, centers G and H, with the angles taken at the circumference (apex B and E) rather than at the base chords.

- Left circle (center G): A (left) and C (right) on the circle, B (top) the apex; chords AB, CB and radii AG, GC drawn.
- A (lower left), B (top), C (lower right), G (center) — left circle.
- Right circle (center H): D (lower left) and F (lower right) on the circle, E (top) the apex; chords DE, FE and radii DH, HF drawn.
- D (lower left), E (top), F (lower right), H (center) — right circle.

Hence it also follows that equal spaces [sectors] at the centers stand on equal arcs. For if the spaces at the centers G, H, standing on the greater arcs AC, DF, are equal, the angles AGC, DHF will also, as we showed, be equal. Therefore, as was

demonstrated in the first part of the proposition, the arcs AC, DF — both the greater and the lesser, ABC, DEF — will be equal to one another.

But if the said angles are unequal, the greater will stand on the greater arc, and the lesser on the lesser. For in equal circles ABC, DEF, let the angle AGC at the center be greater than the angle DHF at the center, and likewise the angle ABC at the circumference greater than the angle DEF at the circumference. I say that the arc AC is greater than the arc DF. For if the angle CGI be made equal to the angle DHF, and the angle CBI equal to the angle DEF, then, as was shown, the arcs CI, DF will be equal. And on that account AC is greater than DF.

Diagram. Two equal circles for the case of unequal angles; the auxiliary point I is added in the left circle.

- Left circle (center G): A (lower left), B (top), C (lower right), G (center), I (bottom) — with the angle CGI cut off.
- Right circle (center H): D (lower left), E (top), F (lower right), H (center).

Moreover, this proposition and the three that immediately follow are to be understood also in one and the same circle. That is: in the same circle equal angles stand on equal arcs, and so on — as is plainly clear from the demonstration of this proposition and of the three following. For the same demonstration that is fitted to two or more circles holds also in one and the same circle.

THEOR. 24. — PROPOS. 27. [Euclid, *Elements* III.27] [Margin: xxvj.]

[Enunciation] In equal circles, the angles that stand on equal arcs are equal to one another, whether they are constructed at the centers or at the circumferences.

[Clavius's demonstration] In equal circles ABC, DEF, whose centers are G, H, let there stand, first, the angles at the centers, AGC and DHF, on the equal arcs AC, DF. I say that the angles AGC and DHF are equal. For if they are not equal, let the angle G be the greater, and let the angle AGI be made equal to the angle DHF. [Margin: a — Elem. III.26] The arcs AI, DF will then be equal. But since the arc AC is assumed equal to the arc DF, the arcs AI, AC will be equal to one another, the part to the whole; which is absurd. Therefore the angles AGC, DHF are equal.

Diagram. Two equal circles for Prop. 27; the auxiliary point I is added in the left circle, where the angle AGI is constructed.

- Left circle (center G): A (lower left), B (top), C (lower right), G (center), I (bottom).
- Right circle (center H): D (lower left), E (top), F (lower right), H (center).

Next let the angles B and E at the circumferences stand on the same equal arcs AC, DF; and these too I say are equal. For if one, say ABC, is the greater, let the angle ABI be made equal to the angle E. [Margin: b — Elem. III.26] Then the arcs AI, DF will be equal. Therefore, as before, the arcs AI, AC will be equal, the part to the whole; which is absurd. Therefore the angles ABC, DEF are equal. In equal circles, then, the angles that stand on equal arcs, and so on. Which was to be demonstrated.

SCHOLIUM [Clavius]

This second part will also be demonstrated thus, when the lines AG, CG; DH, FH form the angles at the centers upon the bases AC, DF. Since the angles ABC and E are halves of the angles AGC and H, [Margin: c — Elem. I.20 (recte III.20)] which we have already shown to be equal, they too will be equal to one another.

In the same way, the spaces [sectors] at the centers G, H, standing on the equal arcs ABC, DEF, will be equal. For when the equal arcs ABC, DEF are taken away from the equal circles, the remaining arcs AIC, DF are made equal; and hence, as we showed, the angles AGC, DHF will be equal. When these are taken away from the spaces about the centers GH — which are equal, since each is equal to four right angles, by corollary 2 of Prop. 15 of Book 1 — the remaining spaces at the centers G, H, standing on the arcs ABC, DEF, will also be equal.

But if the arcs are unequal, the greater angle will stand on the greater, whether at the center or at the circumference, than on the lesser. For in the figure of the preceding scholium, let the arc AC be greater than the arc DF. I say that the angle AGC is greater than the angle DHF, and the angle ABC greater than the angle DEF. For if the arc CI be made equal to the arc DF, and the lines IG, IB be drawn, then, as was shown, both the angles at the center, CGI, DHF, and the angles at the circumference, CBI, DEF, will be equal. Therefore both the angle AGC will be greater than the angle DHF, and the angle ABC greater than the angle DEF.

This is also to be understood of the spaces at the centers G, H, standing on the arcs ABC, DEF. For in the same way it will be demon-

strated that the space at H, standing on the arc DEF, is greater than the space at G, which stands on the arc ABC, if the arc DEF is assumed greater than the arc ABC.

[Corollary — parallel chords] Furthermore, from this proposition we shall infer that two straight lines which in the same circle cut off equal arcs, and do not cut one another, are parallel; and that if they are parallel, equal arcs are cut off by them. For in the circle ABCD let the lines AD, BC cut off the equal arcs AB, DC. I say that AD, BC are parallel. For when the line AC is drawn, since the arcs AB, DC are assumed equal, the angles ACB, CAD standing on them will be equal; [Margin: d — Elem. III.27] and since these are alternate angles, AD, BC will be parallel. [Margin: e — Elem. I.27]

Now let AD, BC be parallel. I say that the intercepted arcs AB, DC are equal. For since AD, BC are parallel, when the line AC is drawn the alternate angles ACB, CAD will be equal; [Margin: f — Elem. I.29] and hence the arcs AB, DC on which they stand will be equal. [Margin: g — Elem. III.26]

Diagram. A circle ABCD with the two parallel chords AD (upper) and BC (lower), and the diagonal AC drawn.

- A (upper left) and D (upper right): endpoints of the upper chord AD.
- B (lower left) and C (lower right): endpoints of the lower chord BC.
- The diagonal AC joins A to C.

It has also seemed not useless to add here the following theorem, bearing on what follows; namely:

[Clavius's added theorem — tangent parallel to chord] A straight line drawn tangent to a circle from the midpoint of some arc is parallel to the straight line that subtends that arc.

[Demonstration] In the circle ABC, whose center is D, from A, the midpoint of the arc BAC, let the line EF be drawn tangent to the circle. I say that EF is parallel to the line BC, which subtends the arc BAC. For draw from the center D to the point of contact A the line DA, cutting the line BC at G, and join the lines DB, DC. Then the angles standing on the equal arcs AB, AC will be equal; [Margin: h — Elem. III.27] and the sides BD, DG of triangle BDG are equal to the sides CD, DG of triangle CDG, each to each. Therefore the angles at G, upon the bases GB, GC, are equal, [Margin: i — Elem. I.4] and hence right angles. Consequently the adjacent angles AGB, AGC are also right. But the angles GAE, GAF are likewise right, [Margin: k — Elem. I.18 (recte III.18)] because DA is perpendicular to EF. Therefore EF, BC are parallel. [Margin: l — Elem. I.28] Which is what was proposed.

From what has been demonstrated in this proposition and the preceding one, there is also inferred this theorem, which we demonstrated at the very beginning of the [treatment of] rectilinear triangles.

Diagram. A circle ABC with center D; the tangent EF is drawn at the arc-midpoint A, and the radius DA cuts the chord BC at G.

- E (top left) — A (top center, point of contact) — F (top right): the tangent line EF touching the circle at A.
- B (left) and C (right): endpoints of the subtended chord BC.
- G (center, on BC): where DA meets BC.
- D (below G, center): center of the circle; radii DB, DC drawn.

[Clavius's added theorem — right angle and quadrant] A right angle at the center stands on a quadrant; an acute angle on an arc less than a quadrant; and an obtuse angle on an arc greater than a quadrant. And conversely, an angle at the center standing on a quadrant is a right angle; standing on an arc less than a quadrant, acute; and on an arc greater than a quadrant, obtuse.

[Demonstration] Let the right angle ABC stand on the arc AC, at the center B of the circle ACDE. I say that the arc AC is a quadrant, and so on. For when the lines AB, CB are produced to D, E, the angles ABE, CBD, adjacent to the right angle ABC, will also be right, by definition 10 of Book 1; and so too the angle DBE, since it is equal to the right angle ABC as vertical to it. [Margin: m — Elem. I.15] Therefore all four angles at the center B are equal, being right; and hence the arcs AC, CD, DE, EA, on which they stand, will be equal. [Margin: n — Elem. III.26] Each of them, then, is a quadrant. And since the line that makes an acute angle at B with AB falls within the arc AC, while the line that makes an obtuse angle at B with the same AB falls within the arc CD, it is plainly established that the acute angle stands on an arc less than a quadrant, and the obtuse on a greater.

Diagram. A circle ACDE with center B, divided into four quadrants by the two perpendicular diameters AD (vertical) and EC (horizontal).

- A (top): upper end of the vertical diameter.
- B (center): center of the circle.
- C (right): right end of the horizontal diameter.
- D (bottom): lower end of the vertical diameter.
- E (left): left end of the horizontal diameter.

But now let the angle ABC at the center B stand on the quadrant AC. I say that the angle ABC is a right angle, and so on. For when the lines AB, CB are again

produced to D, E: since both CAE and ACD, and AED, are semicircles, and AC is a quadrant, then both AE and CD will be quadrants also, and hence DE too, in the semicircle AED, will be a quadrant. Therefore the four arcs AC, CD, DE, EA are equal, and hence the angles at the center B that stand on them will be equal. [Margin: o — Elem. III.27] Since, then, all four together are equal to four right angles, each of them will be a right angle. And because the line drawn with AB that cuts off an arc less than the quadrant AC makes at the center B, with AB, an angle less than the right angle ABC; while the line drawn with the same AB that cuts off an arc greater than the quadrant AC forms at the center B an angle greater than the right angle ABC — it is evident that the angle standing on an arc less than a quadrant is acute, and that standing on a greater, obtuse. Which is what was proposed.

By like reasoning this theorem too is drawn out from what has been demonstrated — the theorem which we also demonstrated in [the doctrine of] sines.

[Clavius's added theorem — a line from the center bisecting a chord] A straight line drawn from the center of a circle and bisecting another straight line not drawn through the center will also bisect the arc that this line subtends. And conversely, if it bisects the arc, it will also bisect the line subtending it.

[Demonstration] From the center A of the circle BCDE, let the line AE, going out, cut the line BD in two equal parts at F. I say that the arc BED is also bisected at E, and so on. For when the lines AB, AD are drawn: since the two sides AB, AF are equal to the two sides AD, AF, each to each, and the base BF is assumed equal to the base DF, the angle BAF will be equal to the angle DAF. [Margin: p — Elem. I.8] Therefore the arc BE will be equal to the arc DE. [Margin: q — Elem. III.26] So the arc BD has been bisected at E.

Diagram. A circle BCDE with center A; the vertical diameter runs C (top) — A — E (bottom), and the chord BD is cut at F.

- A (center): center of the circle.
- B (left) and D (right): endpoints of the chord BD, on the circle.
- C (top): upper end of the vertical line through the center.
- E (bottom): lower end of the vertical line, on the circle.
- F (on BD, below A): point where AE meets and bisects the chord BD.

Now let the line AE bisect the arc BD at E. I say that the line BD is likewise bisected at F, and so on. For when the lines AB, AD are again drawn: since the

arcs BE, DE are assumed equal, the angles BAE, DAE at the center will also be equal. [Margin: r — Elem. III.27] And so, since the two sides AB, AF are equal to the two sides AD, AF, each to each, and they contain equal angles, as we showed, the bases BF, DF will also be equal; [Margin: s — Elem. I.4] and hence the line BD is bisected at F. Which was to be demonstrated.

Nor does it seem right to omit in this place a certain outstanding property of the circle, which Giovanni Battista Benedetti [Giovanni Battista Benedetti], drawing on Cardano [Gerolamo Cardano], *De subtilitate* Book 16, ch. 1, demonstrates; namely:

[Benedetti / Cardano theorem] If in a circle two diameters are drawn cutting one another at right angles, and one of them is produced; and if on one side any number of parallels are drawn dividing each quadrant into equal parts; and finally, if from the endpoint of the other diameter, through the endpoint of the nearest parallel, a straight line is drawn to meet the produced diameter: then the whole line, taken between the point of meeting and the concave circumference of the circle, will be equal to all the parallels together with the diameter that was produced, all taken together.

(continued from the previous page — the demonstration of the corollary theorem appended to Proposition 27)

In the circle ABCD let the diameters AC and BD cut one another at right angles, and let AC be produced beyond C as far as you please. Now divide the quadrant BC into any number of equal parts CF, FG, GB, and the quadrant BA into the same number of parts, and join the straight lines FH and GI. By the scholium of Proposition 27 of this book these will be parallel to AC, since they intercept equal arcs. Finally, from B through G draw the straight line BG extended until it meets AC produced at K. I say that the straight line AK is equal to all the parallels AC, HF, IG taken together.

For draw the straight lines HC and IF and produce them until they meet AK. Since the angles CHF and HFI, standing on the equal arcs CF and HI, are equal, [Margin: (t) *Elem.* III.27] the straight lines HC and IL will be parallel; [Margin: (u) *Elem.* I.27] and FH is also parallel to CL. Therefore CHFL is a parallelogram, [Margin: (x) *Elem.* I.34] and consequently the straight line CL is equal to the straight line FH. By the same reasoning KL will be equal to GI, and so on in turn, if there are more of them. When, therefore, the common segment AC is added, the whole line AK becomes equal to AC, HF, and IG taken together. Which was what was proposed.

Diagram. A circle ABCD with two perpendicular diameters — AC drawn vertically (A at the top, C at the bottom) and BD drawn horizontally (B at the left, D at the right) — crossing at the center E. The upper-left quadrant BA and the lower-left quadrant BC are each subdivided; the chords FH and GI (parallel to the vertical diameter) are drawn. The line BG is produced downward past C to K and L, the whole construction tapering to a point P far below the circle.

- A (top of circle): upper end of the vertical diameter
- B (left of circle): left end of the horizontal diameter
- D (right of circle): right end of the horizontal diameter
- C (bottom of circle): lower end of the vertical diameter
- E (center): intersection of the two diameters
- H (upper-left arc): division point on quadrant BA
- I [?] (upper-left arc, near H): division point on quadrant BA
- G (lower-left arc): division point on quadrant BC
- F (lower-left arc, below E): division point on quadrant BC
- L (below C, on the produced line): point where the construction meets the extended diameter
- P (far bottom): terminal point of the produced line

THEOR. 25. PROPOS. 28.

[Margin: (right) xxvij.]

Enunciation (Euclid). In equal circles, equal straight lines cut off equal arcs, the greater equal to the greater and the lesser equal to the lesser.

Demonstration. In equal circles ABC and DEF, whose centers are G and H, let the straight lines AC and DF be equal. I say that the greater arc ABC is equal to the greater arc DEF, and the lesser arc AC equal to the lesser arc DF. For draw the straight lines AG, GC, DH, HF. Then the sides AG, GC of triangle AGC are equal to the sides DH, HF of triangle DHF; and the bases AC, DF are assumed equal.

Therefore the angles at G and H will be equal, [Margin: (a) *Elem.* I.8] and consequently the arcs AC, DF on which they stand will be equal. [Margin: (b) *Elem.* III.26] These, being taken away from the equal wholes, leave the remainders ABC and DEF equal as well. Therefore in equal circles equal straight lines, and so on. Which was to be demonstrated.

Diagram. Two equal circles side by side, each with an inscribed isosceles triangle whose apex is the center.

- Left circle: B (top of circle), G (center, apex of the triangle), A (lower-left) and C (lower-right), the base AC being a chord.
- Right circle: E (top of circle), H (center, apex), D (lower-left) and F (lower-right), the base DF being a chord.

Scholium. But if the straight lines in equal circles are unequal, the greater line will cut off the greater arc, provided we are speaking of arcs of the circle that are less than a semicircle. For if the discussion concerns arcs greater than a semicircle, the greater line will cut off the lesser arc. For in equal circles ABC, DEF, whose centers are G and H, let the straight line AC be greater than DF. I say that the arc AC, less than a semicircle, is greater than the arc DF, but that the arc ABC is less than the arc DEF. For draw the straight lines AG, GC, DH, HF; then the sides AG, GC of triangle AGC are equal to the sides DH, HF of triangle DHF, while the base AC is assumed greater than the base DF. Therefore the angle AGC will be greater than the angle DHF. [Margin: (c) *Elem.* I.25] Make the angle CGI equal to the angle DHF; then the arc CI will be equal to the arc DF, [Margin: (d) *Elem.* III.26] and consequently the arc AIC will be greater than the arc DF; and therefore the remainder ABC will be less than the remainder DEF.

Diagram. Two equal circles. In the left circle the chord AC (the greater line) is drawn well above the center, and a radius GI is dropped from the center G to the point I on the far arc, marking off the arc CI.

- Left circle: B (top), G (center), A (left) and C (right) on the chord AC, and I (bottom of circle) where the radius GI meets the circumference.
- Right circle: E (top), H (center), D (left) and F (right), with the inscribed triangle DHF.

THEOR. 26. PROPOS. 29.

[Margin: (right) xxvii.]

Enunciation (Euclid). In equal circles, equal arcs are subtended by equal straight lines.

Demonstration. In the same equal circles let equal arcs ABC, DEF be laid down, and likewise AC and DF. I say that the straight lines AC, DF which subtend them are equal. For, the lines being drawn as before, the sides AG, GC of triangle AGC will be equal to the sides DH, HF of triangle DHF; and the angles at G and H are also equal, [Margin: (a) *Elem.* III.27] since they stand on the equal arcs AC, DF. Therefore the bases AC, DF will be equal. [Margin: (b) *Elem.* I.4] Therefore in equal circles equal arcs, and so on. Which was to be shown.

Diagram. Two equal circles, each with an inscribed isosceles triangle apexed at the center.

- Left circle: B (top), G (center, apex), A (lower-left), C (lower-right), base AC.
- Right circle: E (top), H (center, apex), D (lower-left), F (lower-right), base DF.

Scholium. But if the arcs are unequal, the greater line will subtend the greater arc rather than the lesser, provided the discussion concerns arcs less than a semicircle. For if we speak of arcs greater than a semicircle, the lesser line will subtend the greater arc rather than the lesser. For in equal circles ABC, DEF, whose centers are G and H, let the arcs AC, DF be less than a semicircle, and let AC be greater than DF, so that ABC is less than DEF. I say that the line AC is greater than DF. For, drawing the straight lines AG, GC, DH, HF, the angle AGC will be greater than the angle DHF, by the scholium of Proposition 27 of this book. Since, then, the sides AG, GC of triangle AGC are equal to the sides DH, HF of triangle DHF, the base AC will be greater than the base DF, [Margin: (c) *Elem.* I.24] and so on.

Diagram. Two equal circles. In the left circle the chord AC lies high, near the center, giving a shallow (obtuse-apexed) triangle AGC; the right circle has the triangle DHF.

- Left circle: B (top), G (center), A (left), C (right) on the chord AC.
- Right circle: E (top), H (center), D (left), F (right), base DF.

Now the four immediately preceding propositions — 26, 27, 28, and 29 — are to be understood as holding also within one and the same circle, as we noted in the scholium of Proposition 26, and as is clear from the demonstrations given; for those same demonstrations hold good in one and the same circle.

And so these same four propositions will be made more universal if they are stated as follows:

In equal circles, or in the same circle, equal angles stand on equal arcs, whether they are set up at the centers or at the circumferences.

In equal circles, or in the same circle, angles that stand on equal arcs are equal to one another, whether they are set up at the centers or at the circumferences.

In equal circles, or in the same circle, equal straight lines cut off equal arcs, the greater equal to the greater and the lesser to the lesser.

In equal circles, or in the same circle, equal arcs are subtended by equal straight lines.

Scholium. But it will not be without use to demonstrate the following propositions as well.

I. Circles from which equal straight lines cut off similar circumferences are equal.

Demonstration. From the circles ABC, DEF let the equal straight lines AC, DF cut off similar circumferences ABC, DEF. I say that the circles themselves are equal. For if the segments ABC, DEF are similar, then the remaining segments AC, DF will also be similar, as we demonstrated at Proposition 22. Again, since upon the equal straight lines AC, DF similar segments ABC, DEF are set up, these will be equal to one another. [Margin: (d) *Elem.* III.24] By the same reasoning the segments AC, DF, which have been shown to be similar, will be equal. Therefore the whole circles will be equal. In like manner, if the equal straight lines AC, DF are said to cut off similar circumferences AC, DF, then, by what we demonstrated at Proposition 22, the segments ABC, DEF will also be similar. [Margin: (e) *Elem.* III.24] Therefore both the one set and the other will be equal to one another; and consequently the whole circles too will be equal to one another. Which was what was proposed.

Diagram. Two circles side by side, each with an inscribed triangle from the center to a chord (the figure shared by Theorems I–IV).

- Left circle: B (top), G (center), A (lower-left), C (lower-right), base AC.
- Right circle: E (top), H (center), D (lower-left), F (lower-right), base DF.

II. From unequal circles, equal straight lines cut off dissimilar circumferences.

Demonstration. In the same figure let the straight lines AC, DF be equal, but the circles ABC, DEF unequal. I say that the circumferences ABC, DEF are dissimilar. For if the circles themselves were similar, then — as we demonstrated just now — they would be equal; and this conflicts with the hypothesis. Therefore the circumferences ABC, DEF are dissimilar. By the same reasoning the circumferences AC, DF will be dissimilar. Which was to be shown.

III. From unequal circles, the straight lines that cut off similar circumferences are unequal.

Demonstration. In the same figure let the circles be unequal, but the circumferences ABC, DEF similar. I say that the straight lines AC, DF are unequal. For if they were said to be equal, then, by the first theorem, the circles would be

equal — which is absurd, since they are assumed unequal. Therefore the straight lines AC, DF are unequal. In the same way, if the circumferences AC, DF are said to be similar, it will be demonstrated that the straight lines AC, DF are unequal.

IV. Straight lines that cut off similar unequal circumferences from any circles whatever are unequal.

Demonstration. In the same figure let the circumferences ABC, DEF be similar and unequal. I say that the straight lines AC, DF are unequal. For either the circles are equal or they are unequal. Let them first be equal. If, then, the straight lines AC, DE [recte DF] are said to be equal, the circumferences ABC, DEF cut off will be equal, [Margin: (f) *Elem.* III.28] which is contrary to the hypothesis. Therefore the straight lines AC, DF are not equal. Let the circles next be unequal. Then the straight lines AC, DF cutting off the similar circumferences ABC, DEF are unequal, as was demonstrated in the third theorem. We shall show in no other way that the straight lines AC, DF are unequal if the circumferences AC, DF are assumed similar and unequal.

Diagram. A second pair of circles, each with an inscribed triangle from the center to a chord, serving Theorem IV.

- Left circle: B (top), G (center), A (lower-left), C (lower-right), base AC.
- Right circle: E (top), H (center), D (lower-left), F (lower-right), base DF.

V. If in the diameter of a circle a point is taken other than the center, and from it two straight lines falling on the circumference on opposite sides make equal angles with the diameter, then those straight lines will be equal and will cut off equal arcs. And if the lines are equal, they will make equal angles with the diameter and cut off equal arcs. And finally, if they cut off equal arcs, those straight lines will be equal and will make equal angles with the diameter.

(demonstration of Theorem V)

In the circle ABCDE, whose center is F, let a point G be taken on the diameter AD, other than the center, and let first two equal angles CGD, EGD be set up. I say that both the straight lines GC, GE and the arcs CD, ED are equal. For since the two angles CGD, CGF, and likewise the two angles EGD, EGF, are together equal to two right angles, [Margin: (g) *Elem.* I.13] then if the equal angles CGD, EGD are taken away, the remainders CGF, EGF will also be equal. So, drawing the straight lines CF, EF to the center, the two sides FC, FG will be equal to the two sides FE, FG, and the sides opposite the equal angles FGC, FGE will be equal. And each of the remaining angles FCG, FEG is less than a right angle — for if the straight line CD is

drawn, the angle FCD in the isosceles triangle FCD is acute, by the third corollary of Proposition 17 of Book I, and therefore FCG is a fortiori acute; and in the same way, drawing the straight line DE, the angle FEG is acute. Hence, by what we have demonstrated, the base GC will be equal to the base GE, and the angle CFG equal to the angle EFG; [Margin: (h) *Elem.* III.26] and consequently the arcs CD, ED will be equal. The proposition is therefore established.

Diagram. A circle with five points A, B, C, D, E on the circumference, center F, and a point G on the diameter AD below the center. Straight lines run from F and from G to the circumference, forming a star of triangles; the auxiliary point H lies on GB (used in the following theorem).

- A (top): upper end of the diameter AD
- D (bottom): lower end of the diameter AD
- F (center): center of the circle
- G (on the diameter, between F and D): the point taken other than the center
- B (upper-left, on the circle)
- C (left, on the circle)
- E (right, on the circle)
- H (inside, on GB): the point marked off so that GH equals GD (belonging to the following theorem)

Next, let the lines GC, GE be equal. I say that the angles CGD, EGD are also equal, and likewise the arcs CD, ED. For if the lines GC, GE are equal, the two sides GC, GF will be equal to the two sides GE, GF; and the base FC is also equal to the base FE. Therefore the angles FGC, FGE will be equal, [Margin: (i) *Elem.* I.8] and consequently the remainders CGD, EGD, taken from two right angles, will be equal. Again, since the two sides FC, FG are equal to the two sides FE, FG, and the bases GC, GE [recte; printed "GC, CE"] are assumed equal, the angles CFD, EFD will also be equal, [Margin: (k) *Elem.* I.8] and therefore the arcs CD, ED will be equal. [Margin: (l) *Elem.* III.26] Which was what was proposed.

Thirdly, let the arcs CD, ED be equal. I say that the lines GC, GE and the angles CGD, EGD are equal. For if the arcs CD, ED are equal, the angles CFD, EFD will also be equal. [Margin: (m) *Elem.* III.27] Therefore, since the two sides FC, FG are equal to the two sides FE, FG and contain equal angles, the bases GC, GE will be equal, [Margin: (n) *Elem.* I.4] and the angles FGC, FGE; and consequently the remainders CGD, EGD, taken from two right angles. All of which was to be demonstrated.

This theorem, however, is far more universally covered by Lemma 21 of Book I of our *Astrolabe*.

V. [recte VI]

Enunciation. If in the diameter of a circle a point is taken other than the center, and at it two equal angles are set up on the same side, then these equal angles will stand on unequal arcs, and the greater will be that which is farther from the smaller portion of the diameter.

Demonstration. In the same figure just drawn, let there be two equal angles CGD, CGB. I say that the arc BC is greater than the arc CD. For, drawing the straight lines BC, CD: since the straight line GB is greater than the straight line GD, [Margin: (o) *Elem.* III.7] mark off GH equal to GD and join the straight line CH. Since the two sides GH, GC are equal to the two sides GD, GC and contain equal angles, the bases CH, CD will be equal, and the angle GHC equal to the angle GDC. [Margin: (p) *Elem.* I.4] But GDC is acute, being in the isosceles triangle FCD, by the third corollary of Proposition 17 of Book I. Therefore GHC too will be acute; and consequently the remaining angle CHB, taken from two right angles, will be obtuse. Since, then, the two angles CHB, CBH are less than two right angles, [Margin: (q) *Elem.* I.17] the angle CBH will be acute. Therefore the side BC will be greater than the side CH. [Margin: (r) *Elem.* I.19] But CH has been shown equal to CD. Therefore the straight line BC will also be greater than CD; and consequently, by the scholium of Proposition 28 of this book, the arc BC will be greater than the arc CD. Which was to be shown.

PROBL. 4. PROPOS. 30.

[Margin: (right) xxix.]

Enunciation (Euclid). To bisect a given arc.

Demonstration. Let ABC be the arc to be bisected. Draw the straight line AC subtending it, and, this being bisected at D, erect the perpendicular DB, which will bisect the arc ABC at B. For, drawing the straight lines AB, CB, the sides AD, DB of triangle ADB will be equal to the sides CD, DB of triangle CDB; and the angles at D are also equal, namely right angles. Therefore the bases AB, CB will be equal, [Margin: (a) *Elem.* I.4] and consequently the arcs AB, CB will be equal. [Margin: (b) *Elem.* III.28] We have therefore bisected the given arc. Which was to be done.

Diagram. An arc ABC standing on its chord AC; B is the top of the arc, and the perpendicular BD is dropped to the midpoint D of the chord, with the lines BA and BC drawn. A faint full circle is sketched below as construction.

- B (top): the point where the perpendicular meets the arc
- A (lower-left): one end of the chord
- C (lower-right): the other end of the chord
- D (lower-center): the midpoint of the chord AC

Praxis. The practical procedure for dividing a given arc in two does not differ from the procedure for dividing a straight line in two which we set out at Proposition 10 of Book I. For whenever any arc is proposed, such as ABC, one must always understand the straight line AC that subtends it, even if it has not been drawn; and from the points A and C one describes arcs with the same interval, cutting one another above and below the points A and C, and so on, exactly as if the said straight line AC were to be bisected. For the straight line that bisects the line AC will also bisect the given arc in two.

THEOR. 27. PROPOS. 31.

[Margin: (right) xxx.]

Enunciation (Euclid). In a circle, the angle in a semicircle is a right angle; the angle in the greater segment is less than a right angle; and the angle in the lesser segment is greater than a right angle. Moreover, the angle of the greater segment is greater than a right angle, while the angle of the lesser segment is less than a right angle.

Demonstration (begins). For let AC be the diameter of the circle ABC, whose center is D, and let the angle ABC be set up in the semicircle, and let the angle BAC exist in the greater segment CAB. Let the angle BEC also be set up in the lesser segment CEB. I say that the angle ABC in the semicircle is a right angle; that the angle BAC in the greater segment is less than a right angle; and that the angle BEC in the lesser segment is greater than a right angle. Likewise, that the angle of the greater segment, contained by the straight line BC and the arc BAC, is greater than a right angle; but that the angle of the lesser segment, contained by the straight line BC and the arc BEC, is less than a right angle. For draw the straight line BD to the center, and produce AB to F. Since, therefore, the straight lines DA, DB...

Diagram. A circle with horizontal diameter AC and center D. B lies on the upper-left arc and E on the upper-right arc; the lines BA, BC, BD, and BE are drawn, EC is drawn, and AB is produced beyond B to the external point F.

- A (left end of diameter)
- C (right end of diameter)

- D (center)
- B (upper-left, on the circle)
- E (upper-right, on the circle)
- F (outside the circle, upper-left): the point to which AB is produced

Euclid, *Elements*, Book III — Clavius's Commentary

Proposition 31 (continued) — demonstration concluded

[DA and] DB are equal, and so [Margin: a — Elem. I.5] angle DBA is equal to angle DAB. By the same reasoning angle DBC will be equal to angle DCB, [Margin: b — Elem. I.32] and therefore the whole angle ABC will be equal to the two angles BAC and BCA together. But the exterior angle FBC is also equal to those same two interior angles BAC and BCA in triangle ABC. Therefore angles ABC and FBC are equal to one another, and hence each is a right angle. So angle ABC is a right angle — which was the first point.

Now since in triangle ABC [Margin: c — Elem. I.17] the two angles ABC and BAC are together less than two right angles, and angle ABC has been shown to be right, angle BAC (in the greater segment) will be less than a right angle — which was the second point.

Again, since in the quadrilateral ABEC inscribed in the circle [Margin: d — Elem. III.22] the two opposite angles BAC and BEC are equal to two right angles, and angle BAC has been shown to be less than a right angle, angle BEC (the angle in the lesser segment) will be greater than a right angle — which was the third point.

It is enough, however, to have demonstrated that one angle in the semicircle — namely ABC — is right; that one in the greater segment — such as BAC — is less than a right angle; and finally that one in the lesser segment — such as BEC — is greater than a right angle. For since all angles in the same segment are equal to one another, it is clear that if one angle in the semicircle is right, all the angles in the same semicircle are right; if one in the greater segment is less than a right angle, all in the same segment are less than a right angle; and finally, if one in the lesser segment is greater than a right angle, all in the same segment are greater than a right angle. I say this because Euclid does not prove absolutely that any angle whatever in the greater segment BAC is less than a right angle — as is

evident if some other angle were set up in that segment which was not subtended by the diameter, as angle BAC is. Hence any other such angle will likewise be shown to be less than a right angle, because [Margin: e — Elem. III.21] it is equal to angle BAC in the same greater segment, which Euclid has shown to be less than a right angle.

Diagram. A circle with horizontal diameter AC, center D. On the upper-left circumference is B; at the upper right is E. Point F lies outside the circle above and to the left, where the line AB is produced. Chords are drawn from B to E, to C, and to A; the line through F meets B; AC is the diameter; EC is joined.

- A (left end of diameter): terminus of diameter
- C (right end of diameter): terminus of diameter
- D (center): center of the circle
- B (upper-left circumference): vertex of the semicircle angle ABC
- E (upper-right circumference): vertex of the angle BEC in the lesser segment
- F (outside, upper left): point on AB produced

Furthermore, since the right angle ABC is part of the angle of the greater segment BAC (the angle contained by the straight line BC and the arc BAC), the angle of the greater segment will be greater than a right angle — which is the fourth point.

Finally, since the angle of the lesser segment (contained by the straight line BC and the arc BEC) is likewise part of the right angle FBC, the angle of the lesser segment will be less than a right angle — which is the fifth point. Therefore in a circle the angle in the semicircle is right, etc. Which was to be demonstrated.

Another demonstration of this proposition

Diagram. A semicircle on diameter AC with center D; B is the apex on the arc. Straight lines are drawn from B to A, to D, and to C.

- A (left end of base): terminus of the diameter
- D (center of base): center of the semicircle
- C (right end of base): terminus of the diameter
- B (apex on the arc): vertex of the right angle

In the semicircle whose diameter is AC and whose center is D, let ABC be the angle, which I say is right. For when the straight line BD has been drawn, [Margin: f — Elem. I.5] the angles DBA and DAB will be equal, because the straight lines DA and DB are equal. Since, then, the exterior angle BDC [Margin: g

— Elem. I.32] is equal to the two interior angles DBA and DAB in triangle ABD, angle BDC will be double angle DBA. In the same way angle ADB will be double angle DBC; and thus the two angles at D together will be double the whole angle ABC. Since, then, [Margin: h — Elem. I.13] the angles at D are equal to two right angles, angle ABC will be half of them — that is, a right angle. Which was the first point.

Or thus: Since [Margin: i — Elem. I.5] angle DAB equals angle DBA, and angle DCB equals angle DBC, the two angles A and C together will equal angle ABC in triangle ABC. And accordingly angle ABC is half of the three angles A, C, and ABC of that same triangle. Therefore, [Margin: k — Elem. I.32] since the three angles of triangle ABC are equal to two right angles, ABC will be half of two right angles, and hence a right angle.

Diagram. A circle with B at the top; the diameter AE runs from A (left) through the center D to E (upper right on the circumference); C is at the lower right. Chords AB, BC, and BE are drawn, together with AC.

- A (left): one end of diameter AE
- E (upper right circumference): other end of diameter AE
- D (center): center of the circle
- B (top): vertex of the angle in the greater segment
- C (lower right): further point on the circumference

Again, let there be, in the greater segment whose center is D, the angle ABC, which I say is less than a right angle. For when the diameter AE has been drawn and the straight line BE joined, angle ABE will be right, being in a semicircle, as has been demonstrated. Therefore angle ABC, being part of a right angle, will be less than a right angle — which is the second point. And this demonstration fits every angle in the lesser segment [recte "greater segment"? —] — something that could not be said of the earlier demonstration, as we noted there.

Diagram. A figure with B at the top; A at the left, D at the center, C at the lower right, and E below and to the right, outside the original arc, where the diameter AE (produced) meets the circumference. Straight lines AB, BC, and BE are drawn.

- A (left): one end of the diameter
- E (lower right, outside): end of the produced diameter
- D (center): center
- B (top): vertex of the angle in the lesser segment
- C (lower right): point on the circumference

Again, let there be, in the lesser segment whose center is D, the angle ABC, which I say is greater than a right angle. For when the diameter AE has been drawn, meeting the produced circumference at E, and the straight line BE joined, angle ABE will be right, being in a semicircle, as has been demonstrated; and since it is part of angle ABC, angle ABC will be greater than a right angle — which is the third point.

Diagram. A circle with center E. B lies on the upper-left circumference and C at the right, with BC a diameter through the center E. A lies on the lower-left circumference; the line BAF is drawn through A and produced to F outside the circle at the lower left. D lies on the lower circumference; AC is joined.

- B (upper left): one end of diameter BC
- C (right): other end of diameter BC
- E (center): center of the circle
- A (lower left): point where BAF crosses the circumference
- F (outside, lower left): point on BA produced
- D (lower circumference): point in the lesser segment ADC

Now let there be proposed a greater segment ABC and a lesser segment ADC, whose center is E. I say that the angle CAB of the greater segment — namely the one formed by the straight line CA and the arc ABC — is greater than a right angle; but that the angle CAD of the lesser segment, formed by the same straight line CA and the arc ADC, is less than a right angle. For when the diameter CB and the straight line BAF have been drawn, its part BA — joining the two points B and A — will fall [Margin: 1 — Elem. III.2] inside the circle, while the remaining part AF falls outside it; and angle BAC will be right, being in a semicircle, and consequently the adjacent angle FAC will also be right. Since, then, the rectilineal right angle BAC is part of the angle CAB of the greater segment — namely the one contained by the straight line CA and the arc ABC —

— and the angle CAD of the lesser segment, contained namely by the straight line CA and the arc ADC, is likewise part of the right angle FAC, both points are established.

Corollary

Hence it is clear that the angle of a triangle which is equal to the remaining two is a right angle — because the angle adjacent to it (formed when a side is produced beyond the triangle) is equal to those same two, which is evident from the earlier

demonstration; or because it is half of the three angles of the triangle, which are equivalent to two right angles, which is evident from the later demonstration.

From Campanus [Campanus of Novara]

From this proposition it is also clear that those two arguments have no force which we attacked at Proposition 16 of this book. One of them is: "One passes from the greater to the lesser through all intermediate values; therefore through the equal." The other is of this kind: "It is possible to find something greater and something lesser than the same thing; therefore it is possible to find something equal."

Diagram. A circle with center D. A is at the left; the diameter AE runs to E at the right. B lies on the upper-right circumference; from A the line AB is drawn and produced upward beyond the circle. C lies at the lower part of the circumference, and F below and to the left, where AC is produced. Straight lines are drawn from A to B, to E (the diameter), and to C (produced to F).

- A (left): fixed point from which the moving line turns
- D (center): center
- E (right): far end of diameter AE
- B (upper right): point on the circumference; angle ABE in the greater segment
- C (lower circumference): point reached as the line turns to AC
- F (lower left, outside): point on AC produced; angle ACF of the lesser segment

For in the circle ABC, whose center is D and whose diameter is AE, let the straight line AB be drawn. [Margin: m — Elem. III.31] Angle ABE, of the greater segment, will be greater than a right angle. Hence if AB is turned toward AE about the fixed point A, it will always make with the circumference an angle greater than a right angle, until it reaches the diameter AE, [Margin: n — Elem. III.16] where it will make the angle of a semicircle, which is less than a right angle. And if it is turned further, to AC, [Margin: o — Elem. III.31] it will a fortiori make the angle ACF of the lesser segment, which is less than a right angle. Therefore one passes from the angle of the greater segment, which is greater than a right angle, to the angle of the semicircle — or even to that of the lesser segment, each of which is less than a right angle — yet not through an angle equal to a right angle. Since, then, the passage is made through all the intermediate angles, it is clear that the aforesaid inferences are faulty.

Scholium

It is also evident that the converse of this theorem holds — namely: The segment of a circle in which the angle set up is right is a semicircle; the one in which the angle is acute is a greater segment; and the one in which the angle is obtuse is a lesser segment. And the segment whose angle is greater than a right angle is greater than a semicircle; while the one whose angle is less than a right angle is either a semicircle or less than a semicircle.

For, the angle being right, if the segment is not a semicircle it will be either greater — and so the angle will be acute — or lesser — and so the angle in it will be obtuse; both of which conflict with the hypothesis. Again, the angle being acute, if the segment is not greater than a semicircle it will be either a semicircle — and so the angle in it will be right — or less — and so the angle in it will be obtuse; both of which likewise conflict with the hypothesis. Finally, the angle being obtuse, if the segment is not less than a semicircle it will be either a semicircle — and so the angle in it will be right — or greater — and so the angle in it will be acute; which is similarly contrary to the hypothesis. Moreover, when the angle of the segment is greater than a right angle, if the segment is not greater than a semicircle it will be either a semicircle or less than a semicircle, and so its angle will be less than a right angle — which is not what is assumed. But when the angle of the segment is less than a right angle, if the segment is not a semicircle or less than a semicircle it will be greater, and so its angle will in turn be greater than a right angle. Which contradicts the hypothesis.

Indeed, this theorem too is true, namely:

If a straight line is subtended to a right angle — that is, if in a right triangle the side opposite the right angle is bisected, and about that straight line (the side), from the point of division, a circle is described — that circle will necessarily pass through the right angle.

Diagram. The right-angle vertex B stands at the top; below it, on the described circle, is the point E. The base runs A (left) — D (center) — C (right), with the circle AEC drawn on AC and center D. Straight lines join B to A and to C, and E to A and to C. [A second B-like mark appears near the apex — [?].]

- B (top): vertex of the right angle of triangle ABC
- E (below B, on the described circle): point at which the circle would meet, if it did not pass through B
- A (left of base): end of the hypotenuse

- D (center of base): midpoint of the hypotenuse AC
- C (right of base): end of the hypotenuse

For let the straight line AC be subtended to the right angle ABC — or, in the right triangle ABC, let the side AC opposite the right angle B — be bisected at the point D, and from D, with radius DA or DC, let the circle AEC be described, which I say passes through B. For if it were to pass on the near side of B, or beyond it, then when the straight lines AE and CE are drawn so as not to cut AB and AC but to fall either inside or outside them, [Margin: p — Elem. III.31] angle AEC would also be right. Then the right angles B and E would be equal — which is absurd, since angle E must necessarily [Margin: q — Elem. I.21] be either greater or less than angle B. Therefore the circle passes through the point B — which was proposed.

From what has been demonstrated in the earlier part of this proposition, we shall without difficulty draw, from a given point outside a circle, two straight lines — one on each side of the straight line drawn from that same point through the center of the circle — which touch the circle. This is what we promised at Proposition 17 of this book.

Diagram. Two overlapping circles. D is the external point at the left. The line DC runs to C, the center of the given circle, passing through E [printed here as F on the figure — see below], the midpoint of DC. The described circle passes through A (at the top) and B (at the bottom), the points of contact. Straight lines DA and DB are drawn, together with AC and BC.

- D (left, external): the given point outside the circle
- F (on DC, center): midpoint of DC, center of the described circle (*labelled F on the figure; the text calls this point E*)
- C (center-right): center of the given circle
- A (top): point of contact of tangent DA
- B (bottom): point of contact of tangent DB

For let there be a circle AB whose center is C, and a point D outside it. Let the straight line DC be drawn from the given point to the center, and bisected at E; and from E, with radius EC or ED, let the circle ACBD be described, cutting the given circle at the points A and B; and let the straight lines DA and DB be joined. I say that each touches the circle at A and at B. For when the straight lines AC and BC are drawn, [Margin: r — Elem. III.31] each angle A and B will be right, being in a semicircle. Therefore, by the corollary to Proposition 16 of this book, the straight line DA will touch the circle AB at A, and the straight line DB will touch

the same circle at B — since the former is perpendicular to the extremity of the radius AC just as the latter is to the extremity of the radius CB.

But we shall also demonstrate here this not-unwelcome theorem drawn from Pappus of Alexandria [Pappus]:

If through the center of a circle another circle is described, and a straight line is projected through the centers of both circles, and from the point where this line is cut by the latter circle a straight line is drawn in any way whatever, then the portion of it that lies inside the former circle will be bisected by the circumference of the latter.

Diagram — first figure. The larger semicircular arc ABC stands on the base line, with A at the left, D at the center, C toward the right, and F at the right end. A smaller circle DEF sits within it, passing through D, E, and G. The line AF is the base (through both centers); the line FB rises from F and cuts the circumferences at B, E, and G.

- A (left): one end of line AF
- F (right): point where circle DEF meets line AF
- D (center of large circle): also on the small circle
- C (between D and F): point on the base
- B (top left, on large arc): point where FB meets the large circumference
- E (middle, on small circle): point where FB meets the small circumference
- G (right, on large arc): further intersection of FB

Let there be a circle ABC, through whose center D another circle DEF is described, and through the centers of these circles let the straight line AF be drawn; and finally, from the point F, where the circle DEF cuts the straight line AF, let the straight line FB be drawn, cutting the circumferences at B, E, G. I say that BE and GE are equal. For when the straight line DE has been drawn, [Margin: r — Elem. III.31] angle DEF will be right, being in the semicircle DEF. Therefore [Margin: s — Elem. III.3] the straight line DE bisects the straight line BG.

Diagram — second figure. The larger arc AB...C with base A (left) — D (center) — C (right); the inner circle now passes through D, E, and C. The line from B descends through E to C.

- A (left): end of the base
- D (center): center of the large circle
- C (right): point through which the inner circle passes

- B (top): point on the large arc
- E (middle): intersection on the inner circle

But if the latter circle passes through the point C, as in the second figure, then again [Margin: t — Elem. III.31] angle DEC will be right, being in the semicircle DEC; and [Margin: u — Elem. III.3] therefore BC is bisected at E.

Diagram — third figure. The large arc with base A (left) — D (center) — C and G together at the right; the inner circle DEF lies wholly within. The line from B descends through E toward G.

- A (left): end of the base
- D (center): center of the large circle
- C (right): point on the base
- G (right, on arc): intersection of the descending line
- B (top): point on the large arc
- E (middle): intersection on the inner circle

And finally, if the latter circle lies wholly within the former, as in the third figure, then again [Margin: x — Elem. III.31] angle DEF will be right, being in the semicircle DEF. Therefore [Margin: y — Elem. III.3] the straight line DE will bisect the straight line BG. Which was to be shown.

This theorem too can easily be demonstrated in another way.

Additional theorem — the inscribed quadrilateral

A quadrilateral inscribed in a circle, two of whose opposite sides are parallel and equal, is a rectangular parallelogram — that is, either a square or an oblong ["longer on one side"].

Diagram. A circle with an inscribed quadrilateral. A is at the upper left, D at the upper right, B at the lower left, C at the lower right; the diagonals AC and BD cross at the center E.

- A (upper left): vertex
- D (upper right): vertex
- B (lower left): vertex
- C (lower right): vertex
- E (center): intersection of the diagonals AC and BD

In the circle ABCD let a quadrilateral ABCD be inscribed, two of whose opposite sides AD and BC are parallel and equal. I say that it is a parallelogram and a rectangle. For since the straight lines AD and BC are parallel and equal, [Margin: z — Elem. I.33] AB and DC will also be parallel and equal. Therefore ABCD is a parallelogram. And since [Margin: a — Elem. III.28] arc AD equals arc BC (because the straight lines AD and BC are equal), and arc AB equals arc DC (because the straight lines AB and DC are equal), the whole arc BAD will equal the whole arc BCD, and the whole arc ADC will equal the whole arc ABC; and accordingly, when the straight lines BD and AC are drawn, BAD, BCD, ADC, and ABC will be semicircles. Therefore [Margin: b — Elem. III.31] the angles of the quadrilateral will be right. Which was proposed.

Theorem 28. Proposition 32.

[Margin: xxxj — Elem. III.31;]

[*Euclid, Elements III.32 — enunciation.*] If a straight line touches a circle, and from the point of contact a straight line is drawn cutting the circle, then the angles which it makes with the tangent are equal to the angles standing in the alternate segments of the circle.

Diagram — first (upper) figure. The straight tangent line runs A — C — B horizontally across the top, touching the circle at C. Within the circle, D is on the left circumference and G on the right; E is at the bottom, diametrically opposite C, so that CE is a diameter. Straight lines CD, CG, DE, and GE are drawn.

- A (left of tangent): one end of the tangent
- C (tangent point): point of contact
- B (right of tangent): other end of the tangent
- E (bottom): far end of the diameter CE
- D (left circumference): vertex of angle CDE
- G (right circumference): vertex of angle CGE

Diagram — second (lower) figure. Again the tangent A — C — B runs across the top, touching at C. D is on the left circumference, G on the right, E at the lower left, and F at the bottom. Here CE does not pass through the center; CF is the diameter (drawn through the center to F), and EF is joined. Straight lines CD, CG, CE, EF, DE, and GE are drawn.

- A (left of tangent): one end of the tangent
- C (tangent point): point of contact

- B (right of tangent): other end of the tangent
- F (bottom): far end of the diameter CF, drawn through the center
- E (lower left circumference): point on the chord CE
- D (left circumference): vertex of angle CDE
- G (right circumference): vertex of angle CGE

Let the straight line AB touch the circle CDE at the point C, and from C let the straight line CE be drawn, dividing the circle into two segments in which the angles CGE and CDE are formed. I say that angle ACE is equal to angle CGE in the alternate segment, and that angle BCE is equal to angle CDE in the other alternate segment.

First let the straight line CE pass through the center, as in the first figure. [Margin: a — Elem. III.18] Then each angle ACE and BCE will be right; [Margin: b — Elem. III.31] and the angles CGE and CDE, being in semicircles, are also right. Therefore angle ACE is equal to angle CGE, and angle BCE to angle CDE.

Now let CE not pass through the center, as in the latter figure. Then let the straight line CF be drawn through the center, and let the straight line EF be joined. [Margin: c — Elem. III.18] Then CF will be perpendicular to AB, [Margin: d — Elem. III.31] and angle CEF will be right; and therefore the remaining angles ECF and EFC will together equal one right angle — equal to the right angle ACF. So when the common angle ECF is taken away, the remaining angle ACE will equal the remaining angle CFE. [Margin: e — Elem. III.21] But angle CGE is also equal to angle CFE, since both are in the segment CGE. Therefore angle ACE will equal angle CGE. And since in the quadrilateral CDEG [Margin: f — Elem. III.22] the two angles CDE and CGE are equal to two right angles, [Margin: g — Elem. I.13] and the two angles ACE and BCE are also equal to two right angles, then if the equal angles ACE and CGE are taken away, the remaining angle BCE will equal angle CDE. Therefore, if a straight line touches a circle, and from the point of contact, etc. Which was to be shown.

Book III (continued)

Scholium [to Proposition 32]

[Clavius's scholium, giving the converse of the tangent-chord theorem.]

This theorem can be converted, as follows.

[Converse — enunciation.] If a straight line drawn to the extremity of a line that cuts the circle makes with it angles equal to those which stand in the alternate segments of the circle, then the drawn line will be tangent to the circle.

[Demonstration.] In the same figure as before, let the straight line CE first pass through the center, cutting the circle CDE, and let the straight line AB be drawn through C, making the angle ACE equal to the angle CGE. I say that AB touches the circle. Since [Margin: (h) III.31] the angle CGE is a right angle, the angle ACE, being equal to it, will also be a right angle. Therefore, by the corollary to Proposition 16 of this book, AB touches the circle.

Now let CE not pass through the center, and let the latter figure be constructed, as above. Since, then, the angle ACE is assumed equal to the angle CGE in the alternate greater segment, and [Margin: (i) III.21] this in turn is equal to the angle CFE, the angle ACE will also be equal to the angle CFE. Adding therefore the common angle ECF, the angle ACF will equal the two angles EFC and ECF taken together. But the angles EFC and ECF are equal to one right angle, because [Margin: (k) III.31] the angle CEF, being in a semicircle, is a right angle, and [Margin: (l) I.32] the three angles of the triangle CEF are equal to two right angles. The angle ACF is therefore also a right angle; and so, by the corollary to Proposition 16 of this book, AB touches the circle.

In the same way, if the angle BCE is equal to the angle CDE in the alternate lesser segment, the straight line AB will be shown to touch the circle. For since [Margin: (m) I.13] the angles BCE and ACE are equal to two right angles, and likewise [Margin: (n) III.22] the two angles CDE and CGE are equal to two right angles, then if the equal angles BCE and CDE are subtracted, the remaining angles ACE and CGE will be equal. Therefore, as has now been demonstrated, the straight line AB touches the circle.

PROBL. 5. PROPOSITIO 33.

[Margin: xxxij]

[Euclid — enunciation.] On a given straight line to describe a segment of a circle admitting an angle equal to a given rectilineal angle.

[Demonstration.]

First case (the given angle is a right angle). Let the given straight line be AB, and let the given angle first be the right angle C. It is required, then, to describe on AB a segment in which the angle standing will be equal to the given right angle C.

Bisect AB at D, and with center D and radius DA (or DB) describe the semicircle AEB; and what was proposed will be done. For [Margin: (a) III.31] the angle AEB in the semicircle so described is a right angle, and therefore equal to the right angle C.

Diagram. The construction for the right-angle case of Proposition 33. At the far left stands the given right angle, marked C, drawn as two perpendicular arms. To its right, a semicircle is erected on the diameter AB; its apex on the arc is E.

- C (far left): vertex of the given right angle.
- A (left end of the diameter): endpoint of the given line.
- D (midpoint of AB): center of the semicircle.
- B (right end of the diameter): other endpoint of the given line.
- E (apex on the arc): vertex of the inscribed right angle AEB.

Second case (the given angle is acute). Next let the given angle C be acute. At the point A make the angle DAB equal to the acute angle C; and draw AE perpendicular to DA, which will fall above AB. Then make the angle FBA equal to the angle FAB, and let BF cut the straight line AE at F. [Margin: (b) I.6] The straight lines FA and FB will then be equal. Hence, if with center F and radius FA a circle AGB is described, it will pass through B. I say, then, that the angle in the segment AGB, which has been described on AB, is equal to the angle C. For let an angle be made in the said segment AGB. Since AE passes through the center F, and DA is perpendicular to it, the straight line DA touches the circle at A, by the corollary to Proposition 16 of this book. Therefore [Margin: (c) III.32] the angle DAB — that is, the given angle C — will be equal to the angle G in the alternate segment AGB.

Third case (the given angle is obtuse). Third, let the given angle H be obtuse. Again make the angle IAB equal to the angle H, and draw AE perpendicular to IA, which will fall above AB. Let everything else be done as before; and there will be described on AB the segment AKB, in which the angle K is equal to the given obtuse angle H. For the angle IAB — that is, the given angle H — is [Margin: (d) III.32] equal to the angle K in the alternate segment AKB. The demonstration is the same. Thus on the given straight line we have described the segment, and so forth. Which was to be accomplished.

Diagram. The construction for the acute and obtuse cases of Proposition 33, combined in one circle with center F. The chord AB is drawn, with the tangent line passing through A (the point D lies on it below-left, the point I on its extension above-left, for the obtuse case). The major arc carries the point G (segment AGB), the minor arc the point K (segment AKB). At the far left the given

angles are set out (the acute angle and the obtuse angle H). Several small peripheral labels are hard to resolve even at high magnification.

- G (top, major arc): vertex of the inscribed angle for the acute case, segment AGB.
- E (interior, on the perpendicular AE): point where BF meets AE.
- F (center): center of the constructed circle.
- A (left end of chord): tangent point / endpoint of AB.
- B (right end of chord): other endpoint of AB.
- D (lower-left, on the tangent line): point of the tangent DA.
- I [?] (upper-left, on the tangent extended): point of the line IA for the obtuse case.
- K [?] (bottom, minor arc): vertex of the inscribed angle for the obtuse case, segment AKB.
- H [?] (far left): vertex of the given obtuse angle.

[— 3 peripheral labels flagged.]

PROBL. 6. PROPOS. 34.

[Margin: xxxiij]

[Euclid — enunciation.] From a given circle to cut off a segment admitting an angle equal to a given rectilineal angle.

[Demonstration.] Let the given circle be ABC, from which a segment must be cut off in which the angle standing is equal to the given angle D. [Margin: (a) III.17] Draw the straight line EF touching the circle at A. Then make the angle FAB equal to the given angle D. I say, then, that the angle ACB in the cut-off segment ACB is equal to the given angle D. For [Margin: (b) III.32] the angle FAB is equal to the angle C in the alternate segment ACB. Since, then, the angle FAB has been made equal to the given angle D, the angle C will also be equal to the angle D. From the given circle, therefore, we have cut off the segment ACB, and so forth. Which was to be done.

Diagram. To the left stands the given rectilineal angle D (an open V-shape). To the right is the circle carrying the cut-off segment: the tangent line EF touches it at A (top), the chord AB is drawn, and C lies on the arc of the segment ACB.

- D (left): vertex of the given angle.
- A (top, tangent point): where EF touches the circle.

- C (left of circle, on the arc): vertex of the inscribed angle ACB.
- B (lower right, on the circle): other end of the chord.

(The tangent's endpoints E and F, named in the text, are not separately lettered near the point of contact on this small cut; only A, B, C, D are legible.)

THEOR. 29. PROPOS. 35.

[Margin: xxxiv]

[Euclid — enunciation.] If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

[Demonstration.] In the circle ACBD let the straight lines AB and CD cut one another at E. I say that the rectangle contained by the segments AE and EB is equal to the rectangle contained by the segments CE and ED. For either both lines pass through the center, or one only, or neither.

First case (both pass through the center). Let both first pass through the center. Since, then, all four segments are equal to one another, it is plain that the rectangle contained by the two segments of one line is equal to the rectangle contained by the two of the other line, from what we wrote at the beginning of Book II.

Diagram. A circle in which two diameters AB and CD cross at the center E.

- A (top left): end of one diameter.
- D (top right): end of the other diameter.
- E (center): intersection of the diameters.
- C (bottom left): other end of CD.
- B (bottom right): other end of AB.

Second case (only CD passes through the center, bisecting AB). Next let CD alone pass through the center F, and let it first bisect the straight line AB — [Margin: (a) III.3] and therefore at right angles — and let the straight line BF be joined. Since, then, CD is divided into equal parts at F and into unequal parts at E, [Margin: (b) II.5] the rectangle contained by CE and ED, together with the square on EF, will be equal to the square on FD, and therefore to the square on FB, since FD and FB are equal. But the square on FB is [Margin: (c) I.47] equal to the squares on FE and EB. Therefore the rectangle contained by CE and ED, together with the square on EF, will also be equal to the squares on FE and EB. So, when the common square

on FE is taken away, the rectangle contained by CE and ED will remain equal to the square on EB — that is, to the rectangle contained by AE and EB, since AE and EB are equal; and consequently the rectangle contained by them is a square, from what we wrote regarding Definition 1 of Book II.

Diagram. A circle with the diameter CD (vertical) through the center F, and the chord AB (horizontal) which it bisects at E; the triangle FEB is drawn.

- C (top): upper end of the diameter.
- F (center): center of the circle.
- A (left, on the chord): left end of AB.
- E (on AB, at the crossing): foot where CD meets AB.
- B (right, on the chord): right end of AB.
- D (bottom): lower end of the diameter.

Third case, part a (CD passes through the center but does not bisect AB). Now let CD, passing through the center, divide the straight line AB, but not into two equal parts. Let AB therefore be bisected at G, and let the straight lines FG and FB be drawn; [Margin: (d) III.3] and FG will be perpendicular to AB. Now since the rectangle contained by CE and ED, together with the square on FE, [Margin: (e) II.5] is equal to the square on FD — that is, to the square on FB — and the square on FE is [Margin: (f) I.47] equal to the squares on FG and GE, while the square on FB is equal to the squares on FG and GB: the rectangle contained by CE and ED, together with the squares on FG and GE, will also be equal to the squares on FG and GB. So, when the common square on FG is removed, the rectangle contained by CE and ED, together with the square on GE, will remain equal to the square on GB. But the rectangle contained by AE and EB, together with the square on GE, [Margin: (g) II.5] is also equal to the same square on GB, because the straight line AB is bisected at G and cut unequally at E. Therefore the rectangle contained by CE and ED, together with the square on GE, is equal to the rectangle contained by AE and EB, together with the square on the same line GE. Hence, when the common square on GE is taken away, the rectangle contained by CE and ED will remain equal to the rectangle contained by AE and EB. Which was proposed.

Diagram. Two circles side by side (mirror-image variants of the same construction). In each: the diameter CD (C at top, D at bottom) passes through the center F; the chord AB is cut by CD at E and bisected at G, with radii FG and FB drawn.

- C (top of each): upper end of the diameter.
- D (bottom of each): lower end of the diameter.

- F (center of each): center of the circle.
- A (left/right end of the chord): one end of AB.
- B (opposite end of the chord): other end of AB.
- G (on the chord): midpoint of AB.
- E (on the chord): where CD crosses AB.

Third case, part b (neither passes through the center). Third, let neither pass through the center — whether one of them is bisected [by a diameter] or neither. Let the straight line GH be drawn through the center F and the point of section E. Since, then, it has been shown that the rectangle contained by AE and EB is equal to the rectangle contained by GE and EH, whether AB is bisected or not; and likewise that the rectangle contained by CE and ED is also equal to the same rectangle contained by GE and EH, whether CD is bisected or not: the rectangle contained by AE and EB will be equal to the rectangle contained by CE and ED. Which was proposed. If, then, in a circle two straight lines cut one another, and so forth. Which was to be demonstrated.

Diagram. Two small circles (variant configurations) for the case in which neither chord passes through the center. In each, the auxiliary diameter GH is drawn through the center F and the crossing-point E; the two given chords are AB and CD. The labels are very small even at high magnification.

- G (top): one end of the auxiliary diameter.
- H (bottom): other end of the auxiliary diameter.
- F (center): center of the circle.
- E (interior, crossing point): intersection of the two chords.
- A [?], B [?] (on the circle): ends of one chord.
- C [?], D [?] (on the circle): ends of the other chord.

[— small clustered labels, several flagged.]

Scholium

[Clavius's scholium, giving the converse of Proposition 35.]

This theorem can be converted, as follows.

[Converse — enunciation.] If two straight lines cut one another in such a way that the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other, then a circle can be described through

the four extreme points of those lines — that is, a circle described through any three of their extreme points will also pass through the fourth.

[Proof.] Let the straight lines AB and CD cut one another at E, and let the rectangle contained by AE and EB be equal to the rectangle contained by CE and ED. I say that the four points A, D, B, C fall on the circumference of a circle — that is, a circle can be described through them — so that a circle described through the three points A, D, B necessarily passes through the point C as well. For let some circle be described through the three points A, D, and B (how this is done we shall show at Proposition 5 of Book IV); and if it does not pass through C, it will pass either beyond C or short of it, as through F. Since, then, the rectangle contained by FE and ED [Margin: (h) III.35] is equal to the rectangle contained by AE and EB, and the rectangle contained by CE and ED is also assumed equal to the same rectangle contained by AE and EB, the rectangles contained by CE, ED and by FE, ED will be equal — a part and a whole; which is absurd. Therefore the circle will pass through the point C. Which was proposed.

Diagram. A circle with two chords AB and CD crossing at E; the point F (the alternative intersection of the described circle with CD) is marked near C.

- C (upper left, on the circle): end of one chord.
- F (on CD, near C): the point where an incorrectly-placed circle would meet CD.
- A (left): end of chord AB.
- E (interior): crossing point of the chords.
- B (right): other end of AB.
- D (bottom): other end of CD.

THEOR. 30. PROPOS. 36.

[Margin: xxxv]

[Euclid — enunciation.] If some point is taken outside a circle, and from it two straight lines fall on the circle, of which one cuts the circle and the other touches it: the rectangle contained by the whole secant and the part of it taken outside, between the point and the convex circumference, will be equal to the square described on the tangent.

[Demonstration.]

First case (the secant passes through the center). Outside the circle ABC let the point D be taken, from which the line DA is drawn cutting the circle at C, [Margin:

(a) III.17] and the line DB touching the circle at B. I say that the rectangle contained by DA and DC is equal to the square on DB. For let the straight line DA first pass through the center E, and let EB be joined, [Margin: (b) III.18] which will be perpendicular to DB. Since, then, CA is bisected at E, and CD is added to it in a straight line and continuously, [Margin: (c) II.6] the rectangle contained by DA and DC, together with the square on EC — that is, with the square on EB — will be equal to the square on DE. But the square on DE is [Margin: (d) I.47] equal to the squares on EB and BD. Therefore the rectangle contained by DA and DC, together with the square on EB, will be equal to the squares on DB and BE. So, when the common square on EB is taken away, the rectangle contained by DA and DC will remain equal to the square on DB. Which was proposed.

Second case (the secant does not pass through the center). Now let DA, the secant, not pass through the center E. Let AC therefore be bisected at F, and let the straight lines EB, EC, ED, EF be drawn; [Margin: (e) III.18] and EB will be perpendicular to BD, [Margin: (f) III.3] and EF to AC. Since, then, CA is bisected at F, and the straight line CD is added to it, [Margin: (g) II.6] the rectangle contained by DA and DC, together with the square on CF, will be equal to the square on DF. Adding, then, the common square on FE, the rectangle contained by DA and DC, together with the squares on CF and FE, will be equal to the squares on DF and FE. But the squares on CF and FE are [Margin: (h) I.47] equal to the square on EC, and therefore also to the square on EB; and [Margin: (i) I.47] the squares on DF and FE are equal to the square on DE. Therefore the rectangle contained by DA and DC, together with the square on EB, will be equal to the square on DE. Since, then, the square on DE [Margin: (k) I.47] is equal to the squares on DB and BE, the rectangle contained by DA and DC, together with the square on EB, will also be equal to the squares on DB and BE. So, when the common square on BE is taken away, the rectangle contained by DA and DC will remain equal to the square on DB. Which was proposed. If, then, some point is taken outside a circle, and so forth. Which was to be demonstrated.

Diagram. The first case of Proposition 36: the secant passes through the center. From the external point D (upper right) the secant runs through C and the center E to A (bottom); the tangent DB touches the circle at B (upper left), with the radius EB drawn.

- D (upper right, external): the point taken outside the circle.
- B (upper left, tangent point): where DB touches the circle.
- C (on the circle, near the secant's entry): nearer intersection of the secant.
- E (center): center of the circle.

- A (bottom): farther intersection of the secant.

Diagram. The second case (secant not through the center), shown in two side-by-side circles. From the external point D the secant cuts the circle at C and A; the tangent DB touches at B; the center is E, with F the midpoint of AC and the radii EB, EC, ED, EF drawn.

- D (top, external): the external point.
- B (upper left, tangent point): where the tangent touches.
- C (on the circle): nearer intersection of the secant.
- A (left, on the circle): farther intersection of the secant.
- E (center): center of the circle.
- F (on AC): midpoint of AC.

First case (secant through center) and second case (secant not through center) are both illustrated; F appears in the right-hand figure.

COROLLARIUM I

It is clear from this that, if from any point taken outside a circle very many straight lines are drawn cutting the circle, the rectangles contained by the whole lines and their outer parts are equal to one another. For example, if from A the straight lines AC, AD, AE are drawn cutting the circle at F, G, H, the rectangles contained by AC, AF; likewise by AD, AG; and by AE, AH, will be equal to one another. For, when AB is drawn touching the circle, [Margin: (l) III.36] each of those rectangles will be equal to the square on AB; and therefore they will all be equal to one another as well.

Diagram. From the external point A (top) a tangent AB and three secants AC, AD, AE are drawn to the circle. The secants cut the near arc at F, G, H and reach the far points C, D, E; B is the tangent point.

- A (top, external): the external point.
- B (upper left, tangent point): where AB touches the circle.
- F [?], G [?], H [?] (clustered on the near/upper arc): the nearer intersections of the three secants.
- C (left, on the circle): far intersection of one secant.
- D (bottom, on the circle): far intersection of another secant.
- E (lower right, on the circle): far intersection of the third secant.

[— the cluster F, G, H is hard to resolve.]

COROLLARIUM II

It is also established that two straight lines drawn from the same point and touching a circle are equal to one another. For let AB and AC be drawn from A, touching the circle; I say that these are equal to one another. For when the straight line AD is drawn cutting the circle at E, both the square on AB and the square on AC will be [Margin: (m) III.36] equal to the rectangle contained by AD and AE. Therefore the squares on AB and AC will be equal to one another, and consequently the straight lines AB and AC will also be equal.

Diagram. From the external point A (top) two tangents AB and AC touch the circle, and the secant AD cuts it at E.

- A (top, external): the external point.
- B (left, tangent point): where AB touches.
- E (interior, on the secant): nearer intersection of AD.
- C (right, tangent point): where AC touches.
- D (bottom, on the circle): farther intersection of AD.

COROLLARIUM III

It is likewise evident that from the same point taken outside a circle only two lines can be drawn touching the circle. For if, besides the two AB and AC, a third AD could be drawn touching the same circle, then, when the straight lines EB and ED are drawn from the center E, [Margin: (n) III.18] the angles ABE and ADE would be right angles, and therefore equal; which is absurd. For if the straight line AE is drawn, [Margin: (o) I.21] the angle ADE will be greater than the angle ABE.

Otherwise: if a third line AD also touches the circle, then the two tangents AB and AD will be equal, as was shown; which is absurd. For when the straight line AE is drawn to the center E, cutting the circle at F, [Margin: (p) III.8] AD, being nearer to the least line AF, will be less than AB, which is farther from the least line AF. Or thus: if AB and AD are equal, then, adding the equals EB and ED, AB and BE will also be equal to AD and DE; which is absurd, [Margin: (q) I.21] for AB and BE are greater. Therefore only two straight lines can be drawn from the point A touching the circle. Which was proposed.

Diagram. From the external point A (top) the tangents AB and AC touch the circle, and the line AE runs to the center E, cutting the circle at F; D marks the assumed third point of contact. The radii EB and ED are drawn.

- A (top, external): the external point.
- F (interior, on AE): where AE cuts the circle.
- D (interior, left): the assumed third tangent point.
- B (left, tangent point): where AB touches.
- C (right, tangent point): where AC touches.
- E (center): center of the circle.

Euclid's *Elements*, Book III — conclusion (Prop. 36 scholia; Prop. 37) → Book IV (Definitions)

[Printed p. 148 — running header: EVCLID. GEOM.]

COROLLARIUM IV. [Corollary IV — continuing the corollaries of Prop. 36]

Finally, this too is established: if two equal straight lines fall from some point onto the convex circumference, and one of them is tangent to the circle, then the other is tangent to the circle as well. For example, if the two straight lines AB and AC in the preceding figure are equal, and AC is tangent to the circle at C, then AB will likewise be tangent to the same circle at B. For if it is not tangent, let AD be drawn tangent — since two tangents can always be drawn from the same point, as is clear from the scholium of Prop. 31 of this book — and then, by the second corollary, AC and AD will be equal. Since, therefore, AB too is assumed equal to AC, three equal straight lines AB, AC, AD would be drawn, which is absurd. For only two can be drawn.

[Margin: reference letter *r*, marginal citation partly illegible [?]]

ALITER. [An alternative demonstration]

In the figure of the following proposition, let there be two equal straight lines DB and DF, and let DB be tangent to the circle at B. I say that DF too is tangent to the same circle at F. For when the straight lines EB and EF have been drawn from the center, the two sides DB, BE will be equal to the two sides DF, FE, and the base DE is common. [Margin: I.8 (ref. s)] Therefore the angles B and F will be equal.

[Margin: III.18 (ref. t)] But B is a right angle. Therefore F too will be a right angle, and for that reason DF will be tangent to the circle at F, by the corollary of Prop. 16 of this book.

SCHOLIUM. [Scholium to Prop. 36]

We shall demonstrate this proposition with the greatest ease by means of the preceding proposition, in this way.

Diagram. A figure showing an external point A with two concentric-drawn circles (an inner circle BCD and an outer circle AKL sharing the region), a tangent AB, and two secants (AE through the center, and AN not through the center), used for the demonstration below. Points read off the scan under magnification, with position and role where the text makes it plain:

- A (top): the external point from which the tangent and secants are drawn
- K (upper left): on the outer circle AKL
- M (upper right, small): on the tangent/chord near L
- L (right): on the outer circle AKL
- O (left): on the outer circle AKL
- C (center-left): point where the second secant AN meets the inner circle
- D (center): point where the first secant AE meets the inner circle
- B (center-right): point of tangency of AB on the inner circle
- F (center, below C–D): the center of the circle(s)
- N (lower left, inside): on secant AN
- I (right): endpoint on the outer circle after producing AB
- G (lower left, on outer circle): endpoint after producing AN
- E (lower center): endpoint of the first secant on the inner circle
- H (bottom): endpoint of AE produced on the outer circle

From the point A, outside the circle BCD, let the tangent AB and the secant AE be drawn, the secant first passing through the center F and cutting the circumference at D. I say that the rectangle contained by AE, AD is equal to the square on AB. For when a circle AKL has been described from F through A, let AE and AB be produced as far as H and I, and through D let the perpendicular KL be drawn to AH, which will be tangent to the same circle BCD at D, by the corollary of Prop. 16 of this book. And from what has been demonstrated in the scholium of Prop. 3 [?] of this book, AD and HE will be equal; and therefore, with the common DE added, AE and HD will also be equal. Likewise, from the scholium of Prop. 18 of this book, AI and KL will be equal, and they will be bisected at B and D; and therefore the rectangles contained by AB, BI and by KD, DL will be equal to each other — indeed they will be squares, because of the equality of the lines AB, BI, KD, DL. And so, since the rectangle contained by HD, DA — that is, the rectangle

contained by AE (which was shown equal to HD) and AD — [Margin: III.35 (ref. *u/a* [?])] is equal to the rectangle contained by KD, DL, that is, to the rectangle contained by AB, BI, or the square on AB, the proposition is established.

But now let the secant line AN not pass through the center F, and let it cut the circumference at C. I say again that the rectangle contained by AN, AC is equal to the square on AB. For, again, when a circle AKL has been described from F through A, let AN and AB be produced as far as G and I, and through C let OM be drawn tangent to the circle BCD. Then, from the scholium of Prop. 3 [?] of this book, AC and GN will be equal; and therefore, with the common CN added, AN and GC will be equal. Likewise, from the scholium of Prop. 18 of this book, AI and OM will be equal, and they will be bisected at B and C; and therefore the rectangles contained by AB, BI and by OC, CM will be equal to each other — indeed squares — because of the equality of the lines AB, BI, OC, CM. And so, since the rectangle contained by GC, CA — that is, the rectangle contained by AN (which was shown equal to GC) and AC — [Margin: III.35 (ref. *x*)] is equal to the rectangle contained by OC, CM, that is, to the rectangle contained by AB, BI, or the square on AB, once again what was proposed is established.

We shall also convert the first corollary of this proposition in roughly the following way.

[Converse theorem] If two finite straight lines go out from some point, and are cut into two parts in such a way that the rectangles contained by the wholes and the segments near the point are equal, then a circle can be described through the extreme points of the other segments — that is, a circle described through three of the extreme points of the other segments will also pass through the fourth extreme point.

Diagram. A circle with an apex point A above it, from which two lines descend to B and C on the circle; the segments are cut at interior points, with F marked as the alternative (false) intersection.

- A (top): the external point from which AB and AC go out
- D (upper, on AB): a cut point; note that **two "D" labels appear printed near the top of chord AB** [?] (apparently D the cut point and a second D near the circle)
- F (between the two D's): the alternative point (short of D) used for the *reductio*
- E (right, on AC): the other cut point
- B (bottom-left, on circle): extreme point

- C (right, on circle): extreme point

For let two straight lines AB, AC go out from the point A, cut at the points D, E in such a way that the rectangles contained by AB, AD and by AC, AE are equal. I say that the four points B, C, E, D fall on the circumference of a circle — that is, a circle can be described through them — so that a circle described through the three points B, C, E necessarily passes through the remaining point D as well. For let a circle BCE be described through the three points B, C, E (as we shall teach at Prop. 5 of Book 4); and if it be said not to pass through the fourth point D, it will pass either short of D or beyond, as through F. Since, then, the rectangle contained by AB, AF is equal, by the first corollary of this proposition, to the rectangle contained by AC, AE, and the rectangle contained by AB, AD is assumed equal to the same rectangle contained by AC, AE, the rectangles contained by AB, AF and by AB, AD will be equal to each other — a part and the whole — which is absurd. Therefore the circle passes through the point D. Which was to be demonstrated.

It will also be permissible to gather from what has been demonstrated in this proposition a problem of the following kind.

[Problem] Given two straight lines, whether equal or unequal, to adjoin a straight line to one of them, so that the rectangle contained by the whole composite line and the adjoined line is equal to the square of the other.

Diagram. On the left, a vertical line marked (top to bottom) G, B, D, A, with a short separate segment C beside A; on the right, a circle with center A through which the construction is carried out.

- G (top of the line / on the circle, upper right): the extreme point of the adjoined part
- E (top, apex, outside the circle): endpoint of the perpendicular DE
- B (on the line AB): the point to which BG is adjoined
- D (midpoint of AB / on the circle at left): the bisection point
- A (bottom of the line / center of the circle): the center
- C (short segment near A): the given line C
- F (bottom of the circle): endpoint of the secant EF through the center

Let two straight lines AB and C be given, and let a line be adjoined to AB, the smaller, so that the rectangle contained by the whole composite line and the adjoined line is equal to the square of the line C, the larger. Let AB be bisected at D, and with center A [reading of the qualifier "from any center" uncertain [?]] and

radius AD, the half, let a circle DFG be described; and at D let a perpendicular DE be erected to AD, equal to C; and finally, from E through the center A, let the straight line EF be extended, cutting the circumference at G. I say that, if to the line AB there is added BG equal to EG, the rectangle contained by the whole AG and GB is equal to the square of the line C. For since the rectangle contained by FE, EG [Margin: III.36 (ref. y [?])] is equal to the square of the line ED (because ED is tangent to the circle at D, by the corollary of Prop. 16 of this book); and the rectangle contained by FE, EG is equal to the rectangle contained by AG, GB (because the lines FE, EG are equal to the lines AG, GB — for GB was taken equal to EG, and AB was assumed equal to FG); and the square on ED is equal to the square on C (because of the equality of the lines ED and C by construction): therefore the rectangle contained by AG, GB too will be equal to the square on C. Which is what was proposed.

The proposition will be demonstrated in no different way if a line is to be adjoined to C, the larger, so that the rectangle contained by the whole composite line and the adjoined line is equal to the square of the line AB, the smaller.

[Printed p. 149 — running header: LIBER III.]

THEOR. 31. PROPOS. 37.

[Outer-margin numeral: xxxvj (= 36; probably a slip for xxxvij / Euclid III.37 — see report)]

[Enunciation] If some point is taken outside a circle, and from that point two straight lines fall onto the circle, one of which cuts the circle while the other meets it; and if the rectangle contained by the whole secant together with the part assumed outside — between the point and the convex circumference — is equal to the square described on the line that meets the circle: then the line that meets it will be tangent to the circle.

Diagram. Two side-by-side figures for the two cases of the proof (the secant through the center, and the secant not through the center). In each, an external point D lies above a circle, with a tangent DF, a secant DA cutting the circle at C, and radii drawn to the center E.

- D (top): the external point
- B (upper left, on the circle): the point of tangency of DB (the line "meeting" the circle)
- C (upper middle, on the circle): where the secant DA cuts the circle
- F (right, on the circle): the point of tangency of the auxiliary tangent DF

- E (center): the center of the circle
- A (bottom left, on the circle): the far intersection of the secant

Outside the circle ABC, whose center is E, let a point D be taken, from which let the straight line DA be drawn cutting the circle at C, and the straight line DB meeting the circle at the point B; and let the rectangle contained by DA, DC be equal to the square on the line DB. I say that DB is tangent to the circle at B. [Margin: III.17 (ref. *a*)] For let DF be drawn tangent to the circle, and let the straight lines EB, EF be joined. And if the secant DA does not pass through the center E, let the straight line DE also be joined. Since, then, the square on the tangent DF [Margin: III.36 (ref. *b*)] is equal to the rectangle contained by DA, DC, and the square on the line DB is assumed equal to the same rectangle contained by DA, DC, the squares on the lines DF, DB will be equal to each other, and therefore the lines DF, DB will also be equal to each other. And so, since the sides DF, FE of the triangle DFE are equal to the sides DB, BE of the triangle DBE, and the base DE is common, [Margin: I.8 (ref. *c*)] the angles DFE, DBE will be equal. [Margin: III.18 (ref. *d*)] But the angle DFE is a right angle, because DF is tangent to the circle. Therefore the angle DBE too will be a right angle. And for that reason, by the corollary of Prop. 16 of this book, DB will be tangent to the circle — which is what was proposed. If, therefore, some point is taken outside a circle, and so on. Which was to be demonstrated.

SCHOLIUM. [Scholium to Prop. 37]

This theorem is, as is plain, the converse of the preceding theorem. It is worthwhile to demonstrate here the following theorem as well.

[Theorem] If from a given point outside a circle two straight lines are drawn tangent to the circle — or two lines drawn as far as the convex circumference and equal to each other — then the straight line projected from that same point through the center of the circle bisects the angle contained by them. And conversely, the straight line that bisects the angle contained by them will pass through the center of the circle.

In the first figure of this proposition, let the two lines DB, DF be tangent to the circle ABCF at the points B, F — points which, by the second corollary of the preceding proposition, will be equal [in their tangent lengths] — or else let any two equal lines DB, DF be drawn. And through the center E let the straight line DEA be drawn. I say that the angle BDF is bisected by the line DA. For, when the straight lines EB, EF have been joined, since the two sides BD, DE are equal to the

two sides FD, DE, and the base EB is equal to the base EF, [Margin: I.8 (ref. *e*)] the two angles at D will be equal to each other. Which is what was proposed.

Now let the straight line DA bisect the angle BDF. I say that the line DA passes through the center. For if the center is not on DA, let the center E be outside it, and let the straight line ED be joined, as in the later figure of this proposition. Then, as has been demonstrated, the line ED will bisect the angle BDF; and consequently the angles BDA, BDE, being the halves of the angle BDF, will be equal to each other — a part and the whole — which is absurd. Therefore the line DA passes through the center. Which was to be demonstrated.

[Printed p. 150 — chapter opening; no running header]

EVCLIDIS ELEMENTVM QVARTVM. [Euclid's Fourth Book of the Elements]

DEFINITIONES. [Definitions]

I.

A rectilinear figure is said to be inscribed in a rectilinear figure when the individual angles of the figure that is inscribed touch the individual sides of the figure in which it is inscribed.

[Commentary] In treating, in this fourth book, of the various inscriptions of rectilinear figures within a circle and of their circumscriptions about a circle — likewise of the inscriptions of a circle within those same figures and of the descriptions of a circle about them — Euclid sets out in a few definitions what it is for a figure to be inscribed in a figure, or to be described about a figure, beginning with rectilinear figures. Thus, if the angles D, E, F of the inner triangle DEF touch the sides AB, AC, BC of the outer triangle ABC, the triangle DEF will be said to be inscribed in the triangle ABC. But since the angle M of the triangle KLM does not touch the side HI of the triangle GHI, the triangle KLM will not be said to be inscribed in the triangle GHI — even though it lies wholly within it and its two angles K, L touch the two sides GH, GI.

Diagram. Two triangle figures illustrating inscription. Left: an outer triangle ABC with an inner triangle DEF whose vertices touch the sides. Right: an outer triangle GHI with an inner triangle KLM whose vertex M does not reach the side HI.

- A (top), B (bottom-left), C (bottom-right): vertices of the outer triangle (left figure)
- D (left, on AB), E (right, on AC), F (bottom, on BC): vertices of the inscribed triangle
- G (top), H (bottom-left), I (bottom-right): vertices of the outer triangle (right figure)
- K (left), L (right), M (bottom-center, not touching HI): vertices of the interior triangle

II.

Similarly, a figure is said to be described about a figure when the individual sides of the figure that is circumscribed touch the individual angles of the figure about which it is described.

[Commentary] Conversely, the triangle ABC will be said to be described about the triangle DEF, since its individual sides touch the individual angles of the latter. But the triangle GHI will not be said to be described about the triangle KLM, because its side HI does not touch the angle M of the latter. The same is to be understood of the inscriptions and circumscriptions of other rectilinear figures. Moreover, rectilinear figures are, properly speaking, said to be described within rectilinear figures, and to be described about them, when the inscribed and circumscribed figures have their sides equal in number and their angles equal in number — although this is not altogether necessary, since a square can also be described within a triangle, as we shall teach toward the end of Book 6.

III.

A rectilinear figure is said to be inscribed in a circle when the individual angles of the figure that is inscribed touch the circumference of the circle.

[Commentary] Thus, if the angles A, B, C of the triangle ABC touch the circumference of the circle ABC, the triangle will be said to be inscribed in the circle. But if even one only of the angles did not touch the circumference, the triangle would not be said to be inscribed in the circle.

Diagram. A circle with an inscribed triangle ABC, its three vertices touching the circumference.

- A (top): vertex on the circumference
- B (lower left): vertex on the circumference
- C (lower right): vertex on the circumference

IV.

A rectilinear figure, on the other hand, is said to be described about a circle when the individual sides of the figure that is circumscribed touch the circumference of the circle.

Euclid's *Elements*, Book IV (continued)

Definitions (continued)

Definition IV (conclusion). But if the sides of triangle ABC each touch the circumference of circle DEF, the triangle will be said to be described about the circle.

V.

Similarly, a circle too is said to be inscribed in a rectilinear figure when the circle's circumference touches each of the sides of the figure in which it is inscribed.

VI.

A circle, moreover, is said to be circumscribed about a figure when the circle's circumference touches each of the angles of the figure about which it is described.

Clavius's note. Conversely, in the figure of Definition 4 the circle DEF will be said to be inscribed in the triangle ABC; whereas in the figure of Definition 3 the circle ABC will be said to be circumscribed about the triangle ABC. Make the same judgment about the other rectilinear figures that are said to be inscribed in a circle or circumscribed about one, or in which a circle is said to be inscribed, or about which a circle is said to be described.

Diagram. A triangle with an inscribed circle, illustrating Definitions 4–6.

- A (top vertex): apex of the triangle
- B (bottom-left vertex)
- C (bottom-right vertex)
- D (on side AB, upper-left): tangent point of the circle
- E (on base BC, bottom): tangent point
- F (on side AC, upper-right): tangent point

VII.

A straight line is said to be *fitted* [accommodari] or *adapted* [coaptari] in a circle when its extremities lie on the circumference of the circle.

Clavius's note. For example, the straight line AB, since its extremities A and B lie on the circumference of circle ABC, will be said to be adapted or fitted in the said circle; but not the line E, nor CD, because the latter [CD] has only one of its extremities — namely C — on the circumference of the circle, while the former [E] has neither.

Diagram. A small circle illustrating Definition 7, with several chords/segments.

- A (lower-left, on the circle): one extremity of chord AB
- B (lower-right, on the circle): other extremity of chord AB
- C (upper-left, on the circle): extremity of line CD lying on the circumference
- D (upper-right, on/just inside the circumference): the other extremity of line CD
- E (interior, center): a short segment lying wholly within the circle (neither extremity on the circumference)

PROBL. 1. PROPOS. 1.

Enunciation (Euclid). In a given circle, to fit a straight line equal to a given straight line that is not greater than the diameter of the circle.

Demonstration (Clavius). In circle ABC let a straight line be fitted equal to the given straight line D, which however is not greater than the diameter of the given circle. For since the diameter [Margin: a — Elem. III.15] is the greatest of all the straight lines in a circle, if the given line were greater than the diameter, no line equal to it could be fitted in the circle. So draw the diameter BC. Then, if the given line D is equal to the diameter, BC has been fitted, equal to it. But if D is less than the diameter, [Margin: b — Elem. I.3] cut off BE equal to D; and with center B and radius BE describe a circle EA, cutting circle ABC at A. Then, when the straight line BA has been drawn, it will be the line fitted in circle ABC, equal to the given line D. [Margin: c — Def. I.15] For BA is equal to BE, and D is equal to the same BE, by construction. Therefore AB and D will also be equal to one another. So in the given circle we have fitted a straight line, and so on. Which was to be done.

Diagram. Circle ABC with a diameter BC and a second (auxiliary) arc; the given line D stands separately at the left.

- D (left, a short vertical segment): the given straight line
- A (top): the point where the auxiliary arc cuts circle ABC; BA is the fitted chord
- B (center-left): center of the circle / endpoint of the diameter and center of arc EA
- C (right): the other endpoint of the diameter BC
- E (on the diameter, between B and the center): point with BE cut off equal to D

EX FEDERICO COMMANDINO.

(An additional treatment from [Federico Commandino].)

Enunciation. In a given circle, to fit a straight line equal to a given straight line — not greater than the diameter of the circle — and parallel to another given line.

Demonstration. In a given circle ABC, whose center is D, let a straight line be fitted equal to the line EF (not greater than the diameter) and parallel to the line G. [Margin: d — Elem. I.31] Through the center D draw the diameter AC parallel to G. Now if EF is equal to the diameter, what was proposed is already done. But if it is less than the diameter, [Margin: e — Elem. I.11] bisect it at H, and cut off DI equal to HE and DK equal to HF, so that the whole IK is equal to the whole EF. Through I and K draw LM and NO at right angles to AC, and join LN. I say that LN is the line fitted, equal to EF and parallel to G. For since LM and NO are equally distant from the center, [Margin: f — Elem. III.14] they will be equal to one another; and since they are bisected at I and K — because they are cut at right angles by AC, which passes through the center D — [Margin: g — Elem. III.3] their halves LI and NK will also be equal. And since LI and NK are also parallel, [Margin: h — Elem. I.28] [Margin: i — Elem. I.33] LN and IK will likewise be equal and parallel. Therefore, since IK is equal to EF and parallel to G, LN too will be equal to EF [Margin: k — Elem. I.30] and parallel to G. By the same reasoning, if the line MO is drawn, it will be equal to EF and parallel to G. Which is what was proposed.

Diagram. Circle with center D and diameter AC; a rectangle LMNO is inscribed, its vertical sides LM and NO cutting AC at I and K. At the right stand the given line EF (with midpoint H) and the given direction-line G.

- A (left end of diameter AC)
- C (right end of diameter AC)
- D (center of the circle)

- B (top of the circle)
- L (top-left corner of the inscribed rectangle [?])
- M (bottom-left corner)
- N (top-right corner)
- O (bottom-right corner)
- I (on AC, foot of side LM)
- K (on AC, foot of side NO)
- E, H, F (the given line EF, drawn vertically at the right, with midpoint H)
- G (the given direction-line, a separate segment)

THEOR. 2. PROPOS. 2.

Enunciation (Euclid). In a given circle, to inscribe a triangle equiangular to a given triangle.

Demonstration (Clavius). Let there be inscribed in the given circle ABC a triangle equiangular to any given triangle DEF. Draw the line GH tangent to the circle at A, and make the angle GAB equal to the angle F and the angle HAC equal to the angle E; extend the lines AB, AC to the circumference at the points B and C, and join the line BC. The line AC will not fall on the line AB, nor between the lines AB and AG, because the angles GAB and HAC — that is, the angles F and E — [Margin: a — Elem. I.17] are less than two right angles. [Margin: b — Elem. I.13] For they would be equal to two right angles if AC fell on AB, or greater than two right angles if it fell between AB and AG. I say that the triangle ABC inscribed in the given circle is equiangular to the given triangle DEF. For the angle C [Margin: c — Elem. III.32] is equal to the angle GAB, and the angle F is equal to the same angle GAB by construction. Therefore the angles C and F will also be equal to one another. Similarly, since the angle B [Margin: d — Elem. III.32] is equal to the angle HAC, and the angle E is equal to the same angle HAC by construction, the angles B and E will also be equal to one another. Since, then, the two angles B and C of triangle ABC are equal to the two angles E and F of triangle DEF, [Margin: e — Elem. I.32] the remaining angles A and D will also be equal. Therefore the triangle ABC is equiangular to the triangle DEF. So in the given circle we have inscribed a triangle, and so on. Which was to be done.

Diagram. The given triangle DEF at the left (shown twice), and at the right a circle with the inscribed triangle ABC and the tangent line HAG at the top.

- D, E, F: the given triangle

- H (upper-left), A (top), G (upper-right): the tangent line GH touching the circle at A
- A (top vertex of the inscribed triangle, the point of tangency)
- B (lower-right vertex on the circle)
- C (left-center vertex on the circle)

PROBL. 3. PROPOS. 3.

Enunciation (Euclid). About a given circle, to circumscribe a triangle equiangular to a given triangle.

Demonstration (Clavius). About the given circle ABC let a triangle be circumscribed equiangular to the given triangle DEF. Produce the side EF in both directions to G and H; take the center of the circle, I; draw any line AI; and make the angle AIB equal to the angle DEG and the angle BIC equal to the angle DFH. Then from A, B, C draw KL, LM, MK perpendicular to AI, BI, CI; these will touch the circle at the points A, B, C (by the corollary to III.16) and will meet at the points K, L, M. For if the line AC were drawn, the two angles KAC and KCA would be less than two right angles, and therefore AK and CK will meet, and so on. For this line AC, when drawn, would fall outside the lines AI, CI, because these make an angle at I. Since the space about I is equal to four right angles (by the 2nd corollary to I.15) — that is, to the four angles at E and F — and since the two angles AIB, CIB are equal to the two angles DEG, DFH, the remaining space AIC will be equal to the remaining two angles DEF, DFE. But [Margin: a — Elem. I.17] these are less than two right angles. Therefore the space AIC too will be less than two right angles, and so AIC will be a proper angle. Otherwise that space would be either equal to two right angles — namely if AI and CI formed a single straight line — or greater than two right angles, if AI produced fell beyond IC. So of necessity AI produced falls below CI, and therefore the angle AIC is formed toward the side of K; and the line AC, when drawn, will make with AK, CK two angles less than two right angles, and so the lines AK, CK will meet at K. In just the same way we shall show that AL, BL meet at L, and CM, BM at M, because the lines AB, BC, when drawn, will make with AL, BL, CM, BM angles less than two right angles. Therefore there has been circumscribed about the circle the triangle KLM, which I say is equiangular to the triangle DEF. For since all the angles in the quadrilateral AIBL are equal to four right angles — as was shown at I.32 — and the angles IAL, IBL are two right angles, the remaining angles AIB and L will be equal to two right angles. And since [Margin: b — Elem. I.13] the angles DEG, DEF are also equal to two right angles, if the equal angles AIB, DEG are taken away, there will remain the angle L equal to the angle DEF. By like reasoning we shall

show that the angle M is equal to the angle DFE. [Margin: c — Elem. I.32]
 Therefore the remaining angle K will be equal to the remaining angle D, and so the triangle KLM is equiangular to the triangle DEF. Therefore about the given circle we have circumscribed a triangle, and so on. Which was to be accomplished.

Diagram. At the left, the given triangle DEF with its base EF produced to G (beyond E) and H (beyond F). At the right, the circle (center I) with the circumscribed triangle KLM touching it at A, B, C.

- D (apex of the given triangle); E, F (its base); G (extension of the base beyond E); H (extension beyond F)
- I (center of the circle)
- A (upper tangent point on the circle)
- B (lower tangent point)
- C (right-hand tangent point)
- K (top vertex of the circumscribed triangle — the woodcut engraves "V" at the apex)
- L (lower-left vertex)
- M (lower-right vertex)

PROBL. 4. PROPOS. 4.

Enunciation (Euclid). In a given triangle, to inscribe a circle.

Demonstration (Clavius). Let a circle be inscribed in the given triangle ABC. Bisect the two angles ABC, ACB by the straight lines BD, CD, which meet within the triangle at D; and from D draw the perpendiculars DE, DF, DG to the three sides. Now since the two angles DBE, DEB of triangle DBE are equal to the two angles DBF, DFB of triangle DBF, each to each, and the side BD is common, [Margin: a — Elem. I.26] the sides DE, DF will also be equal. By the same reasoning the sides DF, DG in the triangles DCF, DCG will be equal. Since, then, the three lines DE, DF, DG are equal, a circle described with center D and radius DE will pass through the remaining points F and G, and will touch the sides of the triangle at E, F, G (by the corollary to III.16), because the sides are perpendicular to the radii DE, DF, DG. So in the given triangle we have inscribed a circle. Which was to be accomplished.

Diagram. Triangle ABC with an inscribed circle, center D; lines run from D to the vertices and to the three points of tangency.

- A (top vertex)

- B (bottom-left vertex)
- C (bottom-right vertex)
- D (center of the inscribed circle)
- E (tangent point on side AB)
- F (tangent point on base BC)
- G (tangent point on side AC)

SCHOLIVM.

Introduction (Clavius). But since Euclid demonstrated in Book 1 [Margin: b — Elem. I.20] that in every triangle any two sides taken together are greater than the remaining one, it will not be out of place to demonstrate here, with [Giovanni Battista Benedetti], by how much they are greater, by means of this theorem.

Theorem. In every triangle, if a circle is inscribed, any two sides exceed the remaining one by a straight line whose square is four times the rectangle contained by the straight line drawn from the angle contained by those two sides to the concave circumference, together with its outer segment. And if the contained angle is a right angle, those two sides will exceed the side opposite the right angle by the diameter of the circle inscribed in the triangle.

Demonstration. Let there be a triangle ABC, [Margin: c — Elem. IV.4] in which a circle DEF is inscribed, touching the sides of the triangle at the points D, E, F; and from any angle B — whether it be right, obtuse, or acute — let the line BH be drawn, cutting the circle at I. I say that the two sides BA, BC exceed the side AC by a straight line whose square is four times the rectangle contained by BH, BI. For since, by corollary 2 to III.36, the line CF is equal to the line CE, and the line AD to the line AE, the two lines FC, DA will be equal to the side AC. Therefore the sides BA, BC exceed the side AC by the segments BD, BF; and since these are equal (by the same corollary), the same sides BA, BC will exceed the side AC by a line double the segment BD. But the square of the line double the segment BD is four times the square of the line BD, by the scholium to II.4. And [Margin: d — Elem. III.36] the square of the line BD is equal to the rectangle under BH, BI. Therefore the sides BA, BC exceed the side AC by a line whose square is four times the rectangle under BH, BI. Which is what was proposed.

But if the angle A is a right angle, I say that not only do the two sides AB, AC exceed the side BC by a line whose square is four times (*the sentence continues on the next page*)

Diagram. Triangle ABC with an inscribed circle (center G), touching the sides at D, E, F; the line BH is drawn from B, cutting the circle at I. Tick-marks on the sides indicate equal tangent segments.

- A (top vertex)
- B (bottom-left vertex)
- C (bottom-right vertex)
- D (tangent point on side AB)
- E (tangent point on side AC)
- F (tangent point on base BC)
- G (center of the inscribed circle)
- H (upper-right, where line BH meets the far side)
- I (point where BH cuts the near arc of the circle)

Book IV (continued)

Scholium demonstration, continued (Clavius). ... four times the rectangle contained by the line drawn from A into the concave circumference, together with its outer segment, as was shown; but also that this excess is equal to the diameter of the circle DEF. For if from the center G the lines GD, GE are drawn to the points of contact D, E, [Margin: e — Elem. III.18] the angles at D and E will be right. Since, then, the angle A too is assumed to be a right angle, [Margin: f — Elem. I.28] the line AE will be parallel to DG, and AD to EG; and therefore AG will be a parallelogram. [Margin: g — Elem. I.34] Hence AD will be equal to the radius EG, and AE to the radius DG — that is, the two lines AD, AE together will be equal to the whole diameter of the circle. But the two sides AB, AC exceed the side BC by the two lines AD, AE, as was shown. For EC is equal to FC, and DB to FB, by corollary 2 to III.36. Therefore the same sides AB, AC exceed the side BC by the diameter of the inscribed circle. Which is what was proposed.

Corollary within the scholium. And so any two sides exceed the remaining one by the two lines that touch the circle inscribed in the triangle and lie between the angle contained by those two sides and the circle. For it was shown that the sides AB, AC exceed the side BC by the tangent lines AD, AE, and so on.

Application (Clavius). Now that these things have been demonstrated, if the three sides of a triangle are known, we shall find without difficulty the three points at which the circle to be inscribed in the triangle must touch the sides. For since the sides, for example AB, AC, must exceed the side BC by two tangents drawn from A — which are indeed equal — if the side BC is subtracted (known from the known

sides AB, AC), there will remain the two tangents AD, AE, known. So half of this excess will give each of the points D and E. Then the remaining segment EC will give the segment CF, up to the third point F; or the remaining segment DB will give the segment BF, to the same point F. If, then, from the two points D, E perpendiculars DG, EG are erected, they will meet at G, the center of the circle to be inscribed.

For example: let side AB be 6, side AC be 8, and side BC be 10 palms. Subtracting the side BC, 10 palms, from the two sides AB, AC — that is, from 14 palms — there remain 4 palms. So AD, as well as AE, will contain two palms. The segment DB, then, will contain 4 palms, and the segment BF will have as many. But the segment EC, and therefore also CF, will be 6 palms. Likewise, subtracting the side AB, 6 palms, from the sides AC, BC — that is, from 18 palms — the remainder is 12 palms. So each segment CE, CF will be 6 palms, and each of BD, BF, 4 palms. Finally, subtracting the side AC, 8 palms, from the sides AB, CB — that is, from 16 palms — there remain 8 palms. So each segment BD, BF will have 4 palms; but each of CF, CE, 6 palms; and each of AD, AE, 2 palms.

Diagram. The scholium figure, repeated: triangle ABC with an inscribed circle (center G), touching the sides at D, E, F; the line BH is drawn from B, cutting the circle at I. Tick-marks along the sides mark the equal segments used in the numerical example.

- A (top vertex)
- B (bottom-left vertex)
- C (bottom-right vertex)
- D (tangent point on side AB)
- E (tangent point on side AC)
- F (tangent point on base BC)
- G (center of the inscribed circle)
- H (upper-right, where line BH meets the far side)
- I (point where BH cuts the near arc of the circle)

PROBL. 5. PROPOS. 5.

Enunciation (Euclid). About a given triangle, to circumscribe a circle.

Demonstration (Clavius). Let a circle be circumscribed about the given triangle ABC. Bisect two sides AC, AB (which, in a right-angled or obtuse-angled triangle, are for convenience taken about the right or obtuse angle — though this is not strictly necessary, since any two sides can be bisected) at the points D and E; from

these draw DF, EF perpendicular to the said sides, meeting at F. (That they meet is clear: for if the line DE were drawn, the angles FDE, FED would be less than two right angles.) And F will be either within the triangle, or on the side BC, or outside the triangle. Draw the lines FA, FB, FC. Now since the sides AD, DF of triangle ADF are equal to the sides BD, DF of triangle BDF, and the angles at D are right, [Margin: a — Elem. I.4] the bases FA, FB will be equal. In the same way FA, FC will be equal. Since, then, the three lines FA, FB, FC are equal, the circle described with center F and radius FA will pass also through the points B and C. Therefore about the given triangle we have circumscribed a circle. Which was to be done.

Diagram. Three cases of the circumscribing circle are shown side by side, illustrating the three positions of the center F: within the triangle (acute), on the side BC (right), and outside the triangle (obtuse). In each, triangle ABC is inscribed in a circle.

- A (top vertex of the triangle in each case)
- B (left vertex)
- C (right vertex)
- D (midpoint of side AC)
- E (midpoint of side AB)
- F (circumcenter — inside in the first figure, on BC in the second, outside in the third)

COROLLARIUM.

From this it is clear that, if the center falls within the triangle, [Margin: b — Elem. III.31] all the angles are acute, since they all stand in a segment greater than a semicircle; but if it lies on the side BC, [Margin: c — Elem. III.31] the angle BAC opposite that side is a right angle, because it stands in a semicircle; and finally, if it falls outside the triangle, [Margin: d — Elem. III.31] the opposite angle BAC is obtuse, because it stands in a segment less than a semicircle.

Conversely, it is evident that, if the triangle is acute-angled, the center falls within the triangle; if right-angled, on the side opposite the right angle; and finally, if obtuse-angled, outside the triangle. This is easily shown by reducing to some inconsistency or absurdity. For if in an acute-angled triangle the center fell on one side, the angle opposite it would be right; but if outside, that same angle would be obtuse. Likewise, if in a right-angled triangle it fell within, all the angles would be acute; but if outside, the opposite angle would be obtuse. Finally, if in an obtuse-angled triangle it fell on one side, the angle opposite it would be right;

but if within, all the angles would be acute. All of which follow from the earlier part of this corollary, and conflict with the hypothesis.

SCHOLIVM.

It is also gathered from this problem by what art a circle is to be described that passes through three given points not lying on one straight line. For if the given points are joined by three straight lines so as to form a triangle, a circle will easily be described about it, as was taught in this proposition. This, however, will be done more easily by that practical method we handed down at III.25. For let the three given points be A, B, C. From A and B, with any one and the same radius, describe two arcs cutting each other at the points D and E, through which draw the straight line DH. Likewise from A and C, with any equal radius — or, if you like, that former one — draw two other arcs cutting each other at the points F and G, through which draw the line FH, cutting the line DH at H. I say that H is the center of the circle passing through the given points A, B, and C. For if the lines AB, AC, BC were drawn, the sides AB, AC of triangle ABC would be bisected by the lines DH, FH, as was demonstrated in that practical method of proposition III.25. Therefore, as Euclid shows in this fifth problem, H will be the center of the circle described about the triangle ABC. Which is what was proposed.

Diagram. The three-point construction: three given points A, B, C, with pairs of intersecting arcs used to locate the center. The figure is small and schematic (arcs drawn as dotted curves); several labels are faint.

- A (top): a given point
- B (left): a given point
- C (right): a given point
- D (upper-left [?]): intersection of the first pair of arcs
- E (near D [?]): second intersection of the first pair of arcs
- F (upper-right [?]): intersection of the second pair of arcs
- G (right [?]): second intersection of the second pair of arcs
- H (center [?]): meeting point of lines DH and FH — the sought center

PROBL. 6. PROPOS. 6.

Enunciation (Euclid). In a given circle, to inscribe a square.

(The demonstration begins on the following page.)

(Continuation of Proposition 6 — to inscribe a square in a given circle; the enunciation was given at the close of the previous pages.)

Demonstration. Let a square be inscribed in the given circle ABCD, whose centre is E. Draw the two diameters AC and BD, cutting one another at right angles at the centre E, and join the straight lines AB, BC, CD, DA. I say that ABCD is a square inscribed in the given circle. For the sides EA, EB of triangle AEB are equal to the sides EC, ED of triangle CED — since all are radii from the centre — and the contained angles are right angles; therefore the bases AB and CD will be equal [Margin: a — *Elements* I.4]. By the same reasoning the straight lines BC and DA will be equal, likewise CD and DA, and DA and AB. All the sides of the quadrilateral ABCD are therefore equal to one another. We may reach this more briefly as follows. Since the four angles at E are equal — being right angles — the four arcs on which they stand will be equal [Margin: b — *Elements* III.26], and consequently the four chords subtending them will also be equal [Margin: c — *Elements* III.29]. Every side of the quadrilateral ABCD is therefore equal. Moreover its angles are right angles [Margin: d — *Elements* III.31], since they all stand in semicircles. ABCD will therefore be a square, and so we have described a square in the given circle. Which was to be done.

Diagram. A circle with an inscribed square, its diagonals drawn as the two perpendicular diameters through the centre.

- A (top of circle): upper vertex of the square
- B (left of circle): left vertex
- D (right of circle): right vertex
- C (bottom of circle): lower vertex
- E (centre): intersection of the diameters AC and BD

PROBL. 7. PROPOS. 7. [Margin: vij.]

About a given circle to describe a square.

Demonstration. Let a square be described about the given circle ABCD, whose centre is E. Draw the two diameters AC and BD, cutting one another at right angles at the centre E; and through A, B, C, and D draw lines perpendicular to the diameters — FG, FH, HI, IG — meeting at the points F, H, I, G. That AF and BF do in fact meet is clear from this: the straight line AB, when drawn, makes with AF and BF two angles less than two right angles; and the same holds for the others. I say that FHIG is a square described about the given circle. For since the angles AEB and FEB are right angles, FH and AC will be parallel [Margin: a — *Elements*

I.28]; and in the same way GI and AC will be parallel [Margin: b — *Elements* I.28]. Hence FH and GI are also parallel. In the same manner FG and HI are parallel. Since, then, ACHF is a parallelogram, its opposite sides AC and FH will be equal, and its opposite angles ACH and AFH equal [Margin: c — *Elements* I.38]. But ACH is a right angle; therefore AFH will be a right angle. By the same reasoning we shall show that the angles at H, I, and G are right angles, and that the sides HI, IG, GF are equal to the diameters BD and AC. Therefore, since the diameters are equal, the four sides FG, FH, HI, IG will also be equal, and so FGIH will be a square — whose sides indeed touch the circle, by the corollary to Proposition 16 of Book 3. We have therefore described a square about the given circle. Which was to be accomplished.

Diagram. A circle with an inscribed square rotated 45° (vertices at A, B, C, D, the points where the diameters meet the circle) and a larger square FGIH circumscribed about the circle, its sides tangent at A, B, C, D.

- F (top-left corner of the outer square): corner of the circumscribed square
- A (top, tangent point): where the outer side FG touches the circle / top of the inner diameter
- G (top-right corner): corner of the circumscribed square
- B (left, tangent point): where side FH touches the circle
- E (centre): centre of the circle, intersection of diameters AC, BD
- D (right, tangent point): where side GI touches the circle
- H (bottom-left corner): corner of the circumscribed square
- C (bottom, tangent point): where side HI touches the circle
- I (bottom-right corner): corner of the circumscribed square

PROBL. 8. PROPOS. 8. [Margin: viij.]

In a given square to inscribe a circle.

Demonstration. Let a circle be inscribed in the given square ABCD. Bisect the sides at E, F, G, H, and draw the straight lines EG and FH, cutting one another at I. Now since AD and BC are equal and parallel straight lines, their halves AH and BF will also be equal and parallel [Margin: a — *Elements* I.33]. Consequently AB too is parallel and equal to FH. By the same reasoning DC will be parallel and equal to that same FH; and likewise the straight lines AD, BC will be parallel and equal to EG. Therefore AI, IB, CI, ID are parallelograms, and so the straight lines IE, IF, IG, IH will be equal to the straight lines AH, EB, DH, AE. But these last are equal to one another, being halves of the equal lines AD, AB, and so on. Therefore the

straight lines IE, IF, IG, IH will be equal, and so a circle described from I with radius IE will also pass through the points F, G, H. Since it touches the sides AB, BC, CD, DA — by the corollary to Proposition 16 of Book 3, seeing that the angles at E, F, G, H are right angles [Margin: b — *Elements* I.29] — it will be inscribed in the square AC [i.e. square ABCD]. We have therefore described a circle in the given square. Which was to be accomplished.

Diagram. A square with an inscribed circle; the two midlines EG (horizontal) and FH (vertical) are drawn, crossing at the centre I.

- A (top-left corner): vertex of the square
- H (top, midpoint of side AD): tangent point
- D (top-right corner): vertex
- E (left, midpoint of side AB): tangent point
- I (centre): intersection of the midlines EG and FH; centre of the inscribed circle
- G (right, midpoint of side CD): tangent point
- B (bottom-left corner): vertex
- F (bottom, midpoint of side BC): tangent point
- C (bottom-right corner): vertex

PROBL. 9. PROPOS. 9. [Margin: ix.]

About a given square to describe a circle.

Demonstration. Let a circle be described about the square ABCD. Draw the diameters AC and BD, cutting one another at E. Now since the sides AB and AD of triangle ABD are equal, the angles ABD and ADB will be equal [Margin: a — *Elements* I.5]; but the angle BAD is a right angle. Therefore ABD and ADB will be half right angles [Margin: b — *Elements* I.32]. In the same way we shall show that all the remaining angles at A, B, C, D are half right angles, and hence equal to one another. Since, then, the angles [B]DA are equal, the straight lines EA and ED will be equal [Margin: c — *Elements* I.6]. By the same reasoning EA and EB will be equal; and likewise EB and EC, and again EC and ED. Therefore a circle described from E with radius EA will pass through the remaining points B, C, D. We have thus described a circle about the given square. Which was to be done.

Diagram. A square inscribed in a circle, with both diagonals (the diameters AC and BD) drawn to the centre.

- A (top-left): vertex of the square, on the circle

- D (top-right): vertex, on the circle
- E (centre): intersection of the diagonals AC and BD; centre of the circle
- B (bottom-left): vertex, on the circle
- C (bottom-right): vertex, on the circle

SCHOLIUM.

If about a given circle a square be described, and in the same circle another square be inscribed, the circumscribed square will be double the inscribed square. For the side of the circumscribed square is equal to the diameter of the circle, as is established from Proposition 7 of this book — that is, equal to the diagonal of the inscribed square. But the square on the diagonal is double the square of which it is the diagonal, as we showed at Proposition 47 of Book 1. The proposition is thus established.

Diagram. An unlabelled figure illustrating the scholium: an outer square, a circle inscribed in it, and a smaller square inscribed in the circle (rotated 45°, its vertices at the circle's points of tangency), with the inscribed square's diagonals drawn. No point-letters are printed.

PROBL. 10. PROPOS. 10. [Margin: x.]

To construct an isosceles triangle having each of the angles at its base double of the remaining angle.

Demonstration. Take any straight line AB, and let it be divided at C so that the rectangle contained by AB, BC is equal to the square on AC [Margin: a — *Elements* II.11]. Then with centre A and radius AB let a circle be described, and in it let there be fitted a straight line BD equal to AC [Margin: b — *Elements* IV.1], and let AD be joined. Since the straight lines AB and AD are equal, the triangle ABD will be isosceles. I say that each of the angles ABD, ADB is double of the remaining angle A. For let CD be joined, and let a circle DCA be described about the triangle ACD [Margin: c — *Elements* IV.5]. Now since the rectangle contained by AB, BC is equal to the square on the straight line BD, and the straight line AB cuts the circle DCA, the straight line BD will touch that same circle DCA at D [Margin: d — *Elements* III.37]. Therefore the angle BDC is equal to the angle A, in the alternate segment CAD [Margin: e — *Elements* III.32]. Adding the common angle CDA, the whole angle ADB will be equal to the two angles CAD and CDA together; but the external angle BCD is also equal to these same two [Margin: f — *Elements* I.32]. The angle BCD will therefore be equal to the angle ADB — that is, to the angle ABD, since ABD and ADB are equal [Margin: g — *Elements* I.5]; and hence the

straight lines CD and BD will be equal [Margin: h — *Elements* I.6]. But BD was made equal to the straight line AC. Therefore CD too will be equal to CA, and hence the angles CAD and CDA are equal [Margin: i — *Elements* I.5]. The angle ADB, then, which was shown equal to the two angles CAD and CDA, will be double of one of them — namely of the angle A. Therefore the angle ABD too will be double of that same angle A. We have thus constructed an isosceles triangle having, and so on. Which was to be accomplished.

Diagram. A large circle (centre A, radius AB) and a smaller circle DCA circumscribed about the triangle ACD; the isosceles triangle ABD is drawn with its sub-triangle ACD.

- A (top of the inner circle): apex of the isosceles triangle; centre of the large circle
- C (left, on the inner circle): the point dividing AB
- D (lower, on the large circle): base vertex; point of tangency of BD
- B [?] (bottom-left, on the large circle): base vertex of triangle ABD (*glyph abraded, resembles "J"; identified as B from the construction*)

COROLLARIUM.

Since the three angles of triangle ABD are equal to two right angles [Margin: k — *Elements* I.32] — that is, to five fifths of two right angles — it is clear that the angle A is a fifth part of two right angles, while each of B and D is two fifths. Likewise A is two fifths of one right angle, and each of B and D is four fifths, seeing that all three together are equal to two right angles [Margin: l — *Elements* I.32] — that is, to ten fifths of one right angle.

SCHOLIUM.

This problem could have been proposed in the manner of a theorem, thus:

If a straight line be cut as taught in Proposition 11 of Book 2, then an isosceles triangle whose base is equal to the greater segment, and each of whose equal sides is equal to the given line itself, has each of the equal angles at the base double of the remaining angle.

There is no reason for Campanus [Campanus of Novara] and Peletarius [Jacques Peletier] to torment themselves in proving that the straight line BD is so applied to the circle DCA that it in no way cuts it. For, because the square on the straight line BD is equal to the rectangle contained by AB, BC, it was shown in the last proposition of Book 3 that the straight line BD is perpendicular to the semidiameter

of the circle drawn from D. It will therefore touch the circle and will in no way cut it.

As for the art by which an isosceles triangle is to be constructed such that each of its base angles has to the remaining angle any given ratio — not merely the double, as Euclid has done here — this we shall set forth, together with Pappus [Pappus of Alexandria], at the end of Book 6. It is a matter that has hitherto been wanting.

PROBL. 11. PROPOS. 11. [Margin: xj.]

In a given circle to inscribe an equilateral and equiangular pentagon.

Demonstration. Let an equilateral and equiangular pentagon be inscribed in the given circle ABCDE. Construct an isosceles triangle FGH such that each of the angles G, H is double of the remaining angle F [Margin: a — *Elements* IV.10]; and in the circle inscribe a triangle ACD equiangular with the triangle FGH [Margin: b — *Elements* IV.2], and let each of the angles ACD, ADC be bisected by the straight lines CE, DB [Margin: c — *Elements* I.9], and let the straight lines AB, BC, CD, EA be joined. I say that the pentagon ABCDE, inscribed in the given circle, is equilateral and equiangular. For since each of the angles ACD, ADC is double of the angle CAD and is bisected, the five angles ADB, BDC, CAD, DCE, ECA will be equal. Hence the arcs AB, BC, CD, DE, EA, on which they stand, will be equal [Margin: d — *Elements* III.26], and therefore the straight lines AB, BC, CD, DE, EA will be equal as well [Margin: e — *Elements* III.29]. The pentagon ABCDE is therefore equilateral. Again, since the arcs AB and ED are equal, adding the common arc BCD they become the equal arcs ABCD and EDCB. The angles AED and BAE, standing on the said arcs, will therefore be equal [Margin: f — *Elements* III.27]. In the same way the remaining angles will be equal to each of these; for they stand on equal arcs, each of which is composed of three equal arcs. The pentagon ABCDE is therefore equiangular. And since it has also been shown to be equilateral, an equilateral and equiangular pentagon will have been inscribed in the given circle. Which was to be done.

Diagram. A circle with an inscribed equilateral pentagon ABCDE, its five diagonals drawn to form the inscribed pentagram (five-pointed star).

- A (top): pentagon vertex
- B (upper-left): pentagon vertex
- E (upper-right): pentagon vertex
- C (lower-left): pentagon vertex
- D (lower-right): pentagon vertex

Diagram. The auxiliary isosceles triangle FGH, with apex F above and base GH.

- F (apex, top): the vertex whose angle is single
- G (base, left): base angle (double of F)
- H (base, right): base angle (double of F)

COROLLARIUM.

It follows from this that the angle of an equilateral and equiangular pentagon comprises three fifths of two right angles, or six fifths of one right angle [Margin: g — *Elements* III.27]. For since the three angles BAC, CAD, DAE are equal — inasmuch as they stand on the equal arcs BC, CD, DE — and CAD is, by the corollary to the preceding proposition, a fifth part of two right angles (or two fifths of one right angle), the whole angle BAE will be three fifths of two right angles, or six fifths of one right angle.

SCHOLIUM.

If a straight line CD, of given length, be set out, we shall construct upon it an equilateral and equiangular pentagon in this way [Margin: h — Elements IV.10]. Let an isosceles triangle FGH be made, having each of the angles G, H double of the remaining angle F. Then let angles ACD, ADC be constructed equal to the angles G, H, and let the straight lines CA, DA meet at A [Margin: i — Elements I.32]; these will make the angle CAD equal to the angle F, and hence in the triangle ACD each of the angles ACD, ADC will be double of the remaining angle CAD. Now let a circle ABCDE be described about the triangle ACD [Margin: k — Elements IV.5]. Since the triangle ACD is inscribed in it, if the angles ACD, ADC are bisected, there will be inscribed — as before — an equilateral and equiangular pentagon whose side is the straight line CD. Which was the thing proposed.

We shall accomplish the same more easily by this method. Let the given straight line AB, on which the equilateral and equiangular pentagon is to be constructed, be cut at C as taught in Proposition 11 of Book 2, and let it be produced in both directions, and let AD, BE be made equal to the greater segment AC. Then with radius equal to the given line AB, from A and D let two arcs be described cutting one another at F. Likewise from B and E, with the same radius, let two others cut one another at G. Finally from F and G, with the same radius, let two others divide one another at H, and let the straight lines AF, FH, HG, GB be joined. I say that the pentagon ABGHF, described upon the given straight line AB, is equilateral and equiangular. That it is equilateral is clear from the construction, since all the lines taken are equal to AB — that is, all the arcs were described with radius AB. That it is also equiangular is shown thus. Let the straight line DF be drawn; then ADF will

be isosceles, of the kind constructed by Euclid in Proposition 10 — as is clear from the scholium to that same proposition — because its base AD is equal to the greater segment AC, which is equal to the line AB. Hence, by the corollary to that same Proposition 10, the angle DAF will comprise two fifths of two right angles, so that the remaining angle BAF comprises the other three fifths of two right angles. And since, by the corollary to this present proposition, the angle of an equilateral and equiangular pentagon comprises three fifths of two right angles, BAF will be the angle of an equilateral and equiangular pentagon. By the same reasoning ABG will be the angle of an equilateral and equiangular pentagon. From which it follows that the whole pentagon is equiangular. For if it be completed, or conceived to be completed — that is, if upon F and G two other sides be conceived described [Margin: l — Elements I.21] — these will of necessity fall upon the point H. Otherwise, if they were to meet above or below H, they would be either greater or less than the straight lines FH, GH — as is evident, if the base FG were subtended to the angle H — and therefore they would not be equal to the other sides FA, AB, BG, which is absurd. The pentagon ABGHF is therefore both equilateral and equiangular. Which was to be demonstrated.

Diagram. A circle with an inscribed equilateral pentagon and its inscribed pentagram (for the first construction of the scholium), with the auxiliary isosceles triangle FGH drawn to its right.

- A (top of the circle): pentagon apex
- B (upper-left): pentagon vertex
- E (upper-right): pentagon vertex
- C (lower-left): pentagon vertex
- D (lower-right): pentagon vertex
- F (apex of the triangle at right): single-angle vertex
- H (base of the triangle, lower right): base vertex (*the second base vertex G is not separately legible on the leaf [?]*)

Diagram. The pentagon ABGHF constructed on the base line (second construction of the scholium): a collinear base carries the points D, A, C, B, E; F and G stand above, and the apex H is at the top; construction arcs are shown as dotted lines.

- H (apex, top): pentagon apex
- F (upper-left): pentagon vertex
- G (upper-right): pentagon vertex
- D (base, far left): endpoint (AD = greater segment AC)
- A (base): pentagon vertex on the base line

- C (base, centre): the point of section
- B (base): pentagon vertex on the base line
- E (base, far right): endpoint (BE = greater segment AC)

PROBL. 12. PROPOS. 12. [Margin: xij.]

About a given circle to describe an equilateral and equiangular pentagon.

Demonstration. Let an equilateral and equiangular pentagon be described about the given circle ABCDE. In it inscribe an equilateral and equiangular pentagon ABCDE [Margin: a — *Elements* IV.11], and from the centre F draw the straight lines FA, FB, FC, FD, FE; to these draw perpendiculars GH, HI, IK, KL, LG, meeting at G, H, I, K, L. For since the angles GAE, GEA are less than two right angles — being parts, that is, of the right angles FAG, FEG — the straight lines AG, EG will meet toward the parts of G [Margin: b — I.13 [?]], and so with the rest. And since they touch the circle, by the corollary to Proposition 16 of Book 3, an equilateral and equiangular pentagon GHIKL will be described about the circle; which I say to be equilateral and equiangular. For when the straight lines FG, FH, FI, FK, FL are drawn, the squares on both FA, AH and on FB, BH will be equal to the square on the straight line FH [Margin: c — *Elements* I.47]. Therefore the squares on FA, AH will be equal to the squares on FB, BH. Taking away the equal squares on the equal lines FA, FB, there will remain the squares on AH, BH equal; and hence the straight lines AH, BH will be equal. This is also established from the second corollary to Proposition 36 of Book 3, since AH, BH are drawn from the same point H to touch the circle at A and B. Since, then, the sides AF, FH of triangle AFH are equal to the sides BF, FH of triangle BFH, and the base AH is equal to the base BH, as was shown, the angles AFH, BFH will be equal [Margin: d — *Elements* I.8]; and therefore so will the angles AHF, BHF [Margin: e — *Elements* I.4]. The angle AFB is therefore double of the angle BFH, and the angle AHB double of the angle BHF. In the same way we shall show that the angle BFC is double of the angle BFI, and the angle BIC double of the angle BIF. Since, then, the angles AFB, BFC are equal — because they stand on the circumferences AB, BC [Margin: f — *Elements* III.27], which are equal, being subtended by the equal straight lines AB, BC [Margin: g — *Elements* III.28] — their halves BFH, BFI will also be equal. Wherefore, since the two angles BFH, HBF of triangle BFH are equal to the two angles BFI, IBF of triangle IFB, and the side belonging to them adjacent...

Diagram. A circle with an inscribed equilateral pentagon ABCDE (touching the circle at its vertices) and a larger equilateral pentagon GHIKL circumscribed about the circle, its sides tangent to the circle; radii from the centre F to the tangent points are drawn, with many internal lines.

- G (top): apex of the outer (circumscribed) pentagon
- A (upper-left, tangent point): vertex of the inner pentagon
- E (upper-right, tangent point): vertex of the inner pentagon
- H (left): vertex of the outer pentagon
- F (centre): centre of the circle
- L (right): vertex of the outer pentagon
- B (lower-left, tangent point): vertex of the inner pentagon
- D (lower-right, tangent point): vertex of the inner pentagon
- I (bottom-left): vertex of the outer pentagon
- C (bottom, tangent point): vertex of the inner pentagon
- K (bottom-right): vertex of the outer pentagon

Euclid, *Elements*, Book IV — Clavius's Commentary (continued)

Proposition 12 (continued) — to circumscribe an equilateral and equiangular pentagon about a given circle

being adjacent [to it], and with BF common [to both], the sides BH, BI will also be equal, and the angles BHF, BIF equal. Therefore the straight line HI is double the line HB. [Margin: h. Elements I.26] By the same reasoning we shall show that GH is double the line HA. But the lines HB, HA have been shown equal; therefore their doubles HI, HG will also be equal. Similarly we shall demonstrate that the straight lines IK, KL, LG are each equal to any of the lines HI, HG. Therefore the pentagon GHIKL is equilateral.

Again, since it has been shown that the angles BHF, BIF are equal, and are halves of the angles BHA, BIC, their doubles BHA, BIC will also be equal. And by the same reasoning the angles IKL, KLG, LGH will each be equal to any of the angles BHA, BIC. Therefore the pentagon GHIKL is equiangular. And since it has been shown to be equilateral as well, an equilateral and equiangular pentagon will have been circumscribed about the given circle. Which was to be accomplished.

Corollary

[Body heading, p. 157: COROLLARIVM.]

It follows from the demonstration of this problem that, if any equilateral and equiangular figure is described in a circle, and at the ends of the radii drawn

from the center to the angles perpendicular lines are raised, these perpendiculars form another figure, of just as many sides, circumscribed about the circle. For it will always be demonstrated, by the very same reasoning, that these perpendiculars meet, and that they make equal angles — provided that straight lines are drawn from them to the center, as was done in the pentagon — lines which will bisect those same angles, just as was here proved in the pentagon, and so on.

To this belongs the following theorem, bearing on equilateral and equiangular figures, which will be very useful for finding the centers of the circles to be described about those figures, or inscribed within them, as will appear from what follows.

In an equilateral and equiangular figure: if indeed the number of angles is odd, a straight line dropped from any angle and bisecting the opposite side also bisects that angle; and conversely, a straight line bisecting the angle also bisects the opposite side. But if the number of angles is even, a straight line drawn from any angle to the opposite angle bisects both angles; and conversely, a straight line bisecting any angle falls on the opposite angle and bisects it as well.

[Odd-sided figure.] Let there first be an equilateral and equiangular figure of an odd number of sides, ABCDEFG, and from angle A let the straight line AH be dropped, bisecting the opposite side DE. I say that the angle BAG too is bisected. For, drawing from A straight lines to all the non-adjacent angles: since the two sides BA, BC are equal to the two sides GA, GF, and contain equal angles, by hypothesis, [Margin: i. Elements I.4] the lines AC, AF will be equal, and so will the angles BAC, GAF, as well as BCA, GFA. And accordingly, since the whole angles BCD, GFE are assumed equal, the remaining angles ACD, AFE will also be equal. Then again, since the two sides CA, CD are equal to the two sides FA, FE and contain equal angles, as was shown, [Margin: k. Elements I.4] the bases AD, AE will be equal, and so will the angles CAD, FAE, as well as CDA, FEA. And so one must proceed until the opposite side is reached. There, since again the whole angles CDE, FED are equal, the remaining angles ADH, AEH will also be equal. Therefore, since the two sides DA, DH are equal to the two sides EA, EH and contain equal angles, [Margin: l. Elements I.4] the angles DAH, EAH will also be equal. Consequently, since as many angles GAF, FAE, EAH are equal, each to each, the whole angle BAH will also be equal to the whole angle GAH; and therefore the angle BAG is bisected. Which was proposed.

But now let the straight line AH bisect the angle BAG. I say that it also bisects the opposite side DE. For suppose it does not bisect DE: if from A another straight line

be drawn bisecting DE, then — as we have just shown — that line would also bisect the angle BAG. Thus two straight lines would bisect the same angle BAG, which is absurd. Therefore the straight line AH, bisecting the angle BAG, also bisects the side DE.

Diagram. A regular heptagon inscribed in a circle, illustrating the odd-sided case. Straight lines are drawn from the top vertex A to all the non-adjacent vertices, and the line AH descends to H, the midpoint of the base DE.

- A (top vertex): apex from which the lines are drawn
- B (upper left vertex)
- G (upper right vertex)
- C (middle left vertex)
- F (middle right vertex)
- D (lower left vertex)
- H (foot on base, midpoint of side DE)
- E (lower right vertex)

[Even-sided figure.] Next let there be an equilateral and equiangular figure of an even number of sides, ABCDEFGH, and from angle A to the opposite angle E let the straight line AE be drawn. I say that the line AE bisects both the angle BAH and the angle DEF. For, drawing straight lines from A to all the non-adjacent angles, we shall demonstrate, as in the preceding figure, that any angles BAC, CAD, DAE are equal to just as many angles HAG, GAF, FAE, each to each; and therefore that the whole angle BAE is equal to the whole angle HAE, and likewise that the angle DEA is equal to FEA. Therefore each of the angles BAH, DEF is bisected. Which was proposed.

But now let the straight line EA bisect the angle BAH. I say that it falls on the opposite angle E and divides it in half. For if it be denied to fall on E: if from A another straight line be drawn, it will bisect the angle BAH, as we have just shown. Thus two straight lines would bisect the same angle BAH — which is absurd. Therefore the straight line AE, bisecting the angle BAH, falls on E, and consequently, as was just demonstrated, also bisects the angle DEF. Which was proposed.

Diagram. A regular octagon inscribed in a circle, illustrating the even-sided case. Straight lines are drawn from the top vertex A to all the non-adjacent vertices, with the diagonal AE reaching the opposite vertex E.

- A (top vertex): apex from which the lines are drawn

- B (upper left vertex)
- H (upper right vertex)
- C (middle left vertex)
- G (middle right vertex)
- D (lower left vertex)
- E (bottom vertex, opposite A)
- F (lower right vertex)

PROBL. 13. PROPOS. 13.

[Margin: xij.]

To inscribe a circle in a given equilateral and equiangular pentagon.

Let a circle be inscribed in the given pentagon ABCDE. [Margin: a. Elements I.9] Let two of its adjacent angles, BAE and ABC, be bisected by the straight lines AF, BF, which meet at F. Since (by the scholium of the preceding proposition) the straight lines AF, BF bisect the opposite sides CD, DE, the two straight lines AF, BF must necessarily cut one another within the pentagon before they meet the sides CD, DE. Then let the straight lines FC, FD, FE be joined. Now since the sides AB, BF of triangle ABF are equal to the sides CB, BF of triangle CBF — and by construction the angles contained by them, ABF and CBF, are equal — [Margin: b. Elements I.4] the bases AF, CF and the angles BAF, BCF will be equal. Since therefore the angles BAE, BCD are assumed equal, and BAF is half of angle BAE, by construction, BCF too will be half of angle BCD. Thus the angle BCD is bisected. In like manner we shall show that the remaining two angles, CDE and DEA, are bisected. Now let perpendiculars FG, FH, FI, FK, FL be drawn from F to the several sides of the pentagon. Since the two angles FGA, FAG of triangle FAG are equal to the two angles FLA, FAL of triangle FAL, and the side AF, subtending one of the equal angles, is common, [Margin: e. Elements I.26] the straight lines FG, FL will be equal. And similarly the remaining perpendiculars FH, FI, FK will be shown equal to each of these. Therefore a circle described with center F and radius FG will pass through the points H, I, K, L as well. And since the sides of the pentagon touch this circle (by the corollary of *Elements* III.16), because they make right angles with the radii FG, FH, and the rest, the circle will be inscribed in the given pentagon. Which was to be done.

Diagram. An equilateral pentagon ABCDE with an inscribed circle centered at F. The angle-bisectors from the vertices meet at F, from which perpendiculars run

to the five sides; the inscribed circle touches each side at the foot of its perpendicular.

- A (top vertex)
- B (upper left vertex)
- C (lower left vertex)
- D (lower right vertex)
- E (upper right vertex)
- F (center; meeting point of the bisectors, center of the inscribed circle)
- G (foot of perpendicular on side AB, upper left)
- H (foot of perpendicular on side BC, lower left)
- I (foot of perpendicular on side CD, at the bottom) [?]
- K (foot of perpendicular on side DE, lower right)
- L (foot of perpendicular on side EA, upper right)

Scholium

[Body heading, p. 158: SCHOLIVM.]

By this very same art we shall describe a circle in any equilateral and equiangular figure whatever. For we shall always show that, when two adjacent angles have been bisected, if from the point of concurrence of the angle-bisecting lines perpendiculars are drawn to the sides, these are equal to one another; and therefore that that point of concurrence is the center of the circle to be inscribed. Now the two straight lines which bisect the two adjacent angles necessarily meet within the figure — because in a figure of an odd number of sides they bisect the opposite sides, while in a figure of an even number of sides they bisect the opposite angles, as was shown in the scholium of the preceding proposition.

THEOR. 14. PROPOS. 14.

[Margin: xiiij.]

To describe a circle about a given equilateral and equiangular pentagon.

Let a circle be described about the equilateral and equiangular pentagon ABCDE. Let the two angles BAE, ABC be bisected by the straight lines AF, BF, which meet at F within the pentagon, as was demonstrated in the preceding proposition; and joining the straight lines FC, FD, FE, we shall show, as in the preceding problem, that the remaining angles BCD, CDE, DEA are also bisected. Therefore all the half-angles will be equal to one another, since the whole angles are assumed equal.

Now since in triangle AFB the two angles FAB, FBA are equal, [Margin: a. Elements I.6] the straight lines FA, FB will be equal. And by the same reasoning the remaining lines FC, FD, FE will each be equal to these. Therefore a circle described with center F and radius FA will pass through the points B, C, D, E as well. So about the given pentagon [a circle will have been described], and so on. Which was to be done.

Diagram. An equilateral pentagon ABCDE inscribed in its circumscribed circle, centered at F. Straight lines from F run to the five vertices, and the diagonals of the pentagon are drawn, forming a pentagram.

- A (top vertex)
- B (left vertex)
- E (right vertex)
- C (lower left vertex)
- D (lower right vertex)
- F (center of the circumscribed circle)

Scholium

[Body heading, p. 158: SCHOLIVM.]

By the very same device we shall describe a circle about any equilateral and equiangular figure whatever. For we shall always show that, when two adjacent angles have been bisected, the straight lines drawn from the point of concurrence of the angle-bisecting lines to the angles of the figure are equal to one another; and therefore that that point of concurrence is the center of the circle to be circumscribed.

PROBL. 15. PROPOS. 15.

[Margin: xv.]

To inscribe in a given circle a hexagon that is both equilateral and equiangular.

Let there be a given circle ABCDEF, whose center is G, in which an equilateral and equiangular hexagon is to be inscribed. Drawing the diameter AD, let a circle be described with center D and radius DG, which cuts the given circle at the points C and E; and from these, through the center G, let the straight lines CF, EB be extended. Then if the straight lines AB, BC, CD, DE, EF, FA be joined, a hexagon ABCDEF will be inscribed in the given circle; and I say that it is both equilateral

and equiangular. For since the straight line GC is equal to the straight line GD, and the straight line DC is equal to the same straight line DG, by the definition of the circle, the straight lines GC, DC will also be equal to one another; and therefore the triangle CDG will be equilateral. [Margin: a. Elements I.5] Hence its three angles CGD, GDC, DCG will be equal to one another; and since [Margin: b. Elements I.32] they are equal to two right angles, each of them — namely CGD — will be a third part of two right angles. In the same way the angle DGE will be a third part of two right angles. But [Margin: c. Elements I.13] the three angles CGD, DGE, EGF are equal to two right angles. Therefore the remaining angle EGF will also be a third part of two right angles. So the three angles CGD, DGE, EGF are equal to one another; and since [Margin: d. Elements I.15] the vertical angles FGA, AGB, BGC are also equal to them, the six angles at the center G will be equal. [Margin: e. Elements III.26] Therefore the arcs on which they stand, [Margin: f. Elements III.29] and consequently the straight lines AB, BC, CD, DE, EF, FA, will be equal. Wherefore the hexagon ABCDEF is equilateral. Again, since the arc BC is equal to the arc AF, if the common arc CDEF be added, the arcs BCDEF, AFEDC will be equal. Therefore the angles standing on them, BAF and ABC, [Margin: g. Elements III.27] will be equal. And similarly we shall show that the remaining angles BCD, CDE, DEF, EFA are equal to each of these, since each of them stands on an arc composed of four equal arcs — namely, of as many as the inscribed figure has sides, less two. From which it follows that all the angles stand on equal arcs. Therefore the hexagon ABCDEF is also equiangular. So in the given circle we have described an equilateral and equiangular hexagon. Which was to be done.

Diagram. A regular hexagon ABCDEF inscribed in a circle with center G, drawn along the diameter AD. A second circle, with center D and radius DG, cuts the given circle at C and E; the diagonals CF and EB pass through the center G, together with the radii to the vertices.

- A (top vertex; endpoint of the diameter AD)
- B (upper left vertex)
- F (upper right vertex)
- G (center of the given circle)
- C (lower left vertex; intersection of the second circle with the given circle)
- E (lower right vertex; intersection of the second circle with the given circle)
- D (bottom vertex; endpoint of the diameter AD and center of the second circle)

Corollary

[Body heading, p. 158: COROLLARIVM.]

From this it is manifest that the side of a hexagon is equal to the radius of the circle. For DC, the side of the hexagon, is equal to the radius DG, by the definition of the circle.

**

Scholium (continued)

[Body heading, p. 159: SCHOLIVM.]

Moreover, by what has been said about the pentagon in Propositions 12, 13, and 14, we shall describe an equilateral and equiangular hexagon about a given circle. Likewise, in a given equilateral and equiangular hexagon we shall inscribe a circle; and finally, about the same hexagon we shall describe a circle. For, in order that a hexagon be circumscribed about a circle, a hexagon must first be inscribed within the circle, as Euclid has taught in this Proposition 15. But if a circle is to be described either in a hexagon or about a hexagon, the two adjacent angles must be bisected; and the rest is then to be completed, as is delivered in Propositions 12, 13, 14.

Furthermore, from these results we shall also easily demonstrate that there can be given an isosceles acute-angled triangle [triangulum isosceles oxygonium — from Greek ὀξυγώνιον, "acute-angled"] whose third side is greater than either of the equal sides; or, equivalently, in which either of the equal sides is less than the third — a point we noted at Definition 28 of Book I. For let A be the center of the circle BCDE, in which the arc DE is a sixth part and BE a quarter. [Margin: h. Elements IV.15] Then the chord subtending DE will be equal to the radius. And if from any point C, between B and D, a straight line CE be drawn to E, this will be greater than DE (by the scholium of *Elements* III.29) — that is, greater than the radius. Therefore, joining the two radii AC, AE, the triangle ACE will be isosceles, whose third side CE is greater than either of the equal sides AC, AE. I say that the same triangle is also acute-angled. For since [Margin: i. Elements I.5] the angles ACE, AEC are equal, and [Margin: k. Elements I.17] together are less than two right angles, each of them will be acute. And drawing the straight line AB, the angle BAE — standing at the center on the quadrant BE — will be a right angle (by the scholium of *Elements* III.27); and therefore CAE will be acute. Therefore the isosceles triangle ACE, having its third side CE greater than either of the equal sides, is acute-angled. Which was proposed.

Diagram. An arc of a circle centered at A, carrying the points B, C, D, E. The radii AC and AE, together with the chord CE, form the isosceles acute-angled triangle ACE; the radius AB and the short chord DE are also drawn.

- A (apex; center of the circle)
- B (leftmost point on the arc)
- C (point on the arc between B and D)
- D (point on the arc, adjacent to E)
- E (rightmost point on the arc)

PROBL. 16. PROPOS. 16.

[Margin: xvj.]

To describe in a given circle a quindecagon [quintidecagonum — a fifteen-sided figure] that is both equilateral and equiangular.

Let a quindecagon, equilateral and equiangular, be inscribed in the given circle ABC. An equilateral triangle D having been constructed — which, by the corollary of *Elements* I.5, will also be equiangular — [Margin: a. *Elements* IV.2] let an equiangular triangle ABC be inscribed in the given circle, which will also be equilateral, by the corollary of *Elements* I.6. [Margin: b. *Elements* III.26 or 28] And the three arcs AB, BC, CA will be equal — either because the three straight lines AB, BC, CA are equal, or because the three angles A, B, C of triangle ABC are equal. Therefore, of whatever equal parts the whole circumference ABC is fifteen, the arc AB — which is a third part of the whole circumference — will be five such parts. [Margin: c. *Elements* IV.11] Again, let an equilateral and equiangular pentagon AEFGH be inscribed in the given circle, applying one of its angles at the point A; [Margin: d. *Elements* III.28] and the five arcs AE, EF, FG, GH, HA will be equal.

Therefore, of whatever equal parts the whole circumference ABC is fifteen, the arc AE — being a fifth part of the whole circumference — will be three such parts. And so, since the arc AB contains five such parts, and the arc AE three, the remaining arc EB will contain two. [Margin: e. *Elements* III.30] Therefore, the arc EB being bisected at I, the arc BI will be a fifteenth part of the whole circumference. Hence the straight line BI, when drawn, will subtend a fifteenth part of the whole circumference; and if [Margin: f. *Elements* IV.1] fourteen other equal chords be fitted into the circle, an equilateral quindecagon will be inscribed in the circle, [Margin: g. *Elements* III.27] which is also equiangular, since its angles subtend equal arcs — all of them composed, namely, of 13 equal arcs, as is

evident. Therefore in the given circle [we have inscribed] a quindecagon, and so on. Which was to be done.

Similarly, by what has been said about the pentagon above, in Propositions 12, 13, and 14, we shall describe an equilateral and equiangular quindecagon about a given circle. Likewise, in a given equilateral and equiangular quindecagon we shall inscribe a circle; and finally, about the given quindecagon we shall describe a circle.

Diagram. The construction of a regular quindecagon in a circle. An equilateral triangle ABC and a regular pentagon AEFGH are inscribed in the same circle, sharing the vertex A; the arc EB (the difference between the triangle's and the pentagon's arcs) is bisected at I, so that the chord BI subtends one fifteenth of the circumference. A separate small equilateral triangle, labeled D, is the auxiliary figure used to construct the inscribed triangle. The heavy foxing on this page makes several of the lower vertices hard to resolve.

- A (top vertex; shared vertex of the triangle and the pentagon)
- E (upper left vertex, pentagon)
- I (left; midpoint of arc EB)
- B (lower left vertex, triangle)
- F (lower left vertex, pentagon; near the auxiliary triangle D) [?]
- G (lower right vertex, pentagon)
- H (right vertex, pentagon)
- C (lower right vertex, triangle) — not clearly legible under the staining [?]
- D (separate small equilateral triangle serving as the auxiliary construction)

Scholium

[Body heading, p. 159: SCHOLIVM.]

From the structure and demonstration of this problem a method, and a certain art, can be gathered, by which almost infinitely many figures may be inscribed in a given circle. For since the straight line AB is named from the ternary — three — because it is the side of the equilateral triangle, and the straight line AE from the quinary — five — because it is the side of the pentagon; if 3 be multiplied by 5, there result 15. Therefore, from those two sides...

[This opening continues Clavius's scholium to Book IV, Proposition 16 (the quindecagon). Everything in these three pages is Clavius's own added material — a

general theorem on inscribed-polygon sides, followed by two paired "Observations" on the relation between equilateral and equiangular figures inscribed in, or circumscribed about, a circle, each with its demonstrations. Euclid's Book IV has already ended at Prop. 16; Book V does not begin within these pages.]

Book IV — Clavius's scholia to Proposition 16 (continued)

[End of the quindecagon scholium:]

described in the circle, there will be inscribed in the same circle a figure of 15 sides and equal angles, by this reasoning. The denominator of side AB — that is, 3 — is exceeded by the denominator of side AE — that is, by 5 — by the binary [i.e. by 2]. Therefore the arc BE will contain two sides of the aforesaid figure; and so, once the arc BE has been bisected at I, the chord BI will be a side of the figure of 15 sides and equal angles, as was demonstrated. It was by roughly this technique that Euclid proceeded in describing the quindecagon within a circle. From it we may infer a theorem of the following kind.

[Theorem.] If two sides of two equilateral and equiangular figures are inscribed in a circle from the same point, the arc enclosed between those sides will contain as many sides of another figure — to be inscribed in the same circle — as there are units by which the denominators of those two sides differ from one another. And the figure to be inscribed will have as many sides, and equal angles, as there are units in the number produced by multiplying the two denominators together.

Diagram. A circle with point A at the top of the circumference and points B, C, D, E marked in succession down the left-hand arc. From A four chords are drawn — AB, AC, AD, AE — so that the arcs BC, BD, BE, CD, CE, DE between the endpoints can be compared. It illustrates the theorem on the sides of successive inscribed regular figures.

- A (top of circle): common starting point of all the chords
- B, C, D, E (descending the left arc): endpoints of the sides of the hexagon, pentagon, square, and equilateral triangle respectively

[Demonstration by example.] Let there be inscribed in the circle ABCDE — always starting from point A — several sides: namely, of the hexagon the side AB, of the pentagon AC, of the square AD, and finally of the equilateral triangle AE. Since, then, the denominator of side AB — namely 6 — exceeds the denominator of side AC — namely 5 — by a unit, the arc BC enclosed between those sides will contain one side of a figure of 30 sides and equal angles; for from multiplying 5 by 6 there is produced 30. That this is so will be demonstrated as follows. Of such equal

parts as the whole circumference is 30, the arc AB — a sixth part of the circumference — is 5, and the arc AC — a fifth part of the circumference — is 6. Therefore the arc BC will contain one such part.

By like reasoning the arc BD will contain two sides of a figure of 24 sides and equal angles. For the denominator of side AB, namely 6, exceeds the denominator of side AD, namely 4, by the binary [by 2]; and from multiplying 4 into 6 there come 24.

In the same way too the arc BE will comprise three sides of a figure of 18 sides. [For the denominators 6 and 3 differ by 3, and $3 \times 6 = 18$.]

The arc CD, for its part, will embrace one side of a figure of 20 sides. [Denominators 5 and 4 differ by 1; $5 \times 4 = 20$.]

The arc CE, again, two sides of a figure of 15 sides. [Denominators 5 and 3 differ by 2; $5 \times 3 = 15$.]

And finally the arc DE will contain one side of a figure of 12 sides and equal angles. [Denominators 4 and 3 differ by 1; $4 \times 3 = 12$.] By this art and method, then, the sides of almost infinitely many figures are found out.

[First Observation.] It must not be passed over in this place that every equilateral figure **inscribed** in a circle is also equiangular; but that not every equilateral figure **circumscribed** about a circle is necessarily equiangular as well — except when the number of its angles is odd, or, if it is even, when two adjacent angles are equal, or two non-adjacent ones (provided that, one of them being set first, the other occupies any even place whatever, such as the fourth — for if it occupied the second it would be adjacent to the first, which is contrary to the hypothesis — or the sixth, eighth, tenth, and so on). In this matter several have been deceived, thinking that every equilateral figure circumscribed about a circle is necessarily equiangular too; among them is Campanus [Campanus of Novara], no obscure interpreter of Euclid.

Diagram. An equilateral pentagon ABCDE inscribed in a circle, with A at the top, B at the upper left, E at the upper right, C at the lower left, and D at the lower right. It serves the demonstration that an equilateral inscribed figure is equiangular.

- A (top): first vertex
- B (upper left), E (upper right): adjacent vertices
- C (lower left), D (lower right): the remaining vertices

Demonstration. Let there be an equilateral figure ABCDE, of any number of angles, inscribed in a circle. I say that it is necessarily equiangular as well. For since the sides AB, BC, CD are equal, the arcs AB, BC, CD will also be equal [Margin: h — *Elements* III.28]; and therefore the whole arc ABC will be equal to the whole arc BCD. Hence the remaining arcs ADC, BED in the same circle will be equal. Consequently the angles ABC, BCD standing upon them will be equal [Margin: i — *Elements* III.27]. In the same way all the angles, however many there be, will be shown to be equal; and therefore the figure ABCDE is equiangular, whether the number of its angles be even or odd. Which was to be demonstrated.

Diagram. An equilateral pentagon ABCDE circumscribed about a circle whose centre is F. The vertices are A (top), B (upper left), E (upper right), C (lower left), D (lower right). From the centre F radii run to every vertex and to the points of contact G, L, H, K, I on the sides. It serves the demonstration for a circumscribed figure with an odd number of angles.

- A (top), B (upper left), E (upper right), C (lower left), D (lower right): vertices of the pentagon
- F (centre): centre of the inscribed circle
- G (on side AB), L (on side AE), H (on side BC), K (on side DE), I (on side CD): points of contact / feet of the perpendiculars from F

Demonstration. Let there be, again, an equilateral figure ABCDE, of any odd number of angles, circumscribed about a circle whose centre is F. I say that it is necessarily equiangular too. For when straight lines are drawn from the centre F to all the vertices and to the points of contact, these last will be perpendicular to the sides [Margin: k — *Elements* III.18]. And since the tangents AG, AL are equal (from the second corollary of *Elements* III.36), the two sides AG, AF will be equal to the two sides AL, AF. Then, since the bases FG, FL from the centre are also equal, the angles at A will be equal [Margin: l — *Elements* I.8]; and so the angle BAE will be bisected. In just the same way we shall show that all the remaining angles of the figure are bisected. Again, since the two sides AB, BF are equal to the two sides CB, BF, and contain equal angles (shown to be equal), the bases AF, CF and the angles BAF, BCF will also be equal [Margin: m — *Elements* I.4]. Since, then, these angles are the halves of the angles BAE, BCD, as was shown, the whole angles BAE, BCD will likewise be equal. By the same reasoning the line CF will equal the line EF, and the angle BCD will equal the angle DEA. And so on in succession: if the figure has more angles, every third angle will always be equal to the one from which it is third, one being left between them in the middle — that is, the first (and any angle whatever may be set up as the first) will be equal

to the third, the third to the fifth, the fifth to the seventh, the seventh to the ninth, and so on. Thus all the angles set in odd places will be equal to one another; and by the same reasoning all the angles of even places, such as the second, fourth, sixth, eighth, tenth, and so on, will be equal to one another —

Book IV (continued) — printed running header: LIBER IV. (p. 161)

— since the fourth is third from the second, and the sixth also occupies the place third from the fourth, and so on. Therefore, since the proposed figure has an odd number of angles, the first angle — which has been shown equal to all the angles in odd places — will be equal to the last angle, which is odd and adjacent to the first. And this last angle will, by the same reasoning, be equal to the second angle, being the third from the last (the first being left in the middle), and equal to all the others that occupy the place third from the second, such as the fourth, sixth, eighth, tenth, and so on up to the last but one. Wherefore all the angles of the proposed figure will be equal to one another. Which was to be demonstrated.

[Scholium on the even case.] This demonstration does not hold for equilateral figures circumscribed about a circle that have an even number of angles. For from it one may conclude only that the first angle is equal to the third, fifth, seventh, ninth, and the rest occupying odd places; but it will never be proved that the first is equal to the last, because the last occupies an even place — as in a quadrilateral the fourth, in a hexagon the sixth, in an octagon the eighth, and so on. Likewise, from the same demonstration it will only be gathered that the second angle is equal to the fourth, sixth, eighth, tenth, and the rest occupying even places; but it will never be shown that the second angle equals the first, because the first occupies an odd place. And so only the angles of odd places will be proved equal to one another, and only the angles of even places equal to one another.

But if two adjacent angles — or even two non-adjacent ones, provided that one of them is placed in an odd position and the other in an even position — are equal to one another, then at last the equality of all the angles will be concluded by the demonstration above. For if the first angle is equal to the second (since, when two adjacent angles are equal, we may call one of them the first and the other the second), then, since the first has been shown equal to all the angles occupying odd places, and the second to all those in even places, they will all be equal to one another. But if the first angle is equal to some angle occupying an even place (since, when two non-adjacent angles are equal, one lying in an odd place and the

other in an even place, we may call one of them the first and the other the fourth, or sixth, or eighth, and so on, according to how far it stands from the first — but not the second, since the second is adjacent to the first), then, since the first has been shown equal to all the others in odd places, and the other equal to all those in even places — and therefore also to the second angle — the first will also be equal to the second: that is, the two adjacent angles will be equal to one another. Wherefore, as we have just demonstrated, all the angles are equal to one another.

[Objection and counter-example.] But suppose someone should say that, although it cannot be gathered from that demonstration that every equilateral figure whose number of angles is even, circumscribed about a circle, is also equiangular, nevertheless the contrary is not gathered from it either; and that consequently such a figure may indeed be equiangular — not, to be sure, on account of that demonstration, but on account of some other — so that no equilateral figure circumscribed about a circle can be given which is not at the same time equiangular, as Campanus, with some others, asserted. If, I say, anyone should speak so, we reply that infinitely many equilateral figures of an even number of angles, circumscribed about a circle, are **not** equiangular.

Diagram. A hexagon BCDEFG circumscribed about a circle whose centre is A. The circle's circumference is divided by the three diameters BE, CF, DG into six equal parts at the points O, P, Q, R, S, T; six further points H, I, K, L, M, N are the points where the sides of the hexagon touch the circle. The tangent lines drawn at these points meet to form the (equilateral but non-equiangular) hexagon. This is a small-type, densely labelled figure; several peripheral labels could not be resolved with confidence.**

- A (centre): centre of the circle [label not clearly visible in the engraving — [?]]
- B (top vertex), C (left vertex), D (lower-left vertex), F (lower-right / right vertex), G (upper-right vertex): hexagon vertices; the sixth vertex E [?] (right side) could not be confirmed
- O (top), P (left), Q (lower left), R (bottom), S (lower right), T (right): the six equal division points of the circumference
- H (upper left), I (upper right), K (left), N (right), L (lower left), M (lower right / bottom): points of contact of the sides

Demonstration of the counter-example. Let a circle be described about the centre A, and let its circumference be divided by the three diameters BE, CF, DG into six equal parts OP, PQ, QR, RS, ST, TO, in the manner set out in the description of the hexagon. Next let the arc OP be cut unequally at H, so that OH is the greater

segment and PH the lesser. Let arcs OI, QK, QL, SM, SN be taken equal to the arc OH, in such a way that six equal arcs are laid off on both sides of the three non-adjacent semidiameters, one being left in the middle between each pair. There will then be, on both sides of the semidiameters left over, the six remaining arcs PH, PK, RL, RM, TN, TI, all equal. When straight lines have likewise been drawn from the centre A to the points H, K, L, M, N, I, let a perpendicular BC be drawn through H at right angles to HA, cutting the semidiameters AO, AP produced at B and C; this line will touch the circle at H, by the corollary of *Elements* III.16. The lines HB and AB must necessarily meet, because the two angles BHA, BAH are less than two right angles. For BHA is a right angle, and BAH — standing on the arc OH, which is less than a quadrant — is less than a right angle, by the scholium of *Elements* III.27. Let the straight line BI also be joined. And since the two sides AH, AB are equal to the two sides AI, AB, and contain equal angles standing on the equal arcs OH, OI [Margin: n — *Elements* III.27], the bases BH, BI and the angles H, I will also be equal [Margin: o — *Elements* I.4]. Since, then, BHA is a right angle by construction, BIA will be a right angle too; and therefore the line BI will touch the circle at I, and will meet the semidiameter AI produced at G, because the angles AIG, GAI are less than two right angles. For AIG has been shown to be a right angle, while GAI is less than a right angle (by the scholium of *Elements* III.27), on account of the arc TI, which is less than a quadrant. And since the two angles I, A of triangle GAI are equal to the two angles H, A of triangle CAH — for I and H are right angles, and GAI, CAH are equal, standing on the equal arcs TI, PH [Margin: p — *Elements* III.27] — and the sides AI, AH adjacent to them are equal, the sides GI, GA will also be equal to the sides HC, CA [Margin: q — *Elements* I.26]. Since, then, BI has been shown equal to BH, the whole BG will be equal to the whole BC. By the same reasoning, the line GN drawn will touch the circle at N and will meet the semidiameter AF produced at F, and will be equal to GB. Likewise the line FM joined will touch the circle at M, and will meet the semidiameter AR produced at E, and will be equal to FG. Similarly the line EL drawn will touch the circle at L, and, the semidiameter AQ being drawn out, will meet it at D, and will become equal to EF. Finally the line DK joined will touch the same circle at K, and will meet the semidiameter AP produced at C, and will become equal to DE. That DK meets the semidiameter AP at the same point C at which the line BH meets it, is manifest. For if it cut it at another point, then — since the line CK drawn touches the circle at K, as has been demonstrated — two straight lines would touch the circle at the same point K, and thus between the circumference and the tangent a single straight line would be intercepted up to the point of contact, which Euclid demonstrated cannot happen [Margin: r — *Elements* III.16]. Therefore the hexagon BCDEFG circumscribed about the circle is equilateral. But

that this same figure is **not** equiangular we shall demonstrate as follows. Since the figure BCDEFG is equilateral, then, by what we showed a little earlier, the angles of odd places — namely B, D, F — will be equal to one another, and the angles of even places — namely C, E, G — will be equal to one another; and all will be bisected by the semidiameters. But since the three angles of triangle ABH are equal to the three angles of triangle ACH [Margin: s — *Elements* I.32], the right angles at H being taken away, the remaining two will be equal to the remaining two. Now BAH is greater than CAH (by the scholium of *Elements* III.27), because the arc OH is greater than the arc PH. Therefore the remaining angle ABH will be less than the remaining angle ACH; and since these are the halves of the whole angles B and C, as was shown, the whole angle CBG will also be less than the whole angle BCD. In the same way the whole angles D, F will be shown to be less than the whole angles E, G. Therefore the hexagon BCDEFG is not equiangular. Which was to be demonstrated.

The very same construction and demonstration can be carried out in the octagon (the circle being cut by four diameters into 8 parts), in the decagon (by five diameters into 10 parts), and so on.

[*Corollary.*] But if an equilateral figure of any number of angles be circumscribed about a circle by the method of Proposition 12 — a similar figure having first been described within the circle, and lines drawn touching the circle at the angles of the inscribed figure, as was done in the pentagon of Proposition 12 — then that figure will necessarily be equiangular as well, as is clearly established from the demonstration of that same proposition.

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[*Second Observation — the converse.*] It is also worth observing that every equiangular figure **circumscribed** about a circle is also equilateral; but that not every equiangular figure **inscribed** in a circle is necessarily equilateral as well — except when the number of its sides is odd, or, if it is even, when two adjacent sides are equal, or two non-adjacent ones (provided that, one of them being placed first, the other occupies any even place whatever, such as the fourth — for if it occupied the second it would be adjacent to the first, which is contrary to the hypothesis — or the sixth, eighth, tenth, and so on). In this matter, too, one might easily be deceived.

Diagram. An equiangular pentagon ABCDE circumscribed about a circle whose centre is F, with A at the top, B upper left, E upper right, C lower left, D lower

right; radii run from F to the vertices and to the points of contact G, L, H, K, I on the sides. It serves the demonstration that an equiangular circumscribed figure is equilateral.

- A (top), B (upper left), E (upper right), C (lower left), D (lower right): vertices of the pentagon
- F (centre): centre of the inscribed circle
- G (on AB), L (on AE), H (on BC), K (on DE), I (on CD): points of contact / feet of the perpendiculars from F

Demonstration. Let there be an equiangular figure ABCDE circumscribed about a circle whose centre is F. I say that it too is necessarily equilateral. For when straight lines FG, FH, FI, FK, FL are drawn from the centre F to the points of contact — which are perpendicular to the sides of the figure [Margin: t — *Elements* III.18] — and likewise straight lines FA, FB, FC, FD, FE are drawn from the centre F to the vertices of the figure: since the tangents AG, AL are equal (from the second corollary of *Elements* III.36), the two sides AG, AF of triangle AGF will be equal to the two sides AL, AF of triangle ALF; and the bases FG, FL from the centre are also equal. Therefore the angles at A, contained by those sides, are equal [Margin: u — *Elements* I.8]; and so the angle BAE is bisected. In just the same way we shall show that the remaining angles of the figure are bisected; and therefore, since the whole angles are supposed equal, their halves will be equal too. Hence the two angles A, G of triangle AFG will be equal to the two angles B, G of triangle BFG; and the side FG, opposite to one of the equal angles, is common. Therefore the sides AG, BG will also be equal [Margin: x — *Elements* I.26]; and so the side AB is bisected at G. In the same way you will prove that the remaining sides of the figure are bisected. Since, then, AG, AL are equal to their halves, the whole lines AB, AE will also be equal. And by the same reasoning AB, BC, and BC, CD, and CD, DE, and DE, EA will be equal. Wherefore the figure ABCDE is equilateral, whether the number of its sides be even or odd. Which was to be demonstrated.

Diagram. An equiangular pentagon ABCDE inscribed in a circle, with A at the top, B upper left, E upper right, C lower left, D lower right, and a small point marked i near the interior. It serves the demonstration for an inscribed equiangular figure with an odd number of sides.

- A (top), B (upper left), E (upper right), C (lower left), D (lower right): vertices of the inscribed pentagon
- i (interior): auxiliary point

Demonstration. Let there be, again, an equiangular figure ABCDE, of any odd number of sides, inscribed in a circle. I say that it too is necessarily equilateral. For since the angles BAE, ABC are supposed equal, the arcs BCE, ADC on which they stand will be equal [Margin: y — *Elements* III.26]; and when the common arc is taken away from these, the remaining arcs BAE, ABC will likewise be equal. Therefore, the common arc AB being subtracted, the remaining arcs will also be equal. By the same reasoning the arc BG will be equal to the arc DE; and consequently the side AE will be equal to the side BC [Margin: z — *Elements* III.29], and the side BC to the side DE. And so on in succession: if the figure has more sides, every third side will always be equal to the one from which it is third, one being left in the middle — that is, the first (and any side whatever may be set up as the first) will be equal to the third, the third to the fifth, the fifth to the seventh, the seventh to the ninth, and so on. And in this way all the sides set in odd places will be equal to one another. By the same reasoning all the sides of even places, such as the second, fourth, sixth, eighth, tenth, and so on, will be equal to one another — since the fourth is third from the second, and the sixth also occupies the place third from the fourth, and so on. Therefore, since the proposed figure has an odd number of sides, the last side will be equal to the first, to which it is next; and this last side will, by the same reasoning, be equal to the second, being the third from the last, the first being left in the middle; and to all the others that occupy from the second the place third, and so on up to the last but one. Wherefore all the sides of the proposed figure will be equal to one another. Which was to be demonstrated.

[*Scholium on the even case.*] This demonstration, however, does not fit equiangular figures of an even number of sides inscribed in a circle. For from it one may conclude only that the first side is equal to the third, fifth, seventh, and the others occupying odd places; but it will never be demonstrated through it that the first is equal to the last, because the last occupies an even place — as in a quadrilateral the fourth, in a hexagon the sixth, and so on. By like reasoning the same demonstration proves only that the second side is equal to the fourth, sixth, eighth, and the others in even places; but it will never be gathered from it that the second side is equal to the first, because the first lies in an odd place. And so only the sides of odd places will be proved equal to one another, and only the sides placed in even places equal to one another.

But if two adjacent sides, or even two non-adjacent ones — provided one of them is set in an odd place and the other in an even place — are equal to one another, then at last the demonstration above will conclude the equality of all the sides. For if the first side is equal to the second (since, when two adjacent sides are

equal, we may call one of them the first and the other the second), then, since it has been shown that the first is equal to all the sides of odd places, and the second to all those of even places, it is clear that all are equal to one another. And if the first side is equal to some side occupying an even place (since, when two non-adjacent sides are equal, one lying in an odd place and the other in an even place, we may call one of them the first and the other the fourth, or sixth, and so on, according to how far it stands from the first — but not the second, since the second is adjacent to the first), then, since the first has been shown equal to all the others of odd places, and the other to all those of even places, and therefore also to the second side, the first will also be equal to the second: and thus the two adjacent sides will be equal to one another. As, therefore, we have just demonstrated, all the sides will be equal to one another.

[Objection and geometrical proof.] But if anyone should doubt whether an equiangular figure of an even number of sides can be given, inscribed in a circle, which is not equilateral (for although the demonstration above does not prove that all its sides are equal when all its angles are equal, yet the contrary cannot be inferred from it either), we shall demonstrate it geometrically in this way. In the first place, it is agreed that this holds in quadrilateral figures. For a rectangle longer on one side [an oblong] is equiangular — since all its angles are right — but not equilateral. And who would doubt that it can be inscribed in a circle? Next, then, let there be an equilateral and equiangular hexagon ABCDEF described in a circle; and in the arc BC let any point G be taken; and in the third arc from it let an arc DH be taken equal to the arc BG; likewise in the fifth arc from it, an arc FI equal to it; and let the straight lines AG, GC, CH, HE, EI, IA be joined. I say that the hexagon AGCHEI is equiangular, but not equilateral.

Diagram. A circle carrying a regular hexagon ABCDEF (solid), with B upper left, A upper right, C left, F right, D lower left, E lower right; and an inscribed irregular hexagon AGCHEI (dotted), where G lies on the arc BC (left), I on the arc FA (right), and H on the arc DE (bottom). It shows an equiangular but non-equilateral inscribed hexagon.

- B (upper left), A (upper right): adjacent vertices of the regular hexagon
- C (left), F (right): vertices of the regular hexagon
- D (lower left), E (lower right): remaining vertices of the regular hexagon
- G (on arc BC, left), I (on arc FA, right), H (on arc DE, bottom): the added points forming the dotted hexagon AGCHEI

Demonstration (begins). That it is equiangular we shall show thus. The angles B and G are equal, being in the same segment ABC [Margin: a — *Elements* III.21]. By

the same reasoning the angles D, H and F, I are equal. Moreover, since the arcs BG, DH are equal by construction, the common arc GD being added, the whole arcs BCD, GCH will be equal. Therefore the remaining arcs BFD, GFH left in the circle will be equal; and consequently the angles BCD, GCH standing upon them will likewise be equal [Margin: b — *Elements* III.27]. In just the same way we shall show that the angles

(Continuation of the scholium to Book IV, Proposition 16 — Clavius's commentary on inscribing regular polygons.)

that DEF, HEI and FAB, IAG are equal to one another. Therefore the six angles of the hexagon AGCHEI are equal to the six angles of the hexagon ABCDEF, each to each. And so, since this latter is assumed to be equiangular, the former will be equiangular too. That it is not, however, equilateral, we will prove as follows. Side AG is greater than side AB, and side GC is less than side BC, by the scholium to Proposition 29 of Book III. By the same reasoning, sides CH and EI are greater than sides CD and EF, while sides HE and IA are less than sides DE and FA. Since, then, the sides AB, BC, CD, DE, EF, FA are assumed equal, the sides AG, GC, CH, HE, EI, IA will be unequal — although AG, CH, EI are equal to one another, and GC, HE, IA to one another (leaving always one in the middle), as we showed above.

The very same thing will be demonstrated in the same way to be possible in the octagon, the decagon, and the other figures of an even number of sides, provided that first an octagon, decagon, and the other equilateral and equiangular figures are inscribed in the circle.

But if a figure is inscribed in a circle by the method of Propositions 11, 15, and 16, it will always be both equiangular and equilateral, as is clear from the demonstrations of those propositions.

Furthermore, whatever equilateral and equiangular figure we have shown how to inscribe in a circle, we shall likewise know how to describe about a circle, and how to inscribe a circle within it, and how to describe a circle about it, if we imitate the technique that was set out for the pentagon in Propositions 12, 13, and 14.

Again, given any equilateral and equiangular figure inscribed in a circle, there will be inscribed in the same circle a figure having twice as many sides. For if the arcs subtended by the sides are bisected and the bisection points joined by straight lines, the conclusion follows. Thus by means of an inscribed equilateral triangle a hexagon is also inscribed, and hence a dodecagon, a figure of 24 sides,

and so on. Likewise from a square described in a circle an octagon is inscribed, and thence a figure of 16 sides, a figure of 32, 64, 128 sides, and so on.

Moreover, all equilateral and equiangular figures can be inscribed in a circle by means of isosceles triangles, as certain interpreters of Euclid rightly point out at this place.

For figures of an odd number of sides will be inscribed by means of isosceles triangles whose base angles are multiples of their vertex angle. Thus by means of an isosceles triangle each of whose base angles is equal to its vertex angle, described in the circle, there will be inscribed the first figure of odd-sided figures — that is, the equilateral triangle; for an isosceles triangle of this kind will be an equilateral triangle. But if there is inscribed in the circle an isosceles triangle each of whose base angles is double the vertex angle, there will be inscribed the second figure of odd-sided figures, namely the pentagon, provided the two equal angles are bisected, as was shown in Proposition 11. The heptagon, the third figure of odd-sided figures, will be inscribed in the circle by means of an isosceles triangle having each of its base angles triple the vertex angle, if its two equal angles are each divided into three angles equal to the vertex angle. Likewise the fourth figure of odd-sided figures, such as the enneagon, will be inscribed in the circle by means of an isosceles triangle each of whose base angles is quadruple the vertex angle, if each is distributed into four angles equal to the vertex angle, and so on.

This division of the base angles into equal parts is very easy. For if at the vertex angle there are set up in order, along the base, as many equal angles as there are units in the number of the multiple ratio that each base angle bears to the remaining [vertex] angle, the angle will be divided into equal parts. But once such an isosceles triangle has been described in the circle, the figure will be described in the circle without dividing the base angles, provided that straight lines equal to the base are fitted into the circle; for the base is always one side of the figure, as was evident in the case of the pentagon.

Figures of an even number of sides, on the other hand, will be inscribed in a circle by means of isosceles triangles whose base angles are multiples-and-a-half [*sesquilateri*] of their vertex angle. Thus the square, constituting the first figure of even-sided figures, will be inscribed by means of an isosceles triangle each of whose base angles is one-and-a-half times [*sesquialter*] the vertex angle. For the vertex angle stands on a quarter of the circumference: since the two base angles together contain three angles equal to the vertex angle — because each contains it once, and half of it besides — they subtend three parts of the circumference,

and therefore the vertex angle subtends one only. The hexagon, that is, the second figure of even-sided figures, will be inscribed by means of an isosceles triangle each of whose base angles is two-and-a-half times [*duplus sesquialter*] the vertex angle. For the vertex angle stands on a sixth of the circumference, since the remaining angles taken together contain it five times — because each contains it twice, and half of it besides. Likewise the octagon, that is, the third figure of even-sided figures, will be inscribed by means of an isosceles triangle having each of its base angles three-and-a-half times [*triplus sesquilaterus*] the vertex angle.

If, then, a technique were discovered by which isosceles triangles could be constructed having base angles that are multiples of their vertex angle — in the way that Euclid constructed an isosceles triangle having each base angle double the vertex angle — all figures of an odd number of sides could easily be described in a circle. And if their arcs were bisected, all figures of an even number of sides, after the square, would also be inscribed; and thus the circumference of any circle could be divided geometrically into any number of equal parts. This would be of the greatest use to astronomers. For Orontius Finaeus [Oronce Fine] does not rightly claim it for himself in the little book — hitherto, as he himself says, longed for — on the complete description of all rectilinear figures within a circle, and so on, since his demonstrations are false and sophistical, as was shown geometrically by Petrus Nonius Lusitanus [Pedro Nunes] in his book on the errors of Orontius [*De erratis Orontii*].

We ourselves, however, shall at the end of Book 6, following Pappus of Alexandria, describe geometrically a certain curved line [*linea inflexa*] by which not only isosceles triangles whose base angles bear a given ratio to the remaining angle may be constructed — and hence all equilateral and equiangular figures described in a circle — but also the arc of any circle may be distributed into any number of equal parts, whether it be a quadrant or not. Indeed, by means of the same curved line any circle whatever will, by a pleasing operation, be transformed without any trouble into an equal square, as we shall likewise say there. This is a matter that has kept the minds of mathematicians in suspense up to now.

But let us show, if you please, by what method as well any equilateral and equiangular figure may be inscribed in a circle without isosceles triangles whose base angles have a given ratio to the remaining angle. Let, then, an equilateral and equiangular heptagon be inscribed in the circle ABCDEFG. Let its quadrant AH be cut into seven equal parts — that is, into as many as the figure to be

inscribed has sides or angles. You will cut the quadrant AH into the proposed equal parts either by the means we said we would demonstrate at the end of Book 6, or by means of a compass, opening its legs now more, now less, until you find that their span divides the quadrant into the desired parts. For a quadrant is more easily divided into any number of equal parts than the whole circumference, because, by trying again and again and as it were repeating the operation, the work appears more clearly in a small magnitude than in a large one, in that [it is easier to judge] how much must be taken away...

Diagram. A circle with an inscribed equilateral heptagon and the construction for dividing the quadrant. The heptagon vertices are labelled around the circle; a radius runs to the point H on the circumference, and the arc from A toward G is subdivided by small tick-marks (the divided quadrant AH).

- A (top of circle): first vertex / start of the quadrant division
- B (upper left): heptagon vertex
- C (left): heptagon vertex
- D (lower left): heptagon vertex
- E (lower right): heptagon vertex
- F (right): heptagon vertex
- G (upper right): heptagon vertex
- H (right, on circle, at the end of the drawn radius): endpoint of the quadrant AH being subdivided

how much ought to be taken from, or added to, the one part obtained by means of the compass, if that part does not yield the one desired. Once the quadrant has been cut into seven equal parts — or into more, if a figure of more sides is wanted — the straight line AG, which subtends four such parts, will be the side of the figure to be inscribed. For since in the whole circumference there are contained four times as many equal parts as those into which the quadrant is divided, the aggregate of four parts will be contained in the whole circumference as many times as one part is contained in the quadrant: because each part, together with three others (one taken in each of the other three quadrants), makes up an aggregate of four parts. This will also become clear as follows. Since parts have the same ratio as their equimultiples, as Euclid demonstrates in Book 5, Proposition 15 (for it is permissible to apply that proposition here, since it in no way depends on what precedes), one part will be to the quadrant as the quadruple of one part — that is, the arc AG — is to the quadruple of the quadrant, that is, to the whole circumference. Whatever part, then, one of those into which the quadrant is divided is of the quadrant, such a part will the arc AG be of the

whole circumference. And so, if the quadrant is divided into 7 parts, there will be contained 28 such parts in the whole circumference. Therefore the 4 [parts] make $\frac{4}{28}$, that is, $\frac{1}{7}$ of the whole circumference. Likewise, if the quadrant is cut into 11 parts, there will be contained 44 such parts in the whole circumference; and so the 4 make $\frac{4}{44}$, that is, $\frac{1}{11}$ of the whole circumference; and so on for the rest. Therefore, if from the arc AG of four parts equal arcs are continuously cut off, and straight lines are subtended to those same arcs, the proposed figure will be inscribed in the circle, as is clear from what has been demonstrated.

(Praxis: to construct a regular figure on a given straight line.)

Now if on a given straight line any equilateral and equiangular figure whatever is to be described, we shall accomplish it by this method. Let an equilateral and equiangular pentagon be constructed on the given straight line AB. Produce BA to C so that AC is equal to AB; from A, through C, describe an arc of a circle; and, having drawn AD perpendicular to AC so that CD is a quadrant (by the scholium to Proposition 27 of Book III), let the quadrant CD be divided into five equal parts — that is, into as many as the figure to be constructed has sides. Let 4 of these be CE, and join the straight line AE, which will be equal to AB. I say that the angle BAE is equal to the angle of an equilateral and equiangular pentagon. For, taking an arc CF equal to the arc CE, let the straight lines CE, CF be joined. And since CE, subtending 4 parts of the quadrant, is the side of an equilateral and equiangular pentagon to be described in the circle ECF (as we proved just now), CF will be another side; and hence the two sides EC, FC will contain the angle ECF of an equilateral and equiangular pentagon. Now since, when the straight line AF has been drawn, the two sides EC, CA are equal to the two sides FC, CA, and the bases AE, AF are likewise equal, [Margin: (c) Elem. I.8] the angles ECA, FCA will be equal. Since, then, [Margin: (d) Elem. I.5] the angles ACE, AEC are also equal, ACF will be equal to AEC; and, adding the common angle ACE, the whole angle ECF will be equal to the two angles ACE, AEC. But [Margin: (e) Elem. I.32] the exterior angle BAE is equal to the same interior angles ACE, AEC. Therefore ECF and BAE will be equal to one another. Since, then, ECF is an angle of a pentagon, BAE too will be an angle of a pentagon. Consequently, if a circle is described about the three points B, A, E, and straight lines equal to AB, AE are [Margin: (f) Elem. IV.1] fitted into it — BH, HG, GE — an equilateral and equiangular pentagon will have been described in the circle ABHGE, and hence on the given straight line AB. Which was to be done. Nor is there any doubt that straight lines equal to AB, AE can be fitted into the circle so as to complete the whole circumference. For the pentagon's angle BAE stands on three-fifths of the whole circumference, while the individual sides AB, AE each subtend one-fifth; and therefore the individual sides

BH, HG, GE, being taken equal to AB, AE, each subtend one-fifth as well, and so they will complete the whole arc BGE, which contains three-fifths. The same reasoning holds for the other equilateral and equiangular figures.

If perhaps we have at hand some equilateral and equiangular figure to which we wish to construct a similar one on a given straight line — whether that figure is described in some circle or not — it will be enough to construct an angle BAE equal to one of its angles at the end of the given line AB. For if, having placed a straight line AE equal to AB, a circle is described through the three points B, A, E, we shall complete the proposed figure, as we said before.

(Praxis of Ptolemy: to inscribe a pentagon and decagon in a given circle.)

Since, however, that inscription of an equilateral and equiangular pentagon in a circle which Euclid handed down is long and difficult, it has seemed good to append to this fourth book a certain practical construction by which, in one and the same operation, Ptolemy, in Book 1 of the *Almagest* [*magna constructio*], inscribes in a given circle an equilateral and equiangular pentagon and decagon. Let there be a given circle ABC whose center is D. Having drawn the diameter BC, let DA be erected perpendicular to BC. Then, the semidiameter CD having been bisected at E, let the straight line EA be drawn, and let EF be cut off equal to it. And so, if the straight line AF is drawn, AF will be the side of the pentagon, and DF the side of the decagon, to be inscribed in the given circle, as he demonstrates. But since the demonstration of this matter depends on Book 13 of Euclid, it does not seem it should be written here, but rather deferred to its proper place, namely to Book 13.

Nor will it be beyond our plan, or foreign to this book, if we add one further theorem, namely:

Theorem. If the bisection points of the sides of an equilateral and equiangular figure are joined by straight lines, the inscribed figure will likewise be equilateral and equiangular, having the same number of sides as that figure, and having the same center.

Definition. The *center* of an equilateral and equiangular figure is called that point from which a circle is inscribed in the figure, or circumscribed about it — so that the center of the circle and of the figure is the same.

Diagram. The construction of a pentagon on the given line AB. A circle is described through B, A, E; the quadrant CD is divided (with tick-marks along an arc near D–E), and the pentagon ABHGE is fitted in.

- A (center of the figure, at the intersection of the drawn lines): endpoint of the given line and vertex of the constructed angle BAE
- B (lower right): endpoint of the given line AB
- C (left): point on the produced line BA, with $AC = AB$
- D (upper, right of E): endpoint of the perpendicular; CD is the quadrant
- E (upper left): the point marking $4/5$ of the quadrant; $AE = AB$
- F (bottom): point with arc $CF = CE$
- G (top): pentagon vertex
- H (right): pentagon vertex

Diagram. Ptolemy's construction of the pentagon and decagon side in a semicircle. The diameter BC carries the points, with the apex A above and straight lines drawn from A to the base points.

- A (apex, top of the semicircle): pole of the perpendicular DA
- B (left end of the diameter): endpoint of diameter BC
- F (between B and D on the base): point with $EF = EA$; AF is the pentagon side, DF the decagon side
- D (center of the circle, on the base): center; DA is perpendicular to BC
- E (between D and C on the base): midpoint of the semidiameter DC
- C (right end of the diameter): endpoint of diameter BC

(Demonstration of the appended theorem on the inscribed midpoint-figure.)

Let there then be an equilateral and equiangular figure ABCDEF, whose sides are bisected at G, H, I, K, L, M, the straight lines GH, HI, IK, KL, LM, MG being joined. I say that the figure GHIKLM, inscribed in the figure ABCDEF, is also equilateral and equiangular, and has the same center. It will indeed be equilateral, because its sides, subtending equal angles contained by equal straight lines (namely by the halves of equal sides), [Margin: (g) Elem. I.4] are equal. And since the three angles AMG, GML, LMF, as well as the three FLM, MLK, KLE, [Margin: (h) Elem. I.13] are equal to two right angles — while AMG, LMF, FLM, KLE are [Margin: (i) Elem. I.8] equal to one another, since they are contained by equal sides and subtended by equal bases — the remaining angles GML, MLK will be equal. And by the same argument we shall conclude that the remaining angles are equal both to these and to one another. Therefore the figure GHIKLM is equiangular as well.

That it has the same center will be shown as follows. From the center N of the figure ABCDEF let straight lines NG, NH, and so on be drawn to all the angles of

the inscribed figure, and let NA, NB be joined. Since, then, the sides AG, GN of the triangle AGN are equal to the sides BG, GN of the triangle BGN, and the bases AN, BN are equal (being semidiameters of the circle described about the figure), [Margin: (k) Elem. I.8] the angles AGN, BGN will be equal, and therefore right angles. Hence NG is perpendicular to the side AB. In the same way the remaining lines NH, NI, and so on will be perpendicular to the sides BC, CD, and so on. And since, by Definition 4 of Book 3, these show the distances of the equal straight lines AB, BC, and so on from the center N to be [Margin: (l) Elem. III.14] equal to one another, a circle described from N with radius NG will pass through the remaining angles H, I, K, L, M. And therefore N will be the center of the figure GHIKLM, that is, of the circle described about that figure. Which was the proposition.

(Scholium: which regular figures tile the plane.)

Nor indeed should this be passed over: that, among all equilateral and equiangular figures, only the triangle, the square, and the hexagon fill space — that is, only some number of triangles, or squares, or hexagons, can be so fitted together in a plane that there is no gap between their angles, but they constitute a plane surface. For six angles in an equilateral triangle, four in a square, and three in an equilateral and equiangular hexagon are equal to four right angles, which is exactly the space around any point in a plane, as we showed at Proposition 15 of Book 1. For since one angle of an equilateral triangle contains $\frac{2}{3}$ of one right angle, and since all three contain $\frac{6}{3}$ of one right angle — that is, two right angles — six such angles will contain $\frac{12}{3}$ of one right angle, that is, four right angles. Likewise, the four angles in a square are four right angles. Finally, since one angle of an equilateral and equiangular hexagon contains $\frac{4}{3}$ of one right angle, and all six contain $\frac{24}{3}$ of one right angle — that is, eight right angles — three such angles will contain $\frac{12}{3}$ of one right angle, that is, four right angles. This is not seen in the other equilateral and equiangular plane figures. For three angles in an equilateral and equiangular pentagon are equal to $\frac{18}{5}$ of one right angle, that is, to $3\frac{3}{5}$ right angles only, while four are equal to $\frac{24}{5}$ of one right angle, that is, to $4\frac{4}{5}$ right angles — because all five are equivalent to 6 right angles, that is, they constitute $\frac{30}{5}$ of one right angle, and therefore one contains $\frac{6}{5}$ of one right angle. Three pentagons, then, cannot fill space, since the three angles of three fitted together in a plane are less than four right angles: for they make only three right angles with three-fifths. By the same reasoning four pentagons cannot fill space, because the four angles of four fitted together in a plane are greater than four right angles, since they are equal to four right angles and moreover to four-fifths of one right angle, as has been said. By like reasoning space cannot be

filled by heptagons either. For three angles of any equilateral and equiangular heptagon are equal to $30/7$ of one right angle, that is, to $4 \frac{2}{7}$ right angles. For since all seven angles are equal to 10 right angles — that is, comprise $70/7$ of one right angle — it follows that one contains $10/7$ of one right angle, that is, one right angle and $3/7$ of one right angle besides; and hence three contain four right angles and $2/7$ of one right angle. Therefore, three heptagons being fitted together in a plane, their three angles cannot fill space, since they exceed four right angles. A fortiori, neither will four heptagons fill space; nor three octagons, equilateral and equiangular, nor four; nor three or four equilateral and equiangular figures of a greater number of sides than eight — because three angles taken together are always greater than four right angles, being greater than the three angles of an equilateral and equiangular heptagon, which we showed to be greater than four right angles. Therefore only the equilateral triangle, the square, and the equilateral and equiangular hexagon can fill space with their angles, as we have said.

FINIS ELEMENTI QVARTI

(The End of the Fourth Element.)

Diagram. An equilateral, equiangular hexagon ABCDEF with a second hexagon GHIKLM inscribed at the midpoints of its sides; all midpoint-vertices are joined to the common center N by radii, subdividing the figure into triangles.

- A (top-left vertex of the outer hexagon): outer hexagon vertex [?]
- B (left vertex of the outer hexagon): outer hexagon vertex
- C (bottom-left vertex): outer hexagon vertex
- D (bottom-right vertex): outer hexagon vertex
- E (right vertex): outer hexagon vertex
- F (top-right vertex): outer hexagon vertex
- G (upper left): midpoint of side AB, inner-hexagon vertex
- H (lower left): midpoint of side BC, inner-hexagon vertex
- I (bottom center): midpoint of side CD, inner-hexagon vertex
- K (lower right): midpoint of side DE, inner-hexagon vertex
- L (upper right): midpoint of side EF, inner-hexagon vertex
- M (top center): midpoint of side FA, inner-hexagon vertex
- N (center): common center of both hexagons and of the circumscribed circle

EUCLID'S ELEMENTS, BOOK FIVE

DEFINITIONS

I.

[Euclid, Definition 1] A part is a magnitude of a magnitude, the smaller of the larger, when the smaller measures the larger.

[Clavius's commentary] In the preceding four books Euclid treated continuous quantity considered in itself. In the next two he discusses that same quantity not in itself, but insofar as one is related to another — that is, insofar as, when compared with another, it has some ratio to it. In this fifth book he teaches the ratios of continuous quantities in general, without descending to any particular kind of quantity such as a line, a surface, or some body. In the sixth book, by contrast, he shows in particular what ratios lines, angles, circumferences of circles, triangles, and other plane figures have to one another. So, to carry out his plan, he first defines the terms that are used in the demonstrations concerning ratios.

Diagram. A column of six labelled straight-line magnitudes, drawn one above another and used in the commentary on Definitions I and II. Reading down the figure:

- A: a short plain segment (the small measuring magnitude).
- B: a longer segment subdivided by tick marks (per the text, A taken three times measures B, so $B = 3A$).
- C: a still longer segment subdivided by more tick marks (A taken six times measures C, so $C = 6A$).
- F: a long segment with tick marks and a dotted continuation at its right end.
- E: a segment with a tick mark followed by a short dotted stretch.
- D: a short plain segment (the magnitude that measures neither E nor F).

The figure illustrates that A is a part of B and C, while D is not a part of E and F.

And so he says that the smaller magnitude, which measures some larger magnitude, is called a part. For example, since magnitude A, taken three times, measures magnitude B, and taken six times measures magnitude C, magnitude A will be called a part of magnitudes B and C. But since magnitude D does not measure magnitudes E and F — rather, taken twice it exceeds magnitude E, taken

three times it falls short of magnitude F, and taken four times it exceeds that same F — magnitude D will not be called a part of magnitudes E and F.

Among mathematicians a part is of two kinds. One kind measures its whole, so that when repeated a certain number of times it makes up its whole; such is the number 4 set beside 8, 12, 16, 20, and so on. The other kind does not measure its whole, but when taken a certain number of times either exceeds it or falls short of it; of this kind is the number 4 placed beside 6, 7, 9, 10, 18, 38, and so on. The former is usually called an aliquot part, the latter an aliquant part. Here, then, Euclid defines only the aliquot part — both because this alone measures its whole (for an aliquant part is not said to measure its whole), and also because, as will be established from Book 7, an aliquant part is, in numbers, not called by Euclid a "part" but "parts." For the number 4 is not a part of the number 6, but two of its thirds — which are two twos. There is a further reason: in all the demonstrations of this fifth book, "part" is taken by every commentator to mean an aliquot part. It is therefore quite surprising that some interpreters of Euclid — among them Peletier [Jacques Peletier du Mans] — maintain that "part" is here defined so as to embrace every part, both aliquot and aliquant, when nonetheless in the demonstrations they themselves understand by the word "part" only the aliquot part.

II.

[Euclid, Definition 2] A multiple, on the other hand, is the larger of the smaller, when the smaller measures the larger.

[Clavius's commentary] As in the previous example, both magnitude B and magnitude C are a multiple of magnitude A, since A measures each of them. But neither magnitude E nor magnitude F is to be called a multiple of magnitude D, because D measures neither of them. Thus "part" is referred to "multiple," and "multiple" to "part," so that the smaller quantity that measures the larger is called a part of the larger, while the larger, measured by the smaller, is called a multiple of the smaller.

It is, moreover, quite clearly gathered from this definition that a part was defined earlier as that which measures its whole exactly. For if 6 were said to measure 7, as Peletier wishes, then by this definition 7 would be a multiple of 6 — which is absurd. Furthermore, when two smaller magnitudes measure two other larger ones equally — that is, when one smaller magnitude is contained in one larger as many times as the other smaller is contained in the other larger — these two

larger magnitudes are called equimultiples of those two smaller ones. You will say the same if several smaller magnitudes equally measure several larger ones.

III.

[Euclid, Definition 3] A ratio is a certain mutual relation, in respect of quantity, between two magnitudes of the same kind.

[Clavius's commentary] When two quantities of the same kind — such as two numbers, two lines, two surfaces, two solids, and so on — are compared with one another in respect of quantity, that is, in respect of one being greater than the other, or smaller, or equal, this kind of comparison, or mutual relation, is called a ratio, or (as others prefer) a proportion. And so, if some line is compared with some surface, or a number with a line, that comparison will not be called a ratio, because a line and a surface, and a number and a line, are not quantities of the same kind. Likewise, if some line is compared with another line in respect of quality — that is, in respect of one being white and the other black, or one being hot and the other cold, and so on — then, although both are of the same kind, that comparison will not be called a ratio, because it is not made in respect of quantity.

Although a ratio is properly found only among quantities, nevertheless all other things that in some way partake of the nature of quantity — such as times, sounds, voices, places, motions, weights, and powers — are also said to have a ratio, if their relation is considered in respect of quantity. Thus, when we say that one time is greater than another, or smaller, or that two times are equal, and so on, such a relation will be called a ratio, since the times are then considered as being, in a way, quantities.

Furthermore, in every ratio the quantity that is referred to another is called by Euclid, and by other geometers, the antecedent of the ratio; while that to which the other is referred is usually called the consequent of the ratio. Thus, in the ratio of a line of 6 palms to a line of 3 palms, the line of 6 palms is called the antecedent of the ratio, and the line of 3 palms the consequent of the ratio. But if, conversely, the ratio of a line of 3 palms to a line of 6 palms is considered, the line of 3 palms will be called the antecedent, and the line of 6 palms the consequent — and so in the rest.

IV.

[Euclid, Definition 4] Proportion, however, is a likeness of ratios.

[Clavius's commentary] What the translator here calls proportion is what the Greeks call ἀναλογία [Greek: analogia, "proportion"], and what most Latin writers call proportionality. Therefore, just as the comparison of two quantities with one another is called a ratio, so the comparison of two or more ratios with one another is usually named a proportionality. For example, if the ratio of quantity A to quantity B is similar to the ratio of quantity C to quantity D, the relation between these ratios will be called a proportionality. In the same way, if the ratio of E to F is similar to the ratio of F to G, this likeness will be called a proportionality. Now many relations of ratios, or proportionalities (for, when dealing with several, we shall call the comparison of two quantities a "ratio," but the relation of ratios a "proportionality"), are described by writers — especially by Boethius [Boethius] and Jordanus [Jordanus de Nemore]. Among these, the first place has always been held, among the ancients, by the Arithmetic, Geometric, and Musical (or Harmonic) proportionality, of which we shall speak a little later. But Euclid deals only with the Geometric in this book.

Diagram. Two groups of vertical straight-line magnitudes illustrating discrete and continuous proportion, each line labelled with a letter beneath and a number above.

- Left group (discrete proportion, $A : B :: C : D$): $A = 12$, $B = 9$, $C = 4$, $D = 3$ (that is, $12 : 9 :: 4 : 3$, each ratio being $4 : 3$).
- Right group (continuous proportion, $E : F :: F : G$): $E = 16$, $F = 8$, $G = 4$ (that is, $16 : 8 :: 8 : 4$, each ratio being $2 : 1$, with F serving as the mean taken twice).

This Geometric proportionality is indeed of two kinds. One is continuous, in which each of the intermediate quantities is taken twice, so that no interruption of the ratios occurs, but each intermediate quantity is both an antecedent and a consequent — antecedent of the following quantity, consequent of the preceding one. For example, if it is said that as E is to F, so is F to G, this proportion will be called continuous. The other is discrete, or non-continuous, in which each of the intermediate quantities is taken only once, so that an interruption of the ratios does occur, and no quantity is both antecedent and consequent, but either antecedent only or consequent only. For example, if it is said that as A is to B, so is C to D, this proportionality will be called discrete, or non-continuous.

ON THE DIVISIONS OF PROPORTION

I judge it worthwhile to set out briefly here how many kinds of ratio there are among mathematicians, and which are the principal proportionalities and their properties — chiefly for this practical benefit: that what Euclid demonstrates in

these two books about the ratios of magnitudes may be applied to material things whenever needed, and that both what mathematicians say about ratios and what philosophers, following Aristotle [Aristotle], dispute about the ratio of motions may be understood. And so that greater usefulness and pleasure may be gained from the wonderful properties of ratios, we shall undertake a somewhat fuller treatment in this later edition than in the earlier one — while nevertheless deferring almost countless things that might be said to our fuller *Arithmetic*, where we shall pursue all the kinds of proportionalities most diligently.

The proportion defined by Euclid, then, is divided into rational and irrational. A rational proportion is one that can be exhibited in numbers, such as the ratio of a line of 20 palms to a line of 10 palms; for this ratio is displayed in these very numbers, 20 and 10. An irrational proportion is one that cannot be exhibited in numbers, such as the ratio of the diagonal [diameter] of any square to the side of that same square; for this ratio cannot be found in numbers, as is demonstrated by Euclid in Book 10. Others say that a rational proportion is the one that any two commensurable quantities have, and an irrational proportion the one that any two incommensurable quantities have. Now quantities are said to be commensurable

when they have one common aliquot part, or when the same common measure measures them. Of this kind are a line of 20 palms and a line of 8 palms; for a line of 4 palms is an aliquot part of each, and likewise a line of 2 palms. For just as a line of both 4 and 2 palms measures the line of 20 palms, so too the same line, whether of 4 or of 2 palms, measures the line of 8 palms. Not otherwise, all numbers will be called commensurable, because at least unity measures them all. Quantities, on the other hand, are called incommensurable when they have no common aliquot part, or when no common measure can be found for them. Of this kind are the diagonal and the side of the same square. For although each of these lines has infinitely many aliquot parts — namely a half, a third, a fourth, and so on — nevertheless no aliquot part of the one, however small, can measure the other, as Euclid demonstrates in Book 10, the last proposition. In that book many other incommensurable lines besides those two are exhibited. And so, in numbers only a rational proportion is found; but in continuous quantity both a rational and an irrational proportion are contained.

In another way, proportion is usually divided into a proportion of equality and a proportion of inequality. A proportion of equality is between two equal quantities, as between 20 and 20, between 100 and 100, between a line of 10 palms and a line of 10 palms, and so on. A proportion of inequality, by contrast, is

found between two unequal quantities, as between 20 and 10, between 8 and 40, between a line of 6 palms and a line of 2 palms, and so on. Now these two kinds of proportion have this connection with the two above: every proportion of equality is necessarily rational, but not conversely; likewise every irrational proportion is necessarily a proportion of inequality, but not conversely. From this it is clear that some people divide rational proportion less correctly into a proportion of equality and a proportion of inequality. For although every rational proportion is necessarily one of equality or of inequality, still not every proportion of this kind is conversely rational, since many proportions of inequality are irrational. By the same reasoning it is evident that some do not quite correctly distribute the proportion of inequality into a rational and an irrational proportion. For although every proportion of inequality is necessarily rational or irrational, still not every such proportion is conversely a proportion of inequality, since many rational proportions are proportions of equality. In my judgment, therefore, proportion in general is better cut by a twofold division: first into rational and irrational, and second into a proportion of equality and a proportion of inequality. For in this way the dividing members reciprocate with the thing divided (to speak as the logicians do) in each division. I know that a rational proportion can rightly be subdivided into a proportion of equality and of inequality, and likewise a proportion of inequality into a rational and an irrational proportion, provided that in each division the thing divided is understood. But why I have preferred our twofold division — by which proportion is divided in its whole extent — to each of these subdivisions, I have explained above in my reply to Peletier's *Apology*.

Again, the proportion of inequality (for we set aside the proportion of equality, since it cannot be further subdivided, because equal quantities, whether great or small, always have the same proportion of equality) is subdivided into a proportion of greater inequality and one of lesser inequality. A proportion of greater inequality is when a greater quantity is compared with a smaller; such is the proportion of 20 to 10, or of a line of 8 feet to a line of 6 feet, and so on. A proportion of lesser inequality is when a smaller quantity is referred to a greater; such is the proportion of 10 to 20, or of a line of 6 feet to a line of 8 feet, and so on. This division is not empty and superfluous, as many suspect. For the proportion of 4 to 2 is not the same as that of 2 to 4; rather, they differ greatly, since the use of each is very different, as is clear to those who are even moderately versed in geometry and the rule of algebra. These, then, are the general divisions of proportion, insofar as it embraces all proportions, none excluded. Now, however, we shall subdivide both the proportion of greater inequality and that of lesser inequality only insofar as they comprise rational proportions alone; for

concerning quantities that have irrational proportions there will be a discussion in Book 10.

A rational proportion of greater inequality, therefore, is distributed into five kinds: multiple, superparticular, superpartient, multiple-superparticular, and multiple-superpartient. By the same reasoning, a rational proportion of lesser inequality is divided into the same kinds, provided only that to each term the prefix "sub" is prefixed: thus, submultiple, subsuperparticular, subsuperpartient, submultiple-superparticular, and submultiple-superpartient. Of these five kinds, the first three are simple, while the latter two are composed of those three, as is evident. But why there are only these five kinds of rational proportion, both of greater and of lesser inequality, we shall show after the explanation of all these five proportions.

ON MULTIPLE PROPORTION

A multiple proportion is a relation of a greater quantity to a smaller, when the greater contains the smaller a certain number of times — as twice, three times, ten times, a hundred times, and so on — so that the smaller measures the greater; such is the proportion

(This page continues Clavius's scholium "On multiple proportion" [De proportionem multiplici], begun on the preceding page; catchword from the prior page: "proportio.")

the ratio of the number 20 to 4. For 20 contains 4 five times over. Likewise the ratio of a line of 30 feet to a line of 5 feet, and so on.

This kind of ratio contains under itself infinitely many species. For if the greater quantity of a multiple ratio contains the lesser exactly twice, the ratio is called double [dupla]; if three times, triple [tripla]; if ten times, tenfold [decupla]; if a hundred times, hundredfold [centupla], and so on.

From these it will be easy to define all the species of multiple ratio. For an eightfold ratio [proportio octupla] will be nothing other than the relation of a greater quantity to a lesser, when the greater embraces the lesser eight times. In the same way the remaining multiple ratios are to be defined: thus a fivefold ratio [proportio quintupla], such as that of 40 to 8, will be one whose greater quantity contains the lesser five times. Likewise a double ratio, such as that of a line of 10 cubits to a line of 5 cubits, is one in which the greater quantity contains the lesser twice; and so on for the rest.

Furthermore, you will find all multiple ratios in whole numbers — that is, all the numbers that stand in a multiple ratio — in this way. Take the number that indicates how many times the greater number contains the lesser in the given multiple ratio. This number stands to unity in the first multiple ratio of the kind sought: that is, that number and unity are the first and least numbers between which the proposed multiple ratio is found. Now that same number, together with unity, are called by the arithmeticians the *terms* of the multiple ratio in question. Once these terms have been found — the terms between which the first proposed multiple ratio is found — if each is doubled, two numbers will be produced that stand in the second proposed multiple ratio; if each is likewise tripled, numbers of the third proposed multiple ratio will be generated; if each is quadrupled, the numbers of the fourth sought multiple ratio will arise. And so, if the same terms are multiplied by any number whatever, two numbers of the proposed multiple ratio will be produced, holding that place among all the multiple ratios of the species which the multiplying number indicates. Thus if the multiplication is carried out by 100, two numbers will be generated that occupy the hundredth place among all the numbers standing in the proposed ratio.

For example: let all the numbers of the fivefold ratio be sought. Since the greater number contains the lesser five times in a fivefold ratio, the first fivefold ratio will be between 5 and 1; and if these two terms are multiplied by 2, the second fivefold ratio will be found between 10 and 2. If the same terms are multiplied by 3, the numbers 15 and 3 will be produced, of the third fivefold ratio; and so on in succession. Thus if the tenth fivefold ratio is sought, the two first terms found, 5 and 1, are to be multiplied by 10; but if the thousandth is to be found, the same terms are to be multiplied by 1000, and so on.

Likewise all the numbers of any multiple ratio will be found in this way. Let two series of numbers be set out, extending to infinity: let the lower begin from unity and proceed by the natural sequence of numbers; let the upper begin from the number that signifies how many times the greater number contains the lesser in the given ratio, and let it advance by continually adding the same number, first to itself, then to the sum, and so on. For the numbers of the upper order, taken against the numbers of the lower order — the first to the first, the second to the second, the third to the third, and so on — all stand in the multiple ratio of the proposed species. You have examples here in the fivefold, sevenfold, and tenfold ratios.

Quintuple proportions [Proportiones quintuplæ]

Order

Upper	5	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	&c.
		0	5	0	5	0	5	0	5	0	5	0	5	0	5	0	
Lower	1	2	3	4	5	6	7	8	9	1	1	1	1	1	1	1	&c.
										0	1	2	3	4	5	6	

Septuple proportions [Proportiones septuplæ]

Order

Upper	7	1	2	2	3	4	4	5	6	7	7	8	9	9	105	&c.
		4	1	8	5	2	9	6	3	0	7	4	1	8		
Lower	1	2	3	4	5	6	7	8	9	1	1	1	1	1	15	&c.
										0	1	2	3	4		

Decuple proportions [Proportiones decuplæ]

Order

Upper	1	2	3	4	5	6	7	8	9	100	110	120	130	140	&c.
	0	0	0	0	0	0	0	0	0						
Lower	1	2	3	4	5	6	7	8	9	10	11	12	13	14	&c.

ON PROPORTION [DE PROPORTIONE — superparticular]

A superparticular ratio [proportio superparticularis] is the relation of a greater quantity to a lesser, when the greater contains the lesser exactly once and, in addition, one aliquot part of it — namely a half, a third, a fourth, and so on. Such is the ratio of 3 to 2. For 3 contains 2 once, and in addition a unit, which is the half part of the number 2. So too a line of 12 feet to a line of 9 feet has a superparticular ratio, because the former line contains the latter once, and in addition a line of 3 feet, which is the third part of the line of 9 feet, and so on.

This ratio too is divided into infinitely many kinds. For if that aliquot part contained in the greater quantity is the half part of the lesser quantity, the ratio *sesquialter* [sesquialtera] is constituted; if it is the third part, the ratio *sesquitertian* [sesquitertia] arises; if the fourth, *sesquiquartan* [sesquiquarta]; if the thousandth, *sesquimillesimal* [sesquimillesima]; and so on.

Hence, from the very word itself, the definitions of all superparticular ratios will be easy. For a ratio will be *sesquioctave* [sesquioctava] when the greater quantity includes the lesser once and, in addition, the eighth part of the lesser: such as the ratio between 9 and 8. Likewise between 45 and 40. Judge the rest in the same way.

All superparticular ratios — or all the numbers between which any superparticular ratio whatever is found — will be discovered in this way. Take the number that names the aliquot part expressed in the ratio. The number next above it — namely the one that exceeds it by a single unit alone — will have with it the first proposed superparticular ratio, such that in smaller numbers that ratio cannot be found. These two numbers, which are called the *terms* of the given ratio, if they are doubled, tripled, or multiplied by any other number whatever, will generate other numbers standing in the same ratio — namely the second, the third, or that which the multiplying number indicates.

For example: if all the *sesquiseptimal* [sesquiseptimæ] ratios are sought — since here the seventh part is expressed — 7 will be the lesser term of this ratio, and the greater will be 8, exceeding it by a single unit; and between 8 and 7 the first sesquiseptimal ratio exists. If these two terms are doubled, two other numbers, 16 and 14, are produced, between which the second sesquiseptimal ratio is discerned; if they are tripled, the third such ratio will be between 24 and 21. Finally, if the hundredth such ratio is sought, the terms found, 8 and 7, are to be multiplied by 100, so that the numbers 800 and 700 are generated.

Likewise all the numbers of any superparticular ratio will be found, if two orders of numbers are set out: the lower beginning from the number that names the aliquot part, the upper from the number next above it; and each advancing by the continual addition of the number from which it takes its start, first to itself, then to the sum, and so on. We have set out examples here in the sesquialter, sesquiseptimal, and sesquidecimal ratios.

Sesquialter proportions [Proportiones sesquialteræ]

Order

Upper	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
Lower	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30

Sesquiseptimal proportions [Proportiones sesquiseptimæ]

Order

Upper	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120
Lower	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105

Sesquidecimal proportions [Proportiones sesquidecimæ]

Order

Upper	1	2	3	4	5	6	7	8	9	110	121	132	143	154	&c.
	1	2	3	4	5	6	7	8	9						
Lower	1	2	3	4	5	6	7	8	9	100	110	120	130	140	&c.
	0	0	0	0	0	0	0	0	0						

ON PROPORTION

Superpartient [Superpartiente]

A superpartient ratio [proportio superpartiens] is the relation of a greater quantity to a lesser, when the greater contains the lesser exactly once and, in addition, several aliquot parts of it that do not make up a single aliquot part. Such is the ratio of 8 to 5. For 8 contains 5 once, and in addition three units, each of which is an aliquot part — namely a fifth — of this number 5; but the number three composed of them is not any aliquot part of 5. I said that those aliquot parts must not constitute a single aliquot part, on account of the many ratios that at first sight seem to be superpartient when in fact they are superparticular. Of this kind is the ratio between 10 and 8. For although 10 contains 8 once, and in addition two units, each of which is an eighth part of the number 8 — yet, because the number two composed of those units is a fourth part of 8 — that ratio is not to be called superpartient but superparticular, namely *sesquiquartan*. Therefore, for two quantities to be said to have a superpartient ratio, it is necessary that the greater quantity contain the lesser once, together with several aliquot parts of it which, taken together, do not constitute a single aliquot part. Some, failing to notice this, confound the kinds of ratios in a remarkable way.

This superpartient ratio is divided, first, with regard to the number of the aliquot parts, into infinitely many kinds. For if the greater quantity comprehends the lesser once, and is completed by two of its aliquot parts that do not constitute a single one, the ratio is *superbipartient* [superbipartiens]; if by three aliquot parts, *supertripartient* [supertripartiens]; if by ten, *superdecupartient* [superdecupartiens]; and so on.

It is divided, next, with regard to the denomination of the aliquot parts, into yet infinitely many kinds. For a superbipartient ratio between two unequal quantities, of which the greater contains the lesser once and two of its third parts, is called *superbipartient of thirds* [superbipartiens tertias]. But if those two parts are fifths, it will be called *superbipartient of fifths* [superbipartiens quintas]; and likewise for the remaining superbipartient ratios. By the same reasoning, a superdecupartient ratio between two unequal quantities, of which the greater exceeds the lesser by ten eleventh-parts, will be called *superdecupartient of elevenths* [superdecupartiens undecimas]. But if those ten parts are thirteenths, the ratio will be called *superdecupartient of thirteenths* [superdecupartiens decimastertias]; and so on for all the remaining superdecupartient ratios.

Now, lest superpartient ratios be confounded either among themselves or with superparticular ratios — which we have found to have been done by very many — the following points must be carefully considered. First, that in the naming of any superpartient ratio whatever, two numbers are indicated: one of which shows how many aliquot parts of the lesser quantity are left over in the greater; the other indicates what kind those parts are, or how large. Thus in the *supertripartient of eighths* [supertripartiente octauas] ratio, these two numbers, 3 and 8, are denoted; the former of which signifies that the greater quantity of the said ratio contains the lesser once and, further, three of its aliquot parts — this is expressed by that syllable (tri) when it is called *supertripartient*; while the latter, expressed by the word (of eighths), shows that those three aliquot parts are eighth parts of the lesser. Next, that in every superpartient ratio the two aforesaid numbers (which are in fact easily recognized from the very utterance of the ratio, as is clear from the first example) must be such as to have no aliquot part in common except unity — which is indeed an aliquot part of all numbers: that is, they must be prime to one another. For numbers which, besides unity,

admit no other common aliquot part, the arithmeticians, together with [Euclid], call *prime to one another*, as will be established from Book 7. Such are those two numbers 3 and 8, expressed in the superior *supertripartient of eighths* ratio. For unity alone, as is evident, is the common aliquot part of both. Therefore we shall rightly name the ratio between 11 and 8 *supertripartient of eighths* — such as is also the ratio between 22 and 16. But the latter ratio, between 22 and 16, will not rightly be called *supersextupartient of sixteenths* [supersextupartiens sextasdecimas], although the greater contains the lesser once and, in addition, six units, each of which is a sixteenth part of the lesser: it will not, I say, be rightly so called, because the two numbers 6 and 16 expressed in it have the aliquot part 2, by which — as is taught in arithmetic — $6/16$ is reduced to $3/8$; and thus that ratio

is to be called *supertripartient of eighths*. So too the ratio between 9 and 6 will not rightly be called *supertripartient of sixths*, since the two numbers noted in it, 3 and 6, have, besides unity, another common measure, namely 3. For the number three, taken once as itself and then repeated twice, measures the number six; and consequently $3/6$ will be reduced, by the common aliquot part 3, to $1/2$.

Wherefore such a ratio is to be named *sesquialter*, since the greater quantity contains the lesser once and its half part. By the same reasoning, the ratio between 10 and 6 will not rightly be called *superquadripartient of sixths*, because the two numbers noted in it, 4 and 6, have 2 as a common aliquot part besides unity; and thus such a ratio is to be called *superbipartient of thirds*, since the greater quantity contains the lesser once and two of its third parts. From these considerations, then, it will not be difficult for anyone to name conveniently all the superpartient ratios.

It is also clear from what has been said why we divided the superbipartient ratio, a little earlier, into *superbipartient of thirds*, *of fifths*, *of sevenths*, *of ninths*, and so on, passing over *superbipartient of fourths*, *of sixths*, *of eighths*, *of tenths*, and so on. For since these latter, which were omitted, are superparticular — because $2/4$ make $1/2$, $2/6$ constitute $1/3$, $2/8$ yield $1/4$, and finally $2/10$ are equivalent to $1/5$ — the superpartient ratios would be confounded with the superparticular ratios, if these too were reckoned among the number of the superbipartient ratios. But how it is to be recognized whether any two given numbers have, besides unity, some other common aliquot part or not, is taught in arithmetic and will be demonstrated by [Euclid] at the beginning of Book 7.

The finding of all superpartient ratios of any species stands thus. A number as many units greater than the denominator of the aliquot parts named in the ratio as there are parts expressed in the same ratio will have the first ratio of the proposed species to the number denominating those same aliquot parts. These two terms, doubled, tripled, or multiplied by any number whatever, will generate second numbers standing in the same ratio, third numbers, or others, as was said above. For example, if all the numbers of the *supertripartient-of-tenths* ratio are to be sought, the first numbers of this kind will be 13 and 10, because the greater exceeds the lesser — which denominates the tenth parts — by three units, that is, by as many as there are tenth parts mentioned in the supertripartient-of-tenths ratio. Their doubles, which of course constitute the second ratio, are 26 and 20; their triples, 39 and 30. And the hundredth such ratio will be 1300 to 1000, since these numbers are the hundredfold of the terms 13 and 10, which were found in the first place.

You will find these same numbers of all superpartient ratios if you set out two orders of numbers, of which the lower begins from the denominator of the aliquot parts, and the upper from the number that exceeds that former one by the number of the parts named; each order then advancing by the continual addition of the first number of its order to itself and to the sums, as we said above. We have here appended examples of the ratios superbipartient of fifths, superoctupartient of ninths, and superseptupartient of tenths.

Superbipartient-of-fifths proportions [Proportiones superbipartientes quintas]

Order															
Upper	7	14	21	28	35	42	49	56	63	70	77	84	91	98	&c.
Lower	5	10	15	20	25	30	35	40	45	50	55	60	65	70	&c.

Superoctupartient-of-ninths proportions [Proportiones superoctupartientes nonas]

Order													
Upper	17	34	51	68	85	102	119	136	153	170	187	&c.	
Lower	9	18	27	36	45	54	63	72	81	90	99	&c.	

Superseptupartient-of-tenths proportions [Proportiones superseptupartientes decimas]

Order												
Upper	17	34	51	68	85	102	119	136	153	170	187	&c.
Lower	10	20	30	40	50	60	70	80	90	100	110	&c.

ON MULTIPLE-SUPERPARTICULAR PROPORTION [DE PROPORTIONE MULTIPLICI SUPERPARTICULARI]

A multiple-superparticular ratio [proportio multiplex superparticularis] is the relation of a greater quantity to a lesser, when the greater contains the lesser several times — as twice, three times, or four times, and so on — and besides one aliquot part of it. Of this kind is the ratio of 9 to 4. For 9 contains 4 twice (in which respect this ratio agrees with a multiple ratio, being like the double), and in addition they comprehend a unit, which is the fourth part of the lesser number

(in which respect the same proposed ratio is like a superparticular ratio, namely the sesquiquartan). Thus this composite ratio is rightly said to be composed of a multiple and a superparticular.

This ratio, moreover, is divided — with regard to the multiple ratio — into infinitely many kinds, just like the multiple. Thus into *double-superparticular* [dupla superparticularis], *triple-superparticular* [tripla superparticularis], and so on, according as the greater quantity comprehends the lesser twice, or three times, or four times, and so on, together with one aliquot part of the lesser quantity.

Euclid, *Elements*, Book V — Clavius's commentary (continued)

Each one of these kinds is in turn subdivided into infinitely many others, according to the ratio of the superparticular proportion it involves. Take a triple-superparticular proportion, for example: it contains under itself the triple-sesquialterate — that is, the case in which the greater quantity contains the smaller three times and in addition half of it — the triple-sesquitertian, the triple-sesquiquartan, and so on without end.

We shall find all the multiple-superparticular proportions of any species if we attend carefully to the denominator of the aliquot part and the denominator of the multiple proportion mentioned in the proposed proportion. For if we multiply the denominator of the aliquot part by the denominator of the multiple proportion and add one to the product, then that resulting number, set against the denominator of the aliquot part, gives the first proportion of the proposed species. And these two numbers, doubled, tripled, or multiplied by any number whatever, will yield further numbers in the same proportion — the second, the third, or those of any other rank — according to the number of units contained in the multiplier, as has been explained elsewhere.

For example, suppose we are to find all the numbers of the sextuple-sesquinonan proportion $[6 + 1/9]$. We multiply 9, the denominator of the ninth part, by 6, the denominator of the multiple proportion, and to the product, 54, we add 1. The resulting number, 55, set against 9, the denominator of the ninth part, expresses the first sextuple-sesquinonan proportion, in such a way that it cannot be given in smaller whole numbers. The doubles of these numbers, 110 and 18, are the second pair in the same proportion; the triples, 165 and 27, are the third, and so

on — so that their hundredfold values, 5500 and 900, exhibit the hundredth instance of the same proportion.

You will find these same multiple-superparticular proportions if you set up two series of numbers. The lower series begins from the denominator of the named aliquot part; the upper series begins from the number formed by adding one to the product of that same denominator of the aliquot part multiplied into the denominator of the stated multiple proportion. Each series then advances by continually adding the first number of its own series to itself and to the formed number, and so on, as was described above. Here you see worked examples for the double-sesquialterate $[2 + 1/2]$, the triple-sesquiseptima $[3 + 1/7]$, and the decuple-sesquitertian $[10 + 1/3]$ proportions.

Double-sesquialterate proportions

5	10	15	20	25	30	35	40	45	50	55	60	65	70	&c.
2	4	6	8	10	12	14	16	18	20	22	24	26	28	&c.

Triple-sesquiseptima proportions

22	44	66	88	110	132	154	176	198	220	242	&c.
7	14	21	28	35	42	49	56	63	70	77	&c.

Decuple-sesquitertian proportions

31	62	93	124	155	186	217	248	279	310	341	&c.
3	6	9	12	15	18	21	24	27	30	33	&c.

On the multiple-superpartient proportion

[*DE PROPORTIONE MVLTIPlici SVPERPARTIENTE.*]

Finally, a multiple-superpartient proportion is the relation of a greater quantity to a smaller one when the greater contains the smaller several times over and, in addition, several of its aliquot parts that do not add up to a whole — such as the proportion of 11 to 3. I said "that do not add up to a whole" for the reason already stated in the case of the superpartient proportion. For if those aliquot parts did add up to a whole, the proportion would not be multiple-superpartient but

multiple-superparticular. Thus the proportion of 20 to 6 is not to be called triple-superbipartient-sixths, even though 20 contains 6 three times plus two sixths, because two sixths make up one third part. It will instead be called the triple-sesquitercian proportion.

This proportion is subdivided, first, according to the multiple proportion — the multiple falling into double-superpartient, triple-superpartient, and so on.

Next, each of these, according to the number of parts, contains infinite kinds under itself. Thus under the triple-superpartient are contained the triple-superbipartient, the triple-supertripartient, and so on.

Finally, each of these in turn, according to the denomination of the aliquot parts, is cut into still further infinite kinds. Thus the triple-supertripartient is divided into triple-supertripartient-fourths, triple-supertripartient-fifths, and so on. It is not difficult to draw out the definitions and examples of all these from what has been said.

We shall find all the multiple-superpartient proportions of this last division as follows. Multiply the denominator of the proposed aliquot parts by the denominator of the proposed multiple proportion, and to the product add the stated number of aliquot parts. For the resulting number, set against the denominator of those same aliquot parts, has the first of the proportions being sought. And these two numbers, if doubled, tripled, or multiplied by any other number whatever, will give the second numbers, the third, and others in the same proportion. For instance, suppose all the quadruple-superoctupartient-elevenths proportions are sought. We multiply 11, the denominator of the eleventh parts, by 4, the denominator of the quadruple proportion, and to the product, 44, we add 8, the number of the eight parts. The resulting number, 52, set against 11, the denominator of the parts, has the first proportion sought. The second numbers will be 104 and 22; the third, 156 and 33 — namely the doubles, triples, and so on of the first pair. And their hundredfold values, such as 5200 and 1100, have the hundredth instance among all the quadruple-superoctupartient-elevenths proportions.

But if two orders of numbers are set up — the lower beginning from the denominator of the proposed aliquot parts and advancing by continually adding that same denominator; the upper beginning from the number formed by adding, to the product of that same denominator of the parts multiplied into the denominator of the given multiple proportion, the stated number of parts, and then advancing by continually adding that same number, first to itself, then to the

formed number, and so on — we shall likewise obtain all the numbers of the given proportion, as we also said above. Here you see worked examples for the double-superbipartient-thirds, the double-superquintupartient-sixths, and the quintuple-supertripartient-fifths proportions.

Double-superbipartient-thirds proportions [2 + 2/3]

8	16	24	32	40	48	56	64	72	80	88	96	104	&c.
3	6	9	12	15	18	21	24	27	30	33	36	39	&c.

Double-superquintupartient-sixths proportions [2 + 5/6]

17	34	51	68	85	102	119	136	153	170	187	&c.
6	12	18	24	30	36	42	48	54	60	66	&c.

Quintuple-supertripartient-fifths proportions [5 + 3/5]

28	56	84	112	140	168	196	224	252	280	308	&c.
5	10	15	20	25	30	35	40	45	50	55	&c.

On the rational proportions of inequality

[*DE PROPORTIONIBVS RATIONALIBVS INAEQVALITATIS.*]

Everything said so far about the five kinds of rational proportions of greater inequality must be understood equally of the five corresponding kinds of lesser inequality, provided the prefix "sub-" is always prefixed, as has been said. For if, in the examples given, the smaller quantities are compared with the greater, the corresponding proportions of lesser inequality result. Just as the proportion of 100 to 1 is centuple, so the proportion of 1 to 100 is subcentuple. And just as the proportion of 11 to 3 is triple-superbipartient-thirds, so the proportion of 3 to 11 is subtriple-superbipartient-thirds; and so with all the rest.

This is not the place, however, to pass over the remarkable usefulness of the Pythagorean table [tabula Pythagorica] — which we constructed in chapter 4 of our *Arithmetic* — for finding all the rational proportions; and that usefulness is as follows. Once the Pythagorean table has been constructed (and its construction is very easy, since the numbers down the left-hand side form the natural series of

numbers and can run on to infinity, while each crosswise line is generated by continually adding the number on the left-hand side, from which it begins, to itself and to the number so formed, and so on to infinity — so that these crosswise lines are nothing but arithmetic progressions of numbers whose common difference is the first number of each such line), then, first, all the numbers of any multiple proportion whatever are found in this way.

The denominator of a given multiple proportion, taken along the top line, together with the unit on the left-hand side, gives that first multiple proportion. The numbers descending under that same denominator in a straight line, matched one to one with the numbers under the unit on the left-hand side, give the second, third, fourth, and further proportions of the same species, to infinity. Thus the first nonuple proportion will be between the 9 of the top line and the 1 on the left-hand side; the second between the 18 below 9 and the 2 below 1; the third between the 27 of the same line descending from 9 and the 3 on the left-hand side, and so on. The same is to be said of the other multiple proportions.

Next, all the numbers of any superparticular proportion are found in this way. Against the denominator of the aliquot part expressed in the proportion, taken along the top side, the number immediately following gives the first proposed proportion. For the second, and by descending in a straight line from these two numbers, we shall find all the other numbers of the same proportion, as was said of the multiples. For example, the first sesquioctave proportion $[1 + 1/8]$ will be 9 to 8; the others, in order, will lie among the numbers placed beneath them — the second between 18 and 16, the third between 27 and 24, and so on.

Third, you will find all the superpartient proportions of any species in this way. After the denominator of the aliquot parts mentioned in the given proportion, taken along the top side, count off as many numbers as there are parts named in that proportion. For the last of these, set against the denominator of those same parts, will have the first proposed superpartient proportion. The second, third, fourth, and the others in order will be found among the numbers placed beneath those two. For example, if one wants all the numbers of the superoctupartient-elevenths proportion, let him take along the top side the denominator of the parts, namely 11, and after it count off these eight numbers — because of the eight parts — 12, 13, 14, 15, 16, 17, 18, 19. The last, 19, against 11, has the first superoctupartient-elevenths proportion. The number 38 against 22 gives the second, 57 against 33 the third, and the other numbers written beneath 19 and 11 in a straight line give the rest.

Tabula Pythagorica

The multiplication/proportion table. The printed table has no separate header row or column: its top edge (1–22, read left to right) and its left edge (1–14, read top to bottom) are themselves the multiplicands, and each interior cell is their product. Below, the top edge is given as the column header and the left edge as the leading $\times$ column; every interior value is transcribed directly from the leaf, cell by cell, at high magnification.

[illegible]

											1	1	1	1	1	1	1	1	1	1	1	2	2	2
×	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2		
1	1	2	3	4	5	6	7	8	9	1	1	1	1	1	1	1	1	1	2	2	2	2		
1	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9	0	2	3	4		
										0	1	2	3	4	5	6	7	8	9	0	1	2		
1	1	2	3	4	6	7	8	9	1	1	1	1	1	1	1	1	2	2	2	2	2	2		
2	2	4	6	8	0	2	4	6	0	2	3	4	5	6	8	9	0	1	2	4	5	6		
									8	0	2	4	6	8	0	2	4	6	8	0	2	4		
1	1	2	3	5	6	7	9	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2		
3	3	6	9	2	5	8	1	0	1	3	4	5	6	8	9	0	2	3	4	6	7	8		
								4	7	0	3	6	9	2	5	8	1	4	7	0	3	6		
1	1	2	4	5	7	8	9	1	1	1	1	1	1	1	2	2	2	2	2	2	2	3		
4	4	8	2	6	0	4	8	1	2	4	5	6	8	9	1	2	3	5	6	8	9	0		
								2	6	0	4	8	2	6	0	4	8	2	6	0	4	8		

The text resumes beneath the table:

Fourth, you will find all the numbers of any multiple-superparticular proportion by this method. Along the top side take

— the number that stands to the denominator of the aliquot part in exactly the multiple proportion in question. For the number immediately following it will have, to the denominator of the aliquot part, the first proposed proportion; and the numbers placed under those two will have the second and the rest. To find the number that stands to the denominator of the aliquot part in the proposed multiple proportion, take the denominator of the part along the top side and the denominator of the multiple proportion down the left side — or the former denominator down the left side and the latter along the top. In their common corner you will find the multiple number you seek.

For example: all the numbers of the triple-sesquiquinta proportion $[3 + 1/5]$ are to be found. First seek a number that is triple the denominator of the aliquot part — that is, triple 5 — namely 15, which you find in the common corner of the denominator of the triple proportion, which is 3, and the denominator of the aliquot part, which is 5, either of which is taken along the top and the other down the left. Then take this triple number, 15, along the top side. For the number 16, immediately following, set against the denominator of the aliquot part — that is, against 5 — has the first triple-sesquiquinta proportion. Other proportions of the same species you will find among the numbers placed beneath those two, as was said of the others.

Fifth and last, all the numbers of any multiple-superpartient proportion are found in this way. Along the top side take the number that stands to the denominator of the aliquot parts in the proposed multiple proportion — which will likewise be found in the common corner of the denominator of the multiple proportion and the denominator of the aliquot parts. Then, after the multiple number taken, count off as many numbers as there are aliquot parts mentioned. For the last of these, set against the denominator of the aliquot parts, will be found to have the first given multiple-superpartient proportion. Other proportions of the same species you will find beneath that first one, as we said of the others. For example, suppose one wants all the quadruple-supertripartient-fifths proportions. First let him seek the quadruple of the denominator of the parts, which is 5; he will find that number to be 20, since it is written in the common corner of the denominator of the quadruple proportion, which is 4, and the denominator of the fifth parts, which is 5. Then, after finding this number 20, let him count off along the top side these three numbers — 21, 22, 23 — because three fifth parts are named. For the last, 23, against 5 — that is, against the denominator of the parts — has the first quadruple-supertripartient-fifths proportion. The numbers of the other proportions of the same species are contained beneath those two numbers, as has been shown in the other cases.

Once, by this art, all the numbers of any proportion of greater inequality have been found, then, if the smaller are compared with the greater, all the numbers of the same proportion of lesser inequality will also have been found.

Nor should a remarkable property of every rational proportion be passed over in silence here — a property that runs as follows. Once the least, or first, terms and numbers of any proportion have been found, set up an arithmetic progression (an arithmetic proportionality) beginning from a number one unit less than the difference of those least numbers, and advancing by that very same difference of the least numbers. Then, among the first numbers of that progression, there fall as many numbers as there are units in the first least term of the proportion as taken in the arithmetic progression; among the second numbers — which are the doubles of the first — as many as the units contained in the second term of that progression; among the third, as many as the units in the third term; and so on in turn. For example, suppose the double proportion is proposed, whose least numbers are 2 and 1. Set up a progression beginning from 0 — because the difference between 2 and 1 is 1, and the figure 0 is one unit less than 1 — and advancing by the continual addition of one (that is, the difference between 2 and 1): first to 0, then to the formed number 1, then to the completed number 2, and so on, in this way:

0.
 1.
 2.
 3.
 4.
 5.
 6.
 7.
 8.

10.
 11.
 12.
 13.
 14.
 15.&c.

And so, between the first numbers of the double proportion — that is, between 2 and 1 — no middle number falls. Between the second, which are 4 and 2, one middle falls, namely 3. Between the third, 6 and 3, two middles are caught, 5 and 4. Between the fourth, 8 and 4, three middles occur, 7, 6, and 5; and so on. This appears in the figure below, where the extreme numbers of each line stand in all the double proportions — the first, the second, the third, and so on — while the counts of natural-series numbers falling between them correspond to the numbers of the arithmetic proportionality above.

The figure below reproduces the descending double-proportion array (each row runs from its left extreme, twice the right extreme, down to the right extreme; the intervening natural-series numbers are the "middles"). Transcribed from the leaf, cell by cell, at high magnification.

2	1				
4	3	2			
6	5	4	3		
8	7	6	5	4	
10	9	8	7	6	5

12	11	10	9	8	7	6					
14	13	12	11	10	9	8	7				
16	15	14	13	12	11	10	9	8			
18	17	16	15	14	13	12	11	10	9		
20	19	18	17	16	15	14	13	12	11	10	

But if upon the arithmetic proportionality just described you superpose the natural series of numbers beginning from 1, so arranged that each of its terms answers to each term of the arithmetic proportionality now set up, we shall learn faster than can be told how many numbers fall as middles between any two numbers of the double proportion. For the numbers of this natural series will indicate which

(Book V. Clavius's scholia on the arithmetic of ratios — continued from the preceding page. The Euclidean proportion-definitions do not yet begin here; this page continues the series of worked examples showing how many "mean numbers" fall between the terms of a given ratio, and the following pages give Clavius's scholium on the denominators of rational ratios.)

the place which two numbers of the double ratio occupy; while the corresponding numbers, arranged in an arithmetic proportionality, will indicate the multitude of numbers falling between them. Here is an example:

Rank of the double ratios [Ordo proportion. duplarum]											1	1	
											0	1	&c.
Multitude of mean numbers [Multitudo mediorum numerorum]											0	1	&c.

So if you wish to know how many numbers fall between the eleventh terms of the double ratio, take 11 [?] in the natural series of numbers, beneath which the number 10 is placed. You will therefore say that 10 numbers are intercepted between them; and so for the rest.

Next let a tenfold ratio [proportio decupla, 10:1] be proposed, whose least terms are 10 and 1. Their difference is 9; the number one unit less is 8. The arithmetic progression will therefore begin from 8, and will advance by the continual addition of 9 — first to 8, then to the resulting sum 17, and so on in succession, as may be seen here, where we have also written above the natural series of numbers, whose use here is the same as it was above in the double ratio.

Rank of the tenfold ratios [Ordo propor.
decuplarum]

	1	2	3	4	5	6	7	8	&c.
Multitude of mean numbers [Multit. med. numerorum]	8	1	2	3	4	5	6	7	&c.
		7	6	5	4	3	2	1	

Thus between the first terms of the tenfold ratio fall 8 mean numbers; 17 between the second; and 26 between the third. But between the eighth, 71 mean numbers are intercepted, and so for the rest — as this arrangement shows:

10. 9. 8. 7. 6. 5. 4. 3. 2. 1.
20. 19. 18. 17. 16. 15. 14. 13. 12. 11. 10. 9. 8. 7. 6. 5. 4. 3. 2.

Again, let any superparticular ratio [proportio superparticularis] be given. Since its two least numbers differ by a single unit — just as the least numbers of the double ratio do — the same progressions will serve here as in the double ratio, so that the number of mean numbers here is the same as there. This is clear here in the sesquioctave ratio [proportio sesquioctava, 9:8]:

Rank of the sesquioctave ratios [Ordo propor. sesquioct.]											1	1	
	1	2	3	4	5	6	7	8	9	0	1	1	&c.
Multitude of mean numbers [Multit. med. numerorum]	0	1	2	3	4	5	6	7	8	9	1	0	&c.
												0	

The accompanying staircase (each row runs from a multiple of 9 down to the corresponding multiple of 8):

9. 8.
18. 17. 16.
27. 26. 25. 24.
36. 35. 34. 33. 32.
45. 44. 43. 42. 41. 40.
54. 53. 52. 51. 50. 49. 48.
63. 62. 61. 60. 59. 58. 57. 56.

There is likewise offered the superquadripartient-sevenths ratio [proportio superquadripartiens septimas, 11:7], whose least terms are 11 and 7. Their difference is 4. The arithmetic progression will therefore begin from 3 and advance by 4, as appears here, together with the natural series:

Rank of the superquadripartient-sevenths ratios [Ordo propor. superquadr.]	1	2	3	4	5	6	7	8	&c.
Multitude of mean numbers [Multit. med. numerorum]	3	7	1	1	1	2	2	3	&c.
			1	5	9	3	7	1	

Therefore between the first terms of the superquadripartient-sevenths ratio 3 mean numbers are comprehended; 7 between the second; and 23 between the sixth, and so with the rest, as you see here:

11. 10. 9. 8. 7.
 22. 21. 20. 19. 18. 17. 16. 15. 14.
 33. 32. 31. 30. 29. 28. 27. 26. 25. 24. 23. 22. 21.
 44. 43. 42. 41. 40. 39. 38. 37. 36. 35. 34. 33. 32. 31. 30. 29. 28.

Likewise let there be given the double-and-one-sixth ratio [proportio dupla sesquisexta, 13:6], whose least numbers are 13 and 6. Their difference is 7. The progression will therefore begin from 6 [?] and will have a difference of 7 among its numbers, as is seen here, together with the natural series of numbers:

Rank of the double-and-one-sixth ratios [Ordo propor. dup. sesquisex.]	1	2	3	4	5	6	7	8	&c.
Multitude of mean numbers [Multit. med. numeratorum]	6	1 3	2 0	2 7	3 4	4 1	4 8	5 5	&c.

Between the first terms, then, of the double-and-one-sixth ratio fall 6 mean numbers; 13 between the second; and 55 between the eighth, and so on — just as is seen here:

13. 12. 11. 10. 9. 8. 7. 6.
 26. 25. 24. 23. 22. 21. 20. 19. 18. 17. 16. 15. 14. 13. 12.
 39. 38. 37. 36. 35. 34. 33. 32. 31. 30. 29. 28. 27. 26. 25. 24. 23. 22. 21.
 20. 19. 18.

Finally, let it be proposed to test the same thing in the double-and-two-thirds ratio [proportio dupla superbipartiens tertias, 8:3], whose least terms are 8 and 3. Since their difference is 5, the arithmetic progression will take its start from 4 and its numbers will have a difference of 5, as is evident here, together with the natural series of numbers:

Rank of the double-and-two-thirds ratios [Ordo propor. duplarum superbip. tertias]	1	2	3	4	5	6	&c.
Multitude of mean numbers [Multit. med. numeratorum]	4	9	1 4	1 9	2 4	2 9	&c.

Between the first two terms of the double-and-two-thirds ratio, then, will fall 4 mean numbers; and between the seventh, 34, and so on, as is clear here.

The staircase for the double-and-two-thirds ratio, continued from the foot of the previous page (each row runs from a multiple of 8 down to the corresponding multiple of 3):

8. 7. 6. 5. 4. 3.
 16. 15. 14. 13. 12. 11. 10. 9. 8. 7. 6.
 24. 23. 22. 21. 20. 19. 18. 17. 16. 15. 14. 13. 12. 11. 10. 9.
 32. 31. 30. 29. 28. 27. 26. 25. 24. 23. 22. 21. 20. 19. 18. 17. 16. 15. 14.
 13. 12.

It is evident, moreover, that what we have said here about the ratios of greater inequality [proportiones maioris inaequalitatis] is likewise true of the ratios of lesser inequality [proportiones minoris inaequalitatis], since the same numbers are interposed between the greater number and the lesser as between the lesser and the greater.

For the rest, from what we have said about the five species of rational ratio [proportio rationalis], both of greater and of lesser inequality, it is plainly gathered that there can be no more kinds of rational ratio of greater inequality than the five already set forth. For since — as Euclid demonstrates (*Elements* X.5) — any commensurable quantities whatever, between which (as we said) there is a rational ratio, have to one another the ratio that a number has to a number, it follows that every rational ratio of any continuous quantities can be assigned or exhibited in numbers. Either, then, the greater number contains the lesser (to which it is referred) an exact number of times, and by this a multiple ratio [proportio multiplex] is constituted; or it contains it once only, and besides one aliquot part of it, and so we have a superparticular ratio [proportio superparticularis]; or once only, and in addition several aliquot parts of it not making up a whole, and a superpartient ratio [proportio superpartiens] is produced; or several times, and one aliquot part of it, and a multiple-superparticular ratio [proportio multiplex superparticularis] results; or finally several times, and several aliquot parts of it not making up a whole, and a multiple-superpartient ratio [proportio multiplex superpartiens] arises. In no other way can a greater quantity contain a lesser, or a lesser be contained by a greater. By the same reasoning it is established that there are just as many kinds of rational ratio of lesser inequality. Rightly, therefore, have we divided the rational ratio — both of greater and of lesser inequality — into the five kinds we have set forth.

On the Denominators of Rational Ratios [De proportionum rationalium denominatoribus]

Since the denominators [denominatores] of the rational ratios we have expounded so far are of no small use, it will not be out of place to explain briefly by what numbers the individual ratios are denominated. The denominator of any ratio, then, is the number that distinctly and openly expresses the relation of one quantity to another. Thus the denominator of the octuple ratio [proportio octupla, 8:1] is 8; for this number indicates that the greater quantity of the octuple ratio contains the lesser eight times. Likewise the denominator of the sesquiquintan ratio [proportio sesquiquinta, 6:5] is $1\frac{1}{5}$, since this number signifies that the greater quantity of the sesquiquintan ratio contains the lesser once, plus a fifth part of it. And the same must be said of the denominators of the remaining ratios. From this it has come about, I believe, that Euclid in Book 6 [*Elements* VI], and most other mathematicians, call the denominator of any ratio its "quantity" [quantitas]; because the denominator, as we said, shows how great one quantity is in relation to the other with which it is compared, as is clear from the examples given.

From what we have said, moreover, the denominator of each ratio can easily be gathered. For the denominator of a multiple ratio, whatever it may be, is a whole number containing as many units as the number of times the greater quantity is said to contain the lesser in the ratio whose denominator is sought. Thus the denominator of the double ratio is 2; of the ninefold [noncupla], 9; of the hundredfold [centupla], 100; of the thousandfold [millecupla], 1000, and so on. But the denominators of the submultiple ratios that correspond to the multiples are the aliquot parts denominated by the denominators of the multiple ratios to which they answer. Thus the denominator of the subdouble ratio [subdupla] is $\frac{1}{2}$; of the subquintuple [subquintupla], $\frac{1}{5}$; of the subninefold [subnoncupla], $\frac{1}{9}$; of the subhundredfold [subcentupla], $\frac{1}{100}$; of the subthousandfold [submillecupla], $\frac{1}{1000}$. In the same way we shall find the denominators of the other submultiple ratios. Hence the denominator of any submultiple ratio is a fraction whose numerator is always unity, and whose denominator is the number that denominates the corresponding multiple ratio, as is clear from the examples adduced. Nor is there any difficulty at all in finding the denominator of any multiple or submultiple ratio, if what has been said is rightly understood — since the very statement of the ratio offers its denominator, as is manifest from the examples given.

The denominator of any superparticular ratio [proportio superparticularis] is a unit together with that aliquot part which the greater quantity must contain beyond the lesser. Thus the denominator of the sesquialteran ratio [sesquialtera, 3:2] is $1\frac{1}{2}$; of the sesquioctave [sesquioctava, 9:8], $1\frac{1}{8}$; of the sesqui-thousandth [sesquimillesima, 1001:1000], $1\frac{1}{1000}$, and so on. Nor will it be difficult to find the denominator of any superparticular ratio, since the very statement of the ratio expresses the denominator by means of its aliquot part, as is clear from the examples given. But the denominators of the corresponding subsuperparticular ratios are fractions whose numerators are less by just one unit than their denominators. Thus the denominator of the subsesquialteran ratio [subsesquialtera] is $\frac{2}{3}$; of the subsesquioctave [subsesquioctava], $\frac{8}{9}$; of the subsesqui-thousandth [subsesquimillesima], $\frac{1000}{1001}$, and so on. And the denominator of any subsuperparticular ratio will be found if, for the numerator of the fraction, one takes the denominator of the aliquot part expressed in the ratio, and for —

— the denominator of the same fraction, the number greater by a unit. Thus the denominator of the sub-sesqui-tenth ratio [subsesquidecima] is $\frac{10}{11}$; for the numerator of this fraction is the number that denominates the tenth part, namely 10, while the denominator of the same fraction exceeds its numerator by a unit, and so on. We shall also find the denominator of any subsuperparticular ratio in this way: we reduce the denominator of the corresponding superparticular ratio to a single fraction, as we taught in the *Arithmetic* — a fraction whose numerator here always exceeds the denominator by one unit, this denominator being also the denominator of the aliquot part mentioned in the ratio proposed. For if we invert the terms of this fraction, so that the numerator becomes the denominator and the denominator the numerator, we shall have the denominator of the proposed subsuperparticular ratio. For example, suppose the subsesquiseptan ratio [subsesquiseptima] is offered: since the denominator of the sesquiseptan ratio [sesquiseptima, 8:7] answering to it is $1\frac{1}{7}$, which reduces to the fraction $\frac{8}{7}$ — whose numerator is greater by one unit than the denominator of the aliquot part — then, if we invert that fraction thus, $\frac{7}{8}$, we shall say that the denominator of the subsesquiseptan ratio [subsesquiseptima] is $\frac{7}{8}$. Finally, the denominator of any subsuperparticular ratio will perhaps more easily be found if the prime, or least, numbers having the corresponding superparticular ratio are found, as we taught above. For the fraction whose numerator is the smaller of those numbers and whose denominator is the larger will be the denominator of the proposed subsuperparticular ratio: since the prime, or least, numbers having the sesquiseptan ratio are 8 and 7, if the smaller be made the numerator and the

larger the denominator, we shall produce this fraction, $7/8$, as the denominator of the proposed subsesquiseptan ratio [subsesquiseptima], as before.

The denominator of any superpartient ratio [proportio superpartiens] is a unit together with those aliquot parts (not making up a whole) which the greater quantity must contain beyond the lesser. Thus the denominator of the supertripartient-sevenths ratio [supertripartiens septimas, $10:7$] is $1\frac{3}{7}$; of the supertripartient-twentieths [supertripartiens vigesimas, $23:20$], $1\frac{3}{20}$, and so on. Nor is there any difficulty in finding denominators of this kind, because the very statement of the ratio exhibits its proper denominator, as is plainly clear from the earlier examples. But the denominators of the subsuperpartient ratios are fractions whose numerators are less than the denominators of the same fractions by as many units as the number of aliquot parts by which the greater quantity exceeds the lesser. Thus the denominator of the subsupertripartient-sevenths ratio [subsupertripartiens septimas] is $7/10$; of the subsupertripartient-twentieths [subsupertripartiens vigesimas], $20/23$, and so on. And the denominator of any subsuperpartient ratio will be found if, for the numerator of the fraction, one takes the denominator of the aliquot parts expressed in the ratio, and to this is added the number of those same parts, which gives the denominator of the same fraction. [Thus the denominator] of the subsuperquadripartient-elevenths ratio [subsuperquadripartiens undecimas, $15:11$] is $11/15$; for the numerator of this fraction is the number denominating the eleventh parts, namely 11, to which is added the number 4, of the four parts, to make the denominator of the same fraction, [namely] 15. Again, [the denominator] of the subsupertripartient-fifths ratio [subsupertripartiens quintas, $8:5$] is this fraction, $5/8$; for its numerator is the number denominating the fifth parts, namely 5, while its denominator, 8, is made up of that numerator 5 and the number 3, of the three parts. And by the same reasoning we shall find the denominators of the other subsuperpartient ratios. You may also find them in this way: reduce the denominator of the corresponding superpartient ratio to a single fraction, as we taught in the *Arithmetic* — a fraction whose numerator here always exceeds the denominator (which denominates the aliquot parts expressed) by as many units as there are aliquot parts. For the inverted numbers of this fraction — so that the numerator becomes the denominator and the denominator the numerator — will give the denominator of the proposed subsuperpartient ratio. Thus the denominator of the subsuperdecipartient-thirteenthths ratio [subsuperdecupartiens decimastertias, $23:13$] is $13/23$, because the denominator of the superdecipartient-thirteenthths ratio [superdecupartiens decimastertias] is $1\frac{10}{13}$, which reduces to the fraction $23/13$, whose inverted numbers make this fraction, $13/23$. Finally, the denominator of any subsuperpartient ratio will perhaps more easily be found if

the prime, or least, numbers having the corresponding superpartient ratio are found, as we handed down above. For the fraction whose numerator is the smaller of those numbers, and whose denominator is the larger, will be the denominator of the proposed subsuperpartient ratio. For example, suppose the subsuperquadripartient-ninths ratio [subsuperquadripartiens nonas, 13:9] is proposed: since the least numbers having the superquadripartient-ninths ratio are 13 and 9, we shall make the fraction $9/13$ the denominator of the subsuperquadripartient-ninths ratio; and so for the rest.

The denominator of any multiple-superparticular ratio [proportio multiplex superparticularis] is the whole number that denominates the multiple ratio expressed, together with that aliquot part which the greater quantity must contain beyond the lesser quantity. Thus the denominator of the triple-and-one-seventh ratio [trippla sesquiseptima, 22:7] is $3 \frac{1}{7}$; of the quintuple-and-one-ninth [quintupla sesquinona, 46:9], $5 \frac{1}{9}$, and so on. So that there is no labor at all in exhibiting the denominator of any multiple-superparticular ratio; since the very statement of the ratio distinctly expresses both the denominator of the multiple ratio and the aliquot part, as the examples set out declare. But the denominators of the submultiple-superparticular ratios are fractions whose numerators are the numbers denominating the aliquot parts expressed in the ratios. Thus the denominator of the subtriple-and-one-seventh ratio [subtrippla sesquiseptima] is $7/22$; of the subquintuple-and-one-ninth [subquintupla sesquinona], $9/46$, and so on. And the denominator of any submultiple-superparticular ratio will be found if, for the numerator of the fraction, one takes the denominator of the aliquot part; and if this is multiplied by the denominator of the multiple ratio, and a unit is added to the product, the denominator of the same fraction is obtained. Thus the denominator of the subquadruple-and-one-sixth ratio [subquadrupla sesquisexta, 25:6] is $6/25$; for...

[Scholium (Clavius), continued — on the denominators of proportions. The whole of this opening (printed pp. 178–180) belongs to Clavius's commentary on the arithmetic of ratios; the core Euclidean definitions of Book V have not yet begun.]

*(A note on vocabulary. Throughout this passage Clavius uses "denominator" in a special technical sense: the number or fraction that **names** a ratio — namely the quotient of its antecedent divided by its consequent, written as a whole number, a mixed number, or a fraction. This is not the modern arithmetical "denominator." The ratio-names themselves follow the traditional (Boethian) system: a "multiple" [multiplex] ratio is one like $n:1$; a "superparticular" [superparticularis] ratio exceeds by a single aliquot part, like $(n+1):n$; a "superpartient" [superpartiens] ratio*

exceeds by several such parts; and the prefix "sub-" marks the inverse ratio. The compound names combine these — e.g. "quadrupla sesquisexta" means $4 + 1/6$, i.e. the ratio 25:6. I keep Clavius's Latin ratio-names in italics and give the numeric ratio in parentheses.)

so that the numerator of this fraction, 6, denominates sixth-parts, and it is multiplied into 4, the denominator of the quadruple proportion; and to the product, 24, unity is added, so that the denominator of the same fraction is produced as 25 — and so on.

The same denominators of *submultiple-superparticular* proportions will be found if the denominator of the corresponding *multiple-superparticular* proportion is reduced to a single fraction, just as we taught in the Arithmetic: namely by multiplying the denominator of the multiple proportion by the denominator of the fraction attached to it, and then adding to the product unity, that is, the numerator of that same fraction. For if the terms of this fraction exchange their order with one another, the result is the denominator of the proposed proportion.

For example, suppose a *subquadrupla sesquisexta* proportion is given. Since the denominator of the corresponding *quadrupla sesquisexta* proportion (25:6) is $4 \frac{1}{6}$, we multiply 4 — that is, the denominator of the multiple proportion — into 6, the denominator of the attached fraction, and to the resulting number, 24, we add 1, namely the numerator of the same fraction; so that we reduce the whole denominator $4 \frac{1}{6}$ to this fraction, $25/6$. And if the terms of this fraction exchange their order, there results this fraction, $6/25$, to serve as the denominator of the *subquadrupla sesquisexta* proportion. One must proceed in the same way in the remaining cases.

Finally, you will perhaps find the denominator of any *submultiple-superparticular* proportion more easily if you find the two prime — that is, least — numbers of the corresponding *multiple-superparticular* proportion, as was explained above. For the fraction whose numerator is the smaller of those numbers, and whose denominator is the larger, will be the denominator of the proposed proportion. Thus, if a *subtrippla sesquiseptima* proportion is offered: since the prime (or least) numbers of the *trippla sesquiseptima* proportion (22:7) are 22 and 7, there will be made from them this fraction, $7/22$, to serve as the denominator of the *subtrippla sesquiseptima* proportion; and so for the rest.

The denominator of any *multiplex superpartiens* (multiple-superpartient) proportion is a whole number denominating the multiple proportion expressed within it, together with those aliquot parts — not making up a whole one — by

which the greater quantity must exceed the lesser. Thus the denominator of the *tripla superquincupartiens octavas* proportion (29:8) is $3\frac{5}{8}$; that of the *quadrupla superbipartiens quintas* proportion (22:5) is $4\frac{2}{5}$; and so on. For there is no difficulty in this method of finding the denominators in multiple-superpartient proportions, since in each of them there is openly and distinctly expressed both the denominator of the multiple proportion contained in it and the aliquot parts, as plainly appears in the examples set out.

The denominators of *submultiple-superpartient* proportions, however, are fractions whose numerators are the numbers that denominate the aliquot parts expressed in the proportions. Thus the denominator of the *subtripla superquincupartiens octavas* proportion is $\frac{8}{29}$; that of the *subquadrupla superbipartiens quintas* proportion is $\frac{5}{22}$; and so on. The denominator of any submultiple-superpartient proportion will be found if for the numerator of the fraction one takes the denominator of the aliquot parts, and then — multiplying this by the denominator of the multiple proportion and adding to the product the number of the aliquot parts — one obtains the denominator of that same fraction. Thus the denominator of the *subdupla superoctupartiens decimastertias* proportion is $\frac{13}{34}$; for the numerator of this fraction, 13, denominates thirteenth-parts, and if it is multiplied into 2, the denominator of the double proportion, and to the resulting number, 26, is added the number 8, the number of the eight parts, there is produced the denominator of the same fraction, 34. And so on.

You will also find the denominator of any submultiple-superpartient proportion in this way. Reduce the denominator of the multiple-superpartient proportion that corresponds to the proposed one to a single fraction, as we prescribed in the Arithmetic — namely by multiplying the denominator of the multiple proportion by the denominator of the fraction attached to it, and adding to the product the numerator of that same fraction. For if the terms of this fraction exchange their order between themselves, they will give a fraction which will be the denominator of the submultiple-superpartient proportion. For instance, if a *subquintupla supertripartiens decimas* proportion is proposed, we reduce the denominator of the corresponding *quintupla supertripartiens decimas* proportion — that is, $5\frac{3}{10}$ — to this fraction, $\frac{53}{10}$ (which is done by multiplying 5 into 10 and adding 3 to the resulting number, so as to make the numerator 53, under which the same denominator 10 is to be set). For if this fraction exchanges its terms, there results the denominator of the *subquintupla supertripartiens decimas* proportion, $\frac{10}{53}$; and so on.

But perhaps you will obtain the denominator of any submultiple-superpartient proportion more easily if you find the prime — that is, least — numbers of the corresponding multiple-superpartient proportion, and construct a fraction from them, taking the smaller for numerator and the larger for denominator of the proposed proportion. For if a *subquintupla supertripartiens decimas* proportion is proposed: since the least numbers in the *quintupla supertripartiens decimas* proportion are 53 and 10, the denominator of the proposed proportion will be constructed from them as this fraction, $10/53$; and so on.

Finally, the denominator of a proportion of equality is always unity: because in that proportion one quantity must be equal to the other, and hence one contains the other exactly once and nothing besides — which is precisely what unity signifies.

From what we have said about the denominators of proportions, it is, I think, clear that the denominators of proportions of the greater inequality denominate those proportions of greater inequality both in reality and in name, whereas the denominators of proportions of the lesser inequality denominate the proportions of the lesser inequality in reality only, and not also in word or name. This is evident from the examples already brought forward. For the denominator, for instance, of the *tripla superquadrupartiens nonas* proportion ($31:9$), which is $3\frac{4}{9}$ — that is, three whole units and four ninth-parts — denominates that proportion both in reality and in word, since it distinctly and openly indicates to us that in it the greater quantity contains the lesser three times, and four ninth-parts of it besides. But the denominator of the corresponding *subtripla superquadrupartiens nonas* proportion, namely $9/31$ — that is, nine thirty-first parts — does indeed in reality denominate the *subtripla superquadrupartiens nonas* proportion, since it truly signifies that in it the lesser quantity is nine

thirty-first parts of the greater (which is altogether necessary, so that the lesser quantity may have to the greater the *subtripla superquadrupartiens nonas* proportion); but in word, or name, it in no way denominates that proportion: since "nine thirty-first parts" and the name "*subtripla superquadrupartiens nonas* proportion" seem to have nothing in common, but to differ from it entirely.

Therefore, if the denominator of some proportion of the lesser inequality is expressed in numbers — namely by a fraction — so that you may know how the proportion it denominates ought to be pronounced, look at what proportion the denominator of that fraction has to its numerator. You will know this if you divide the denominator of the fraction by its numerator. For the quotient will denominate the proportion that the denominator of the fraction has to its

numerator. For by the same name you will pronounce the proportion of the given denominator, only with the syllable "sub-" prefixed, by which that proportion of the denominator's fraction to its numerator is called. Thus, if the denominator $1/8$ is given, its proportion will be called *suboctupla* (1:8). For when the numerator of the fraction is unity, the proportion of the given denominator will be a *submultiple*, denominated from the denominator of that same fraction. But if the given denominator is, say, $20/327$, then dividing the denominator 327 by the numerator, the quotient is $16 \frac{7}{20}$. Therefore the denominator of the fraction, 327, has to its numerator, 20, the *sedecupla superseptupartiens vigesimas* proportion (327:20); and consequently the proportion whose denominator is $20/327$ is to be called *subsedecupla superseptupartiens vigesimas*; and so on.

But since not all proportions can conveniently be called by their proper names — for who would conveniently call the proportion 37:1 "thirty-seven-fold," or "thirty-times-sevenfold," or by any name of that sort? and who would readily call the proportion 48:29 "*supernovemdecupartiens vigesimasnonas*"? and so on — geometers are generally accustomed, in their writings, to express any such proportion by the prime, or least, numbers of that proportion. Thus they call the proportion 13:1 (or 39:3, or 390:30) the proportion of 13 to 1, rather than "thirteenfold"; and conversely they call the proportion 1:13 (or 3:39, or 30:390) the proportion of 1 to 13, rather than "*subtredecupla*." So too they call the proportion 52:40 the proportion which is 13 to 10, which would otherwise be called *supertripartiens decimas*; and the proportion 40:52 they call the one which is 10 to 13, which by its proper name would be called *subsupertripartiens decimas*; and so with the rest.

Although, however, any proportion can be expressed by its least numbers, as we have said, it is nonetheless entirely necessary to know the denominator of that same proportion, so that we may recognize the relation of one number to another. For even though someone may say that the proportion of the number 1700 to 400 is the same as that of 17 to 4, I still shall not fully understand what that proportion is unless I know its denominator, which is $4 \frac{1}{4}$. For this openly declares that the greater number contains the lesser four times, and a fourth part of it besides. And this denominator is more easily perceived from the least numbers of some proportion than from numbers that are not least. But if it should sometimes happen that the least numbers of some proportion cannot easily be recognized, one must have recourse to the rule which I shall shortly add, and by which the denominator of the proportion between any two numbers whatever — whether they be the least numbers or not — is made known.

But if mathematicians express any proportion by its least numbers, it will be far more convenient for us to explain the proportion of any two numbers by its denominator: so that we call the proportion 100:20 the one whose denominator is 5; likewise the proportion 48:38 the one whose denominator is $1\frac{5}{19}$, which would be called *superquintupartiens decimasnonas*; so too the proportion 20:100 can be called the one which has the denominator $\frac{1}{5}$; and the proportion 38:48 the one whose denominator is $\frac{19}{24}$; and so with the rest — since, plainly, the denominator most clearly displays the relation of one number to another, as has been said.

Moreover, the denominator of a proportion between any two numbers will be found thus. The antecedent number is divided by the consequent. For the quotient, once the fraction (if there be any) has first been reduced to its least numbers, as we have handed down in the Arithmetic, will be the denominator of that proportion. For example, given the proportion 39:3 (or 390:30), we shall say that its denominator is 13 — that is, the quotient, namely the number of the division of 39 by 3, or of 390 by 30. But if, conversely, the terms be taken in reverse, the quotient of the division will be this fraction, $\frac{3}{39}$ or $\frac{30}{390}$. For when a smaller number is divided by a larger, the quotient is always a fraction whose numerator is the smaller number and whose denominator is the larger, as we explained in the Arithmetic. And because both fractions, if their numbers are reduced to the least, come back to the same, we shall say that the denominator of the proportion 3:39 (or 30:390) is $\frac{1}{13}$. Again, given the proportion 52:40, the fraction is reduced to this, $1\frac{3}{10}$. Therefore the denominator of the proportion 52:40, from the division of 52 by 40, will be found to be the denominator, or quotient, $\frac{52}{40}$ — that is, in the least numbers, $\frac{13}{10}$. But if the proportion 40:52 is offered, [its denominator, in the least numbers, is] $\frac{10}{13}$. Furthermore, the proportion 100:20 will have as denominator 3 [recte: 5]; for, dividing 100 by 20, the quotient is 5. But the denominator of the proportion 20:100 will be $\frac{1}{5}$, because, dividing 20 by 100, the quotient is $\frac{20}{100}$ — that is, in the least numbers, $\frac{1}{5}$. Next, the denominator of the proportion 48:10 will be $4\frac{4}{5}$, which comes from dividing 48 by 10; the quotient is $4\frac{4}{5}$. But the denominator of the proportion 10:48 will be $\frac{10}{48}$ — that is, in the least numbers, $\frac{5}{24}$. And because the denominator of the fraction, $\frac{5}{24}$, has to its numerator the *quadrupla superquadrupartiens quintas* proportion (since, dividing 24 by 5, the quotient comes to $4\frac{4}{5}$), the proportion 10:48 will be called *subquadrupla superquadrupartiens quintas*. For, so that a proportion of the lesser inequality may be duly expressed, one must first denominate the proportion of the denominator of the fraction to its numerator, as we said above.

Once the denominator of the proportion of two numbers has been found in the least numbers, as has been said, and reduced to a single fraction (as was handed down in the Arithmetic), the numerator and denominator of the fraction will be the least numbers between which that proportion is found. In a multiple proportion, however — namely when the denominator found has no fraction attached to it — the denominator found and unity will be the least numbers of that proportion. For instance, since the denominator of the proportion 348:120 is $2\frac{9}{10}$, if it is reduced to this single fraction, $\frac{29}{10}$ (which is done by multiplying the whole number 2 into the denominator of the fraction, 10, and adding to the resulting number, 20, the numerator of the same fraction, 9), the numbers 29 and 10 will be the least numbers having the same proportion as 348:120. So too, since the denominator of the proportion 824:1133 is $\frac{8}{11}$, the numbers 8 and 11 will be the least numbers of the proportion 824:1133. Likewise, since the denominator of the proportion 84:12 is 7, the least numbers in that proportion will be 7 and 1. And conversely, since the denominator of the proportion 12:84 is $\frac{1}{7}$, its least numbers will be 1 and 7. And so, given any two numbers whatever, if the least numbers having the same proportion as they are sought, one must find the denominator of their proportion, as has been said, and reduce it to a single fraction. For the numerator and denominator of this fraction will be the least numbers sought. Thus, if the least numbers of the proportion which 400 has to 90 are sought: dividing the antecedent by the consequent term, the quotient comes to $4\frac{4}{9}$ in the least numbers. Reducing this to the fraction $\frac{40}{9}$, we shall say that the least numbers of the proportion 400:90 are 40 and 9. And the same reasoning holds for the rest.

Now if, two proportions being proposed, you wish to know which of them is the greater or the lesser, you will call the greater the one whose denominator is greater, and the lesser the one whose denominator is less. But if the denominators are equal, you will declare the proportions also to be equal. Thus, if these two proportions are proposed — *superoctupartiens decimasquintas* (23:15) and *superdecupartiens decimasseptimas* (27:17), whose denominators are $1\frac{8}{15}$ and $1\frac{10}{17}$ — you will say that the former is less than the latter: because the denominator $1\frac{8}{15}$ is less than the denominator $1\frac{10}{17}$. By what art you may recognize which of two fractions is the greater or the lesser, or whether they are equal, we have handed down in the Arithmetic. By the same reasoning, if two proportions are offered whose denominators are $10\frac{1}{1000}$ and $9\frac{100}{101}$, we shall say the former is the greater. For although the fraction $\frac{1}{1000}$ is less than the fraction $\frac{100}{101}$, still the whole number 10 is greater than the whole number 9. So, if two proportions are given — one 17:11, the other 18:12 — you will say the

former is greater than the latter, because the denominator of the former, $1\frac{6}{11}$, is greater than the denominator of the latter, $1\frac{1}{2}$.

Furthermore, when two proportions are proposed in numbers, we shall easily discern which of them is greater, even without inquiring into their denominators, by this method. Multiply the antecedent term of the former proportion into the consequent term of the latter, and the antecedent term of the latter into the consequent term of the former; the proportion whose antecedent term has produced the greater number is the greater proportion; and if two equal numbers have been generated, the proportions will be equal to each other. Thus, if the two proportions 17:11 and 18:12 are proposed, we shall find that the former is greater, because the antecedent of the former, 17, multiplied into the consequent of the latter, 12, produces 204, whereas the antecedent of the latter, 18, into the consequent of the former, 11, produces only 198. Again, if two proportions 18:12 and 108:72 are given, the same number, 1296, will be produced both from the antecedent of the former, 18, multiplied into the consequent of the latter, 72, and from the antecedent of the latter, 108, multiplied into the consequent of the former, 12. Therefore the proportions themselves will be equal to each other. The reason for this operation is that in this way, when four numbers are proposed, the two extremes are multiplied together, and likewise the two means, as is evident in the given examples. Therefore, if equal numbers are produced, the proportion of the first to the second will be the same as that of the third to the fourth, as Euclid demonstrates in Book 7, proposition 19. Hence it follows that, if the first multiplied into the fourth produces a greater number than the second into the third, the first is too great to be able to have the same proportion to the second as the third has to the fourth: since it would have to be smaller in order to produce the same number, and thereby to have the same proportion.

Sometimes two numbers in some given proportion are wanted — whether they be prime numbers or not. To find them, then, we shall multiply, or divide, any assumed number by the denominator of the proposed proportion. For the number produced will have the proposed proportion to the assumed number that was multiplied, or the number assumed and divided will have it to the quotient. For example, if two numbers in the *quintupla* proportion (5:1) are to be found, we shall multiply any number, say 100, into the denominator 5. For the number produced, 500, will have to the multiplied number, 100, the quintuple proportion. Or we shall divide the same number, 100, by the denominator 5. For the divided number, 20, will likewise have to the quotient, 20, the given quintuple proportion. So too, if two numbers in the *dupla sesquitertia* proportion (7:3),

whose denominator is $2\frac{1}{3}$, are to be found, we shall multiply any number, say 24, by $2\frac{1}{3}$. For the number produced, 56, has to the multiplied number, 24, the proportion denominated by $2\frac{1}{3}$. Likewise, if we divide the number 24 by $2\frac{1}{3}$, the divided number, 24, will have to the quotient, $10\frac{2}{7}$, the given *dupla sesquitertia* proportion.

But in order to avoid fractions, if one wishes to do the business by multiplication, the number to be multiplied should be one that is numbered by the denominator of the aliquot part, or of the several parts, mentioned in the proportion. In a multiple proportion, however — since no part is mentioned — any number whatever may be assumed. Thus, if two numbers of the *tripla superbipartiens nonas* proportion (29:9) are wanted, the number to be taken should be one numbered by 9, such as 18, or 27, 36, 45, and so on. For any of these, multiplied into the denominator $3\frac{2}{9}$, will produce a whole number: as when 45 is multiplied into $3\frac{2}{9}$, the number 145 results, which has to the assumed number, 45, the given proportion. So too, if someone desires a *subquadrupla* proportion (1:4), whose denominator is $\frac{1}{4}$, the number to be taken should be one numbered by 4, such as 12. For this, multiplied into the denominator $\frac{1}{4}$, will produce 3, a whole number, which has to the assumed number, 12, the *subquadrupla* proportion. But if the business is to be done by division, and fractions are to be avoided, the number to be taken in a multiple proportion should be one numbered by the denominator of the proportion.

In other proportions, however, the denominator of the proportion is to be reduced to a single fraction, and the number to be taken should be one numbered by the numerator. Thus, if a *quintupla* proportion is sought, one should take, say, the number 30, numbered by 5. For this, divided by the denominator 5, makes the quotient 6, to which it has the quintuple proportion. But if a proportion is desired whose denominator is $4\frac{3}{7}$, then, reducing it to the fraction $\frac{31}{7}$, the number to be taken should be one numbered by 31, such as 62. For this, divided by $\frac{31}{7}$, makes the quotient 14, to which the number taken, 62, has the proportion denominated by $4\frac{3}{7}$, namely the *quadrupla supertripartiens septimas* proportion (31:7).

Moreover, if two numbers are proposed, and someone wishes to find two others in the same proportion, it will be enough if he multiplies, or divides, each of them by any number whatever. For the numbers produced, or the quotients, will have the same proportion as the two proposed numbers. Likewise, if, given any number, another is to be found which has a given proportion to it, or to which it has a given proportion, you will obtain the first if you multiply the proposed

number by the denominator of the proportion; and you will obtain the second if you divide the same number by the denominator of the proportion. But if a proportion is given in two numbers, and some third number is proposed: if a fourth is to be found to which the third given number has the same proportion as the first given number has to the second — then, if the denominator of the proportion of the two given numbers is known, the third given number is to be divided by the denominator. For the quotient will be the number we seek. Or certainly, whether the denominator is known or not, the third number is to be multiplied by the second given number — that is, by the consequent term of the former proportion — and the number produced is to be divided by the first, or by the antecedent term. Thus, if the denominator is $4/7$ and a third number, 20, is likewise given: divide the number 20 by the denominator $4/7$, and the fourth number sought, 35, will be produced. Or multiply the third number, 20, by the second, 7, and divide the number produced, 140, by the first, 4. For to the quotient, 35, the third number, 20, will have the same proportion as 4 to 7.

But if a fourth is to be found which has to the third the same proportion as the two given numbers have — then, if the denominator is known, it is to be multiplied into the third number. For the number produced will be the one you seek. Or, whether the denominator is known or not, the third number is to be multiplied by the first given number — namely by the antecedent term — and the number produced is to be divided by the second, that is, by the consequent term. Thus, given the same two numbers 4 and 7, and a third number 20, multiply 20 by the denominator $4/7$, and the number sought, $11 \frac{3}{7}$, will be generated. Or multiply the third number, 20, into the first, 4, and divide the number produced, 80, by the second, 7. For to the third number, 20, the quotient, $11 \frac{3}{7}$, will have the same proportion as 4 to 7.

Now since we have said that the proportion whose denominator is greater is the greater proportion, it follows that among the multiple proportions the least given is the *dupla*, denominated by 2, but not the greatest: because the number 2, among all the denomi- ...is the smallest [denominator] of the multiple ratios, whereas a greatest cannot be given, since the numbers increase without limit. It also follows that among the superparticular ratios a greatest can be found — namely the sesquialter [sesquialtera, 3:2], denominated by $1\frac{1}{2}$ — but not a least: because the number $1\frac{1}{2}$ is the greatest of all the denominators of superparticular ratios, while a least cannot be given, since the denominators of fractions can proceed to infinity. This is established from what we set out in the Arithmetic: that a fraction whose denominator is larger is smaller, if the numerator is the same; whereas a fraction whose denominator is smaller, the numerator being the

same, is larger. Since therefore the denominator 2 is the smallest of all, the fraction $\frac{1}{2}$ will be the greatest of all those that have 1 for numerator (such as are the fractional denominators of the superparticular ratios).

Finally, it is inferred that in the other ratios of greater inequality neither a least nor a greatest can be given. This is to be understood of the last species. For if the discussion is not about the last species, then among the superpartients the superbipartient [superbipartiens] will be the least, while the superpartient of thirds [superpartiens tertias] will be the greatest. Among the multiple superparticulars the greatest will be the multiple sesquialter [multiplex sesquialtera]; the least, the double superparticular [dupla superparticularis]. Among the multiple superpartients, finally, the greatest will be the multiple superpartient of thirds; the least, the double superbipartient [dupla superbipartiens]. Conversely, in the submultiple ratios a greatest is given — namely the subdouble [subdupla] — but not a least; among the subsuperparticulars, on the other hand, a least is found — such as the subsesquialter [subsesquialtera] — but not a greatest; and in the remaining ratios of lesser inequality, finally, neither a greatest nor a least can be given. All of this will be clear if the denominators of the ratios are compared with one another.

It is also gathered that this is the connection among the ratio of greater inequality, the ratio of equality, and the ratio of lesser inequality: that every ratio of greater inequality is greater than the ratio of equality, and every ratio of lesser inequality is less than it. The reason is that every denominator of a ratio of greater inequality, however small, is greater than the unity that is the denominator of the ratio of equality — since it cannot be less than unity together with one aliquot part — whereas the denominator of a ratio of lesser inequality is always less than unity, being a fraction whose numerator is exceeded by its denominator, as is plain from what we said above. The ratio of equality is therefore the mean between the ratio of greater inequality and the ratio of lesser inequality.

Here a point worth noting is this: the ratio of greater inequality, by decreasing, draws ever nearer and nearer, to infinity, to the ratio of equality, yet never reaches it; and conversely the ratio of lesser inequality, by increasing, comes ever nearer and nearer, to infinity, to that same ratio of equality, yet never attains it. For given any multiple ratio, a smaller one can be given, down to the double, which is the least of all multiples. Then a smaller than this is given — namely some superparticular or superpartient — and since a least of these cannot be given, and every ratio of this kind is greater than the ratio of equality, it plainly

follows that a ratio of greater inequality cannot arrive at the ratio of equality, even though by decreasing to infinity it approaches it. Again, since every ratio of lesser inequality is smaller than the ratio of equality, and yet a greatest cannot be found among them, it is likewise manifest that a ratio of lesser inequality cannot reach the ratio of equality, although by increasing to infinity it approaches it.

But so that both may be learned by an example too, note the following. If between any two non-adjacent numbers some intermediate number is interposed — greater, that is, than one of them and smaller than the other — then the ratio of the greater number to the smaller is divided into two smaller ratios, of which one is that of the greater number to the interposed middle, the other that of the middle to the smaller. Likewise the ratio of the smaller number to the greater is divided into two larger ratios, of which one is that of the smaller number to the interposed middle, the other that of the middle to the greater.

For example, if between the two numbers 6 and 3 the middle 4 is placed, thus, 6. 4. 3., the ratio 6 to 3, which is double, will be divided into the sesquialter 6 to 4 and the sesquitertian [sesquitertia] 4 to 3, each of which is smaller than the double ratio 6 to 3. Conversely, the ratio 3 to 6, which is subdouble, will be divided into the subsesquitertian [subsesquitertia] 3 to 4 and the subsesquialter [subsesquialtera] 4 to 6, each of which is larger than the subdouble ratio 3 to 6. The same happens if between the same numbers 6 and 3 the middle 5 is interposed, thus, 6. 5. 3. For both the ratio 6 to 5, a sesquiquintan [sesquiquinta], and the superbipartient-of-thirds ratio 5 to 3, are smaller than the double ratio 6 to 3; and conversely both the superbipartient-of-thirds ratio 3 to 5 and the subsesquiquintan [subsesquiquinta] 5 to 6 are larger than the subdouble ratio 3 to 6. All of this can be proved from the denominators.

But that both may be demonstrated — the greater [ratio] compared with the greater, and the greater with the less — the greater given term will have to the smaller given term a greater ratio than the same greater has to the middle, and than the middle has to the smaller; while the smaller [given term] to the greater will have a smaller ratio than the same smaller has to the middle, and than the middle has to the greater, as is evident either from what we explained there, or from Proposition 8 of this Book 5, where Euclid [Euclid] demonstrates that, given two unequal quantities, the greater has a greater ratio to some third than the smaller does; and, on the contrary, that the same third quantity has a smaller ratio to the greater than to the smaller [At è contrario, tertiam eandem quantitatem ad maiorem habere minorem proportionem, quam ad maiorem — see note]. From this follows what we said: namely, given these three numbers 6.

4. 3., that the ratio of the same number 6 to the smaller 3 is greater than to the middle 4; likewise that the ratio of the greater number 6 to 3 is greater than that of the smaller 4 to the same 3. Conversely, that the ratio of the same number 3 to the smaller 4 is greater than to the greater 6; likewise that the ratio of the greater number 4 to 6 is greater than that of the smaller 3 to the same 6. And the reasoning is the same in the remaining cases.

These things established, the double ratio 4 to 2 will be divided here — 4.3.2 — into two smaller ratios. And just as the ratio 8 to 6, which is the same as 4 to 3, is divided here — 8.7.6 — so the ratio 6 to 4, which is the same as 3 to 2, is divided here — 6.5.4 — into two smaller ratios. Likewise, just as the ratio 16 to 14, which is the same as 8 to 7, is divided here — 16.15.14 — so the ratio 14 to 12, which is the same as 7 to 6, is divided here — 14.13.12 — and the ratio 12 to 10, which is the same as 6 to 5, here — 12.11.10 [?] [read 12.11.10; the first term breaks the monotone chain 16.15.14 / 14.13.12 / 12.11.10 / 10.9.8 — see Numerical impossibles] — and the ratio 10 to 8, which is the same as 5 to 4, here — 10.9.8 — into two smaller ratios; and so it can be done to infinity.

Conversely, the subsesquialter ratio 4 to 6 will be divided here — 4.5.6 — into two larger ratios. And just as the ratio 8 to 10, which is the same as 4 to 5, is divided here — 8.9.10 — so the ratio 10 to 12, which is the same as 5 to 6, here — 10.11.12 — into two larger ratios. Likewise, just as the ratio 16 to 18, which is the same as 8 to 9, is divided here — 16.17.18 — so the ratio 18 to 20, which is the same as 9 to 10, here — 18.19.20 — and the ratio 20 to 22, which is the same as 10 to 11, here — 20.21.22 — and the ratio 22 to 24, which is the same as 11 to 12, here — 22.23.24 — into two larger ratios; and of this increase there will never be an end. The same may be tried in any other ratios whatever. Thus the decuple ratio 10 to 1 is divided here — 10.6.1 — into two smaller ratios; and the sub-superseptupartient-elevenths ratio [subsuperseptupartiens undecimas] 11 to 18 is divided here — 11.15.18 — into two larger ratios; and so on.

On Proportionalities [De Proportionalitatibus]

Proportionality, as defined by Euclid, is divided into several kinds, as may be seen in Boethius [Boethius], Jordanus [Jordanus Nemorarius], and other arithmeticians. But the principal proportionalities — which the aforesaid authors call "means" [medietates] — are these three: the Arithmetic, the Geometric, and the Musical or Harmonic.

Arithmetic proportionality, or mean, is when three or more numbers proceed by the same difference. For example, these numbers 4. 7. 10. 13. 16., each of which

exceeds its antecedent by the ternary [i.e. by 3], are said to constitute Arithmetic proportionality. It is of two kinds, continuous and discrete. It is continuous when in the progression of numbers there is no interruption, but each is compared with its nearest neighbor, as happens in the given example. It is discrete when in the progression of numbers there is an interruption, so that only pairs are compared among themselves, and not each with its nearest antecedent, as happens in these numbers 4. 7. 8. 11. 30. 33. For the same difference holds between the pairs 4 and 7, and 8 and 11, and 30 and 33, but not between 4 and 7 and 7 and 8, and so on.

Geometric proportionality, or mean, is when three or more numbers have the same ratio — which indeed Euclid defined. For this is properly called proportionality, or analogy; the others are so called improperly, since the ratio is not always the same among their terms, so that they are more rightly called "means," on account of the middle terms which, by a certain fixed rule, are interposed between the extremes. For example, these numbers 2. 6. 18. 54., since each has the same triple ratio to its antecedent, constitute Geometric proportionality. This too is of two kinds, continuous and discrete, as we explained in the fourth definition of this book. The continuous is seen in the numbers given [2. 6. 18. 54], the discrete in these six, 2. 3. 12. 18. 20. 30. For only the pairs 2 and 3, and 12 and 18, and 20 and 30 have the same sesquialter ratio, but not each to its nearest antecedent.

Musical or Harmonic proportionality is when three numbers are so ordered that the ratio of the greatest to the least is the same as the ratio of the difference between the two greater to the difference between the two smaller — so that there is neither the same difference between them, as in the Arithmetic, nor the same ratio, as in the Geometric. For example, these three numbers 3. 4. 6.: since the ratio of the difference between the greatest 6 and the middle 4 [i.e. 2], to the difference between the middle 4 and the least 3, that is, to 1, is the same (for in both cases the ratio is double), they constitute Musical or Harmonic proportionality. For the terms themselves have neither the same difference nor the same ratio, as is plain. So too these three numbers 42. 12. 7. constitute Harmonic proportionality, because the ratio of the greatest 42 to the least 7 is the same as the ratio of the difference between the greatest 42 and the middle 12 — that is, of the number 30 — to the difference 5 between the middle 12 and the least 7, both ratios being sextuple.

This kind of proportionality is called Musical, or Harmonic, because for the most part its numbers have those ratios in which the musical consonances consist.

Thus in the earlier example, between 6 and 4 there is the sesquialter ratio, which constitutes the consonance called the diapente, or the Fifth; likewise between 4 and 3 there is the sesquitercian ratio, which constitutes the consonance they call the diatessaron, or the Fourth; and finally between the extremes 6 and 3 the double ratio is seen, which constitutes the diapason consonance, or the octave. And in the same way the same is observed in most other cases.

Some Properties of the Three Proportionalities, or Means, which we have explained [*Proprietates aliquot trium proportionalitatum, sive medietatum, quas explicavimus*]

I.

In three numbers of Arithmetic proportionality the ratio of the greatest to the middle is smaller than that of the middle to the least. As is clear here: 2. 4. 6. For the ratio 6 to 4 is sesquialter, and 4 to 2 double.

But in three numbers of Musical proportionality the ratio of the greatest to the middle is greater than that of the middle to the least. As you see here: 3. 4. 6. For the ratio 6 to 4 is sesquialter, and 4 to 3 sesquitercian.

Finally, in three numbers of Geometric proportionality the ratio of the greatest to the middle is the same as that of the middle to the least, as is clear from its definition, and appears in these three numbers 3. 6. 12. For both 12 to 6 and 6 to 3 have the double ratio. In this respect Geometric proportionality holds the middle place between the Arithmetic and the Harmonic: since in the Arithmetic the ratio between the greater numbers is smaller than between the smaller; conversely in the Harmonic the ratio between the greater numbers is greater than between the smaller; while in the Geometric it is the same between the greater as between the smaller, as has been explained.

II.

Arithmetic proportionality has the differences of its terms equal, but their ratios unequal. Geometric, on the contrary, has the differences of its terms unequal, but their ratios equal. Harmonic, finally, has neither the differences nor the ratios of its terms equal. This is clear from the examples above.

III.

In three numbers of Arithmetic proportionality the middle exceeds the smaller by the same [fractional] part, or parts, of itself by which it is exceeded by the greater. As here: 3. 7. 11. The middle 7 exceeds the smaller 3 by four units, which

make $\frac{4}{7}$ of the middle 7 itself. Likewise the same middle 7 is exceeded by the greater 11 by four units also, which constitute $\frac{4}{7}$ of the same middle. Again here, 15. 20. 25.: the middle 20 exceeds the smaller 15 by five units, which make $\frac{1}{4}$ of the middle 20 itself; and the same middle 20 is exceeded by the greater 25 by five units, which constitute $\frac{1}{4}$ of the same middle 20.

But in three numbers of Geometric proportionality the middle exceeds the smaller by the same part, or parts, of itself by which it is exceeded by the greater as a part of the greater. As here, 4. 6. 9.: the middle 6 exceeds the smaller 4 by the binary [2], which is $\frac{1}{3}$ of the middle, and is exceeded by the greater 9 by the ternary [3], which is likewise $\frac{1}{3}$ of the greater. So too here, 9. 15. 25.: the middle 15 exceeds the smaller 9 by six units, which make $\frac{2}{5}$ of the middle 15 itself; and the same middle 15 is exceeded by the greater 25 by ten units, which likewise make $\frac{2}{5}$ of the greater 25 itself.

Finally, in three numbers of Harmonic proportionality the middle exceeds the smaller by the same part, or parts, of the smaller by which it is exceeded by the greater as a part of the greater. As here, 3. 4. 6.: the middle 4 exceeds the smaller 3 by a unit, which is $\frac{1}{3}$ of that same smaller, and is exceeded by the greater 6 by the binary [2], which is likewise $\frac{1}{3}$ of that same greater. Again, in this Harmonic proportionality 42. 12. 7.: the middle 12 exceeds the smaller 7 by five units, which are $\frac{5}{7}$ of the smaller 7 itself; and the same middle 12 is exceeded by the greater 42 by thirty units, which are $\frac{30}{42}$ — that is, in lowest terms, $\frac{5}{7}$ — of the greater 42 itself.

III.

In three numbers of Arithmetic proportionality the sum of the extremes is double the middle. As here, 3. 7. 11.: the sum of the extremes, 14, is double the middle 7.

But in three numbers of proportionality — Geometric as well as Harmonic — the sum of the extremes exceeds the double of the middle by the number by which the difference of the greater exceeds the difference of the smaller. As here, 4. 6. 9. and 3. 4. 6.: just as the sum 13 of the extremes 4 and 9 exceeds the double of the middle 6, that is, 12, by a unit — by which difference the greater difference, namely 3, exceeds 2, the smaller difference — so the sum 9 of the extremes 3 and 6 exceeds the double of the middle 4, that is, 8, by a unit, by which the greater difference, namely 2, exceeds the smaller difference, which is 1. So too here, 9. 15. 25., where there is continuous superbipartient-of-thirds proportion: the sum 34 of the extremes 9 and 25 exceeds the double of the middle 15, that is, 30, by the number 4, by which same [amount] the greater difference, 10, exceeds the

smaller difference, 6. Likewise in this Harmonic proportionality, 42. 12. 7.: the sum 49 of the extremes exceeds the double of the middle 12, that is, 24, by the number 25, by which the greater difference, 30, exceeds the smaller difference, 5.

V.

In three numbers of Arithmetic proportionality the extremes multiplied together produce a number which is exceeded by the square of the middle by a number equal to the product of the difference of the smaller into the difference of the greater. As here, 3. 7. 11.: the number 33, made from 3 into 11, is exceeded by 49, the square of the middle, by the number 16, which is made from the difference 4 into the difference 4.

But in three numbers of Geometric proportionality the number generated from the first into the third is equal to the square of the middle, as here, 4. 6. 9.: the extremes multiplied together make 36, the square of the middle.

Finally, in three numbers of Harmonic proportionality the number generated from the multiplication of the extremes together exceeds the square of the middle by a number equal to the product of the difference of the smaller into the difference of the greater. As here, 42. 12. 7.: the number 294, made from 42 into 7, exceeds 144, the square of the middle, by the number 150, which is made from the difference 30 into the difference 5. In this respect too Geometric proportionality holds the middle place between the Arithmetic and the Harmonic: for in the Arithmetic less is produced from the first into the third than from the middle into itself, in the Harmonic more, and in the Geometric the same number is generated.

VI.

In three numbers of Arithmetic proportionality the sum of the extremes multiplied into the middle produces a number which exceeds the double of the product of the first into the third by a number equal to the product of the difference of the smaller into the doubled difference of the greater. As here, 3. 7. 11.: the sum of the extremes, 14, into the middle 7 makes 98, which number exceeds the number 66 — the double of the product 33 of the first into the third — by the number 32, which is made from the difference 4 into the doubled difference 4, that is, into 8.

But in three numbers of Geometric proportionality the sum of the extremes multiplied into the middle generates a number which exceeds the double of the product of the first into the third by a number equal to the product of the

difference of the smaller into the difference of the greater. As here, 4. 6. 9.: the sum 13 of the extremes into the middle 6 makes 78, which number exceeds the double of the product of 4 into 9, that is, 72, by the number 6, generated from the difference 2 into the difference 3.

Finally, in three numbers of Harmonic proportionality, from the sum of the extremes into the middle there is produced a number double the product of the first into the third. As here, 3. 4. 6.: from the sum of the extremes, 9, into the middle 4 comes the number 36, double the product 18 of 3 into 6.

VII.

Given four numbers geometrically but not continuously proportional, if the second is the mean between the extremes in Arithmetic proportionality, the third will be the mean between the same extremes in Harmonic proportionality; and if the second is the mean in Harmonic proportionality, the third will be the mean in Arithmetic. As here: as 6 is to 9, so is 8 to 12. And 6. 9. 12. are Arithmetically proportional, since they have the same excess 3; you see that 6. 8. 12. are Harmonically proportional, since the ratio 12 to 6 is the same as that of the difference of the greater, 4, to 2, the difference of the smaller. Likewise here, 6. 8. 9. 12., since as 6 is to 8, so is 9 to 12: and 6. 8. 12. are Harmonically proportional, as we said, while you see that 6. 9. 12. are Arithmetically proportional, since they have the same excess 3.

Again, given four numbers of which one of the two middle terms is the mean between the extremes in Arithmetic proportionality, and the other in Harmonic, the four given numbers will be geometrically (though not continuously) proportional. Thus, given these four numbers 4. 6. 8. 12., the third, 8, is the Arithmetic mean between the extremes 4 and 12, and the second, 6, is the Harmonic mean between the same extremes 4 and 12; you see that as 4 is to 6, so is 8 to 12, and that as 4 is to 8, so is 6 to 12.

VIII.

Given any number of numbers continuously proportional, whether Arithmetically, Geometrically, or Harmonically (though several numbers are said to continue in Harmonic proportionality when the first three are Harmonically proportional, likewise the three next following once the first is left aside, and, leaving aside the first two, another three that follow, and again, the first three left aside, the next three, and so on) — then those numbers will be constituted in the same proportionality that are placed at positions both odd and even, and also partly at odd and partly at even alternating positions, provided that an equal

multitude of numbers is interposed alternately between any odd and even, or between the partly odd and partly even. Such are the first, third, fifth, seventh, and so on; likewise the second, fourth, sixth, and so on; furthermore the first, fourth, seventh, and so on; moreover the first, fifth, ninth, and so on; and so onward. Which is evident in the examples that follow.

Euclid's *Elements*, Book V — Clavius's excursus *On Proportionalities* (continued)

The demonstration of the preceding property is completed by the three tables below, which display arithmetic, geometric, and harmonic proportionalities and the sub-progressions that can be drawn from each. In each table the top row is the full progression; each following row picks out, in its own aligned columns, a further progression of the same kind (taking every second, every third, every fourth term, and so on).

PROPORTIONALITATES ARITHMETICAE [Arithmetic Proportionalities]

3	7	11	15	19	23	27	31	35	39	43	47	51	55	59
3		11		19		27		35		43		51		59
	7		15		23		31		39		47		55	
3			15			27			39			51		
	7			19			31			43			55	
3				19				35				51		

Proportionalitates Geometricae [Geometric Proportionalities]

	1	2	4	8	16	32	64							
5	0	0	0	0	0	0	0	1280	2560	5120	10240	20480	40960	
5		2		8		32		1280		5120		20480		
		0		0		0								
	1		4		16		64		2560		10240		40960	
	0		0		0		0							
5			4			32			2560			20480		
			0			0								
	1			8			64			5120			40960	

	1	2	4	8	16	32	64							
5	0	0	0	0	0	0	0	1280	2560	5120	10240	20480	40960	
	0			0			0							
5				8				1280				20480		
				0										

Proportionalitates Harmonicae [Harmonic Proportionalities]

252	280	315	360	420	504	630	840	1260	2520
252		315		420		630		1260	
	280		360		504		840		2520
252			360			630			2520
	280			420			840		
252				420				1260	

IX.

However many proportional numbers are given — whether in arithmetic, geometric, or harmonic proportionality — if each is multiplied by some one and the same number, or divided by it, then the product-numbers, or the quotients, will likewise have arithmetic, geometric, or harmonic proportionality. As you see here, with each term multiplied by 4, and likewise divided by 4:

Arithmetica proportional.		Geometrica proportional.			Harmonica proportional.		
20	2 3	20	40	8	60	8	140
	8 6			0		4	
<i>Producti</i> [Products]			<i>Produ</i> <i>cti</i>			<i>Produ</i> <i>cti</i>	
80	1 1	80	160	3	240	3	560
	1 4			2		3	
	2 4			0		6	
<i>Quotientes</i> [Quotients]			<i>Quoti</i> <i>entes</i>			<i>Quoti</i> <i>entes</i>	
5	7 9	5	10	2	15	2	35
				0		1	

For the rest: in geometric and harmonic proportionality the product-numbers and the quotients keep the very same kind of ratio that the multiplied or divided numbers have. But in arithmetic proportionality they do not keep the same difference. Instead, the product-numbers have a difference that is a multiple of the difference of the multiplicands, denominated by the multiplying number — in the given example, quadruple. For the difference between 20 and 28 is 8, whereas between 80 and 112 it is 32. The difference of the quotients, by contrast, has to the difference of the divided numbers a submultiple ratio denominated by the divisor, with the prefix "sub-" set before it — in the given example, subquadruple. For the difference between 20 and 28 is 8, whereas between 5 and 7 it is 2. The same ratio of the differences — between the products and the multiplicands, and between the quotients and the divided numbers — is found in geometric and harmonic proportionality as well. Thus you see that the differences of the numbers 80, 160, 320, and likewise of the numbers 240, 336, 560, are quadruple the differences among the numbers 20, 40, 80 and among the numbers 60, 84, 140; while the differences of the numbers 5, 10, 20 and of the numbers 15, 21, 35 are subquadruple of those same differences.

DE PROPORTIONALITATE ARITHMETICA [On Arithmetic Proportionality]

I.

Given any two numbers whatever, if you add their difference to the greater, you will have a third term, greater than both given numbers, in arithmetic proportionality. And if you add the same difference to this third term, you will make a fourth, greater still, in the same proportionality. Going on in this way you will find infinitely many others, always greater, so long as you keep adding the difference to the last one found. Thus, given the two numbers 4 and 11, whose difference is 7, this arithmetic proportionality is set up, which can be extended to infinity:

$4 \cdot 11 \cdot 18 \cdot 25 \cdot 32 \cdot 39 \cdot 46 \cdot 53 \cdot 60 \cdot 67 \cdot \&c.$

But if you subtract the same difference from the lesser number, when it can be subtracted, you will again have a third term, less than both, in the same proportionality. And if you take the same difference from this third term, the remainder becomes a fourth, smaller still, in the same proportionality. Going on in this way you will find others, smaller, so long as you keep taking the difference

from the last one found, until the subtraction can no longer be made. Thus, given the numbers 30 and 34, whose difference is 4, this arithmetic proportionality is set up, running down as far as 2, beyond which it cannot proceed, since 4 can no longer be subtracted from 2:

$$34 \cdot 30 \cdot 26 \cdot 22 \cdot 18 \cdot 14 \cdot 10 \cdot 6 \cdot 2$$

The same proportionality can be extended even if the difference is not assumed, in this way. Given two numbers, from the doubled greater subtract the lesser: the remainder will be a third term greater than both given numbers. If from the double of this term you subtract the one just preceding it, you will have a fourth; and so on to infinity. Thus, given the two numbers 4 and 15: if from the doubled 15 — that is, from 30 — you subtract 4, the remainder is the third term, 26. From its double, 52, if you also take away 15, you will have the fourth term, 37; and so on in succession, as appears here:

$$4 \cdot 15 \cdot 26 \cdot 37 \cdot 48 \cdot \&c.$$

Again, given two numbers: if from the double of the lesser you subtract the greater, a third number, less than both, will be left. And if from the double of this you take away the one just preceding, you will leave a fourth term, smaller still; and so on in succession, until the subtraction can no longer be made. Thus, given the two numbers 20 and 27: if from the doubled 20 — that is, from 40 — you subtract 27, the remainder becomes the third term, 13. From its double, 26, if you also take away 20, the fourth term, 6, will remain. And since from its double, 12, the preceding term, 13, cannot be subtracted, the proportionality cannot proceed further, as you see here:

$$27 \cdot 20 \cdot 13 \cdot 6$$

II.

When the number of terms is odd, the sum of the extremes is equal to the sum of any two middle terms equally distant from the extremes, and equal to double the middle term from which the extremes are equally distant. As here:

$$4 \cdot 7 \cdot 10 \cdot 13 \cdot 16 \cdot 19 \cdot 22 \cdot 25 \cdot 28$$

The sum of 4 and 28, likewise of 7 and 25, likewise of 10 and 22, likewise of 13 and 19, and likewise double the middle term 16, is always 32.

When the number of terms is even, the sum of the extremes is always equal to the sum of any two middle terms equally removed from the extremes. As here:

$$4 \cdot 7 \cdot 10 \cdot 13 \cdot 16 \cdot 19 \cdot 22 \cdot 25 \cdot 28 \cdot 31$$

The sum of 4 and 31, likewise of 7 and 28, of 10 and 25, of 13 and 22, and of 16 and 19, is always 35.

And so, when the number of terms is odd, the sum of all the terms will have, to the middle term equally distant from the extremes — that is, to half the sum of the extremes, or of any two terms equally distant from the extremes — a multiple ratio denominated by the number of terms. For since the sum of the extremes, or of any two terms equally spaced from the extremes, is double the middle term, the middle term (that is, half the sum of the extremes) is contained in the sum of all the terms as many times as there are terms: once in the middle term itself, and twice in each sum of two terms equally distant from the middle. Hence the sum of all the terms has, to half the aggregate of the extremes — that is, to the middle term — a multiple ratio whose denominator is the number of terms. As in this series of nine numbers:

$$6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 21 \cdot 24 \cdot 27 \cdot 30$$

The sum of all the terms will contain the middle term 18 — that is, half the sum of the extremes 6 and 30 — as many times as there are units in 9, the number of terms; that is, the sum of all the terms will have to 18, half the sum of the extremes (or to the middle term), a ninefold ratio. And the reasoning is the same in the other cases.

Therefore we will easily find the sum of all the terms of an arithmetic proportionality whose number of terms is odd, if we multiply half the aggregate of the extremes (which is always even when the number of terms is odd) by the number of terms. As in this series of 11 terms:

$$7 \cdot 19 \cdot 31 \cdot 43 \cdot 55 \cdot 67 \cdot 79 \cdot 91 \cdot 103 \cdot 115 \cdot 127$$

The sum of the extremes is 134. Its half, 67 — which does not differ from the middle term, 67 — multiplied by 11, the number of terms, produces 737, the sum of all the terms. This same procedure works also for a proportionality whose number of terms is even; but when the sum of the extremes is odd (for when the number of terms is even the sum of the extremes can be odd, not always even) its half will be a whole number plus $\frac{1}{2}$.

When, however, the number of terms is even, the sum of all the terms will have, to the sum of the extremes (or of any two terms equally distant from the extremes), a multiple ratio denominated by half the number of terms. For since the sum of any two terms equally distant from the extremes equals the sum of the

extremes, this sum of the extremes is contained in the sum of all the terms as many times as unity is contained in half the number of terms — once, that is, in each sum of two terms equally spaced from the extremes. Therefore the sum of all the terms has to the sum of the extremes a multiple ratio denominated by half the number of terms. As in this series of 10 terms:

$$6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 21 \cdot 24 \cdot 27 \cdot 30 \cdot 33$$

The sum of all the terms will contain the sum of the extremes, 6 and 33 (which is 39), five times; that is, the sum of all the terms will have to 38 [recte: 39], the sum of the extremes, a fivefold ratio. And so with the others.

Therefore we will obtain the sum of all the terms of an arithmetic proportionality whose number of terms is even, if we multiply the sum of the extremes by half the number of terms (which number, being even, is halved without any fraction of a unit). As in this series of 10 terms:

$$6 \cdot 11 \cdot 16 \cdot 21 \cdot 26 \cdot 31 \cdot 36 \cdot 41 \cdot 46 \cdot 51$$

The sum of the extremes is 57, which multiplied into 5, half the number of terms, makes 285, the sum of all the terms. This procedure works also for a proportionality whose number of terms is odd; but half the number of terms will then always be a whole number plus $\frac{1}{2}$, since the number of terms is taken to be odd.

You see, then, that it is enough for finding the sum that the extreme terms, together with the number of terms, be known. And how, from one known extreme, the number of terms, and the difference, we can arrive at knowledge of the other extreme, we shall teach at once.

III.

If in an arithmetic proportionality, of however many terms, we know one of the extremes, the number of terms, and the difference, we shall find the other extreme in this way. We will multiply the number next-less than the number of terms by the given difference, and add the product to the known lesser extreme; or subtract the same product from the known greater extreme.

**

For in the former case we make the greater extreme, which is sought, while in the latter the remainder becomes the unknown lesser extreme. For example: if the lesser extreme 4 is given, the number of terms 12, and the difference 5, we will

multiply 11 (the number next-less than the number of terms) by the given difference 5, and to the product 55 we will add the lesser extreme 4. The sum, 59, will be the greater extreme, as is plain in these 12 terms, whose difference is 5:

$$4 \cdot 9 \cdot 14 \cdot 19 \cdot 24 \cdot 29 \cdot 34 \cdot 39 \cdot 44 \cdot 49 \cdot 54 \cdot 59$$

But if the greater extreme 59 is given, and the same number of terms 12, with the same difference 5, we will take away the number 55 (the product of 11, the number next-less than the number of terms, into the difference 5) from the known greater extreme 59. The remaining number, 4, will be the lesser extreme, as is clear in the same example.

Now it is necessary, if the lesser extreme is sought, that the number produced from the number next-less than the number of terms, multiplied into the difference, be greater than the known greater extreme, so that the subtraction can be made. Otherwise the question will be impossible: that is, it cannot happen that the given number be the greater extreme in any progression whose difference and number of terms are as proposed. For example, if someone were to give the greater extreme 20, the number of terms 6, and the difference 5, the lesser extreme could not be found, because the number 25, produced from 5 (which is next-less than the number of terms) multiplied into the difference 5, is greater than the given greater extreme 20. But should it happen that the product equals the greater extreme, so that after the subtraction 0 is left, the question will indeed be possible, but the lesser extreme will be 0. Thus, if in this last example the greater extreme were proposed as 25, this arithmetic proportionality would result:

$$0 \cdot 5 \cdot 10 \cdot 15 \cdot 20 \cdot 25$$

III.

If the extreme terms are known, together with the number of terms, we shall find the difference of the numbers in this way. Having taken the lesser extreme away from the greater, we will divide the remaining number by the number next-less than the number of terms. For the quotient will be the difference sought. Thus, if someone says there are 10 terms of some arithmetic proportionality whose extreme numbers are 7 and 52: taking the lesser extreme 7 from the greater 52, we will divide the remaining number 45 by 9, the number next-less than the number of terms. The quotient, 5, will be the difference sought, as appears here:

$$7 \cdot 12 \cdot 17 \cdot 22 \cdot 27 \cdot 32 \cdot 37 \cdot 42 \cdot 47 \cdot 52$$

But so that fractions may be avoided, and the proposed progression fall among whole numbers, it is necessary that, once the lesser extreme has been taken from the greater, the remaining number be measurable by the number next-less than the number of terms. For if it is not so measured, the difference found will be either a fraction or a whole number with a fraction — though the question is still possible. Thus, if someone proposes the extreme terms 2 and 38, and the number of terms 9: taking 2 from 38, and dividing the remaining number 36 by 8, the difference is found to be $4\frac{1}{2}$, as is clear in this example [printed: "templo"; recte "exemplo"]:

$$2 \cdot 6\frac{1}{2} \cdot 11 \cdot 15\frac{1}{2} \cdot 20 \cdot 24\frac{1}{2} \cdot 29 \cdot 33\frac{1}{2} \cdot 38$$

V.

If the two extreme terms, together with the difference, are known, we shall elicit the number of terms by this method. Having taken the lesser extreme from the greater, we will divide the remaining number by the difference. For if we add 1 to the quotient, we will make up the number of terms sought. Thus, given the two extremes 10 and 40, with the difference 3: taking 10 from 40 and dividing the remaining number 30 by 3, the quotient is 10; adding 1, the number of terms becomes 11, as may be seen here:

$$10 \cdot 13 \cdot 16 \cdot 19 \cdot 22 \cdot 25 \cdot 28 \cdot 31 \cdot 34 \cdot 37 \cdot 40$$

But so that the proposed question be possible — that is, so that some progression may really exist in the nature of things, having all the proposed conditions — it is necessary that, the lesser [?] extreme having been subtracted from the greater, the remaining number be measured by the difference. For if it is not so measured, the quotient will not be a whole number, and hence cannot indicate the number of terms, even if 1 is added. For it cannot happen that any progression be given whose number of terms is not a whole number.

[The word "lesser" renders the sense required by the parallel rules III–VIII and by the worked example "10 from 40"; the printed word here is unclear and may read maggiore ("greater").]

VI.

If the two extreme terms, together with the sum of all the terms, are known, we shall discover both the number of terms and the difference by this art. We will divide the sum of all the terms by the sum of the extremes. The quotient will give half the number of terms, and doubled it will indicate the whole number. Then,

the number of terms having been found, since the two extreme terms are also known, we will find the difference as was handed down in the fourth rule — namely, the lesser extreme being taken from the greater, we divide the remaining number by the number next-less than the number of terms found. For example: if the sum is proposed as 515, and the extreme terms as 20 and 83, we will divide the sum 515 by 103, the sum of the extremes. The quotient, 5, doubled, will give the number of terms, 10; and we will obtain their difference if, the lesser extreme 20 being taken from the greater 83, we divide the remaining number 63 by 9, the number next-less than the number of terms found. For the quotient, 7, will be the difference, as you discern here:

$$20 \cdot 27 \cdot 34 \cdot 41 \cdot 48 \cdot 55 \cdot 62 \cdot 69 \cdot 76 \cdot 83$$

You see, then, that for the question to be possible — that is, for some progression really to exist in which the sum of all the terms and the two extremes stand as proposed — it is necessary that the given sum of all the terms be measured by the sum of the extremes. Otherwise the number of terms cannot be drawn out, if the quotient is not a whole number. Or at least, if the given sum of all the terms is not measured by the sum of the extremes, it is necessary that, once the sum of all the terms is divided by the sum of the extremes, the quotient be a whole number plus $\frac{1}{2}$, so that when doubled it can yield a whole number of terms.

VII.

If the extremes of three terms have a double ratio, the middle term will have to the difference a triple ratio. But if the ratio of the extremes is greater than double, the ratio of the middle term to the difference will be less than triple; and if, finally, the ratio of the extremes is less than double, the ratio of the middle term to the difference will be greater than triple. Thus here, 8 · 12 · 16: the ratio of the extremes 16 and 8 is double; and that of the middle term 12 to the difference 4 is triple. But here, 8 · 14 · 20: the ratio between the extremes 20 and 8 is *dupla sesquialtera* ($2\frac{1}{2} : 1$), namely greater than double; and that of the middle term 14 to the difference 6 is *dupla sesquitertia* ($2\frac{1}{3} : 1$), namely less than triple. And here, finally, 8 · 11 · 14: the extremes 14 and 8 have the ratio *supertripartiens quartas* ($7 : 4$, i.e. $1\frac{3}{4} : 1$), which is less than double; and the middle term 11 to the difference 3 has the ratio *trippla superbipartiens tertias* ($11 : 3$, i.e. $3\frac{2}{3} : 1$) — that is, greater than triple.

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[Clavius's own excursus on proportionality, appended to Book V — not Euclid's text. This opening continues the section De Proportionalitate Arithmetica, whose rules I–VII appeared earlier; rules VIII–X follow here, then the parallel section* De Proportionalitate Geometrica begins.]*

VIII.

Between any two numbers you will place the arithmetic proportional mean if you take half of their sum. Thus, given the two numbers 6 and 30, their sum is 36; so half of this, 18, stands as the arithmetic proportional in the middle position between 6 and 30, as is plain here: 6, 18, 30.

And so, if the middle term to be set up is to be a whole number, both of the given numbers must be either even or [both] odd, so that the sum of the extremes is always even — that is, so that it can be halved without a fraction of a unit. For when the sum of the extremes is odd — which happens when one of the extremes is even and the other odd — the middle term will be a whole number plus half a unit. Thus, given the two numbers 8 and 25, their sum is 33; so half of this is $16\frac{1}{2}$, and the middle term will fall out this way: 8, $16\frac{1}{2}$, 25.

IX.

Between any two numbers, however, you will place as many arithmetic proportional means as anyone directs, in this way. Subtract the smaller from the larger. Then divide the remainder by the number that is next larger than the number of means to be placed. The quotient will be the common difference of the arithmetic proportionality; add it to the smaller given number and you produce the first mean, add the same difference to that and you make the second, and so on. Alternatively, if you subtract that difference from the larger given number, the last mean is left; take the same difference away from that, and the second-to-last remains, and so on. For example, if eleven middle terms are to be placed between 6 and 114, we subtract 6 from 114 and divide the remainder, 108, by 12 (the number next larger than the number of means to be placed). The quotient, 9, will be the common difference of the terms. So the eleven middle terms between 6 and 114 will stand thus:

6.
 15.
 24.
 33.
 42.

51.
60.
69.
78.
96.
105.

But since usually the two numbers proposed are such that whole proportional means cannot fall between them — because the quotient found for the difference is not always a whole number — if you wish to avoid fractions, the two proposed numbers must be of such a kind that, when the smaller is subtracted from the larger, the remainder is measured by the number next larger than the number of means to be placed. You will easily find such numbers as follows. Take any number for the smaller extreme, and add it to some number that nine (which is next larger than the number of means to be established) measures — namely to 72 — and you will produce the larger extreme, 79. You will then find the difference, as was said, by subtracting 7 from 79 and dividing the remainder, 72, by 9. For the quotient, 8, will be the difference, as you see here:

7.
15.
23.
31.
39.
47.
55.
63.
71.

But if someone specifies both the smaller extreme and the common difference of the proportionality, you will find the larger extreme — such that between the smaller and the larger fall as many middle numbers as you wish, all having that difference — by multiplying the number next larger than the number of means to be placed by the given difference, and adding the smaller extreme to the product. For instance, if one wishes to set up between 5 and some larger number 9 middle terms whose common difference is 4, then 10 must be multiplied by 4, and to the product, 40, the smaller extreme, 5, must be added. For between the resulting

number, 45, and the given smaller extreme, 5, nine middle terms are contained, with the given difference 4, as is plain here:

5.
 9.
 13.
 17.
 21.
 25.
 29.
 33.
 37.

By this other pleasant procedure too you will set up as many proportional means as you wish between two given numbers. Divide each of the two given numbers by the number next larger than the number of means to be placed, and set the quotients each beneath its own [number]. From those same quotients, ascend by continually adding them — adding each first to itself, then to the resulting number, and so on — so that you build two arithmetic proportionalities that begin from the quotients and advance by those same [quotients], each having as many terms (apart from the two given numbers) as there are middle terms desired. Finally, add each term of the one series to the term of the other opposite it — that is, the greatest of the one to the least of the other, then the next below it to the one just above this, and so on. For the combined numbers will give the desired means. For example, let 4 arithmetic proportional middle terms be placed between 15 and 100. Dividing each by 5 (that is, by the number next larger than the number of means), the quotients become 3 and 20. So you will set up these two series of 4 arithmetically proportional terms beneath 15 and 100 by continually adding those quotients: adding the first, 3, to 3 to make 6; likewise 3 to 6 to make 9; likewise 3 to 9 to make 12. Again, adding 20 to 20 to make 40; likewise 20 to 40 to make 60; likewise 20 to 60 to make 80, as you see here:

15	32	49	66	83	100
12					80
9					60
6					40
3					20

From 12, the top of the first order, and 20, the bottom of the second order, comes 32 for the first mean. From 9 and 40 comes 49 for the second mean. From 6 and 60 comes 66 for the third mean. Finally, from 3 and 80 comes 83 for the fourth mean. And so, in the first order one must always descend from top to bottom, but in the second order ascend from bottom to top.

It is worth noticing that the difference of the quotients — which are the smallest numbers of the two series set up — is also the common difference of the proportionality being constructed. Thus you see that in the proposed example the difference is 17, and this same difference etiam

is also [the difference] between the quotients 3 and 20. Therefore, once the quotients are found, if the smaller is subtracted from the larger, and the remainder is added to the given smaller extreme, and to this combined number that same remainder is again added, and so on, the very same middle terms will be established, even if those two series are not set up.

You see, then, that each proposed number must be measured by a number exceeding the number of means by one unit, if fractions are to be avoided; or else the two extremes must be found, as we prescribed in the earlier part of this rule. For in that case too fractions are avoided, even if the number next larger than the number of means does not measure them — because when the smaller quotient is subtracted from the larger, a whole number is always left, as appears in the examples of that earlier rule. For in the third of them, for instance, 9 middle terms are to be placed between 5 and 45. So if each is divided by 10, the quotients become $5/10$ and $45/10$ — that is, $\frac{1}{2}$ and $4\frac{1}{2}$ — whose difference, 4, is a whole number, from whose continual addition to the given smaller extreme the desired middle terms are produced. These same terms will be found if, from the quotients $\frac{1}{2}$ and $4\frac{1}{2}$, two ascending series are set up, as was said, by the continual addition of those quotients, etc.

X.

We will distribute any proposed number into as many parts as one directs, preserving arithmetic proportionality, in this way. Dividing the given number by half the number of terms, we will cut the quotient into two unequal parts, however chosen, to serve as the extreme terms of the proportionality. Then, since two extreme terms have now been given along with the number of terms, we will subtract the smaller extreme from the larger and divide the remainder by the number next smaller than the number of terms. For thus the common difference will come out in the quotient, as is clear from Rule 4. If this difference is then

added to the smaller assumed extreme, and again to the resulting number, and so on, the given number of terms will be established, whose sum equals the proposed number; and accordingly the given number will have been cut into a number of arithmetically proportional parts. For example, if the number 830 is to be distributed into 10 arithmetically proportional parts, we will divide it by 5, half the number of terms, and cut the quotient, 166, into 20 and 146, the extreme terms of the proportionality to be established. Then, subtracting 20 from 146, we will divide the remainder, 126, by 9, the number next smaller than the number of terms. For the quotient, 14 (as was said in Rule 4), will be the common difference of the proportionality, as you see here:

20.
 34.
 48.
 62.
 76.
 90.
 104.
 118.
 132.

Again, if the number 10 is to be divided into 10 arithmetically proportional parts, we will divide it by 5, half the number of terms, and cut the quotient, 2, into two unequal parts, $\frac{1}{2}$ and $1\frac{1}{2}$, to serve as the extreme terms of the proportionality. Then, subtracting the smaller extreme, $\frac{1}{2}$, from the larger, $1\frac{1}{2}$, we will divide the remainder, 1, by 9, the number next smaller than the number of parts to be established. For the quotient, $\frac{1}{9}$, will be the common difference to be added continually to the smaller extreme, $\frac{1}{2}$, and to the resulting number, and so on. To make this addition easier, we will reduce the smaller extreme, $\frac{1}{2}$, and the difference, $\frac{1}{9}$, to the same denomination — namely to $\frac{9}{18}$ and $\frac{2}{18}$. So the number 10 will be divided into 10 arithmetically proportional parts, of which the first is $\frac{1}{2}$, or $\frac{9}{18}$, the last is $1\frac{1}{2}$, or $1\frac{9}{18}$, and the difference is $\frac{2}{18}$:

$\frac{9}{18}$. $\frac{11}{18}$. $\frac{13}{18}$. $\frac{15}{18}$. $\frac{17}{18}$. $1\frac{1}{18}$. $1\frac{3}{18}$. $1\frac{5}{18}$. $1\frac{7}{18}$. $1\frac{9}{18}$.

It is certain that in this way the given number can be divided in various ways into the proposed number of proportional parts — according, namely, as the quotient (when the given number is divided by half the number of terms) is cut into different pairs of unequal parts for the extreme terms.

We will accomplish the same thing by the following method. Take the given number as if it were the sum of all the terms to be established; divided by the given number of terms, the doubled quotient will give the sum of the extremes — because half of this sum of the extremes (that is, the quotient found), multiplied by the number of terms (that is, by the divisor), produces the sum of all the terms, namely the given number, as was said in Rule 2. So if we cut the doubled quotient into two unequal numbers for the extreme terms, and, subtracting the smaller from the larger, divide the remainder by the number next smaller than the given number of terms, the quotient will give the common difference, as we said in Rule 4. And if we add this to the smaller extreme just formed, and again to the resulting number, and so on, the required arithmetic proportionality will be established. For instance, if the number 198 is proposed to be cut into 9 arithmetically proportional parts, we will divide it by 9, the number of terms, so that the quotient becomes 22, which doubled gives 44, the sum of the extremes. So, setting the smaller extreme at 10 and hence the larger at 34, if we subtract 10 from 34 and divide the remainder, 24, by 8 (the number next smaller than the number of terms), the common difference 3 is produced. So the 9 terms of the arithmetic proportionality making up the sum 198 will stand thus:

10.
 13.
 16.
 19.
 22.
 25.
 28.
 31.

Here too it is plain that the given number can be distributed in various ways into the given number of parts — according, namely, as the first doubled quotient (which we said is the sum of the extremes) is cut into different pairs of parts for the two extreme terms.

But perhaps you will accomplish the same thing more easily (though the operation is somewhat longer) in this way. Take as many arbitrary arithmetically proportional numbers as the number of parts into which the given number is to be distributed, multiply each by the given number, and divide the products by the sum of all the terms taken, found by Rule 2. For the quotients will give the parts you seek. For example, let the number 525 be divided into 10 arithmetically proportional parts. Take any 10 arithmetically proportional numbers, 4, 7, 10, 13,

16, 19, 22, 25, 28, 31, whose sum is 175. Multiplying each by the given number, 525, these 10 numbers are created: 2100, 3675, 5250, 6825, 8400, 9975, 11550, 13125, 14700, 16275. Dividing each of these by 175, the sum of your numbers taken in arithmetic proportionality, the 10 parts you seek are produced:

12.
 21.
 30.
 39.
 48.
 57.
 66.
 75.
 84.

You will obtain the same result if you divide the given number by the sum of the assumed numbers, and multiply the quotient into each of the assumed [numbers]. assu

multiply into each of the assumed numbers. For the products will give the parts you seek. As in the given example: if the given number, 525, is divided by 175, the sum of the assumed numbers, the quotient is 3; and multiplying this into each of the assumed numbers, 4, 7, 10, etc., the same parts as before are produced: 12, 21, 30, etc.

To make the operation easier and shorter, it is enough to multiply the first two numbers of the assumed proportionality by the given number and divide the products by the sum of all the terms of that same assumed proportionality. Or it is enough, once the given number has been divided by the sum of the assumed numbers, to multiply the quotient into the first two assumed numbers. For thus the first two parts of the given number are found, and the difference between them, added to the larger, makes the third; and the same difference added to the third makes the fourth, and so on. And I think it is clearly established here too that the given number can be distributed in various ways into as many proportional parts as one directs — according, namely, as different numbers of the same arithmetic proportionality are assumed, or [numbers] whose difference is different.

By this method the parts found have, in order, the same ratios among themselves as are found, in the same order, among the assumed terms of the proportionality. And each number of the assumed proportionality always has exactly the same

ratio to each of the parts found — the first to the first, the second to the second, and so on. Thus you see that between the first parts found, 12 and 21, there is the ratio of 7 to 4 [supertripartiens quartas], just as there is between the first numbers taken, 4 and 7, etc. Likewise, 4 to 12, and 7 to 21, and 10 to 30, etc., have exactly the same ratio, namely the subtriple ratio [1 to 3].

And so, if an arithmetic proportionality is taken whose first two numbers have the double ratio [2 to 1] — so that the first is the common difference of all the terms — then the first parts too will have the double ratio, and therefore the first part will also be the common difference by whose addition the other parts are accumulated. For instance, if in the above example you had taken these 10 terms, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, whose sum is 55; or these 10 numbers, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, whose sum is 165; and had multiplied each first term by the given number, 525, and divided the products, 525 and 1575, by their respective sums — the former by 55 and the latter by 165 — you would have found the first part to be $9\frac{6}{11}$, which would also be the common difference. Hence all 10 parts constituting the arithmetic proportionality would have been these:

$9\frac{6}{11}$. $19\frac{1}{11}$. $28\frac{7}{11}$. $38\frac{2}{11}$. $47\frac{8}{11}$. $57\frac{3}{11}$. $66\frac{9}{11}$. $76\frac{4}{11}$. $85\frac{10}{11}$. $95\frac{5}{11}$.

DE PROPORTIONALITATE Geometrica.

[On Geometric Proportionality.]

I.

Given any two numbers, if you take the denominator of the ratio they have [denominator proportionis] by comparing the larger with the smaller — that is, if, dividing the larger by the smaller, you multiply the quotient into the larger — you will generate a third term in geometric proportionality, larger than both given numbers. And if you multiply the same denominator, or quotient, into this third term, you will produce a fourth, still larger, in the same proportionality. And so on, you will establish infinitely many others, always larger, if you keep multiplying the denominator, or quotient, by the last one found. Thus, given the two numbers 4 and 12, the denominator of the ratio of 12 to 4 is 3, since dividing 12 by 4 makes the quotient 3. If, then, you multiply 3 into 12, and again into the product, and so on to infinity, you will establish this geometric proportionality, which can be extended to infinity:

4.

12.

36.
 108.
 324.
 972.
 2916.
 8748. &c.

But if you compare the smaller with the larger and take the denominator of the ratio they have — that is, dividing the smaller by the larger — and multiply the quotient into the smaller; or, what is the same, divide the smaller by the denominator of the ratio that the larger has to the smaller, a third proportional number will be created, smaller than both. From this, by the same method, you will find a fourth still smaller, and so on to infinity. Thus, given the two numbers 8 and 16, the denominator of the ratio of 8 to 16 is $\frac{1}{2}$, since dividing 8 by 16 makes the quotient $\frac{1}{2}$. But the denominator of the ratio of 16 to 8 is 2, since dividing 16 by 8 makes the quotient 2. If, then, you multiply $\frac{1}{2}$ into 8 — or divide 8 by 2 — and do the same with the resulting number, this geometric proportionality will be established, progressing to infinity toward smaller numbers, just as that other one progressed to infinity toward larger numbers:

16.
 8.
 4.
 2.
 1. $\frac{1}{2}$. $\frac{1}{4}$. $\frac{1}{8}$. $\frac{1}{16}$. $\frac{1}{32}$. $\frac{1}{64}$.

The same proportionality will also be extended to infinity even if the denominator of the ratio is unknown, in this way. Given two numbers, multiply the larger by itself and divide the product by the smaller. For the quotient will be a third term larger than both. If you multiply this by itself and divide the product by the immediately preceding term, the quotient will give a fourth term, and so on. Thus, given the two numbers 3 and 6: multiplying 6 by itself makes 36, which divided by 3 gives 12, the third term. Again, multiplying 12 by itself makes 144, which divided by 6 gives 24, the fourth term; and so on to infinity, as you see here: 3. 6. 12. 24. 48. &c.

Again, given two numbers, if you multiply the smaller by itself and divide the product by the larger, the quotient will give a third term smaller than both. If you multiply this in turn by itself and divide the product by the immediately preceding term, the quotient will give a fourth term still smaller, and so on. Thus,

given the two numbers 6 and 18: multiplying 6 by itself makes 36, which divided by 18 gives 2, the third term. Again, multiplying 2 by itself makes 4, which divided by 6 gives $\frac{4}{6}$, or $\frac{2}{3}$ in lowest terms, for the fourth term; and so on to infinity, as appears here:

18.

6.

2. $\frac{2}{3}$. $\frac{2}{9}$. $\frac{2}{27}$. &c.

Now, if you wish to extend any non-multiple ratio (for in a multiple ratio there is no difficulty, if you follow the prescribed rule) into whole numbers, to as many larger terms as you like, you will accomplish it by this pleasant and easy procedure. Take as many continuously proportional terms of that multiple ratio, beginning from 1, whose denominator names the aliquot part or parts mentioned in the given ratio — take, I say, as many terms as in the proposed ratio propor

(Rule I, continued from the preceding page.)

you want terms in a proportionality that is not a multiple one. For the last of them will be the first term of your proportion, which is not a multiple. Therefore, if you multiply that term by the denominator of the given proportion, and again multiply the resulting product by the same denominator, and so on in turn, you will establish the terms you want. And here is the remarkable thing: if the required terms are found in this way, the proportionality cannot be extended beyond the proposed terms without a fraction. For example, suppose seven terms in sesquialterate proportion [proportio sesquialtera] (3:2) are to be found. Since the denominator of the fraction $\frac{1}{2}$ — which was just mentioned — is 2, one must take seven terms in double proportion [dupla] (2:1), namely 1. 2. 4. 8. 16. 32. 64. If, then, you begin from 64, you will establish seven terms of sesquialterate proportionality, and no more — these, of course:

64.

96.

144.

216.

324.

486.

These terms will easily be found if to the first, 64, you add its half, and to the second, 96, now produced, its half likewise, and so on. In the same way too, if six terms of the double-superbipartient-fifths proportion [dupla superbipartiens

quintas] (12:5) are wanted, one must take six terms of the quintuple proportion [quintupla] (5:1) — because of the denominator of the fifth-parts, which is 5 — namely 1. 5. 25. 125. 625. 3125. For if you begin from the last, 3125, you will find these six terms in double-superbipartient-fifths proportion:

3125.
 7500.
 18000.
 43200.
 38144.

These terms will easily be found if you add two fifths of the first to that same first doubled, and again two fifths of the second, now produced, to that same second doubled, and so on. And the numbers found in this way are always the smallest in their proportion — so much so that it is utterly impossible for as many other terms in the same continued proportion to be found which are smaller than these, unless we are willing to admit fractions. The same terms are found more easily if, once the two smallest terms in the given proportion have been discovered, the first term found from that multiple proportion is multiplied by the greater [of the two smallest], and the product is divided by the smaller; for in this way the second term will be found. And if this in turn is multiplied by the greater of the smallest terms, and the product divided by the smaller, the third term will be found; and so on. Thus in the latter example, the smallest terms of the double-superbipartient-fifths proportion are 5 and 12. So if the sixth term of the quintuple proportion, 3125 — which must be the first of the six terms sought — is multiplied by 12, and the product 37500 is divided by 5, the second term 7500 will result; and if this in turn is multiplied by 12, and the product 90000 divided by 5, the third term 18000 will be produced; and so on. Moreover, in any multiple proportion whatever, however many smallest terms there are, they always begin from 1. Thus the three smallest terms in continued quadruple proportion are these: 1. 4. 16.

II.

When the number of terms in continued proportion is odd, the number produced by multiplying the extremes by each other is equal to the number created by multiplying together any two terms equally distant from the extremes, and also equal to the number produced by the middle term multiplied into itself. As in these five numbers of sesquialterate proportion:

16.

24.

36.

54.

the number 1296 is produced as much from 16×81 as from 24×54 , and from $36 \times$ itself.

When, however, the number of proportional terms is even — even if they are not continuously proportional, provided that the pairs interrupting the continued proportion have one and the same ratio among themselves (that is, provided that the second and third; likewise the fourth and fifth; and also the sixth and seventh, and so on, the places where the proportion is interrupted, are themselves also non-continuously proportional, yet in a proportion different from the one that the first has to the second, the third to the fourth, and the fifth to the sixth, and so on) — when, I say, the number of terms is even, the number produced by multiplying one extreme into the other is always equal to the number generated by the multiplication of any two terms equally distant from the extremes. As here, in six numbers of continued double proportion:

3.

6.

12.

24.

48.

From 3×96 , and from 6×48 , and from 12×24 , the same number 288 is always produced.

Likewise in this non-continuous sesquitertian [sesquitertia] (4:3) sequence of eight terms, where the pairs interrupting the continued proportion have one and the same ratio — namely the triple, which is different from the sesquitertian:

3.

4.

12.

16.

48.

64.

192.

the very same number 768 arises from 3×256 , and from 4×192 , and from 12×64 , and from 16×48 .

HENCE it follows that, when the number of terms is odd, the number produced by multiplying one extreme into the other, or by the multiplication together of any two terms equally distant from the extremes or from the middle, is a square number. (A number is called *square* when it is produced by the multiplication of some number into itself; and the number producing it is called its *side* or *root*.) Its root, or side, is the middle number — because, of course, the same number is generated from the middle term into itself, and therefore the middle term is its square root. But the number produced from three terms multiplied together — of which two are either the extremes or equally removed from the extremes, and the third is the middle — is a cube. (That number is called a *cube* which is produced from some number multiplied into itself, and then into the resulting product; and the number which, multiplied into itself and again into the product, produces the cube, is called its *cubic side* or *cube root*.) Its side, or root, is the middle number — because, of course, it is generated from the middle term multiplied cubically into itself, that is, first into itself, then into the product of the other two multiplied together. For the middle term, multiplied into itself, makes the product of the other two; and therefore, multiplied again into this result, it produces the cube.

IT ALSO FOLLOWS that, when there are four numbers proportional either continuously or non-continuously, the number produced by the mutual multiplication of them all (three or more numbers are said to *multiply one another continuously* when one is multiplied into another, and the resulting number into a third, and this product into a fourth, and so on, until all the numbers are

(Rule II, continued.)

multiplied together) is a square, whose side or root is the number generated from the first into the fourth, or from the second into the third. For from the mutual multiplication of all four numbers together the same number is created which is produced when the number formed from the first into the fourth is multiplied by the product of the second into the third (which is equal to the product of the first into the fourth). And when there are six numbers, proportional either continuously or non-continuously — provided that the pairs of numbers interrupting the continued proportion are also non-continuously proportional — a cube arises from the mutual multiplication of all six numbers together, whose root or side is the number produced from the multiplication of the extremes, or

of any two equally distant from the extremes. This is generated from the number formed by the two extremes multiplied together, multiplied cubically into itself: that is, once into itself — namely into the product of the two nearest the extremes — and again into the product, when this resulting number is multiplied into the product of the two middle terms; since indeed these three products are equal to one another.

III.

IF, given any number of terms in continued proportion, the smaller extreme is subtracted from the greater, and the remaining number is divided by a number one unit less than the denominator of the proportion (which each of the proposed numbers has to the smaller number nearest it), and finally the quotient is added to the greater extreme, the sum of all the terms will be produced. As here, in continued triple proportion:

2.
 6.
 18.
 54.
 162.
 486.
 1458.
 4374.

With the smaller extreme 2 subtracted from the greater 13122, and the remaining number 13120 divided by 2 — that is, by a number one unit less than the denominator 3 of the triple proportion, which each of these numbers has to the immediately preceding smaller one — the quotient becomes 6560; and if the greater extreme 13122 is added to it, the sum of all becomes 19682. Likewise in this continued sesquitercian sequence:

243.
 324.
 432.
 576.
 768.

With the smaller extreme 243 subtracted from the greater 1024, and the remaining number 781 divided by $\frac{1}{3}$ (a number one unit less than the

denominator $1\frac{1}{3}$), the quotient becomes 2343. Therefore, with the greater extreme 1024 added, the sum of all the terms becomes 3367.

And so, as you see, to find the sum of any number of terms in geometric proportionality it is enough that the two extremes be known, together with the denominator of the proportion. But by what device the last term of any geometric proportionality of however many terms is to be investigated — even if we do not know the middle numbers — we have set out abundantly in the sections on progressions in our *Practical Arithmetic* [Arithmetica practica], so that it would be superfluous to repeat it here.

IV.

Given any number of numbers exceeding one another equally — that is, arithmetically proportional — if just as many other numbers are taken having the same ratios as those first ones taken in the same order, these later numbers will also exceed one another equally; that is, they will be arithmetically proportional, just like the first. For example, if these six numbers, 4. 7. 10. 13. 16. 19., are set out as arithmetically proportional, and just as many others are taken with the same ratios — of which sort are these, 20. 35. 50. 65. 80. 95. — you will find that they too are arithmetically proportional, since indeed they have a common difference of 15. But beware of thinking that this property can be converted: for it is not the case that, if any number of arithmetically proportional numbers are given, and just as many others are also taken as arithmetically proportional, then — even if the common difference is the same in both — these will necessarily be proportional to those. This is clear in these two arithmetic progressions:

A diagram showing 18 numbered points arranged in a circular pattern, representing the vertices of a regular 18-sided polygon. The points are numbered 1 through 18 in a clockwise direction starting from the top.

9.

11.

13.

15.

17.

19.

Here the first two numbers of the former, 2 and 4, have the double ratio, whereas the first two of the latter, 3 and 5, have the superbipartient-thirds ratio (5:3). Likewise the first number 3 in the latter order has to the first number 2 in the former order the sesquialterate ratio, while the second, 5, has to the second, 4, the sesquiquartan ratio (5:4), and so on.

V.

Given any two numbers, if one of them is divided into any number of parts, and the individual parts are multiplied into the other, and the individual resulting numbers are divided by the former number, all the quotients taken together will constitute the latter number, and they will have among themselves, in order, exactly the same ratios that the parts of the former number have among themselves in the same order. And the individual quotients — that is, the parts of the latter number — will have to the individual parts of the former (as the first to the first, the second to the second, and so on) one and the same ratio throughout. For instance, given the two numbers 57 and 285: if the former is cut into parts which together make it up, 2. 5. 10. 40., and each of them is multiplied into the latter number 285, these numbers will be generated: 570. 1425. 2850. 11400.; and when these are severally divided by the former number 57, the quotients become 10. 25. 50. 200., which gathered into one sum constitute the latter number 285, and have the same ratios that the parts 2. 5. 10. 40. have — namely double, sesquialterate, double, and quadruple. And the individual quotients to the individual parts — as 10 to 2, and 25 to 5, and 50 to 10, and 200 to 40 — have altogether one and the same ratio, namely quintuple.

You will obtain the same parts of the latter number if you divide it by the former (which was distributed into parts), and multiply the resulting quotient into the individual parts of that same former number. As in the same example: dividing the number 285 by 57 gives the quotient 5, and this multiplied into the parts 2. 5. 10. 40. of the number 57 gives the parts of the number 285 — these: 10. 25. 50. 200. — as before.

From this it is easily gathered how any two numbers may be cut into component parts equal to the numbers themselves, in such a way that the parts of the one are proportional to the parts of the other, comparing both the parts of each number with one another and the parts of the one with the parts of the other. And upon this property rests the third method of distributing a proposed number into any number of arithmetically proportional parts,

(Rule V, continued.)

proportional parts, which we prescribed in Rule 10 of arithmetic proportionality. For when just as many arithmetically proportional numbers are taken, their sum will be the number divided into those numbers as into the parts composing it. Therefore, if these individual parts are multiplied into the given number, and the resulting numbers are divided by that sum; or if the given number is divided by the sum of the assumed numbers, and the quotient is multiplied into each of those numbers — there will be produced just as many parts of the proposed number in the same ratios. And therefore they too will be arithmetically proportional, as is evident from the preceding Rule 4.

VI.

GIVEN any number of numbers in continued proportion, their differences too will be, in the same proportion, continuously proportional. As in the adjacent example both the numbers and their differences have the continued sesquialterate proportion:

Differences	32	48	72	108	162	243	
Terms	64	96	144	216	324	486	729

By the same reasoning, in this other example which follows, both the numbers and their differences have the double proportion. Here you see that the differences do not differ from the numbers of the double proportion of which they are the differences. This happens for the double proportion alone, whatever it may be — because in it any greater number exceeds the immediately preceding smaller one by that very number immediately preceding, of which it is the double, that is, which it contains twice.

Differences	7	14	28	56	112	224	448	896	
Terms	7	14	28	56	112	224	448	896	1792

VII.

GIVEN any number of numbers in continued proportion, if their sum is divided by each of them severally, the quotients will be, in reverse order, continuously proportional in the same proportion. And if the sum of the quotients is divided severally by the quotients themselves, exactly the same quotients will be created in reverse order; and so on to infinity. Moreover, the sum of those quotients is equal to the number produced from the first quotient into the last, or from the multiplication of any two equally distant from the extremes, or finally — if the number of terms is odd — from the middle multiplied into itself. This may indeed seem incredible. For example: if these four numbers in continued double proportion, 3. 6. 12. 24., are given, and their sum 45 is divided by each severally, the quotients will be 15. $7\frac{1}{2}$. $3\frac{3}{4}$. $1\frac{1}{8}$., continuously proportional in reverse order in the same double proportion. And if their sum $28\frac{1}{8}$ is divided by these quotients, exactly the same quotients will result in reverse order, and so on to infinity. And the sum itself, $28\frac{1}{8}$, is equal to the number produced from $15 \times 1\frac{1}{8}$, or from $7\frac{1}{2} \times 3\frac{3}{4}$. So too, given these five terms in continued triple proportion, 2. 6. 18. 54. 162., if we divide their sum severally by them, we shall produce these five quotients: 121. $40\frac{1}{3}$. $13\frac{4}{9}$. $4\frac{13}{27}$. $1\frac{40}{81}$., in the same continued triple proportion. Their sum, $180\frac{61}{81}$, is produced either from $121 \times 1\frac{40}{81}$, or from $40\frac{1}{3} \times 4\frac{13}{27}$, or finally from $13\frac{4}{9}$ into itself. Here two things are worth considering. First, that when the number of terms is odd, the sum of the quotients is a square number whose root is the middle term — since indeed it is produced by the middle term multiplied into itself, as has been said. Second, that the same quotients are always generated by dividing the sum of any number of terms of the same continued proportion, however great or small those terms may be, provided the number of terms is always the same. For instance, if there are three continuously doubled terms, whether great or small, the quotients will always be 7. $3\frac{1}{2}$. $1\frac{3}{4}$. If there are four, the quotients will be 15. $7\frac{1}{2}$. $3\frac{3}{4}$. $1\frac{1}{8}$. In five continuously tripled numbers the quotients will always be 121. $40\frac{1}{3}$. $13\frac{4}{9}$. $4\frac{13}{27}$. $1\frac{40}{81}$. And finally, in five numbers having the continued sesquialterate proportion, these quotients will be found: $13\frac{3}{16}$. $8\frac{19}{24}$. $5\frac{31}{36}$. $3\frac{49}{54}$ [?]. $2\frac{49}{81}$ [?].

FROM this property we shall find any number of terms in a given proportion, continuously proportional, whose sum is equal to the number produced from the extremes multiplied together, or from any two middle terms equally distant from the extremes, or even from the middle term multiplied into itself, if the number of terms is odd. For if as many terms as need to be found are taken in the given proportion, continuously proportional, whether great or small, and their sum is divided separately by them, the quotients will be the numbers that are sought. But if the sum is to be a square number, terms must be taken whose number is

odd and equal to the given odd number of terms, and so on. This may be reckoned as something of a marvel: that in a given proportion any number of continuously proportional terms one pleases can be furnished, whose sum is generated by the multiplication of the first term into the last — whereas we find the contrary in all continued proportions of inequality whose terms are whole numbers, and in all others too whose numbers have fractions mixed in, unless these quotient numbers are the products of dividing the sum of some number of continuously proportional terms by the proportional terms themselves, as we have said.

IN THE SAME WAY we shall satisfy the question, if anyone asks for two numbers having a given proportion whose sum is equal to the number produced from the multiplication of one into the other. For if any two numbers in the given proportion are taken, and their sum is divided by them severally, the proposed question will be satisfied by the quotients found. For example, if someone wants two numbers between which is found the triple-superbipartient-ninths proportion (29:9), we shall take either these two numbers, 29 and 9, or 87 and 27, having the given proportion, or any others whatever. For the sum of the first pair, 38, divided by 29 and by 9 — or the sum of the latter, 114, divided by 87 and 27 — will give the quotients $1\frac{9}{29}$ and $4\frac{2}{9}$, having the same proportion as 29 and 9, or 87 and 27, though in reverse order; and their aggregate is $5\frac{139}{261}$ [?], which number is likewise generated from the multiplication of $1\frac{9}{29}$ into $4\frac{2}{9}$.

VIII.

BETWEEN any two numbers whatever we shall establish a geometric mean proportional [medium Geometrice proportionale], if we multiply them by each other, and [take the root of what is] produced...

so that we take the square root of the number produced. For example, given the two numbers 6 and 96, multiplying one by the other gives 576, whose square root, 24, will be the mean proportional between the given numbers, thus: 6, 24, 96. For 96 stands to 24 in the quadruple ratio, and 24 stands to 6 in the same ratio.

Accordingly, if the mean number to be set between two given numbers is to be rational — that is, expressible in numbers — then the number produced by multiplying one into the other must be a perfect square. For unless it is a square, it will have no root among the numbers; rather its root will be a number that the arithmeticians call *surd* [surdum] or *irrational* [irrationalem], and which they express in this way: the square root of such-and-such a number. For instance, given the two numbers 7 and 9, multiplying one by the other yields 63, which has no root. We shall therefore say that the square root of 63, which cannot be

expressed in numbers, is the mean proportional between the given numbers 7 and 9, thus:

[Boxed inset: 7 | R.q. 63. | 9.] (*that is, 7, $\sqrt{63}$ [radix quadrata], 9*)

This root is greater than 7, since the square of 7 is 49, but less than 8, since the square of 8 is 64. Nor can 7-plus-some-fraction, which lies between 7 and 8, be the root, even if unity is imagined to be divided to infinity; for every such number-with-a-fraction produces a number-with-a-fraction, as we shall prove at Definition 8 — and 63 is not a number of that kind.

IX.

Should anyone wish to find three numbers in arithmetic proportionality such that between each pair of them a single geometric mean proportional falls, he will achieve it by the following method. Take any three numbers of which the second is five times the first and the third seven times the first. Their squares will then be arithmetically proportional, and the first number multiplied into the second will produce the geometric mean between the first square and the second, in the ratio of the first assumed number to the second; while the second assumed number multiplied into the third will generate the mean proportional between the second square and the third, in the ratio that the second and third assumed numbers bear to each other. This is plainly seen in the following example.

Tres num. assump.

[the three assumed numbers]	Primus. 8. [first, 8]	quintup. pri. 40. [5× first, 40]	Septup. pri. 56. [7× first, 56]
Quad. Arit. pro. [their arithmetically-proportional squares]	64	1600	3136.
Medij Geomet. [geometric means]	320	2240	

For the difference between the three squares is 1536, and the mean proportionals between them, 320 and 2240, are produced from 8 into 40 and from 40 into 56.

From this you will easily find three square numbers that exceed one another by equal differences, if for their roots you take three numbers of which the second is five times the first and the third seven times the first, as we said.

By the same reasoning, given any square, two larger squares will be found that together with it constitute an arithmetic proportionality. For if the root of the given square is set as the first number, and two other numbers are taken of which one is five times that root and the other seven times it, then the squares of these two numbers are the ones we seek. Thus if the square number 64 is given, its root 8 will be the first number, 40 the second, and 56 the third, and so on.

Once three arithmetically-proportional squares have been found, if you multiply them by any one and the same number, or divide them by any aliquot part they may have, the resulting products or quotients will likewise be arithmetically proportional, and between each pair of them a single geometric mean will again be intercepted. These means are found in the same way: the means between the discovered squares are simply multiplied by that same number, or divided by that same aliquot part. Thus, if the squares above, together with their mean proportionals, are doubled, these numbers arise:

Arithmet. proportio.	128	3200	6272
Medij Geometr.	640	4480	

But if the same numbers are divided by 8, an aliquot part common to the squares (and any aliquot parts of the squares are also aliquot parts of the means), these numbers result:

Arithmet. proportio.	8	200	392
Medij Geometr.	40	280	

Numbers found in this way will always keep the same ratios that the squares and their means have among themselves. Thus if you divide these — 8, 200, 392 — by 4, you find another set, 2, 50, 98, arithmetically proportional, between which the geometric means 10 and 70 fall. And if finally you divide these — 2, 50, 98 — by 2, you obtain yet another set, 1, 25, 49, between which the geometric means are 5 and 35.

Here, too, a point seems worthy of wonder: whenever between three arithmetically-proportional numbers — whether they are squares or not — two geometric mean proportionals fall, one between the first and the second and the other between the second and the third, then the square of the first mean, the square of the second number of the arithmetic proportionality, and the square of the second mean constitute an arithmetic proportionality. As in the latter example: the squares of the three numbers 40, 200, 280 are 1600, 40000, 78400 — arithmetically proportional, having the same common difference 30400 [recte:

38400; for $40000 - 1600 = 38400$ and $78400 - 40000 = 38400$, so the printed 30400 appears to be a compositor error] — and so with the rest.

X.

Between any two numbers whatever we shall find however many continued mean proportionals we please, by an operation both pleasing and ingenious. To explain it more clearly, a few things must first be said about the kinds of roots and about the way numbers are generated from their roots. In every continued proportion beginning from 1, then, the second term is the root of all the numbers that follow. For this root, set down twice and multiplied into itself, produces the third term, which is a square number, and the root itself is called its square root. The same root, set down three times — multiplied first into itself and then again into the product — generates the fourth term, which is a cube,

and is therefore called its cube root. Finally, if the same root is set down four times and three multiplications are performed — first into itself, then into the product, and thirdly again into the last product — the fifth term results; and if it is set down five times, the sixth term arises, and so on to infinity. But how all these roots ought to be named and denoted is not the place here to explain; the whole matter belongs to algebra, where we shall also teach by what art they are to be extracted from a given number. For now, so that the matter at hand may be understood, three continued proportionalities are to be noted, written below: the first two of them are arithmetic — namely two natural series of numbers beginning from 1 — while the third is geometric, likewise taking its start from 1. Here it will be plainly apparent what an excellent use the natural series of numbers has in the matter proposed.

Numerus mediorum. [number of means]		1	2	3	4	5	6	&c.
Numeri significantes radices. [numbers signifying the kinds of roots]	1	2	3	4	5	6	7	&c.
Numeri Geometr. propor. [geometrically-proportional numbers]	1	2	4	8	16	32	64	128 &c.

The first row comprises the number of means to be established; the second, the numbers that signify the kinds of roots — that is, how many times the second term of the third proportionality (which we called the root of all that follow) must be set down in order to produce, when multiplied, the corresponding numbers of the geometric proportionality. For example, the unit of the middle row signifies that the number 2 of the third row, which is the root of all the others, set down once, is not to be multiplied, but is itself the root of them all, produced by no

multiplication of itself. But the number 2 of that same second row indicates that the same root 2 of the third row, set down twice thus, 2 2, and so multiplied into itself, generates 4, the third term of that same third row. Likewise the number 3 of the middle row denotes that the same root 2 of the lowest row, set down three times thus, 2·2·2, and so multiplied, produces 8, the fourth term of the lowest row. For from the root 2 in the first place, into the same 2 in the second place, comes the number 4; and from this product into the root 2 in the third place is generated the number 8 — namely the fourth term of the geometric proportionality, standing under the number 3 of the second row. You may say the same of the others. Thus, for example, the number 7 of the second proportionality shows that the same root 2 of the third row, set down seven times in this manner, 2·2·2·2·2·2·2, and multiplied continuously into itself six times, produces 128, the eighth number of the geometric proportionality, placed under that number 7. For from 2 in the first place into 2 in the second place comes 4; from 4 into 2 in the third place comes 8; from 8 into 2 in the fourth place comes 16; from 16 into 2 in the fifth place comes 32; from 32 into 2 in the sixth place comes 64; and finally from 64 into 2 in the seventh place comes 128, and so on. And this holds not only in the proposed continued double proportion, but in all others beginning from unity, whatever number occupies the second place as the root of the rest and denominates the proportion.

Accordingly, if you wish to set however many means you like between two proposed numbers, take the number of means to be established in the first row. For from each given number the root is to be extracted which, set down in order and continuously multiplied, produces each given number as many times as there are units in the number of the second row that is written beneath the number of means to be established (taken in the top row). Place the roots you have found beneath the given numbers, each under its own, and from them ascend by the continuous multiplication of each into itself (multiplying first each root into itself, then into the resulting product, and so on in succession), so as to set up two multiplicative geometric proportionalities beginning from the roots and denominated by them, of as many terms — excluding the two given numbers — as there are means sought. After this, multiply each term of the one row into the opposite and facing terms of the other row — that is, the greatest of the one into the least of the other, then the next below it into the next above this, and so on. Thus in the one row you always descend from the highest to the lowest, while in the other you ascend from the lowest to the highest, as will be clear in the examples. For the numbers so produced will give the means we seek. For example, if a single mean proportional is to be found between 9 and 144, we shall take 1 in the first row, and under this term, in the second row, the number 2,

which indicates that from each given number a square root is to be extracted, which set down twice, by its multiplication, produces each — as we explained. Now the square roots found are 3 and 12. And because only a single mean term is wanted, there is no need to set up geometric proportionalities beginning from the roots that have more terms; rather the roots themselves are to be multiplied together. For the number produced, 36, will be the mean proportional between 9 and 144, as you see here: 9, 36, 144. So too, if a single mean term is to be set between 18 and 288: since these numbers have no square roots, we shall mark the former thus, R.q. 18 [$\sqrt{18}$], and the latter thus, R.q. 288 [$\sqrt{288}$]. These roots multiplied together make 72, the mean proportional between 18 and 288, thus: 18, 72, 288. Here, too, it is worthy of wonder that two numbers, neither of which can be expressed, multiplied together generate a rational number — the demonstration of which is to be sought from Book 10. But how two surd numbers, which of course cannot be expressed, such as $\sqrt{18}$ and $\sqrt{288}$, are to be multiplied together, we shall teach in the Algebra.

Again, if two numbers 16 and 625 are given, between which three means are to be established, we take in the first row the number of means, 3, beneath which in the second row is written the number 4. Therefore from each given number the root is to be extracted which, set down four times and continuously multiplied, produces each — the root that is usually called the *square of the square* [quadrati quadrata] or *zenzizenzic* [Zensizensica]; in the given example these roots are 2 and 5. You will therefore set up these rows of three terms each beneath 16 and 625, geometrically proportional in a multiple ratio, beginning from those same roots 2 and 5, from which the ratios of these rows are denominated, as appears here:

16	40	100	250	625
8				125
4				25
2				5

From 8, the highest of the first row, into 5, the lowest of the second row, comes 40, the first mean. From 4 into 25 comes 100, the second mean. Finally, from 2 into 125 comes 250, the third mean.

It is also worth knowing that the means found, together with the two given extremes, contain the same continued proportion that is found between the roots of the given numbers. For thus in the given example you see that 625 is to 250, and 250 to 100,

and 100 to 40, and 40 to 16, as 5 is to 2. Therefore, once the roots are found, if the greater is divided by the lesser, and the smaller given number is multiplied by the quotient (which is the denominator of the ratio of the roots found), and the resulting number is again multiplied by the same quotient, and so on in succession, the same mean terms will be set up between the two given extremes, even without setting up those two rows of multiple ratios denominated by the roots. As in the given example: dividing the root 5 by the root 2 gives the quotient $2\frac{1}{2}$, namely the denominator of the ratio of 5 to 2. If, then, you multiply the smaller given number, 16, by $2\frac{1}{2}$, you will generate 40, the first mean; and if you again multiply 40 by $2\frac{1}{2}$, you will produce 100, the second mean; and finally if you multiply 100 by $2\frac{1}{2}$, you will produce 250, the third mean.

It is clear, however, that between any two given numbers it is not always possible to establish the proposed number of terms as rational — because the given numbers do not always have the roots that must be extracted, or at any rate do not stand in the same ratio as some two numbers from which those roots could be extracted (which would also suffice). For just as between 16 and 625, which have square-of-square roots — that is, roots which set down four times produce them by their multiplications — three mean terms fall, so too between 48 and 1875, which are proportional to those 16 and 625, three mean terms fall (although they have no similar roots), as appears here: 48, 120, 300, 750, 1875.

Hence, if anyone wants two numbers between which however many rational mean terms you please fall in a given ratio, two numbers must be taken that somehow have the given ratio. For if each of them is set down as many times as there are mean terms to be sought, plus once more, and multiplied continuously, the last two numbers produced will be the ones we seek. Between them the mean terms will be established in the way we have set out — that is, either by setting up beneath them two rows of multiple ratios which the assumed numbers denominate, of as many terms as there are means sought; or by multiplying the smaller of them by the denominator of the given ratio, which holds between the two accepted numbers, and so on. For example, if one wants two numbers between which four rational mean terms can be set in the superbipartient-thirds ratio, whose denominator is $1\frac{2}{3}$, we take the two numbers 6 and 10, which have the given superbipartient-thirds ratio, and set each down five times thus, $6\cdot6\cdot6\cdot6\cdot6$ and $10\cdot10\cdot10\cdot10\cdot10$. For from the continuous multiplication of each these numbers arise, 7776 and 100000, between which you will set four mean proportional numbers, as you see here:

7776	12960	21600	36000	60000	100000	---	---	---	---	---	---	1296
	10000	216		1000	36		100	6		10		

These mean terms are found either from 1296 into 10, from 216 into 100, from 36 into 1000, and from 6 into 10000; or from the denominator $1\frac{2}{3}$ of the superbipartient-thirds ratio into the smaller number found, 7776, and then into the resulting product, and so on.

XI.

We shall distribute any given number into however many geometrically-proportional parts we please, in a given ratio, in this way. Take as many continued proportionals in the given ratio — whether the least ones or not the least — as the parts into which the given number is to be distributed, and divide the given number by their sum. For the quotient, multiplied into each of the terms taken in the given ratio, will produce the parts of the given number, keeping the given ratio among themselves. For instance, if the given number 756 is to be cut into 6 continued proportional parts in the double ratio: if six continued double terms are taken, 3, 6, 12, 24, 48, 96, whose sum is 189, and the given number 756 is divided by this sum, while the quotient 4 is multiplied into each of the accepted numbers, these six continued double numbers will be produced, which collected together into one sum make up the given number 756, namely:

12.				
	24.			
		48.		
			96.	
				192.

The same parts will be found if these other six continued double numbers are taken, $1/100$, $2/100$, $4/100$, $8/100$, $16/100$, $32/100$, whose sum is $63/100$; or these other six, 100, 200, 400, 800, 1600, 3200, whose sum is 6300. For the given number 756, divided by the former sum $63/100$, makes 1200; and this quotient, multiplied into the first accepted number $1/100$, will give the first part 12, as before, and so on. And the same given number 756, divided by the latter sum 6300, makes the quotient $756/6300$, which multiplied into the first accepted number 100 produces the same first part 12, and so with the rest.

But if you want a number that can be cut into however many proportional parts you please without any fraction (for unless the rule we are about to give is kept,

we shall generally fall into fractions), you will achieve your aim if you take any multiple of the sum of the same proportional terms assumed. For example, if one seeks a number that can be distributed into 5 whole numbers having a continued sesquialter [3:2] ratio: having taken any 5 continued terms in the sesquialter ratio, such as 64, 96, 144, 216, 324, whose sum is 844, we take any multiple of this sum — say the tenfold, 8440. For this, by the rule handed down, will be cut into 5 whole numbers in a continued sesquialter ratio, since, when it is divided by the sum of the accepted numbers, the quotient is necessarily a whole number.

Hence we shall with no trouble satisfy the question in which we are bidden to find, in a given continued ratio, however many numbers that, collected into one sum, make up any given number. For if the given number is cut into as many proportional parts of the given ratio as we have taught, the question will be solved. For example, if one wishes to find 4 numbers in a continued triple ratio whose sum is 100: this number 100 is to be cut into 4 numbers of triple ratio, as in $2\frac{1}{2}$, $7\frac{1}{2}$, $22\frac{1}{2}$, $67\frac{1}{2}$. For these make up the given number 100. But if they are to make up the sum 400, the numbers sought in continued triple ratio will be these: 10, 30, 90, 270. For their sum is 400. And if finally they are to make up a sum not exceeding 1, the numbers will be of this kind: $\frac{1}{40}$, $\frac{3}{40}$, $\frac{9}{40}$, $\frac{27}{40}$. For their sum is $\frac{40}{40}$, that is, 1.

On Harmonic Proportionality

[Clavius's treatise appended to Euclid, Book V. Section headings "I."–"V." are Clavius's own numbered rules.]

I.

Three numbers in harmonic proportion may be found from any three numbers in arithmetic proportion, by multiplying the first into the second and into the third, and the second into the third — as the examples set out below show.

Arit.	1. 2. 3.	3. 7. 11.	4. 6. 8.	10. 60. 110.
Har.	2. 3. 6.	21. 33. 77.	24. 32. 48.	600. 1100. 6600.

For in every case the first term of the arithmetic proportion, multiplied into the second and into the third, produces the first and second terms of the harmonic proportion; and the second multiplied into the third produces the third.

There is another way — which comes to the same thing — in which harmonic proportion is generated from arithmetic. The middle term of the arithmetic proportion, multiplied into the extremes, produces the extremes of the harmonic; while the extremes of that same arithmetic proportion, multiplied by each other, produce the middle term of the harmonic. This is evident in the very same examples.

From this it follows that the extreme terms of a harmonic proportion — and therefore also their differences — stand in the same ratio as the extremes of the arithmetic proportion from which the harmonic arose. For the extremes of the arithmetic proportion, multiplied by the same middle term, produced the extremes of the harmonic.

Therefore, if three numbers in harmonic proportion are to be found whose extremes — and hence whose differences — have a given ratio, take two numbers standing in the given ratio, set an arithmetic mean between them, and then from these three terms find the three harmonic proportionals, as we have taught. For example, suppose three numbers are sought whose extremes have the sesquitercian ratio [4:3]. Take 3 and 4, between which that ratio holds. But since their sum, 7, is odd, no whole-number arithmetic mean can be placed between them; so multiply both by some even number, namely by 2. Between the products 6 and 8 a whole-number arithmetic mean does fall — namely 7, half of their sum. From these three arithmetic proportionals 6, 7, 8, then, arise these three harmonic proportionals, 42, 48, 56, whose extremes — and therefore whose differences — have the given sesquitercian ratio.

It also follows that the pairs of terms of a three-term harmonic proportion are proportional, in reversed order, to the pairs of the arithmetic proportion from which the harmonic arose: that is, in the harmonic the second is to the first as, in the arithmetic, the third is to the second; and the second is to the first [as the third is to the second]. For the first two terms of the harmonic were produced from the last two of the arithmetic multiplied by the same first term, while the last two of the harmonic were produced from the first two of the arithmetic multiplied into the third. This is clear from the examples above.

Conversely, if the first number of a harmonic proportion is multiplied into the second and the third, and the second into the third, three numbers in arithmetic proportion will be produced. Or (what is the same) if the middle term of the harmonic is multiplied into the two extremes, the two extremes of the arithmetic will be generated; while the two extremes of the harmonic, multiplied by each other, will produce the middle term of the arithmetic — in exactly the same way

in which, we said, the harmonic arises from the arithmetic. Thus you see that from this harmonic proportion 2, 3, 6 there is generated this arithmetic one, 6, 12, 18. For whether you multiply the first, 2, into the second, 3, and the third, 6, and likewise the second, 3, into the third, 6, you will produce 6, 12, 18; or whether you multiply the middle, 3, into the extremes 2 and 6, you make the extremes 6 and 18, and from the first, 2, into the third, 6, the middle, 12.

A harmonic proportion is said to be continued beyond three terms — whether by advancing toward greater numbers or by returning toward smaller ones — when, the first three numbers being harmonic proportionals, two more are joined to the third which together with it likewise constitute a harmonic proportion, and in the very same ratio; then to the last, which is the fifth, two others are joined in the same way; and to the last, the seventh, two others again; and so on to infinity. This is done, advancing toward greater numbers, if the last two numbers are multiplied by the denominator of the ratio which the third bears to the first. Thus, given these three harmonic proportionals 2, 3, 6, if the last two, 3 and 6, are multiplied by 3, the denominator of the ratio of 6 to 2, they become 9 and 18, and there will now be five terms, 2, 3, 6, 9, 18, harmonic proportionals. And if the last two, 9 and 18, are again multiplied by 3, the denominator of the ratio, these seven terms, 2, 3, 6, 9, 18, 27, 54, will be continuous harmonic proportionals. In this way we can proceed to infinity, so that the harmonic proportion is always continued through the terms in the odd places — the third, fifth, seventh, ninth, and so on. But if one wishes to proceed toward smaller terms, the two smaller numbers must be divided by the denominator of the ratio of the extremes; and the two smaller numbers thus found must again be divided by the same denominator, and so on to infinity. Thus, given these three numbers, 18, 9, 6, if the two smaller, 9 and 6, are divided by 3, the denominator of the ratio between the extremes 18 and 6, they become 3 and 2, and there will now be five terms, 18, 9, 6, 3, 2. And if the last two found, 3 and 2, are again divided by the same denominator, 3, two others will be joined to the last term, 2, in harmonic proportion, thus: 18, 9, 6, 3, 2, 1, $\frac{2}{3}$; and so one may go on to infinity.

In another way — and indeed more properly — a harmonic proportion is continued, whether advancing toward greater numbers or toward smaller, when the first three numbers are harmonic proportionals; likewise, leaving out the first, the next three; and, leaving out two, the next three; and so on. But in this kind of continuation the ratio between the extremes of one triple will never be the same as that between the extremes of another triple. Thus, among these four numbers, 3, 4, 6, 12, the harmonic proportion is said to be continued, since both

the three 3, 4, 6 and the three 4, 6, 12 are harmonic proportionals; yet the extremes of the first, 3 and 6, have the double ratio [2:1], while the extremes of the second, 4 and 12, have the triple [3:1]. Likewise a harmonic proportion is said to be continued in these five terms, 10, 12, 15, 20, 30, since in the three 10, 12, 15, and in the three 12, 15, 20, and in the three 15, 20, 30, a harmonic proportion is found, even though the extremes of the several triples do not have the same ratios. In what way a harmonic proportion is to be continued in both directions in this manner, when it can be continued (for it cannot always be extended in this way by advancing toward greater numbers), and in what way any desired quantity of continuous harmonic proportionals, as we have set out, may be found — all this will become clear from what follows.

II.

Between any two given numbers you will set a harmonic mean in this way. Take the number produced by multiplying the difference of the two given numbers into the smaller of them, divide it by their sum, and add the quotient to the smaller number. The number so assembled will be the mean you seek. Or else multiply the same difference of the given numbers into the greater, and divide the product by their sum: for if you subtract this quotient from the greater given number, what remains will be the mean sought.

Thus, to find the mean between the two numbers 15 and 60, multiply their difference, 45, into the smaller, 15; divide the product, 675, by their sum, 75. For if you add the quotient, 9, to the smaller, 15, you will make up the middle term, 24, as is plain here: 15, 24, 60. You will find the same mean if you multiply the same difference, 45, into the greater, 60, and divide the product, 2700, by their sum, 75. For the quotient, 36, subtracted from the greater, 60, leaves the same middle term, 24.

You see, then, that whenever the sum of the two given numbers does not exactly measure the product of their difference into the smaller (or into the greater) of them, the mean found must necessarily be a whole number with a fraction. For example, to find the mean between 7 and 10, multiply their difference, 3, into the smaller, 7, and divide the product, 21, by their sum, 17. For the quotient, $1\frac{4}{17}$, added to that same smaller number, 7, makes the middle term $8\frac{4}{17}$, as you see here: 7, $8\frac{4}{17}$, 10. For just as 10 has to 7 the supertripartient-of-sevenths ratio [10:7], so does the difference of the greater terms, $1\frac{13}{17}$, have to the difference of the smaller terms, $1\frac{4}{17}$. Or else multiply their difference, 3, into the greater, 10, and divide the resulting number, 30, by their sum, 17: for the quotient, $1\frac{13}{17}$, subtracted from the greater, 10, will leave the same middle term, $8\frac{4}{17}$.

Others give the finding of the middle term between two given numbers in harmonic proportion in this way. Having found the two least numbers in the ratio of the two given numbers, they direct that the quotient obtained by dividing the difference of the two given numbers by the sum of the two least numbers found be multiplied into the smaller number found, and the resulting number subtracted from the greater given number; or that the same quotient be multiplied into the greater number found, and the resulting number added to the smaller given number. For always either the number so assembled, or this remainder, will give the middle term sought. Thus, to find the mean between 20 and 30: the least numbers in the ratio of 20 to 30 are 2 and 3. If, then, the difference of the given numbers, 10, is divided by the sum of those found, 5, the quotient is 2; and if we multiply this into the smaller found, 2, and add the product, 4, to the smaller given number, 20 — or if we multiply the same quotient, 2, into the greater found, 3, and take the product, 6, from the greater given number, 30 — the middle term, 24, will always be found, as appears here: 20, 24, 30.

The middle term will be found more briefly thus. Divide the difference of the given numbers by the number that exceeds by one unit the denominator of the ratio which the given numbers have. The quotient, added to the smaller term, will give the mean. Thus, given the numbers 10 and 40, which have the quadruple ratio: if their difference, 30, is divided by 5 — the number that exceeds the denominator of the ratio by one unit — the quotient becomes 6, which, added to the smaller term, 10, produces the middle term, 16, thus: 10, 16, 40.

But to aid the memory we shall perhaps find it more convenient (though by a somewhat longer operation) to find the middle term between two given extremes in this way. If the two extremes are not both odd, or both even, but one even and the other odd, double them, and between the doubled numbers set an arithmetic mean, which will always be half their sum. Then, through these three arithmetic proportionals, seek three harmonic proportionals by Rule I. For the middle term found in this way will lie between two numbers that have the same ratio to each other as the two given extremes. If, then, it is arranged that as the first number found is to the second found, so the first given number is to another — that is, if the number made by multiplying the second found into the first given is divided by the first found — the mean between the two given extremes will be found. For example, to find the mean between 7 and 10, double them so that they become 14 and 20. Their sum, 34, is even, and its half, 17, is the arithmetic mean between 14 and 20, thus: 14, 17, 20. From these three there will be found these three harmonic proportionals, 238, 280, 340. If, then, it is arranged that as the first

found, 238, is to the second, 280, so the first given number, 7, is to another, there will be found the number $8\frac{4}{17}$, which is the harmonic mean between 7 and 10 that were proposed from the beginning.

III.

Given two numbers, we shall find a third greater than both in harmonic proportion — when this can be done (and the working itself will nicely show when it cannot) — in this way. We shall divide the number generated by multiplying one into the other by the number that remains when the difference of the two is subtracted from the smaller given term. For the quotient will be the third term, greater than both, which we seek. But whenever the divisor turns out to be 0, or whenever the difference of the two cannot be subtracted from the smaller term, it is impossible for a greater third term to be joined to the two given numbers in harmonic proportion. Thus, if the two smaller terms 12 and 16 are given, we shall divide the number produced from them, 192, by 8 — the number that remains when the difference of the two, 4, is subtracted from the smaller, 12. For the quotient, 24, together with the two given numbers, constitutes this harmonic proportion, 12, 16, 24. We shall extend it if to the two terms 16 and 24 we join a third, namely by dividing 384 — the number produced from 16 into 24 — by 8, the number that remains once the difference of the two, which is 8, has been subtracted from the smaller, 16. For the number 48 will be found, so that these four terms will stand in harmonic proportion: 12, 16, 24, 48. But if we try to join yet another, greater term to these, we shall labor in vain. For with the two last numbers 24 and 48 given, the divisor is found to be 0; since the difference of the two, which is 24, subtracted from the smaller, 24, leaves 0.

If someone proposes these two numbers, 10 and 12, there will be joined to them a third greater than both, 15. To these two, 10 and 11, there will be added the third $12\frac{2}{5}$ [?]. And to the two 90 and 99, the third 110. But to 3 and 6, none can be joined: for the difference between them, which is 3, taken from the smaller [term], 3 [?], leaves 0. And in the same way, whenever the given numbers have the double ratio, no proportional third can be joined to them, since the difference of the two is always equal to the smaller term. Much less, when the given numbers have a ratio greater than double, will any greater proportional third be found for them in harmonic proportion, since the difference of the two is then always greater than the smaller term, and therefore the subtraction cannot be made. Thus, if the numbers 3 and 7 are given, whose ratio is double-sesquitercian [7:3] — that is, greater than double — you see that their difference, 4, is greater than the smaller term, 3. Hence no proportional third can be joined to them.

Others investigate the third term thus. They multiply the difference of the given numbers into the greater, divide the resulting number by the number that remains when the same difference is subtracted from the smaller number, and finally add the quotient to the greater, to make the third. Thus, given the two numbers 12 and 16: if their difference, 4, is multiplied into 16, and the product, 64, is divided by 8 (the number that remains once the same difference, 4, is subtracted from the smaller, 12), the quotient is 8, which, added to the greater, 16 [?], makes the third, 24, as before.

THEREFORE, in order that a greater proportional third may be joined to two numbers, their ratio must be either superparticular or superpartient — that is, less than double. From this it plainly follows that a harmonic proportion cannot proceed to infinity toward greater numbers, if we speak of continuation taken in the second manner, which is its proper sense, as

IV. [continued from the preceding — running head: EVCLID. GEOM., p. 198]

we said in the first rule. For as soon as one arrives at two numbers whose ratio is double, or greater, there the proportion necessarily halts, and will not be carried further.

It also follows, from the earlier rule, that when the smaller term exceeds the difference of the two by just one unit, the third is generated by multiplying the first into the second: for this product, divided by that one unit, gives the third number, as we have said. And it is agreed that when a number is divided by one unit, the quotient is the same number divided. Thus, if two numbers 6 and 11 are given, since the smaller, 6, exceeds the difference of the two, 5, by one unit, the third proportional will be the number 66, which is produced from 6 into 11, as you see here: 6, 11, 66. The same is seen in these numbers, 2, 3, 6 and 10, 19, 190. For the third is always produced from the first into the second, because the first exceeds the difference of the two prior terms by one unit alone.

From the later rule it follows that when the smaller term exceeds the difference of the two by one unit alone, the third is generated if the product of the difference of the two into the greater is added to the greater: for that product, divided by that one unit, does not yield a quotient different from the product itself, and so on. Thus, given the same numbers, 6 and 11: if the difference, 5, is multiplied into 11 and the product, 55, is added to the greater, 11, the third term becomes 66, as before.

Hence three harmonic proportionals can also easily be found: if for the first any number is taken; for the second, its double with one unit removed beforehand; and for the third, the number produced from the first into the second. Thus, if the first is 8, the second will be 15, and the third 120; and so with the rest. And the ratio both of the extremes so found and of the differences will always be a multiple denominated by the middle term, which is invariably odd. Thus in these three, 8, 15, 120, the ratio is fifteenfold.

Hence, if three least whole numbers in harmonic proportion are sought in any multiple ratio: if the denominator is an odd number, we shall set that denominator in the middle, and half of the number next greater than it in the first place; and in the third place we shall put the number produced from the first into the second. Thus, if three whole and least numbers are wanted in the sevenfold ratio, we shall place 7 in the middle, and 4 (that is, half of 8, the number next greater than 7) in the first place, and in the third the number 28, produced from 4 into 7, as you see here: 4, 7, 28. But if the denominator is even, we shall set the number next greater in the first place; in the third we shall put the number produced from that next-greater number into the denominator; and in the middle we shall place the doubled denominator. Thus, if three least whole numbers are to be found in the eightfold ratio, we shall set 9, the number next greater than the denominator 8, in the first place; in the third, the number 72, generated from 9 into 8; and in the middle, finally, the double of the denominator, namely 16, as appears here: 9, 16, 72.

But if three numbers in harmonic proportion are to be found in a given ratio, whether they are least and whole or not, we must proceed in this way. In the middle we shall set the denominator of the given ratio; in the first, half of the number that exceeds the denominator by one unit; and in the third place we shall put the number made by multiplying the first into the middle. For example, if three are sought in the given double-supertripartient-of-sevenths ratio [$2\frac{3}{7}$ to 1]: the denominator of this ratio will be $2\frac{3}{7}$, to be set in the middle; and in the first, the number $1\frac{5}{7}$, namely half of $3\frac{3}{7}$, the number that exceeds the denominator $2\frac{3}{7}$ by one unit. The third, finally, will be $4\frac{8}{49}$, made from the first, $1\frac{5}{7}$, into the middle, $2\frac{3}{7}$. Thus the three numbers found will stand as $1\frac{5}{7}$, $2\frac{3}{7}$, $4\frac{8}{49}$; and these will be reduced to the same denominator — $\frac{84}{49}$, $\frac{119}{49}$, $\frac{204}{49}$ — if each is first reduced to a single fraction. The numerators, 84, 119, 204, will be whole numbers in harmonic proportion, and the ratio of the extremes, and therefore of the differences, will be double-supertripartient-of-sevenths, which was in fact the given ratio.

IV.

Given two numbers, we shall find a third smaller than both in harmonic proportion, in this way. We shall divide the number produced from one into the other by the sum gathered from the greater given number and the difference of the two. For the quotient will be the third term sought. Thus, if the two numbers 6 and 12 are given: if we divide the number 72, made from 6 into 12, by 18, the sum gathered from 12 and the difference of the two, 6, we shall find the third, 4, smaller than both and proportional to them, as you see here: 4, 6, 12. We shall extend this, going back toward smaller numbers, if to the two numbers 4 and 6 we join a smaller third, 3, thus: 3, 4, 6, 12; the number 3 being found by dividing the number 24, made from 4 into 6, by 8 — namely the sum gathered from 6 and the difference of the two, 2. In the same way, to the two smaller numbers 3 and 4 there will be joined a smaller third, $2\frac{1}{2}$ [?]. And in this manner any harmonic proportion will decrease continuously to infinity.

Here one may observe a truly admirable variety among these three proportions — the arithmetic, the harmonic, and the geometric — a variety that is explained by these three propositions:

I. Arithmetic proportion increases to infinity, but does not decrease to infinity.

II. Harmonic proportion, by contrast, decreases to infinity, but cannot be increased to infinity. This is to be understood if it must be continued so that any three numbers have a harmonic proportion — that is, so that the first, second, and third are harmonic proportionals; likewise the second, third, and fourth; likewise the third, fourth, and fifth; and so on. For otherwise it can be extended in both directions to infinity, like the geometric, as we said above in Rule I.

III. Geometric proportion, finally, both increases and decreases to infinity.

It is also worth noting that in a cube there are found four continuous terms in harmonic proportion which, variously compared with one another, express the chief and perfect musical consonances. For its 6 square faces, 8 solid angles, 12 edges, and 24 plane angles constitute these four terms, 6, 8, 12, 24, continuous harmonic proportionals. Now the ratio of 8 to 6 is sesquitercian [4:3], which makes the consonance Diatessaron, or the Fourth. The ratio of 12 to 8 is sesquialter [3:2], containing the consonance Diapente, or the Fifth. The ratio next, of 12 to 6, or of 24 to 12, is double [2:1], setting out the consonance Diapason, or the Octave. But the ratio of 24 to 8 is triple [3:1], producing the consonance of

Diapason and Diapente together — that is, the Twelfth. Finally, the ratio of 24 to 6 is quadruple [4:1], exhibiting the consonance Disdiapason, or the Fifteenth.

V.

Given any number of numbers in arithmetic proportion, if any number measured by them — whether the least or not the least — is divided by each of them singly, the quotients will be, in reversed order, harmonic proportionals. And conversely, given any number of numbers in harmonic proportion, if the number they measure is divided by each singly, the quotients will be, in reversed order, arithmetic proportionals. For example, let there be arithmetic proportionals 1, 2, 3, 4, 5, 6. The number they measure is 720, generated by their continuous multiplication into one another. But the least number measured by them is 60, which you will find by the method we set out in the *Practical Arithmetic*, chapter 10. So, the first [720] being divided by each of the given numbers, there will be found these in continuous harmonic proportion:

720.
360.
240.
180.
144.

And when the latter number [i.e. the least, 60] is divided by the same given numbers, these are found:

60.
30.
20.
15.
12.

Now these numbers that have been found have, two by two, the same ratios as the given numbers of the arithmetic progression, but in reverse order: namely, as in the harmonic series the first is to the second, so in the arithmetic the second is to the first; and as in the harmonic the second is to the third, so in the arithmetic the third is to the second, and so on. Moreover, the numbers found by way of the least number measured [60] are the least in their respective ratios. Conversely, if the number measured by the numbers found is divided by each of them singly, an equal count of arithmetic proportionals will be produced; and if indeed the least number they measure is divided, the very same arithmetic proportionals

will come out, in the same order in which they were given — but in the reverse order when they are compared with the harmonic proportionals found. For it is always the case that if in one progression the numbers proceed from smaller to greater, in the other the progression runs from greater to smaller, and vice versa.

Therefore, if anyone wants any number of terms in continued harmonic proportion, he must take an equal number of terms arithmetically proportional in any way whatever, and divide the number measured by them (and indeed the least such number, if the least terms are wanted) by each one singly, as we have said.

VI.

Given three numbers in harmonic proportion, the product of the difference of the two earlier terms into the third term equals the product of the difference of the two later terms into the first term. As you see here: for both 2 into 12 and 4 into 6 make the number 24. The reason is this: since, as the first term 12 is to the second 6, so the third term 4 is to the fourth 2, the number produced from the first, 12, into the fourth, 2, will equal the number produced from the second, 6, into the third, 4 — as is clear from the second rule of geometric proportion. The same holds for more terms, if they are taken three at a time together with their differences, as appears in these six terms.

Differences	2 4	
Harmonic prop.	6 8	

Differences	30	10	5	3	2
Harmonic prop.	60	30	20	15	12

For 30 into 20 and 10 into 60 both make 600. Likewise 10 into 15 and 5 into 30 both make 150. Again, 5 into 12 and 3 into 20 both make 60. And finally 3 into 10 and 2 into 15 both make 30. And so, as you see, by this pleasant operation you can test whether any given number of terms preserve continued harmonic proportion or not. For if you take them three at a time with their differences, and if the differences, taken in interchanged order, are multiplied into the first and second terms and produce equal numbers on both sides, then all the terms are in continued harmonic proportion; but if not, by no means.

VII.

Now suppose someone wishes to find five least numbers in which all three kinds of proportionality occur — namely, so that the first three numbers have arithmetic proportion; then, dropping the first, the three following have geometric proportion; and the last three have harmonic proportion; and finally the first, third, and fifth have a given continued geometric ratio. Or suppose someone wishes to find three numbers continuously proportional in a given ratio, such that between the first and second falls an arithmetic mean, between the second and third a harmonic mean, and finally the middle of the three given numbers is the geometric mean between the two means found — and such that the five numbers so found are the least of all in which this can occur. This will be accomplished as follows.

Let the given ratio be triple, in which the three least terms are taken, 9, 3, 1. Between the first two [9 and 3] let the arithmetic mean be placed, 6. By this [6] divide the square of the assumed middle number 3 [that is, 9], and let the quotient $1\frac{1}{2}$ be placed between 3 and 1, as appears in the first order set out here. But since this quotient is a whole number with a fraction, we shall reduce it to the single fraction $\frac{3}{2}$, as you see in the second order. And if we take its numerator 3, leaving the denominator, and multiply the other numbers of the second order by the denominator 2, we shall produce the five least numbers sought, as you discern in the third order. For 18, 12, 6 have arithmetic proportion; 12, 6, 3 [recte: printed "12.3.6"] geometric; and 6, 3, 2 harmonic; and finally 18, 6, 2 the given triple ratio. But if the five least terms in these three continued proportionalities need not be found, one may take any three numbers having the given continued ratio, and with them the two means — the second and fourth — are to be investigated, as has been said.

9.	6.	3.	$1\frac{1}{2}$.	1.
9.	6.	3.	$\frac{3}{2}$.	1.
18.	12.	6.	3.	2.

If the smaller terms are to be compared with the greater, and the given ratio is subtriple, we shall find, by the same device, from these three least numbers 1, 3, 9, these five subtriples: 2, 4, 6, 9, 18. For 2, 4, 6 keep arithmetic proportion; 4, 6, 9 geometric; 6, 9, 18 harmonic; and 2, 6, 18 the proposed subtriple ratio. You see the method of finding them in these three orders set alongside.

1.	2.	3.	$4\frac{1}{2}$.	9.
1.	2.	3.	$9/2$.	9.
2.	4.	6.	9.	18.

Again, let the given ratio be sesquialter (3:2), in which the three least terms are 9, 6, 4. Between 9 and 6 the arithmetic mean is $7\frac{1}{2}$, as in the first order. Let it be reduced to the single fraction $15/2$, as in the second order. Taking the numerator 15 for the second number, and leaving the denominator, let the other numbers in the second order be multiplied into the denominator 2 — as is done in the third order. And let the square of the number 12, namely 144, be divided by 15, and let the quotient $9\frac{3}{5}$, in least terms, be placed as the mean between 12 and 8 of that same third order. This number $9\frac{3}{5}$ we shall reduce to the single fraction $48/5$, as in the fourth order. Finally, taking the numerator 48 for the penultimate number and leaving the denominator, we multiply the other numbers of the fourth order into the denominator 5; and the five least numbers sought will be found, as appears in the fifth order.

9.	$7\frac{1}{2}$.	6.		4.
9.	$15/2$.	6.		4.
18.	15.	12.	$9\frac{3}{5}$.	8.
18.	15.	12.	$48/5$.	8.
90.	75.	60.	48.	40.

If smaller numbers are to be compared with greater, the five least terms in the given subsequalter ratio will be found, as you see here in the fourth order. And the same reasoning applies in the rest.

You can also, if you like, set out all five numbers in the first order, reduce the fractions, if there are any, to a single least denomination, and then reduce each to a single fraction. For if you substitute the numerators for them, keeping the common denominator, and multiply the remaining whole numbers by the common denominator, you will produce the five least whole numbers sought. For example, if a subquadruple ratio [1:4] is proposed, the three least terms will be 1, 4, 16. Adding 1 and 4 together, half the sum is $2\frac{1}{2}$, the arithmetic mean between them. And if the square of the number 4 [namely 16] is divided by $2\frac{1}{2}$, the quotient is $6\frac{2}{5}$, the harmonic mean between 4 and 16. The two fractions are then reduced to the same denomination, as in the second order. And when each fraction is reduced to a single fraction, the third order is formed. Then, taking the numerators, keeping the common denominator, and multiplying the whole

numbers by the common denominator, the five least numbers sought are formed, as you see in the fifth order. And in this way, into however many terms fractions are mixed, you will reduce them to as many whole terms whenever the matter requires.

4.		6.		9.
4.	5.	6.	$7\frac{1}{5}$.	9.
4.	5.	6.	$36/5$.	9.
20.	25.	30.	36.	45.

1.	$2\frac{1}{2}$.	4.	$6\frac{2}{5}$.	16.
1.	$2\frac{5}{10}$.	4.	$6\frac{4}{10}$.	16.
1.	$25/10$.	4.	$64/10$.	16.
10.	25.	40.	64.	16 [recte: 160]

VIII.

When four numbers are so arranged that they are geometrically proportional (though not continuously), while one of the means keeps arithmetic proportion with the extremes and the other keeps harmonic proportion — so that in those four numbers all three proportionalities we have spoken of are found — those four numbers are said to constitute the greatest harmony. But when four numbers are so arranged that only two proportionalities are found in them — either geometric and arithmetic, or geometric and harmonic, or harmonic and arithmetic — those four numbers are said to constitute a lesser harmony.

You will construct the greatest harmony, in which the extreme terms also have a given ratio, thus. Let the given ratio be that of 6 to 1, which the extreme terms must have. Since their sum is odd, I take their doubles, 12 and 2, between which let the arithmetic mean be placed, 7, thus: 12, 7, 2. But if their sum had been even, I would have taken the arithmetic mean between them at once. From these three arithmetic proportionals I find, by the first rule, 84, 24, 14, whose extremes have the given ratio — the same as that of 12 to 2, or 6 to 1. So if between the extremes 84 and 14 (which, being found in this way, will always be even) an arithmetic mean, 49, is also placed, the four numbers sought will be 84, 49, 24, 14. For the extremes have the given ratio of 6 to 1. Next, all three proportionalities are found in them. For among the four there is a non-continuous geometric proportion, since the ratio of 84 to 49 is the same as that of 24 to 14, and that of 84 to 24 the

same as that of 49 to 14 — as is clearly evident from the seventh property of these three proportionalities. Between 84, 49, 14 there is arithmetic proportion; and finally between 84, 24, 14 harmonic proportion, by construction, as you see here.

	84	49	24	14	
Three proportionalities together	.	.	.	.	
Geometric prop.	84	49	24	14	Vel: 84. 24. 49. 14.
	.	.	.	.	
Arithmetic prop.	84	49		14	
	.	.		.	
Harmonic prop.	84		24	14	
	.		.	.	

Again, let the given ratio be that of 3 to 2. Since again their sum is odd, let their doubles be 6 and 4, between which let the arithmetic mean be placed, 5, thus: 6, 5, 4. From these, by the first rule, I find the harmonic proportionals 30, 24, 20. And placing the arithmetic mean 25 between 30 and 20, by the same reasoning the four numbers sought will be 30, 25, 24, 20, since they have all three proportionalities, as appears here.

Geometric prop.	30	25	24	20	Vel: 30 24 25 20
Arithmetic prop.	30.	25.		20.	
Harmonic prop.	30.		24.	20.	

We shall find the same four numbers of the greatest harmony if, for the extremes, we take any two numbers having the given ratio, whether even or odd. For if between these the arithmetic mean is placed, and it is arranged so that as the found term is to the first extreme, so the other extreme is to another term, then this found term will be the harmonic mean between the same extremes. For when, in four geometrically proportional numbers — such as the four found — one of the means is the mean in arithmetic proportion, the other of the means is the mean in harmonic proportion between the same extremes, as we said in the seventh property of these three proportionalities. So when these four numbers found are whole numbers, what is proposed will be accomplished; but if fractions are mixed in, we shall reduce them to whole numbers, as we wrote at the end of the preceding rule 7. For instance, if the given ratio is double, we shall set the extremes as 12 and 6. Half of their sum, 9, will be the middle term between them, arithmetically proportional, thus: 12, 9, 6. If, then, it is arranged that as 9 is to 6, so 12 is to another term — or as 9 is to 12, so 6 is to another — the number 8 will

always be found between 12 and 6, in the harmonic proportion of the greatest harmony: 12, 9, 8, 6. Finally, if in the same double ratio of the extremes we take 20 and 10, we shall find the arithmetic mean 15 and the harmonic mean $13\frac{1}{3}$, in least numbers, thus: 20, 15, $13\frac{1}{3}$, 10. So if we reduce the third number to the single fraction $\frac{40}{3}$, and, taking its numerator 40 with the denominator left aside, set it as a term, others will be found if we multiply the ones found by the denominator 3: 60, 45, 40, 30.

We shall find the lesser harmony thus. If in four numbers only geometric proportionality together with arithmetic is to be found, take three continuously geometrically proportional numbers whose extremes are both even or both odd. For if you place between the extremes an arithmetic mean, you will have the four numbers sought. For example, if these three sesquialter numbers are taken, 16, 24, 36, and between the extremes the arithmetic mean 26 is found, the four numbers sought will be 16, 24, 26, 36, of which the three 16, 24, 36 are geometrically proportional, and the three 16, 26, 36 arithmetically proportional, by construction.

But if only geometric together with harmonic is desired, likewise take any three continuously geometrically proportional numbers. For if between the extremes you find a mean in harmonic proportion, you will have the four numbers that are sought.

For example, if the same three sesquialter numbers are taken, 16, 24, 36, the harmonic mean between the extremes will be found, $22\frac{2}{13}$. So the four numbers found will stand thus: 16, $22\frac{2}{13}$, 24, 36 — which are reduced to these whole numbers: 208, 286 [recte: 288], 312, 468. And if you had taken those three doubled numbers, 20, 10, 5, you would have found the harmonic mean between the extremes, 8, thus: 20, 10, 8, 5.

Finally, if you want harmonic together with arithmetic proportionality alone, take any three harmonic proportionals. For if between the middle term and either of the extremes you place an arithmetic mean, the four numbers we seek will be found. For example, if these three are taken, 6, 8, 12, the arithmetic mean between the first two will be 7. So the four numbers found will be these: 6, 7, 8, 12. But between the latter two the arithmetic mean will be 10. So the four numbers found will stand thus: 6, 8, 10, 12 — which are all (as happens by some chance in this example) arithmetically proportional, and not only the three 8, 10, 12 by construction; whereas only the three actually taken, 6, 8, 12, have harmonic proportionality.

IX.

If any number of continuously arithmetically-proportional numbers are to be found, between each successive pair of which a single whole harmonic mean falls, it will be accomplished as follows. Take as many arithmetically-proportional terms as are sought, and take the least terms of the ratio of each neighbouring pair. If the least number measured by the sums of the pairs of least terms taken is found, and this is multiplied into each of the arithmetically-proportional numbers taken, the numbers sought will be produced. For example, suppose four are to be found. Take any four arithmetically-proportional numbers, 3, 6, 9, 12; and 1, 2, the least terms of the ratio of 3 to 6; and 2, 3, the least of the ratio of 6 to 9; and 3, 4, the least in the ratio of 9 to 12. The three sums of the pairs of least terms will be 3, 5, 7; and the least number measured by these is 105. So if you multiply this separately into each of the numbers taken, 3, 6, 9, 12, you will produce the four arithmetically-proportional numbers you desire: 315, 630, 945, 1260. For between the first and second the harmonic mean is 420; between the second and third, 756; between the third and fourth, 1080 — as you discern here.

Arithmetic prop.	315	630	945	1260
Harmonic means	420	756	1080	

And so, if you want two numbers in the given ratio of two numbers, between which a single whole harmonic mean falls, take the two least numbers in the given ratio and multiply their sum into the two terms of the given ratio. For instance, if the given ratio is triple, between 4 and 12, the least terms will be 1, 3, whose sum 4, if you multiply it into 4 and 12, you will make 16 and 48, the numbers sought. The harmonic mean between them will be 24, as you see here: 16, 24, 48. If the least numbers of the given ratio had been 1 and 3, we would have multiplied each into their sum 4, and these two numbers would have been made, 4 and 12, between which the harmonic mean 6 falls.

X.

But if you want any number of terms in a given continued geometric proportion, between each successive pair of which two whole means fall — one in arithmetic proportion and the other in harmonic — you will achieve it by this device. Take as many least continuously-geometrically-proportional numbers as are desired, and multiply each by the sum of the two least numbers in the same ratio. And if the products are all even, they will be the ones we seek; if not, their doubles will give the numbers sought. For example, suppose three in sesquialter ratio are to

be found. Take as many least numbers in the given ratio, 4, 6, 9; and the two least in the same ratio, 2, 3. Multiply their sum, 5, into each of the numbers taken; and, since the products are not all even, double them, and you will have the proposed numbers 40, 60, 90. For between 40 and 60 the harmonic mean is 48 and the arithmetic 50; and between 60 and 90 the harmonic mean is 72 and the arithmetic 75. As you see here.

Geometric prop.	40	60	90
Harmonic means	48	72	
Arithmetic means	50	75	

From this you will easily find two numbers in a given ratio, between which two means are enclosed — one according to arithmetic proportion, the other according to harmonic. For, having taken the least terms of the given ratio, if you multiply their sum either into those two least terms or into any two other numbers having the given ratio, the products will be numbers: if they are even, the ones sought; but if either is not even, their doubles are the ones we seek. For instance, if the given ratio is sesquitertian [4:3], whose least numbers are 3 and 4, and their sum 7: if you multiply 7 into 3 and 4, or into 12 and 16 (which also have the given ratio), you will produce either 21, 28 or 84 and 112. And because the first two are not even, their doubles, 42 and 56, are the ones we seek. But the latter two, because they are even, will be the ones sought. For between both the one pair and the other fall the two means we seek, as is manifest here.

Numbers found	42	56	84	112
Harmonic means	48	Vel	96	
Arithmetic means	49		98	

XI.

But if someone should ask to be given three numbers in harmonic proportion, between each pair of which two means fall — one in arithmetic proportion and the other in geometric — we shall carry it out by this method. By the ninth rule of geometric proportion, let three arithmetically-proportional numbers be found, 2, 50, 98, between the first pair of which the geometric mean is 10, and between the latter pair, 70; and let them be arranged in order thus: 2, 10, 50, 70, 98. Let the least number measured by these five be 2450. Dividing it by the same five numbers, let the quotients be 1225, 245, 49, 35, 25. So if all the quotients are [either all] even or [all] odd, the first, third, and fifth — namely 1225, 49, 25 — will be the numbers sought. For they are harmonic proportionals, as is

established from the fifth rule, since they are the quotients produced from dividing the number 2450 (which they measure) by the same numbers 2, 50, 98. And between each pair fall geometric proportionals: 245, the second quotient, and 35, the third quotient. For 1225, 245, 49 have the same ratios, in reverse order, that the continued proportionals 2, 10, 50 have, as is established...

[Rule XI, continued]

as is established from Rule 7 of geometric proportion. In the same way, 49, 35, 25 will have the same ratios—taken in reverse order—as the continued proportionals 50, 70, 98, since both sets are quotients produced by dividing the same number, 2450, by 2, 10, 50 and by 50, 70, 98. It is certain, moreover, that between each pair—between 1225 and 49, and between 49 and 25—a single arithmetic proportional [mean] likewise falls, provided all the quotients are made either all even or all odd. If they are not all even, all five quotients must be doubled so that they become even numbers; these will yield exactly what we seek, since they have the same ratios as their halves. Consequently, just as the second term is the geometric mean between the first and third, so the fourth term is the geometric mean between the third and fifth. Therefore, since arithmetic means also fall between the same terms, the proposition is established. You see the example set out here before your eyes.

[row]

Tres Har.prop.inuenti. [Three harmonic proportionals found]	1225	4 9	2 5
Medii Arith. [Arithmetic means]	637	3 7	
Medii Geometr. [Geometric means]		245	3 5

XII.

Again, if three numbers in arithmetic proportion are sought, between each pair of which two proportional means fall—one geometric and one harmonic—we shall satisfy the question in this way. Once more, by Rule 9 of geometric proportion, find three arithmetically-proportional numbers, 2, 50, 98, between each pair of which a single mean falls in geometric proportion. Take the two least numbers, 1 and 25, in the ratio of the first two, and likewise the two least, 25 and 49, in the ratio of the last two; their sums, 26 and 74, measure the least number 962, into which the three numbers found—2, 50, 98—are to be multiplied. For the

products, 1924, 48100, 94276, will be the numbers we seek. Since they have the same ratios as their submultiples 2, 50, 98, it follows that between each pair of them single means are intercepted in the same geometric proportion. Then, because the sum of the least numbers 1, 25 and 25, 49—in the ratios 2-to-50 and 50-to-98 (which three, 2, 50, 98, are arithmetically proportional)—measures the least number 962, which, multiplied into those three, 2, 50, 98, produced 1924, 48100, 94276, single harmonic proportional means will fall between each pair of these three, as was set out in Rule 9. The example, with its means, will therefore stand thus.

[row]

Arit.prop.nu. [Arithmetically-proportional numbers]	192 4	4810 0	9427 6
Medii Har. [Harmonic means]	370 0	6370 0	
Medii Geo. [Geometric means]		962 0	6734 0

XIII.

From what we have said, one can find three proportional numbers—whether arithmetic, geometric, or harmonic—between each pair of which three means fall: one arithmetic, another geometric, and the remaining one harmonic. Suppose first that three numbers of arithmetic proportion are to be found. By the preceding Rule 12, find three numbers in arithmetic proportion between each pair of which two means fall, one in geometric and the other in harmonic proportion. If these three numbers are all even or all odd, single arithmetic proportionals will also fall between each pair, and so they will satisfy the question. But if they are not all even or all odd, their doubles will be the numbers we seek. Here you have an example in the three numbers of the preceding rule, which are all even, as appears here.

[row]

Arith.prop. [Arithmetical proportionals]	19 24	481 00	942 76
Medii Har. [Harmonic means]	37 00	637 00	
Med.Geo. [Geometric means]	96 20		673 40

[row]

Med.Arith. [Arithmetic means]	250 12	711 88

Next, suppose three numbers in geometric proportion are to be found, between each pair of which three means fall—one of each kind—and let the given ratio that the geometric means must produce be the sesquialter [3:2]. Take the five least continued proportionals of the given ratio, 16, 24, 36, 54, 81, and the two least numbers, 4 and 9, in the ratio of the first (16) to the third (36) and of the third (36) to the fifth (81). Multiply the sum of these least numbers, 13, into that same first (16), third (36), and fifth (81). The products, 208, 468, 1053—doubled, because they are not all even or all odd—namely 416, 936, 2106, so that single arithmetic means can fall between each pair, are the numbers we seek. For between each pair fall two means, one harmonic and the other arithmetic, as is clear from Rule 10; and between the same terms geometric means also fall—namely the doubles of those produced by multiplying that same sum, 13, of the two least numbers into the second (24) and the third [recte: fourth] (54), reckoned from the beginning—as is evident here.

[row]

Geo.prop. [Geometrical proportionals]	41 6	93 6	210 6
Med.Ha. [Harmonic means]	57 6	129 6	
Med.Geo. [Geometric means]	62 4	140 4	
Med.Ar. [Arithmetic means]		67 6	152 1

Finally, suppose three numbers in harmonic proportion are to be found. Having found three in harmonic proportion, 1225, 49, 25, between each pair of which two means fall—one arithmetic, the other geometric—as we taught in Rule 11: let the two least numbers of the ratio of the first two be 25 and 1, and the two least of the ratio of the last two be 49 and 25, whose sums, 26 and 74, measure the least number 962. If you multiply this into the three numbers found, you will generate the three sought: 1178450, 47138, 24050. For between each pair two means will fall, one arithmetic and the other geometric, just as among their submultiples 1225, 49, 25—namely the numbers produced from the same 962 [multiplied] into the means found above. Likewise between the same pairs single harmonic means

will fall, as is clear from Rule 12, since they are produced from the least number measured by the sums of the least terms, [multiplied] into all three. Here you have the example.

[row]

Har.pro. [Harmonic proportionals]	11784 50	471 38	240 50
Med.Ar. [Arithmetic means]	6127 94	355 94	
Me.Geo. [Geometric means]	2356 90	336 70	
Me.Ha. [Harmonic means]		906 50	318 50

XIV.

By the same reasoning we shall find two numbers whose ratio is the double [square] of a given ratio, between which three means fall: one in arithmetic proportion, another in geometric, and the third in harmonic. Let the given ratio be 6 to 3, and the three least numbers in it, 4, 2, 1; then, by Definition 10 of this book, the ratio 4 to 1 is the double [square] of the given ratio 2 to 1, or 6 to 3. Multiply the sum 5—collected from 4 and 1—into any two numbers whose ratio is the double of the given ratio, say into 12 and 3. If the resulting products, 60 and 15, are not both even or both odd, double them: the doubles, 120 and 30, are what we seek. But if both were even or both odd, they would themselves be the sought numbers, and doubling would not be necessary. The three means between 120 and 30 stand as appears here.

[row]

Duo numeri inuenti. [The two numbers found]	120	30
Medius Arithmet. [Arithmetic mean]	75	
Medius Geometr. [Geometric mean]	60	
Medius Harmon. [Harmonic mean]		48

XV.

Finally, if one wishes to distribute any given number into any number of parts that preserve harmonic proportion, you will accomplish it by this method. By Rule 5, take as many harmonically-proportional numbers as the number of parts

into which the number is to be distributed. For if you divide the numbers produced by multiplying the given number into the numbers taken, by their sum, the quotients will be the parts you seek, and the numbers taken will be proportional in the same order. Or, if you divide the given number by the sum of the numbers taken and multiply the quotient into each of the numbers taken, you obtain the numbers sought. For example, if the number 130 is to be distributed into three harmonically-proportional parts: take the three numbers 6, 8, 12 found by Rule 5, and divide the numbers 780, 1040, 1560—which arise from multiplying the given number 130 into the assumed numbers 6, 8, 12—by their sum, 26; the quotients, 30, 40, 60, will be the parts we seek. Or, if you divide the given number 130 by 26, the sum of the numbers taken, and multiply the quotient, 5, into each of the numbers taken, 6, 8, 12, the numbers produced, 30, 40, 60, will be the same parts we seek. For they make up the given number 130 and are harmonically proportional, just as the assumed numbers 6, 8, 12. In the same way the number 10 is divided into these three parts, $2\frac{4}{13}$, $3\frac{1}{13}$, $4\frac{8}{13}$, proportional in the same manner. And the same reasoning holds for more parts.

In what manner, from the ratio of equality, all rational ratios of inequality arise

[QVO PACTO EX PROPORTIONE aequalitatis omnes inaequalitatis proportiones rationales oriantur]

Given three equal terms—whether units or numbers—it is remarkable to say how easily and pleasantly, by varying their addition among themselves, all the rational ratios of inequality are generated within three terms, in the following order.

Equal terms, or numbers, produce double numbers [duplus, 2:1]; doubles produce triples; triples produce quadruples; quadruples produce quintuples; and so on to infinity.

Next, from the multiple terms so found, if they invert their order, all the superparticular ratios [superparticularis, $(n+1):n$] are generated, preserving this order: inverted doubles produce sesquialters [3:2]; inverted triples produce sesquitertians [4:3]; inverted quadruples produce sesquiquartans [5:4]; and so on in succession.

Third, from the superparticular terms found, if they too invert their order, all the superpartient ratios [superpartiens] are produced, in this series: inverted sesquialters yield superbipartients [superbipartiens, 5:3]; inverted sesquitertians

yield supertripartients [supertripartiens, 7:4]; inverted sesquiquartans yield superquadrupartients [9:5]; and so on.

Fourth, from those same superparticular terms, in the order in which they were found, arise all the multiple-superparticular ratios, in this order: sesquialters give double-sesquialters; double-sesquialters give triple-sesquialters; triple-sesquialters give quadruple-sesquialters, and so on. Likewise sesquitertians produce double-sesquitertians; double-sesquitertians triple-sesquitertians; triple-sesquitertians quadruple-sesquitertians, and so on. Likewise sesquiquartans yield double-sesquiquartans; double-sesquiquartans triple-sesquiquartans; triple-sesquiquartans quadruple-sesquiquartans, and so on.

Fifth, from the superpartient terms found, all the multiple-superpartient ratios are elicited, in this order: superbipartients generate double-superbipartients; double-superbipartients triple-superbipartients; triple-superbipartients quadruple-superbipartients, and so on. Supertripartients likewise bring forth double-supertripartients; double-supertripartients triple-supertripartients; triple-supertripartients quadruple-supertripartients, and so on. Superquadrupartients likewise generate double-superquadrupartients and triple-superquadrupartients; double-superquadrupartients quadruple-superquadrupartients; and so with the rest.

The method by which all these ratios are generated from three equal terms, in the order we have set out, is single and easy—namely this. The sum collected from the first term of the given proportion (from which another is to be generated), taken once, plus the second taken twice, plus the third taken once, gives the first term of the geometric proportion to be constructed.

The second term of the geometric proportion to be constructed arises from the second and third terms of the given proportion (from which the other must arise), each taken once.

The third term of the geometric proportion to be constructed is furnished by the third term of the given proportion (from which the other must be derived), taken once.

This threefold precept is set before the eyes in the appended diagram, where it is plainly clear which terms of the given proportion are to be taken once, or twice, or omitted, so that the terms of the other proportion may be constructed.

Diagram of the generation of one geometric proportion from another geometric one

[*TYPVS PROCREATIONIS VNIVS PROPORTIONIS GEOMETRICÆ ex alia Geometrica*]

Order of the terms of the given proportion	First	Second	Third
The first [of the new] is constituted if the given terms are taken thus	Once	Twice	Once
The second is constituted if the given terms are taken thus		Once	Once
The third is constituted if the given terms are taken thus			Once

There now follow examples of the generation of all ratios from the ratio of equality. We set out two ratios of equality, one in units and the other in twos [binaries]. Here it seems worth noting that from the units are always generated the least terms of each three-term proportion.

GENERATIO MVLTIPPLICIVM [Generation of the multiple ratios]

Row						
Æqualitatis proportio. [Ratio of equality]	1	1	1	2	2	2
Dupli ex æqualibus creati. [Doubles created from equals]	4	2	1	8	4	2
Tripli ex duplis. [Triples from doubles]	9	3	1	18	6	2
Quadrupli ex triplis. [Quadruples from triples]	16	4	1	32	8	2
Quintupli ex quadruplis. [Quintuples from quadruples]	25	5	1	50	10	2

GENERATIO SVPERPARTICVLARIVM [Generation of the superparticular ratios]

Row						
Dupli inuerso ordine. [Doubles in inverted order]	1	2	4	2	4	8
Sesquialt. ex duplis inuersis. [Sesquialters from inverted doubles]	9	6	4	18	12	8
Tripli ordine conuerso. [Triples in reversed order]	1	3	9	2	6	18

Row						
Sesquiter. ex triplis inuersis. [Sesquitertians from inverted triples]	16	12	9	32	24	18
Quadrupli ordine conuerso. [Quadruples in reversed order]	1	4	16	2	8	32
Sesquiquar. ex quad. conuersis. [Sesquiquartans from reversed quadruples]	25	20	16	50	40	32

GENERATIO SVPERPARTIENTIVM [Generation of the superpartient ratios]

Row						
Sesquialteri inuersi. [Inverted sesquialters]	4	6	9	8	1 2	18
Superb. ex sesquial. inuersis. [Superbipartients from inverted sesquialters]	2 5	1 5	9	50	3 0	18
Sesquiter. ordine inuerso. [Sesquitertians in inverted order]	9	1 2	1 6	18	2 4	32
Supertr. ex sesquit. inuersis. [Supertripartients from inverted sesquitertians]	4 9	2 8	1 6	98	5 6	32
Sesquiquart. conuerso ordine. [Sesquiquartans in reversed order]	1 6	2 0	2 5	32	4 0	50
Superquadrupar. ex sesquiquar. conuersis. [Superquadrupartients from reversed sesquiquartans]	8 1	4 5	2 5	162	9 0	60 [?]

GENERATIO MVLTIPPLICIVM. Superparticularium [Generation of the multiple-superparticular ratios]

Row						
Sesquialteri directo ordine. [Sesquialters in direct order]	0 [?]		6	4	18	12 8
Dupli sesquialteri ex sesquialt. [Double-sesquialters from sesquialters]	25		1 0	4	50	20 8
Tripli sesquial. ex dupl. sesquial. [Triple-sesquialters from double-sesquialters]	49		1 4	4	98	28 8
Quadrupl. sesqual. ex tripl. sesquial.	81		1	4	162	36 8

Row						
[Quadruple-sesquialters from triple-sesquialters]		8				
Sesquitertii ordine recto. [Sesquitertians in direct order]	16	1	9	32	24	1
		2				8
Dupli sesquiter. ex sesquit. [Double-sesquitertians from sesquitertians]	49	2	9	98	42	1
		1				8
Tripli sesquit. ex dup. sesquit. [Triple-sesquitertians from double-sesquitertians]	100	3	9	200	60	1
		0				8
Quadr. sesquit. ex trip. sesquit. [Quadruple-sesquitertians from triple-sesquitertians]	169	3	9	338	78	1
		9				8
Sesquiquarti ordine recto. [Sesquiquartans in direct order]	25	2	1	50	40	3
		0	6			2
Dupli sesquiquar. ex sesquiqua. [Double-sesquiquartans from sesquiquartans]	81	3	1	162	72	3
		6	6			2
Tripl. sesquiq. ex trip. sesquiq. [Triple-sesquiquartans from triple-sesquiquartans]	169	5	1	338	104	3
		2	6			2
Quad. sesquiq. ex trip. sesquiq. [Quadruple-sesquiquartans from triple-sesquiquartans]	289	6	1	578	136	3
		8	6			2

Liber V (continued)

Generatio Multiplicium Superpartientium

Generation of the Multiple-Superpartient ratios.

This completes the Generation table begun on the preceding page. Each row gives a ratio in its own name (in direct or inverted order), followed by its continued-proportion triple stated "in units" (the first three number columns) and then the same triple "in binaries" — its exact double (the last three number columns), which serves as a built-in check.

Ratio (name)	(units)			(binaries — double)		
Superbipartient, in direct order	25	1	9	50	3	1
		5			0	8
Double superbipartient, from the superbipartient	64	2	9	128	4	1
		4			8	8

Ratio (name)	(units)	(binaries — double)
Triple superbipartient, from the double superbipartient	121 3 9 3	242 6 1 6 8
Quadruple superbipartient, from the triple superbipartient	196 4 9 2	392 8 1 4 8
Supertripartient, in direct order	49 2 1 8 6	98 5 3 6 2
Double supertripartient, from the supertripartient	121 4 1 4 6	242 8 3 8 2
Triple supertripartient, from the double supertripartient	225 6 1 0 6	450 1 3 2 2 0
Quadruple supertripartient, from the triple supertripartient	361 7 1 6 6	722 3 3 5 2 2
Superquadripartient, in direct order	81 4 2 5 5	162 9 5 0 0
Double superquadripartient, from the superquadripartient	196 7 2 0 5	392 1 5 4 0 0
Triple superquadripartient, from the double superquadripartient	361 9 2 5 5	722 1 5 9 0 0
Quadruple superquadripartient, from the triple superquadripartient	576 1 2 2 5 0	1152 1 5 4 0 0

The generating formula

Beneath the table Clavius sets out the "adjoined formula" showing how the last triple (576, 120, 25) is built by adding up the three preceding terms (361, 95, 25):

361

95

95	95	
25	25	25
576	120	25

Here you can plainly see that any three terms are generated from the three immediately preceding them, according to the threefold rule given above. Take, for example, these last three: 576, 120, 25. They are produced from the three preceding terms 361, 95, 25. The first, 576, is amassed from the first term 361 once, the second 95 twice, and the third 25 once. The second, 120, is composed of the second term 95 once and the third 25 once. The third, finally — 25 — is simply the third term, 25, taken once, as the adjoined formula shows. In this same way all the others are generated, beginning from equal terms, 1.1.1 or 2.2.2.

How, conversely, every ratio of inequality is brought back to a ratio of equality

(Body heading, p. 205: "QVO PACTO VICISSIM OMNIS PROPORTIO INAEQVALITATIS AD AEQVALITATIS PROPORTIONEM REVOCETUR.")

Given, once again, three unequal terms in any ratio whatever, a ratio of equality can in turn be restored from them — by a procedure no less pleasant than easy — through a varied subtraction of the unequal terms, one from another. The matter is managed by this threefold rule:

[Rule 1.] Let the *smaller* of the three given unequal terms be set as one extreme of the ratio to which the reduction is made.

[Rule 2.] With that *smallest* term subtracted from the middle one, let the remaining number be placed in the middle position of the ratio to which the reduction is made.

[Rule 3.] Let the sum gathered from the extreme just established (taken once) together with the middle term (taken twice) be subtracted from the largest of the three given numbers; and let the number left over become the other extreme of the new ratio.

By this method any ratio of three unequal terms is carried back to another ratio — always by comparing the larger numbers with the smaller — and that ratio again, in the same way, to yet another, and so on in succession until three equal terms are reached. And if the three given terms stand in a continued double ratio,

they are reduced to equality at the very first operation. The examples that follow will make this plain.

First example — quintuple ratio reduced to equality:

Ratio			
Quintuple ratio	150	30	6
Quadruple	96	24	6
Triple	54	18	6
Double	24	12	6
Ratio of equality	6	6	6

This example shows the quintuple ratio brought back to the quadruple, the quadruple to the triple, the triple to the double, and finally the double to equality. In this second example, by contrast, you see the supertripartient-of-fourths ratio reduced to the sesquitertian, the sesquitertian to the triple, the triple to the double, and the double at last to equality.

Second example — supertripartient-of-fourths ratio reduced to equality:

Ratio			
Supertripartient-of-fourths ratio	46 [recte: 49]	2	1
		8	6
Sesquitertian		9	1
		2	6
Triple		9	3
Double		4	2
Ratio of equality		1	1

But this too seems not unworthy of notice: that every ratio is, in the first instance, reduced to the very one from which it took its rise. Thus in the first example the quintuple is reduced to the quadruple, from which it arose; the quadruple to the triple, since the triple begot it; and so with the rest, all the way to equality. In the latter example, by contrast, the supertripartient ratio is called back to the sesquitertian in inverted order — for it is established that sesquitertian numbers, in reversed order, generate supertripartient ones, as we said above. Likewise this sesquitertian, by comparing the larger numbers with the smaller, is reduced to the triple in inverted order, just as the triple inverted generates the sesquitertian.

Then the triple, by again comparing the larger numbers with the smaller, is brought down to the double, and this to equality, and so on.

Nor do I think it should be passed over in silence that from three terms of any geometric proportionality whatever — whether it be of equality or of inequality — any one of the three proportionalities we explained above can be generated, if the terms are combined among themselves in various ways. That a geometric proportionality of three terms produces another geometric proportionality of three terms — provided the first term is taken once, the second twice, and the third once for the first term; the second and third once each for the second; and finally the third alone once for the third — is evident from what we wrote a little earlier about the origin of all ratios of inequality out of the ratio of equality.

But to generate an *arithmetic* proportionality of three terms from the same geometric proportionality, you will observe this threefold rule:

[Rule 1.] Make the sum collected from the first term, plus the second taken twice, plus the third once, the first term of the arithmetic proportionality.

[Rule 2.] The sum made from the first, second, and third each taken once, set in the second place.

[Rule 3.] Finally, make the third term the third term.

You see the pattern of this production here.

Pattern of the production of the arithmetic ratio from the geometric.

(Table heading, p. 206: "*TYPVS PROCREATIONIS PROPORTIONIS Arithmeticae ex Geometrica.*")

	First	Second	Third
The order of the terms of both proportionalities	First	Second	Third
The first is made from the given terms combined thus	Twice	Twice	Once
The second is made thus	Once	Once	Once
The third, finally, in this way			Once

It makes no difference at all whether you make the smallest number the first term of the geometric proportionality or the largest — as will be clear from the examples that follow.

Generation of the arithmetic proportionality from the geometric.

(Table heading, p. 206: "*GENERATIO ARITHMETICAE Proportionalitatis ex Geometrica.*")

Geometric prop.	1	1	1	4	4	4	9	6	4	4	6	9
Arithmetic prop.	5	3	1	2	1	4	3	1	4	2	1	9
				0	2		4	9		9	9	

Now this is most worthy of consideration: the difference of the arithmetic proportionality so constituted is always equal to the sum of the first two terms of the geometric proportionality from which it is built. Thus you see that in the first example the difference is 2, the sum of 1 and 1; in the second, 8, namely the sum of 4 and 4; in the third, 15, which sum is made up from 9 and 6; and finally in the fourth, 10, that is, the sum gathered from 4 and 6 — and so on in the rest.

And therefore, if an arithmetic proportionality is constituted out of a ratio of equality, its difference will be double one of the terms of that equality, as is plain in the first two examples.

Finally, that a *harmonic* proportionality also arises from a geometric one will be made clear by this other threefold rule:

[Rule 1.] The sum collected from the first taken twice, the second three times, and the third once — set this in the first place of the harmonic proportionality.

[Rule 2.] The sum made from the second taken twice and the third once — in the second place.

[Rule 3.] The sum, finally, of the second and third — in the third place.

Here too it makes no difference whether the smallest term is called the first or the largest. You see the pattern of this generation set out here.

Pattern of the production of the harmonic ratio from the geometric.

(Table heading, p. 206: "*TYPVS PROCREATIONIS HARMONICAE proportionalitatis ex Geometrica.*")

	First	Second	Third
The order of the terms of the proportionality	First	Second	Third
The first is made from the three given	Twice	Three times	Once

	First	Second	Third
terms gathered into one thus			
The second thus		Twice	Once
The third, finally, in this way		Once	Once

Generation of the harmonic proportionality from the geometric.

(Table heading, p. 206: "GENERATIO PROPORTIONALITATIS Harmonicae ex Geometrica.")

Geometric prop.	1	1	1	4	4	4	9	6	4	4	6	9
Harmonic prop.	6	3	2	2	1	8	4	1	1	3	2	1
				4	2		0	6	0	5	1	5

Now conversely, any harmonic proportionality whatever will be reduced to equality in this way:

[Rule 1.] Subtract the smallest term from the middle one, and make what is left the middle term of the proportionality to be constituted.

[Rule 2.] With this middle term found and taken away from the smallest term of the given harmonic proportionality, the remaining number will be one of the extremes of the proportionality.

[Rule 3.] Finally, the smallest given term taken once and the middle term just found taken twice, collected into one sum and subtracted from the largest given term, will yield double the other extreme. Half of this, therefore, will be that other extreme itself.

By this method a harmonic proportionality will be brought back either to equality, or to some geometric proportionality that is then reduced to equality, as we taught above. And the proposed harmonic proportionality always arises from that geometric one — if the earlier rule is applied — to which it is brought back by this rule. You see the examples here:

				2	1		4	1	1	3	2	1
Harmonic prop.	6	3	2	4	2	8	0	6	0	5	1	5
Geometric prop.	1	1	1	4	4	4	9	6	4	4	6	9

Further examples

(Body heading, p. 207: "ALIA EXEMPLA.")

Harmonic prop.	3	4	6	6	4	3	2	1	1
				0	0	0	4	6	2
Geometric prop.	2	1	$\frac{1}{2}[\?]$	5	1	2	2	4	8
					0	0			
Equality	$\frac{1}{2}[\?]$	$\frac{1}{2}[\?]$	$\frac{1}{2}[\?]$	5	5	5	2	2	2

But an arithmetic proportionality will be brought back to equality, or to the geometric proportionality by which it is constituted, in this way:

[Rule 1.] Make the smallest term one of the extremes. Then, with that same smallest term subtracted from the middle, the remaining number will be one compounded from the middle term of the proportionality to be constituted and the other extreme. Therefore, if this remaining number is double the smallest, then, cutting it in two, the half will be both the middle term and the other extreme; and the given arithmetic proportionality will be reduced to the equality from which it took its origin.

Example (margin): Let the arithmetic proportionality 3.9.15 be given. If, in constituting the geometric one, you make 3 one extreme — from the smallest term of the given arithmetic proportionality — and take that same number away from the given middle term 9, there remain 6; and this number is double the smallest, 3. Therefore the given proportionality is reduced to this equality, 3.3.3, from which it arises according to the earlier rule.

3	9	15
3	3	3

[Rule 2.] But if that remaining number is not double the smallest, it must be cut into two parts which, together with the smallest, constitute a continued geometric proportionality. To this the arithmetic one will be brought back.

Example (margin): Let the arithmetic proportionality 34.19.4 be given. Making 4 one geometric extreme — the smallest term of the arithmetic proportionality — and subtracting it from the middle, 19, there is left the number 15, which is not double the smallest. So, cutting it into 6 and 9, that arithmetic proportionality will be reduced to the geometric 9.6.4, from which in turn it takes its rise, according to the rule set out above.

34	19	4
9	6	4

But since that remaining number cannot always be divided into two such parts as, together with the smallest, form a continued geometric proportionality — how that division is to be made, and when it cannot be done in [whole] numbers, we shall teach at Proposition 17 of Book VI (Euclid VI.17) — unless surd, irrational numbers are admitted, we shall bring back any arithmetic proportionality directly to the equality from which it arose (namely the one from which, if equal terms are combined with the given difference, it takes its origin), in this way:

[Rule 1.] Let the smallest given term become one of the two extremes. Then, with the difference of the proportionality subtracted from the middle, let the remaining number become the middle term; and with that same difference doubled and taken away from the largest term, let the remaining number become the other extreme.

Example (margin): Let the arithmetic proportionality 20.27.34 be given, whose difference is 7. If 20 is made one of the extremes, and the difference 7 is taken from 27, and the same doubled [14] from 34, the two remaining terms will be equal: 20, 20. From this equality, once the difference 7 is known — which the terms of the arithmetic proportionality must have — the arithmetic proportionality itself is constructed, in this manner:

20	27	34
20	20	20

[Rule 2.] Let one of the equal terms become the lesser extreme. To it let the given difference be added, so as to make the second term. And to that same lesser extreme let the given difference, doubled, be adjoined, so as to produce the greater extreme.

Example (margin): In the given example the lesser extreme will be 20; if the difference 7 is added to it, the middle term 27 results; and if the same difference 7, doubled, is added to that same least extreme 20, the greater extreme 34 is obtained. In the same way, given this equality 15.15.15, if from it an arithmetic proportionality is to be constituted whose difference is 20, add this difference to 15, one of the equal terms, to have the second term 35; and if the same difference doubled — that is, the number 40 — is added to the same equal term, the third term 55 results. The first is 15, one of the equal terms, as you see here:

Difference 20.

15	15	15
15	35	55

And let it suffice to have said these few things here, out of the infinitely many that could be said — by countless books and to their advantage — about the immense power and nature of ratios and proportionalities. For we shall explain more, and things indeed most delightful to know, in our fuller *Arithmetic* [*Epitome Arithmeticae Practicae*], God willing. Now let us return to interpreting Euclid.

V.

(Return to Euclid, Book V — definition.)

Magnitudes are said to **have a ratio** to one another when, being multiplied, they are capable of exceeding one another.

Clavius's commentary continues, on the preceding definition (that two magnitudes "have a ratio").

Since Euclid, in the third definition, had called the relation of two magnitudes of the same kind a "ratio" [ratio] — which we, with other authors, call a "proportion" [proportio] — he now explains, by this fifth definition, what two quantities of the same kind require in order to be said to have a ratio. For not all lines, nor even all plane angles, have a ratio to one another, though they be quantities of the same kind, as we shall shortly say. He says, then, that those magnitudes are said to have a ratio to one another whichever of which, when multiplied, is so increased that it at last surpasses the other; so that if either of them, however much multiplied, could never exceed the other, they would by no reasoning be said to have a ratio.

For example, the diameter and the side of the same square are said to have a ratio (an irrational one, indeed, which can be expressed by no number), because the side multiplied by 2 — that is, taken twice — exceeds the diameter. For since two sides of a square together with the diameter form an isosceles triangle, [Margin: 20. primi (Euclid, *Elements* I.20)] two sides of the square will be greater than its diameter. In the same way, too, the circumference of a circle and its diameter have a ratio (although this is not yet explored and known to us), because the diameter multiplied by 4 — that is, taken four times — exceeds the circumference; for every circumference of a circle, as Archimedes proves,

contains the diameter of the same only three times, together with a further portion a little less than a seventh part of the diameter.

In the same way many curvilinear figures will have a ratio to rectilinear ones, since both equality and inequality is found between them — for Hippocrates of Chios [Hippocrates Chius] squared a certain lunule [lunula] (a figure contained by two arcs of circles, like the new Moon when it is seen to be sickle-shaped), and Archimedes squared a parabola: that is, the former found a square equal to a certain lunule, the latter one equal to the parabola. Hence it follows that a square may be given greater than that lunule and than the parabola, and, conversely, a lunule and parabola greater than that square. To this add what Archimedes proved in his book *On Spirals* [*De Spiralibus*]: that some straight line is equal to the circumference of a circle, another double it, another triple, and so on; and consequently that some square is equal to a circle, as he proved in the little book *On the Measurement of the Circle* [*De dimensione circuli*]. On the equality of the straight and the circular line, and of the square and the circle, we shall treat at the end of Book 6.

Add, further, that by Proclus [Proclus] too an equality between rectilinear and curvilinear angles is demonstrated. For in Book 3 of his commentary on the first book of Euclid, at the 12th Axiom (which is his 4th Postulate), he shows that to a right angle, as well as to an obtuse and an acute one, a curvilinear angle equal to it can be exhibited. For let ABC be a right angle, contained by the equal straight lines AB, BC, about which let the semicircles ADB, BEC be described. Since, then, the angles of the semicircles ABD, CBE are equal, by adding the common mixed angle ABE the whole curvilinear angle DBE becomes equal to the whole right angle ABC.

Diagram. A horizontal row of three woodcut constructions illustrating a rectilinear angle set equal to a curvilinear (mixed) angle — for the right, the obtuse, and the acute case respectively.

- First construction (right angle): A (apex, top of the vertical side), B (vertex at the foot of the vertical) [?], with a right angle ABC; D (point on the left semicircular arc ADB), E (point on the middle arc BEC). The curvilinear angle DBE equals the right angle ABC.
- Second construction (obtuse angle): F (apex), G (vertex, foot), H (right-hand foot); I (arc point) [?], K (arc point). The curvilinear angle IGK equals the obtuse angle FGH.

- Third construction (acute angle): L (apex), M (vertex, foot), N (right-hand foot); O (arc point), P (arc point). The curvilinear angle OMP equals the acute angle LMN.

[Diagram is a worn woodcut; the first-construction vertex letter and the label I are not cleanly resolved]

Similarly you will show that the curvilinear angle IGK is equal to the obtuse angle FGH; and likewise the curvilinear OMP to the acute LMN, provided that here you [?] take away the common mixed angle contained by the straight line LM and the curve MP. From all this it is established that either of the angles, when multiplied, can exceed the other; and therefore they will have a ratio to one another.

But on the other hand a finite line will not have a ratio to an infinite line, because a finite line, however it be multiplied, can never surpass an infinite one. So too neither a line with a surface, nor a surface with a body, will, for the same reason, have any ratio. Finally, an angle of contact [angulus contactus] will not be reckoned to have a ratio to a rectilinear angle, because an angle of contact, however much multiplied, always remains less than any rectilinear angle whatever, even the smallest, as we demonstrated at proposition 16 of Book 3.

And so, that Euclid might more openly explain what magnitudes of the same kind are said to have a ratio — that is, what magnitudes of the same kind he understood in the third definition — he wished, by this definition, that those be understood which have this condition: that either, multiplied, can surpass the other; but not others, even though they seem to be comprised under the same kind of quantity, such as a finite and an infinite line, a rectilinear angle and an angle of contact, and so on. For this reason, in most demonstrations of ratios, he bids one multiply so many times one of the proposed magnitudes said to have a ratio to one another, until it exceeds the other. This he does also at proposition 1 of Book 10, and in very many other propositions.

Let those, then, be silent who think that, by "magnitudes of the same kind" in the definition of proportion (which Euclid called "ratio"), one must understand those contained under the same proximate or lowest genus. For on this reasoning there would be no ratio between rectilinear and curvilinear angles, nor between rectilinear and curvilinear figures, since they are not contained under the same proximate genus — which we have said to be false. Let those too be silent who suppose that one must understand magnitudes in the same genus of quantity, or in the same subaltern genus (as the logicians speak), so that it suffices, for two quantities to be said to have a ratio, that they be either lines, or surfaces, or

bodies, or angles, or numbers. For on that reasoning there would be a ratio between a rectilinear angle and an angle of contact, since they are contained in the genus of angle; and likewise a finite line and an infinite line would have a ratio to one another, since they exist in the genus of line — both of which are plainly false, as is clear from this definition.

Diagram. Four vertical magnitude-lines labelled at the foot A, B, C, D, illustrating the equimultiples C (of A) and D (of B). The two longer lines, C and D, are marked with the numerals 16 and 12 respectively; the numerals on the two shorter lines A and B are not legibly resolved.

- A (leftmost, shorter): value [numeral: ?]
- B (shorter): value [numeral: ?]
- C: 16
- D: 12

[Only 16 and 12 are cleanly legible]

It is clear from all this how ineptly and how falsely Orontius [Oronce Fine] expounded this definition. For he says that Euclid does not define, or teach, what magnitudes are said to have a ratio, but what sort of ratio any two proposed quantities have. And so, he says, Euclid means that if a magnitude A is referred to a magnitude B, and both are multiplied equally — that is, if any equimultiples of both are taken, C being as much a multiple of A as D is of B — then the magnitudes A and B have to one another the same ratio as their equimultiples C and D. But he fails to notice this: that what he states is not a definition, but the fifteenth theorem of this fifth book, where it is demonstrated by Euclid that parts are in the same ratio as their like multiples — that is, that the magnitudes A, B have the same ratio as their equimultiples C, D. This definition, therefore, is not to be understood in that way, especially since the ratio between C and D is as unknown as that between A and B, seeing that it is always the same. Hence Euclid would not rightly lead us to a knowledge of the ratio between A and B. The sense of this definition, therefore, is the one we have set out, as is plainly established from the words of Euclid.

Definition VI

Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when the equimultiples of the first and third, compared with the equimultiples of the second and fourth according to whatever multiplication, either both alike fall short of, or both alike are

equal to, or both alike exceed them — if those be taken which correspond to one another.

Clavius's commentary.

Here Euclid explains what conditions magnitudes require, among geometers, in order to be said to have the same ratio. To accomplish this he is obliged to resort to their equimultiples [aeque multiplicia], so as to embrace all ratios of magnitudes, both rational and irrational. Let there be, then, four magnitudes: A the first, B the second, C the third, and D the fourth. Take any equimultiples whatever of the first and third — E of A, and F of C. Likewise take other equimultiples, any whatever, of the second and fourth — G of B, and H of D — whether these latter two are multiples of the second and fourth in the same degree as the former two are of the first and third, or not.

Now if the equimultiples taken are compared with one another in their corresponding pairs — the multiple of the first with the multiple of the second, that is, E with G; and likewise the multiple of the third with the multiple of the fourth, that is, F with H — and it is found invariably that they stand thus: that whenever E, the multiple of the first magnitude A, is less than G, the multiple of the second magnitude B, then F too, the multiple of the third magnitude C, is less than H, the multiple of the fourth magnitude D; or, if E is equal to G, then F too is equal to H; or, finally, if E is greater than G, then F too is greater than H — which is precisely "each together falls short of each, or is equal together, or exceeds together" — in such a way that in no class of multiples can the contrary be found: that is, that E is never less than G without F also being less than H; that E is never equal to G without F too being equal to H; and finally that E is never greater than G without F too being greater than H —

Diagram. Eight vertical magnitude-lines in two groups, illustrating the four magnitudes and their equimultiples. Upper group (left to right): E, A, B, G. Lower group: F, C, D, H. E and G (the equimultiples of A and B) are drawn longer than A and B; likewise F and H longer than C and D. No numerals are attached.

- E: equimultiple of A
- A: first magnitude
- B: second magnitude
- G: equimultiple of B
- F: equimultiple of C
- C: third magnitude
- D: fourth magnitude

- H: equimultiple of D

if, I say, it is found that whatever equimultiples are taken always behave as stated, then the ratio of the first magnitude A to the second B will be said to be the same as the ratio of the third magnitude C to the fourth D. But if it should ever be found, even in a single class of multiples, that the multiple E [?] falls short of the multiple G while the multiple F [?] does not fall short of the multiple H; or that E is equal to G while F is not equal to H; or, finally, that E exceeds G while F does not exceed H — then, however much the aforesaid condition may be found in infinitely many other multiples, the proposed quantities will by no reasoning be said to have the same ratio, but different ratios, as will become clear from Definition 8.

And so, in order that by some demonstration, through this sixth definition, it may be concluded that four quantities have the same ratio, it will have to be shown — as Euclid diligently observes both in this fifth book and elsewhere — that any equimultiples whatever of the first and third, compared with any equimultiples whatever of the second and fourth, always have the aforesaid condition of shortfall, equality, or excess, so that its contrary can never be found. Similarly, if four quantities are granted to have the same ratio, it must also necessarily be granted that any equimultiples of the first and third, compared with any equimultiples of the second and fourth, have the same condition of shortfall, equality, or excess; for the definition and the thing defined must be convertible.

But to see clearly in what way, given four magnitudes having the same ratio, some equimultiples of the first and third magnitude fall short together of certain equimultiples of the second and fourth magnitude, while others are equal together, and yet others exceed together — when those are taken which correspond to one another — it is well to offer one example in numbers. Let the four numbers, then, be 3, 2, 6, 4. Take equimultiples of the first and third, namely their quadruples, 12 and 24. Likewise take other equimultiples of the second and fourth, say their septuples, 14 and 28. You see, then, that 12, the multiple of the first, falls short of 14, the multiple of the second, just as 24, the multiple of the third, falls short of 28, the multiple of the fourth. Again, take other equimultiples of the first and third, namely their sextuples, 18 and 36; likewise other equimultiples of the second and fourth, say their ninefold multiples, 18 and 36. You see, therefore, that 18, the multiple of the first, is equal to 18, the multiple of the second, just as 36, the multiple of the third, equals 36, the multiple of the fourth. Finally, take other equimultiples of the first and third, namely their triples, 9 and 18; likewise other equimultiples of the second and fourth, say their

doubles, 4 and 8. You see, then, that 9, the multiple of the first, exceeds 4, the multiple of the second, just as 18, the multiple of the third, surpasses 8, the multiple of the fourth. If, then, in all equimultiples, taken in whatever multiplication, one of these three cases is always found to hold true, the ratio 3 to 2 will be said to be the same as the ratio 6 to 4 — otherwise not.

Diagram. A set of nested (concentric) ellipses with a central rectangle, tabulating the worked example on the four numbers 3, 2, 6, 4 and their equimultiples. The central rectangle holds the four base numbers; the surrounding rings hold, on the left, the multiples of 3 and 6, and on the right, the multiples of 2 and 4.

- Central rectangle: 3 (top-left), 2 (top-right); 6 (bottom-left), 4 (bottom-right)
- Left side, top row / bottom row (multiples of 3 and 6): 9 / 18 (triples); 18 / 36 (sextuples); 12 / 24 (quadruples)
- Right side, top row / bottom row (multiples of 2 and 4): 14 / 28 (septuples); 18 / 36 (ninefold); [4?] / 8 (doubles)

Campanus [Campanus of Novara] and Orontius [Oronce Fine], however, expound this definition quite differently. For they say that Euclid means that four magnitudes have the same ratio only when the equimultiples of the first and third, relative to the equimultiples of the second and fourth, either both together fall short proportionally — that is, in the same ratio — or are both together equal, or both together exceed proportionally, if those be taken which correspond to one another. More plainly, as Campanus puts it: when their equimultiples are proportional — that is, when the multiple of the first has the same ratio to the multiple of the second as the multiple of the third has to the multiple of the fourth.

But who does not see that, if the definition is understood in this way, Euclid is defining a thing by itself? Which is plainly absurd. Besides, if Euclid means that those magnitudes are in the same ratio whose equimultiples (if they are taken and compared with one another in the order stated) stand in the same ratio, then why, I ask, does he prove in the 4th theorem of this book that, if four magnitudes are in the same ratio, their equimultiples taken in the order we mentioned also have the same ratio? Indeed, since that theorem is shown by means of this definition, it is evident that the same is being demonstrated by the same — which is ridiculous, as we shall point out in that place. There is this too: if we interpret the definition in that way, very many theorems of this fifth book cannot be demonstrated, as we shall note in the proper places. The definition, therefore, is to be understood as we have set it out: namely, that given four magnitudes, whenever the multiple of the first falls short of the multiple of the second, or is

equal to it, or exceeds it, then necessarily the multiple of the third likewise falls short of the multiple of the fourth, or is equal to it, or exceeds it — whatever that shortfall or excess may be, without considering whether it is proportional or not; so that it never happens that the multiple of the first falls short of the multiple of the second while the multiple of the third does not fall short of the multiple of the fourth; or that the multiple of the first is equal to the multiple of the second while the multiple of the third is not equal to the multiple of the fourth; or, finally, that the multiple of the first is greater than the multiple of the se-

-cond, while the multiple of the third is not greater than the multiple of the fourth. Given, I say, four magnitudes whose equimultiples, taken as stated, always preserve that condition of shortfall, equality, and excess, those four magnitudes are said to have the same ratio — whatever be the case regarding the ratio of their equimultiples, which is not now considered; but it will be demonstrated later, in the fourth theorem, that this shortfall or excess is proportional.

Furthermore, Campanus tries to show that this definition must be understood of a proportional shortfall and excess. For if it were understood of any shortfall whatever, then, he says, these four numbers 4, 3, 6, 5 would be in the same ratio. For if equimultiples of the first and third — namely of 4 and 6 — are taken, say their doubles, 8 and 12; and likewise equimultiples of the second and fourth, namely of 3 and 5, also their doubles, 6 and 10 — then 8, the multiple of the first, exceeds 6, the multiple of the second, just as 12, the multiple of the third, exceeds 10, the multiple of the fourth. And the same is seen if the quadruples of the first and third are taken, 16 and 24, while the triples of the second and fourth are taken, 9 and 15. If, then, it were enough that the equimultiples taken exceed one another together in any way whatever, and it were not required that they surpass one another proportionally, then the ratio 4 to 3 would be the same as the ratio 6 to 5 — which is false, since the ratio 4 to 3 is sesquitertian [4:3], whereas the ratio 6 to 5 is sesquiquintan [6:5]. The shortfall or excess of the equimultiples, therefore, must be understood as proportional; for in that way it will follow that the ratio 4 to 3 is not the same as 6 to 5, since their equimultiples do not exceed one another proportionally, as is evident.

Diagram. The first of two nested-ellipse figures illustrating Campanus's example on the numbers 4, 3, 6, 5. The central rectangle holds the base numbers; the left rings hold multiples of 4 and 6, the right rings multiples of 3 and 5.

- Central rectangle: 4 (top-left), 3 (top-right); 6 (bottom-left), 5 (bottom-right)

- Left side, top / bottom (multiples of 4 and 6): 16 / 24 (quadruples); 8 / 12 (doubles)
- Right side, top / bottom (multiples of 3 and 5): 6 / 10 (doubles); 9 / 15 (triples)

Diagram. The second nested-ellipse figure on the same numbers 4, 3, 6, 5, giving the counter-instances (quadruples/quintuples and triples/quadruples).

- Central rectangle: 4 (top-left), 3 (top-right); 6 (bottom-left), 5 (bottom-right)
- Left side, top / bottom (multiples of 4 and 6): 12 / 18 (triples); 16 / 24 (quadruples)
- Right side, top / bottom (multiples of 3 and 5): 15 / 25 (quintuples); 12 / 20 (quadruples)

Yet it must be said that Campanus was mistaken in a remarkable way. For although the equimultiple numbers he adduced do exceed one another together, still very many others will be found in which the multiple of the first indeed exceeds, or is equal to, the multiple of the second, while the multiple of the third falls short of the multiple of the fourth. For if, in his example, the quadruples of the first and third are taken, 16 and 24, but the quintuples of the second and fourth are taken, 15 and 25, then 16, the multiple of the first, does exceed 15, the multiple of the second; but 24, the multiple of the third, does not exceed 25, the multiple of the fourth — rather it falls short. And if the triples of the first and third are taken, 12 and 18, but the quadruples of the second and fourth are taken, 12 and 20, then 12, the multiple of the first, will be equal to 12, the multiple of the second; but 18, the multiple of the third, will not be equal to 20, the multiple of the fourth — rather it will fall short of it. Since, therefore, it is not found that all equimultiples whatever of the said numbers behave thus — that if the multiple of the first exceeds the multiple of the second, the multiple of the third necessarily also exceeds the multiple of the fourth — even though this does happen with some equimultiples, the said numbers will not, according to this sixth definition, be said to have the same ratio, as will become still clearer from the eighth definition.

For the rest, this definition also embraces three magnitudes having the same ratio, provided only that the second be counted among them so that four are had. For example: the ratio 9 to 6 will be said to be the same as 6 to 4, because any equimultiples whatever taken of 9 and 6 either together fall short of the equimultiples taken of 6 and 4, or are equal to them, or together exceed them, and so on.

Definition VII

And let magnitudes having the same ratio be called proportional.

Clavius's commentary.

Thus, if among the magnitudes A, B, C, D the ratio of A to B is the same as that of C to D, those magnitudes will be called proportional. In the same way, if the ratio of E to F is the same as that of F to G, the magnitudes E, F, G will be called proportional.

Diagram. Seven horizontal magnitude-lines with tick-marks, each labelled with a letter at the left and a numeral at the right, illustrating discrete proportionality (A, B, C, D) and continuous proportionality (E, F, G).

- A — 12
- B — 4
- C — 9
- D — 3
- E — 16
- F — 8
- G — 4

(A : B = 12 : 4 = C : D = 9 : 3; and E : F = 16 : 8 = F : G = 8 : 4.)

Now some proportional magnitudes are continuous, among which a continuous proportionality is found, such as the magnitudes E, F, G; while others are proportional not continuously but discretely, such as the magnitudes A, B, C, D. For in the latter there is an interruption of the proportions, but in the former none at all, as was said in the fourth definition.

Definition VIII

But when, of equimultiples, the multiple of the first magnitude exceeds the multiple of the second, while the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a greater ratio to the second than the third has to the fourth.

Clavius's commentary.

Here Euclid sets out what condition four magnitudes must satisfy for the first to be said to have a greater ratio to the second than the third has to the fourth. You would put it thus. If equimultiples of the first and third are taken, and likewise

other equimultiples of the second and fourth, and it is found on some occasion (though not always) that the multiple of the first is greater than the multiple of the second, while the multiple of the third is not greater than the multiple of the fourth but either less or equal, then the ratio of the first magnitude to the second will be said to be greater than that of the third to the fourth.

Diagram. Four short horizontal magnitude-lines with tick-marks, set in two facing columns, for the worked example of this definition: the triples E, F (of the first A and third C) and the quadruples G, H (of the second B and fourth D). The individual letter-labels are printed small and are not cleanly legible on the scan.

- Line 1: E [?] — triple of the first magnitude A
- Line 2: F [?] — triple of the third magnitude C
- Line 3: G [?] — quadruple of the second magnitude B
- Line 4: H [?] — quadruple of the fourth magnitude D

[Labels not cleanly resolved]

And since E, the multiple of the first, is indeed greater than H [recte: G], the multiple of the second, but F, the multiple of the third, is not greater than H, the multiple of the fourth — indeed less — the ratio of A, the first magnitude, to B, the second, will be said to be greater than that of C, the third, to D, the fourth.

It is not, however, necessary, in order for the first of four magnitudes to be said to have a greater ratio to the second than the third has to the fourth, that the equimultiples in every multiplication behave so that the multiple of the first exceeds the multiple of the second while the multiple of the third does not exceed the multiple of the fourth; rather it is enough that in some one multiplication they behave thus. For it can sometimes happen that the multiple of the first is greater than the multiple of the second, just as the multiple of the third is greater than the multiple of the fourth; and likewise that the multiple of the first is less than the multiple of the second, and the multiple of the third less than the multiple of the fourth. Yet this does not occur in every multiplication; rather, on some occasion the multiple of the first does surpass the multiple of the second, while the multiple of the third is either less than or equal to the multiple of the fourth. For that reason the first magnitude will be said to have a greater ratio to the second than the third has to the fourth — not the same ratio — as is evident in the accompanying example.

Diagram. A nested-ellipse figure for the example of Definition VIII, on the numbers 3, 2, 4, 3 (the first-to-second ratio 3:2 greater than the third-to-fourth

ratio 4:3). The central rectangle holds the base numbers; the left rings hold multiples of 3 and 4, the right rings multiples of 2 and 3.

- Central rectangle: 3 (top-left), 2 (top-right); 4 (bottom-left), 3 (bottom-right)
- Left side, top / bottom (multiples of 3 and 4): 15 / 20 (quintuples); 9 / 12 (triples); 12 / 16 (quadruples)
- Right side, top / bottom (multiples of 2 and 3): 10 / 15 (quintuples); 18 / 12 [?] (middle ring); 14 / 21 (septuples)

And so, for four magnitudes to be called proportional, it is necessary that their equimultiples, taken according to any multiplications whatever, either together fall short, or together are equal, or together exceed, as was set out in the sixth definition. But for the first to be said to have a greater ratio to the second than the third has to the fourth, it is enough that, in some one multiplication, the multiple of the first...

the multiple of the first exceeds the multiple of the second, while the multiple of the third does not exceed the multiple of the fourth — even though, under countless other multiplications, the equimultiples of the first and third together exceed the equimultiples of the second and fourth.

But if, conversely, on some occasion the multiple of the first falls short of the multiple of the second, while the multiple of the third does not fall short of the multiple of the fourth, then the first magnitude will be said to have a *smaller* ratio to the second than the third has to the fourth — even though, under very many other multiplications, the equimultiples of the first and third together fall short of the equimultiples of the second and fourth. Thus, in the same numbers of the proposed example, the ratio of 2 to 3 will be said to be smaller than that of 3 to 4, and so on.

Why Euclid, in Definitions VI and VIII, defined four proportional and non-proportional magnitudes by means of their equimultiples

It might seem strange to someone why Euclid said that four magnitudes have the same ratio — that is, that the first is to the second as the third is to the fourth — only in the following case: that the equimultiples of the first and third (taken according to any multiplication whatever) and the equimultiples of the second and fourth (taken likewise according to any multiplication whatever) stand so that, whenever the multiple of the first is greater than the multiple of the second, the multiple of the third is likewise greater than the multiple of the fourth; whenever the multiple of the first equals the multiple of the second, the multiple of the third likewise equals the multiple of the fourth; and whenever, finally, the

multiple of the first is less than the multiple of the second, the multiple of the third is likewise less than the multiple of the fourth. And likewise it might seem strange why he held that, of four magnitudes, the first has a greater ratio to the second than the third has to the fourth only when some equimultiples of the first and third, and other equimultiples of the second and fourth, taken thus, stand so that the multiple of the first is greater than the multiple of the second, but the multiple of the third is not greater than the multiple of the fourth — being instead either equal to or less than it. Since, I say, it might seem strange to someone that from that excess, equality, and defect of the equimultiples taken in that order the likeness of the ratios (or their unlikeness) does not immediately appear, it seems necessary to explain here why Euclid chose to define magnitudes — whether proportional or not proportional — in this way.

To this difficulty I reply that Euclid chose to call those magnitudes proportional whose equimultiples behave as has been said, because he had nothing better known through which he could explain magnitudes — whether incommensurable or commensurable — that have the same ratio or do not have it. Nor indeed is it necessary that a reason be given why some thing is defined in this way or that; it is enough that the thing defined is never asserted to belong to something unless it is first demonstrated that the given definition fits it. This is seen in other definitions too. For just as Euclid, in Definition 10 of Book 1, called a right angle the one made by a line falling upon another line so as to make equal angles — so that it must never be granted that some angle is a right angle unless it is first shown that it is produced by such a line; and just as, in the last definition of Book 3, he defined similar segments of circles to be those in which the angles are equal, so that segments are to be granted similar only when one has proved that the angles standing in them are equal — so too, in Definitions 6 and 8 of this book, he called those four magnitudes proportional (and not proportional) whose equimultiples have the condition he set out, so that it must never be believed, when he calls four magnitudes to have the same ratio or not the same, unless he has first demonstrated that this condition fits those magnitudes. But although this reply is true and proper, nevertheless, because from that definition it does not seem possible to conclude that magnitudes whose equimultiples have that condition are *truly* proportional or not proportional — even if Euclid wishes to call only these proportional or not proportional — we shall explain a little more carefully that Euclid rightly defined proportional magnitudes in this way, and that therefore it can be granted by anyone, without any doubt, that those to which Definition 6 applies truly have the same ratio, and those to which Definition 8 applies do not have the same ratio.

To make this clearer, we must recall that ratio is twofold: **rational**, which exists between commensurable quantities and so can always be found in numbers; and **irrational**, which exists between incommensurable quantities and can in no way be found in numbers. Therefore, had Euclid set up his discussion only about rational ratios, he could have defined four proportional magnitudes in the way he defined four proportional numbers in Book 7, Definition 20 — namely: *Magnitudes are proportional when the first is the same equimultiple of the second, or the same part, or the same parts, as the third is of the fourth*; or, at any rate (as we added to that definition), *when the first contains the second equally as often, and in addition the same part or parts of it, as the third contains the fourth*. And non-proportional magnitudes he could have defined thus: *Magnitudes are non-proportional* — that is, the first has a greater ratio to the second than the third has to the fourth — *when the first is a greater multiple of the second, or a greater part, or greater parts, than the third is of the fourth*; or, at any rate, *when the first contains the second as often as the third contains the fourth, whether the same part or parts are left over on both sides or not; or when the first contains the second as often as the third contains the fourth, but the first includes in addition a greater part or greater parts of the second than the third does of the fourth*. Euclid could, I say, have defined proportional and non-proportional magnitudes in this way, if he were dealing only with rational ratios, because all ratios of magnitudes could be exhibited in numbers and hence defined — like the ratios of numbers — as the same or different, as we said. For the ratio of numbers (as we added, from Campanus [Campanus of Novara] and other writers, at Definition 24 of Book 7) is a certain relation of one number to another, according as it is a multiple of it, or a part, or parts of it; or, at any rate, according as it contains it once, or several times, and in addition some part or parts of it. All of this is clear both from the definitions of the five species of rational ratio — of both greater and lesser inequality — which we treated above, and from what we wrote at Definitions 20 and 24 of Book 7.

But since Euclid wished to embrace not only rational ratios but irrational ones as well, he could not define four proportional and non-proportional magnitudes in that way. For in an irrational ratio the greater magnitude can neither be a multiple of the greater [*maioris*; recte *minoris*? — "of the smaller"], nor contain it once or several times together with some part or parts of it in addition; and likewise the smaller cannot be a part of the greater, nor parts of it — since magnitudes having an irrational ratio are incommensurable, and so can have no common aliquot part, however small. He was therefore compelled to turn to the rational ratios of magnitudes — that is, to the ratios of numbers — since every rational ratio (that is, of commensurable magnitudes) is as the ratio of a number

to a number, as he proved in Book 10, Proposition 5. He was compelled, I say, to seek something certain to belong to any four numbers, or commensurable magnitudes, that have the same ratio or not the same, so that — if that same thing were demonstrated to belong to four magnitudes, even incommensurable ones — those four magnitudes could with full right also be called proportional or not proportional, since they have the same property that any four proportional (or non-proportional) numbers have; indeed, the same property that any four commensurable magnitudes having the same ratio (or not the same) are necessarily shown to have. For we cannot otherwise know or ex-

-plain that incommensurable magnitudes are proportional or non-proportional except through something that is certain to belong (as we said) to any numbers whatever, or to commensurable magnitudes, having the same ratio or not the same — magnitudes in which the likeness or unlikeness of the ratios is clearly discerned. For example: when the equal angles standing in segments of circles are commensurable with four right angles — that is, when they are the same part or parts of four right angles — then the segments themselves are likewise the same part or parts of their circles, as we shall demonstrate at the end of Book 6; so that they are rightly called similar. And for that reason all other segments too, in which the angles are equal, are called similar even if they are not commensurable with four right angles. For when the angles are commensurable with four right angles, those segments are truly similar — that is, the same parts of their circles — even though this is not discerned in segments where the angles standing in them are incommensurable with four right angles, since then the segments can be neither the same part nor the same parts of the circles, but stand altogether incommensurable with the circles themselves. For we can have no other indication that segments are similar or dissimilar than the equality or inequality of the angles standing in them — from which the true similarity or dissimilarity of the segments commensurable with the circles is indicated. And therefore, just as no one calls this similarity of segments into doubt — even though the similarity does not appear as clearly in segments incommensurable with their circles as in segments commensurable with them — so he would not act rightly who called into doubt the similarity of ratios in incommensurable magnitudes, when he finds that there belongs to them what is certainly known to belong to all magnitudes having the same rational ratio, even though this similarity of ratios is not so evident in incommensurable magnitudes.

And so, given four proportional numbers, and the equimultiples of the first and third taken according to any multiplication whatever, and likewise the equimultiples of the second and fourth according to any multiplication whatever,

it is always true (as we shall shortly demonstrate) that if the multiple of the first is greater than the multiple of the second, the multiple of the third is necessarily also greater than the multiple of the fourth; if the former is equal, this too is equal; and if less, less. And conversely: given four numbers, and the equimultiples of the first and third taken according to any multiplication, and likewise the equimultiples of the second and fourth according to any multiplication — if, the multiple of the first being greater than the multiple of the second, the multiple of the third is also greater than the multiple of the fourth; and, the former being equal, this too is equal; and, the former being less, this less — it is invariably true that the four numbers are proportional. Again, given four numbers of which the first has a greater ratio to the second than the third has to the fourth, and the equimultiples of the first and third, and likewise of the second and fourth, taken — it necessarily happens sometimes that the multiple of the first is greater than the multiple of the second, while the multiple of the third is not greater than the multiple of the fourth. And conversely: given four numbers, with the equimultiples of the first and third, and of the second and fourth, taken — if it sometimes happens that the multiple of the first is greater than the multiple of the second, but the multiple of the third is not greater than the multiple of the fourth, then it is beyond all doubt true that the ratio of the first to the second is greater than that of the third to the fourth.

Since, I say, matters always stand thus in numbers — and therefore also in commensurable magnitudes — so that the contrary is never found, it is with good reason that any four magnitudes whatever, even incommensurable ones, will be said to have the same ratio when, the equimultiples of the first and third being taken according to any multiplication, and likewise those of the second and fourth according to any multiplication, it is invariably found that the multiple of the third is greater than the multiple of the fourth whenever the multiple of the first is greater than the multiple of the second; that the multiple of the third is equal to the multiple of the fourth when the multiple of the first is equal to the multiple of the second; and, finally, that the multiple of the third is less than the multiple of the fourth by this very fact, that the multiple of the first is less than the multiple of the second. By the same reasoning, any four magnitudes whatever — commensurable as well as incommensurable — will with full right be said not to have the same ratio, but rather that the ratio of the first to the second is greater than that of the third to the fourth, when, the equimultiples being taken as described, it sometimes happens that the multiple of the first is greater than the multiple of the second, while the multiple of the third is not greater than the multiple of the fourth. For the former condition is invariably found in all proportional numbers, and the latter in non-proportional ones, as we shall

demonstrate presently; and we can have no other indication by which to recognize whether incommensurable magnitudes are proportional or not — especially since incommensurable magnitudes are often shown, by another route, to have the same ratio to other magnitudes (which are proved proportional in this way) as a number has to a number, so that there can be no doubt at all that all magnitudes, even incommensurable ones, whose equimultiples behave as stated in the sixth definition, are proportional. For if this were not true, some manifest absurdity would doubtless follow at some point from that proportionality — which, however, has not so far occurred; on the contrary, those things which are demonstrated from magnitudes proportional in that sense are for the most part shown, by another route (as we said), to be perfectly true, as we shall also demonstrate later. This being so, I consider it clearly established why Euclid, in those two definitions, employed (and rightly) the equimultiple magnitudes of the proposed magnitudes. But now let us demonstrate geometrically what we promised, having called in some propositions of Book 7 — which in no way depend on the sixth and eighth definitions, or on the demonstrations of this fifth book, so that they can most conveniently be demonstrated before Book 5, and thus be assumed here as already demonstrated. This we shall accomplish by the four propositions that follow.

I.

Given four proportional numbers, and equimultiples of the first and third taken according to any multiplication, and likewise equimultiples of the second and fourth taken according to any multiplication whatever: if the multiple of the first is greater than the multiple of the second, the multiple of the third will also be greater than the multiple of the fourth; if the multiple of the first is equal to the multiple of the second, the multiple of the third will also be equal to the multiple of the fourth; and if, finally, the multiple of the first is less than the multiple of the second, the multiple of the third will also be less than the multiple of the fourth.

Demonstration. Let the first number A have to the second B the same ratio that the third C has to the fourth D; and let E, F be equimultiples of the first A and the third C, and G, H equimultiples of the second B and the fourth D, whatever this multiplication may be. I say that if E, the multiple of the first A, is greater than G, the multiple of the second B, then F, the mul-

-tiple of the third C, is likewise greater than H, the multiple of the fourth D. And if E is equal to G, then F is also equal to H. And if, finally, E is less than G, then F is also less than H. For since A is to B as C is to D, by alternation A will also be to C as

B is to D. [Margin: (a) Elements VII.13] Now A is to C as E is to F, because the same number, multiplying A and C, produced E and F — since E and F were taken as equimultiples of A and C. [Margin: (b) Elements VII.17] And for the same reason, B is to D as G is to H. Therefore, by the lemma of Proposition 14 of Book 7, E will also be to F as G is to H. And by alternation, E is to G as F is to H. [Margin: (c) Elements VII.13] Consequently, if E is greater than G, F will also be greater than H, as in the first example. But if F [recte: E] is equal to G, F will also be equal to H, as in the second example. And if, finally, E is less than G, F will also be less than H, as in the third example. Which was to be demonstrated.

First example:

E, 9	A, 3	C, 6	F, 18
G, 4	B, 2	D, 4	H, 8

Second example:

E, 18	A, 3	C, 6	F, 36
G, 18	B, 2	D, 4	H, 36

Third example:

E, 12	A, 3	C, 6	F, 24
G, 14	B, 2	D, 4	H, 28

II.

Given four non-proportional numbers, such that the ratio of the first to the second is greater than that of the third to the fourth: if equimultiples of the first and third, and likewise equimultiples of the second and fourth, are taken, it can happen that the multiple of the first is greater than the multiple of the second, while the multiple of the third is not greater than the multiple of the fourth.

Demonstration. Let the first number A have to the second B a greater ratio than the third C has to the fourth D. I say that it can happen, when equimultiples of the first A and the third C are taken, and likewise equimultiples of the second B and the fourth D, that the multiple of A, the first, is greater than the multiple of B, the second, while the multiple of C, the third, is not greater than the multiple of D, the

fourth. For let the numbers B and D, multiplying each other, make E. And since D measures E (by Proposition 7 of Book 7), and E measures the product of C by E (by the same proposition), D will also measure this same product of C by E (by Proposition 11 of Book 7). Let D measure the product of C by E by F; and thus, from D multiplied by F, there will arise the number which D measures by F (by Proposition 9 of Book 7) — that is, the same number that arises from C multiplied by E. Since, therefore, the same number is generated from D (the first) multiplied by F (the fourth) as from C (the second) multiplied by E (the third), then D, the first, will be to C, the second, as E, the third, is to F, the fourth. [Margin: (a) Elements VII.19] And by conversion, C is to D as F is to E.

Again, since B measures E (by Proposition 7 of Book 7), and E measures the product of A by E (by the same proposition), B will also measure this same product of A by E (by Proposition 11 of Book 7). Let B measure the product of A by E by FG; and thus, from B multiplied by FG, there arises the number which B measures by FG (by Proposition 9 of Book 7) — that is, the same number that arises from A multiplied by E. Since, therefore, the same number is generated from B (the first) multiplied by FG (the fourth) as from A (the second) multiplied by E (the third), then B, the first, will be to A, the second, as FG, the third, is to E, the fourth. [Margin: (b) Elements VII.19] And by conversion, A is to B as FG is to E. Since, therefore, FG is to E as A is to B, and the ratio of A to B was assumed greater than that of C to D, the ratio of FG to E will also be greater than that of C to D. But it was shown that C is to D as F is to E. Therefore the ratio of FG to E will also be greater than the ratio of F to E, and hence the number FG will be greater than the number F — all of which follows plainly from what we wrote at Definition 20 of Book 7. Let FG, then, exceed F itself by the number G.

Now let there be taken equimultiples K, H, I of F, G, C, under this condition: that H be greater than E, while I be not less than D. Then (by the scholium to Proposition 5 of Book 7) the whole KH will be the same multiple of the whole FG as K is of F, or I of C. Again let there be taken equimultiples L, M, N of D, E, under this condition: that L be the multiple of D next greater than I — that is, such equimultiples that, when the number D is subtracted from L, the remaining number is not greater than I, but either equal to it (as happens in the second example) or less (as happens in the first example). And since it was shown that D is to C as E is to F, and, by construction, C is to I as F is to K (because I, K were taken as equimultiples of C, F), then, *ex aequali*, D will also be to I as E is to K. [Margin: (a) Elements VII.14]

Furthermore, since it was shown that D is to C as E is to F, by alternation D will also be to E as C is to F. Now D is to E as L is to MN, because the same number, multiplying D and E, made L and MN — since L, MN were taken as equimultiples of D, E. And for the same reason, C is to F as I is to K. Therefore, by the lemma of Proposition 14 of Book 7, L will also be to MN as I is to K; and by alternation, L is to I as MN is to K.

Since, therefore, the whole L is to I as the whole MN is to K, and, as D subtracted from L is to that same I, so is E subtracted from MN to that same K, then (by Theorem 6 of the scholium to Proposition 22 of Book 7) the remainder of L will be to I as the remainder of MN is to K:

Worked example (the same figure is set beside both halves of the demonstration on this page):

First example:

K, 56	H, 7	O, 21	I, 28
F, 8	G, 1	A, 3	C, 4
	E, 6	B, 2	D, 3
M, 54	N, 6	P, 20	L, 30

Second example:

K, 160	H, 440	O, 60	I, 20
F, 8	G, 22	A, 3	C, 1
	E, 80	B, 8	D, 10
M, 160	N, 80	P, 24	L, 39

Now the remainder from L is not greater than I, but either smaller—as in the first example—or equal—as in the second; this is because the remainder from L, together with D, makes up L itself, the multiple of D that is next greater than I, by construction. Therefore the remainder from MN will not be greater than K either; and so, if N (equal to E) is subtracted from MN, the remainder M will be either smaller than K, as in the first example, or equal to it, as in the second. Since, then, H, the multiple of G, is greater than E (or than N) by construction, the whole KH will be greater than the whole MN.

Finally, take O to be the same multiple of A that KH is of FG (or that I is of C); and likewise take P to be the same multiple of B that MN is of E (or that L is of D). Since it has been shown that FG is to E as A is to B, and since KH and O have been taken as equimultiples of FG and A—the first and the third—while MN and P have been taken as equimultiples of E and B—the second and the fourth—it follows from the preceding proposition that if KH is greater than MN, then O too is greater than P. Since, therefore, KH has been shown to be greater than MN, O will also be greater than P. And so, since O and I are equimultiples of A and C (the first and the third), while P and L are equimultiples of B and D (the second and the fourth), and since it has been demonstrated that O is greater than P while I is smaller than L by construction, it can happen that—when the ratio of the first, A, to the second, B, is greater than the ratio of the third, C, to the fourth, D, and equimultiples have been taken as described—the multiple of the first (namely O) is greater than the multiple of the second (P), yet the multiple of the third (namely I) is not greater than the multiple of the fourth (L). Which was to be demonstrated.

But if the ratio of the first to the second is smaller than the ratio of the third to the fourth, and equimultiples of the first and third, and likewise of the second and fourth, are taken, it can also happen that sometimes the multiple of the first is smaller than the multiple of the second, while the multiple of the third is not smaller than the multiple of the fourth.

For in the same example the ratio of C, the first, to D, the second, is smaller than the ratio of A, the third, to B, the fourth; and it has been demonstrated that I, the multiple of the first C, is smaller than L, the multiple of the second D, while O, the multiple of the third A, is greater than P, the multiple of the fourth B.

III.

Given four numbers, and taking equimultiples of the first and third according to any multiplication whatever, and likewise equimultiples of the second and fourth according to any multiplication whatever: if, whenever the multiple of the first is greater than the multiple of the second, the multiple of the third is necessarily also greater than the multiple of the fourth; and whenever the former is equal, the latter too is always equal; and finally whenever the former is smaller, the latter too is perpetually smaller—then the ratio of the first to the second will be the same as the ratio of the third to the fourth.

First example:

E, 9	A, 3	C, 6	F, 18
G, 4	B, 2	D, 4	H, 8

Second example:

E, 18	A, 3	C, 6	F, 36
G, 18	B, 2	D, 4	H, 36

Third example:

E, 12	A, 3	C, 6	F, 24
G, 14	B, 2	D, 4	H, 28

Let there be four numbers A, B, C, D, and let equimultiples E and F of the first, A, and the third, C, be taken—of whatever kind—and likewise equimultiples G and H of the second, B, and the fourth, D, whatever this multiplication may be. I say that if E and F, the multiples of the first and third, are always either both greater than G and H, the multiples of the second and fourth, or both equal, or both smaller, then A, the first, is to B, the second, as C, the third, is to D, the fourth. For if the ratio of A to B were greater or smaller than the ratio of C to D, it could happen—as was demonstrated in the preceding proposition—that E, the multiple of the first, would sometimes be greater or smaller than G, the multiple of the second, while F, the multiple of the third, would not be greater (or not smaller) than H, the multiple of the fourth. But that is contrary to the hypothesis. Therefore A is to B as C is to D. Which was to be shown.

III.

Given four numbers, and taking equimultiples of the first and third, and likewise equimultiples of the second and fourth: if it should ever happen that the multiple of the first is greater than the multiple of the second, while the multiple of the third is not greater than the multiple of the fourth, then the ratio of the first to the second will be greater than the ratio of the third to the fourth.

E, 21	A, 3	C, 4	F, 28
G, 20	B, 2	D, 3	H, 30

Let there be four numbers A, B, C, D; and taking equimultiples E and F of A and C (the first and third), and equimultiples G and H of B and D (the second and fourth), let E, the multiple of the first A, be greater than G, the multiple of the second B, while F, the multiple of the third C, is not greater than H, the multiple of

the fourth D. I say that the ratio of the first, A, to the second, B, is greater than that of the third, C, to the fourth, D. For since E is greater than G, while F is not greater than H—the former being a ratio of greater inequality, the latter one of equality or of lesser inequality—it follows, by Theorem 10 of the scholium to Proposition 22 of Book 7, that by alternation the ratio of E to F is likewise greater than that of G to H. [Margin: (a) *Elements* VII.17 ("17. septimi")] But E is to F as A is to C, because the same number that multiplies A and C has produced E and F, since E and F are equimultiples of A and C; and for the same reason G is to H as B is to D. Therefore the ratio of A to C will also be greater than that of B to D; and by alternation, again by Theorem 10 of the scholium to Proposition 22 of Book 7, the ratio of A to B will also be greater than that of C to D. Which was to be shown.

But if the multiple of the first is smaller than the multiple of the second, while the multiple of the third is not smaller than the multiple of the fourth, then the ratio of the first to the second will be smaller than the ratio of the third to the fourth.

E, 28	A, 4	C, 3	F, 21
G, 30	B, 3	D, 2	H, 20

For the demonstration is entirely the same, provided that throughout one says "smaller ratio" in place of "greater ratio," with nothing else changed. This is evident in the present example, where the four numbers are A, B, C, D, and E, the multiple of the first A, is smaller than G, the multiple of the second B, while F, the multiple of the third C, is not smaller than H, the multiple of the fourth, and so on.

But let us now demonstrate that in magnitudes which are incommensurable with others, and which are proportional by Definition 6, one can generally find a ratio that can be exhibited in numbers—so that magnitudes to which Definition 6 applies are truly proportional, since from magnitudes taken as proportional in that sense a true conclusion follows, and nothing absurd can be inferred from it. This too we undertook a little earlier to demonstrate. Let AB, for example, be double the line CD, and on these lines let the squares ABEF and CDGH be described, with diagonals BF and DH.

Diagram. Two squares are drawn side by side, each with a single diagonal. The larger square on the left is set on side AB; the smaller square on the right on side CD.

- F (top-left of the larger square): vertex; the printed glyph is faint, resembling "T" [?]
- E (top-right of the larger square): vertex

- A (bottom-left of the larger square): vertex; the base A–B is labeled 4
- B (bottom-right of the larger square): vertex
- Diagonal from B to F is labeled **R. 32** (the square root of 32)
- H (top-left of the smaller square): vertex
- G (top-right of the smaller square): vertex
- C (bottom-left of the smaller square): vertex; the base C–D is labeled 2
- D (bottom-right of the smaller square): vertex
- Diagonal from D to H is labeled **R. 8** (the square root of 8)

Now, by the scholium to Proposition 47 of Book 1, the square on the diagonal BF is double the square on the side AB (that is, double the square ABEF), and likewise the square on the diagonal DH is double the square on the side CD (that is, double the square CDGH). So if the side AB is set at 4 and the side CD at 2, the square ABEF will be 16, and the square on the diagonal BF, its double, will be 32. Hence the diagonal BF itself will be the square root of the number 32 [*radix quadrata*], which cannot be expressed in numbers, since it is greater than 5 and less than 6, and no number intermediate between 5 and 6—that is, a number composed of 5 and some fraction of a unit—can, when multiplied by itself, produce the whole number 32, as will become clear from the following lemma. The square CDGH will be 4, and the square on the diagonal DH, its double, will be 8; hence the diagonal DH itself will be the square root of the number 8, which likewise cannot be expressed in numbers, since it is greater than 2 and less than 3, and so on. Thus the ratio both of AB to BF and of CD to DH is irrational, and therefore the diagonal BF is incommensurable [*incommensurabilis*] with the side AB, and the diagonal DH with the side CD. You see, then, that what Euclid demonstrated in the last proposition of Book 10—from what was proved in Book 6 about proportional magnitudes by Definition 6 of this Book 5—namely that the diagonal of a square is incommensurable with the side of that same square, is in fact perfectly true; for we ourselves have here shown the same thing by another route, without ratios.

Again: since the diagonals BF and DH bisect the right angles of the squares—as we showed in the scholium to Proposition 34 of Book 1 and in Corollary 2 to Proposition 4 of Book 2—the angles ABF, AFB, CDH, and CHD will be half right angles, and therefore equal to one another. And since the right angles at A and C are likewise equal, the triangles ABF and CDH will be equiangular. Therefore [Margin: (a) *Elements* VI.4 ("4. sexti")] AB will be to BF as CD is to DH; and by alternation, AB will be to CD as BF is to DH. But the line AB was posited as double CD; therefore BF too will be double DH.

And so, since by Definition 6 of this book we have been led to believe that the diagonal BF is double the diagonal DH—even though each diagonal has an irrational ratio to its own side—let us see whether we can learn by another route that the diagonal BF really is double the diagonal DH, as was concluded from Definition 6, by this very fact, that the side AB is posited as double the side CD, even though the ratio both of the side AB to the diagonal BF and of the side CD to the diagonal DH is irrational. (For Proposition 4 of Book 6, by which we showed that AB is to BF as CD is to DH, depends on Proposition 2, and this on Proposition 1 of that same Book 6; and the first proposition of Book 6 derives its force from Definition 6 of this Book 5. Likewise the alternation of ratio, which we have also used here, cannot be demonstrated for incommensurable magnitudes without that same Definition 6, as is clear from Proposition 16 of Book 5.)

This we shall easily learn, without ratios, as follows. Since the side AB is double the side CD, by the scholium to Proposition 4 of Book 2 the square AE will be quadruple the square CG. So if the square AE is set at 4, the square CG will be 1. And since the square on the diagonal BF is double the square AE, and the square on the diagonal DH double the square CG—by the scholium to Proposition 47 of Book 1—the square on the diagonal BF will be 8 and the square on the diagonal DH will be 2; and therefore the former will be quadruple the latter. Hence, by the scholium to Proposition 4 of Book 2, the line BF will be double the line DH. You see, then, once again, that if we proceed by what has been demonstrated from Definition 6 of this book concerning proportional magnitudes, we have arrived at a true conclusion—namely that the diagonals of two squares have a double ratio (2 : 1) to each other whenever the side of one is double the side of the other. So there is no doubt that magnitudes whose equimultiples fulfill the condition prescribed by Definition 6 are truly proportional.

Moreover, from the rules for irrational numbers taught in algebra it is also established that the diagonal BF—that is, $\sqrt{32}$ —has a double ratio to the diagonal DH, that is, to $\sqrt{8}$. For whether $\sqrt{32}$ is divided by $\sqrt{8}$, giving a quotient of $\sqrt{4}$ (that is, the number 2, which names the ratio of the diagonal BF to the diagonal DH), or $\sqrt{8}$ is multiplied by 2, the result is $\sqrt{32}$. And in this way everything that is demonstrated from Definition 6 of this book concerning the ratios of incommensurable magnitudes can be explained through the rules of irrational numbers. This furnishes a further argument that the magnitudes to which Definition 6 of this book applies are truly proportional; for the calculus of irrational numbers is found to agree perpetually with the demonstrations that depend on that definition—as is very well known to those versed in the precepts of algebra.

LEMMA.

That no number composed of 5 and some fraction of a unit—however far the unit be divided—can generate the whole number 32, we shall demonstrate easily as follows.

Let there be the number 5 together with any fraction, $\frac{3}{7}$, thus: $5\frac{3}{7}$; and let it be reduced, as we taught in the *Arithmetic*, to this single fraction, $\frac{38}{7}$. Now it is certain that the denominator 7 does not measure the numerator 38; otherwise, on dividing 38 by 7, an integer without a fraction would result, which is contrary to the hypothesis. Let the fraction $\frac{38}{7}$ now be multiplied by itself (which is done, as we said in the *Arithmetic*, by multiplying both the numerator 38 by itself and the denominator 7 by itself), and let the fraction $\frac{1444}{49}$ be generated, so that its numerator 1444 is the square of that numerator 38, and its denominator 49 the square of the denominator 7. And since the root 7 does not measure the root 38, as we showed, [Margin: (a) *Elements* VIII.16 ("16. Octaui")] the square 49 will likewise not measure the square 1444. Therefore, on dividing the square 1444 by the square 49, the quotient will not be a whole number, but some fraction will cling to it; otherwise the square 49 would measure the square 1444, the contrary of which we have demonstrated. And by the same reasoning it will be shown that any whole number together with any fraction, multiplied into itself, generates a whole number together with a fraction. But if a fraction whose numerator is smaller than its denominator is multiplied by itself, there will always be produced a fraction whose numerator is likewise smaller than its denominator. For since the numerator of the product-fraction is the square number of the numerator of the given fraction, and the denominator the square number of the denominator, then if the numerator of the given fraction is smaller than its denominator, its square number will also be smaller than the square of the latter. Thus from $\frac{2}{3}$ multiplied by itself there is generated

the broken number $\frac{4}{9}$, whose numerator is smaller than its denominator. And so any fraction multiplied by itself generates a number that is not whole—except when the numerator of the given fraction is measured by the denominator, in which case that fraction is rather a whole number: such as the fraction $\frac{20}{5}$, which constitutes 4 whole units.

IX.

A proportion consists, at the very fewest, in three terms.

For since every analogy or proportionality—which the translator, as has been said, calls a "proportion"—is a likeness of two or more ratios, and since every

ratio has both an antecedent and a consequent term, it is necessary that in every proportionality there be found, at the least, two antecedent terms and two consequent terms. Hence, if the proportionality is discontinuous, at least four terms (or magnitudes) will be required; but if it is continuous, there will be at the fewest three terms, since the middle term is taken twice—being the consequent of one ratio and the antecedent of the other. And this is the smallest number of terms of a proportionality. For in any two terms whatever, only a ratio is found, not a proportionality.

Diagram. A set of vertical line-segments in two groups, illustrating the fewest terms of a proportionality: a discontinuous proportion on the left (four terms) and a continuous proportion on the right (three terms, the middle taken twice). The small numerals printed at the tops of the segments are hard to resolve at native resolution; the clearly legible ones are given, the rest flagged. (≥ 3 uncertain labels).*

- Left group, labeled **A, B, C, D** (a discontinuous proportion, four terms): heights read as $A = 12$, $B = 9$, $C = 4$, $D = [\text{numeral: ?}]$ (short segment; glyph resembles "1" or "3")
- Right group, labeled **E, F, G** (a continuous proportion, three terms): heights read as $E = 16$, $F = 8$, $G = 4$

X.

When three magnitudes are proportional, the first is said to have to the third the *duplicate* ratio of that which it has to the second. And when four magnitudes are proportional, the first is said to have to the fourth the *triplicate* ratio of that which it has to the second. And so on continually, one more each time, as long as the proportion continues.

Diagram. Five vertical line-segments of decreasing height, labeled left to right, illustrating five magnitudes in continued proportion (common ratio 2 : 3).

- A: height **81**
- B: height **54**
- C: height **36**
- D: height [numeral: ?] (glyph ambiguous, resembling "74"; the continued proportion 81, 54, 36, __, 16 of ratio 2 : 3 requires **24**)
- E: height **16** (the base letter is faint, resembling "I" [?])

For example, let there be magnitudes A, B, C, D, E in continued proportion, so that A is to B as B is to C, as C is to D, and as D is to E. The ratio of A, the first

magnitude, to C, the third, is called the duplicate of the ratio that A, the first magnitude, has to B, the second; because between A and C two ratios are set, each equal to the ratio of A to B—namely the ratio of A to B and that of B to C—so that the ratio of A to C in a way encompasses the ratio of A to B taken twice, that is, set out twice in order. But the ratio of A, the first magnitude, to D, the fourth, is called the triplicate of the ratio that A, the first magnitude, has to B, the second; because between A and D three ratios are found, each equal to the ratio of A to B—namely the ratios of A to B, of B to C, and of C to D—and therefore the ratio of A to D in a way includes the ratio of A to B taken three times, that is, set out three times in order. Likewise the ratio of A to E is called the quadruplicate of the ratio of A to B, because four ratios are interposed between A and E, each equal to the ratio of A to B, and so on.

But if, conversely, E is to D as D is to C, as C is to B, and as B is to A, then the ratio of E to C will be called the duplicate of the ratio of E to D; the ratio of E to B will be called the triplicate of the ratio of E to D; and likewise the ratio of E to A will be called the quadruplicate of the ratio of E to D, and so on.

Some commentators infer from this definition that, if several quantities in continued proportion are set out, the ratio of the first quantity to the third is *double* the ratio of the first quantity to the second, on the ground that Euclid calls it the *duplicate* ratio of the latter. In the same way they hold that the ratio of the first quantity to the fourth is *triple* the ratio that the first quantity has to the second, and so on. This, however, is in no way to be granted. For Euclid did not mean this; he taught only that the ratio of the first quantity to the third is *called* the duplicate of the ratio that the first quantity has to the second, because between the first and the third quantity there is found, as it were, the ratio of the first quantity to the second taken twice—since between the first quantity and the third are interposed two ratios equal to the ratio that the first quantity has to the second—and so on for the rest, as we have said. But he does not understand the one to be double the other; otherwise he would be proposing a theorem that anyone might rightly refuse to grant. For who will assert that, among these numbers in continued proportion—25, 5, 1—the ratio of 25 to 1 is double the ratio of 25 to 5, when one would rather call it quintuple? Yet no one will deny that the former is called the duplicate of the latter in the sense explained, because the ratio of 25 to 5 is set down twice and continuously. Again, how will the ratio of 1 to 25 be double the ratio of 1 to 5, when the former is smaller and the latter greater? For by Proposition 8 of this book, the quantity 1 has a greater ratio to the quantity 5 than to the quantity 25, because 25 is greater than 5. Yet the former will be called the duplicate of the latter, for the reason already explained, even

though it is a fifth part of it. Therefore, although the ratio of 25 to 1 is called the duplicate of the quintuple ratio, it is nevertheless the tenfold ratio that is the double of that same ratio. Just as the octuple ratio, likewise, is the double of the quadruple ratio, while nonetheless the duplicate of the quadruple is the sixteenfold ratio, as is evident here: 16, 4, 1. Finally, among three equal magnitudes, or among three equal numbers—4, 4, 4—and hence in continued proportion, how can the ratio of the first to the third be double the ratio of the first to the second, when it is entirely the same? Yet the one will be called the duplicate of the other, because the latter is set down twice in order, continuously, between the first number and the third.

But those authors who insist that the ratio of the first quantity to the third is the *double* of the ratio which the first quantity has to the second — among whom, in this passage, is also Federico Commandino [Federicus Commandinus], which greatly astonishes me, since he is otherwise a most excellent mathematician — say that this definition requires unequal terms, and that the first must be the greater; so that the definition must necessarily be understood of a continued proportion of the greater inequality [proportio maioris inæqualitatis]. Hence, they say, it is no wonder that the ratio 1 to 25 is not the double of the ratio 1 to 5, nor the ratio 4 to 4 the double of the ratio 4 to 4. But this interpretation is not merely untrue; it runs directly counter to Euclid, who in very many places assumes that a triplicate ratio is found in a proportion of the lesser inequality [proportio minoris inæqualitatis]. The clearest instance — to pass over the others — is Proposition 12 of Book 12, where he proves that similar cones and cylinders are in the triplicate ratio of the diameters of their bases. For if the greater cone or cylinder is compared with the smaller, then in the second part of that demonstration Euclid necessarily assumes it to have been proved, in Proposition 8 of the same Book 12, that similar pyramids — even when the smaller is compared with the greater — are in the triplicate ratio of their homologous sides, as we noted in that place. Add to this that the propositions in which certain figures are shown to have the duplicate or triplicate ratio of their homologous sides — such as 19 and 20 of Book 6, and 11, 12, 18, 19 of Book 8, and 33 of Book 11, and 12 and 18 of Book 12 — would not be universal if they were to be understood only of the greater inequality. This definition, then, must be explained as we have explained it.

And so in this passage Euclid only explains what is to be understood by the name of duplicate, triplicate ratio, and so on, so that the demonstrations of the following books may be grasped and applied to material things. He does not, however, determine which ratio is the double, triple, or quadruple of another.

For example: since it is established from Proposition 20 of Book 6 that the ratio of a square to a square is the duplicate of the ratio which the side of the former square has to the side of the latter, it will follow that, if the ratio of the sides is continued through three terms, the ratio of square to square is the same as the ratio of the first term to the third. Thus, if the side of the former square is three spans and that of the latter one span, the former square will have to the latter the ratio 9 to 1, so that it contains the latter nine times. For the ratio 9 to 1, which is ninefold, is called by this definition the duplicate of the triple ratio 9 to 3, or 3 to 1, as is plain in these numbers 9. 3. 1. But we must not infer that the ratio of square to square is twice as great as the ratio of side to side. Likewise the latter square will have to the former the ratio 1 to 9, so that it is a ninth part of it, because the ratio 1 to 9 is called the duplicate of the ratio 1 to 3, as is evident in these same numbers 1. 3. 9. By the same reasoning, since in the last proposition of Book 12 it is demonstrated that spheres have to one another the triplicate ratio of their diameters, it will follow that a sphere whose diameter contains three spans has to a sphere whose diameter is one span only the ratio 27 to 1. For this is called the triplicate of the triple ratio, as appears here: 27. 9. 3. 1. And so, given two spheres whose diameters have a tenfold ratio, the spheres themselves will have a thousandfold ratio, since this is the triplicate of the tenfold, as is clear here: 1000. 100. 10. 1. But who would ever say that a thousandfold ratio is only three times greater than a tenfold ratio, rather than that a thirtyfold ratio is the triple of a tenfold?

Furthermore, given any rational ratio, if its denominator [denominator] is multiplied into itself, there results the denominator of the ratio that is called the duplicate of the given ratio. For example, since from multiplying 4 — the denominator of the quadruple ratio — into itself there is produced 16, the sixteenfold ratio is called the duplicate of the quadruple ratio, as is seen here: 16. 4. 1; and likewise here: 48. 12. 3. Conversely, since from multiplying $\frac{1}{4}$ — the denominator of the one-fourth ratio [proportio subquadrupla] — into itself there is produced $\frac{1}{16}$, the one-sixteenth ratio [subsedecupla] is called the duplicate of the one-fourth ratio. And if the denominator is again multiplied into that product, the denominator of the triplicate ratio will be produced. So, in the former example, from multiplying 4 into 16 there are produced 64, the denominator of the ratio which, being quadruple, is called triplicate, as you see here: 64. 16. 4. 1; and likewise here: 192. 48. 12. 3. Again, if the same denominator is multiplied into this latter product, the denominator of the quadruplicate ratio will be found; and so on for the rest.

Thus the denominator of a duplicate ratio is produced from the denominator of the given ratio set down twice and multiplied in this way. For instance, the number denominating the duplicate of the triple ratio is produced from 3, the denominator of the triple ratio, set down twice, in this manner 3. 3., and so multiplied: for three times three make 9, the denominator of the ninefold ratio, which is called the duplicate of the triple, as can be seen here: 9. 3. 1; and likewise here: 18. 6. 2. But the denominator of a triplicate ratio is generated from the denominator of the given ratio set down three times and multiplied thus. As in the given example, the denominator of the triplicate ratio, namely 27, is produced from 3 set down three times, thus 3. 3. 3., and so multiplied, by saying "three times three, three times over," and so on. In the same way a quadruplicate ratio will arise from the denominator set down four times, a quintuplicate from the same set down five times and multiplied likewise, and so forth.

And so the duplication, triplication, quadruplication, and the like, of any ratio treated of in this definition is nothing else than the multiplication of the denominators of the intermediate ratios among themselves. For from this multiplication is produced the denominator of the ratio which the extreme terms have to one another; and this ratio is therefore called the duplicate, triplicate, or quadruplicate of that ratio whose denominator has been multiplied — according as the denominator has been set down twice, three times, or four times, and so multiplied, as we have explained. For example, given these four numbers in continued proportion in the quadruple ratio, the first, 192, has to the fourth, 3, a ratio denominating by 64, which is called the triplicate of the quadruple ratio; because the denominator 64 of the ratio of the extremes 192 and 3 is produced from the denominator 4 of the quadruple ratio set down three times — on account of the three quadruple ratios interposed between the extremes — and multiplied thus, by saying "four times four are 16, and four times 16 are 64."

Denominatores [Denominators]	4	4	4	
Num. propor. [Numbers in proportion]	192	48	12	3

Likewise, with the same numbers inverted, so that they stand in continued proportion in the one-fourth ratio, the first, 3, has to the fourth, 192, a ratio denominating by $1/64$, which is called the triplicate of the one-fourth ratio; because the denominator $1/64$ of the ratio of the first number 3 to the fourth 192 is produced from the denominator $1/4$ of the one-fourth ratio set down three times and so multiplied, on account of the three one-fourth ratios interposed between the extremes. For $1/4$ into $1/4$ makes $1/16$, and $1/16$ into $1/4$ makes $1/64$.

Denominatores [Denominators]	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	
Num. proport. [Numbers in proportion]	3	12	48	192

There is therefore nothing to prevent the ratio 3 to 192, denominated by $\frac{1}{64}$, from being called the triplicate of the one-fourth ratio 3 to 12, even though it is smaller than the latter: because the denominator $\frac{1}{64}$ has been produced from the denominator $\frac{1}{4}$ set down three times and so multiplied, as we have said. Yet the ratio 3 to 192, denominated by $\frac{1}{64}$, cannot be said to be *three times greater* than the one-fourth ratio 3 to 12, because the denominator $\frac{1}{4}$ tripled does not make the denominator $\frac{1}{64}$, but $\frac{3}{4}$; and so the ratio is sub-

—sesquitercian, the triple of the one-fourth ratio [subquadrupla]. Just so, again, it is not the ratio denominated by 64, but rather the twelvefold ratio, that is the triple of the quadruple ratio; because the denominator 4 tripled does not produce the denominator 64, but 12. Moreover, when any number of proportional numbers are continued — whether the greater are compared with the less, or the less with the greater — the denominator of the ratio of the first to the last is generated from the denominators of the intermediate ratios multiplied among themselves, as we shall demonstrate at Definition 5 of Book 6. And that one ratio is at last the double, triple, and so forth, of another, when the denominator of the one is the double, triple, etc., of the denominator of the other — so that the tenfold ratio is the double of the fivefold, and the sixfold the triple of the double — we shall show at the end of Book 9; and this subject has been treated at length by Rodolfo Volumnio [Rodolphus Volumnius] in his *Disputation on the Proportion of Proportions* [*Disputatio de proportionibus proportionum*]. For now let it suffice to confirm this very point by the common judgment of mankind, drawn from sensible things.

Suppose, then, that some Agent [Agens] has to a Patient [Patiens] a tenfold ratio, say, so that the Agent is 10 and the Patient 1. Who would be so out of his senses as not to grasp at once that, if the same Agent is increased so as to become 20 while the Patient remains 1, the Agent then has twice the power with respect to the same Patient as before? So the twentyfold ratio, whose denominator 20 is the double of the denominator 10, is the double of the tenfold ratio — but not the hundredfold ratio, as the authors of the contrary opinion would have it. Nevertheless this hundredfold ratio is called the duplicate of the tenfold, on account of the multiplication of the denominator 10 into itself, and on account of the two tenfold ratios interposed between the numbers that have the hundredfold ratio, as appears here: 100. 10. 1. Likewise, on the contrary, suppose some Agent has to a Patient, say, a one-tenth ratio [subdecupla], so that the Agent

is 1 and the Patient 10. Who is so dull and unschooled as not to understand that an Agent which becomes 2 is twice as powerful with respect to the same Patient 10 as an Agent of 1? Since, then, the Agent 2 has to the Patient 10 a one-fifth ratio [subquintupla], whose denominator $1/5$ (or $2/10$) is compounded from the denominator $1/10$ taken twice, the one-fifth ratio will indeed be the double of the one-tenth ratio — not the one that is one-hundredth [subcentupla], as the aforesaid authors would have it, since this last is far smaller than the one-tenth. Yet the one-hundredth ratio will be called the duplicate of the one-tenth ratio, for the reason often explained, as appears here: 1. 10. 100.

From what we have said, it can be gathered without obscurity that a ratio of two quantities between which no mean is interposed shares, in a manner, the nature of a line, since it is produced from no other ratio; that a ratio whose quantities enclose a single mean in continued proportionality has the character of a square surface, since it is generated from two equal ratios, just as a square is formed from two equal lines; and finally that a ratio between whose quantities two means in continued proportionality are interposed takes on the nature of a solid, indeed of a cube, since it arises from three equal ratios, just as a cube too is raised from three equal lines. But on these matters you will find more in the arithmeticians who have expounded the rule of algebra [Algebræ regula].

XI.

Homologous [homologæ], or similar-in-ratio, magnitudes are called: the antecedents to the antecedents, and the consequents to the consequents.

He defined proportionality above as a similarity of ratios. Now he teaches that not only in any proportionality are the ratios said to be similar, but that the terms themselves — the quantities — are also called similar, or homologous. He says that the antecedent magnitudes are called homologous, or similar in ratio, to one another, and likewise the consequents to one another, so that in very many demonstrations we may understand which sides of the figures, compared with one another, ought to be the antecedents of the ratios and which the consequents — as will become clear in Book 6. If, then, there is the ratio A to B which is C to D, the quantity A will be called similar to the quantity C, and B similar to D. For, on account of the similarity of the ratios, each antecedent magnitude must be either equal to each consequent, or greater, or smaller, in the same manner; otherwise each antecedent would not have the same ratio to each consequent. You have an example in the magnitudes set out, in which the antecedents are greater than the consequents in the same way — namely half as large again. Another example you see in the magnitudes E, F, G in continued proportion, where E and F are

homologous just as F and G are, as is plain. And it is for this reason that Euclid, in Definitions 6 and 8, directed that equimultiples be taken of the first and third magnitudes — that is, of the antecedents — and other equimultiples of the second and fourth magnitudes, namely the consequents. For these are similar in proportional magnitudes, as is clear from this definition, but dissimilar in magnitudes that are not proportional.

Diagram. A row of vertical line-segments standing on a common baseline, illustrating the homologous magnitudes of Definition XI. The four leftmost segments carry a numeral above each and a letter beneath:

- A: line of magnitude 9 (an antecedent)
- B: line of magnitude 6 (a consequent)
- C: line of magnitude 12 (an antecedent)
- D: line of magnitude 8 (a consequent)

Here $A : B = 9 : 6$ and $C : D = 12 : 8$, each ratio being $3 : 2$ (the antecedent half as large again as its consequent), so A and C are homologous antecedents, B and D homologous consequents. To the right stand three further segments, labelled E, F, G at their feet, drawn in continued proportion and marked with short internal cross-strokes; no numerals are printed on them [numeral: none printed].

Oronce Finé [Orontius] and some other interpreters explain this definition quite differently. For they suppose that Euclid means to teach that, among proportional magnitudes, both the antecedent magnitudes and the consequents can be compared with one another in various ways — as will appear in the following definitions. But if the words of the definition are weighed more carefully, and its use in Book 6 is considered, no one will doubt that our exposition is to be preferred to this one.

XII.

An alternate ratio [alterna ratio] is the taking of antecedent to antecedent, and of consequent to consequent.

Here he explains certain modes of arguing about ratios, whose use among geometers is most frequent. These are six in number. The first is called the alternate or permuted ratio [proportio alterna, siue permutata]. The second, the inverse, or ratio by contraries [inversa, seu proportio è contrario]. The third, composition of the ratio, or conjunct proportionality [compositio rationis, seu coniuncta proportionalitas]. The fourth, division of the ratio, or disjunct proportionality [diuisio rationis, vel disiuncta proportionalitas]. The fifth,

conversion of the ratio, or inverted proportionality [conuersio rationis, siue euersa proportionalitas]. The sixth, finally, is called the ratio from equality, or equal proportion [proportio ex æqualitate, seu æqua proportio]. The alternate, or permuted, ratio, then, he says, occurs when, four proportional magnitudes being given, it is inferred that the ratio of the antecedent of the first proportion to the antecedent of the second is the same as the ratio which the consequent of the first has to the consequent of the second. Thus, if we suppose the ratio A to B which is C to D, and therefore conclude that the ratio A to C is the same as the ratio B to D, we shall be said to argue from a permuted ratio. Greek writers, in this kind of argument, use more or less this manner of speaking: As A is to B, so C is to D; therefore, alternately, or by exchange [permutando, seu vicissim], there will also be,

—there will also be, as A to C, so B to D. That such an inference is sound will be demonstrated at Proposition 16 of this book. However, in this mode of arguing it is necessary that all four magnitudes be of the same kind, so that a ratio may hold between any two of them taken together. For this is not correctly inferred: "As line A is to line B, so number C is to number D; therefore, alternately, as line A is to number C, so line B is to number D" — since there is no ratio of a line to a number, or conversely, as is evident from Definition 5. In the other modes of arguing that follow, however, the former magnitudes may be of one kind of magnitude and the latter of another kind, as will be established from the demonstrations of this Book 5.

Diagram. Four vertical line-segments standing on a common baseline, illustrating the alternate ratio of Definition XII. Each carries a numeral above and a letter beneath:

- A: line of magnitude 18
- B: line of magnitude 15
- C: line of magnitude 12
- D: line of magnitude 10

Here $A : B = 18 : 15 = C : D = 12 : 10$ (each $6 : 5$); alternately, $A : C = 18 : 12 = B : D = 15 : 10$ (each $3 : 2$), as the text states.

XIII.

An inverse ratio [inuersa ratio] is the taking of the consequent, as antecedent, to the antecedent, as consequent.

For instance, if from the fact that A is to B as C is to D we infer that B is to A as D is to C — that is, if we refer the consequents to the antecedents — we shall be said to argue from an inverse ratio. In this argument the authors speak more or less thus: As A is to B, so C is to D; therefore, by conversion, or by contraries [conuertendo, vel è contrario], B will also be to A as D is to C. That this mode of arguing is sound will be shown in the corollary to Proposition 4 of this book. Moreover, the two former magnitudes may be of one kind and the latter of another. For it will be correct to infer: as line A is to line B, so is triangle (or number) C to triangle (or number) D; therefore, inverting, as line B is to line A, so is triangle (or number) D to triangle (or number) C, as will be established from the corollary to Proposition 4.

Diagram. Four vertical line-segments standing on a common baseline, illustrating the inverse ratio of Definition XIII. Each carries a numeral above and a letter beneath:

- A: line of magnitude 9
- B: line of magnitude 6
- C: line of magnitude 12
- D: line of magnitude 8

Here $A : B = 9 : 6 = C : D = 12 : 8$; inversely, $B : A = 6 : 9 = D : C = 8 : 12$.

XIV.

Composition of a ratio [compositio rationis] is the taking of the antecedent together with the consequent, as one, to the consequent itself.

Let there be the ratio AB to BC, which is DE to EF. If from this it is gathered that the ratio of the whole AC — namely, of the antecedent together with the consequent — to BC, the consequent, is the same as the ratio which the whole DF — the antecedent, again, together with the consequent — has to EF, the consequent, then such an argument will be called composition of the ratio, because from the antecedent and the consequent another, new antecedent is compounded. This manner of speaking you will find among Greek writers in this argument: As AB is to BC, so DE is to EF; therefore, by composition [componendo], AC will also be to BC as DF is to EF. This mode of arguing will be demonstrated in Proposition 18 of this book.

To this mode of arguing by composition of the ratio, two others can be added. The first may be called the *converse* composition of the ratio: that is, when the antecedent and the consequent are taken as one, which is then compared with

the antecedent. Thus, if it is "as AB is to BC, so DE is to EF," we infer: therefore, as AC, formed from the antecedent and the consequent, is to the antecedent AB, so is DF, composed from the antecedent and the consequent, to the antecedent DE. That this argument is valid we shall demonstrate at Proposition 18 of this book; and in it we may use this manner of speaking: "therefore, by converse composition of the ratio." The other may be called the *contrary* composition of the ratio: that is, when the same antecedent magnitude is referred to the antecedent and the consequent taken as one. Thus, if it is "as AB is to BC, so DE is to EF," we infer, by contrary composition of the ratio: therefore, as AB, the antecedent, is to the whole AC, composed of the antecedent and the consequent, so is DE, the antecedent, to DF, composed of the antecedent and the consequent. And that this argumentative formula is valid we shall show at Proposition 18 of this book.

Diagram. Two horizontal line-segments, one above the other, illustrating the composition of Definition XIV. The upper line runs from A through B to C, divided at B; the lower from D through E to F, divided at E:

- Upper line — A to B: 12; B to C: 8 (whole AC = 20)
- Lower line — D to E: 6; E to F: 4 (whole DF = 10)

Here $AB : BC = 12 : 8 = DE : EF = 6 : 4$ (each 3 : 2); by composition, $AC : BC = 20 : 8 = DF : EF = 10 : 4$ (each 5 : 2).

XV.

Division of a ratio [diuisio rationis] is the taking of the excess by which the antecedent surpasses the consequent, to the consequent itself.

For instance, if it is said that the ratio of the whole AB to CB is the same as that of the whole DE to FE, then AC — the excess by which the antecedent surpasses the consequent — will be to CB, the consequent, as DF — the excess by which the antecedent exceeds the consequent — is to FE, the consequent. In this division of the ratio the authors speak thus: therefore, by division [diuidendo], and so on. This inference will be shown at Proposition 17 of this book.

To this mode of arguing two others can also be joined. The first we may call the *converse* division of the ratio: that is, when the consequent is referred to the excess by which the antecedent surpasses the consequent. Thus, if it is "as AB is to CB, so DE is to FE,"

Diagram. Two horizontal line-segments, one above the other, illustrating the division of Definition XV. The upper line runs from A through C to B, divided at C; the lower from D through F to E, divided at F:

- Upper line — A to C: 12; C to B: 4 (whole AB = 16)
- Lower line — D to F: 6; F to E: 2 (whole DE = 8)

Here $AB : CB = 16 : 4 = DE : FE = 8 : 2$ (each 4 : 1); by division, $AC : CB = 12 : 4 = DF : FE = 6 : 2$ (each 3 : 1).

so is DE to FE — we conclude, by the *converse* division of the ratio. Therefore it will be: as CB, the consequent, is to AC, the excess by which the antecedent surpasses the consequent, so is FE, the consequent, to DF, the excess by which the antecedent surpasses the consequent. That this line of argument holds good we shall show at Proposition 17 of this book.

It is clear that, for both of these arguments by division of a ratio to hold — Euclid's as well as our own — the antecedent must be greater than the consequent; otherwise the division could not be carried out.

The other mode may be called the *contrary* division of a ratio: namely, when the antecedent is compared with the excess by which the consequent surpasses the antecedent. As when we say: AC is to AB as DF is to DE. Then also, by the contrary division of the ratio, it will be: as AC, the antecedent, is to CB, the excess by which the consequent surpasses the antecedent, so is DF, the antecedent, to FE, the excess by which the consequent surpasses the antecedent. This mode of arguing too we shall demonstrate at Proposition 17 of this book.

Moreover it is evident that in this contrary division of a ratio the consequent must be greater than the antecedent, so that the excess by which the consequent surpasses the antecedent can be taken.

XVI.

Conversion of a ratio is the taking of the antecedent to the excess by which the antecedent surpasses the consequent itself.

Diagram. Two stacked horizontal number-lines illustrating conversion of a ratio, subdivided by tick marks.

- Upper line: A (left end) — C (dividing point) — B (right end). The segment CB is marked 4; the segment AC carries a small numeral I cannot resolve, [numeral: ?].
- Lower line: D (left end) — F (dividing point) — E (right end). The segment DF is marked 12; the segment FE is marked 8.

We conclude in this way: as the whole magnitude AB is to CB, so is the whole DE to FE; therefore this same AB will also be to AC — the excess by which the antecedent surpasses the consequent — as DE is to DF. We shall be said to argue by conversion of the ratio; and this is roughly how the writers put it: "Therefore, by conversion of the ratio, and so on." This mode of arguing will be confirmed in the corollary to Proposition 19 of this book.

It is likewise plain that in this mode of arguing by conversion of a ratio the antecedent must surpass the consequent, so that the excess by which the antecedent surpasses the consequent can be taken.

XVII.

There is a ratio *ex aequali* when there are more than two magnitudes, and another set equal to them in number, taken two by two and in the same ratio, such that, as among the first magnitudes the first is to the last, so among the second magnitudes the first is to the last.

Or otherwise: the taking of the extremes by the removal of the means.

Diagram. Six vertical bars of graded height, subdivided by tick marks, illustrating a ratio *ex aequali* in ordered arrangement. Reading left to right, each bar carries a numeral at its top and a letter at its foot:

- A: 18
- B: 12
- C: 6
- D: 12
- E: 8
- F: 4

(Here $A : B = D : E = 3 : 2$, and $B : C = E : F = 2 : 1$, so that $A : C = D : F = 3 : 1$.)

Let there be several magnitudes, more than two — A, B, C — and just as many, D, E, F, in the same proportion two by two: that is, A is to B as D is to E, and B is to C as E is to F. If, then, it is inferred that on this account the proportion of A to C — the first to the last among the first magnitudes — is the same as that of D to F — the first to the last among the second magnitudes — this form of argument will be said to be taken *ex aequo*, or from equality: that is, one in which the extreme magnitudes, once the means have been removed, are found to have one and the same proportion between themselves, as is expressed in the other definition. And since one may argue *ex aequali* in proportions in two ways — the one when we take the magnitudes two by two in the same proportion, proceeding in order, the

other when the order is inverted — Euclid explains in the two following definitions what ordinate proportion is, and what perturbed proportion is.

XVIII.

Ordinate proportion is when, as an antecedent is to a consequent, so is an antecedent to a consequent; and further, as a consequent is to something else, so is a consequent to something else.

Diagram. Six vertical bars, subdivided by tick marks, illustrating ordinate proportion. Left to right, each with a top numeral and a foot letter:

- A: 12
- B: 6
- C: 4
- D: 6
- E: 3
- F: [numeral: ?] (the sixth numeral is damaged/illegible)

For example, when A is to B as D is to E; and again, as B, the consequent, is to something else, namely to C, so is E, the consequent, to F, something else — such a proportion will be called ordinate, because the same order is kept both in the first three magnitudes and in the second three, since in each set the first is first compared with the second, then the second with the third. When, therefore, ordinate proportion is preserved in a mode of arguing *ex aequali*, it is demonstrated at Proposition 22 of this book that the argument is sound.

XIX.

Perturbed proportion is when, three magnitudes being set out, and others equal to them in number, as in the first magnitudes the antecedent is to the consequent, so in the second magnitudes the antecedent is to the consequent; but as in the first magnitudes the consequent is to something else, so in the second magnitudes something else is to the antecedent.

Diagram. Six vertical bars, subdivided by tick marks, illustrating perturbed proportion. Left to right, each with a top numeral and a foot letter:

- A: 12
- B: 8
- C: 4

- D: 12
- E: 6
- F: 4

(Here $A : B = E : F = 3 : 2$, and $B : C = D : E = 2 : 1$.)

If it be that, as A is to B, so is E to F; and then, as in the first magnitudes B, the consequent, is to C, something else, so in the second magnitudes something else, D, is to E, the antecedent magnitude — such a proportion will be called perturbed, because the same order is not preserved in the proportions of the magnitudes. For in the first magnitudes the first is compared with the second, but in the second set the second with the third; and then in the first the second with the third, but in the second the first with the second. When, therefore, perturbed proportion is preserved in a mode of arguing *ex aequali*, it is demonstrated at Proposition 23 of this book that the argument is sound.

Moreover, both perturbed and ordinate proportion always yield, *ex aequali*, the same proportion of the extremes, even if more than three magnitudes are set out, as will become clear from Propositions 22 and 23 of this book.

The commentators on Euclid make use — in this book, and in others where the proportions of magnitudes are treated — of a certain axiom, which we judged it would not be useless to append here. It runs as follows.

Whatever proportion one magnitude has to another, any given magnitude will have the same to some other; and some other magnitude will have the same to any given magnitude.

Diagram. Five horizontal segments of unequal length, stacked, each marked off with a tick, illustrating the axiom of a fourth proportional. Top to bottom the segments are labelled A, B, C, D, and a fifth whose letter is imperfectly printed (it reads as an F; the commentary's fifth magnitude is E) [?].

For example, whatever proportion A has to B, the given magnitude C will have the same to some other, namely to D; likewise some other magnitude E will have the same to the given C. For although we may sometimes not know what that fourth magnitude is, there is nevertheless no doubt that it can exist in the nature of things, since this implies no contradiction — as the philosophers say — nor does anything absurd follow from it.

THEOR. 1. PROPOS. 1.

[Margin: j.]

If there be any number of magnitudes, each of them the same multiple of each one of an equal number of other magnitudes, then whatever multiple one magnitude is of one, the same multiple will all be of all.

Let there be any number of magnitudes AB, CD, equimultiples of as many magnitudes E, F. I say that AB and CD together are the same multiple of E and F together as AB is of E, or as CD is of F.

For since AB and CD are equimultiples of E and F, let AB be divided into the magnitudes AG, GH, HB equal to E, and CD likewise into the magnitudes CI, IK, KD equal to F. (Each can be divided into wholly equal parts, since AB and CD are equimultiples of E and F, so that E is contained exactly in AB as many times as F in CD, as is clear from what we wrote in Definition 2 of this book.) Then the magnitudes AG, GH, HB will be equal in number to the magnitudes CI, IK, KD. Now since AG and E are equal to each other, if the equals CI and F be added to them, [Margin: a — 2. *pron.* (the second common notion — "if equals be added to equals...")] AG and CI together will be equal to E and F together. In the same way GH and IK together will be equal to E and F together, and likewise HB and KD to the same E and F. Therefore, as many times as E is contained in AB, or F in CD, so many times are E and F together contained in AB and CD together. And so, whatever multiple AB is of E, the same multiple are AB and CD together of E and F together, as is clear from what we wrote in Definition 2 of this book. Therefore, if there be any number of magnitudes equimultiples of any number of others, and so on. Which was to be demonstrated.

Diagram. Horizontal segments illustrating Proposition 1. Two short reference segments are set at the top, labelled E and F. Below them: the magnitude AB, divided into three equal parts by the points A, G, H, B (so $AG = GH = HB = E$); and the magnitude CD, divided into three equal parts by the points C, I, K, D (so $CI = IK = KD = F$).

Scholium. This same result will be demonstrated in general at Proposition 12, for every kind of proportion, rational as well as irrational. But it was necessary to demonstrate it first here, in multiple proportion, because from it other propositions must be demonstrated before Proposition 12 can be proved.

THEOR. 2. PROPOS. 2.

[Margin: ij.]

If a first magnitude be the same multiple of a second as a third is of a fourth, and a fifth also be the same multiple of the second as a sixth is of

the fourth, then the sum of the first and the fifth will be the same multiple of the second as the sum of the third and the sixth is of the fourth.

Diagram. Horizontal segments illustrating Proposition 2. The magnitude AG is drawn as one line divided at B into AB and BG (points A, B, G); a short segment marks the magnitude C. Below, the magnitude DH is drawn divided at E into DE and EH (points D, E, H); a short segment marks the magnitude F.

Let the first magnitude AB be the same multiple of the second, C, as the third, DE, is of the fourth, F; and again, let BG, the fifth, be the same multiple of C, the second, as EH, the sixth, is of F, the fourth. I say that AB, the first, composed with the fifth BG — that is, the whole AG — is the same multiple of the second, C, as DE, the third, composed with the sixth EH — the whole DH — is of the fourth, F.

For since AB and DE are equimultiples of C and F, there will be in AB as many magnitudes equal to C as there are in DE equal to F. By the same reasoning, there will be in BG as many equal to C as there are in EH equal to F. If, then, to the equal multitudes AB, DE are added the equal multitudes BG, EH, the whole multitudes AG, DH will be equal in number. Therefore C is contained in AG as many times as F in DH; and so the whole AG (the first composed with the fifth) [print: AB] is the same multiple of C, the second, as DH (the third composed with the sixth) is of F, the fourth. If, then, a first magnitude be a multiple of a second, and so on. Which was to be shown.

Scholium. But if the first magnitude and the third be equal to the second and the fourth, while the fifth and sixth are equimultiples of the second and fourth; or if, on the other hand, the first and third be equimultiples of the second and fourth, while the fifth and sixth are equal to the second and fourth — then by the same reasoning the whole AG (the first and the fifth) will be the same multiple of the second, C, as the whole DH (the third and the sixth) is of the fourth, F. For AG, a multitude of magnitudes equal to C, will always be shown to be equal to DH, a multitude of magnitudes equal to F, as is clear in the accompanying figures. But if both the first and third, and the fifth and sixth, are supposed equal to the second and fourth, it is clearer than daylight that the first and fifth together, and the third and sixth together, are equimultiples — namely doubles — of the second and fourth magnitudes.

Diagram. The first of two figures accompanying the scholium. A tall bar for the composite AG, marked off at B (points A at top, B intermediate, G at foot); a short

bar **C**; a tall bar for the composite DH, marked off at E (points D at top, E intermediate, H at foot); and a short bar **F**.

Diagram. The second, companion figure of the scholium, of the same construction : a tall bar AG marked at B (points A, B, G); a short bar **C**; a tall bar DH marked at E (points D, E, H); a short bar **F** — illustrating the alternative case in which the divisions fall differently.

This too will be concluded by Euclid, for every kind of proportion in general, at Proposition 24; but it was worthwhile to demonstrate it first in multiple proportion, so that what follows may be demonstrated.

THEOR. 3. PROPOS. 3.

[Margin: iij.]

If a first magnitude be the same multiple of a second as a third is of a fourth, and equimultiples be taken of the first and third, then *ex aequo* each of the magnitudes taken will be the same multiple of each — the one of the second, the other of the fourth.

Diagram. Vertical bars illustrating Proposition 3. One bar for E, divided into three equal parts by the points E (foot), G, H, and I (top) [top letter I: ?]; a companion bar for F, divided into three equal parts by the points F (foot), K, L, and M (top). Beside them stand the four shorter bars for the magnitudes A, B, C, D. Foot letters, left to right: E, A, B, F, C, D.

Let the first magnitude A be the same multiple of the second, B, as C, the third, is of the fourth, D; and let E, F be taken as equimultiples of the first and third, A and C. I say that *ex aequo* E is the same multiple of B, the second, as F is of D, the fourth.

For since E and F are equimultiples of A and C, if E and F are distributed into magnitudes equal to A and C — namely into EG, GH, HI, and FK, KL, LM — there will be as many parts in E equal to A as there are in F equal to C. Now since EG, FK are equal to A and C, and A and C are, by hypothesis, equimultiples of B and D, EG and FK will also be equimultiples of the same B and D. By like reasoning, so will GH and KL; and likewise HI and LM will be equimultiples of the same B and D. Therefore, since EG, the first magnitude, is the same multiple of the second, B, as FK, the third, is of the fourth, D; and likewise GH, the fifth, is the same multiple of the same second, B, as KL, the sixth, is of the same fourth, D; [Margin: a — 2. *quinti* (Proposition 2 of Book V)] EH, composed of the first and the fifth, will be the same multiple of the second, B, as FL, composed of the third and the sixth, is of

the fourth, D. Again, since EH, the first, is the same multiple of the second, B, as FL, the third, is of the fourth, D — as was just demonstrated — and HI, the fifth, is the same multiple of the second, B, as LM, the sixth, is of the fourth, D; [Margin: b — 2. *quint.* (Proposition 2 of Book V)] EI, composed of the first and the fifth, will be the same multiple of the second, B, as FM, composed of the third and the sixth, is of the fourth, D. And the reasoning is the same if there are more parts in E and F. If, then, a first magnitude be the same multiple of a second as a third is of a fourth, and so on. Which was to be shown.

Scholium. This theorem will be shown at Proposition 22 not only for equimultiple magnitudes, but also for all that, taken two by two, have the same proportion, whether rational or irrational; yet it was necessary to demonstrate it first here, in multiple proportion, so that the following Proposition 4 may be demonstrated.

THEOR. 4. PROPOS. 4.

[Margin: iiij.]

If a first magnitude have the same ratio to a second as a third has to a fourth, then equimultiples of the first and third will also have, to equimultiples of the second and fourth — taken by any multiplication whatever — the same ratio, provided they are taken so as to correspond to one another.

Let the proportion of A to B be the same as that of C to D; and let E and F be taken as equimultiples of the first, A, and the third, C; likewise let G and H be equimultiples of the second, B, and the fourth, D, by any multiplication whatever — whether E, F be such multiples of...

the equimultiples of A and C, and likewise G and H the equimultiples of B and D — whether or not [these correspond]. Given this, it follows from definition 6 of this book that if E falls short of G, then F likewise falls short of H; if E is equal to G, then F is likewise equal to H; and finally, if E exceeds G, then F likewise exceeds H. Otherwise, by definition 6, the ratio of A to B would not be the same as that of C to D, if their equimultiples did not always behave in this way.

Now I say that the multiples of the first and third not only fall short of, together equal, or together exceed the multiples of the second and fourth — as we have said — but also have the same ratio to one another: that is, E, the multiple of the first A, is to G, the multiple of the second B, as F, the multiple of the third C, is to H, the multiple of the fourth D. That is to say: if again E is set as the first

magnitude, G as the second, F as the third, and H as the fourth, and if any equimultiples whatever of E and F are taken, and likewise any equimultiples of G and H, then the multiples of E and F will together fall short of, together equal, or together exceed the multiples of G and H — as definition 6 requires. For it is the same thing to say that four magnitudes have the same ratio and to say that their equimultiples (as we have said) together fall short of, together equal, or together exceed one another.

For let I, K again be taken as equimultiples of E, F; likewise L, M as equimultiples of G, H. Now since E (the first) is the same multiple of A (the second) as F (the third) is of C (the fourth), and I, K have been taken as equimultiples of E, F, the first and third, [Margin: b — Elements V.3] then *ex aequali* I, K will also be equimultiples of A, C, the second and fourth. By the same reasoning, L, M will be equimultiples of B, D. And since the ratio of A, the first, to B, the second, is posited to be the same as that of C, the third, to D, the fourth, and I, K have been shown to be equimultiples of the first and third, A, C, while L, M are equimultiples of the second and fourth, B, D, [Margin: c — Elements V, definition 6] it comes about that if I, the multiple of the first, falls short of L, the multiple of the second, then K, the multiple of the third, necessarily falls short of M, the multiple of the fourth; and if I is equal to L, then K too is necessarily equal to M; and finally, if I exceeds L, then K too necessarily exceeds M. The same will be shown for any equimultiples whatever of the magnitudes E and F, and likewise of the magnitudes G and H — because these equimultiples, whatever they may be, will always [Margin: d — Elements V.3] be equimultiples of the magnitudes A, C and B, D as well. And so, since I and K are equimultiples of the first E and the third F, and L and M are equimultiples of the second G and the fourth H, and it has been shown that if I, the multiple of the first, is less than L, the multiple of the second, then K, the multiple of the third, is also less than M, the multiple of the fourth, and so on — and that this holds under any multiplication whatever — [Margin: e — Elements V, definition 6] then E, the first, will be to G, the second, as F, the third, is to H, the fourth. So then, if the first has the same ratio to the second [that the third has to the fourth], and so on. Which was to be demonstrated.

Diagram. A schematic set of twelve parallel vertical magnitude-bars, arranged in an upper and a lower rank (six bars each), illustrating the equimultiples used in the proof of Proposition 4. Each upper label pairs with the lower label directly beneath it. The bars vary in length to indicate the relative sizes of the magnitudes.

- Upper rank (left to right): I, E, A, B, G, L

- Lower rank (left to right): K, F, C, D, H, M
- Vertical pairings: I/K, E/F, A/C, B/D, G/H, L/M (I, K are equimultiples of E, F; L, M of G, H; E, F of A, C; G, H of B, D).

COROLLARY

From this the inverse ratio, which Euclid explained in definition 13, is easily demonstrated — that is, if four magnitudes are proportional, they are also proportional conversely, or by the inverse ratio. For let A be to B as C is to D. I say that, by conversion, B is to A as D is to C. For let E, F be taken as equimultiples of A, C, the first and third; likewise let G, H be taken as equimultiples of B, D, the second and fourth. Since A, the first, stands to B, the second, as C, the third, stands to D, the fourth, [Margin: f — Elements V, definition 6] it necessarily follows that if E, the multiple of the first, is less than G, the multiple of the second, or equal to it, or greater, then F, the multiple of the third, is likewise less than, equal to, or greater than H, the multiple of the fourth. And it is clear that, conversely, if G is greater than E, or equal to it, or less, then H too will be greater than, equal to, or less than F — under whatever multiplication these equimultiples are taken. For if both E, F are less than both G, H, then conversely both G, H will be greater than both E, F; and if both E, F are equal to both G, H, then conversely both G, H will also be equal to both E, F; and finally, if both E, F are greater than both G, H, then reciprocally both G, H will be less than both E, F. And so, since G, H have been taken as equimultiples of the first B and the third D, and E, F as equimultiples of the second A and the fourth C, and it has been shown that G, H together exceed, together equal, or together fall short of E, F — under whatever multiplication these multiples are taken — [Margin: g — Elements V, definition 6] then B, the first, will be to A, the second, as D, the third, is to C, the fourth. Which was to be demonstrated.

Diagram. Four horizontal magnitude-lines, one above another, illustrating the corollary on inverse ratio. Each line carries two labels marking a magnitude and its equimultiple.

- Line 1: E — F (E, F, the equimultiples of A, C)
- Line 2: A — C (the first and third)
- Line 3: B — D (the second and fourth)
- Line 4: G — H (G, H, the equimultiples of B, D)

SCHOLIUM

This proposition, together with its corollary, is true whether or not the two magnitudes A, B are of the same kind as the two magnitudes C, D, as is clear from the demonstration.

Orontius [Oronce Fine] attempts to demonstrate this fourth theorem as follows. After showing, from proposition 3 of this book, that I, K are equimultiples of the magnitudes A, C, and likewise that L, M are equimultiples of the magnitudes B, D, he immediately infers, by the converse of definition 6 (according to his own exposition), that I is to L as K is to M. And therefore, he says, E will also be to G as F is to H, by the same definition. This demonstration is thoroughly faulty. First, because on this reading the same thing is proved by the same thing: namely, that when four proportional magnitudes are given, the equimultiples of the first and third have the same ratio to the equimultiples of the second and fourth — inferred from the fact that (by definition 6, as he himself explained it) the equimultiples of the first and third have the same ratio to the equimultiples of the second and fourth; which is absurd. And second, because in the same manner he might have inferred, right from the start, that E is to G as F is to H by definition 6, without taking on any new equimultiple magnitudes — for there is no more reason in the one case than in the other. This kind of demonstration must therefore be rejected, together with the exposition of definition 6, as we have said above. But according to our interpretation of the same definition, all that is established is that if I is less than L, or equal to it, or greater, then K too is less than M, or equal to it, or greater — just as the same is established for the equimultiples E, F, G, H, and for any others whatever, if they are taken as they correspond to one another. Hence from definition 6 it is rightly concluded that E, the first, is to G, the second, as F, the third, is to H, the fourth — since definition 6 applies to them.

Diagram. A second set of twelve parallel vertical magnitude-bars in an upper and lower rank of six, accompanying the scholium's discussion of Orontius's demonstration; the layout repeats that of the figure for Proposition 4.

- Upper rank (left to right): I, E, A, B, G, L
- Lower rank (left to right): K, F, C, D, H, M

THEOR. 5. PROPOS. 5. [Margin: v.]

If one magnitude is the same multiple of another that a part taken away is of a part taken away, then the remainder too will be the same multiple of the remainder that the whole is of the whole.

So then: let the whole AB be the same multiple of the whole CD that the part taken away, AE, is of the part taken away, CF — whether AE and CF are commensurable with the wholes AB, CD (as in the first figure) or incommensurable (as in the second figure); and likewise whether AE, CF are composed of the same parts of which the wholes AB, CD are composed (as in the former figure) or not of the same parts (as in the latter figure). I say that the remainder EB is the same multiple of the remainder FD that the whole AB is of the whole CD. For let EB be set as such a multiple of some magnitude — namely of GC — as AE is of CF, or as the whole AB is of the whole CD. Now since AE and EB are equimultiples of CF and GC respectively, [Margin: a — Elements V.1] the whole AB will be the same multiple of the whole GF that AE is of CF — that is, all of them of all, as one of one. But AB was also assumed to be the same multiple of CD as it is of GF; [Margin: b — 6. pron. (common notion 6)] and therefore GF and CD are equal. So, when the common part CF is taken away, GC and FD will be equal. Therefore EB will be the same multiple of FD that it is of GC. But EB was set as the same multiple of GC that AE is of CF — that is, as the whole AB is of the whole CD. Therefore the remainder EB is the same multiple of the remainder FD that the whole AB is of the whole CD, which is the thing proposed.

Diagram. The first (commensurable) figure for Proposition 5: two horizontal magnitude-lines, tick-marked into equal parts. The upper line AB carries the point E; the lower line runs G–C–F–D (the auxiliary magnitude GC together with CF, FD).

- Upper line (left to right): A, E, B
- Lower line (left to right): G, C, F, D

Diagram. The second (incommensurable) figure for Proposition 5: two horizontal magnitude-lines of the same lettering as the first figure but with parts not composed of the same units.

- Upper line (left to right): A, E, B
- Lower line (left to right): G, C, F, D

ALTERNATIVELY. Let the whole AB be the same multiple of the whole CD that the part taken away, AE, is of the part taken away, CF. I say that the remainder EB is the same multiple of the remainder FD that the whole is of the whole. For let GA be set as such a multiple of FD as AE is of CF, or as the whole AB is of the whole CD. Since AE and GA are equimultiples of CF and FD respectively, [Margin: c — Elements V.1] the whole GE will be the same multiple of the whole CD that AE is of CF. But AB too is, by hypothesis, the same multiple of that same CD that AE is

of CF. Therefore GE and AB are equimultiples of CD, [Margin: d — 6. pron. (common notion 6)] and hence equal to one another. So, when the common part AE is taken away, GA and EB will be equal, and therefore equimultiples of FD — since GA was set as a multiple of FD. But GA was set as the same multiple of FD that AB is of CD. Therefore the remainder EB too will be the same multiple of FD that the whole AB is of the whole CD; which is the thing proposed. And so, if one magnitude is the same multiple of another, and so on. Which was to be demonstrated.

Diagram. The pair of figures for the alternative proof of Proposition 5: horizontal magnitude-lines carrying the auxiliary segment GA together with AE, EB, CF, FD. The engraved letters are faint and partly render mirror-wise on the scan; the readable labels are listed, the rest flagged. []

- Upper line: A, E, B; leftmost point G[?]
- Lower line: C[?], F, D

Diagram. The second figure of the alternative proof, of the same construction as the preceding; labels faint and partly mirror-rendered. []

- Upper line: A[?], E[?], B[?]
- Lower line: F[?], D[?]

SCHOLIUM

In general, this same thing will be proved in proposition 19, for magnitudes of any ratio whatever, and not only for magnitudes in a multiple ratio, as has been done here.

THEOR. 6. PROPOS. 6. [Margin: vj.]

If two magnitudes are equimultiples of two magnitudes, and certain [magnitudes] taken away from them are equimultiples of the same, then the remainders too are either equal to those same [magnitudes] or equimultiples of them.

Let the magnitudes AB, CD be equimultiples of E, F; and let AG, CH, taken away from them, be equimultiples of the same E, F. I say that the remainders GB, HD are either equal to those same E, F, or else equimultiples of them. For since AB is a multiple of E, and AG, which is taken away, is likewise a multiple of the same E, the remainder GB will be either equal to E or a multiple of it; for otherwise a magnitude unequal [to E] and not a multiple, added to a multiple, would compose a multiple — which is absurd. Suppose first, then, that GB is equal to E. I say that

HD too is equal to F. For let CI be set equal to F. And since the first, AG, is the same multiple of the second E that the third, CH, is of the fourth F; and the fifth, GB, is equal to the second E, just as the sixth, CI, is equal to the fourth F; [Margin: a — Elements V.2] then AB, the first together with the fifth, will be the same multiple of the second E that HI, the third together with the sixth, is of the fourth F. But CD was also the same multiple of F that AB is of E. Therefore HI and CD are equimultiples of F, [Margin: b — 6. pron. (common notion 6)] and hence equal to one another. So, when the common part CH is taken away, the remainders CI, HD will be equal. And since CI was set equal to F, HD too will be equal to that same F. Which is the thing proposed.

Diagram. The first figure for Proposition 6: horizontal magnitude-lines. The line AB carries the point G; the line CD carries the point H; the magnitudes E and F are drawn as separate shorter segments.

- Line AB (left to right): A, G, B
- Segment: E
- Line CD (left to right): C, H, D
- Segment: F

Suppose next that GB is a multiple of E. I say that HD is likewise a multiple of F. For let CI be set as the same multiple of F that GB is of E; [Margin: c — Elements V.2] then, as before, AB will be the same multiple of E that HI is of F. [Margin: d — 6. pron. (common notion 6)] Hence again HI and CD will be equal; and so, when the common part CH is taken away, the remainders CI, HD will also be equal. But CI is, by hypothesis, the same multiple of F that GB is of E. Therefore HD too will be the same multiple of F that GB is of E — which is the thing proposed. And so, if two magnitudes are equimultiples of two magnitudes, and so on. Which was to be shown.

Diagram. The second figure for Proposition 6 (the case where the remainder is a multiple): horizontal magnitude-lines with the line AB carrying the point C, the lower CD-line carrying the points C, H, and the separate segments E and F.

- Line AB (left to right): A, C, B
- Segment: E
- Lower line (left to right): I[?], C, H, D
- Segment: F

SCHOLIUM [continued]

We shall likewise prove this in general in proposition 24, for every kind of ratio. We shall demonstrate the whole proposition more briefly as follows. Since AB and CD are equimultiples of E and F, there will be in AB as many magnitudes equal to E as there are magnitudes in CD equal to F. Again, since AG and CH are equimultiples of the same E and F, there will likewise be in AG as many magnitudes equal to E as there are magnitudes in CH equal to F. If, then, from the equal multitudes AB, CD there are taken away the equal multitudes AG, CH, the multitudes GB, HD that remain will be equal. Therefore E will be contained in GB as many times as F is contained in HD; and accordingly, if GB is equal to E, HD too will be equal to F; but if GB is a multiple of E, then HD will be the same multiple of F that GB is of E — since E is contained in GB as many times as F is found in HD, as has been shown.

THEOR. 7. PROPOS. 7. [Margin: vij]

Equal magnitudes have the same ratio to the same magnitude; and the same magnitude has the same ratio to equal magnitudes.

Let A, B be two magnitudes equal to one another, and C any third magnitude. I say that A and B have the same ratio to C; and likewise that C, conversely, has the same ratio to A and to B. Let D, E be taken as equimultiples of the equal magnitudes A, B; [Margin: a — 6. pron. (common notion 6)] then D, E will be equal to one another. Again, let F be taken as any multiple of C. Now since D, E are equal, it happens that each of them is either less than F, or equal to it, or greater — whatever multiplication is used in taking these multiples. Therefore, since D, E, the equimultiples of the first A and the third B, are together less than F, the multiple of the second and fourth C (for C stands in place of two magnitudes, and so on), or equal to it, or greater, [Margin: b — Elements V, definition 6] the ratio of the first A to the second C will be the same as that of the third B to the fourth C.

In the same way we shall show that F is either less than both D, E, or equal to both, or greater. Therefore, since F, the multiple of the first and third C, together falls short of, or is equal to, or is greater than D and E, the equimultiples of the second A and the fourth B, [Margin: c — Elements V, definition 6] the ratio of the first C to the second A will also be the same as that of the third C to the fourth B; which is the thing proposed. This second part could be shown more briefly by the corollary to proposition 4, from the inverse ratio. For since it has already been shown that A is to C as B is to C, then by conversion C will be to A as C is to B. Therefore equal magnitudes have the same ratio to the same magnitude, and the

same magnitude has the same ratio to equal magnitudes. Which was to be demonstrated.

Diagram. The figure for Proposition 7: a set of separate horizontal magnitude-lines. The equal magnitudes A, B are drawn as equal segments; C is a third magnitude; D, E are the equimultiples of A, B, and F the multiple of C.

- Segments (as labelled on the lines): C, A, B, E, F

SCHOLIUM

It is clear that nothing can be inferred in this proposition from definition 6 according to the exposition of Campanus and Orontius [Oronce Fine]. For the demonstration does not establish that both multiples D and E have the same ratio to the multiple F, but only that, since they are equal, each of them is either less than, or equal to, or greater than the multiple F. And the same is seen in almost all the propositions that draw their conclusion by means of definition 6 — so that this exposition has rightly been rejected by us.

In almost the same way we shall show that magnitudes equal to one another have the same ratio to other magnitudes equal to one another, if in place of the single multiple F two equimultiples are taken — which Euclid, for the sake of ease, omitted, though he sometimes uses it in what follows, just as if it had been demonstrated in this proposition 7. For let A and B be equal to one another, and C and D likewise equal to one another. I say that A is to C as B is to D. For let equimultiples E, F of A and B, the first and third, be taken; likewise let G, H be taken as equimultiples of C and D, the second and fourth; [Margin: d — 6. pron. (common notion 6)] then E and F will be equal to one another, just as G and H are equal to one another. Therefore, if E, the multiple of the first, falls short of G, the multiple of the second, then F, the multiple of the third, will also fall short of H, the multiple of the fourth; and if it is equal, it will be equal; and if it exceeds, it will exceed. [Margin: e — Elements V, definition 6] Therefore the ratio of A, the first, to C, the second, is the same as that of B, the third, to D, the fourth. Which is the thing proposed.

Diagram. The figure for the scholium to Proposition 7: eight parallel vertical magnitude-bars in an upper and a lower rank of four, showing the two pairs of equal magnitudes and their equimultiples.

- Upper rank (left to right): E, A, C, G
- Lower rank (left to right): F, B, D, H

THEOR. 8. PROPOS. 8. [Margin: viij.]

Of unequal magnitudes, the greater has a greater ratio to the same magnitude than the less has; and the same magnitude has a greater ratio to the less than it has to the greater.

Let there be unequal magnitudes: AB the greater and C the less; and let D be any third magnitude. I say that the ratio of AB to D is greater than the ratio of C to D; and conversely, that the ratio of D to C is greater than the ratio of D to AB. In AB, the greater magnitude, let a magnitude AE be understood equal to the lesser C, so that EB is the remainder. Let both EB and AE then be equally multiplied, on this rule: that GF, the multiple of EB, is greater than D, while HG, the multiple of AE, is not less than that same D, but either greater or equal. In the former figure it was necessary to take GF, HG as triples of EB, AE, because the double of AE would have been less than D. Instead of the triples, any other greater equimultiples could have been taken. In the latter figure, however, it is enough to take GF, HG as the doubles of EB, AE, because each of GF, HG is greater than D; yet in place of the doubles any other greater equimultiples could have been taken. Now since the two FG, GH are equimultiples of the two BE, EA, [Margin: a — Elements V.1] the whole FH will also be the same multiple of the whole AB that GH [printed: FH; recte GH] is of AE — that is, of C, since C and AE were set equal. Let there also be taken a multiple IK of D that is the next greater than HG, namely its double, as in the former figure. But if the double is not greater than HG, let the triple, or quadruple, and so on, be taken. In the latter figure IK has been taken as the quadruple of D, because both the double and the triple are less than HG, whereas the quadruple is now greater. Therefore, when LK is cut off equal to D, IL will not be greater than HG — otherwise IK would not be the multiple of D next greater than HG, but IL too would be greater than GH. But if IK is the double of D, it is clear that IL is not greater than HG, since...

Diagram. The figures for Proposition 8 (a "former" and a "latter" figure): horizontal magnitude-lines showing AB with the point E (AE equal to C, remainder EB), the third magnitude D, and the constructed multiples FH (=FG+GH), HG, IK, LK. The lower part of the page is heavily stained and several labels are faint or mirror-rendered on the scan; readable labels are listed and the rest flagged. []

- Line AB with dividing point: A, E, B
- Further labelled points/segments read: H, C[?], F, L[?], K, D[?]

Proposition 8 (continued) — demonstration

), and therefore HG will be either equal to IL itself or greater. And since FG has been posited greater than D, while LK is equal to that same D, FG too will be greater than LK. Since, then, HG is not less than IL — as has been demonstrated — but either equal or greater, the whole FH will be greater than IK.

Diagram. A set of labelled horizontal line-segments (magnitudes) illustrating the proof of Proposition 8, arranged in a left group and a right group. The page is foxed across the figure; the readings below are taken from the scan at high magnification.

- F ... G ... H (top line): the multiple FH, divided at G into FG and GH, with tick-marks showing the repetitions.
- K ... L ... I (second line): the multiple IK, marked at L.
- A ... E ... B (right group): the magnitude AB, divided at E [?] (the point-letter is partly lost to foxing).
- C: a single magnitude-segment.
- D: a single magnitude-segment.

Accordingly, since FH and HG are equimultiples of the first, AB, and the third, C — while IK is a multiple of D, which stands in the place of the second and the fourth — and since FH, the multiple of the first, is greater than IK, the multiple of the second, whereas HG, the multiple of the third, is not greater than IK, the multiple of the fourth (indeed it is less, by hypothesis, for IK was taken as the multiple of D that is greater than HG): [Margin: (b) — *Elements* V, def. 8] therefore AB, the first, will have a greater ratio to D, the second, than C, the third, has to D, the fourth.

But conversely, since IK, the multiple of the first D — for let D now be posited as the first and the third, C as the second, and AB as the fourth — is greater than HG, the multiple of the second C, whereas IK, the multiple of the third D, is not greater than FH, the multiple of the fourth AB (indeed it is less, since FH is greater than IK, as has been shown): [Margin: (c) — *Elements* V, def. 8] therefore D, the first, will have a greater ratio to C, the second, than D, the third, has to AB, the fourth — which is what was proposed. Therefore, of unequal magnitudes the greater has [a greater ratio] to the same, and so on. Which was to be shown.

SCHOLIUM

Pedro Nunes [Petrus Nonius — Pedro Nunes], treating of proportion in his *Algebra*, rightly observed that this eighth proposition cannot be used to prove that a right angle has a greater ratio to any other angle than the angle of a semicircle [angulus semicirculi] has to that same angle — even though a right angle is greater than the angle of a semicircle. For the excess of the right angle over the angle of a semicircle, which is the angle of contact [angulus contingentiae], can never, however much it be multiplied, surpass that third angle; yet this is exactly what the demonstration of the present proposition requires. Indeed, as was made plain, FG, the multiple of the excess EB, must surpass the third magnitude D.

Nor should this seem surprising to anyone. For a right angle is not properly said to have a ratio to the angle of a semicircle, since it exceeds it by the angle of contact [angulus contactus], which is not of the same nature as either of the two angles, so that it cannot be so multiplied as at last to surpass either of them. Consider an analogy: if a point could be added to a line one span long, so as to make a line exceeding the span-long line by a single point, the composite line would not properly be called greater than the span-long line — because the excess is indivisible and of another nature. In the same way, if a line one span long were set alongside any plane figure, one would not properly say that this magnitude, composed of diverse magnitudes, has to that plane figure a ratio of greater inequality — because the excess is not of the same nature as that plane figure. So likewise a right angle does not properly have to the angle of a semicircle a ratio of greater inequality; for since it is by the addition of the angle of contact to the angle of a semicircle that the right angle is produced, the right angle therefore exceeds the angle of a semicircle by a quantity of a different nature, as has been said.

Accordingly, it is not only those magnitudes that do not properly have a ratio of which neither, when multiplied, can surpass the other — as we said at definition 5 of this book [*Elements* V, def. 5]; but neither, as this same Pedro Nunes noted, are those properly said to have a ratio whose excess is of a nature different from theirs, such as are a right angle and the angle of a semicircle. So that magnitudes are properly said to have a ratio to one another only when either of them, multiplied, can surpass the other, and when their excess, multiplied, can likewise surpass both. I confess this matter is exceedingly obscure and can scarcely be rightly understood; but it arises from the indivisibility of the angle of contact by a straight line. There are several other things of this kind in geometry which,

though most true because of an evident demonstration, nevertheless present on another side such a difficulty that the understanding does not easily extricate itself from it. For example, it is most true that a sphere touches a plane at a point, as Theodosius [Theodosius] demonstrates in Proposition 3 of Book 1 [*Sphaerica* I.3]. But by what account it comes about that it does not follow from this — if a globe be moved continuously over a plane — that a straight line composed of points is described, no one to this day has so explained that the difficulty is entirely removed.

THEOR. 9. PROPOS. 9. [Margin: ix.]

Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are likewise equal to one another.

Let A and B first have the same ratio to C. I say that A and B are equal to one another. For suppose, if it be possible, that one of them, namely A, is greater and B less. [Margin: (a) — *Elements* V, 8] Then A, the greater, will have a greater ratio to C than B, the less, has to that same C — which is contrary to the hypothesis. Therefore A and B are not unequal, but equal.

Next, let C have the same ratio to A and to B. I say again that A and B are equal. For if one of them, namely A, were greater and B less, [Margin: (b) — *Elements* V, 8] then C would have a greater ratio to B, the less, than to A, the greater — which is contrary to the hypothesis. Therefore A will not be greater than B, but equal. Therefore magnitudes which have the same ratio to the same magnitude, and so on. Which was to be demonstrated.

SCHOLIUM

This ninth proposition converts both parts of theorem 7, as is manifest.

THEOR. 10. PROPOS. 10. [Margin: x.]

Of magnitudes having a ratio to the same magnitude, that which has the greater ratio is the greater; and that to which the same magnitude has the greater ratio is the less.

Let A first have a greater ratio to C than B has to that same C. I say that A is greater than B. For if A were equal to B, [Margin: (a) — *Elements* V, 7] A and B would have the same ratio to C; and if A were less than B, [Margin: (b) — *Elements* V, 8] then B, the greater, would have a greater ratio to C than A, the less, has to

that same C — which is contrary to the hypothesis. Therefore A is not equal to, nor less than,

B, but greater.

Secondly, let C have a greater ratio to B than to A. I say that B is less than A. For B will not be equal to A: [Margin: (c) — *Elements* V, 7] otherwise C would have the same ratio to A and to B, which is contrary to the hypothesis. Nor indeed will B be greater than A: [Margin: (d) — *Elements* V, 8] for otherwise C would have a greater ratio to A, the less, than to B, the greater — which is even more contrary to the hypothesis. Therefore B is less than A, which is what was proposed. Therefore of magnitudes having a ratio to the same magnitude, and so on. Which was to be demonstrated.

SCHOLIUM

This proposition too converts both parts of theorem 8, as is plain.

THEOR. 11. PROPOS. 11. [Margin: xj.]

Ratios that are the same as the same ratio are also the same as one another.

Let the ratios A to B and C to D each be the same as the ratio E to F. I say that the ratios A to B and C to D are also the same as one another — according to the sixth definition [*Elements* V, def. 6]; that is, that when equimultiples of A and C are taken, and likewise equimultiples of B and D, it will always happen that the multiples of A and C either together fall short of, or together are equal to, or together exceed the multiples of B and D.

Diagram. Three groups of labelled horizontal line-segments (magnitudes), one group per proportion. Each group pairs an antecedent and consequent with their equimultiples:

- Group 1: G (equimultiple of A), A, B, K (equimultiple of B).
- Group 2: H (equimultiple of C), C, D, L (equimultiple of D).
- Group 3: I (equimultiple of E), E, F, M (equimultiple of F).

Take, then, any equimultiples whatever, G, H, I, of all the antecedents A, C, E; and any others, K, L, M, equimultiples of all the consequents B, D, F. Since A, the first, is posited to be to B, the second, as E, the third, is to F, the fourth, [Margin: (a) — *Elements* V, def. 6] it comes about that, if G, the multiple of the first, falls short of K, the multiple of the second, then I, the multiple of the third, also falls short of M,

the multiple of the fourth; and if G is equal to K, or greater, then I is likewise equal to M, or greater. [Margin: (b) — *Elements* V, def. 6] But (as will be shown in the same way) if I is less than M, or equal, or greater, then H too is less than L, or equal, or greater — because E, the first, is posited to be to F, the second, as C, the third, is to D, the fourth. Hence if G, the multiple of the first A, falls short of K, the multiple of the second B, then H, the multiple of the third C, will also fall short of L, the multiple of the fourth D; and if G is equal to, or greater than, K, then H too will be equal to, or greater than, L. And the same will be shown to occur in any other equimultiples whatever. [Margin: (c) — *Elements* V, def. 6] Therefore A, the first, will be to B, the second, as C, the third, is to D, the fourth. Therefore ratios that are the same as the same ratio are also the same as one another. Which was to be shown.

SCHOLIVM

"The same as one another." Ratios that are the same as the same ratios are also the same as one another. Or if it be that as A is to B, so is C to D, and moreover E is to F as A is to B, and G is to H as C is to D: then E will also be to F as G is to H. For since the ratios E to F and C to D are the same as the ratio A to B, [Margin: (d) — *Elements* V, 11] E to F will be as C to D. And again, since the ratios E to F and G to H are the same as the ratio C to D, [Margin: (e) — *Elements* V, 11] E to F will likewise be as G to H.

A	B	C	D	E	F	G	H
3	2	6	4	9	6	12	8

Again, if the ratios A to B, C to D, and E to F are the same as one another, and moreover as A is to B so is G to H, and as C is to D so is I to K, and as E is to F so is L to M: then the ratios G to H, I to K, and L to M will also be the same as one another. For, as was shown just now in the case of four ratios, G is to H as I is to K, because these ratios are similar to the ratios A to B and C to D.

A	B	C	D	E	F	G	H	I	K	L	M
3	2	6	4	9	6	12	8	18	12	30	20

In the same way, I will be to K as L is to M, since the ratios I to K and L to M are the same as the ratios C to D and E to F, which are equal. Likewise G will be to H as L is to M, since these ratios are the same as the ratios A to B and E to F, which are posited equal. The same reasoning holds for more terms.

THEOR. 12. PROPOS. 12. [Margin: xij.]

If there be any number of proportional magnitudes, then as one of the antecedents is to one of the consequents, so will all the antecedents together be to all the consequents together.

What he demonstrated in Proposition 1, concerning multiple proportion, he here shows for every kind of proportion, the irrational included. Let there be, then, any number of magnitudes A, B, C, D, E, F, proportional — that is, let A be to B as C is to D and as E is to F. I say that as one of the antecedents is to one of the consequents, namely A to B, so are all the antecedents together, A, C, E, to all the consequents together, B, D, F.

Diagram. Labelled horizontal line-segments (magnitudes) arranged in four rows:

- Top row: G, H, I (the equimultiples of the antecedents).
- Second row: A, C, E (the antecedents).
- Third row: B, D, F (the consequents).
- Bottom row: K, L, M (the equimultiples of the consequents).

For when G, H, I are taken as equimultiples of the antecedents, and K, L, M as equimultiples of the consequents, [Margin: (a) — *Elements* V, 1] then all of G, H, I together will be multiples of all of A, C, E together in the same way as one of them is a multiple of one, namely as G is of A; and all of K, L, M together will be multiples of all of B, D, F together in the same way as one of them is a multiple of one, namely as K is of B. Now since A, the first, is posited to be to B, the second, as C, the third, is to D, the fourth, and as the further magnitude E, the third, is to the further F, the fourth, [Margin: (b) — *Elements* V, def. 6] it comes about that, if G, the multiple of the first, falls short of K, the multiple of the second, then H, the multiple of the third, also falls short of L, the multiple of the fourth, and I falls short of M; and if G is equal to K, or greater, then H too is equal to L, and I to M, or greater. And accordingly, whether G is less than, or equal to, or greater than K, all of G, H, I together are together less than, or equal to, or greater than all of K, L, M. [Margin: (c) — *Elements* V, def. 6] Therefore, as A, the first, is to B, the second, so will A, C, E, the third, be to B, D, F, the fourth. Therefore, if there be any number of proportional magnitudes, and so on. Which was to be demonstrated.

THEOR. 13. PROPOS. 13. [Margin: xij.]

If a first magnitude has to a second the same ratio that a third has to a fourth, and the third has to the fourth a greater ratio than a fifth has to a

sixth: then the first will also have to the second a greater ratio than the fifth has to the sixth.

Diagram. Labelled horizontal line-segments (magnitudes) arranged in four rows:

- Top row: G, H, I (the equimultiples of the antecedents).
- Second row: A, C, E (the antecedents).
- Third row: B, D, F (the consequents).
- Bottom row: K, L, M (the equimultiples of the consequents).

Let A, the first, be to B, the second, as C, the third, is to D, the fourth; and let the ratio of C, the third, to D, the fourth, be greater than that of E, the fifth, to F, the sixth. I say that the ratio of A, the first, to B, the second, is also greater than that of E, the fifth, to F, the sixth — according to the eighth definition [*Elements* V, def. 8]; that is, that when equimultiples of A and E are taken, and likewise equimultiples of B and F, it can happen that the multiple of A exceeds the multiple of B, while the multiple of E does not exceed the multiple of F.

For when G, H, I are taken as equimultiples of the antecedents, and K, L, M as equimultiples of the consequents, then, since A, the first, is to B, the second, as C, the third, is to D, the fourth, [Margin: (a) — *Elements* V, def. 6] it comes about that, if G, the multiple of the first, exceeds K, the multiple of the second, then H, the multiple of the third, also exceeds L, the multiple of the fourth, and so on. But when H exceeds L, [Margin: (b) — *Elements* V, def. 8] I does not necessarily exceed M; rather it will sometimes be equal, or less — because the ratio of C, the first, to D, the second, is posited greater than that of E, the third, to F, the fourth. Therefore if G exceeds K, I does not necessarily exceed M. [Margin: (c) — *Elements* V, def. 8] Therefore the ratio of A, the first, to B, the second, is greater than that of E, the third, to F, the fourth. Hence, if a first has to a second the same ratio that a third has to a fourth, and so on. Which was to be shown.

SCHOLIUM

But if the ratio of C, the third, to D, the fourth, be less than that of E, the fifth, to F, the sixth, then the ratio of A, the first, to B, the second, will also be less than that of E, the fifth, to F, the sixth. For if the ratio of C to D is less than that of E to F — that is, if the ratio of E, the first, to F, the second, is greater than that of C, the third, to D, the fourth — [Margin: (d) — *Elements* V, def. 8] it comes about that, if I exceeds M, H does not necessarily exceed L, but sometimes falls short of it, or is equal to it. [Margin: (e) — *Elements* V, def. 6] But if H falls short of L, or is equal, then G too will fall short of K, or be equal to it, because C, the first, is posited to be

to D, the second, as A, the third, is to B, the fourth. Hence if I exceeds M, G will not necessarily exceed K; [Margin: (f) — *Elements* V, def. 8] and therefore the ratio of E, the first, to F, the second, will be greater than that of A, the third, to B, the fourth — that is, the ratio of A to B will be less than that of E to F. Which is what was proposed.

In the same way: if a first has to a second a greater ratio than a third has to a fourth, and the third has to the fourth a greater ratio than a fifth has to a sixth, then the first will have to the second a still greater ratio than the fifth has to the sixth.

And if a first has to a second a smaller ratio than a third has to a fourth, and the third has to the fourth a smaller ratio than a fifth has to a sixth, then the first will have to the second a still smaller ratio than the fifth has to the sixth.

THEOR. 14. PROPOS. 14. [Margin: xiv.]

If a first magnitude has to a second the same ratio that a third has to a fourth, and the first is greater than the third, then the second will also be greater than the fourth. But if the first is equal to the third, the second will also be equal to the fourth; and if less, less.

Let A, the first, be to B, the second, as C, the third, is to D, the fourth. I say that if A is greater than C, then B will also be greater than D; that if A is equal to C, then B is also equal to D; and finally that if A is less than C, then B is also less than D.

Diagram. Four labelled horizontal line-segments (magnitudes), for the case A greater than C: A, B, C, D.

Let A first be greater than C. [Margin: (a) — *Elements* V, 8] Then, on that account, the ratio of A, the greater, to B will be greater than that of C, the less, to that same B. Now since C, the first, is to D, the second, as A, the third, is to B, the fourth — and this is greater, as we showed, than the ratio of C, the fifth, to B, the sixth — [Margin: (b) — *Elements* V, 13] the ratio of C, the first, to D, the second, is greater than that of C, the fifth, to B, the sixth. [Margin: (c) — *Elements* V, 10] Therefore D is less than B; and so B will be greater than D. Which is what was proposed.

Diagram. Four labelled horizontal line-segments (magnitudes), for the case A equal to C: A, B, C, D.

Next, let A be equal to C. [Margin: (d) — *Elements* V, 7] Then, on that account, A will be to B as C is to B. Now since the ratios C to D and C to B are the same as the ratio A to B, [Margin: (e) — *Elements* V, 11] the ratios C to D and C to B will also be

the same as one another; [Margin: (f) — *Elements* V, 9] and therefore B and D will be equal. Which is what was proposed.

Diagram. Four labelled horizontal line-segments (magnitudes), for the case A less than C: A, B, C, D.

Thirdly, let A be less than C. [Margin: (g) — *Elements* V, 8] Then, on this account, the ratio of C, the greater, to B is greater than that of A, the less, to that same B. Now since C, the first, is to D, the second, as A, the third, is to B, the fourth — and the ratio of A, the third, to B, the fourth, is less than that of C, the fifth, to B, the sixth — [Margin: (h) — *Elements* V, 13] the ratio of C, the first, to D, the second, will also be less than that of C, the fifth, to B, the sixth; [Margin: (i) — *Elements* V, 10] and therefore B will be less than D, which is what was proposed. Therefore, if a first has to a second the same ratio, and so on. Which was to be demonstrated.

SCHOLIVM

But if the second be greater than, or equal to, or less than the fourth, then by the same reasoning the first will be greater than, or equal to, or less than the third. For let B first be greater than D, as in the first figure. I say that A too is greater than C. For since B is greater than D, [Margin: (k) — *Elements* V, 8] the ratio of C to D will be greater than that of C to B [...]. Since, then, as the first A is to the second B, so the third C is to the

(Continuation of the SCHOLIVM to Proposition 14.)

fourth, D. But the ratio of C, the third, to D, the fourth, has been shown to be greater than the ratio of C, the fifth, to B, the sixth [Margin: (l) — *Elements* V, 13]; and therefore the ratio of A, the first, to B, the second, will also be greater than the ratio of C, the fifth, to B, the sixth [Margin: (m) — *Elements* V, 10]. Consequently A will be greater than C, which is what was proposed.

NEXT, let B be equal to D, as in the second figure. I say that A too is equal to C. For since B is equal to D, C will be to B as C is to D [Margin: (n) — *Elements* V, 11]; but A is also to B as C is to D. Therefore A is to B as C is to B as well [Margin: (o) — *Elements* V, 9]; and consequently A is equal to C. Which is what was proposed.

THIRD, let B be less than D, as in the third figure. I say that A too is less than C. For since B is less than D, the ratio of C to D will be less than the ratio of C to B [Margin: (p) — *Elements* V, 8]. Now since A, the first, is to B, the second, as C, the third, is to D, the fourth — and the ratio of C, the third, to D, the fourth, has been shown to be less than the ratio of C, the fifth, to B, the sixth [Margin: (q) — *Elements* V, 13] — the ratio of A, the first, to B, the second, will also be less than

the ratio of C, the fifth, to B, the sixth [Margin: (r) — *Elements* V, 10]. Therefore C will be greater than A, and consequently A will be less than C. Which is what was proposed.

MORE BRIEFLY: If B is greater than D, and A is said to be not greater than C but either less than or equal to it, then B too will be less than or equal to D [Margin: (s) — *Elements* V, 14], which is contrary to the hypothesis. And the same reasoning holds in the other cases.

OR THUS: If A is to B as C is to D, then inversely [convertendo] B will be to A as D is to C. Therefore if B is greater than, less than, or equal to D, A too will be greater than, less than, or equal to C [Margin: (t) — *Elements* V, 14].

Euclid, however, did not demonstrate that if the first is greater than, equal to, or less than the second, the third is likewise greater than, equal to, or less than the fourth — though most geometers, both ancient and modern, do argue in this way — because it is evident from the similarity of the ratios. For it follows that if both ratios are of greater inequality, each antecedent magnitude (that is, the first and the third) is greater than each consequent (that is, the second and the fourth); if both ratios are of equality, each antecedent magnitude is equal to each consequent; and finally, if both ratios are of less inequality, each antecedent magnitude is less than each consequent.

For example, if A is to B as C is to D, then each ratio will be of greater inequality, or of equality, or of less inequality. Hence if A, the first, is greater than B, the second, then C, the third, will also be greater than D, the fourth; if equal, equal; and if less, less. Which is what was proposed. This, however, we shall demonstrate geometrically, along with Federico Commandino [Federico Commandino], at Proposition 16 of this book — though it is not necessary.

Diagram. The figure for the scholium, at the right margin. Four magnitudes are drawn as line-segments and paired by brackets: A over C on the left, B over D on the right, representing the four magnitudes A, B, C, D whose ratios $A : B$ and $C : D$ are compared. Visible labels: A, B, C, D.

- A (upper left): first magnitude
- B (upper right): second magnitude
- C (lower left): third magnitude
- D (lower right): fourth magnitude

[Margin: xv.]

THEOR. 15. PROPOS. 15.

Parts have the same ratio as their equimultiples, provided they are taken as they correspond to one another.

Let CD and EF be equimultiples of the parts A and B. I say that CD is to EF as A is to B. For since CD and EF are equimultiples of A and B, A will be contained in CD as many times as B is contained in EF. Divide CD, then, into the parts CG, GH, HD equal to A, and EF into the parts EI, IK, KF equal to B; then CG will be to EI as A is to B [Margin: (a) — *Elements* V, 7], since CG and A are equal to one another, as are EI and B. By the same reasoning GH will be to IK, and HD to KF, as A is to B; and therefore CG, GH, HD will have the same ratio to EI, IK, KF [Margin: (b) — *Elements* V, 11]. Hence, as CG is to EI — that is, as A is to B — so CD will be to EF [Margin: (c) — *Elements* V, 12], namely all of CG, GH, HD together to all of EI, IK, KF together, which is what was proposed. Parts, therefore, have the same ratio as their equimultiples, and so on. Which was to be demonstrated.

Diagram. Two vertical magnitude-lines at the right margin. The taller line is CD, running down from the top endpoint D and marked off by the intermediate points into the segments CG, GH, HD; clearly legible labels on it are D, H, G (the foot-label C is faint). Beside it stands the shorter line EF, divided into the parts EI, IK, KF; the labels E, I, K, F are only partly resolvable in the scan.

- D (top of taller line): endpoint of CD
- H, G (division points on CD): marking the parts CG, GH, HD
- E, I, K, F (on the shorter line): endpoints and division points of EF, marking EI, IK, KF [partly legible]

[Margin: xvj.]

THEOR. 16. PROPOS. 16.

If four magnitudes are proportional, they will also be proportional alternately.

Here Euclid demonstrates the alternate, or interchanged, proportion — the ratio explained in Definition 12. Let A be to B as C is to D. I say that alternately [vicissim], or by permutation [permutando], A is also to C as B is to D. For let E, F be taken as equimultiples of A, B, the first and second; and likewise G, H as equimultiples of C, D, the third and fourth. Then E will be to F as A is to B [Margin: (a) — *Elements* V, 15], since E and F are equimultiples of the parts A and

B. By the same reasoning G will be to H as C is to D. Now since the ratios $E : F$ and $G : H$ are the same as the ratio $A : B$, they will also be the same as one another [Margin: (b) — *Elements* V, 11]. Again, since the ratios $E : F$ and $G : H$ are the same as the ratio $C : D$, they too will be the same as one another [Margin: (c) — *Elements* V, 11]; that is, as E, the first, is to F, the second, so G, the third, will be to H, the fourth. Hence if E, the first, is greater than G, the third, or equal to it, or less than it, F, the second, will also be greater than H, the fourth, or equal, or less [Margin: (d) — *Elements* V, 14], in whatever multiplication the equimultiples E, F and the equimultiples G, H have been taken. Therefore A, the first, is to C, the second, as B, the third, is to D, the fourth [Margin: (e) — *Elements* V, def. 6] (since E and F are equimultiples of the first A and the third B, while G and H are equimultiples of the second C and the fourth D, and the former fall short of, are equal to, or exceed the latter all together, and so on) — which is what was proposed. Therefore if four magnitudes are proportional, they will also be proportional alternately. Which was to be shown.

Diagram. The figure for Proposition 16, at the lower right. It shows the four magnitudes A, B, C, D together with their equimultiples E, F, G, H, drawn partly as bracketed vertical segments (A and C above, B and E beside them, with I, K, F below) and partly as a stack of short horizontal magnitude-lines whose left-hand labels read, top to bottom, E, A, B, F, G, C, D, H. The thin lines and small labels do not all resolve individually in this scan.

- Vertical group: A, C (upper), B, E (lower), and I, K, F below
- Horizontal magnitude-lines (labels, top to bottom): E, A, B, F, G, C, D, H

[— several of the small segment-labels in this figure are not individually resolvable]

(Continuation of the SCHOLIVM to Proposition 16.)

Moreover, the demonstration of this proposition holds only when the four magnitudes are of the same kind. For if two, A and B, were of one kind and two, C and D, of another, then the multiples E, F would also be of one kind (namely that in which A, B are), and the multiples G, H of another (namely that in which C, D are). Hence it could not be said that E is greater than, equal to, or less than G; and consequently nothing could be inferred from Definition 6 of this book. Therefore alternate proportion is to be employed only among four magnitudes of the same kind. Some philosophers, failing to notice this, fall into grave errors when they apply it to things of different kinds.

From this that too will be demonstrated which, at the end of the scholium to Proposition 14, we showed from the mere similarity of the ratios, and which we undertook not to demonstrate at that place — namely:

If a first has to a second the same ratio as a third to a fourth, and the first is greater than the second, then the third will also be greater than the fourth; if equal, equal; and if less, less.

Although what is here proposed is self-evident, as we said at Proposition 14, we shall nonetheless demonstrate it, along with Federico Commandino, in this way. Let A, the first, be to B, the second, as C, the third, is to D, the fourth. I say that if A, the first, is greater than B, the second, then C, the third, is greater than the fourth D; if equal, equal; and if less, less. For alternately [permutando] A will be to C as B is to D [Margin: (f) — *Elements* V, 16]. Therefore if A, the first, is greater than B, the third, then C, the second, will also be greater than D, the fourth [Margin: (g) — *Elements* V, 14]; and if equal, equal; and if less, less. Which is what was proposed.

But this demonstration too holds only when the four magnitudes are of the same kind. Hence it is preferable to prove the point from the nature of ratios, as we did at Proposition 14. For in that way what is proposed will always be true, even if the magnitudes A, B are contained in one kind and C, D in another — indeed, even if A, B are continuous quantities and C, D are numbers, and so on.

[Margin: xvij.]

THEOR. 17. PROPOS. 17.

If magnitudes taken jointly are proportional, they will also be proportional when taken separately.

Here Euclid demonstrates the separation of a ratio [divisio rationis], which he set out in Definition 15. Let the composite magnitudes AB, CB and DE, FE be proportional — that is, let AB be to CB as DE is to FE. I say that, when separated, they too are proportional; that is, that AC is to CB as DF is to FE, in the sense we explained in Definition 6. For let equimultiples GH, HI, KL, LM of AC, CB, DF, FE be taken in the same order; then GI will be the same multiple of AB as GH is of AC [Margin: (a) — *Elements* V, 1] — that is, as KL is of DF — and likewise KM will be the same multiple of DE [Margin: (b) — *Elements* V, 1]. Therefore GI, KM are equimultiples of AB, DE. Again, let IN, MO be taken as equimultiples of CB, FE. Now since HI, the multiple of the second CB, is the same multiple as LM, [the multiple] of the fourth FE, and likewise IN, the fifth, is as great a multiple of the

second CB as MO, the sixth, is of the fourth FE, then HN will be the same multiple of the second CB as LO is of the fourth FE [Margin: (c) — *Elements* V, 2]. And so, since AB, the first, is to CB, the second, as DE, the third, is to FE, the fourth, and equimultiples GI, KM have been taken of the first and third, and equimultiples HN, LO of the second and fourth, it follows [Margin: (d) — *Elements* V, def. 6] that if GI, the multiple of the first AB, falls short of HN, the multiple of the second CB, then KM, the multiple of the third DE, likewise falls short of LO, the multiple of the fourth FE; if equal, equal; and if it exceeds, it exceeds. But if both GI falls short of HN and KM of LO, then, the common parts being taken away, GH will also fall short of IN, and KL of MO. And if GI is equal to HN and KM to LO, then, the common parts HI, LM being taken away, GH will be equal to IN and KL to MO. And finally, if GI exceeds HN and KM exceeds LO, then, the common parts HI, LM being taken away, GH will also exceed IN and KL will exceed MO. For this reason, since GH, KL have been taken as equimultiples of the first and third AC, DF, and IN, MO as equimultiples of the second and fourth CB, FE, and it has been shown that (in whatever multiplication those equimultiples were taken) the equimultiples of the first and third alike fall short of, are equal to, or exceed the equimultiples of the second and fourth by one and the same amount — AC, the first, will be to CB, the second, as DF, the third, is to FE, the fourth [Margin: (e) — *Elements* V, def. 6]. Therefore if composite magnitudes are proportional, and so on. Which was to be demonstrated.

Diagram. The figure for the demonstration of Proposition 17, at the left of the column. It consists of several parallel vertical magnitude-lines, each divided by tick-marks into the segments named in the proof (the magnitudes AC, CB, DF, FE and their equimultiples GH, HI, KL, LM, IN, MO, and the composite multiples GI, KM, HN, LO). Legible point-labels, reading down the several lines, include N, O, I, M, H, L, B, C, F, G, A, D, K (the letter E is also present); the thin rules and small italic labels do not all resolve cleanly.

- Vertical lines carrying (across the several columns) the labels N, O, I, M, H, L, B, C, F, G, A, D, K, E

[— more than two labels in this dense line-figure are not individually resolvable]

SCHOLIUM.

From these things we shall easily demonstrate that mode of arguing which, at Definition 15, we called the converse separation of a ratio [divisio rationis conversa]: namely, that if AB is to CB as DE is to FE, then also CB is to AC as FE is to DF. This is evident as follows. Since AB is to CB as DE is to FE, by separation

[dividendo] AC will be to CB as DF is to FE [Margin: (f) — *Elements* V, 17]. Therefore by conversion [convertendo] CB will also be to AC as FE is to DF. Which is what was proposed.

With no trouble at all, that mode of arguing will also be demonstrated which we called the contrary [mode], and in which the antecedent magnitude is less than the consequent — not greater, as in the separation of a ratio which Euclid defined — like the one we have just demonstrated. Let AC be to CB as DF is to FE. Since AB is to AC as DE is to DF, then by separation [dividendo] CB will be to AC as FE is to DF [Margin: (g) — *Elements* V, 17]; and consequently, by conversion, AC will be to CB as DF is to FE. Which is what was proposed.

Diagram. A numeric line-figure for the scholium. Two horizontal lines, each divided by a point into two segments bearing numerals. The upper line runs A—C—B, with the segment AC marked **12** and the segment CB marked **4**. The lower line runs D—F—E, with the segment DF marked **6** and the segment FE marked **2**.

- Upper line: A ... [12] ... C ... [4] ... B
- Lower line: D ... [6] ... F ... [2] ... E

THEOR. 18. PROPOS. 18.

If separated magnitudes are proportional, they will also be proportional when taken jointly.

Here Euclid demonstrates the composition of a ratio [compositio rationis], which he described in Definition 14. For let the separated magnitudes AB, BC and DE, EF be proportional — that is, AB is to BC as DE is to EF. I say that, taken jointly, they too are proportional; that is, that AC is to BC as DF is to EF. For if AC is not to BC as DF is to EF, then DF will have to some magnitude...

(Continuation of the demonstration of Proposition 18.)

smaller than EF, or to some larger one, the same ratio which AC has to BC. First let DF have to GF — smaller than EF, if that be possible — the same ratio which AC has to BC. Since, then, AC is to BC as DF is to GF, by separation [dividendo] AB will also be to BC as DG is to GF [Margin: (a) — *Elements* V, 17]. But AB is to BC as DE, which was assumed, is to EF. Therefore DG, the first, will also be to GF, the second, as DE, the third, is to EF, the fourth [Margin: (b) — *Elements* V, 11]. Since, then, DG, the first, is greater than DE, the third, GF, the second, will also be greater than EF, the fourth [Margin: (c) — *Elements* V, 14] — the part greater than the whole. Which is absurd.

Next let DF have to HF — greater than EF, if that be possible — the same ratio which AC has to BC. Since, then, AC is to BC as DF is to HF, by separation [dividendo] AB will also be to BC as DH is to HF [Margin: (d) — *Elements* V, 17]. But AB is to BC as DE, which was likewise assumed, is to EF. Therefore DH, the first, will also be to HF, the second, as DE, the third, is to EF, the fourth [Margin: (e) — *Elements* V, 11]. Since, then, DH, the first, is less than DE, the third, HF, the second, will also be less than EF, the fourth — the whole less than the part. Which is absurd. Therefore DF will have, to a magnitude neither smaller than EF nor greater, the same ratio which AC has to BC. Hence DF will be to EF itself as AC is to BC, which is what was proposed. Therefore if separated magnitudes are proportional, and so on. Which was to be demonstrated.

SCHOLIUM.

Some, along with Campanus [Campanus], demonstrate this proposition by almost the same reasoning by which Euclid demonstrated the preceding one — in this way, namely. Let the separated magnitudes AC, CB, DF, FE be proportional — that is, AC to CB as DF to FE. I say that, taken jointly, they too are proportional, so that AB is to CB as DE is to FE. For let there be taken, as in the preceding proposition, equimultiples GH, HI, KL, LM of AC, CB, DF, FE; and likewise other equimultiples IN, MO of CB, FE. Then again GI, KM will be equimultiples of AB, DE [Margin: (f) — *Elements* V, 1], and HN, LO equimultiples of CB, FE [Margin: (g) — *Elements* V, 2], as was shown in the preceding theorem. Now since AC, the first, is assumed to be to CB, the second, as DF, the third, is to FE, the fourth, and GH, KL have been taken as equimultiples of the first and third AC, DF, and IN, MO as equimultiples of the second and fourth CB, FE, it follows [Margin: (h) — *Elements* V, def. 6] that if GH, the multiple of the first AC, falls short of IN, the multiple of the second CB, then KL, the multiple of the third DF, likewise falls short of MO, the multiple of the fourth FE; if equal, equal; and if it exceeds, it exceeds. But if both GH falls short of IN and KL of MO, then, adding the common parts HI, LM, GI will also fall short of HN, and KM of LO. And if GH is equal to IN and KL to MO, then, adding the common parts HI, LM, GI will also be equal to HN, and KM to LO. And finally, if GH exceeds IN and KL exceeds MO, then, adding the common parts HI, LM, GI will also exceed HN, and KM will exceed LO. Since, therefore, if GI, the multiple of the first AB, falls short of HN, the multiple of the second CB, then KM, the multiple of the third DE, also falls short of LO, the multiple of the fourth FE; and if equal, equal; and if it exceeds, it exceeds — AB, the first, will be to CB, the second, as DE, the third, is to FE, the fourth [Margin: (i) — *Elements* V, def. 6]. Which is what was proposed. Therefore if separated magnitudes are proportional, and so on. Which was to be shown.

But this demonstration does not correctly infer what is proposed from Definition 6, because HN, LO are not multiples of CB, FE in the same way that the multiples IN, MO of those same CB, FE were taken. Hence one might justly doubt whether, under any multiplication whatever, the equimultiples preserve that condition of falling short of, being equal to, and exceeding, which Euclid required in Definition 6 — since in this demonstration one is not free to assume any equimultiples whatever of CB, FE, but only such as arise from the multiples HI, IN and LM, MO already taken. This objection cannot be raised against the demonstration of the preceding proposition, since there IN, MO were taken as any equimultiples whatever of CB, FE, and they remain the same equimultiples of those same CB, FE in the separated proportionality. For this reason Euclid's demonstration is to be preferred to this one of Campanus. Yet I have chosen to explain it too, lest the studious reader, abandoning Euclid's, should seize upon it as a good one — especially since it is direct, whereas Euclid's leads us to what cannot be [i.e., proceeds by reduction to the impossible].

Diagram. The figure for the Campanus demonstration in the scholium, at the left of the column. As in the Proposition 17 figure, it is a set of parallel vertical magnitude-lines divided by tick-marks (the magnitudes AC, CB, DF, FE and their equimultiples GH, HI, KL, LM, IN, MO, with the composite multiples GI, KM, HN, LO). Legible point-labels down the several lines include N, O, I, M, H, L, B, C, E, F, G, A, D, K; the thin rules and small italic labels do not all resolve cleanly.

- Vertical lines carrying (across the several columns) the labels N, O, I, M, H, L, B, C, E, F, G, A, D, K

[— more than two labels in this dense line-figure are not individually resolvable]

From these things we shall also easily confirm the two modes of arguing which we described at Definition 14. The former we called the converse composition of a ratio. Let AB be to BC as DE is to EF. I say that, by the converse composition of a ratio, AC is also to AB as DF is to DE. For since AB is to BC as DE is to EF, by conversion [convertendo] BC will be to AB as EF is to DE. Therefore, by composition [componendo], AC will be to AB as DF is to DE [Margin: (k) — *Elements* V, 18]. Which is what was proposed.

The latter mode we called the contrary composition of a ratio. Let it again be that AB is to BC as DE is to EF. I say that, by the contrary composition of a ratio, AB is also to AC as DE is to DF. For since AB is to BC as DE is to EF, by conversion BC will be to AB as EF is to DE. Therefore, by composition, AC will be to AB as DF is to DE

[Margin: (l) — *Elements* V, 18]; and consequently, by conversion once more, AB will be to AC as DE is to DF. Which is what was proposed.

Diagram. Two numeric line-figures for the scholium (one for each of the two argument-modes; both carry the same lettering and numerals). Each has an upper line A—B—C, with the segment AB marked **12** and the segment BC marked **8**, and a lower line D—E—F, with the segment DE marked **6** and the segment EF marked **4**.

- Upper line: A ... [12] ... B ... [8] ... C
- Lower line: D ... [6] ... E ... [4] ... F

THEOR. 19. PROPOS. 19.

If, as a whole is to a whole, so a part taken away is to a part taken away, then the remainder will also be to the remainder as the whole is to the whole.

What was demonstrated at Proposition 5 for multiple proportion is here demonstrated for every proportion, even an irrational one. Let the whole AB be to the whole CD as the part taken away AE is to the part taken away CF. I say that the remainder EB is also to the remainder FD as the whole AB is to the whole CD. For since AB is to CD as AE is to CF, then alternately [permutando] AB will also be to AE as CD is to CF [Margin: (a) — *Elements* V, 16]. Therefore, by separation [dividendo], EB will be to AE as FD is to CF [Margin: (b) — *Elements* V, 17]. Hence, alternately once more, EB will be to FD as AE is to CF [Margin: (c) — *Elements* V, 16] — that is, as the whole AB is to the whole CD, since AB was assumed to be to CD as AE is to CF. Therefore if, as a whole is to a whole, and so on. Which was to be demonstrated.

Diagram. The figure for Proposition 19. Two horizontal lines, each divided by an interior point. The upper line runs A—E—B (whole AB, part taken away AE, remainder EB). The lower line runs C—F—D (whole CD, part taken away CF, remainder FD).

- Upper line: A (start) — E (division point) — B (end)
- Lower line: C (start) — F (division point) — D (end)

COROLLARIVM [Corollary — to Proposition 19]

From this, that method of arguing in proportions which is drawn from the conversion of ratio [*convertendo*] can easily be demonstrated, in accordance with definition 16.

Let AB be to CB as DE is to FE. I say that, by conversion of ratio, it is likewise the case that AB is to AC as DE is to DF. For since AB is to CB as DE is to FE, then by separation [*dividendo*] AC will also be to CB as DF is to FE [Margin: (d) — V.17]. Therefore also by conversion, CB is to AC as FE is to DF; and consequently by composition [*componendo*] as well, AB is to AC as DE is to DF [Margin: (e) — V.18]. Which was to be demonstrated.

Diagram. Two horizontal straight lines representing the magnitudes of the corollary, drawn one above the other and subdivided by tick-marks.

- Upper line: points **A, C, B** in order (A at the left end, B at the right). The number **4** is printed over the segment CB.
- Lower line: points **D, F, E** in order (D at the left end, E at the right). The number **12** is printed over the segment DF, and **8** over the segment FE.

SCHOLIVM [Scholium]

All the interpreters of Euclid demonstrate the conversion of ratio in this way. Since AB is to CB as DE is to FE, then by alternation the whole AB will be to the whole DE as CB, the part taken away, is to FE, the part taken away [Margin: (f) — V.16]. Therefore, as the whole AB is to the whole DE, so too the remainder AC will be to the remainder DF [Margin: (g) — V.19]; and consequently, by alternation once more, AB is to AC as DE is to DF. Which was to be demonstrated.

But who does not see that this demonstration holds only for magnitudes of the same kind? For it employs the alternate or permuted proportion, which has force only among magnitudes of the same kind — as we noted both in definition 12 and in proposition 19. Therefore, since Euclid and the other geometers apply this method of arguing from the conversion of ratio to all magnitudes, even those not of the same kind, we have rejected this common demonstration of the interpreters and devised another of our own that fits all magnitudes. For it holds even if the first two quantities, AB and CB, are of one kind — namely lines — while the latter two, DE and FE, are of another kind — namely surfaces, or angles, or solids, or finally numbers — because in it the alternate or permuted proportion was not assumed.

[Margin: xx.]

THEOR. 20. PROPOS. 20.

If there are three magnitudes, and others equal to them in number, taken two by two and in the same ratio, then, *ex aequali*, if the first is greater than the third, the fourth will also be greater than the sixth. But if the first is equal to the third, the fourth will also be equal to the sixth; and if the first is less, the fourth too will be less.

Let there be three magnitudes A, B, C, and as many others D, E, F, and let A be to B as D is to E, and B to C as E is to F. Suppose first that A, the first, is greater than C, the third. I say that D, the fourth, is also greater than F, the sixth. For since A is greater than C, the ratio of A to B is greater than that of C to B [Margin: (a) — V.8]. But A is to B as D is to E; therefore the ratio of D to E is likewise greater than that of C to B [Margin: (b) — V.13]. Now since B is to C as D is to E, by conversion C is to B as F is to E. The ratio of D to E is therefore greater than that of F to E as well; and hence D is greater than F [Margin: (c) — V.10]. Which was to be demonstrated.

Next, let A be equal to C. I say that D is also equal to F. For since A is equal to C, A will be to B as C is to B [Margin: (d) — V.7]. But A is to B as D is to E; therefore D is to E as C is to B [Margin: (e) — V.11]. Now C is to B as F is to E, by inverse ratio, as before. Therefore D is to E as F is to E; and hence D and F are equal [Margin: (f) — V.9]. Which was to be demonstrated.

Third, let A be less than C. I say that D is also less than F. For since A is less than C, the ratio of A to B is less than that of C to B [Margin: (g) — V.8]. But A is to B as D is to E; therefore the ratio of D to E is likewise less than that of C to B [Margin: (h) — V.13]. Now by conversion, as before, C is to B as F is to E. Therefore the ratio of D to E is also less than that of F to E; and for that reason D is less than F [Margin: (i) — V.10]. Which was to be demonstrated. And so, if there are three magnitudes and others equal to them in number, and the rest as stated.

Diagram. In the left column, three stacked groups of six parallel horizontal line-segments (magnitudes), illustrating the three cases of the demonstration (A greater than, equal to, and less than C). Each group is labelled, from top to bottom, A, B, C, D, E, F; the segments are of differing lengths and are subdivided by tick-marks. The marginal citation letters a–i beside them belong to the demonstration text, not to the figure.

SCHOLIUM [Scholium]

Moreover, in proposition 22 Euclid will show that the magnitudes A and D are not only greater than, equal to, or less than the two magnitudes C and E — as he has demonstrated here — but also that they have the same ratio to them *ex aequali*; and this he could not demonstrate unless he had first established the present theorem, as will be clear from that proposition 22. [?]

[Margin: xxj.]

THEOR. 21. PROPOS. 21.

If there are three magnitudes, and others equal to them in number, taken two by two and in the same ratio, but their proportion is perturbed, then, *ex aequali*, if the first is greater than the third, the fourth will also be greater than the sixth. But if the first is equal to the third, the fourth will also be equal to the sixth; and if the first is less, this too will be less.

Let there be three magnitudes A, B, C, and as many others D, E, F, taken two by two and in the same ratio, and let their proportion be perturbed — that is, let A be to B as E is to F, and B to C as D is to E. Suppose first that A, the first, is greater than C, the third. I say that D, the fourth, is also greater than F, the sixth. For since A is greater than C, the ratio of A to B is greater than that of C to B [Margin: (a) — V.8]. But A is to B as E is to F; therefore the ratio of E to F is greater than that of C to B [Margin: (b) — V.13]. Now since B is to C as D is to E, by conversion C is to B as E is to D. The ratio of E to F is therefore greater than that of E to D as well; and hence D is greater than F [Margin: (c) — V.10]. Which was to be demonstrated.

Next, let A be equal to C. I say that D is likewise equal to F. For since A is equal to C, A will be to B as C is to B [Margin: (d) — V.7]. But A is to B as E is to F; therefore C is to B as E is to F [Margin: (e) — V.11]. Now by inverse ratio C is to B as E is to D, as before. Therefore E is to F as E is to D; and for that reason D is equal to F [Margin: (f) — V.9]. Which was to be demonstrated.

Third, let A be less than C. I say that D is also less than F. For since A is less than C, the ratio of A to B is less than that of C to B [Margin: (g) — V.8]. But A is to B as E is to F; therefore the ratio of E to F is less than that of C to B [Margin: (h) — V.13]. Now, as before, by inverse ratio C is to B as E is to D; therefore the ratio of E to F is also less than that of E to D; and hence D is less than F. Which was to be demonstrated [Margin: (i) — V.10]. And so, if there are three magnitudes and others equal to them in number, and the rest as stated. Which was to be shown.

Diagram. In the right-hand column, stacked groups of six parallel vertical line-segments (magnitudes) for the three cases of this demonstration. Each group is labelled **A, B, C, D, E, F**; the segments are of differing lengths and subdivided by tick-marks.

SCHOLIVM [Scholium]

Moreover, in proposition 23 Euclid will show that the two magnitudes A and D are not only greater than, equal to, or less than the two magnitudes C and F, but also that they have the same ratio to them *ex aequali*; and this he could not demonstrate without the aid of the present theorem, as will be evident from that proposition 23.

[Margin: xxij.]

THEOR. 22. PROPOS. 22.

If there are any number of magnitudes whatever, and others equal to them in number, taken two by two in the same ratio, then, *ex aequali*, they too will be in the same ratio.

Here Euclid demonstrates the method of arguing in proportions *ex aequali*, when the proportion is ordered. Let there first be three magnitudes A, B, C and three others D, E, F, and let A be to B as D is to E, and B to C as E is to F. I say that, *ex aequali* as well, A is to C as D is to F. For let G, H be taken as equimultiples of A, D; likewise I, K as equimultiples of B, E; likewise L, M as equimultiples of C, F. Then, since A, the first, is to B, the second, as D, the third, is to E, the fourth, G, the multiple of the first A, will be to I, the multiple of the second B, as H, the multiple of the third D, is to K, the multiple of the fourth E [Margin: (a) — V.4]. By the same reasoning, since B, the first, is to C, the second, as E, the third, is to F, the fourth, I, the multiple of the first B, will be to L, the multiple of the second C, as K, the multiple of the third E, is to M, the multiple of the fourth F [Margin: (b) — V.4]. Since, therefore, there are three magnitudes G, I, L and three others H, K, M, taken two by two in the same proportion, it follows that if G, the first, exceeds L, the third, then H, the fourth, necessarily also exceeds M, the sixth; and if equal, equal; and if it falls short, it falls short [Margin: (c) — V.10]. And so, since G and H, the equimultiples of the first A and the third D, either both fall short of L and M, the equimultiples of the second C and the fourth F, or are both equal to them, or both exceed them — whatever multiplication be taken in forming these multiples — A, the first, will be to C, the second, as D, the third, is to F, the fourth [Margin: (d) — V def. 6]. Which was to be demonstrated.

Next, let there be more magnitudes than three, so that also C is to N as F is to O. I say that it still holds that A is to

N as D is to O. For since it has already been shown, among three magnitudes, that A is to C as D is to F, and it is now posited that C is to N as F is to O, there will be three magnitudes A, C, N and three others D, F, O, taken two by two in the same ratio. Therefore, from what was shown *ex aequali* among three magnitudes, it again follows that A is to N as D is to O. And in the same way the same thing will be shown among five magnitudes by means of four, just as it was demonstrated among four by means of three; and so for more. And so, if there are any number of magnitudes whatever, and the rest as stated. Which was to be shown.

SCHOLIVM [Scholium]

Moreover, it does not seem right to pass over in this place a certain theorem very familiar to the ancient mathematicians, although — so far as I know — no one has yet demonstrated it. It is of the following kind.

If the first has the same ratio to the second as the third has to the fourth, then equimultiples of the first and third will have the same ratio to the second and fourth; likewise equimultiples of the second and fourth will have the same ratio to the first and third. And conversely, the second and fourth will have the same ratio to equimultiples of the first and third; likewise the first and third will have the same ratio to equimultiples of the second and fourth.

Let A, the first, be to B, the second, as C, the third, is to D, the fourth; and let E, F be taken as equimultiples of A, C, and G, H as equimultiples of B, D. I say that E is to B as F is to D; likewise that G is to A as H is to C. And conversely, that B is to E as D is to F; likewise that A is to G as C is to H.

For since E is to A as F is to C, by construction (the multiple ratio being the same in each case), and it is posited that A is to B as C is to D, then *ex aequali* E will be to B as F is to D [Margin: (e) — V.22]. Again, since G is to B as H is to D — because in each case, by construction, the multiple ratio is the same — and B is to A as D is to C (for since it is posited that A is to B as C is to D, by conversion B is to A as D is to C), then *ex aequali* G will be to A as H is to C [Margin: (f) — V.22]. Next, since B is to A as D is to C, by inverse ratio, and A is to E as C is to F — because by construction the submultiple ratio is the same in each case — then *ex aequali* B will be to E as D is to F [Margin: (g) — V.22]. Again, since it is posited that A is to B as C is to D, and B is to G as D is to H [?], because by construction the submultiple

ratio is the same in each case, then *ex aequali* A will be to G as C is to H [Margin: (h) — V.22]. Which was to be demonstrated.

Diagram. Four parallel horizontal line-segments (magnitudes), one above the other and subdivided by tick-marks. From top to bottom the lines carry the labels **E F, A C, B D, G H** — E, F being equimultiples of A, C, and G, H equimultiples of B, D.

From this is evident the mode of argument that geometers use most frequently — above all Archimedes, Apollonius of Perga [Apollonius Pergaeus], Theon, and others. Namely: A is to B as C is to D. Therefore, as E — the double, or triple, or quadruple, etc., of A — is to B, so also will F — the double, or triple, or quadruple, etc., of C — be to D. Likewise, A is to B as C is to D. Therefore, as A is to the double, or triple, or quadruple, etc., of B — namely to G — so also will C be to the double, or triple, or quadruple, etc., of D — namely to H.

[Margin: xxii.]

THEOR. 23. PROPOS. 23.

If there are three magnitudes, and others equal to them in number, taken two by two in the same ratio, but their proportion is perturbed, then, *ex aequali*, they too will be in the same ratio.

Here the ratio is demonstrated *ex aequali*, when the proportion is perturbed. Let there be three magnitudes A, B, C and three others D, E, F, and let their proportion be perturbed — that is, let A be to B as E is to F, and B to C as D is to E. I say that, *ex aequali* as well, A is to C as D is to F. For let G, H, I be taken as equimultiples of A, B, D; likewise K, L, M as equimultiples of C, E, F. Then A will be to B as G is to H, since G, H are equimultiples of A, B [Margin: (a) — V.15]. But A is to B as E is to F; therefore G is to H as E is to F as well [Margin: (b) — V.11]. But E is to F as L is to M, since L, M are equimultiples of E, F [Margin: (c) — V.15]. Therefore G is to H as L is to M [Margin: (d) — V.11]. Again, since B, the first, is to C, the second, as D, the third, is to E, the fourth, H, the multiple of the first B, will be to K, the multiple of the second C, as I, the multiple of the third D, is to L, the multiple of the fourth E [Margin: (e) — V.4]. Since, therefore, there are three magnitudes G, H, K and three others I, L, M, taken two by two in the same ratio, and their proportion is perturbed — it having been shown that G is to H as L is to M, and that H is to K as I is to L — it follows that if G, the first, exceeds K, the third, then I, the fourth, also exceeds M, the sixth; and if equal, equal; and if it falls short, it falls short [Margin: (f) — V.21]. And so, since G and I, the equimultiples of the first A and the third D, either both fall short of K and M, the

equimultiples of the second C and the fourth F, or are both equal to them, or both exceed them, A, the first, will be to C, the second, as D, the third, is to F, the fourth [Margin: (g) — V def. 6]. Which was to be demonstrated. And so, if there are three magnitudes, and the rest as stated. Which was to be shown.

Diagram. In the left column, two rows of parallel vertical line-segments (magnitudes), subdivided by tick-marks. The upper row is labelled **A, B, C, N, D, E, F, O**; the lower row **G, H, K, I, L, M** — G, H, I being equimultiples of A, B, D, and K, L, M equimultiples of C, E, F.

SCHOLIVM [Scholium]

But if there are more than three magnitudes and their proportion is perturbed — that is, if A is to B as F is to O, and B to C as E is to F, and C to N as D is to E — I say that it still holds that A is to N as D is to O. And although, among geometers, the *ex aequali* proportion is not in use for more magnitudes than three — which was the reason why Euclid framed the preceding proposition [22] for any number of magnitudes, but this one for three only, even though it is true for more as well — it is proved thus. Since it has already been shown, among three magnitudes, that A is to C as F is to E, there will be three magnitudes A, C, N and three others D, E, O, taken two by two in the same proportion, and their proportion is perturbed. Therefore, again from what was shown *ex aequali* among three magnitudes, A will be to N as D is to O. And in the same way the same thing will be shown among five magnitudes by means of four, just as it was demonstrated among four by means of three; and so for more. Which was to be demonstrated.

Euclid, *Elements* — Book V (continued)

THEOR. 24. PROPOS. 24. [Margin: xxiv.]

If a first has the same ratio to a second as a third has to a fourth, and a fifth also has the same ratio to the second as a sixth has to the fourth: then the first together with the fifth will also have the same ratio to the second as the third together with the sixth has to the fourth.

What Euclid demonstrated in Proposition 2 for multiple proportion alone, he here demonstrates for every proportion, even the irrational. For let AB be the first, standing to C the second as DE the third stands to F the fourth; likewise let BG be the fifth, standing to C the second as EH the sixth stands to F the fourth. I say that AG, composed of the first and the fifth, stands to the second C as DH, composed of the third and the sixth, stands to the fourth F. For since BG is to C as

EH is to F, then by conversion [convertendo] C will be to BG as F is to EH. Therefore, since AB is to C as DE is to F, and C is to BG as F is to EH, [Margin: a — *Elements* V.22] there will be, ex aequali, AB to BG as DE to EH. [Margin: b — *Elements* V.18] By composition [componendo], then, as the whole AG is to BG, so the whole DH is to EH. And so, since again AG is to BG as DH is to EH, and BG is to C as EH is to F, [Margin: c — *Elements* V.22] there will be, ex aequali, AG to C as DH to F. Which is the thing proposed. Therefore, if a first has the same ratio to a second, etc. Which was to be demonstrated.

Diagram. A single long vertical line at the left, subdivided by tick marks, with two short vertical bars set off to its right. The long line represents the composed magnitudes AG (= AB + BG) and DH (= DE + EH), drawn continuously; the short bars are the second and fourth magnitudes.

- G (top of the long line): endpoint of AG
- B (below G): division point; BG = the fifth magnitude, AB = the first
- A (below B): shared point (AG runs from A up to G)
- D (below A): endpoint of DH
- E (below D): division point; DE = the third magnitude, EH = the sixth
- H (bottom of the long line): endpoint of DH
- C (short bar, upper right): the second magnitude
- F (short bar, lower right): the fourth magnitude

SCHOLIUM

This proposition is true whether or not the magnitudes AB, BG, and C are of the same kind as the magnitudes DE, EH, and F — as is clear from the demonstration.

In almost the same way there will be shown, for every kind of proportion, what was demonstrated in Theorem 6 of this book for multiple magnitudes only.

Namely:

If two magnitudes have the same ratio to two magnitudes, and certain magnitudes taken away from them have the same ratio to the same two: then the remainders also will have the same ratio to the same two.

For let AG, DH have the same ratio to C and F — that is, let AG be to C as DH is to F — and let the magnitudes taken away, AB and DE, have the same ratio to the same C, F — that is, let AB be to C as DE is to F. I say that the remainders BG, EH also have the same ratio to the same C, F — that is, that BG is to C as EH is to F. For since AB is to C as DE is to F, then by conversion C will be to AB as F is to DE.

[Margin: d — *Elements* V.22] There will therefore be, ex aequali, AG to AB as DH to DE. [Margin: e — *Elements* V.17] By division [dividendo], then, BG will be to AB as EH is to DE. And so, since again BG is to AB as EH is to DE, and AB is to C as DE is to F, [Margin: f — *Elements* V.22] there will be, ex aequali, BG to C as EH to F. Which is the thing proposed.

THEOR. 25. PROPOS. 25. [Margin: xxv.]

If four magnitudes are proportional: the greatest and the least together will be greater than the remaining two together.

For let AB be to CD as E is to F, and let AB be the greatest of all and F the least. I say that the two AB and F together are greater than the two CD and E together. For let there be taken from AB a magnitude AG equal to E, and from CD another, CH, equal to F. Then AG will be to CH as E is to F — that is, as AB is to CD. Therefore, since the whole AB is to the whole CD as the part taken away AG is to the part taken away CH, [Margin: a — *Elements* V.19] it will also be that as the whole AB is to the whole CD, so the remainder GB is to the remainder HD. But AB (since it is the greatest of all) is greater than CD. Therefore GB too will be greater than HD. Now since AG and E are equal, if to them be added the equals F and CH — namely F to AG, and CH to E — then AG and F together will be equal to E and CH together. When, therefore, the unequals GB and HD are added, AB and F together will become greater than E and CD together, since GB is greater than HD. Which is the thing proposed. Therefore, if four magnitudes are proportional, etc. Which was to be demonstrated.

Diagram. Two vertical lines with tick-mark subdivisions, and two short vertical bars set alongside. The left line is AB; the right line is CD; the two short bars are the third and fourth magnitudes E and F.

- B (top of left line): endpoint of AB [reading of this topmost letter as B rather than E is slightly uncertain]
- G (below B on the left line): division point; AG (from A up to G) = E
- A (bottom of left line): endpoint of AB
- D (top of right line): endpoint of CD
- H (below D on the right line): division point; CH (from C up to H) = F
- C (bottom of right line): endpoint of CD
- E (short bar): the third magnitude
- F (short bar): the fourth magnitude

SCHOLIUM

It necessarily follows that, if the antecedent magnitude of one proportion is the greatest of all, the consequent of the other is the least of all — as may be seen in the example proposed. For since AB is to CD as E is to F, and AB the first is greater than E the third, [Margin: b — *Elements* V.14] CD the second will also be greater than F the fourth. Likewise, because AB is greater than CD, E too will be greater than F, on account of the same proportion of AB to CD and of E to F, as we showed in the scholium to Proposition 14. But if, on the contrary, the antecedent of one proportion is the least of all, the consequent of the other will be the greatest of all — as is evident, if one states that F is to E as CD is to AB. All four magnitudes must also be of the same kind; otherwise one magnitude could not be composed from the greatest and the least — indeed, not even from the remaining two. Federicus Commandinus [Federico Commandino] adds here another theorem not much unlike this twenty-fifth, namely:

If three magnitudes are proportional: the greatest and the least together will be greater than double the remaining one.

Let A be to B as B is to C, and let A be the greatest and C the least. I say that A and C together are greater than double B. For, taking D equal to B, we shall have A to B as D to C. Therefore A and C together will be greater than B and D together, [Margin: b — *Elements* V.25] as was just now demonstrated — that is, than double B. Which is the thing proposed.

Diagram. Four vertical bars of decreasing length, standing side by side; A is the longest and C the shortest, with B and D (equal) between.

- A: the greatest magnitude
- B: the middle (remaining) magnitude
- D: magnitude taken equal to B
- C: the least magnitude

Here Euclid puts an end to the fifth book. But since Campanus [Campanus of Novara] and some others append certain further propositions — which the weightiest writers, such as Archimedes, Apollonius [of Perga], Johannes Regiomontanus [Regiomontanus], and others, very often use, and cite as though they were Euclid's — I have thought it well to attach them to this fifth book, to demonstrate them with the greatest brevity possible, and also to bring them into the number and series of Euclid's propositions. They are all set forth concerning

magnitudes not in proportion [magnitudinibus impropotionalibus], of which the first is this:

THEOR. 26. PROPOS. 26. [Margin: xxvj.]

If a first has to a second a greater ratio than a third has to a fourth: then, by conversion [convertendo], the second will have to the first a smaller ratio than the fourth has to the third.

For let A have to B a greater ratio than C has to D. I say that the ratio of B to A is smaller than the ratio of D to C. For let E be conceived such that E is to B as C is to D; then the ratio of A to B will also be greater than the ratio of E to B, [Margin: a — *Elements* V.10] and on that account A will be greater than E. [Margin: b — *Elements* V.8] Hence the ratio of B to A (the greater) will be less than the ratio of B to E (the less). But as B is to E, so, by conversion, is D to C. Therefore the ratio of B to A is also smaller than the ratio of D to C. Which is the thing proposed.

Diagram. Five short vertical bars with tick marks, arranged as an upper row (A, B, E) with two bars (C, D) set below A and B. They represent the magnitudes of the two given ratios $A : B$ and $C : D$, together with the conceived magnitude E.

- A (upper left): first magnitude
- B (upper middle): second magnitude
- E (upper right): the conceived magnitude, $E : B = C : D$
- C (below A): third magnitude
- D (below B): fourth magnitude

SCHOLIVM

In almost the same way we shall demonstrate that, if a first has to a second a smaller ratio than a third has to a fourth, then by conversion the ratio of the second to the first is greater than that of the fourth to the third — provided we change the word "greater" into the word "less," and conversely.

For let the ratio of A to B be smaller than that of C to D. I say that, by conversion, B to A has a greater ratio than D to C. For let E be conceived such that E is to B as C is to D; then the ratio of A to B will also be smaller than that of E to B, [Margin: c — *Elements* V.10] and on that account A will be less than E. [Margin: d — *Elements* V.8] Hence the ratio of B to A (A being the less) will be greater than the ratio of B to E (E being the greater). But as B is to E, so, by conversion, is D to C. Therefore the ratio of B to A is also greater than that of D to C. Which is the thing proposed.

Alternatively. Since the ratio of A to B is smaller than that of C to D, the ratio of C to D will be greater than that of A to B. [Margin: e — *Elements* V.26] Therefore, by conversion, the ratio of D to C will be smaller than that of B to A; and consequently the ratio of B to A is greater than that of D to C. Which is the thing proposed.

Diagram. The same schematic arrangement as the preceding figure: an upper row of bars (A, B, E) with two bars (C, D) beneath A and B, serving the scholium's case in which $A : B$ is the smaller ratio.

- A (upper left): first magnitude
- B (upper middle): second magnitude
- E (upper right): the conceived magnitude, $E : B = C : D$
- C (below A): third magnitude
- D (below B): fourth magnitude

THEOR. 27. PROPOS. 27. [Margin: xxvij.]

If a first has to a second a greater ratio than a third has to a fourth: then it will also, alternately, have to the third a greater ratio than the second has to the fourth.

For let A have to B a greater ratio than C has to D. I say that, alternately [permutando], the ratio of A to C is also greater than that of B to D. For let E be conceived such that E is to B as C is to D; then the ratio of A to B will be greater even than that of E to B. [Margin: a — *Elements* V.10] And so A will be greater than E. [Margin: b — *Elements* V.8] Hence the ratio of A to C will be greater than that of E to C. [Margin: c — *Elements* V.16] But since, alternately, E is to C as B is to D (because E was set to be to B as C is to D), the ratio of A to C will therefore also be greater than that of B to D. Which is the thing proposed.

Diagram. Three vertical lines with tick subdivisions; C is set below A, and D below B, with E standing to the right. They represent the ratios $A : B$ and $C : D$ and the conceived magnitude E.

- A (upper left): first magnitude
- C (below A): third magnitude
- B (middle): second magnitude
- D (below B): fourth magnitude
- E (right): the conceived magnitude, $E : B = C : D$

SCHOLIUM

Similarly we shall show that, if a first has to a second a smaller ratio than a third has to a fourth, then alternately the first has to the third a smaller ratio than the second has to the fourth. For let the ratio of A to B be smaller than that of C to D. I say that, alternately, the ratio of A to C is also smaller than that of B to D. For let E be conceived such that E is to B as C is to D; then the ratio of A to B will also be smaller than that of E to B, [Margin: d — *Elements* V.10] and on that account A will be less than E. [Margin: e — *Elements* V.8] Hence the ratio of A to C will be smaller than that of E to C. [Margin: f — *Elements* V.16] But, alternately, E is to C as B is to D (because E was set to be to B as C is to D). Therefore the ratio of A to C will also be smaller than that of B to D. Which is the thing proposed.

Alternatively. Since the ratio of A to B is smaller than that of C to D, the ratio of C to D will be greater than that of A to B. [Margin: g — *Elements* V.27] Therefore, alternately, the ratio of C to A will also be greater than that of D to B; [Margin: h — *Elements* V.26] and consequently, by conversion, the ratio of A to C will be smaller than that of B to D. Which is the thing proposed.

Diagram. The same schematic arrangement as the preceding figure: vertical lines for A (with C below) and B (with D below), and E to the right, serving the scholium's case in which $A : B$ is the smaller ratio.

- A (upper left): first magnitude
- C (below A): third magnitude
- B (middle): second magnitude
- D (below B): fourth magnitude
- E (right): the conceived magnitude, $E : B = C : D$

THEOR. 28. PROPOS. 28. [Margin: xxvii.]

If a first has to a second a greater ratio than a third has to a fourth: then also, by composition [componendo], the first together with the second will have to the second a greater ratio than the third together with the fourth has to the fourth.

Let the ratio of AB to BC be greater than that of DE to EF. I say that, by composition, the ratio of AC to BC is also greater than that of DF to EF. For let GB be conceived such that GB is to BC as DE is to EF; then the ratio of AB to BC will also be greater than that of GB to BC. [Margin: a — *Elements* V.10] And so AB is greater than GB. Adding, therefore, the common BC, AC will become greater than GC; [Margin: b — *Elements* V.8] and on that account the ratio of AC to BC will be

greater than that of GC to BC. But, by composition, GC is to BC as DF is to EF (because GB was set to be to BC as DE is to EF). [Margin: c — *Elements* V.18] Therefore the ratio of AC to BC will also be greater than that of DF to EF. Which is the thing proposed.

Diagram. A composite figure of two straight lines. On the left, a vertical line runs from C (top) through B down to A (bottom), with a second line slanting from the upper part down to G at the lower left (representing GB, and $GC = GB + BC$). On the right, a parallel vertical line runs from F (top) through E down to D (bottom), representing DF (= DE + EF).

- C (top of left line): endpoint of AC (AB runs from A up to B, BC from B to C)
- B (on left line): division point of AC; AB above, BC below
- G (lower left): endpoint of GB, so that $GB : BC = DE : EF$
- A (bottom of left line): endpoint of AC
- F (top of right line): endpoint of DF
- E (on right line): division point; DE above, EF below
- D (bottom of right line): endpoint of DF

SCHOLIUM

By the same reasoning we shall show that, if the ratio of the first to the second is smaller than that of the third to the fourth, then the ratio of the first and second together to the second is also smaller than that of the third and fourth together to the fourth.

These pages continue Euclid's **Book V** with Clavius's annexed propositions (the "greater/smaller ratio" and *ex aequali* theorems, Props 28–32). The Book V→VI transition does **not** occur here; the running header remains "LIBER V." throughout. The magnitudes are represented by the labelled straight-line schematics standard to Clavius's Book V. Enunciations are set as blockquotes; Clavius's demonstrations and scholia follow as prose. Marginal citation letters key to *Elements* Book V propositions and are rendered as [Margin: ...].

[Conclusion of the Scholium to Prop. 28]

(continued from the previous page; catchword "SIT")

For let AB have a smaller ratio to BC than DE has to EF. I say that *componendo* as well, AC has a smaller ratio to BC than DF has to EF. For let GB be to BC as DE is to EF. Then AB will also have a smaller ratio to BC than GB has to BC,[Margin: a — *Elements* V.10] and so AB will be smaller than GB. Therefore, when the common

BC is added, AC becomes smaller than GC,[Margin: b — Elements V.8] and for that reason AC has a smaller ratio to BC than GC has to BC. But *componendo*, as GC is to BC, so is DF to EF — since GB was assumed to be to BC as DE is to EF. Therefore AC will also have a smaller ratio to BC than DF has to EF. Which was to be proved.

Alternatively. Since AB has a smaller ratio to BC than DE has to EF, DE will have a greater ratio to EF than AB has to BC.[Margin: c — Elements V.28] Therefore *componendo*, DF will also have a greater ratio to EF than AC has to BC, and for that reason AC will have a smaller ratio to BC than DF has to EF. Which was to be proved.

Diagram. A pair of adjacent vertical magnitude-lines with division points marked by ticks.

- Left line (top to bottom): C, B, A, G
- Right line (top to bottom): F, E, D

THEOR. 29. PROPOS. 29.

[Margin: xxix.]

If the first compounded with the second has a greater ratio to the second than the third compounded with the fourth has to the fourth: then *dividendo* the first will also have a greater ratio to the second than the third has to the fourth.

Let AC have a greater ratio to BC than DF to EF. I say that *dividendo* AB too has a greater ratio to BC than DE to EF. For let GC be to BC as DF to EF; then AC has a greater ratio to BC than GC has to BC,[Margin: a — Elements V.10] and so AC is greater than GC. Therefore, when the common BC is taken away, AB is greater than GB,[Margin: b — Elements V.8] and for that reason AB has a greater ratio to BC than GB has to BC.[Margin: c — Elements V.17] But *dividendo*, as GB is to BC, so is DE to EF (since GC was assumed to be to BC as DF to EF). Therefore AB also has a greater ratio to BC than DE to EF. Which was to be proved.

Diagram. Two adjacent vertical magnitude-lines with tick-marked division points.

- Left line (top to bottom): C, B, G, A
- Right line (top to bottom): F, E, D

SCHOLIUM

But if the first compounded with the second has a smaller ratio to the second than the third compounded with the fourth has to the fourth, then *dividendo* the first will also have a smaller ratio to the second than the third has to the fourth.

For let AC have a smaller ratio to BC than DF to EF. I say that *dividendo* AB too has a smaller ratio to BC than DE to EF. For let GC be to BC as DF to EF; then AC has a smaller ratio to BC than GC has to BC,[Margin: d — Elements V.10] and so AC is smaller than GC. Therefore, when the common BC is taken away, AB is smaller than GB,[Margin: e — Elements V.8] and for that reason AB has a smaller ratio to BC than GB has to BC.[Margin: f — Elements V.17] But *dividendo*, as GB is to BC, so is DE to EF (since GC was assumed to be to BC as DF to EF). Therefore AB also has a smaller ratio to BC than DE to EF. Which was to be proved.

Alternatively. Since AC has a smaller ratio to BC than DF to EF, DF will have a greater ratio to EF than AC to BC.[Margin: g — Elements V.29] Therefore *dividendo* too, DE has a greater ratio to EF than AB to BC; and consequently AB has a smaller ratio to BC than DE to EF. Which was to be proved.

Diagram. Two adjacent vertical magnitude-lines with tick-marked division points.

- Left line (top to bottom): C, B, A, G
- Right line (top to bottom): F, E, D

THEOR. 30. PROPOSITIO 30.

[Margin: xxx.]

(Heading verbatim: "PROPOSITIO" is spelled out here, unlike the abbreviated "PROPOS." of the neighbouring theorems.)

If the first compounded with the second has a greater ratio to the second than the third compounded with the fourth has to the fourth: then, by conversion of ratio, the first-and-second together will have a smaller ratio to the first than the third-and-fourth together has to the third.

Let AC have a greater ratio to BC than DF to EF. I say that, by conversion of ratio, AC has a smaller ratio to AB than DF to DE. For since AC has a greater ratio to BC than DF to EF,[Margin: a — Elements V.29] *dividendo* AB will also have a greater ratio to BC than DE to EF.[Margin: b — Elements V.26] Therefore *convertendo*, BC

has a smaller ratio to AB than EF to DE;[Margin: c — Elements V.28] and for that reason *componendo* too, the whole AC has a smaller ratio to AB than the whole DF has to DE. Which was to be proved.

Diagram. Two adjacent vertical magnitude-lines with tick-marked division points.

- Left line (top to bottom): C, B, A
- Right line (top to bottom): F, E, D

SCHOLIVM (to Prop. 30)

By a not dissimilar argument we shall show that, if the first compounded with the second has a smaller ratio to the second than the third compounded with the fourth has to the fourth, then, by conversion of ratio, the ratio of the first-and-second together to the first is greater than that of the third-and-fourth together to the third.

For let AC have a smaller ratio to BC than DF to EF. I say that, by conversion of ratio, AC has a greater ratio to AB than DF to DE. For since AC has a smaller ratio to BC than DF to EF,[Margin: d — Elements V.29] *dividendo* AB will also have a smaller ratio to BC than DE to EF.[Margin: e — Elements V.26] Therefore *convertendo*, BC has a greater ratio to AB than EF to DE;[Margin: f — Elements V.28] and hence *componendo* too, AC has a greater ratio to AB than DF to DE. Which was to be proved.

Alternatively. Since AC has a smaller ratio to BC than DF to EF, DF will have a greater ratio to EF than AC to BC.[Margin: g — Elements V.30] Therefore, by conversion of ratio, DF has a smaller ratio to DE than AC to AB — that is, AC has a greater ratio to AB than DF to DE. Which was to be proved.

Diagram. Two adjacent vertical magnitude-lines with tick-marked division points.

- Left line (top to bottom): C, B, A
- Right line (top to bottom): F, E, D

THEOR. 31. PROPOS. 31.

[Margin: xxxj.]

If there are three magnitudes, and others equal to them in number, and the ratio of the first of the first set to the second is greater than the ratio

of the first of the latter set to the second; and likewise the ratio of the second of the first set to the third is greater than the ratio of the second of the latter set to the third: then also, *ex aequali*, the ratio of the first of the first set to the third will be greater than the ratio of the first of the latter set to the third.

Let there be three magnitudes A, B, C, and three others D, E, F, and let A have a greater ratio to B than D to E, and likewise B a greater ratio to C than E to F. I say that *ex aequali* A also has a greater ratio to C than D to F. For let G be to C as E to F; then B has a greater ratio to C than G to C,[Margin: a — Elements V.10] and so B is greater than G. Therefore[Margin: b — Elements V.8] the ratio of A to the smaller G is greater than the ratio of A to the greater B; but A to B is posited greater than D to E. Much more, then, is A to G greater than D to E. Again, let H be to G as D to E; then A has a greater ratio to G than H to G,[Margin: c — Elements V.10] and so A is greater than H.[Margin: d — Elements V.8] Therefore the greater magnitude A will have a greater ratio to C than the smaller magnitude H has to the same C.[Margin: e — Elements V.22] But H to C is, *ex aequali*, as D to F (since $D:E = H:G$ and $E:F = G:C$). Therefore A also has a greater ratio to C than D to F. Which was to be proved.

Diagram. A row of five vertical magnitude-lines with tick-marked division points; the first three carry a second label at their foot.

- Line 1: A (over D)
- Line 2: B (over E)
- Line 3: C (over F)
- Line 4: G
- Line 5: H

SCHOLIVM

By the same reasoning we shall show that, if the ratio of A to B is the same as that of D to E, while B to C is greater than E to F; or conversely, if the ratio of A to B is greater than that of D to E, while B to C is the same as E to F — then *ex aequali* the ratio of A to C is likewise greater than that of D to F.

For let A first be to B as D to E, but let B have a greater ratio to C than E to F. Let G be to C as E to F; then B has a greater ratio to C than G to C,[Margin: f — Elements V.10] and therefore A has a greater ratio to G than to B.[Margin: g — Elements V.8] But A to B is posited as D to E; therefore A to G is also greater than D to E. Again, let H be to G as D to E; then A has a greater ratio to G than H to G,[Margin:

h — Elements V.10] and so A is greater than H.[Margin: i — Elements V.8]
Therefore A has a greater ratio to C than H to C.[Margin: k — Elements V.22] But
H to C is, *ex aequali*, as D to F (since $D:E = H:G$ and $E:F = G:C$). Therefore A too has
a greater ratio to C than D to F. Which was to be proved.

Next, let A have a greater ratio to B than D to E, but B to C as E to F. Let G be to C
as E to F; then B to C is also as G to C,[Margin: l — Elements V.9] and so B is equal
to G.[Margin: m — Elements V.7] Therefore A to G is as A to B. But A to B is posited
greater than D to E; therefore A to G is also greater than D to E. Again, let H be to
G as D to E; then A has a greater ratio to G than H to G,[Margin: n — Elements
V.10] and so A is greater than H.[Margin: o — Elements V.8] Therefore A has a
greater ratio to C than H to C.[Margin: p — Elements V.22] But H to C is, *ex aequali*,
as D to F (since $D:E = H:G$ and $E:F = G:C$). Therefore A too has a greater ratio to C
than D to F. Which was to be proved.

By a like demonstration we shall show that, if the ratios of the first magnitudes
are smaller, the ratio of the outermost terms is also smaller.

Diagram. A row of five vertical magnitude-lines with tick-marked division points;
the first three carry a second label at their foot.

- Line 1: A (over D)
- Line 2: B (over E)
- Line 3: C (over F)
- Line 4: G
- Line 5: H

(scholium to Prop. 31, continued)

The same will also result if A is to B as D to E, but the ratio of B to C is smaller
than that of E to F; or, conversely, if A to B is smaller than D to E, while B to C is as
E to F. For the demonstration will always be the same, provided only that you
everywhere change the word "greater" into the word "smaller," as is clear.

But if there are more magnitudes than three — whether all the ratios in one set of
magnitudes are greater (or smaller) than all the ratios in the other set, or only
one is greater than one, or two, and so on — we shall show that the ratio of the
first of the first set to the last is likewise greater (or smaller) than that of the first
of the latter set to the last, by the method we set out in Proposition 22: using
Proposition 22 in place of the present Proposition 31 whenever three magnitudes
of one set are proportional to three magnitudes of the other set, and using this

scholium whenever just one ratio among three magnitudes is equal to just one ratio among the three magnitudes of the other set.

Diagram. A boxed list of two ordered sets of four magnitudes, in two columns.

- Left column: A, B, C, D
- Right column: E, F, G, H

For if A has a greater ratio to B than E to F, and B a greater ratio to C than F to G, and C a greater ratio to D than G to H,[Margin: a — Elements V.31] then *ex aequali* A has a greater ratio to C than E to G. And since, again, A has a greater ratio to C than E to G, and C a greater ratio to D than G to H,[Margin: b — Elements V.31] then *ex aequali* A also has a greater ratio to D than E to H.

But if the three magnitudes A, B, C are proportional to the three E, F, G, while C has a greater ratio to D than G to H,[Margin: c — Elements V.22] then *ex aequali* A is to C as E to G. Hence, by this scholium, A will have a greater ratio to D than E to H.

But if A has a greater ratio to B than E to F, while the three B, C, D are proportional to the three F, G, H, then by this scholium A will have a greater ratio to C than E to G; and again, A a greater ratio to D than E to H.

Diagram. A boxed list of two ordered sets of four magnitudes, in two columns.

- Left column: A, B, C, D
- Right column: E, F, G, H

But if A has a greater ratio to B than E to F, while B to C is as F to G, and C has a greater ratio to D than G to H; or if A to B is as E to F, while B has a greater ratio to C than F to G, and C to D is as G to H — then in every case, by this scholium, A will have a greater ratio to C than E to G, and consequently again a greater ratio to D than E to H, and so on.

You will conclude the same if all the ratios of one set are smaller than all those of the other, whether one or two of them; and likewise if there are more magnitudes, as we said in Proposition 22.

THEOR. 32. PROPOS. 32.

[Margin: xxxij.]

If there are three magnitudes, and others equal to them in number, and the ratio of the first of the first set to the second is greater than the ratio

of the second of the latter set to the third; and likewise the ratio of the second of the first set to the third is greater than the ratio of the first of the latter set to the second: then also, *ex aequali*, the ratio of the first of the first set to the third will be greater than the ratio of the first of the latter set to the third.

Let there be three magnitudes A, B, C, and three others D, E, F, and let A have a greater ratio to B than E to F, and likewise B a greater ratio to C than D to E. I say that *ex aequali* A also has a greater ratio to C than D to F. For let G be to C as D to E; then B has a greater ratio to C than G to C, [Margin: a — Elements V.10] and so B is greater than G. [Margin: b — Elements V.8] Therefore the ratio of A to the smaller G is greater than the ratio of the same A to the greater B; but A to B is greater than E to F. Much more, then, is A to G greater than E to F. Again, let H be to G as E to F; then A has a greater ratio to G than H to G, [Margin: c — Elements V.10] and so A is greater than H. [Margin: d — Elements V.8] For this reason the greater A will have a greater ratio to C than the smaller H has to the same C. [Margin: e — Elements V.23] But H to C is, *ex aequali*, as D to F (since $D:E = G:C$ and $E:F = H:G$). Therefore A also has a greater ratio to C than D to F. Which was to be proved.

Diagram. A row of five vertical magnitude-lines with tick-marked division points; the first three carry a second label at their foot.

- Line 1: A (over D)
- Line 2: B (over E)
- Line 3: C (over F)
- Line 4: G
- Line 5: H

SCHOLIVM

By the same reasoning — if A to B is as E to F, while B to C is greater than D to E; or conversely, if A to B is greater than E to F, while B to C is as D to E — we shall show that *ex aequali* A has a greater ratio to C than D to F, as is seen in the figure set out here.

Not otherwise shall we show that, if the ratios of the first magnitudes are smaller, the ratio of the extremes is also smaller, and so on.

And if there are more magnitudes than three, we shall prove that the ratio of the first [of the first set to the last is likewise greater or smaller...]

Diagram. A stack of five horizontal magnitude-lines (the "figure set out here" cited in the scholium); the third and lowest two lines carry paired end-labels.

- Line 1 (top): H
- Line 2: G
- Line 3: C, F
- Line 4: B, E
- Line 5 (bottom): A, D

the first of the antecedents has to the last a greater ratio than the first of the consequents has to the last — by the same method we employed at Proposition 23, and so on. All of this is clear, provided the demonstrations of the scholium to the preceding proposition are examined with care.

THEOR. 33. PROPOS. 33.

[Margin: xxxii.]

If the ratio of a whole to a whole is greater than that of a part taken away to a part taken away, then the ratio of the remainder to the remainder will also be greater than that of the whole to the whole.

Let the ratio of the whole AB to the whole CD be greater than that of the part removed, AE, to the part removed, CF. I say that the ratio of the remainder EB to the remainder FD is also greater than that of the whole AB to the whole CD.

For since AB has to CD a greater ratio than AE to CF, [a] [Margin: a — *Elements* V.27] by alternation the ratio of AB to AE will also be greater than that of CD to CF. [b] [Margin: b — *Elements* V.30] Therefore, by conversion of the ratio, the ratio of AB to EB will be smaller than that of CD to FD. [c] [Margin: c — *Elements* V.27] By alternation, then, the ratio of AB to CD will likewise be smaller than that of EB to FD; that is, the remainder EB will have to the remainder FD a greater ratio than the whole AB has to the whole CD. Which was what was proposed.

Diagram. Two stacked horizontal magnitude-lines with tick divisions, illustrating the proposition. The upper line is the whole AB, divided at E into the removed part AE and the remainder EB; the lower line is the whole CD, divided at F into the removed part CF and the remainder FD.

- Upper line, left end: A
- Upper line, interior division point: E
- Upper line, right end: B

- Lower line, left end: C
- Lower line, interior division point: F
- Lower line, right end: D

SCHOLIUM

But if the whole has to the whole a smaller ratio than the part removed has to the part removed, then the remainder will also have to the remainder a smaller ratio than the whole has to the whole — as is evident from the manner of demonstration, provided one everywhere substitutes the word "smaller" for the word "greater," and the word "greater" for the word "smaller."

THEOR. 34. PROPOS. 34.

[Margin: xxxiv.]

If there are any number of magnitudes, and others equal to them in number, and the ratio of the first of the first set to the first of the second is greater than that of the second to the second, and this in turn greater than that of the third to the third, and so on: then all the first magnitudes taken together will have to all the second taken together a greater ratio than all the first, the first omitted, have to all the second, the first likewise omitted; but a smaller ratio than the first of the first set has to the first of the second; and finally a greater ratio than the last of the first set has to the last of the second.

Let there first be three magnitudes A, B, C, and three others D, E, F. Let the ratio of A to D be greater than that of B to E, and likewise the ratio of B to E greater than that of C to F. I say that the ratio of A, B, C together to D, E, F together is greater than that of B, C together to E, F together; smaller, however, than the ratio of A to D; and, lastly, greater than the ratio of C to F.

For since A has to D a greater ratio than B to E, [a] [Margin: a — *Elements* V.27] by alternation A will have to B a greater ratio than D to E. [b] [Margin: b — *Elements* V.28] Therefore, by composition, the ratio of A, B together to B will be greater than that of D, E together to E. [c] [Margin: c — *Elements* V.27] By alternation again, then, the ratio of A, B together to D, E together will be greater than that of B to E. And so, since the whole A, B has to the whole D, E a greater ratio than the removed part B has to the removed part E, [d] [Margin: d — *Elements* V.33] the remainder A will also have to the remainder D a greater ratio than the whole A, B has to the whole D, E. By the same reasoning, the ratio of B to E is greater than

that of the whole B, C to the whole E, F; therefore the ratio of A to D is much greater than that of the whole B, C to the whole E, F. [e] [Margin: e — *Elements* V.27] By alternation, then, the ratio of A to B, C is greater than that of D to E, F; [f] [Margin: f — *Elements* V.28] and therefore, by composition, the ratio of the whole A, B, C to B, C is greater than that of the whole D, E, F to E, F. [g] [Margin: g — *Elements* V.27] And again, by alternation, the ratio of all of A, B, C together to all of D, E, F together is greater than that of B, C to E, F. Which is the first point.

And so, since the whole A, B, C has to the whole D, E, F a greater ratio than the removed part B, C has to the removed part E, F, [h] [Margin: h — *Elements* V.33] the ratio of the remainder A to the remainder D will also be greater than that of the whole A, B, C to the whole D, E, F. Which is the second point.

Since, moreover, the ratio of B to E is greater than that of C to F, [i] [Margin: i — *Elements* V.27] by alternation the ratio of B to C will also be greater than that of E to F; [k] [Margin: k — *Elements* V.28] and, by composition, the ratio of the whole BC to C is greater than that of the whole E, F to F; [l] [Margin: l — *Elements* V.27] and again, by alternation, the ratio of B, C to E, F is greater than that of C to F. But the ratio of A, B, C to D, E, F is greater than that of B, C to E, F, as we have shown; therefore the ratio of all of A, B, C to all of D, E, F is much greater than that of the last, C, to the last, F. Which is the third point.

Next, let there be four magnitudes on each side, under the same hypothesis — that is, let the ratio of the first, A, to D be greater than that of the second, B, to E; and this greater than the ratio of the third, C, to F; and this in turn greater than the ratio of the fourth, G, to H. I say that the same results follow. For, just as has now been shown for three, the ratio of B, C, G to E, F, H is greater [than that of G to H]; therefore the ratio of A to D will be much greater than that of B, C, G to E, F, H. [m] [Margin: m — *Elements* V.27] By alternation, then, the ratio of A to B, C, G will be greater than that of D to E, F, H; [n] [Margin: n — *Elements* V.28] and, by composition, the ratio of A, B, C, G to B, C, G is greater than that of D, E, F, H to E, F, H; [o] [Margin: o — *Elements* V.27] and, by alternation, the ratio of A, B, C, G to D, E, F, H is greater than that of B, C, G to E, F, H. Which is the first point.

And so, since the whole A, B, C, G has to the whole D, E, F, H a greater ratio than the removed part B, C, G has to the removed part E, F, H, [p] [Margin: p — *Elements* V.33] the ratio of the remainder A to the remainder D will also be greater than that of the whole A, B, C, G to the whole D, E, F, H. Which is the second point.

Since, moreover, as was demonstrated for three, the ratio of B, C, G to E, F, H is greater than that of G to H, and the ratio of A, B, C, G to D, E, F, H is greater than that of B, C, G to E, F, H, as was shown: the ratio of A, B, C, G to D, E, F, H will be much greater than that of the last, G, to the last, H. Which is the third point.

By the same method you will conclude that the same follows for five magnitudes taken by fours, for six taken by fives, for seven taken by sixes, and so on — just as we have shown for four taken by threes. The whole theorem, therefore, is established, and so on.

FINIS ELEMENTI QVINTI. (*End of the Fifth Element.*)

EVCLIDIS ELEMENTVM SEXTVM

Euclid's Elements, Book Six

DEFINITIONES · Definitions

I.

Rectilinear figures are **similar** when they have their several angles equal, each to each, and also have the sides about the equal angles proportional.

Thus triangles ABC and DEF are said to be similar if they are equiangular — that is, if angle A is equal to angle D, B to E, and C to F — and if, moreover, the sides about the equal angles are proportional, that is: as AB is to AC, so is DE to DF; and as AB is to BC, so is DE to EF; and as AC is to CB, so is DF to FE.

But if the angles of the one are equal to the angles of the other, each to each, while the sides about the equal angles are not proportional — or the reverse — such figures will not be called similar. Of this sort are the square and the oblong (the figure longer on one side). For these figures do indeed have their angles equal, being right angles, yet the sides of the one are not proportional to the sides of the other, since the sides of the square about any right angle stand in a ratio of equality, whereas the sides of the oblong about any right angle stand in a ratio of inequality. From this it is clear that all equiangular and equilateral rectilinear figures which have their angles and sides equal in number are similar, however unequal in size they may be among themselves. Of this sort are the equilateral triangles GHI and KLM. For, on account of the equality of their sides, GH will be to GI as KL to KM; likewise GH to HI as KL to LM, and GI to IH as KM to ML, since the ratio is always one of equality. The same is to be said of squares, pentagons,

hexagons, heptagons, octagons, and other figures of that kind — rectilinear, equiangular, and equilateral.

Diagram. Two similar right triangles illustrating the definition of similar figures. Each has its right angle at the lower-left vertex; the two are of different size but the same shape.

- A (apex, upper left of first triangle)
- B (lower-left vertex, right angle, first triangle)
- C (lower-right vertex, first triangle)
- D (apex, upper part of second triangle)
- E (lower-left vertex, right angle, second triangle)
- F (lower-right vertex, second triangle)

Diagram. Two equilateral triangles of unequal size (the "similar though unequal" example).

- G (apex, first triangle)
- H (lower-left vertex, first triangle)
- I (lower-right vertex, first triangle)
- K (apex, second triangle)
- L (lower-left vertex, second triangle)
- M (lower-right vertex, second triangle)

II.

Figures are **reciprocally related** when in each figure there are antecedent and consequent terms of the ratios.

Thus, if in the parallelograms ABCD and EFGH the sides AB and BC are proportional to the sides EF and FG in such a way that in each figure there is both an antecedent and a consequent of different ratios — that is, so that the ratio of AB to EF is the same as that of FG to BC, or, equivalently, that AB is to FG as EF is to BC (for in either way AB is the antecedent of one ratio and BC the consequent of another in the figure ABCD, just as, in the first way, EF is the consequent of one and FG the antecedent of another; or, in the second way, FG is the consequent and EF the antecedent, in the figure EFGH) — such parallelograms will be called reciprocally related, even though they are not similar. Likewise the triangles IKL and MNO will be reciprocally related if IK is to MN as MO is to IL, or if IK is to MO as MN is to IL. Nor do I recall having found among the geometers any use of reciprocally related figures other than in parallelograms and triangles.

Diagram. Two upright parallelograms and two triangles set in a row, illustrating reciprocally related figures.

- A (upper-left, first parallelogram); B (upper-right, first parallelogram); C (lower-right, first parallelogram); D (lower-left, first parallelogram)
- E (upper-left, second parallelogram); F (upper-right, second parallelogram); G (lower-right, second parallelogram); H (lower-left, second parallelogram)
- I (apex, first triangle); K (lower-left vertex, first triangle); L (lower-right vertex, first triangle)
- M (apex, second triangle); O (lower-left vertex, second triangle); N (lower-right vertex, second triangle)

III.

A straight line is said to be cut in **extreme and mean ratio** when, as the whole is to the greater segment, so the greater is to the less.

If any straight line AB is divided at C unequally, so that, as the whole AB is to the greater segment AC, so the greater segment AC is to the lesser segment CB, it will be said to be cut in extreme and mean ratio. Euclid will teach this division at Proposition 30 of the present book, and he has already taught it in other words at Book 2, Proposition 11, as will become clear at Proposition 30 of this book. There are almost countless valuable properties and uses of a line divided in this way, as will appear from the books of solid geometry, especially Book 13 — so that not without reason has it been called by some the **divine proportion**, the proportion into which a line is divided in this manner.

Diagram. A single straight line divided at an interior point.

- A (left end)
- C (interior point of division)
- B (right end)

IIII.

The **altitude** of any figure is the perpendicular line drawn from the vertex to the base.

If from the vertex A of triangle ABC a perpendicular AG is drawn to the base BC, this perpendicular will be called the altitude of triangle ABC; so that the triangle is said to have an altitude as great as the perpendicular AG. In the same way, the perpendicular DH, drawn from D, the vertex of triangle DEF, to the base EF produced toward E, will be called the altitude of triangle DEF. And so, if the

perpendiculars dropped from the vertices of two figures to their bases (whether these are produced or not) are equal, such figures will be said to have the same altitude. Now these perpendiculars will be equal when the bases of the figures, and their vertices, are set upon the same parallels — as are the perpendiculars AG, DH of triangles ABC, DEF, set upon the same parallels. For since the angles AGH and DHG, the interior angles on the same side, are equal to two right angles — indeed are two right angles — [a] [Margin: a — *Elements* I.28] the straight lines AG and DH will be parallel; and AD and GH are also parallel, since the triangles are supposed to be set upon the same parallels. Therefore ADHG will be a parallelogram; [b] [Margin: b — *Elements* I.34] and consequently the opposite sides AG and DH will be equal. These triangles will therefore be said to have the same altitude. But if in the same triangles the vertices are taken to be C and F, and the bases AB and DE, they will not have the same altitude; for the perpendicular drawn from F to the base DE produced is not equal to the perpendicular drawn from C to the base AB, since the triangles themselves cannot in that case be set upon the same parallels, as is manifest.

Diagram. Two triangles standing on a common baseline, with their apexes on an upper line parallel to it, each with its altitude drawn. The construction shows the parallelogram ADHG formed by the two apexes and the two perpendicular feet.

- A (apex of first triangle, on the upper parallel)
- D (apex of second triangle, on the upper parallel)
- Baseline, left to right: B (foot of first triangle) — G (foot of perpendicular AG) — C (foot of first triangle) — H (foot of perpendicular DH, on EF produced) — E (base vertex of second triangle) — F (base vertex of second triangle)
- AG (perpendicular altitude of triangle ABC)
- DH (perpendicular altitude of triangle DEF, dropped to base EF produced to H)

Euclid has rightly defined the altitude of any figure by means of the perpendicular line drawn from the vertex to the base — because, as Ptolemy [Ptolemæus] writes in his little treatise *On the Analemma*, and as Simplicius [Simplicio] reports in his book *On Measurement*, the measure of anything ought to be fixed and determinate, not indefinite; and among all straight lines, by which geometers, like ordinary people, rightly measure everything, the perpendicular alone is fixed and determinate, while all the others are uncertain and indeterminate. On this matter we have written more fully at the beginning of the

commentaries which we published on the *Sphere* of Johannes de Sacrobosco [Ioan. de sacro bosco].

V.

A ratio is said to be compounded of ratios when the quantities of the ratios, multiplied together, produce some ratio.

Since the denominator of any ratio expresses how great the antecedent magnitude is with respect to the consequent — as the denominator of the quadruple ratio, namely 4, shows that in any quadruple ratio the antecedent magnitude contains the consequent four times; while the denominator of the subquadruple ratio, namely $\frac{1}{4}$, indicates that the antecedent is the fourth part of the consequent, and so on — the denominator is therefore usually called by geometers the "quantity of the ratio," so that "quantity of a ratio" signifies the same as "denominator." This definition, then, means that some ratio is compounded of two or more ratios when the denominators of these, that is, their quantities, multiplied together (or, as Zamberti [Zambertus] renders it, when they have produced the quantity of that ratio, that is, its denominator).

Thus the duodecuple ratio (12:1) is said to be compounded of the double (2:1) and the sextuple (6:1); for the denominator of the duodecuple ratio, namely 12, is produced by multiplying the denominator of the double ratio, namely 2, into the denominator of the sextuple, that is, into 6. In the same way the same duodecuple ratio is said to be compounded of the triple (3:1) and the quadruple (4:1); for from the multiplication of 3 into 4 the same denominator 12 of the duodecuple ratio is produced. By the same reasoning the thirty-fold ratio (30:1) is said to be compounded of the double, the triple, and the quintuple (5:1); for the denominators of these, 2, 3, and 5, multiplied together, produce 30, its denominator. So too the double ratio (2:1) will be said to be compounded of the sesquialterate (3:2) and the sesquitertian (4:3); because the denominator of the sesquialterate, $1\frac{1}{2}$, multiplied into $1\frac{1}{3}$, the denominator of the sesquitertian, produces 2, the denominator of the double. Likewise the same double ratio will be compounded of the sesquiseptimal (8:7) and the supertripartient-of-fourths (7:4); for the denominators of these ratios, $1\frac{1}{7}$ and $1\frac{3}{4}$, multiplied together, produce 2, the denominator of the double. Again, the same double ratio will be said to be compounded of the subsesquiquartan (4:5) and the double-sesquialterate (5:2), because the denominators of these, $\frac{4}{5}$ and $2\frac{1}{2}$, multiplied together, likewise produce the same denominator 2 of the double ratio. And so the double ratio will be said to be compounded of infinitely many other ratios — which may be said of any other ratio as well, as will appear a little later.

Furthermore, just as, among magnitudes continuously proportional, the ratio of the first to the last is said to be compounded of the ratio of the first to the second, of the second to the third, of the third to the fourth, and so on — since that ratio is composed of these intermediate ones, and its denominator is produced from their denominators multiplied together, as we set out in Book 5, Definition 10 — so that, if there are two equal intermediate ratios out of which it is said to be compounded, the ratio of the first magnitude to the last is called the duplicate of the ratio of the first to the second; if three, the triplicate, and so on. Likewise, among any magnitudes whatever, set in any order, the ratio of the first to the last will be said to be compounded of the ratio of the first to the second, of the second to the third, of the third to the fourth, and so on, until the ratio is complete; because the denominator of the ratio of the first magnitude to the last arises from the denominators of the intermediate ratios multiplied together. This first point we shall confirm by a certain induction of Theon of Alexandria [Theonis Alexā[n]drini], which he adduces in this place; and then we shall establish the same by two demonstrations, one of which is handed down by Eutocius of Ascalon [Eutocio Ascalonita] in his commentary on Book 2 of Archimedes, *On the Sphere and Cylinder*, Theorem 4, and in Book 1 of Apollonius of Perga [Apollonij Pergæi], *On the Elements of Conics*, Proposition 11; the other we shall confirm from Witelo [Vitellione], Book 1, Proposition 13, of his *Perspective*.

(Continuation of Clavius's commentary on Euclid, Book VI, Definition V — the ratio compounded of ratios. Theon's induction, followed by the demonstrations of Eutocius and Witelo.)

THEON therefore attempts to settle the matter as follows.

Let AB have to CD a given ratio — say double, or triple, or any other whatever — and let CD likewise have to EF a given ratio. I say that the ratio of AB to EF is made up of the ratio of AB to CD and of CD to EF; or that the size [*quantitas*] of the ratio of AB to CD, multiplied into the size of the ratio of CD to EF, produces the size of the ratio of AB to EF.

First, let AB be greater than CD, and CD greater than EF; and let AB be double CD, and CD triple EF. Since, then, CD is triple EF, while AB is double CD, AB will be sextuple EF — for if we double the triple of something, sextuple results, and this is precisely the compounding. Or thus: since AB is double CD, divide AB into parts equal to CD, that is, into AG and GB. Now since CD is triple EF, and AG is equal to CD, AG too is triple EF; and for the same reason GB is also triple EF. Therefore the whole AB is sextuple EF. Hence the ratio of AB to EF is compounded,

through the middle term CD, out of the ratio of AB to CD and that of CD to EF.

Diagram. Three vertical line-segments illustrating the double/triple case. The leftmost and tallest, AB, is divided at G into two equal parts AG and GB; the middle segment is CD; the rightmost and shortest is EF.

- A (top of first segment)
- G (division point of AB)
- B (foot of first segment)
- C (top of second segment)
- D (foot of second segment)
- E (top of third segment)
- F (foot of third segment)

In like manner the same conclusion is gathered even if CD is less than each of AB and EF. For again let AB be triple CD, but let CD be half of EF. Since CD is half EF, while AB is triple CD, AB will be sesquialterate to EF [3:2] — for if we triple the half of something, it will contain that thing once and a half. And since AB is triple CD, and CD is half EF: of such parts as AB contains three, equal to CD, EF contains two. Therefore AB is sesquialterate to EF. So the ratio of AB to EF is connected, through the middle term CD, and compounded out of the ratio of AB to CD and of CD to EF.

Diagram. Three vertical line-segments illustrating the case where CD is less than both : AB (tall), CD (short), EF (medium).

- A (top), B (foot) — first segment
- C (?) (top), D (foot) — second segment
- E (top), F (foot) — third segment

But now again let CD be greater than each of AB and EF, and let AB be half of CD, and CD sesquitercian [4:3] to EF. Since, then, of such parts as AB is two, CD is four, and of such as CD is four, EF is three, therefore of such as AB is two, EF is three. So again the ratio of AB to EF is connected through the middle term CD, and it is that of two to three. And likewise with more terms and in the remaining cases. It is also evident that if, from any given compound ratio, one of the composing ratios is taken away — one of the simple ratios being cast out — what remains will be the ratio compounded of the rest.

These things we have taken word for word from Theon, following the translation of Zambertus [Bartolomeo Zamberti].

Diagram. Three vertical line-segments illustrating the case where CD is the greatest ($AB = 2$, $CD = 4$, $EF = 3$) : CD tallest, EF of middle height, AB shortest.

- A (top), B (foot) — first segment
- C (top), D (foot) — second segment
- E (top), F (foot) — third segment

The demonstration of Eutocius runs as follows.

Let there be three magnitudes A, B, C. I say that the ratio of A to C is compounded of the ratios of A to B and of B to C. This will be easily demonstrated once the following principle is assumed: that the size, or denominator, of any ratio, multiplied into the consequent magnitude of the same ratio, produces the antecedent. For since the denominator indicates the relation of the antecedent to the consequent, the consequent taken according to the denominator — that is, multiplied by the denominator — must necessarily restore the antecedent. Thus, because 12 to 3 has a quadruple ratio, 3 multiplied by 4 produces 12. Likewise, because 4 to 20 has a subquintuple ratio, whose denominator is $\frac{1}{5}$, it comes about that $\frac{1}{5}$ taken into the consequent 20 yields 4; and so on.

Let, then, the denominator of the ratio of A to B be D, and let the denominator of the ratio of B to C be E, and let D multiplying E produce F. I say that F is the size, or denominator, of the ratio of A to C — that is, that if F is multiplied into C, A is produced. For let G be produced by the multiplication of F into C; I say that G is equal to A, and hence that A too arises from F into C. For since D is the denominator of the ratio of A to B, if D is multiplied into B, A will be produced. By the same reasoning, if E is multiplied into C, B will be produced. Since, then, F and E, multiplying C, produce G and B (for from F into C comes G, and from E into C comes B, as was said), [Margin: Elements VII.18] it will be, as F is to E, so G is to B. Again, since D, multiplying E and B, produces F and A, [Margin: Elements VII.17] it will be, as E is to B, so F is to A; and by alternation, as E is to B, so F is to A; and inversely again, as F is to E, so A is to B. But it was shown that as F is to E, so is G to B. Therefore, as A is to B, so is G to B; and hence the magnitudes A and G are equal. Therefore, since G is produced from F into C, A too will be produced from F into C; and consequently F will be the size of the ratio of A to C. Which was to be proved.

Diagram. Eutocius's construction. A row of vertical line-segments labelled, along the foot, A B C D E F G (the three magnitudes A, B, C; the denominators D, E, F; and

the product G), together with taller construction lines at the far left and toward the right. Numeric annotations 4, 3, and 6 stand beside individual segments, and two segments carry a small mark I read as $Z [?] / 2 [?]$. The tall left-hand line's apex bears a bold letter that reads as K [?], not referenced in the text. **

The illustrative values keyed to this figure are:

A	B	C	D	E	F	G
100	20	5	5	4	20	100

(Here $A:B = 100:20 = 5$, denominator $D = 5$; $B:C = 20:5 = 4$, denominator $E = 4$; $A:C = 100:5 = 20$, denominator $F = D \times E = 20$; and $G = F \times C = 20 \times 5 = 100 = A$.)

The like reasoning holds for more magnitudes. For the ratio of the first to the last will always be compounded of the ratios of the first to the second, of the second to the third, of the third to the fourth, and so on. Thus, if there are four magnitudes A, B, C, D, the ratio of A to D will be compounded of the ratios A to B, B to C, and C to D. For if three magnitudes A, C, D are considered, the ratio of A to D will be compounded of the ratios A to C and C to D, as has been shown. But by the same reasoning the ratio of A to C is compounded of the ratios A to B and B to C. Therefore the ratio of A to D is likewise compounded of the ratios A to B, B to C, and C to D. Which was to be proved. You will see the same in 5, 6, 7, or however many magnitudes.

Diagram. Four vertical line-segments labelled A B C D along the foot, illustrating the generalization to more magnitudes, with a taller construction line whose apex reads as K [?] and numeric annotations 4 and 8 beside the segments. **

Finally, Witelo [Vitellio] offers a demonstration of the following kind.

Let there be three magnitudes A, B, C. I say that the ratio of A to C is compounded of the ratios of A to B and of B to C. This will be demonstrated by assuming the same principle of Eutocius — namely, that the denominator of a ratio, multiplied into the consequent magnitude of the ratio, produces the antecedent magnitude. For let D be the denominator of the ratio of A to B, and E the denominator of the ratio of B to C; and let F be the denominator of the ratio of A to C. It must then be shown that F is produced from D into E, which will come about thus. Since what is produced from F, the first quantity, into C, the fourth, is equal to what is generated from D, the second, into B, the third — because A is always produced (for from F, the denominator, into C, the consequent, the antecedent A is produced; and likewise from D, the denominator, into B, the consequent, the antecedent A is produced) — [Margin: Elements VII.19] it will be, as F, the first, is

to D, the second, so B, the third, is to C, the fourth. Since, then, E is the denominator of the ratio of B to C, E will also be the denominator of the ratio of F to D. Therefore E, multiplying the consequent D, will produce the antecedent F. Which was to be proved. So Witelo.

The illustrative values for this demonstration are:

A	B	C	D	E	F
100	20	5	5	4	20

The same will be shown for more magnitudes, as before from Eutocius. For instance, given six magnitudes A, B, C, D, E, F, the ratio of A to F will be compounded of the ratios A to B, B to C, C to D, D to E, and E to F. For, as has been shown, the ratio of A to F is compounded of the ratio A to B and the ratio B to F. But this ratio B to F is compounded of the ratios B to C and C to F; and this ratio C to F of the ratios C to D and D to F; and finally this ratio D to F is compounded of the ratios D to E and E to F. Therefore the ratio of A to F is generated from all the intermediate ratios; that is, the denominator of the ratio of A to F is generated from the five denominators of the five intermediate ratios multiplied among themselves.

Diagram. Six vertical line-segments labelled A B C D E F along the foot, for the six-magnitude compound, their heights proportional to the values listed alongside.

A	B	C	D	E	F
12	6	3	2	1	4

(A:B = 2, B:C = 2, C:D = 3:2, D:E = 2, E:F = 1:4; the compound A:F = 12:4 = 3.)

From these things there is clearly established what we taught with respect to Definition 10 of Book 5: namely, that when any number of quantities are continued in continued proportion, the ratio of the first to the last is compounded of all the equal intermediate ratios. That is, the denominator of the ratio which the first has to the last is produced from the denominator of the ratio which the first has to the second, multiplied into itself, and then this into the number so produced, and so on, until as many multiplications are made as — less one — there are ratios interposed between the first quantity and the last. So that the duplicate ratio of some ratio arises when the denominator of that ratio is set down twice (on account of the two equal ratios placed between the first quantity and the third) and is thus multiplied into itself; and the triplicate, when the

denominator is likewise set down three times (on account of the three equal ratios placed between the first quantity and the fourth) and is thus multiplied, first into itself, then again into the product. And so it must be said of the quadruplicate, quintuplicate, and the rest, without end. For the demonstration is the same in magnitudes that are in continued proportion and in magnitudes not proportional: only, in the former the denominators will be equal. Thus, if A, B, C are set down in continued proportion, the denominators D and E will be equal, and F will be the denominator of the duplicate ratio, and so on.

Although the demonstration both of Eutocius and of Witelo strictly fits only rational ratios — since each rests on the propositions of Book 7 of Euclid — nevertheless, because what is demonstrated concerning ratios of numbers holds also for incommensurable magnitudes, that is, for irrational ratios, either demonstration may be said to apply to all ratios.

But even if it were not established — given several magnitudes — that the denominator of the ratio which the first has to the last is produced from the multiplication of the denominators of the intermediate ratios among themselves (as we have demonstrated), the demonstrations in which the compounding of ratios is employed would nonetheless be no less certain, since in them this multiplication of denominators is not used. For just as Euclid said, in Definition 10 of Book 5, that when several magnitudes are set out in continued proportion, the first has to the third the duplicate ratio of that which the first has to the second — that is, one compounded of two equal intermediate ratios — and that the first has to the fourth the triplicate ratio of that which the first has to the second, i.e. one compounded of three equal intermediate ratios, and so on, with no mention made of any multiplication of denominators: so too, if several magnitudes of the same kind are set out in order but not in continued proportion, the first will be said to have to the last a ratio compounded of all the intermediate ratios — even though these are not all equal to one another (whether some be of greater inequality, some of equality, and some of lesser inequality, or whether all be of the same kind) — for this reason alone: that those intermediate ratios are interposed between the two extreme magnitudes. It is just as, in Definition 10 of Book 5, the ratio of the first to the third was called duplicate for this reason alone, that two equal ratios are interposed between the two extreme magnitudes. So that there is no other difference between this compounding of ratios and that duplication, triplication, etc. explained in Book 5, than this: that in the duplication, triplication, etc. of ratios the interposed ratios are all equal, whereas in the compounding of ratios the interposed ratios need not be equal. The multiplication of denominators is nonetheless useful, so that we may know what

that ratio is which is called the duplicate, triplicate, etc. of another, or which is said to be compounded of given ratios.

For example. To know what ratio is called the duplicate of the decuple ratio, we set down 10, the denominator of the decuple ratio, twice, thus: 10, 10, and multiply one into the other. For the number generated, 100, is the denominator of the ratio that is the duplicate of the decuple. And to have the denominator of the ratio that is called the triplicate of the same decuple, we set down the denominator 10 three times, thus: 10, 10, 10, and multiply the first into the second, and the product 100 into the third. For the number thus produced, 1000, is the denominator of the triplicate of the decuple ratio. The same must be said of other ratios as well. But we shall learn this also without multiplying denominators. For if three numbers are continued in a given ratio — as we taught in the treatise on ratios, when dealing with geometric proportionality — the two extreme numbers will have the duplicate of the given ratio; and if four numbers are continued, the two extremes will have the triplicate of the given ratio. But the greater must be compared with the lesser when the given ratio is one of greater inequality, and the lesser with the greater when the given ratio is one of lesser inequality. For instance, if the duplicate of the decuple ratio is sought, we shall

continue three numbers in decuple ratio, thus: 1, 10, 100; or 3, 30, 300. For the ratio of 100 to 1, or of 300 to 3, which is centuple, will be called the duplicate of the decuple. In the same way, the ratio of 1 to 100, or of 3 to 300, which is subcentuple, is called the duplicate of the subdecuple ratio, 1 to 10, or 3 to 30.

By the same method, to learn what ratio it is that is said, for example, to be compounded of the triple, double, sesquialterate, and sesquitertian, we set down their denominators in order, thus: 3, 2, $1\frac{1}{2}$, $1\frac{1}{3}$, and multiply them among themselves: first 3 into 2; then the product 6 into $1\frac{1}{2}$; and this number produced, 9, again into $1\frac{1}{3}$; and so on, if there were more denominators. The last number produced, 12, is the denominator of the ratio that is said to be compounded of the given ratios; so that the duodecuple ratio is said to be compounded of the triple, double, sesquialterate, and sesquitertian.

But we shall understand this even without such multiplication of denominators. For if the component ratios are continued in numbers, so that the first is to the second in the first component ratio, the second to the third in the second, the third to the fourth in the third, the fourth to the fifth in the fourth, and so on, then the ratio which the first number has to the last will be compounded of the given ratios. As in the preceding example the given ratios — namely the triple, double,

sesquialterate, and sesquitercian — are continued in these numbers: 36, 12, 6, 4, 3; or in these: 108, 36, 18, 12, 9; or in these: 12, 4, 2, $1\frac{1}{3}$, 1. For in every case the first number has to the second a triple ratio, the second to the third a double, the third to the fourth a sesquialterate, and the fourth to the fifth a sesquitercian. Therefore the ratio of the first to the last, which is duodecuple, is said to be compounded of those four ratios, as before. And in what way any number of ratios are to be continued in the least whole numbers, Euclid teaches in Book 8, Proposition 4.

Therefore, when Euclid demonstrates, in Proposition 23 of this book, that equiangular parallelograms have a ratio compounded of the two ratios which the two sides about one angle of the one bear to the two sides about the equal angle of the other, he means nothing else than this: that if those two ratios of the sides are continued in three quantities, the parallelograms have between themselves the ratio which the first quantity has to the third. Thus, if the ratio of one side of the first parallelogram to one side of the second is as 8 to 3, and the ratio of the other side to the other side is as 1 to 2, nothing else is to be understood than this: if three numbers are taken, say 24, 9, 18, of which the first is to the second as 8 to 3 and the second to the third as 1 to 2, then the ratio of the first parallelogram to the second is the same as that which the first number, 24, has to the third, 18 — which is sesquitercian. For the ratio of 24 to 18 is compounded, as has been said, of the ratios of 24 to 9 and of 9 to 18 — that is, of the ratios 8 to 3 and 1 to 2, which we said obtained between the sides.

Furthermore, just as certain interpreters of Euclid wished, at Definition 10 of Book 5, that the duplicate, triplicate ratio, etc. be truly greater than that of which it is called the duplicate or triplicate — namely double or triple it, etc. — and therefore that this definition should be understood of magnitudes in continued proportion in a ratio of greater inequality (which, however, we there showed to be false): so the same men wish, in this compounding of ratios too, that the ratio of the extremes, which is said to be compounded of the intermediate ratios, be truly greater than any of the intermediate components — as being formed from the addition of all the intermediate ratios together — and consequently that all the intermediate ratios must be less than the ratio of the extremes. But this is false, and altogether contrary to Euclid, as is clear from the preceding example, which we drew from Proposition 23 of this book. For, given these three numbers, 24, 9, 18, which have among themselves in order the same ratios which the sides of one parallelogram have to the sides of the other, the ratio of parallelogram to parallelogram will be the same as that of 24 to 18, as Euclid demonstrates. But who does not see that the ratio of 24 to 18, which is sesquitercian, is much less

than the ratio of 24 to 9, which is double-and-two-thirds [*dupla superbipartiens tertias*, i.e. 8:3]? But on this matter we shall write more at the end of Book 9.

Since this is so, if someone wishes to have any number of ratios composing a given ratio — that is, whose denominators, multiplied among themselves, generate the denominator of the given ratio — then between two numbers of the given ratio (whatever they be) there must be set as many middle numbers (whatever they be) as there are component ratios desired, less one. For instance, if someone wants three ratios out of which the centuple ratio is compounded, the question is satisfied by placing two middle numbers between 100 and 1, thus: 100, 50, 10, 1. For the ratio of the extremes, 100 to 1, is compounded of the three intermediate ratios — that is, of the double, quintuple, and decuple. Likewise, if you set other middle numbers, thus: 100, 10, 3, 1, the same ratio 100 to 1 will be compounded of the decuple, the triple-sesquitercian, and the triple. So too you see here, in 2, 3, 1, 10, 1, that the double ratio is compounded of four — namely of the subsesquialterate, the triple, the subdecuple, and the decuple.

But if someone asks for any number of ratios composing a given ratio, all of which are also given except one, we shall continue from one extreme of the given ratio all the given component ratios in order. For these, together with the ratio which the last middle number has to the other extreme of the given ratio, will compose the given ratio. For example, if someone says: give me five ratios, four of which are the double, triple, sesquialterate, and quintuple-sesquitercian, composing the sesquialterate ratio between 9 and 6 — we shall set these four middle numbers, giving 9, $4\frac{1}{2}$, $1\frac{1}{2}$, 1, $\frac{3}{16}$ [?], 6, so that 9 to $4\frac{1}{2}$ has the double ratio, $4\frac{1}{2}$ to $1\frac{1}{2}$ the triple, $1\frac{1}{2}$ to 1 the sesquialterate, and 1 to $\frac{3}{16}$ [?] the quintuple-sesquitercian. For out of these four, together with the ratio of $\frac{3}{16}$ [?] to 6 — which is the same as 1 to 32 — the sesquialterate ratio 9 to 6 is compounded. The same will result if the four given ratios are continued from the lesser extreme, 6, thus: 6, 12, 36, 54, 288, 9, so that 12 to 6 has the double ratio, 36 to 12 the triple, 54 to 36 the sesquialterate, and 288 to 54 the quintuple-sesquitercian. For the fifth ratio, 9 to 288, is the same as the earlier $\frac{3}{16}$ [?] to 6, which is as 1 to 32.

From these things we shall, with no trouble, satisfy the question in which someone demands to be given any number of numbers which, multiplied among themselves, produce some given number. For if two numbers are taken having a ratio denominated by the given number, and between them are placed as many numbers (whatever they be), less one, as the numbers desired, then the denominators of the intermediate ratios will be the numbers sought. For these,

multiplied among themselves, produce the denominator of the ratio of the extremes, as has been demonstrated — that is, the given number. For instance, if someone asks for three numbers from whose mutual multiplication 100 is generated, we shall place, between two numbers having a centuple ratio (500 and 5), two middle numbers whatever they be, thus: 500, 250, 50, 5. And the three numbers sought will be 2, 5, 10 — namely the denominators of the ratios 500 to 250, 250 to 50, and 50 to 5. For from 2 into 5 comes 10, and from 10 into 10 comes 100.

But if someone demands a single number which, together with any number of others already given, when they are multiplied among themselves, produces some given number: again two numbers must be taken whose ratio is denominated by the given number that is to be produced, and from one of the two the ratios denominated by the other given numbers must be continued. For the denominator of the ratio between the other number and the last of those continued will be the number we seek. Thus, if these four numbers are given — 2, 3, $1\frac{1}{2}$, $5\frac{1}{3}$ — and a fifth is sought which, multiplied into the number produced from their mutual multiplication, generates this given number, $1\frac{1}{2}$: we shall take two numbers, 9 and 6, having the sesquialterate ratio denominated by the given number $1\frac{1}{2}$; and from 9 we shall continue the four ratios denominated by 2, 3, $1\frac{1}{2}$, $5\frac{1}{3}$, thus: 9, $4\frac{1}{2}$, $1\frac{1}{2}$, 1, $\frac{3}{16}$ [?], 6. For the fifth number sought will be the denominator of the ratio of $\frac{3}{16}$ [?] to 6 — namely $\frac{1}{32}$. For these five numbers — 2, 3, $1\frac{1}{2}$, $5\frac{1}{3}$, $\frac{1}{32}$ — of which the first four are given and the fifth is found, when multiplied among themselves produce the given number $1\frac{1}{2}$. The same fifth number, $\frac{1}{32}$, will be found if the four ratios denominated by the given numbers are continued from the number 6, thus: 6, 12, 36,

54.

288.

9. The denominator of the ratio of 9 to 288 is $\frac{1}{32}$, as before.

The same holds too if four numbers are sought which, multiplied together, produce 1. We set down, between any two equal numbers whose ratio of equality is denominated by 1, any three middle numbers we please — as you see here: 4, 2, 1, 3, 4. For the denominators of the ratios 4 to 2, 2 to 1, 1 to 3, and 3 to 4 — namely 2, 2, $\frac{1}{3}$, $\frac{3}{4}$ [?] — are the four numbers sought, since when multiplied together they generate 1.

Such problems are solved **more easily** if you multiply together as many arbitrary numbers, less one, as are required, and then divide the given number that is to be generated by the last number produced. For the quotient, together with the

numbers assumed and multiplied among themselves, will be the numbers we are looking for. Thus in the previous problem: if these three numbers, for example, 4, 5, 6, are multiplied together, and 1 (which is to be produced) is divided by the number they yield, 120, the quotient will be $1/120$. The four numbers sought are therefore 4, 5, 6, and $1/120$. For 4 into 5 makes 20, and 20 into 6 makes 120, and 120 into $1/120$ makes 1.

Finally, this too must not be passed over — namely: just as, when any number of terms are set down in order, the denominator of the ratio of the extremes is produced from all the denominators of the intermediate ratios (as we have shown), so also, when any number of terms are set down in order, yet in such a way that each following term is greater than the one before it, the difference of the extremes is accumulated from all the differences of the intermediate terms. For example, here: 3, 7, 12, 20, 30, 100, 713. The difference between 3 and 713 is 710. But the difference between 3 and 7 is 4; between 7 and 12 is 5; between 12 and 20 is 8; between 20 and 30 is 10; between 30 and 100 is 70; and finally between 100 and 713 is 613. All these differences — 4, 5, 8, 10, 70, 613 — together make up 710, the difference of the extremes.

And let this be what we have to say about the five definitions set down by Euclid in this sixth book. To them we judge that the following sixth should be added, which will do much to make propositions 27, 28, 29, and 30 of this book — and a great many others of the tenth book — more easily understood. It is as follows:

VI.

A parallelogram applied to a certain straight line is said to *fall short of* [deficere] a parallelogram when it does not occupy the whole line; and to *exceed* [excedere] when it occupies a greater line than the one to which it is applied — yet in such a way that the deficient or exceeding parallelogram has the same altitude as the parallelogram applied, and together with it constitutes one whole parallelogram.

Let there be a given straight line AB, upon which the parallelogram ACDE is constructed, which does not occupy the whole of AB but falls short by CB. Draw BF parallel to CD until, meeting ED extended, it reaches F, so that the whole parallelogram ABFE is completed. The parallelogram AD, applied along the straight line AB, is then said to fall short of the parallelogram DB — so that DB is called the *defect* [defectus].

Again, let there be a given straight line AC, upon which the parallelogram ABFE is constructed, having a side AB greater than the given straight line AC; and let CD

be drawn parallel to BF. The parallelogram AF, applied along the straight line AC, is then said to exceed by the parallelogram DB — so that DB is called the *excess* [excessus].

Now this defect or excess DB may, in the case of right-angled figures, be either a square or an oblong [figure longer on one side]; and in the case of non-right-angled figures, either a rhombus or a rhomboid, as is plain.

Diagram. A leaning parallelogram (rhomboid) illustrating the applied, deficient, and exceeding parallelograms of Definition VI. The base runs left to right; the top is parallel to it, and an internal vertical divides the whole into two parts.

- A (bottom-left corner): start of the base line
- C (bottom, middle): division point on the base
- B (bottom-right corner): end of the base line
- E [?] (top-left corner): above A
- D (top, middle): above C, foot of the internal dividing line
- F (top-right corner): above B
- (Whole figure ABFE; ACDE is the applied parallelogram, CBFD the defect/excess.)

THEOR. 1. PROPOS. 1.

Triangles and parallelograms which have the same altitude are to one another as their bases.

Let there be two triangles ABC and DEF having the same altitude, whose bases are BC and EF; likewise two parallelograms CG and EH of the same altitude, whose bases are the same BC and EF. I say that triangle ABC is to triangle DEF, and parallelogram CG is to parallelogram EH, as the base BC is to the base EF. That is: if the base BC is set as the first magnitude and the base EF as the second, while triangle ABC (or parallelogram CG) is the third and triangle DEF (or parallelogram EH) the fourth, then equimultiples of the first and third must, together, either fall short of, or be equal to, or exceed the equimultiples of the second and fourth — as definition 6 of Book 5 requires.

For let the triangles and the parallelograms alike be placed between the same parallels GH and LN. (For, as was said in definition 4, triangles and parallelograms will have the same altitude only when they are constructed between the same parallels, since the perpendiculars dropped from the vertices to the bases will then be equal.) On BL take any number of straight lines BI, IK, KL

equal to BC; likewise on FN mark off any number of straight lines FM, MN equal to EF. Then from A and from D draw the straight lines AI, AK, AL and DM, DN.

[Margin: a — I.38] The triangles AIB, AKI, ALK, constructed upon equal bases and between the same parallels, are equal to one another; and by the same reasoning the triangles DEF, DFM, DMN will be equal. Therefore, whatever multiple the straight line CL is of the straight line BC, the triangle ACL will be the same multiple of the triangle ABC; and whatever multiple the straight line EN is of the straight line EF, the triangle DEN will likewise be that multiple of the triangle DEF — since the whole triangles ACL and DEN are divided into just as many equal triangles as the whole straight lines CL and EN were cut into equal parts. [Margin: b — I.38] Now if the base CL is equal to the base EN, the triangle ACL is necessarily equal to the triangle DEN; and accordingly, if CL is greater than EN, then necessarily ACL is greater...

is greater than DEN; and if less, less. Therefore the equimultiples of the first magnitude BC and the third ABC — namely the straight line CL and the triangle ACL — will, together with the equimultiples of the second EF and the fourth DEF — namely the straight line EN and the triangle DEN — either be equal, or the one exceed the other, if those are taken which correspond to each other. [Margin: a — V, def. 6] Therefore the ratio that the first magnitude BC has to the second EF (base to base) is the same as that which the third ABC has to the fourth DEF (triangle to triangle). And so, as base is to base, so is triangle to triangle. Which was what was proposed.

Now [Margin: b — V.15] since triangle ABC is to triangle DEF as parallelogram CG [Margin: c — I.34] (which is double triangle ABC) is to parallelogram EH [Margin: d — I.34] (which is double triangle DEF), it is clear [Margin: e — V.11] that parallelogram is likewise to parallelogram as base is to base. This too can be confirmed by the same argument we used for the triangles, if first, from the points I, K, L, straight lines are drawn parallel to BG, and likewise from the points M, N, parallels to FH, and so on. Therefore triangles and parallelograms which have the same altitude are to one another as their bases. Which was to be demonstrated.

Diagram. The equimultiples figure for Proposition 1: a horizontal base line with equal segments marked off on either side, and two apex points from which lines fan down to the division points, all set between two parallels. (The same figure is repeated at the top of the following page.)

- Top line (left to right): G, A, D, H

- A (upper): common vertex of triangles ALK, AKI, AIB, ABC (over the left base)
- D (upper): common vertex of triangles DEF, DFM, DMN (over the right base)
- Base line (left to right): L, K, I, B, C, E, F, M, N
- BI, IK, KL equal to BC; FM, MN equal to EF

Diagram. The same equimultiples figure for Proposition 1, repeated at the top of this page.

- Top line (left to right): G, A, D, H
- A (upper): vertex over the left base (triangles ALK, AKI, AIB, ABC)
- D (upper): vertex over the right base (triangles DEF, DFM, DMN)
- Base line (left to right): L, K, I, B, C, E, F, M, N

SCHOLIUM.

But we shall also demonstrate the converse of this. Namely:

Triangles and parallelograms which are to one another as their bases have equal altitudes, or the same altitude.

Let triangle ABC be to triangle DEF, and parallelogram AGBC to parallelogram DEFH, as the base BC is to the base EF. I say that their altitudes — namely the perpendiculars AI and DK — are equal. For if they are not equal, let AI, if possible, be greater than DK. Cut off IL equal to DK, draw LM parallel to BC, cutting the side AC at N, and join the straight line BN. [Margin: a — VI.1] Then triangle NBC will be to triangle DEF as the base BC is to the base EF, since the altitudes LI and DK are assumed equal. But it was also the case that ABC is to DEF as BC is to EF. [Margin: b — V.11] Therefore the triangles NBC and ABC have the same ratio to triangle DEF; [Margin: c — V.9] and accordingly they are equal to one another — the part and the whole. Which is absurd. The altitudes AI and DK are therefore not unequal, but equal. The same reasoning holds for parallelograms: for by a similar argument we shall show that the parallelograms NMBC and AGBC are equal, if the altitudes AI and DK are said to be unequal while LI and DK are equal.

Here Federico Commandino [Federicus Commandinus] adds another theorem, which we shall demonstrate more briefly. Namely:

Diagram. Two figures for the scholium's first converse: on the left a parallelogram AGBC with the inscribed triangle NBC and the construction lines of

the altitude; on the right a parallelogram DEFH with its altitude DK. Several small interior labels are hard to resolve. []

- Left figure — G (top-left), A (top-right), B (bottom-left), C (bottom-right)
- I [?] (base, foot of perpendicular AI)
- M [?], L [?], N [?] (interior points of the construction LM/IL/N)
- Right figure — D (top-left), H (top-right), K (bottom-left, foot of DK), E, F (base)

Triangles and parallelograms whose bases are equal, or the same, are to one another as their altitudes.

Let there be two triangles ABC and DEF, and parallelograms AGBC and DEFH, having equal bases BC and EF. I say that triangle ABC is to triangle DEF, and parallelogram AGBC to parallelogram DEFH, as the altitude AI is to the altitude DK. For if the straight lines IL and KM are taken equal to the bases BC and EF, and the straight lines LA and MD are drawn, [Margin: d — I.38] then triangle ALI will be equal to triangle ABC, since they stand on equal bases LI and BC and between the same parallels AG and IB. In the same way triangle DKM will be equal to triangle DEF. [Margin: e — V.7] Therefore, as ABC is to DEF, so will ALI be to DKM. [Margin: f — VI.1] But as ALI is to DKM, so is AI to DK. (For if AI and DK are set as the bases, the equal straight lines LI and KM will be the altitudes.) Therefore ABC will be to DEF as AI is to DK. Which was what was proposed.

And [Margin: g — V.15] since ABC is to DEF as parallelogram AGBC [Margin: a — I.34] (which is double triangle ABC) is to parallelogram DEFH [Margin: b — I.34] (which is double triangle DEF), [Margin: c — V.11] AGBC will also be to DEFH as AI is to DK. This too can be confirmed in the same way if the straight lines LG and MH are drawn. The same will follow if the triangles and parallelograms have the same base.

This too we shall convert, in the following manner.

Diagram. Two figures for the scholium's second converse: on the left a triangle ABC with the parallelogram AGBC and its equal-base construction; on the right a triangle DEF with the parallelogram DEFH.

- Left figure — G (top-left), A (top-right), B (bottom-left), L, C, I (base, left to right)
- Right figure — D (top-left), H (top-right), K, F, M (base, left to right)

Triangles and parallelograms which are to one another as their altitudes have equal bases, provided they do not have one and the same base.

Let triangle ABC be to triangle DEF, and parallelogram AGBC to parallelogram DEFH, as the altitude AI is to the altitude DK. I say that their bases BC and EF are equal. For if they are not equal, let BC, if possible, be greater than EF. Cut off BL equal to EF and draw the straight line LA. Then, as we have just demonstrated, ABL will be to DEF as AI is to DK. But AI is to DK, by hypothesis, as ABC is to DEF. [Margin: d — V.11] Therefore ABL and ABC have the same ratio to DEF; [Margin: e — V.9] and accordingly they are equal to one another — the part and the whole. Which is absurd. The bases BC and EF are therefore not unequal, but equal. The same reasoning applies to parallelograms: for by a similar argument, drawing LM parallel to GF, we shall show that the parallelograms MGBI and AGBC are equal, if the bases BC and EF are said to be unequal while BL and EF are equal.

In all these cases the demonstration will not vary even if the triangles and parallelograms are right-angled — so that their sides are the altitudes — or if one is right-angled and the other not. We have taken the more difficult case, namely when neither is right-angled; for in that case the demonstration sometimes requires a larger construction.

Diagram. Two figures for the scholium's third converse: on the left a parallelogram AGBC with the inscribed parallelogram MGBI on the reduced base; on the right the parallelogram DEFH.

- Left figure — G (top-left), M (top), A (top-right); B, L, C (base, left to right); I (?) (base, foot of the altitude)
- Right figure — D (top-left), H (top-right); K, E, F (base, left to right)

THEOR. 2. PROPOS. 2.

[Margin: ij.]

If a straight line is drawn parallel to one side of a triangle, it will cut the sides of the triangle proportionally. And if the sides of a triangle are cut proportionally, then the straight line joined to the points of section will be parallel to the remaining side of the triangle.

In the triangle ABC let there first be drawn the straight line DE parallel to the side BC. I say that the sides AB and AC are cut proportionally at D and E — that is, that AD is to DB as AE is to EC. For when the straight lines CD and BE are drawn, [Margin: a — I.37] the triangles DEB and DEC, constructed on the same base DE and between the same parallels DE and BC, are equal to one another. [Margin: b — V.7] Therefore, as triangle ADE is to triangle DEB, so is the same triangle ADE to triangle DEC. [Margin: c — VI.1] But as triangle ADE is to triangle DEB, so is the

base AD to the base DB (since these triangles are of the same altitude, as is evident if a straight line parallel to AB is drawn through E); and by the same reasoning, as triangle ADE is to triangle DEC, so is the base AE to the base EC. [Margin: d — V.11] Therefore AD is to DB as AE is to EC (since these two ratios are the same as the ratio of triangle ADE to triangle DEB, and of the same triangle ADE to triangle DEC). Which was what was proposed.

Next, let the straight line DE cut the sides AB and AC proportionally. I say that DE is parallel to the remaining side BC. For when the straight lines CD and BE are drawn again, [Margin: e — VI.1] the base AD will be to the base DB as triangle ADE is to triangle DEB, since they are of the same altitude. But it is assumed that AD is to DB as AE is to EC. [Margin: f — V.11] Therefore triangle ADE will be to triangle DEB as AE is to EC. But again [Margin: g — VI.1] the base AE is to the base EC as triangle ADE is to triangle DEC, since they are of the same altitude. [Margin: h — V.11] Therefore triangle ADE is to triangle DEB as the same triangle ADE is to triangle DEC. [Margin: i — V.9] The triangles DEB and DEC are therefore equal; and consequently, since they have the same base DE, [Margin: k — I.39] they will be placed between the same parallels. Therefore DE is parallel to BC. Which was what was proposed. And so, if a straight line is drawn parallel to one side of a triangle, and so on. Which was to be shown.

Diagram. Triangle ABC with DE drawn parallel to the base BC, and the two cross-diagonals CD and BE drawn (crossing inside the triangle).

- A (apex, top): vertex of the triangle
- E (left, mid-height): point on side AC
- D (right, mid-height): point on side AB, on line DE
- C (bottom-left): base vertex
- B (bottom-right): base vertex

THEOR. 3. PROPOS. 3.

[Margin: iij.]

If an angle of a triangle is bisected, and the straight line bisecting the angle also cuts the base, the segments of the base will have the same ratio as the remaining sides of the triangle. And if the segments of the base have the same ratio as the remaining sides of the triangle, then the straight line drawn from the vertex to the point of section bisects the angle of the triangle itself.

In the triangle ABC let the straight line AD first bisect the angle BAC. I say that BA is to AC as BD is to DC. For through B draw the straight line BE parallel to AD, until it meets CA extended at E. (BE and CA will certainly meet, because the angles C and CBE are less than two right angles. For since C and CDA [Margin: a — I.17] are less than two right angles, [Margin: b — I.29] and the angle CDA is equal to the angle CBE, exterior to interior, the angles C and CBE will also be less than two right angles.) [Margin: c — I.29] And the angle EBA will be equal to the alternate angle BAD, and the angle E to the exterior angle DAC. Since, then, the two angles BAD and DAC are assumed equal, the angles EBA and E will also be equal to one another; [Margin: d — I.6] and therefore the straight lines BA and EA are equal to one another. [Margin: e — V.7] Therefore EA is to AC as BA is to the same AC. [Margin: f — VI.2] But EA is to AC as BD is to DC, since in the triangle BCE the straight line AD is parallel to the side BE. [Margin: g — V.11] Therefore BA is to AC as BD is to DC. Which was what was proposed.

Next, let BA be to AC as BD is to DC. I say that the straight line AD bisects the angle BAC. For again, through B, draw the straight line BE parallel to AD, meeting CA extended at E. Since, then, BA is to AC as...

Diagram. Triangle for Proposition 3: the bisector AD from the vertex to the base, with BE drawn from B parallel to AD meeting CA extended at E.

- E (apex, top): meeting point of BE with CA extended
- A (right, mid-height): vertex whose angle BAC is bisected
- B (bottom-left): base vertex
- D (bottom, middle): point where the bisector meets the base
- C (bottom-right): base vertex

THEOR. 3. PROPOS. 3. — conclusion of the demonstration

as AC, so is BD placed to DC. [Margin: Elements VI.2] But BD is to DC as EA is to AC — for in triangle BCE the line AD is drawn parallel to side BE. [Margin: Elements V.11] Therefore BA is to AC as EA is to the same AC; [Margin: Elements V.9] and so BA and EA [printed: "EB"; recte EA] are equal to one another. [Margin: Elements I.5] Consequently the angles ABE and E [that is, AEB] will also be equal. [Margin: Elements I.29] Since, then, angle ABE equals the alternate angle BAD, and angle E equals the exterior angle DAC, the two angles BAD and DAC will likewise be equal to one another — which is the proposition. And so, if an angle of a triangle is bisected, and so on. Which was to be demonstrated.

THEOR. 4. PROPOS. 4.

[Margin: iij.]

In equiangular triangles the sides about the equal angles are proportional, and those are the corresponding (homologous) sides which subtend the equal angles.

Let ABC and DCE be equiangular triangles, with angle ABC equal to angle DCE, angle ACB equal to angle DEC, and angle BAC equal to angle CDE. I say that AB is to BC as DC is to CE; that BC is to CA as CE is to ED; and finally that AB is to AC as DC is to DE. For this is what it means for the sides about the equal angles to be proportional, and for the corresponding sides to be those which subtend the equal angles — that is, all the antecedents look toward equal angles, and the consequents likewise.

Let the sides BC and CE be set along a single straight line, so that the exterior angle DCE is equal to the interior angle ABC, and likewise the exterior angle ACB to the interior angle DEC. Since the two angles ABC and ACB [Margin: Elements I.17] are less than two right angles, and angle DEC is equal to angle ACB, the angles B and E will also be less than two right angles. [Margin: 13. prop.] Therefore the straight lines BA and ED, produced toward the sides A and D, will meet. Let them be produced, then, and let them meet at F.

Now since the exterior angle DCE is equal to the interior and opposite angle ABC, [Margin: Elements I.28] the lines CD and BF will be parallel; [Margin: Elements I.34] and by the same reasoning CA and EF will be parallel, because the exterior angle ACB is equal to the interior angle DEC. ACDF is therefore a parallelogram, and consequently AF equals CD, and CA equals DF.

Since, then, in triangle BEF the line AC is parallel to side EF, [Margin: Elements VI.2] AB will be to AF — that is, to DC (which is equal to AF) — as BC is to CE. [Margin: Elements V.16] Therefore, alternately, AB is to BC as DC is to CE. Again, since in the same triangle BEF the line CD is parallel to side BF, [Margin: Elements VI.2] BC will be to CE as FD — that is, as CA (which is equal to FD) — is to ED. [Margin: Elements V.16] Therefore, alternately, BC is to CA as CE is to ED. Since, then, AB is to BC as DC is to CE, and BC is to CA as CE is to ED, [Margin: Elements V.22] it will follow, ex aequali, that AB is to CA as DC is to ED. Which is the proposition. Therefore in equiangular triangles the sides are proportional, and so on. Which was to be demonstrated.

Diagram. The figure for Proposition 4. Along the base lie B (left), C (middle), and E (right) on one straight line. From B the left side rises through A to the apex F; from E the right side rises through D to F. The segment AD is joined across; CA and CD are drawn, forming the parallelogram ACDF.

- F (apex): meeting-point of the produced sides BA and ED
- A (on BF): vertex of triangle ABC
- D (on FE): vertex of triangle DCE
- B (base, left): endpoint
- C (base, middle): shared vertex of the two triangles
- E (base, right): endpoint

COROLLARIUM.

From this it follows that a straight line drawn parallel to one side of a triangle cuts off a triangle similar to the whole. For in triangle ABC let DE be drawn parallel to side BC. I say that triangle ADE is similar to triangle ABC. They are equiangular, [Margin: Elements I.29] since the angles ADE and AED are equal to the exterior angles ABC and ACB, and angle A is common. Therefore, as has been demonstrated, [Margin: Elements VI.4] they have the sides about the equal angles proportional; and consequently, by the definition, they are similar.

Diagram. The figure for the corollary. Triangle ABC with apex A at the top; B at the lower left and C at the lower right form the base BC. The line DE is drawn parallel to BC, with D on side AB and E on side AC, cutting off the similar triangle ADE.

- A (apex): common vertex
- D (on AB): left end of the parallel
- E (on AC): right end of the parallel
- B (base, left): endpoint
- C (base, right): endpoint

SCHOLIUM.

I think it not foreign to our purpose to demonstrate here two theorems. The first of these Federico Commandino [Federico Commandino] proves in his little book on Archimedes' *On Floating Bodies* [Archimedes, *De iis quae vehuntur in aqua*]; the second, in his commentary on Apollonius's *Conics* [Apollonius, *Conica*]. The first is as follows.

If from two points of any straight line — one of them an endpoint, the other lying within the line — two mutually parallel lines are drawn toward the same side, so that they have the same ratio as the segments of the straight line enclosed between them and the other endpoint, then the straight line joining the extremity of the one to the endpoint of the original line will pass through the extremity of the other line.

Let AB be a straight line, and from the points A and C let two parallels AD and CE be drawn, having the same ratio as AB has to BC — so that AD is to CE as AB is to BC; or CE is to AD as BC is to AB. I say that the straight line joining the extremities B and E passes through the point D. Likewise that the straight line joining the extremities B and D passes through the point E. For if the straight line BE does not pass through D, let it meet AD at the point F, which will be either above D or below it, as in the first figure. Since, then, by the corollary of this proposition the triangles BAF and BCE are similar, [Margin: Elements VI.4] AB will be to AF as BC is to CE; and, alternately, [Margin: Elements V.16] AB is to BC as AF is to CE. But AB is to BC as AD is to CE. [Margin: Elements V.11] Therefore AF is to CE as AD is to CE, [Margin: Elements V.9] and consequently the lines AF and AD will be equal — the part and the whole. Which is absurd. Therefore the straight line BE passes through the point D. Which was the first point.

Diagram. The first case ("prima figura") of the scholium's first theorem. Along the base lie A (left), C (middle) and B (right). From A the line AD rises; from C the parallel CE rises. Lines are drawn from B to D and from B to E; where BE meets AD (produced) the auxiliary point F is marked above D. Labels legibly read: A, B, C, D, E, F. The precise placement of the two interior labels (E and the auxiliary F) is hard to resolve at this resolution.

- A (base, left): endpoint of AB
- C (base, middle): interior point of AB
- B (base, right): endpoint
- D (top of AD): extremity of the first parallel
- E (on CE): extremity of the second parallel [?]
- F (above D): intersection of BE with AD produced [?]

[\[continuation on the following page\]](#)

Again, if the straight line BD does not pass through E, let it pass through F, a point either above E or below it, as in the second figure. Since, then, by the corollary of this proposition the triangles BAD and BCF are similar, [Margin: Elements VI.4] AB will be to AD as BC is to CF; and, alternately, [Margin: Elements V.16] AB is to BC

as AD is to CF. But AB was to BC also as AD is to CE. [Margin: Elements V.11] Therefore AD is to CF as AD is to CE; [Margin: Elements V.9] and consequently the lines CF and CE will be equal — the part and the whole. Which is absurd. Therefore the straight line BD passes through E. Which was the second point.

Diagram. The second case ("secunda figura") of the scholium's first theorem. Along the base lie A (left), C (middle) and B (right). From A rises AD (with D at the top), from C rises the parallel CE; lines BD and BE are drawn, and the auxiliary point F falls below D on the line AD. Labels legibly read: A, B, C, D, E, F; the two interior labels (E and the auxiliary F) are hard to resolve.

- A (base, left): endpoint
- C (base, middle): interior point
- B (base, right): endpoint
- D (top of AD): extremity of the first parallel
- E (on CE): extremity of the second parallel [?]
- F (below D): intersection point [?]

This also holds when the two parallels are drawn to opposite sides, having the same ratio as the segments of the straight line lying between the neighbouring parallels and one and the same point within the given line. Thus, as in the figure of Problem 7 of the scholium to Proposition 17: if the parallels AE and FC are drawn to opposite sides, and AE is to CF as AG is to CG, then the straight line EF will pass through G — that is, the two lines EG and FG will form a single straight line. For since AE is to CF as AG is to CG, then, alternately, AE is to AG as CF is to CG. And since the alternate angles A and C are equal, [Margin: Elements VI.6] the angles AGE and CGF will also be equal. Therefore, since AC is a straight line, EG and FG will likewise form one straight line — by what we demonstrated from Proclus [Proclus] on Proposition 15 of Book 1. That was the first theorem; the second is as follows.

Diagram. A crowded construction borrowed from the figure of Problem 7 of the scholium to Proposition 17. Around a central region are grouped many lettered points; those legibly read are D (top), K (below D, left), F (upper middle), H (below F), I (below H), A (left), M (right of A), G (below M), C (right of G), B (far right), L (below A), E (bottom). The dotted and solid lines form several overlapping triangles and parallels.

- D (top): vertex
- K (upper left): vertex [?]
- F (upper middle): vertex

- H (below F): interior point [?]
- I (below H): interior point [?]
- A (left): vertex
- M (right of A): interior point [?]
- G (lower middle): interior point
- C (right of G): vertex
- B (right): vertex
- L (lower left): vertex [?]
- E (bottom): vertex

If in any triangle a straight line is drawn parallel to one side, and from any point of that side a straight line is drawn to the opposite angle, then the parallel line and that side will be divided in the same ratios.

In triangle ABC let DE be drawn parallel to side BC, and from any point F let the straight line FA be extended to angle A, cutting DE at G. I say that BF is to FC as DG is to GE. For since triangles AFB and AGD are similar (by the corollary of this proposition), [Margin: Elements VI.4] AF will be to BF as AG is to DG; [Margin: Elements V.16] and, alternately, AF is to AG as BF is to DG. By the same argument we shall conclude that AF is to AG as FC is to GE. [Margin: Elements V.11] Therefore BF is to DG as FC is to GE; and, alternately, BF is to FC as DG is to GE. Which was the proposition.

Alternatively: since triangles ABF and ADG are similar, and likewise triangles AFC and AGE (by the corollary of this proposition), [Margin: Elements VI.4] BF will be to FA as DG is to GA; likewise FA is to FC as GA is to GE. [Margin: Elements V.22] Therefore, ex aequali, BF is to FC as DG is to GE. Which was to be demonstrated.

Diagram. The figure for Clavius's added theorem. Triangle ABC with apex A; the base runs B (left) — F (middle) — C (right). The line DE is drawn parallel to BC, with D on side AB and E on side AC; the line FA rises from F to the apex A, cutting DE at G.

- A (apex): opposite angle
- D (on AB): left end of the parallel
- G (on DE): point of section on the parallel
- E (on AC): right end of the parallel
- B (base, left): endpoint
- F (base, middle): point of section on the side
- C (base, right): endpoint

THEOR. 5. PROPOS. 5.

[Margin: v.]

If two triangles have their sides proportional, the triangles will be equiangular, and will have equal those angles under which the corresponding sides are subtended.

Let triangles ABC and DEF have their sides proportional, so that AB is to BC as DE is to EF, BC is to CA as EF is to FD, and finally AB is to AC as DE is to DF. I say that the triangles are equiangular — namely that angle A equals angle D, angle B equals angle E, and angle C equals angle F; for in this way equal angles subtend the corresponding sides.

On EF construct angle FEG equal to angle B, and angle EFG equal to angle C, with the lines EG and FG meeting at G. [Margin: Elements I.32] Then the remaining angle G will be equal to the remaining angle A. Triangles ABC and GEF are therefore equiangular. [Margin: Elements VI.4] Hence AB is to BC as GE is to EF; but AB is to BC as DE is to EF (by assumption). [Margin: Elements V.11] Therefore GE is to EF as DE is to EF; [Margin: Elements V.9] and consequently GE and DE will be equal. Again, [Margin: Elements VI.4] since BC is to CA as EF is to FG — but BC is to CA as EF is to FD (by assumption) — [Margin: Elements V.11] EF is to FG as the same EF is to FD; [Margin: Elements V.9] and therefore FG and FD will be equal.

And so, since the sides EG and FG are equal to the sides DE and DF, each to each, and the base EF is common, [Margin: Elements I.8] the angles G and D will be equal; [Margin: Elements I.4] and consequently the remaining angles GEF and GFE will also equal the remaining angles DEF and DFE. Therefore, since angle G equals angle A, angle D too will equal the same angle A; and in the same way angle DEF will equal angle B, and angle DFE will equal angle C. Which is the proposition. If, then, two triangles have their sides proportional, and so on. Which was to be shown.

Diagram. The figure for Proposition 5. On the left, triangle ABC (A apex, B lower left, C lower right). On the right, the constructed figure: D at the top, E at the left and F at the right (with EF horizontal), and G below, forming triangle GEF beneath DEF.

- A (apex, left triangle): vertex
- B (lower left): vertex
- C (lower right): vertex

- D (top, right figure): vertex
- E (left): vertex of the base EF
- F (right): vertex of the base EF
- G (below EF): apex of the constructed triangle GEF

THEOR. 6. PROPOSITIO 6.

[Margin: vj.]

If two triangles have one angle equal to one angle, and the sides about the equal angles proportional, the triangles will be equiangular, and will have those angles equal under which the corresponding sides are subtended.

Let angle B of triangle ABC be equal to angle E of triangle DEF, and let the sides AB, BC be proportional to the sides DE, EF — that is, let AB be to BC as DE is to EF. I say that the remaining angles are equal to the remaining angles: namely angle A to angle D, and angle C to angle F; for in this way the equal angles subtend corresponding sides.

On EF construct angle FEG equal to angle B, and angle EFG equal to angle C; then, as was stated in the preceding proposition, triangle GEF will be equiangular with triangle ABC. [Margin: Elements VI.4] Hence AB is to BC as GE is to EF; but AB is to BC as DE is to EF (by assumption). [Margin: Elements V.11] Therefore DE is to EF as GE is to the same EF; [Margin: Elements V.9] and so DE and GE will be equal.

And so, since the sides DE, EF are equal to the sides GE, EF, and the angles contained by them are also equal — for angle DEF is set equal to angle B, to which angle FEG was made equal, and therefore the angles DEF and GEF are equal to one another — [Margin: Elements I.4] the remaining angles D and EFD will be equal to the remaining angles G and EFG. Then, since angle G equals angle A and angle EFG equals angle C, the angles D and EFD will also equal angles A and C; and for that reason triangles ABC and DEF are equiangular. Which is the proposition. If, then, two triangles have one angle equal to one angle, and so on. Which was to be demonstrated.

Diagram. The figure for Proposition 6. On the left, triangle ABC (A apex, B lower left, C lower right). On the right, the constructed figure: D at the top, E at the left and F at the right (EF horizontal), and G below, forming triangle GEF beneath triangle DEF.

- A (apex, left triangle): vertex

- B (lower left): vertex
- C (lower right): vertex
- D (top, right figure): vertex
- E (left): vertex of base EF
- F (right): vertex of base EF
- G (below EF): apex of the constructed triangle GEF

THEOR. 7. PROPOSITIO 7.

If two triangles have one angle equal to one angle, the sides about other angles proportional, and each of the remaining angles either both less than a right angle or both not less than a right angle, the triangles will be equiangular and will have equal those angles about which the sides are proportional.

Let angle A of triangle ABC be equal to angle D of triangle DEF, and let the sides AC, CB about angle ACB be proportional to the sides DF, FE about angle F — that is, let AC be to CB as DF is to FE — yet under this condition, that each of the remaining angles B and E is either less than a right angle or not less than a right angle. I say that the triangles are equiangular: that the angles ACB and F, about which the sides are proportional, and the angles B and E, are equal.

First let both B and E be less than a right angle. On this supposition, if the angles ACB and F are not equal, let ACB be the greater; and construct angle ACG equal to F. Then, since angle A is also assumed equal to angle D, [Margin: Elements I.32] the remaining angle AGC will equal the remaining angle E, and so triangles AGC and DEF will be equiangular. [Margin: Elements VI.4] Hence AC is to CG as DF is to EF; but DF is to FE as AC is to CB (by assumption). [Margin: Elements V.11] Therefore AC is to CG as the same AC is to CB; [Margin: Elements V.9] and consequently CG and CB will be equal, [Margin: Elements I.5] and the angles CBG and CGB equal. Now since angle B is assumed less than a right angle, CGB too will be less than a right angle, and therefore its adjacent angle AGC will be greater than a right angle, [Margin: Elements I.13] since AGC and CGB together equal two right angles. But angle AGC was shown to equal angle E. Angle E therefore is also greater than a right angle; yet it was assumed to be less than a right angle. Which is absurd.

Next let both B and E be not less than a right angle. Then, as before, angle B will equal angle CGB; and consequently the angles CBG and CGB in triangle BCG will not be less than two right angles, but either greater than or equal to two right angles. Which is absurd, [Margin: Elements I.17] for they are less than two right

angles. Therefore the angles ACB and F are not unequal, but will be equal — which is the proposition — and [Margin: Elements I.32] consequently the remaining angles B and E will also be equal. If, then, two triangles have one angle equal to one angle, and so on. Which was to be demonstrated.

Diagram. The figure for Proposition 7. On the left, triangle ABC (A apex, B lower left, C lower right); from a point G on the left side a line is drawn to C, marking off the auxiliary triangle used in the reductio. On the right, triangle DEF (D apex, E lower left, F lower right).

- A (apex, left triangle): vertex, equal angle
- G (on side AB, lower): auxiliary point
- B (lower left): vertex
- C (lower right): vertex about which the sides are proportional
- D (apex, right triangle): vertex
- E (lower left): vertex
- F (lower right): vertex

SCHOLIVM.

Euclid added that each of the remaining angles B and E must be either less than a right angle or not less than a right angle. For otherwise, with the whole hypothesis remaining, it would not follow that the triangles are equiangular. Suppose, in the same triangle ABC, that CG is made equal to CB [Margin: Elements I.19] — which can happen when angle B is acute and greater than angle A, for then side CA will be greater than side CB. If, then, a circle is described with center C and radius CB, it will cut the straight line AB before it cuts AC, which is longer than CB. And this circle will necessarily cut AB, because, owing to the acute angle B, the line AB falls within it — since it is drawn below the tangent, which at B makes a right angle with CB, as is clear from Proposition 16 of Book 3 (Elements III.16). In that case the two triangles ABC and AGC will have one angle equal to one angle — indeed angle A in common — and about the other angles ACB and ACG the sides proportional; that is, [Margin: Elements V.7] AC is to CB as the same AC is to CG, since the lines CB and CG are assumed equal — and yet the triangles ABC and AGC are in no way equiangular, as is evident. This happens because it is not the case that both angles ABC and AGC are less than a right angle, or both not less. On the contrary, ABC is indeed less than a right angle, whereas AGC is greater than a right angle. For since CG and CB are equal sides, [Margin: Elements I.5] and therefore the angles CBG and CGB equal, [Margin: Elements I.17 (or I.32)]

each of them will be less than a right angle, [Margin: Elements I.13] and consequently AGC will be greater than a right angle.

[Margin: viij.]

THEOR. 8. PROPOS. 8.

If in a right-angled triangle a perpendicular is drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole triangle and to one another.

In triangle ABC let angle BAC be a right angle, and from it let the perpendicular AD be drawn to the base. I say that the triangles ADB and ADC are similar both to the whole triangle ABC and to one another. For since in triangles ABC and DBA the angles BAC and ADB are right angles, and angle B is common, [Margin: Elements I.32] the remaining angles ACB and DAB [printed: "DAE"; recte DAB] will also be equal. Triangle DBA is therefore equiangular with triangle ABC, [Margin: Elements VI.4] and consequently they will have the sides about the equal angles proportional, and so on; that is, CB is to BA as BA is to BD; BA is to AC as BD is to DA; and BC is to CA as BA is to AD. For in this way the corresponding sides are set opposite the equal angles, as Proposition 4 of this book requires. Hence triangle ADB is similar to the whole triangle ABC.

In the same way it is shown that triangle ADC is similar to the same triangle ABC. For the angles BAC and ADC are right angles, and angle C is common, [Margin: Elements I.32] and consequently the remaining angles ABC and CAD are equal. Hence [Margin: Elements VI.4] BC is to CA as CA is to CD; CA is to AB as CD is to DA; and CB is to BA as CA is to AD. For thus too the corresponding sides are set opposite the equal angles, as prescribed by Proposition 4 of this book. In just the same way it will be demonstrated that triangles ADB...

Diagram. The figure for Proposition 8. A right-angled triangle with the right angle at the apex A. The base runs C (lower left) — D (lower middle) — B (lower right); AD is the perpendicular dropped from A to the base, dividing the triangle into ADC (left) and ADB (right).

- A (apex): the right angle BAC
- C (base, left): vertex
- D (base, middle): foot of the perpendicular AD
- B (base, right): vertex

Book VI (continued)

Proposition 8 — demonstration concluded

[triangles] ADB and ADC [are similar to one another], since the angles ADB and ADC are right, the angles ABD and CAD have been shown equal, and likewise the angles BAD and ACD. And therefore [Margin: c. *Elements* VI.4] BD is to CA as DA is to DC; DA is to AB as DC is to CA; and AB is to BD as CA is to AD. If, then, in a right-angled triangle a perpendicular is drawn from the right angle to the base, and so on. Which was to be demonstrated. [Margin: d. *Elements* VI.4]

COROLLARIUM (Corollary)

From this it is evident that the perpendicular dropped in a right triangle from the right angle to the base is the mean proportional [*media proportionalis*] between the two segments of the base; and likewise that each of the sides containing the right angle is the mean proportional between the whole base and that segment of the base which is adjacent to that side.

For it has been shown that BD is to DA as DA is to DC, and therefore that DA is the mean proportional between BD and CD. Likewise, that CB is to BA as BA is to BD, and therefore that BA is the mean proportional between CB and BD. Finally, that BC is to CA as CA is to CD, and therefore that CA is the mean proportional between BC and CD. Which was proposed.

PROBL. 1. PROPOS. 9. (Problem 1, Proposition 9)

[Margin: ix. [?]]

To cut off a prescribed part from a given straight line.

Suppose we are required to cut off a third part from the line AB. From A draw a straight line AC at any angle, forming the angle CAB. On AC mark off as many equal parts, of any size whatever, as is the number of the part to be taken from AB — in the present example three: AD, DE, EF. Then from F to B draw the straight line FB, and through D draw DG parallel to it. I say that AG is the required third part of the straight line AB.

For since in the triangle ABF the straight line DG is parallel to the side FB, [Margin: a. *Elements* VI.2] FD is to DA as BG is to GA. Therefore, by composition [*componendo*], [Margin: b. *Elements* V.18] FA is to DA as BA is to GA. But FA is triple AD by construction; therefore BA is also triple AG, and so AG is the third

part of AB, which was required. We have therefore cut off from a given straight line the prescribed part. Which was to be done.

Diagram. The figure for Proposition 9: a triangle with the base AB running along the bottom and an oblique upper side AC.

- A (bottom-left vertex): apex of the angle CAB.
- G (on base AB, between A and B): the point marking off the required third part AG.
- B (bottom-right vertex): far end of the base.
- D (on AC, first division from A): AD is the first equal part.
- E (on AC, second division): DE is the second equal part.
- F (on AC, third division, near C): EF is the third equal part.
- C (top-right vertex): far end of the oblique side.
- FB is drawn, and DG is drawn parallel to it.

SCHOLIUM (Scholium)

But if the part to be cut off from AB is not an aliquot part, but one comprising several aliquot parts that do not make up a single one — for instance, one containing four elevenths of AB — then eleven equal parts must be marked off along AC as far as the point D; from D draw the straight line DB to B, and parallel to it draw EF from E, the end of the fourth part. Then AF will be the required part.

For again, DA is to AE as BA is to AF. Therefore, by conversion [*convertendo*], AE is to AD as AF is to AB. But AE, by construction, is the part containing four elevenths of AD; therefore AF will be the same part of the straight line AB. Which was proposed. In just the same way there may be cut off from AB any part comprising any number of aliquot parts of the line that do not make up a single one.

Diagram. The figure for the Scholium: a long, shallow triangle with base AB and an oblique upper side AD marked off into eleven equal parts (indicated by tick marks).

- A (bottom-left vertex): apex of the angle.
- F (on base AB): foot of the parallel EF.
- B (bottom-right vertex): far end of the base.
- E (on AD, at the end of the fourth part from A): from E the line EF is drawn parallel to DB.
- D (upper-right, end of the eleven divisions): from D the line DB is drawn.

- A short mark labelled "q" [?] appears just beyond D at the upper right (uncertain — possibly a reference letter or line-extension).

PROBL. 2. PROPOS. 10. (Problem 2, Proposition 10)

[Margin: x. [?]]

To divide a given undivided straight line in the same way as another given straight line has already been divided.

Let AB be the straight line to be cut in the same way as the straight line AC has been cut at D and E — that is, into parts proportional to the parts AD, DE, EC. Join the two given lines at A, forming any angle BAC, and draw the connecting straight line BC. Then from D and E draw DF and EG parallel to BC. I say that the straight line AB is cut at F and G in the same way as AC is cut at D and E.

[Margin: a. *Elements* VI.2] For AD is to DE as AF is to FG; therefore the parts AF, FG are proportional to the parts AD, DE. And if DH is drawn parallel to FB, cutting EG at I, then again [Margin: b. *Elements* VI.2] DE is to EC as DI is to IH — that is, as FG is to GB — [Margin: c. *Elements* I.34] since FG is equal to DI, and GB to IH. Therefore the parts FG, GB will also be proportional to the parts DE, EC. And the same reasoning holds for more parts, if from E and C lines are drawn parallel to AB, and so on. Thus we have cut the given undivided straight line in the same way as the other given straight line was cut. Which was to be done.

Diagram. The figure for Proposition 10: a triangle with base AB, apex above, formed by the two given lines meeting at A.

- A (bottom-left vertex): apex of the angle BAC.
- F (on base AB): first division point.
- G (on base AB): second division point.
- B (bottom-right vertex): far end of the base.
- D (on the upper side): division point, from which DF is drawn parallel to BC.
- I (interior): intersection of DH with EG.
- H (upper right): endpoint of the line DH drawn parallel to FB.
- C (apex) [?] and E (further division on the upper side) [?]: named in the construction but not clearly resolved on the leaf.

(*Scholium to Proposition 10, continued.*) ...as is evident from the demonstration of this problem. And since AH has been divided into five equal parts, AB too will be

divided into the same number of equal parts. In just the same way the same straight line AB may be divided into more equal parts.

[The preceding scholium had run:] From this problem one can derive a very easy method and procedure for dividing a given straight line into any number of equal parts — which we undertook to do at proposition 10 of Book I, and which we have also demonstrated by another route at proposition 40 of Book I. For let AB be the straight line to be divided into five equal parts. Draw AC making any angle CAB with AB, and mark off along it five equal parts AD, DE, EF, FG, GH. Now since the straight line AH has been divided in this manner, if from H to B the line HB is drawn, and parallel to it, from the points D, E, F, G, the lines DI, EK, FL, GM are drawn, then AB will be divided in the same way as AH.

To this method of dividing a straight line into any number of equal parts, others, not unpleasing, may be added — as we promised at proposition 10 of Book I, and demonstrated at proposition 40 of Book I. For let AB again be the given straight line, to be divided into five equal parts. From B draw the straight line BC at any angle to AB. Then, from an assumed point D — whether above B or below it — draw DE parallel to AB, and mark off along it five equal parts DF, FG, GH, HI, IE, subject to this condition: that if D is above B, the line DE, made up of the five equal parts, is less than AB; but greater, if D lies below B. Finally, from A through E draw the straight line AC, meeting BC at the point C; and from C, through the points F, G, H, I, draw the lines CK, CL, CM, CN, which I say divide the straight line AB into five equal parts.

For since DE is parallel to AB, the lines EH and AM in the triangle CAM will be divided proportionally, by the scholium to proposition 4 of this book. Therefore, since EH is bisected at I, AM too will be bisected at N. By the same reasoning the line NL in triangle CNL, the line MK in triangle CMK, and the line LB in triangle CLB will each be bisected. Therefore AN is equal to NM, NM to ML, ML to LK, and LK to KB; and consequently AB is divided into five equal parts. Which was proposed.

Diagram. A large triangle with its apex at the top, from which lines fan down to the base. Two horizontal lines cross it.

- Apex (top): the meeting point C [?] (label not clearly resolved).
- Base row (bottom): A, N, M, L, K, B — the line AB divided at N, M, L, K.
- Middle row (a chord parallel to the base): E, I, H, G, F, D (left to right) — the line DE divided at I, H, G, F.
- A further row of labels A, N, M, L, K, B appears higher in the figure [?].

ALITER (Another method)

From the extreme points A and B draw two straight lines BC and AD parallel to one another — that is, make the angles A and B equal. From BC mark off four equal parts BE, EF, FG, GH, so that there are as many parts, less one, as the number into which the line is to be divided; and from AD cut off the same number of equal parts, AI, IK, KL, LM. Then, drawing the straight lines EM, FL, GK, HI, which cut AB at N, O, P, Q, I say that AB is divided into five equal parts.

For since GH, IK are equal and parallel, [Margin: d. *Elements* I.33] HI and GK will also be parallel; and by the same reasoning GK, FL, EM will be parallel. Therefore, since AM is divided into four equal parts, AQ too will be divided into four equal parts, as is evident from the demonstration of this Proposition 10. By the same reasoning BN will be divided into four equal parts, because BH is divided into the same number of equal parts. Therefore, since both AN and BQ are each equal to the individual parts NO, OP, PQ, all five parts AN, NO, OP, PQ, QB will be equal to one another. Which was proposed.

Diagram. A lattice figure: two oblique lines AD and BC parallel to one another, with the line AB crossed by the connecting lines to form a network of parallelograms.

- H, G, F, E (along BC, descending toward B): the divisions BE, EF, FG, GH.
- A, I, K, L, M (along AD): the divisions AI, IK, KL, LM.
- N, O, P, Q (on AB): the intersection points where EM, FL, GK, HI cross AB.
- B (right end of AB).

[Margin: d. *Elements* I.33]

ALITER (Another method — by an instrument)

First prepare an instrument suited to this purpose in the following way. Draw two straight lines CD, EF parallel to one another at any distance; from each mark off any number of equal parts — at least as many as the number of parts into which the line is to be divided — and join the corresponding pairs of points by straight lines. Then, with a compass, take the length of the line AB to be divided, and from some point of the line CE — say from E — transfer it onto the instrument to the point I, that is, to that transverse line which bounds as many spaces as the number of parts into which the line is to be divided; in the example this is the line GH, for it encloses five spaces. Finally, from E to I draw the straight line EI, which I say is divided into five equal parts.

For since GK, HL are equal and parallel, [Margin: e. *Elements* I.33] GH and KL will also be parallel; and in the same way all the lines between CD and EF will be shown parallel to one another. Therefore, as is clear from the demonstration of this Proposition 10, just as EG is divided into five equal parts, so likewise EI will be divided into the same number. For if the point taken is not the same as E or C, IE must be produced until it meets EF or DC; for then, as was proved in this Proposition 10, the lines EI, EG will be similarly divided. But if perhaps IE meets neither FE nor DC, but is parallel to both, [Margin: f. *Elements* I.34] the individual parts of the line EI will be equal to the individual parts of the line EG, because of the parallelograms enclosed between the parallels. Therefore, since all the parts of EG are equal, all the parts of EI will also be equal. If, then, by means of the compass the individual parts of the line EI are transferred onto the proposed line AB, equal to EI, then AB too will be divided into five equal parts. Which was proposed.

It is sometimes necessary to extend the parallel lines beyond the instrument. Thus, if the same length is transferred from M as far as N, a point on the line GH when it is produced, then MN too will be divided into five equal parts — as is clear, if from M a line is drawn parallel to EG as far as the line GH, or if NM is produced until it meets GE produced. If the line to be divided is very short, smaller portions of the parallels CD, EF should be taken, and so on. But if the line is somewhat long and is to be divided into few parts, it may be transferred across twice as many spaces, or three times, or four times as many, and so on — for two, three, or four spaces will then give one part. For instance, if it is to be divided into three parts and is transferred across six spaces, two spaces will give one part; if it is transferred across fifteen spaces, five spaces will give one part, because five spaces make a third of fifteen. Likewise, if a line to be divided into two equal parts is transferred across eighteen spaces, nine spaces will give one part, since nine spaces make half of eighteen; and so with the rest.

Diagram. The dividing instrument: two long, nearly parallel lines CD and EF, connected by a series of transverse "rungs" that enclose equal spaces (a ladder-like lattice).

- C, A (top): ends of the upper transverse line region.
- M (upper, on the top rung).
- L, K (on an intermediate rung).
- N, H, I, G, B (along the lower part): division points and the endpoints of the segments used.
- E, F, G, D: the frame lines CD, EF and the working line EG / EI.

[Margin: e. *Elements* I.33] [Margin: f. *Elements* I.34]

In the same way, on the analogy of this proposition, we may easily demonstrate the following problem, which is not seldom employed by geometers, and which is of this kind:

To cut a given straight line into two parts having any given ratio.

Let AB be the straight line to be cut into two parts having between them the ratio that the lines C and D have. From A draw a line AE making any angle A, and on it mark off AF equal to C, and FG equal to D. Then draw GB, and through F draw FH parallel to it. I say that AB is cut at H according to the ratio of C to D. This is manifest, [Margin: g. *Elements* VI.7 [?]] since, as AF is to FG — and therefore as C is to D — so is AH to HB.

Diagram. A triangle with two short reference segments (C and D) at the left, and the oblique construction line AE.

- A (top): apex of the angle.
- C, D (two short vertical segments at the left): the two given lines whose ratio is prescribed.
- H (on AB): the point of section.
- B (bottom-left): lower end of the line AB being cut.
- F (on AE): with AF equal to C.
- G (on AE, lower right): with FG equal to D; GB is drawn, and FH is drawn parallel to GB.
- E (bottom-right): far end of the oblique line AE.

Finally, we insert here a certain theorem, also bearing on the sections of lines, taken from Federico Commandino[’s commentary] on Archimedes’s little book on the things carried in water [Archimedes, *On Floating Bodies*]. Namely:

If two straight lines are each cut at two points proportionally, then the intermediate sections will also be in the same ratio as any two of the segments.

(*Commandino’s theorem, demonstration.*) Let the straight lines AB and CD be cut proportionally, each at two points, E, F and G, H, so that AE is to EB as CG is to GD, and likewise AF is to FB as CH is to HD. I say that the intermediate sections EF, GH are also proportional to any two of the segments FB, HD — or to the two AE, CG — that is, that FB is to EF as HD is to GH, and likewise that AE is to EF as CG is to GH.

For since AE is to EB as CG is to GD, by composition AB will be to EB as CD is to GD. Likewise, since AF is to FB as CH is to HD, by composition AB will be to FB as CD is to HD, and by conversion FB will be to AB as HD is to CD. Therefore, since FB is to AB as HD is to CD, and AB is to EB as CD is to GD, [Margin: h. *Elements* V.22] *ex aequali* FB will be to EB as HD is to GD; and by conversion EB will be to FB as GD is to HD; and by conversion of the ratio [*per conversionem rationis*] EB will be to EF as GD is to GH; and by division [*dividendo*] FB will be to EF as HD is to GH. Which was the first point.

Again, since AE is to EB as CD [recte: CG] is to GD, and EB is to EF as GD is to GH, as has been shown, *ex aequali* AE will be to EF as CG is to GH. Which was the second point.

The same will be shown more briefly in this way. Let the two points A and C be brought together, and let the straight lines DB, HF, GE be joined. Now since it is assumed that AE is to EB as CG is to GD, [Margin: i. *Elements* VI.2] GE will be parallel to DB. Again, since it is assumed that AF is to FB as CH is to HD, [Margin: k. *Elements* VI.2] HF too will be parallel to DB. [Margin: l. *Elements* I.30] Therefore GE and HF will be parallel to one another; and on that account [Margin: m. *Elements* VI.2] AE will be to EF as CG is to GH. Which was the second point.

In the same way, if the extreme points B, D are brought together into one, and so on, we shall show that BF is to EF as HD is to GH; and, by conversion, that EF is to FB as GH is to HD. Which was the first point.

The same holds even if the magnitudes AB, CD are of different kinds — as, for example, if AB is a surface and CD a line or a solid — as is clear from the earlier demonstration.

Diagram. Two related figures. First, two horizontal straight lines set one above the other; second, a triangle version (with A and C brought together).

Two parallel lines:

- Upper line: A — E — F — B, cut at E and F.
- Lower line: C — G — H — D, cut at G and H.

Triangle (A and C coincident):

- D (apex, top): from which DB descends.
- H, G (on the side AD, descending): with HF and GE drawn.
- A / C (bottom-left, coincident): apex of the triangle.
- E, F (on the base AB): the section points.

- B (bottom-right): far end of the base.

[Margin: h. *Elements* V.22]

Nor does it seem right to omit here the theorem that follows:

If a straight line is cut into any number of arithmetically proportional parts, and another straight line is cut into the same number of parts having the same ratios among themselves as those parts, then the parts of this [second] line too will be arithmetically proportional.

Let the straight line AB be cut into arithmetically proportional parts AC, CD, DB — that is, into parts having the same excess [*excessus*, i.e. common difference].

[Margin: n. *Elements* VI.10] Let another straight line EF be cut into parts EG, GH, HF proportional to those. I say that these parts too have the same excess — that is, that they are arithmetically proportional.

For let CI be equal to AC, and DK equal to CD, so that ID is the excess between CD and AC, and KB the excess between DB and CD, and therefore equal to ID. Let GL likewise be equal to EG, and HM equal to GH, so that LH is the excess between GH and EG, and MF the excess between HF and GH. It must be proved that the excesses LH, MF are equal.

Now since AC is to CD as EG is to GH, then by alternation AC is to EG as CD is to GH. Likewise, since CD is to DB as GH is to HF, then also by alternation CD is to GH as DB is to HF. And so, however many the parts, the parts of line AB will always have to the parts of line EF, each to each, the same ratio — even though the parts of a single line are not continuously proportional. Since, then, CD is to GH as AC is to EG — that is, as CI is to GL — [Margin: o. *Elements* V.19] the remainder ID will also be to the remainder LH as the whole CD is to the whole GH. Again, since DB is to HF as CD is to GH — that is, as DK is to HM — [Margin: p. *Elements* V.19] the remainder KB will also be to the remainder MF as the whole DB is to the whole HF, or as the subtracted DK is to the subtracted HM; that is, ID is to LH as CD is to GH. [Margin: q. *Elements* V.11] Therefore ID is to LH as KB is to MF. But ID is equal to KB; [Margin: r. *Elements* V.14] therefore LH will also be equal to MF. And on that account EG, GH, HF are arithmetically proportional, since they have equal excesses. Which was proposed.

Diagram. Two vertical scaled lines, side by side, marking the parts and their excesses.

- Left line (bottom to top): A, C, I, D, K, B — the line AB with parts AC, CD, DB, and the auxiliary points I (with CI = AC) and K (with DK = CD).

- Right line (bottom to top): E, G, L, H, M, F — the line EF with parts EG, GH, HF, and the auxiliary points L (with GL = EG) and M (with HM = GH).

[Margin: o. *Elements* V.19] [Margin: p. *Elements* V.19] [Margin: q. *Elements* V.11]
[Margin: r. *Elements* V.14]

This theorem cannot be converted in any way. For it does not follow, if two lines are arithmetically proportional, that the parts of the one have the same ratios as the parts of the other. For suppose that in line EF the second part is set as GM, so that the excess between EG and GM is LM; then, beyond M, take a portion equal to GM and add to it the same excess LM — three arithmetically proportional parts will be formed in the line EF, and yet they will not have the same ratios among themselves as the parts of line AB have: [Margin: s. *Elements* V.8] since the ratio of EG to GM is less than that of EG to GH — that is, than that of AC to CD.

From this another theorem is inferred:

If two unequal lines have the same ratio to two others — the greater to the greater — then the excess of the first pair will also be to the excess of the second pair as one of the first is to one of the second.

As in the figure above: since AC was to EG — that is, CI to GL — so was the excess ID to the excess LH, as CD is to GH. Therefore, if AC, CD are double (or half) the parts EG, GH, then ID too will be double (or half) of LH, and so on.

From these results we shall also solve this problem:

To cut a given straight line into any number of arithmetically proportional parts.

Let the straight line EF be cut into three arithmetically proportional parts. Take three lines AC, CD, DB, arithmetically proportional in any way, composing the straight line AB. Then if the given line EF is [Margin: t. *Elements* VI.10] cut similarly at G, H, as AB is cut at C, D, the parts EG, GH, HF will be arithmetically proportional, as has been demonstrated.

All of this is true also for numbers, and for any other magnitudes whatsoever, since the demonstration is always the same. And from these results we derived, in the tenth rule of arithmetical proportionality, the final method of distributing a given number into any number of arithmetically proportional parts. For by it a given number is divided into parts having the same ratios as the assumed numbers of the arithmetical proportion, since it is always arranged that, as the sum of the assumed numbers is to the given number, so each of the assumed

numbers is to another [i.e. to the corresponding part]. From this it follows that the assumed numbers have the same ratio to the parts of the given number that are found. Therefore, as has been shown here, the parts found are arithmetically proportional.

PROBL. 3. PROPOSITIO 11. (Problem 3, Proposition 11)

[Margin: xi. [?]]

Given two straight lines, to find a third proportional.

PROBL. 3. PROPOS. 11. (construction and demonstration, continued)

(The enunciation — "Given two straight lines, to find a third proportional" — was set out at the end of the preceding pages; the construction opens here.)

Let AB and AC be two straight lines so arranged as to form any angle A, and let a third proportional to them be sought, such that as AB is to AC, so AC is to the third. Extend AB — which we wish to be the antecedent — and take BD equal to AC, which is to be the consequent, that is, the mean. Then, having drawn the straight line BC, from D draw DE parallel to it, meeting AC extended at E. I say that CE is the third proportional; that is, that as AB is to AC, so AC is to CE. For since in triangle ADE the straight line BC is drawn parallel to the side DE, [Margin: a — 2. sexti (*Elements* VI.2)] it follows that as AB is to BD, so AC is to CE. [Margin: b — 7. quinti (*Elements* V.7)] But as AB is to BD, so is the same AB to AC, since BD is equal to AC. Therefore as AB is to AC, so AC is to CE — which was what was proposed. So, given two straight lines, we have found a third proportional. Which was to be done.

Diagram. The construction figure for Proposition 11. On the horizontal base lie A (left), B (middle), and D (right); E is the apex above, and C lies on the slanting line AE.

- A (bottom-left): vertex of the given angle
- B (base, between A and D): endpoint marking $BD = AC$ laid off along the extended base
- D (bottom-right): terminus of the extended line AB
- E (apex): where DE (parallel to BC) meets AC extended
- C (on line A–E): the given point; CE is the third proportional found
- Lines drawn: AE (with C on it), ED, and BC (with $BC \parallel DE$)

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We shall prove the same thing another way, as follows. Let the two given straight lines AB and BC be set at a right angle ABC, and let the straight line AC be joined. Then, with AB extended as the antecedent, draw from C the perpendicular CD to AC, meeting AB extended at D. I say that BD is the third proportional. For since in triangle ACD the angle ACD is right, and from it the perpendicular CB is dropped to the base AD, then by the corollary to Proposition 8 of this book, BC will be the mean proportional between AB and BD; that is, as AB is to BC, so will BC be to BD. Which was what was proposed.

Diagram. The alternative (Scholium) figure. A right triangle standing on base AD with its right angle at the apex C.

- A (bottom-left): endpoint of base AD
- B (base, between A and D): foot of the perpendicular CB
- D (bottom-right): where the perpendicular CD (from C to AC) meets AB extended
- C (apex): right angle ACD; $CB \perp AD$
- Lines drawn: AC, CD, and the perpendicular CB

Once the third continuously-proportional line has been found, if you drop the first and find a third to the other two, you will have four continuously proportional lines. For example, if to the lines A and B a third proportional C is found, and to the two lines B and C a third proportional D, then A, B, C, D will be four continuously proportional lines. By the same technique a fifth proportional will be obtained — a sixth, a seventh, an eighth — and so on without limit.

Diagram. Four separate straight-line segments of decreasing length are drawn one above another and labelled A, B, C, D — the sample lines of the continued proportion just described.

- A, B, C, D: four line segments in continued proportion

A more expeditious method. We shall find any number of lines continuously proportional in a given ratio, in the following way. Let the given ratio first be that of the larger line A to the smaller B. About the straight line CD, taken equal to A, describe the circle CEDF, and in it fit the chords DE and DF, each equal to B. With a ruler applied to the points E and F, draw the straight line EG, which will be perpendicular to CD. For since the straight line DC, passing through the centre, bisects the arc ECF (indeed, when the equal arcs DE and DF are removed from the equal semicircles DEC and DFG, the remaining arcs CE and CF will be equal), the

same line will also bisect the drawn straight line EF, by what we showed in the scholium to Proposition 27 of Book 3; [Margin: c — 3. tertij (*Elements* III.3)] and therefore it does so at right angles. And since, if the straight line EC were drawn, [Margin: d — 31. tertij (*Elements* III.31)] the angle at E would be right, being in a semicircle, it follows from the corollary to Proposition 8 of this book that DE is the mean proportional between CD and DG; and hence DG will be the third proportional to CD and DE — that is, to the two given lines A and B. Now if from D, through G, an arc is described cutting the straight lines DE and DF at H and I, and a ruler is applied to the points H and I and the straight line HK is drawn, then DK will be the fourth proportional. For on account of the angles HDG and IDG (which are equal, [Margin: e — 27. tertij (*Elements* III.27)] because the arcs EC and FC are equal), [Margin: f — 26. tertij (*Elements* III.26)] the arcs HC and IG will be equal; and therefore, by the scholium to Proposition 27 of Book 3, the drawn straight line HI will be bisected at K, [Margin: g — 3. tertij (*Elements* III.3)] and hence at right angles. [Margin: h — 28. primi (*Elements* I.28)] Therefore EG and HK are parallel. [Margin: i — 2. vel 4. sexti (*Elements* VI.2 or VI.4)] So then, as the straight line ED is to DG, so will DH — that is, DG — be to DK. Therefore the four lines CD, DE, DG, DK are continuously proportional. And if from D, through K, another arc is described cutting the straight lines DE and DF, a fifth proportional will be found by the same technique, and so on without limit.

Diagram. A composite construction figure serving the two "expeditious" methods. A circle (CEDF) stands on a horizontal diameter running to D at the right; a straight line slants down from the upper left through the points T, R, P, E toward the circle, and several chords, arcs, and perpendiculars are drawn. At the far left are short vertical reference segments labelled A and B (the two given lines); a baseline below carries the marks M, Q, O and I.

- On the slanting upper line (top to bottom): T, R, P, E, H
- On the horizontal diameter: S, C, N, G, K, and D (right endpoint)
- Circle passing through C, E, D, F, with F at the bottom
- I: point within the lower part of the figure
- A, B (far left): the two given reference segments; I: a taller vertical reference line
- M, Q, O (lower baseline): division/tick marks
- [Several small peripheral vertex-letters on the outer margin are at the resolution floor — this figure]

Next, let the given ratio be that of the smaller line B to the larger A. Having drawn the two straight lines DC and LM perpendicular to DL, cut off from them

the straight lines DN and LO, each equal to the smaller line B; and with a ruler applied to the points O and N, draw the straight line NP, [Margin: k — 29. primi (*Elements* I.29)] which will be perpendicular to DC, [Margin: l — 33. primi (*Elements* I.33)] since it is parallel to DL. Then, taking the straight line DC equal to the larger line A, describe from D, through C, an arc cutting the straight line NP at P; and extend from D, through P, the straight line DPT; then DP will be equal to DC — that is, to A. Cut off also LQ equal to DC, and with the ruler applied to the points Q and C, draw the straight line CR, which by the same reasoning will be perpendicular to DC and parallel to NP. [Margin: m — 2. vel 4. sexti (*Elements* VI.2 or VI.4)] And since, as ND is to DP — that is, as B is to A (because ND and DP have been taken equal to B and A) — so is CD to DR, that is, so is DP to DR; the three straight lines ND, DP, DR will be continuously proportional, and hence DR will be the third proportional to B and A. And if from D, and the straight line DS equal to LM being cut off, a ruler is applied to the points M and S and the straight line ST is drawn, then DT will be the fourth proportional. For by the same reasoning ST will be parallel to CR and perpendicular to DC. [Margin: n — 2. vel 4. sexti (*Elements* VI.2 or VI.4)] Therefore it will follow that, as CD is to DR — that is, as A is to DR — so is SD, that is, DR, to DT. So then the four lines ND, DP, DR, DT are continuously proportional. And if from D, through T, another arc is described cutting the straight line DC produced, a fifth proportional will be found by the same technique, and so on without limit.

From these methods we shall, without great labour, find two mean proportionals between two given straight lines — not indeed by a strictly geometrical procedure, but as it were by trial, repeating the practical operation again and again until we attain what we seek. For instance, if two mean proportionals are to be found between CD and DK, then, having described about the larger line CD the circle CEDF, we erect from K the perpendicular KH to CD, and draw DH under this condition: that from the point E (where DH meets the circumference) a perpendicular EG dropped to DC cuts off the line DG equal to the line DH — that is, so that the arc described from D through H passes through the point G. When this is done, DE and DH will be the two mean proportionals between CD and DK. For, as we have shown, as CD

is to DE, so is DE to DG, by the corollary to Proposition 8 of this book. [Margin: a — 2. vel 4. sexti (*Elements* VI.2 or VI.4)] And as DE is to DG, so is DH — that is, DG — to DK. Therefore the four straight lines CD, DE, DH, DK are continuously proportional, and hence DE and DH are the two mean proportionals between CD and DK. But if the straight lines DH and DG are found not to be equal, then, if DH

turns out to be the smaller, DH must be raised higher; whereas if it is the larger, it must be lowered further, until DH and DG are found equal.

PROBL. 4. PROPOS. 12. [Margin: xj. [?]]

Given three straight lines, to find a fourth proportional.

Let AB, BC, AD be three straight lines to which a fourth proportional is to be found, namely such that as AB is to BC, so AD is to the fourth. Arrange the first two, AB and BC, along one straight line, which is AC; and let the third, AD, make with the first line AB any angle A whatever. Then from B to D draw the straight line BD, and through C draw CE parallel to it, meeting AD extended at the point E. I say that DE is the fourth proportional. For since in triangle ACE the line BD is drawn parallel to the side CE, [Margin: a — 2. sexti (*Elements* VI.2)] it follows that as AB is to BC, so AD is to DE. Therefore DE is the fourth proportional; and consequently, given three straight lines, we have found a fourth proportional. Which was to be done.

Diagram. The construction figure for Proposition 12. Along the base lie A (left), B (middle), C (right); D is on the slanting line above, and E is the apex at upper right.

- A (bottom-left): vertex of the angle
- B (base, between A and C): junction of AB and BC along the straight line AC
- C (bottom-right): endpoint of AC
- D (on the slant A–E): endpoint of the third given line AD
- E (apex, upper right): where CE (parallel to BD) meets AD extended
- Lines drawn: AE (with D on it), BD, and CE (with $CE \parallel BD$)

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From this we readily gather how, given two straight lines, two others may be found in the same ratio with them. For if two straight lines A and B are given in any ratio, and any third line C is taken and a fourth proportional D is found to it, so that as A is to B, so C is to D, then what is proposed will have been accomplished. By the same technique six lines will be found, or eight, ten, twelve, and so on, of which every consecutive pair keeps the same ratio.

Diagram. Four separate straight-line segments of decreasing length, drawn one above another and labelled A, B, C, D — the sample lines of the scholium.

- A, B, C, D: four line segments in the same continued ratio

We shall also demonstrate, with Pappus, the following problem. Namely:

Given three straight lines, to find a fourth, which is to the third as the first is to the second.

Let AB, BC, CD be three straight lines, and let it be required to find a fourth which is to the third, CD, as AB, the first, is to BC, the second. Arrange the first two, AB and BC, in a straight line so that they make the line AC; and let the third, CD, make with the second, BC, any angle C whatever. Then from B to D draw the straight line BD, and through A draw AE parallel to it, meeting CD extended at E. I say that ED is the fourth — that is, that ED is to DC, the third, as AB, the first, is to BC, the second. This is manifest, [Margin: b — 2. sexti (*Elements* VI.2)] since AC and EC are cut proportionally at the points B and D.

ALTERNATIVELY: Let the second, CB, be made the first, and the first, AB, be made the second. [Margin: c — 12. sexti (*Elements* VI.12)] Then let a fourth proportional DE be found to the three lines CB, BA, CD, so that as the second CB is to the first BA, so the third CD is to the fourth DE. For then, by conversion, as AB, the first, is to BC, the second, so DE, the fourth, is to CD, the third.

Diagram. The figure for the Pappus problem. Along the base lie C (left), B (middle), A (right); D is on the slanting line above, and E is the apex at upper right.

- C (bottom-left): endpoint of CA
- B (base, between C and A): junction of the first two lines
- A (bottom-right): vertex of the angle
- D (on the slant C–E): endpoint of the third given line CD
- E (apex): where AE (parallel to BD) meets CD extended
- Lines drawn: CE (with D on it), BD, and AE (with AE $\parallel$ BD)

PROBL. 5. PROPOS. 13. [Margin: xij. [?]]

Given two straight lines, to find a mean proportional.

Let AB and BC be two straight lines to which a mean proportional is to be found, set along the straight line AC. Bisect AC at E, and from E as centre, with radius EA or EC, describe the semicircle ADC. Then from B erect the perpendicular BD to AC, up to the circumference. I say that BD is the mean proportional between AB and BC. For, the straight lines AD and CD being drawn, [Margin: a — 31. tertij (*Elements* III.31)] the angle ADC will be right, being in a semicircle. Since, then, from the right angle ADC of the right-angled triangle ADC the perpendicular DB is dropped to the base AC, it follows, by the corollary to Proposition 8 of this book,

that BD is the mean proportional between AB and BC. So, given two straight lines, we have found a mean proportional. Which was to be done.

Diagram. The construction figure for Proposition 13. A semicircle on the diameter AC, with apex D above.

- A (bottom-left): endpoint of the diameter
- E (base, centre): midpoint of AC, centre of the semicircle
- B (base, between E and C): the division point of AC into AB and BC; foot of the perpendicular
- C (bottom-right): endpoint of the diameter
- D (apex, on the circumference): the perpendicular DB meets the arc here
- Lines drawn: the semicircle ADC, the chords AD and DC, and the perpendicular DB

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From this it becomes clear that a straight line drawn within a circle, from any point of the diameter, perpendicular to that diameter as far as the circumference, is the mean proportional between the two segments of the diameter made by the perpendicular. For let a semicircle ABC be given, and from the point D of the diameter AC let the straight line DB be drawn to the circumference, perpendicular to AC. I say that DB is the mean proportional between AD and DC — which is plainly established by the demonstration of this problem. For if the straight lines AB and CB are drawn, [Margin: b — 31. tertij (*Elements* III.31)] the angle ABC will be right. Therefore, by the corollary to Proposition 8, the proposition is established. By the same reasoning the perpendicular EF will be the mean proportional between AE and CE; likewise GH between AG and GC; and the same is to be said of any others whatever that are drawn perpendicular to the diameter from any points of the diameter.

Diagram. The scholium figure. A semicircle on the diameter AC, with three perpendiculars rising to the circumference.

- A (bottom-left): endpoint of the diameter
- D (base): foot of the perpendicular DB
- E (base): foot of the perpendicular EF
- G (base): foot of the perpendicular GH
- C (bottom-right): endpoint of the diameter
- B (circumference, over D): top of perpendicular BD
- F (circumference, over E): top of perpendicular FE

- H (circumference, over G): top of perpendicular HG
- Lines drawn: the semicircle, the chords AB and CB, and the three perpendiculars BD, FE, HG

(*Scholium, continued.*) When the smaller given line is very small, the perpendicular BD will cut the circumference at D too obliquely. Therefore, so that the point D may then be found more precisely, we shall multiply the smaller line BC as many times as we please. For example, we shall take BE as the quadruple of BC, and shall cut the larger line AB into as many equal parts as BE contains multiples of BC, letting one part be BF, so that AB has the same ratio to BF as BE has to BC. And at length, about EE let a circle be described cutting BD at D: then BD will be the mean proportional between BF and BE, as we have demonstrated in this scholium. I say that the same line is the mean proportional between [AB and BC]. For since, as the first AB is to the second BF, so the third BE is to the fourth BC, [Margin: c — 16. sexti (*Elements* VI.16)] the rectangle contained by AB, the first, and BC, the fourth, will be equal to the rectangle contained by BF, the second, and BE, the third. [Margin: d — 17. sexti (*Elements* VI.17)] But this is equal to the square on BD. Therefore so is the former. [Margin: e — 17. sexti (*Elements* VI.17) [?]] Hence BD is the mean proportional between AB and BC.

But you need not be troubled that we have cited Propositions 16 and 17 of this book, which have not yet been demonstrated; for this demonstration, together with its construction, can be deferred until after Proposition 17 of this book, so that there is no danger of "begging the question," as the philosophers put it.

FROM PELETIER [Jacques Peletier]

Given a straight line, to cut another straight line — one that is not smaller than double the former — in such a way that the given straight line is the mean proportional between the segments of this one.

Let AC be the given straight line, and let the straight line AB be proposed for division (one that is not smaller than double AC itself, but either double or larger), in such a way that AC is the mean proportional between its segments. Set the straight lines AB and AC at a right angle BAC, and, having bisected AB at D, describe from D as centre, with radius DA or DB, the semicircle AEB. Then through C draw CE parallel to AB, cutting or touching the circumference at the point E. (Now CE will necessarily either cut or touch the circumference. For if it is less than half of AB — that is, less than the semidiameter erected at right angles from D — it will cut the circumference; whereas it will touch that same circumference if it is equal to half of AB, that is, equal to the semidiameter drawn

at right angles from D; and if it is less than half of AB, it will necessarily cut the circumference, or touch it.) From E drop the perpendicular EF to AB. I say that AC is the mean proportional between the segments AF and FB. For, the straight lines AE and BE being drawn, it will follow — as has already been demonstrated, from the corollary to Proposition 8 of this book — that EF is the mean proportional between AF and FB. [Margin: f — 34. primi (*Elements* I.34) [?]] Since, then, EF is equal to AC, because ACEF is a parallelogram (for CE is parallel to AF by construction, [Margin: g — 28. primi (*Elements* I.28)] and AC is parallel to EF on account of the right angles CAF and EFA), AC too will be the mean proportional between AF and FB. Which was what was proposed.

Diagram. The figure for the continuation of the Proposition-13 scholium (the "when the smaller line is very small" construction). A large semicircle on the diameter A...E, with apex D, overlapped by a smaller semicircle sharing the apex D.

- A (bottom-left): endpoint of the larger diameter
- F (base): endpoint of the part BF cut off on AB
- B (base): division point
- C (base): endpoint of the smaller line
- E (bottom-right): endpoint of the larger diameter
- D (apex, shared by both arcs): top of the perpendicular BD
- Lines drawn: the large semicircle A–D–E, the smaller semicircle, and the perpendicular BD

Diagram. The Peletier-problem figure. A semicircle on the diameter AB, with the parallelogram ACEF standing on AF.

- C (top-left): corner of the parallelogram ACEF
- E (upper, on the circumference): where CE meets the semicircle; corner of the parallelogram
- A (bottom-left): endpoint of the diameter; vertex of the right angle
- F (base): foot of the perpendicular EF
- D (base, centre): midpoint of AB, centre of the semicircle
- B (bottom-right): endpoint of the diameter
- Lines drawn: the semicircle AEB, the parallelogram A–C–E–F (with $AC \perp AB$ and $CE \parallel AB$), the chords AE and BE, and the perpendicular EF

THEOR. 9. PROPOS. 14. [Margin: xiiij.]

In equal parallelograms which have one angle equal to one angle, the sides about the equal angles are reciprocally proportional. And parallelograms which have one angle equal to one angle, and whose sides about the equal angles are reciprocally proportional, are equal.

Let ABCD and BEFG be two equal parallelograms having the angles ABC and EBG equal. I say that the sides about these angles are reciprocally proportional — that is, that as AB is to BG, so EB is to BC. For let the parallelograms be joined at the equal angles, so that AB and BG make one straight line. When this is done, since the angles ABC and EBG are equal, EB and BC will also form one straight line, as we demonstrated from Proclus at Proposition 15 of Book 1. Now let DC and FG be extended until they meet at H. Since, then, the parallelograms DB and BF are equal, [Margin: a — 7. quinti (*Elements* V.7)] it follows that as DB is to BH, so BF is to the same BH. But as DB is to BH, [Margin: b — 1. sexti (*Elements* VI.1)] so is AB, the base, to the base BG, because the parallelograms are of the same altitude; and likewise as BF is to BH, so is EB to BC. Therefore as AB is to BG, so is EB to BC. Which was what was proposed.

CONVERSELY, let the sides about the equal angles ABC and EBG now be reciprocally proportional — that is, as AB is to BG, so EB is to BC. I say that the parallelograms DB and BF are equal. For, the same construction being made, since as AB is to BG, so DE is to BH, [Margin: c — 1. sexti (*Elements* VI.1)] and as EB is to BC, so BF is to the same BH, it will also follow that as DB is to BH, so BF is to the same BH. [Margin: d — 9. quinti (*Elements* V.9)] And therefore the parallelograms DB and BF will be equal. So then, of equal parallelograms which have one angle equal to one angle, etc. Which was to be demonstrated.

Diagram. The figure for Proposition 14. Two parallelograms joined at B, the first (ABCD) lying above and the second (BEFG) below.

- D (top-left): corner of parallelogram ABCD
- C (top, middle): corner of ABCD
- H (top-right): where DC and FG extended meet
- A (middle row, left): corner of ABCD
- B (middle row, centre): the shared vertex at the equal angles
- G (middle row, right): corner of BEFG
- E (bottom-left): corner of BEFG
- F (bottom, middle): corner of BEFG

- Lines drawn: parallelogram A–B–C–D, parallelogram B–E–F–G, and DC, FG extended to H

THEOR. 10. PROPOS. 15. [Margin: xiv.]

In equal triangles which have one angle equal to one angle, the sides about the equal angles are reciprocally proportional. And triangles which have one angle equal to one angle, and whose sides about the equal angles are reciprocally proportional, are equal.

Let ABC and DBE be two equal triangles having the angles at B equal. I say that the sides about these angles are reciprocally proportional — that is, that as AB is to BE, so DB is to BC. For let the triangles be joined at the equal angles, so that AB and BE make one straight line. When this is done, since the angles ABC and DBE are equal, DB and BC will also form one straight line, as was demonstrated from Proclus at Proposition 15 of Book 1. Now, the straight line CE being drawn, since the triangles ABC and DBE are equal, [Margin: a — 7. quinti (*Elements* V.7)] it follows that as ABC is to BCE, so DBE is to the same BCE. But as triangle ABC is to triangle BCE, [Margin: b — 1. sexti (*Elements* VI.1)] so is the base AB to the base BE, because these triangles are of the same altitude; and likewise as DBE is to BCE, so is the base DB to BC. Therefore as AB is to BE, so is DB to BC. Which was what was proposed.

Diagram. The figure for Proposition 15. Two triangles, ABC and DBE, meeting at the common vertex B.

- A (top-left): vertex of triangle ABC
- D (top-right): vertex of triangle DBE
- B (centre): the shared vertex at the equal angles
- C (bottom-left): base vertex of triangle ABC
- E (bottom-right): base vertex of triangle DBE
- Lines drawn: triangle A–B–C, triangle D–B–E, and the straight line CE joining the two bases

LIBER VI (continued)

THEOR. 11. PROPOS. 16. [Margin prop-no.: XV.]

(Conclusion of the converse of Prop. 15, begun in the preceding pages.)

Now conversely, let the sides about the equal angles at B be reciprocally proportional — that is, let AB be to BE as DB is to BC. I say that the triangles ABC

and DBE are equal. For, the same construction having been made, since AB is to BE as DB is to BC; and since AB is to BE as triangle ABC is to triangle BCE [a]; and DB is to BC as triangle DBE is to the same triangle BCE; it will follow that ABC is to BCE as DBE is to that same BCE, and therefore the triangles ABC and DBE will be equal [b]. So, of triangles that are equal and have one angle equal to one angle, and so on. Which was to be shown.

[Margin: a — *Elements* VI.1] [Margin: b — *Elements* V.9]

THEOR. 11. PROPOS. 16. [Margin prop-no.: XV.]

If four straight lines are proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means. And if the rectangle contained by the extremes is equal to the rectangle contained by the means, then those four straight lines will be proportional.

Let the four proportional lines be AB, FG, EF, BC — so that AB is to FG as EF is to BC. Let ABCD be the rectangle contained by the extremes AB and BC, and EFGH the rectangle contained by the means EF and FG. I say that the rectangles AC and EG are equal. For since the right angles B and F are equal, and AB is to FG as EF is to BC, the sides about the equal angles B and F are reciprocally proportional. Therefore the parallelograms AC and EG will be equal [a]. Which is what was proposed.

[Margin: a — *Elements* VI.14]

Conversely, let the rectangles AC and EG now be equal. I say that the four straight lines AB, FG, EF, BC are proportional — that is, that AB is to FG as EF is to BC. For since the rectangles AC and EG are equal and have equal angles (namely the right angles B and F), the sides about these angles will be reciprocally proportional [b]: AB will be to FG as EF is to BC. And so, if four straight lines are proportional, and so on. Which was to be shown.

[Margin: b — *Elements* VI.14]

Diagram. At the left, four line segments representing the four proportional lines are drawn one above another; at the right, two rectangles.

- Line segments (left column): A—B; F—G; E—F; B—C.
- Upper rectangle (right): A (top-left), B (top-right), C (bottom-right), D (bottom-left) — the rectangle AC contained by the extremes.
- Lower rectangle (right): E (top-left), F (top-right), G (bottom-right), H (bottom-left) — the rectangle EG contained by the means.

SCHOLIUM.

The same is true even if the parallelograms AC and EG are not right-angled, provided only that they are equiangular — that is, provided the angles contained by the said straight lines are equal. This is evident from the figure, for the demonstration is exactly the same.

Diagram. The same four line segments at the left; at the right, two oblique (equiangular) parallelograms in place of the rectangles.

- Line segments (left): A—B; F—G; E—F; B—C.
- Upper parallelogram: A (top-left), B (top-right), C (bottom-right), D (bottom-left).
- Lower parallelogram: E (top-left), F (top-right), G (bottom-right), H (bottom-left).

THEOR. 12. PROPOS. 17. [Margin prop-no.: xvj.]

If three straight lines are proportional, the rectangle contained by the extremes is equal to the square described on the mean. And if the rectangle contained by the extremes is equal to the square described on the mean, then those three straight lines will be proportional.

Let the three straight lines AB, EF, and BC be proportional — so that AB is to EF as EF is to BC. Let ABCD be the rectangle contained by the extremes AB and BC, and let EFGH be the square on the mean EF. I say that the rectangle AC and the square EG are equal. For, taking a line FG equal to EF itself, the four lines AB, EF, FG, BC will be proportional — AB being to EF as FG is to BC; and the square EG will be the rectangle contained by the means EF and FG, because EF and FG are equal. Therefore the rectangle AC, contained by the extremes AB and BC, is equal to the square EG — that is, to the rectangle contained by the means EF and FG [a]. Which is what was proposed.

[Margin: a — *Elements* VI.16]

But now let the rectangle AC and the square EG be equal. I say that AB is to EF as EF is to BC. For since the rectangles AC and EG are equal, AB will be to EF as FG is to BC [b]; and as FG is to BC, so is EF — which equals FG — to the same BC [c]. Therefore AB is to EF as EF is to BC. So, if three straight lines are proportional, and so on. Which was to be demonstrated.

[Margin: b — *Elements* VI.16] [Margin: c — *Elements* V.7]

Diagram. At the left, three line segments (the three proportional lines); at the right, a rectangle above and a square below.

- Line segments (left): $A-B$; $E-F$; $B-C$.
- Upper rectangle: A (top-left), B (top-right), C (bottom-right), D (bottom-left) — the rectangle AC.
- Lower square: E (top-left), F (top-right), G (bottom-right), H (bottom-left) — the square EG on the mean.

COROLLARIVM.

From the latter part of this theorem it follows that any straight line is the mean proportional between any two other straight lines that contain a rectangle equal to the square on it. For from the fact that the lines AB and BC contain a rectangle equal to the square on the line EF, it was shown that AB is to EF as EF is to BC. Therefore EF is the mean proportional between AB and BC.

SCHOLIVM.

The same conclusions follow entirely even if the parallelograms are not right-angled, provided they are equiangular — so that EG is a rhombus and AC a rhomboid. For the demonstration in these cases will not be different, as the figure shows.

Diagram. At the top, line segments; at the right, two oblique parallelograms — one a rhombus (EG), the other a rhomboid (AC).

- Line segments: $A-B$; $F-G$; $E-F$; $B-C$.
- Upper parallelogram: A (top-left), B (top-right), C (bottom-right), D (bottom-left).
- Lower parallelogram: E (top-left), F (top-right), G (bottom-right), H (bottom-left).

I should like, at this point, to demonstrate together with Peletier [Jacques Peletier du Mans] a problem not without use, bearing on proportional lines. It is of this kind:

A straight line having been cut into two parts in any way, to divide one of them again into two parts, so that all three parts are in continued proportion.

Let AB be cut at C in any way, and suppose the part CB is to be so divided into two further parts that all three parts are in continued proportion. Let AC be bisected

at F, and FC again bisected at I, so that CI is a quarter of the first part AC. From B and I erect two perpendiculars BE and IK equal to the first part AC, and draw the line EK, so that the rectangle IE is contained by AC (the first part) and the line IB (made up of the other part BC together with CI, the quarter of the first part). Now since AC is bisected at F, and CB is added to it in a straight line, the rectangle contained by AB, BC together with the square on CF is equal to the square on BF [c].

[Margin: c — *Elements* II.6]

But the rectangle IE is less than the rectangle contained by AB, BC together with the square on CF. (For, CD being drawn perpendicular to AB, the rectangle CE contained by AC, CB will be less than the rectangle contained by AB, BC; while the rectangle CK is equal to the square on CF, since each is a quarter of the square on AC — the rectangle CK because CK is to the square on AC (that is, to the rectangle contained by AC, CD) as CI is to CA [d]; and so, since CI is a quarter of the line AC, CK will also be a quarter of the square on AC. And the square on CF is a quarter of the same square on AC, by the scholium to Prop. 4 of Book II.) The rectangle IE will therefore also be less than the square on BF.

[Margin: d — *Elements* VI.1]

If, then, a square is made equal to the rectangle IE [e], its side will be less than FB but greater than FC — since the square on FC has been shown equal to the rectangle CK, and hence less than the rectangle IE. Let that side be FL. I say that the part CB is so cut at L that the three parts AC, CL, LB are in continued proportion.

[Margin: e — *Elements* II.14]

For let the square FM be described on FL, and let the line DC be extended to O and the side ML to P; and, taking two lines FG and LQ each equal to CF, draw the line GQ cutting CO at H. Then CG will be the square on the line CF, because CF and FG are equal. And since, when the equals FC and FG are taken from the equals FL and FN, the remainders CL and GN are equal [f]; and CL is equal to HQ and GN to HO; the remainders HQ and HO will also be equal, and therefore HM will be the square on CL.

[Margin: f — *Elements* I.34]

Again, since the rectangle CQ is contained by CL and CH (that is, CF, the half of AC), and the rectangle GO by GN (that is, CL) and GH (that is, CF, the half of the same AC, that is CD), the two rectangles CQ and GO together will be equal to the

rectangle CP contained by the same CL and CD (which equals CH and GH together) [g]. And so, since the rectangle IE is equal to the square FM, and the rectangle CK to the square CG, and the rectangle CP to the two rectangles CQ, GO together, the remaining rectangle LE — contained by BE (that is, AC) and BL — will be equal to the remaining square HM. Therefore, since with the three given lines AC, CL, LB the rectangle LE, contained by the extremes AC and BL, is equal to the square HM on the mean CL [h], the three lines AC, CL, LB will be in continued proportion. Which is what was proposed.

[Margin: g — *Elements* II.1] [Margin: h — *Elements* VI.17]

Diagram. A compound square-and-rectangle construction on the base line AB, gridded by the perpendiculars and the squares FM, CG, HM.

- Base line AB: A (left), I (between F and C, the quarter-point mark), B (right).
- Top edge: K, D, P, E.
- Upper band: F, C, L.
- Lower band: G, H, Q.
- Bottom edge: N, O, M.

The same problem will be carried out in numbers as follows. To the latter part of the proposed number add a quarter of the first part, and multiply the number so formed into the first part. Then extract the square root of the resulting product. For if from the root found you subtract half the first part, the remainder will be the second proportional part; and if you subtract this from the latter part of the given number, the remainder will be the third part.

For example, let the line AB be set at 76, the part AC at 36, and CB at 40. We add CI — that is, 9, a quarter of the first part 36 — to CB, that is to the latter part 40, and multiply the number so formed, 49 (namely the line IB), into the first part 36 (that is, into the line AC). From the product 1764 (that is, the rectangle IE) we extract the square root, 42, for the line FL. If from this we subtract CF, half of the part AC (namely 18), the remaining part CL will be 24. This, taken from the whole latter part CB, 40, leaves the third part LB, 16. So the three parts AC, CL, LB of the number AB, 76 — these three, 36, 24, 16 — are in continued proportion. From this it is clear that when the square root cannot be extracted from that product, the problem cannot be carried out in numbers. And from this too will be understood what we wrote near the end of Book V, in the treatment of proportions, where we dealt with the origin of geometric proportionality out of arithmetic proportionality.

But since so far we have discussed proportional lines with Euclid, it will not be foreign to our purpose to treat of them also with Pappus of Alexandria [Pappus], who, given two of the three terms of any mediety whatever — arithmetic, geometric, or harmonic — seeks the third; and, in the case of geometric proportionality, does so otherwise than Euclid did. We shall carry this out in the following propositions, of which let this be the first.

I.

Given two straight lines, to find the mean proportional in arithmetic proportionality.

Let there be two given lines AB and C (here represented by units set out lengthwise), between which the mean in arithmetic proportionality is to be found. Set FH equal to the lesser, C, and from the greater, AB, cut off another part AD equal to it. Bisecting the segment BD at E, take HG equal to DE itself. I say that FG is the arithmetic mean proportional between AB and C. For AB exceeds FG by the excess BE, and FG exceeds C by the excess GH (that is, DE). Since, then, the two excesses BE and DE are equal by construction, it plainly follows that the lines AB, FG, and C are in arithmetic proportion.

Diagram. Three vertical dotted segments representing the lines AB, FG, and C.

- Line AB (left): A (bottom), D, E, B (top).
- Line FG (middle): F (bottom), H, G (top).
- Line C (right): C (bottom point of the short segment).

(*Conclusion of Prop. I.*) Hence it follows that the mean line FG is half of the sum formed from the extremes. For since both AD and C are equal to FH, the sum of AD and C is double FH; and DB is likewise double HG [a]. Therefore the sum of AB and C is double the whole FG; and consequently FG is half the sum of the extremes. Which is what was proposed.

[Margin: a — *Elements* V.1]

II.

Given two straight lines, to find the lesser extreme in arithmetic proportionality.

In the same figure, let the two given lines be AB and FG, for which the lesser third term in the arithmetic mediety is to be found. From the lesser, FG, cut off a line GH equal to the excess BE by which AB exceeds FG; and set C equal to FH. I say that C is the lesser third proportional. For AB exceeds FG by the excess BE, and FG

exceeds C by the excess GH — which two excesses BE and GH are equal by construction.

III.

Given two straight lines, to find the greater extreme in arithmetic proportionality.

In the same figure, let the two given lines be C and FG, for which the greater third term in the arithmetic mediety is to be found. Take AE equal to FG, and to it add EB equal to the excess GH by which FG exceeds C; so that the three lines AB, FG, and C are in arithmetic proportion. For AB exceeds FG by the excess BE, and FG exceeds C by the excess GH. Since, then, the two excesses BE and GH are equal by construction, the proposition is evident.

And so, given two unequal numbers 30 and 16: if half of the excess — namely 7 — is added to the lesser, 16, the arithmetic mean proportional 23 is produced, as here appears: 30, 23, 16. Or the middle term is obtained by taking half the sum of the extremes, as is evident in the same example — for the sum of the extremes is 46, and its half, 23, constitutes the middle term.

But if from the lesser, 16, the excess 14 is subtracted, the remainder will be the lesser extreme, 2, in the arithmetic mediety, as you see here: 30, 16, 2.

And finally, if to the greater, 30, the excess 14 is added, the greater extreme 44 is produced, in arithmetic proportionality, as is evident here: 44, 30, 16.

IIII.

Given two straight lines, to find the mean proportional in geometric proportionality.

Let the given lines be AB and BC, having the same endpoint B, between which the geometric mean proportional is to be found. Describing the semicircle ADB on the greater line AB, erect from C the perpendicular CD to AB; and from B, through D, describe an arc cutting AB at E. I say that BE is the mean proportional between AB and BC in geometric proportionality. For, the lines AD and BD being drawn, the angle ABD [recte: ADB — angle in a semicircle] will be right [a]. Therefore, by the corollary of Prop. 8 of this book, the line BD — that is, BE, which is equal to it — will be the mean proportional between AB and BC. Which is what was proposed.

[Margin: a — *Elements* III.31]

Diagram. A semicircle described on the diameter AB, with the perpendicular CD and the arc from B cutting AB at E.

- Diameter (left to right): A, E, C, B.
- Apex on the semicircle: D.
- Chords/lines drawn: AD, DB, DC (perpendicular to AB at C), DE.

V.

Given two straight lines, to find the lesser extreme in geometric proportionality.

In the same figure, let the given lines be AB and BE, possessing the same endpoint B, for which the lesser third geometric proportional is to be found. Describing the semicircle ADB on the greater line AB, describe from B, through E, an arc cutting the circumference ADB at D — the point from which the perpendicular DC is dropped to AB. I say that BC is the third proportional to AB and BE. For, the lines AD and BD being drawn, the angle ADB will be right [a]. Therefore, by the corollary of Prop. 8 of this book, BD — that is, BE, which is equal to it — will be the mean proportional between AB and BC; that is, AB will be to BE as BE is to BC. Which is what was proposed.

[Margin: a — *Elements* III.31 [?] (printed digit at the outer-margin resolution floor; reads "21", but the geometry and Prop. III require III.31)]

VI.

Given two straight lines, to find the greater extreme in geometric proportionality.

In the same figure, let the given lines be CB and BE, possessing the same endpoint B, for which the greater third geometric proportional is to be found. From C, the endpoint of the lesser, erect the perpendicular CD to EB, and let the arc described from B through E cut it at D. Then, the line BD being drawn, erect to it at D the perpendicular DA, cutting BE produced at A. I say that AB is the third proportional to CB and BE. For since the angle ADB is right, by the corollary of Prop. 8 of this book, BD — that is, BE, which is equal to it — will be the mean proportional between BC and AB; that is, CB will be to BE as BE is to AB. Which is what was proposed.

VII.

Given two straight lines, to find the mean proportional in harmonic proportionality.

VII.

(Construction and demonstration of the added proposition whose enunciation — "Given two straight lines, to find the harmonic mean proportional" — ended the previous section.)

Let there be given two straight lines AB and BC sharing the same endpoint B, between which the harmonic mean proportional is to be found. Through A, draw a perpendicular DE to AB, and mark off on it equal segments AD and AE of any length whatever. From C also raise a perpendicular CF to AB, cutting the line BD (once drawn) at the point F; and from F draw a line FE to E, cutting AB at G. I say that BG is the harmonic mean proportional between AB and BC — that is, that AB, the first term, is to BC, the third, as AG (the excess between the first term AB and the mean BG) is to GC (the excess between the mean BG and the third term BC). [Margin: m — Elements V.28]

For since CF is parallel to DE, on account of the right angles at A and C, by the scholium to Proposition 4 of this book the triangle BCF is similar to the triangle BAD. Hence AB is to AD as BC is to CF; and by alternation, AB is to BC as AD is to CF, that is, as AE is to CF. Again, because of the similarity of the triangles AEG and CFG, [Margin: n — Elements VI.4] AE is to AG as CF is to CG; and by alternation, AE is to CF as AG is to GC. But it was shown that AE is to CF as AB is to BC. Therefore AB is also to BC — the first line to the third — as AG (the excess between the first term AB and the mean BG) is to GC (the excess between the mean BG and the third term BC). Which is what was proposed.

Diagram. The construction for the harmonic mean. A horizontal base line runs from A (left) to B (right). Through A a vertical line DE is drawn, with D above and E below A. A point C lies on the base between A and B, and the perpendicular CF rises from C to F. The line from B to D and the line FE from F down to E cross the base at the interior points; G lies on the base. Additional equal marks (AK = AL) with K above-left and L below-left support the "these magnitudes are not determined" argument, in which BK meets CF at H and the line LH is drawn; M is an alternative point on the base.

- D (top, above A): upper endpoint of the perpendicular at A
- E (bottom, below A): lower endpoint of the perpendicular at A
- A (left end of base): foot of the perpendicular DE
- B (right end of base): shared endpoint of the two given lines
- C (on base, between G and B): foot of the perpendicular CF

- F (upper region, right of centre): top of the perpendicular CF / intersection of BD produced with CF
- G (on base, centre): point where FE cuts AB; foot of the mean BG
- K (upper left): endpoint of the alternative equal segment AK
- L (lower left): endpoint of the alternative equal segment AL
- H (centre): intersection of BK with the perpendicular CF
- M (on base, right of A): the alternative interior point used in the reductio
- I (centre, below H): interior intersection point [?]

These magnitudes are not determined. The magnitudes of the straight lines AD and AE are not fixed: whether they are taken larger or smaller, the same mean BG is found. For let there be other equal segments AK and AL. Draw the line BK, cutting the perpendicular CF at H, and join the line LH — which I claim passes through G. For if it were to pass through some other point, say through M, between A and G, we would show in the same way that AB is to BC as AM is to MC. For the triangles ALM and MHC would again be equiangular, provided HML is a straight line. But AB is to BC as AG is to GC, as was shown. Therefore AM would also be to MC as AG is to GC [printed "AC"] — which is absurd. [Margin: o — Elements V.8] For the ratio of AM to GC is less than that of AG to GC; and the ratio of AM to MC is still less than that to GC. Much less, therefore, will the ratio of AM to MC equal that of AG to GC. So AM is not to MC as AG is to GC; and consequently HL will not cut AG at M. In the same way it will not cut GC. It therefore passes through G; and for that reason the same mean BG is found by way of the equal lines AK and AL.

VIII.

Given two straight lines, to find the lesser extreme in the harmonic mean.

Let there be given, in the same figure, two straight lines AB and BG sharing the same endpoint B, for which the lesser third harmonic proportional is to be found. Draw DE perpendicular to AB, and mark off equal segments AD and AE of any length whatever. Then, having joined the line BD, draw from E through G a line cutting BD at the point F, and from F let a perpendicular FC be dropped to AB. I say that BC is the harmonic proportional to AB and BG — that is, that AB, the first term, is to BC, the third, as AG (the excess between the first and the second) is to GC (the excess between the second and the third). This will indeed be demonstrated just as in the preceding proposition.

These magnitudes are not determined here either. The magnitudes of AD and AE are not fixed here either, for whether they are taken larger or smaller, the same third term BC is found. For let there be other equal segments AK and AL. Draw the line BK, cutting the perpendicular CF at H, and join the line LG, which I claim falls at H. For if it should fall below, say at I, we would show in the same way that AB is to BC as AK is to CH, that is, as AL is to CH. But AB was to BC as AG is to GC. Therefore AG is to GC as AL is to CH. And since, if LGI is a straight line, [Margin: p — Elements VI.4] by the similarity of the triangles ALG and CIG, AL is to AG as CI is to GC, then by alternation AL is to CI as AG is to GC. [Margin: q — Elements V.9] Since, then, it has been shown that AG is to GC as AL is to CH, AL will be to CH as it is to CI. Therefore CH and CI are equal — a whole and its part — which is absurd. So LG will not fall below H; and in the same way we shall show that it does not fall above. It therefore falls at H, where BK cuts the perpendicular CF; and consequently the perpendicular HC dropped from H to AB will exhibit the same lesser third term BC, the harmonic proportional.

Corollary. From this it follows that, given three straight lines in harmonic proportion, the least is greater than the excess by which the mean exceeds the least. For since AB is to BC as AG is to GC, and since AB, the first, is greater than AG (the excess between the first term AB and the mean BG), [Margin: r — Elements V.14] BC too, the third, will be greater than GC (the excess between the mean BG and the least term BC).

IX.

Given two straight lines, to find the greater extreme in harmonic proportionality.

Let there be given two straight lines AB and BC sharing the same endpoint B, for which the greater third harmonic proportional is to be found. Through A, draw a perpendicular DE to BC, mark off equal segments AD and AE of any length whatever, and draw the line BD; let EC, produced, cut this line produced at the point F (that BD and EC do in fact meet we shall prove shortly). From F let a perpendicular FG be dropped to BC produced. I say that GB is the harmonic proportional to CB and BA — that is, that GB, the first term, is to BA, the third, as GC (the excess between the first term GB and the mean BC) is to CA (the excess between the mean CB and the third term BA). [Margin: s — Elements I.28]

For since GF is parallel to AD, on account of the right angles at G and A, by the scholium to Proposition 4 of this book the triangle BAD is similar to the triangle BGF. Hence GB is to GF as BA is to AD; and by alternation, GB is to BA as GF is to

AD, that is, as GF is to AE. Again, because of the similarity of the triangles FGC and EAC, [Margin: t — Elements VI.4] FG is to GC as EA is to CA; and by alternation, FG is to AE as GC is to CA. But it was shown that FG is to AE as GB is to BA. Therefore GB, the first term, is also to BA, the third, as GC (the excess between the first term GB and the mean BC) is to CA (the excess between the mean BC and the third term BA). Which is what was proposed.

Diagram. The construction for the greater harmonic extreme. From an apex F at the top, lines descend to a base line that runs (left to right) through G, C, A, and B, with H just to the right of A; K lies below the base and E at the bottom. The line BD rises to D at the upper right, and EC produced meets BD produced at F; the perpendicular FG falls to G. L (upper left), M (mid-left), and I (centre) support the "these magnitudes are not determined" argument.

- F (apex, top): meeting point of BD produced and EC produced
- D (upper right): endpoint of the perpendicular through A (with E below)
- E (bottom): lower endpoint of the perpendicular through A
- G (base, left): foot of the perpendicular FG
- C (base, left of centre): foot of the mean
- A (base, centre): foot of the perpendicular DE
- H (base, just right of A): the point AH cut off equal to AC
- B (base, right end): shared endpoint of the given lines
- L (upper left): endpoint of the alternative equal segment (AI = AK version)
- M (mid-left): the alternative interior point used in the reductio
- I (centre): interior intersection point [?]
- K (below base): endpoint of the alternative equal segment

That BD and EC do meet is proved as follows. Since BC is set as the mean and BA as the third or least term, BA (the third) will be greater than CA — the excess by which the mean BC exceeds the least BA — as we showed at the end of the preceding problem. For unless AB were greater than AC, no greater third term could be found in harmonic proportionality, because EC would not meet BD. Cut off, then, AH equal to AC, and join the line HD. Since, therefore, the sides AE and AC are equal to the sides AD and AH, [Margin: u — Elements I.4] and they contain equal angles (right angles, or vertical angles at A), the angles ACE and AHD will also be equal; and since these are alternate angles, [Margin: b — Elements I.27] the lines EC and HD will be parallel. [Margin: c — Elements I.29] Consequently the interior angles DHC and HCF are equal to two right angles. Since, then, [Margin: d — Elements I.16] the angle DHC is greater than the interior angle B, the two

angles BCF and CBF will be less than two right angles. [Margin: e — Axiom 13 (13. pronunciatum; Clavius's form of Euclid's parallel postulate — that lines meet when the interior angles are less than two right angles)] Therefore BD and EC will meet, as was proposed.

These magnitudes are not determined here either. The magnitudes of AD and AE are not fixed here either, for whether they are taken larger or smaller, the same third term BG is found. For let there be other equal segments AI and AK. Draw the line BI, which, produced, cuts the perpendicular FG at L, and draw KC, which I claim, when produced, falls at L. For if it should fall below, say at M, we would show in the same way that GB is to BA as GL is to AI, that is, as GL is to AK. But GB was to BA as GC is to CA. Therefore GC is to CA as GL is to AK. And since, if KCM is a straight line, [Margin: f — Elements VI.4] by the similarity of the triangles GMC and AKC, GM is to GC as KA is to AC, then by alternation GM is to AK as GC is to CA. Since, then, it has been shown that GL is to AK as GC is to CA, GM too will be to AK as GL is to AK. [Margin: g — Elements V.9] Therefore GM and GL are equal — a part and the whole — which is absurd. So KC will not fall below L; and by the same reasoning it will not fall above. It therefore falls at L, where BI produced cuts the perpendicular FG; and consequently the perpendicular LG dropped from L to GB falls at G and exhibits the same greater third term BG, the harmonic proportional.

From this it is clear that when two numbers have a double ratio, or one greater than double, no greater third term can be added to them in harmonic proportionality, as we stated in the treatment of proportions, in the third rule of harmonic proportionality. For since we showed that the least line BA is greater than the excess AC by which the mean BC exceeds the least AB, AB will be greater than half of BC; and consequently BC to AB — the mean to the least — will have a ratio less than double. But if BC to AB were double, or greater than double, then AB would be either equal to AC or less than it. Therefore, as was shown, no greater third term could be found in harmonic proportionality, because EC would not meet BD.

But let us also demonstrate, following Giovanni Battista Benedetti [Io. Baptista Benedictus], by what method the same terms of harmonic proportionality may be found among numbers. For from this the rationale of certain procedures will become clear which we set out in the treatment of proportions in Book 5. We shall use several propositions of Book 7, since they do not depend on Books 5 and 6 and can therefore be demonstrated before those books. Here, then, we shall take our beginning.

X.

From any two given numbers, to find three numbers in harmonic proportion whose extremes have the same ratio as the two given numbers.

On any straight line AB describe the square ABCD; and having taken two equal parts BE and EF on the side AB, draw EG and FH parallel to AD and BC. Likewise, having taken a segment AI equal to BE (or EF), draw IK parallel to AB and DC, cutting FH at L. The rectangle AG will be contained by AB (that is, by the sum of the lines AE and EB, [Margin: a — Elements I.34] since EG is equal to BC and therefore to AB) and by AE (namely the greater part of the sum AB). The rectangle EC, however, will be contained by AB (that is, by the same sum of the lines AE and EB) and by EB (namely the lesser part of the same sum AB). Finally, both the rectangle AK and the rectangle GL will be contained under the parts AE and EB, because AI has been taken equal to EB, and HL, LK are equal to AE, EB. [Margin: b — Elements I.34] For since FH is equal to AD, that is, to AB, and FL is equal to AI, that is, to EB, the remainder HL will be equal to the remainder AE; [Margin: c — Elements I.34] and KL is equal to EF, that is, to EB. From this it follows that the two rectangles AK and KH together contain twice the rectangle contained under the parts AE and EB.

I say that the rectangle AG, the sum of the two rectangles AK and KH, and the rectangle EC constitute a harmonic proportionality — that is, that as AG (the first) is to EC (the third), so IH (the excess between the first, AG, and the second, HLIAEG) is to AL (the excess between the second, HLIAEG, and the third, EC, or FG, [Margin: d — Elements I.36] which is equal to EC). For since HL is equal to AE and LF is equal to EB, as we showed, [Margin: e — Elements V.7] AE will be to EB as HL is to LF. But AE is to EB [Margin: f — Elements VI.1] as AG is to EC; and HL is to LF as HI is to HL. Therefore AG, the first, is also to EC, the third, as HI (the excess between the first and the middle term) is to AL (the excess between the middle term and the third). Which is what was proposed.

Diagram. The square ABCD, arranged for the numerical harmonic construction. The square has D at the top-left, C at the top-right, A at the bottom-left, B at the bottom-right. Along the top edge lie H and G (between D and C); along the bottom edge lie F and E (between A and B). The verticals EG and FH run from the bottom edge to the top edge. A horizontal line IK crosses from the left edge (at I) to the right edge (at K), cutting FH at the central point L. These lines partition the square into the rectangles AG, EC, AK, KH, and GL discussed in the proof.

- D (top-left corner)
- C (top-right corner)
- A (bottom-left corner)
- B (bottom-right corner)
- H (top edge): top of the vertical FH
- G (top edge, right of H): top of the vertical EG
- F (bottom edge): foot of the vertical FH
- E (bottom edge, right of F): foot of the vertical EG
- I (left edge): left end of the horizontal IK
- K (right edge): right end of the horizontal IK
- L (centre): intersection of IK with FH

Thus, if AE is set as 6 and EB as 4, the sum AB will be 10; and if we multiply this into AE and EB separately, AG becomes 60 and EC becomes 40. And twice the product of AE by EB — that is, HLIAEG — becomes 48, the harmonic mean. Therefore the three numbers in harmonic proportion will be 60, 48, 40, whose extremes have the same ratio as the given numbers 6 and 4. For since the same sum 10, gathered from 6 and 4, multiplying each of 6 and 4 has produced the extremes 60 and 40, [Margin: g — Elements VII.17] 60 will be to 40 as 6 is to 4. This is also evident from the figure: [Margin: h — Elements VI.1] for AG, that is 60, is to EC, that is 40, as AE, that is 6, is to EB, that is 4.

By this technique, if the sum of any two given numbers is multiplied into each of them separately, the extreme terms of a harmonic proportionality will be generated having the same ratio as the two given numbers; while the middle term will be the number that is double the product of the two given numbers multiplied together.

From this we shall also easily find three numbers in harmonic proportion whose extremes have a given ratio, and whose middle term equals a given number. For example, if the extremes are to have a quintuple ratio and the middle is to be 30: take 5 and 1 in the given ratio, and their sum 6 is to be multiplied into each, so as to make 30 and 6, between which the middle term 10 is set — which is double the product of the given numbers 5 and 1 multiplied one into the other. And 30, 10, 6 will be in harmonic proportion, as has been said. Then let the given middle 30 be divided by the found middle 10, and let the quotient 3 be multiplied into the three found numbers 30, 10, 6. In this way the three required numbers 90, 30, 18 will result.

Another method.

Between the two given numbers 6 and 4, let the arithmetic mean proportional be set — namely 5, half of their sum — thus: 6, 5, 4. Then let the middle term 5, multiplied into the extremes 6 and 4, produce 30 and 20; and between these let the middle term 24 be set, which the extremes produce when multiplied together, thus: 30, 24, 20. I say that these three are in harmonic proportion. For the three numbers 60, 48, 40 — which arise from the sum 10 of the given numbers 6 and 4 taken into the given numbers 6 and 4, and from 4 into 6 — are the doubles of the three numbers here found (since these were produced from half the sum of the given numbers into the given numbers, and from 4 into 6 once), as is clear. Therefore the excess between 60 and 48 is double the excess between 30 and 24, just as the excess between 48 and 40 is double the excess between 24 and 20, as we showed at Proposition 10. Since, then, the three numbers 60, 48, 40 are in harmonic proportion, as we demonstrated in the preceding problem, their halves too, 30, 24, 20, [Margin: i — Elements V.15] which have the same ratios, will be in harmonic proportion — inasmuch as their excesses also have the same ratio as the excesses of the numbers 60, 48, 40, seeing that the excesses of the numbers 30, 24, 20 are the halves of the excesses of these numbers 60, 48, 40, as has been said. [Margin: 60. 48. 40 / 30. 24. 20]

XI.

Given two numbers, to find the harmonic mean proportional.

Let there be two given numbers AB, 15, and BC, 10, between which the harmonic mean proportional is to be found. From the greater, AB, 15, subtract the number BD equal to the lesser BC, 10; and let the remaining number AD, 5 — namely the difference of the given numbers — be multiplied into the lesser BC, 10; and let the product, 50, divided by AC, 25, the sum of the given numbers, yield the quotient DE, 2.

[Number line: A ····· E · D ····· B ····· C]

Let this, added to the lesser BD, 10, make BE, 12. I say that BE is the harmonic mean proportional between AB and BC, thus: 15, 12, 10. For since the number arising from AD into BC, divided by AC, gives the quotient DE, the same number will be reproduced by multiplying the quotient DE into the divisor AC, as is clear from the definition of division. Since, then, the same number arises from AC (the first) into DE (the fourth) as from BC (the second) into AD (the third), [Margin: b — Elements VII.19 or VI.16] AC (the first) will be to BC (the second) as AD (the third) is to DE (the fourth); and by separation, AB is to BC (the first given to the

third given) as AE (the excess between the first term AB and the found middle BE) is to DE (the excess between the middle BE and BD, that is, the third term BC). Therefore AB, BE, BC — namely 15, 12, 10 — are in harmonic proportion. Which is what was proposed.

And so, if the difference of the given numbers is multiplied into the lesser, and the product divided by the sum of the given numbers, and finally the quotient added to the lesser, the middle term of the harmonic proportionality will be formed.

XII.

Given two numbers, to find the lesser extreme in harmonic proportionality.

Let there be two given numbers AB, 15, and AC, 12, for which the lesser harmonic proportional is to be found. Add BD equal to BC, the excess between the given numbers. By this sum AD, 18 — gathered from the greater number and the difference of the given numbers — divide the number 36 (made from the lesser given number AC into BC, the difference of the given numbers); and let the quotient CE be subtracted from the lesser given number AC. I say that the remainder AE, 10, is the required lesser third term, thus: 15, 12, 10.

[Number line: A ······· E · C ··· B ··· D]

For since the number arising from AC into BC, divided by AD, gives the quotient CE, the same number will be reproduced by multiplying the quotient CE into the divisor AD, as is clear from the definition of division. Since, then, the same number is generated from AD (the first) into CE (the fourth) as from BC (the second) into AC (the third), [Margin: a — Elements VII.19 or VI.16] AD (the first) will be to BC (the second) — that is, to BD — as AC (the third) is to CE (the fourth). And by separation, AB is to BD, that is to BC, as AE is to CE; and by alternation, AB (the first given) is to AE (the third, found) as BC (the excess between the first term AB and the middle AC) is to CE (the excess between the middle AC and the third term AE). Therefore the three numbers AB, AC, AE — that is, 15, 12, 10 — are in harmonic proportion. Which is what was proposed.

And so, if the number produced from the lesser number into the difference of the given numbers is divided by the sum gathered from the greater number and the difference of the given numbers, and the quotient is subtracted from the lesser given number, the remainder will be the lesser extreme in harmonic proportionality.

Another method.

Again let there be two given numbers AB, 15, and AC, 12, for which the lesser is to be found in the harmonic mean. To the greater, AB, add BD equal to the difference BC, 3, of the given numbers. And by the sum AD, 18, divide the number 180 produced from the multiplication of the given numbers AB, AC — that is, 15 and 12 — into each other, and let the quotient be AE, 10. I say that AE is the lesser extreme in the harmonic mean.

[Number line: A ······· E ·· C ··· B ··· D]

For since the number generated from AB into AC, divided by AD, gives the quotient AE, the same number will be reproduced by multiplying the quotient AE into the divisor AD, as is clear from the definition of division. Since, then, the same number arises from AD (the first) into AE (the fourth) as from AB (the second) into AC (the third), [Margin: b — Elements VII.19 or VI.16] AD (the first) will be to AB (the second) as AC (the third) is to AE (the fourth); and consequently, since AD is greater than AB, AC too will be greater than AE. Hence also, by separation, BD — that is, BC — is to AB as CE is to AE; and by conversion, AB is to BC as AE is to CE; and by alternation, AB (the first) is to AE (the third, found) as BC (the excess between the first term AB and the middle AC) is to CE (the excess between the middle AC and the third term AE). Therefore the three numbers AB, AC, AE — that is, 15, 12, 10 — are in harmonic proportion. Which is what was proposed.

And so, if the number produced from the multiplication of the given numbers into each other is divided by the sum gathered from the greater number and the difference of the given numbers, the quotient will be the lesser extreme term of the harmonic mean.

Euclid, *Elements*, Book VI — Clavius's added propositions on means; Euclid Prop. 18

(Continuation of Clavius's inserted series of propositions on the arithmetic, geometric, and harmonic means, following Euclid VI.15–17; then Euclid's own Proposition 18.)

XIII.

Given two numbers, to find the greater extreme in harmonic proportionality.

Let there be two given numbers, $AB = 10$ and $AC = 12$, for which a third and greater term in the harmonic mean is to be found. From the lesser number AB cut off BD , equal to the difference $BC = 2$ of the given numbers. Then, by the remaining number $AD = 8$, divide the number 24, which is produced by multiplying the greater number $AC = 12$ into the difference $BC = 2$ (that is, BD) of the given numbers. Let the quotient $CE = 3$ be added to the greater number $AC = 12$. I say that the sum $AE = 15$ is the greater term in the proportionality, or harmonic mean — that is, that the three numbers AE , AC , AB , namely 15, 12, 10, are harmonically proportional.

Diagram. A number-line laying out the five points in order from left to right, with their computed positions:

- A: left end (0)
- D: at 8 ($AD = 8$, the remainder after cutting off $BD = 2$ from B)
- B: at 10 (the given number AB)
- C: at 12 (the given number AC ; $BC = 2$)
- E: right end, at 15 ($CE = 3$ added to C)

For when the number produced from AC into BD is divided by AD and the quotient is CE , the same number will be reproduced by multiplying the quotient CE into the divisor AD . Since, then, the same number is generated from AC (the first) into BD (the fourth) as from CE (the second) into AD (the third), AC (the first) will be to CE (the second) as AD (the third) is to BD (the fourth). [Margin: a. Elements VII.19; or VI.16] And by composition, AE is to CE as AB is to BD , that is, to BC . And by alternation, AE (the first) is to AB (the third) as CE — the excess between the first term AE and the mean AC — is to BC , the excess between the mean AC and the third term AB . Therefore the three numbers AE , AC , AB , namely 15, 12, 10, are harmonically proportional. Which is what was proposed.

And so, given two numbers: if their difference is subtracted from the lesser, and the number produced from the greater number into the difference of the given numbers is divided by the remainder, and the quotient is added to the greater number, the greater extreme term of the harmonic mean will be formed.

ALITER.

Again, let there be two given numbers, $AB = 10$ and $AC = 12$, for which the greater term in the harmonic mean is to be found.

From the lesser AB take away BD , equal to the excess BC of the given numbers. Then, by the remaining number $AD = 8$, divide the number 120, produced by

multiplying the given numbers $AB = 10$ and $AC = 12$ together; and let the quotient be $AE = 15$. I say that AE is the greater term in harmonic proportionality.

Diagram. The same number-line as above, points in order A, D, B, C, E (positions 0, 8, 10, 12, 15).

For when the number produced from AB into AC is divided by AD and the quotient is AE , the same number will be reproduced by multiplying the quotient AE into the divisor AD , by the definition of division. Since, then, the same number is generated from AB (the first) into AC (the fourth) as from AD (the second) into AE (the third), AB (the first) will be to AD (the second) as AE (the third) is to AC (the fourth). [Margin: b. Elements VII.19; or VI.16] Consequently, since AB is greater than AD , AE will also be greater than AC . And since AE is to AC as AB is to AD , then by conversion of the ratio AE will be to CE as AB is to BD , that is, to BC . And by alternation, AE — the first term found — is to AB (the third) as CE , the excess between the first term AE and the mean AC , is to BC , the excess between the mean AC and the third term AB . Therefore the three numbers $AE = 15$, $AC = 12$, $AB = 10$ are harmonically proportional. Which is what was proposed.

And so, given two numbers: if their difference is subtracted from the lesser, and the number generated by multiplying the given numbers together is divided by the remainder, the quotient will give the greater extreme term of the harmonic mean.

Finally, following Pappus, I wish also to demonstrate those procedures by which, at the end of the treatment of proportions, we showed that from a proportion of equality all rational proportions arise, and conversely that from a proportion of inequality any equality arises; and finally that from any geometric proportionality there arise a geometric, an arithmetic, and a harmonic proportionality. Thus from a geometric proportionality — whether of equal or of unequal terms — there is generated another geometric proportionality of unequal terms.

XIV.

Given three terms in geometric proportion, whether equal or unequal:
the sum made up of the first once, the second twice, and the third once;
the sum composed of the second and third once each; and the third taken once — are geometrically proportional.

Let A, B, C be three terms continuously proportional in the geometric sense, and let D be the sum collected from A once, B twice, and C once; let E be the sum

composed of B and C once each; and finally let F be equal to C. I say that D, E, F are likewise geometrically proportional. For since A is to B as B is to C, then by composition A + B together is to B as B + C together is to C. [Margin: a. Elements V.12] Therefore, as all the antecedents together — that is, the term D — are to all the consequents together, namely B + C, so is one antecedent, that is B + C, to one consequent, C. But D equals A + B and B + C together, E equals B + C, and F equals C. Therefore D is to E as E is to F; and consequently D, E, F are geometrically proportional. Which is what was proposed.

	first	second	third	first	second	third
A B C	1	1	1	12	6	3
D E F	4	2	1	27	6 [?]	3

From this it is clear that if the terms A, B, C are equal, then D, E, F are doubles [i.e. in continued double proportion]. For B and C together are double C, and therefore E is double F, since E equals B + C and F equals C. Since, then, D, E, F are continuously proportional, they will be in double proportion. And if the equal terms A, B, C are units, a double proportionality D, E, F is generated in the least terms 4, 2, 1, because F is equal to C, that is, to unity.

But if A, B, C are doubles [in double proportion], the resulting D, E, F are triples. For if B is double C, the sum of B and C — that is, E — is triple C, that is, triple F; and so D, E, F will have a continued triple proportion. In the same way, from triples arise quadruples, from quadruples quintuples, and so on without end.

But if the terms A, B, C are doubles in reverse order, the resulting D, E, F are sesquialterate [3 : 2]. For since B is half of C, the sum of B and C — that is, E — is sesquialterate to C, that is, to F. In the same way, if A, B, C are triples in reverse order, the resulting D, E, F are sesquitertian [4 : 3]. For B will be a third part of C; therefore the sum of B and C — that is, E — contains C (or F) once and, in addition, a third part of it; and so D, E, F will have a continued sesquitertian proportion. By the same reasoning, from inverse quadruples arise sesquiquartans [5 : 4], and from inverse quintuples sesquiquintans [6 : 5], and so on to infinity. In the same manner we shall show that the generations of other proportions from other proportions have been correctly prescribed in the treatment of proportions; and likewise that any proportion of inequality can rightly be brought back to equality — since in that reduction we in some way retrace the operation by which we showed a proportion of inequality to be generated from equality.

(The third example — 3, 20, 1 — shows terms that are not geometrically proportional, yet still yield an arithmetic mean 47, 24, 1.)

XVI.

Given three terms, whether equal or unequal, in geometric proportion: the sum collected from the first twice, the second three times, and the third once; the sum composed of the second twice and the third once; and, finally, the sum gathered from the second once and the third once — are harmonically proportional.

Let A, B, C be three geometrically proportional terms, and let D be the sum collected from A twice, B three times, and C once; let E be the sum composed of B twice and C once; and finally let F be the sum gathered from B once and C once. I say that D, E, F constitute a proportionality, or harmonic mean. For since A is to B as B is to C, then, by the scholium to Proposition 22 of Book V, twice A is to B as twice B is to C. Therefore by composition, twice A plus once B is to B as twice B plus once C is to C. [Margin: a. Elements V.12] Hence, as all the antecedents together — namely twice A, three times B, and once C, that is, the term D — are to all the consequents together, namely to the sum of B and C, that is, to F, so is one antecedent, namely twice A plus once B, to B. Now twice A plus once B is the excess by which twice A, three times B, and once C — that is, the term D — exceeds the mean E [twice B plus once C]; while B is the excess by which E [twice B plus once C] exceeds the third term F. Therefore D, the first, will be to F, the third, as the excess between the first and the mean is to the excess between the mean and the third; and so D, E, F will be harmonically proportional. Which is what was proposed.

	first	second	third
A B C	12	6	3
D E F	45	15	9

The least terms D, E, F are always produced when the given terms A, B, C are units, as is clear in the examples set out. When, however, A, B, C are not units — even if they are the least terms in their proportion — the least terms D, E, F are not always produced. This is plainly established by the following examples. For in the first example, from the least terms of the sesquialterate proportion; in the second, from the least terms of the subquadruple proportion; and in the third, from the least terms of the subsesquitertian proportion, there are generated three terms in harmonic proportionality that are not the least. But if the terms

are inverted, then at last — remarkably — the least terms of the harmonic mean are produced, as these examples show.

Terms **not** least (given terms not units):

	first	second	third	first	second	third	first	second	third
A B C	9	6	4	1	4	16	9	12	16
D E F	40	16	10	30	24	20	70	40	28

The same terms inverted, now yielding the **least** harmonic terms:

	first	second	third	first	second	third	first	second	third
A B C	4	6	9	16	4	1	16	12	9
D E F	35	21	15	45	9	5	77	33	21

PROBL. 6. PROPOS 18.

[Margin: xix.]

On a given straight line, to describe a rectilinear figure similar and similarly situated to a given rectilinear figure.

Let AB be the given straight line, on which a rectilinear figure similar and similarly situated to the rectilinear figure CDEFG is to be described. From any one angle — say from F — draw lines to each of the opposite angles, resolving the figure into the triangles CDF, DEF, FGC. Next, on AB make angle BAI equal to angle DCF, and angle ABI equal to angle CDF, so that the lines AI and BI meet at the point I. (They will certainly meet, because the two angles IAB and IBA are less than two right angles, being equal to the two angles FCD and FDC, which [Margin: a. Elements I.17] are less than two right angles.) [Margin: b. Elements I.32] Then the remaining angle AIB will be equal to the remaining angle CFD, and the whole triangle AIB will be equiangular with the whole triangle CFD. Again, make angle IBH equal to angle FDE, and angle BIH equal to angle DFE. And since [Margin: c. Elements I.17] the two angles EDF and EFD are less than two right angles, the two angles HBI and HIB will also be less than two right angles; and therefore the lines

BH and IH will meet. Let them meet at the point H; and by the same reasoning triangle BHI will be equiangular with triangle DEF. Furthermore, make angle AIK equal to angle CFG, and angle IAK equal to angle FCG. And since [Margin: d. Elements I.17] the two angles GCF and GFC are less than two right angles, the two angles KAI and KIA will also be less than two right angles; and therefore the lines AK and IK will meet at some point. Let them meet at K; and triangle AKI will likewise be equiangular with triangle CGF. Proceed in this way until all the triangles of the proposed figure are exhausted, if there are more of them. I say, then, that the rectilinear figure ABHIK is similar and similarly situated to the rectilinear figure CDEFG.

Diagram. Two pentagons side by side. On the left is the constructed figure ABHIK; on the right is the given figure CDEFG, each triangulated by lines drawn from an apex to the base corners.

- Left pentagon (constructed, ABHIK):
- A: lower-left vertex (base left)
- B: lower-right vertex (base right)
- H: right vertex
- I: apex (top) — printed faint [?]
- K: upper-left vertex
- Right pentagon (given, CDEFG):
- C: lower-left vertex (base left)
- D: lower-right vertex (base right)
- E: upper-right vertex
- F: apex (top)
- G: upper-left vertex

For since angle IAB has been made equal to angle FCD, and angle IAK equal to angle FCG, the whole angle BAK will be equal to the whole angle DCG. By the same reasoning angle ABH will be equal to angle CDE, and the remaining angles equal to the remaining ones, as is clear from the construction, since the several parts of the one have been made equal to the several parts of the other. The rectilinear figure ABHIK will therefore be equiangular with the rectilinear figure CDEFG. And since [Margin: e. Elements VI.4] AB is to BI as CD is to DF, and BI is to BH as DF is to DE, the sides about the equal angles ABH and CDE are proportional, [Margin: f. Elements V.22] just as the sides about the equal angles H and E are proportional, because the triangles BHI and DEF are equiangular. [Margin: g. Elements VI.4] Again, HI is to IB as EF is to FD, and IB is to IA as FD is to FC, and IA

is to IK as FC is to FG; [Margin: h. Elements VI.4] and so, *ex aequali*, HI is to IK as EF is to FG, [Margin: i. Elements V.22] and therefore the sides about the equal angles HIK and EFG will also be proportional, and so on for the rest. Since, then, the figures are equiangular and have the sides about their equal angles proportional, they are similar and similarly described. Thus on the given straight line we have described a rectilinear figure similar and similarly situated to the given rectilinear figure. Which was to be done.

SCHOLIUM.

Rectilinear figures described upon straight lines are said to be similar and similarly situated when equal angles are set up upon those straight lines, and both the remaining angles are equal and the sides are proportional, following one another always in the same order. For example, the triangles ABC and DEF will not only be similar but also similarly described upon the straight lines BC and EF, if angles B and C are equal to angles E and F, and AB is to BC as DE is to EF, and so on. But upon the straight lines BC and DE they will not be said to be similarly described (although they are similar), since angles B and C are not equal to angles D and E. Likewise the rectangles AC and EG, being similar, will be said to be similarly described upon the straight lines DC and HG, because as AD is to DC, so is EH to HG, and so on. But the rectangles AC and IG will not be said to be similarly described upon the straight lines DC and HG, although they are similar, as is evident. They will, however, be similarly described upon the straight lines DC and IH, or upon the straight lines AD and GH.

Diagram. Two similar triangles. On the left the smaller triangle ABC (A at the apex, B at lower-left, C at lower-right); on the right the larger triangle DEF (D at the apex, E at lower-left, F at lower-right).

- A: apex of left triangle
- B: lower-left of left triangle
- C: lower-right of left triangle
- D: apex of right triangle
- E: lower-left of right triangle
- F: lower-right of right triangle

Diagram. Three rectangles illustrating similar and similarly-situated placement. A wide rectangle EFGH (E top-left, F top-right, G bottom-right, H bottom-left); beside it a tall rectangle IKGH (I top-left, K top-right, G bottom-right, H bottom-left) standing on the same base HG; and below, a wide rectangle ABCD (A top-left, B top-right, C bottom-right, D bottom-left).

- Rectangle "AC": vertices A (top-left), B (top-right), C (bottom-right), D (bottom-left); base DC, left side AD
- Rectangle "EG": vertices E (top-left), F (top-right), G (bottom-right), H (bottom-left); base HG, left side EH
- Rectangle "IG": vertices I (top-left), K (top-right), G (bottom-right), H (bottom-left); the tall rectangle on base HG

Moreover, all figures described according to the construction of this problem are necessarily similarly situated, as is clear in the rectilinear figures ABHIK and CDEFG described upon the lines AB and CD. Likewise all equilateral and equiangular figures are similarly situated upon any of their sides.

Perhaps, however, we shall accomplish the proposed problem more expeditiously in the following way. Let a rectilinear figure similar and similarly situated be described upon the given straight line G (or another equal to it), on the model of the given rectilinear figure ABCDEF. Producing the two sides AB and AF about angle A, draw from A through all the other angles the lines AC, AD, AE, extended as far as you like. Then from AB cut off AH equal to the given line G — or from AB itself, further produced, if G happens to be greater than AB. After this, through H draw the line HI parallel to the side BC, and through I the line IK parallel to the side CD, and so on in succession, until all the sides have their parallels, except the two produced sides AB and AF. What is proposed will then be accomplished: that is, the figure AHIKLM upon the straight line AH...

SCHOLIVM (to Prop. 18, continued)

Diagram. A composite construction figure illustrating the "more expeditious" method of the preceding scholium: the model figure ABCDEF and the similar figure AHIKLM constructed upon the line AH, sharing the common apex A, with the diagonals drawn from A through the several angles and the given line G marked (as short vertical segments) along the right of the figure. Many lines radiate from the shared apex to the successive vertices of the two overlapping figures. Legible vertex-labels, with approximate positions:

- L (top peak): apex vertex of the constructed/model outline
- K (upper right): vertex
- E (upper centre-left): model-figure vertex
- D (upper right, below K): model-figure vertex
- N (left): vertex/endpoint on a produced side
- F (left, below N): model-figure vertex

- M (lower left): constructed-figure vertex
- I (centre-right): constructed-figure vertex
- H (lower centre): constructed-figure vertex on the base
- B (lower right): model-figure vertex on the base
- G (vertical segments, right side): the given line, marked at several points
- [?] interior label near centre (reads as a second "L" or "K") — unresolved
- [?] label at centre-right (reads as "Q" or "G") — unresolved
- [?] label at far-right foot (reads as "II", possibly a doubled point) — unresolved
- A (shared apex): expected at the lower convergence of the radiating lines, not clearly lettered on the tile [?]

[Margin: k. *Elements* I.29] [Margin: l. *Elements* VI.4]

[the figure AHIKLM] will be similar to the proposed figure ABCDEF and similarly situated, being constructed upon the straight line AB — that is, upon AH, equal to the given straight line G. For the angle HAM is common to both; and since the angles AHI, AML are equal to the angles ABC, AFE [Margin: k. *Elements* I.29], and likewise the angles AIH, AIK are equal to the angles ACB, ACD — that is, the whole angle HIK is equal to the whole angle BCD — and in the same way the angles IKL, KLM are equal to the angles CDE, DEF: the rectilinear figures ABCDEF, AHIKLM will be equiangular. Moreover, the sides about the equal angles are proportional. For since the triangles AHI, AIK, AKL, ALM are similar, by the corollary of Prop. 4 of this book, to the triangles AEC [recte: ABC?], ACD, ADE, AEF, it follows [Margin: l. *Elements* VI.4] that as AB is to BC, so is AH to HI; again, as BC is to CA, so is HI to IA; and as CA is to CD, so is IA to IK; and consequently, *ex aequali*, as BC is to CD, so is HI to IK, and so on. Therefore, by the definition, the rectilinear figures ABCDEF and AHIKLM are similar, and indeed similarly situated, by the very method we described just above in this scholium.

THEOR. 13. PROPOS. 19.

[Margin: xvij.]

Similar triangles are to one another in the duplicate ratio of their corresponding [homologous] sides.

Clavius's demonstration. Let ABC, DEF be similar triangles, having the angles at B and E equal, likewise those at C and F, and so on; and let AB be to BC as DE is to EF, and so forth. I say that the triangles have to one another a ratio that is the duplicate of the ratio which the corresponding sides BC and EF have — or AB and

DE, or AC and DF. That is: if a third proportional BG be found to the corresponding sides BC, EF, then the triangle ABC is to the triangle DEF as the straight line BC is to the straight line BG. Consequently, since by Definition 10 of Book 5 the ratio of BC to BG is called the duplicate of the ratio of BC to EF, the ratio of triangle to triangle is likewise said to be the duplicate of the ratio of the corresponding sides BC, EF.

So that this amounts to nothing else than this: two triangles, or any two similar and similarly situated figures, have the duplicate of the ratio of two corresponding sides — that is, triangle is to triangle, or figure to figure, as the first line is to the third, when three lines are continually proportional in the ratio of the two corresponding sides. Such here are the three straight lines BC, EF, BG, continually proportional in the ratio of the corresponding sides BC, EF.

Diagram. Two similar triangles, with a separate line-segment below.

- A (apex of first triangle)
- B (foot, left) — C (foot, right): base BC
- D (apex of second triangle)
- E (foot, left) — F (foot, right): base EF
- B — G (a separate horizontal segment beneath the triangles): the third proportional BG laid out on the line B...G

First, then, let the sides BC, EF be equal, and consequently let the third proportional BG be equal to them, so that the ratio of BC to BG — which is called the duplicate of the ratio of the side BC to the side EF — is a ratio of equality. Now since the triangles ABC, DEF likewise have a ratio of equality [Margin: a. *Elements* I.26], because they are equal to one another (their angles B, C being equal to the angles E, F, and the sides BC, EF, which lie between those angles, being equal), the triangle will be to the triangle as the straight line BC is to the straight line BG. Since, then, this ratio is called the duplicate of the ratio of the corresponding sides BC, EF, the ratio of triangle ABC to triangle DEF will also be called the duplicate of the ratio which the side BC has to the side EF.

This can also be established as follows. Since, as has been said, the triangles ABC, DEF are equal — that is, have a ratio of equality — just as the corresponding sides BC, EF also do; and since a duplicated ratio of equality produces only a ratio of equality (for, three equal magnitudes being set out, the first is said to have to the third the duplicate of the ratio which the first has to the second, as is clear from Definition 10 of Book 5, even though the first has to the third a ratio of equality, just as the first has to the second): the triangle ABC will have to the triangle DEF

the duplicate of the ratio which the side BC has to the side EF. Which was the thing proposed.

Diagram. Two triangles for the case where BC exceeds EF, the larger triangle having its base cut at G.

- A (apex of larger triangle)
- B (foot, left) — G (point on the base) — C (foot, right): base BC with G between B and C, so that BG is the third proportional; AG is joined
- D (apex of smaller triangle)
- E (foot, left) — F (foot, right): base EF

[Margin: b. *Elements* VI.11] [Margin: c. *Elements* V.11] [Margin: d. *Elements* VI.15]
[Margin: e. *Elements* V.7] [Margin: f. *Elements* VI.1]

Next, let the side BC be greater than the side EF; and from BC let there be cut off [Margin: b. *Elements* VI.11] BG, a third proportional to the straight lines BC, EF, and let AG be joined. Now since AB is to BC as DE is to EF, then, by alternation, AB is to DE as BC is to EF. But as BC is to EF, so, by construction, is EF to BG. Therefore [Margin: c. *Elements* V.11] AB is to DE as EF is to BG. Since, then, the triangles ABG, DEF have the sides about the equal angles B, E reciprocally proportional, they will be equal to one another [Margin: d. *Elements* VI.15]; and therefore, as triangle ABC is to triangle DEF, so [Margin: e. *Elements* V.7] will the same triangle ABC be to triangle ABG. But as triangle ABC is to triangle ABG — being of the same altitude — so [Margin: f. *Elements* VI.1] is the base BC to the base BG. Therefore, as triangle ABC is to triangle DEF, so is BC to BG. And since the three lines BC, EF, BG are continually proportional, the ratio of the first, BC, to the third, BG, is called the duplicate of the ratio of BC, the first, to EF, the second. Therefore the triangle ABC also has to the triangle DEF the duplicate of the ratio of the side BC to the side EF. Therefore similar triangles are to one another [in the duplicate ratio of their corresponding sides], and so on. Which was to be demonstrated.

COROLLARIVM

From this it is manifest that, if three straight lines be proportional, then as the first is to the third, so is the triangle described upon the first to the triangle similarly and similarly-situatedly described upon the second.

For let there be three proportional straight lines A, B, C; and upon the first, A, and the second, B, let there be constructed similar triangles A and B, similarly described. I say that, as the straight line A, the first, is to the straight line C, the

third, so is the triangle A to the triangle B. For the ratio of the straight line A to the straight line C is, by definition, the duplicate of the ratio of the straight line A to the straight line B [Margin: g. *Elements* VI.19]. Since, then, the triangle A also has to the triangle B the duplicate of the ratio of the straight line A to the straight line B, it follows that as the straight line A is to the straight line C, so is the triangle A to the triangle B.

Diagram. Two similar right triangles with a short line-segment.

- A (the larger right triangle, right angle at its lower left)
- B (the smaller right triangle, right angle at its lower left)
- C (a short horizontal segment to the right): the third proportional line

In the same way you will show that the triangle upon the second is to the triangle similarly described upon the third as the first line is to the third. For let the three proportionals be C, B, A, and upon B, the second, and A, the third, let there be constructed similar and similarly-situated triangles B and A. I say that, as the straight line C is to the straight line A — [which ratio is] the duplicate of the ratio of C to B, that is, of the straight line B to the straight line A [Margin: h. *Elements* VI.19] — so is the triangle B to the triangle A. Since, then, the triangle B also has to the triangle A the duplicate of the ratio of the straight line B to the straight line A (because B and A are the corresponding sides): as the straight line C is to the straight line A, so will the triangle B be to the triangle A.

SCHOLIVM (to Prop. 19)

And so, if the ratio of the side BC to the side EF is known in numbers — say, as 6 to 5 — we shall also learn from this Proposition 19, in numbers, what ratio the triangle ABC has to the triangle DEF. For if the ratio 6 to 5 be continued in three numbers, thus: 36, 30, 25, the triangle will be to the triangle as 36 to 25 — that is, it will have a ratio whose denominator [multiplier] is $1\frac{11}{25}$. For since the ratio 36 to 25 is the duplicate of the ratio 36 to 30 (or 6 to 5), which the corresponding sides are supposed to have; and since the triangles too have the duplicate of the ratio of the corresponding sides, as has been shown in this theorem: the triangle will be to the triangle as 36 to 25.

We shall find the same ratio of the triangles if we multiply into itself the denominator of the ratio of the corresponding sides — namely $1\frac{1}{5}$. For the resulting number $36/25$, that is, $1\frac{11}{25}$, will be the denominator of the ratio which is the duplicate of the ratio of the sides, and consequently also the

denominator of the ratio of the triangles, since this too is the duplicate of the same ratio of the sides.

So too, if the corresponding sides had the ratio of 10 to 1, the triangles would have the ratio of 100 to 1, since the latter is the duplicate of the former, as appears here: 100, 10, 1. Or again, because the denominator of the tenfold ratio — that is, 10 — multiplied into itself begets 100, the denominator of the ratio which is the duplicate of the tenfold ratio.

THEOR. 14. PROPOS. 20.

[Margin: xviiij.]

Similar polygons are divided into similar triangles, equal in number and corresponding to the wholes; and the polygons have to one another the duplicate of the ratio which a corresponding side has to a corresponding side.

Clavius's demonstration. Let ABCDE, FGHIK be similar polygons, having the angles BAE, GFK equal, likewise the angles B, G, and so on in order; and let the sides about the equal angles be proportional — so that as AB is to BC, so is FG to GH; and as BC is to CD, so is GH to HI, and so forth. I say first that these polygons are divided into similar triangles, equal in number. For from the angles BAE, GFK let straight lines be drawn to the several opposite angles, namely AC, AD and FH, FI; and the polygons will be divided into triangles equal in number.

Now since the angle B is equal to the angle G by hypothesis, and the sides about them are proportional, the triangles ABC, FGH will be equiangular [Margin: a. *Elements* VI.6], having the angles BAC, GFH equal, and likewise the angles ACB, FHG, which lie opposite the corresponding sides. Therefore [Margin: b. *Elements* VI.4] they will have the sides about the equal angles proportional, and consequently they will be similar to one another. By the same reasoning the triangles AED, FKI will be similar, having the angles EAD, KFI equal, and the angles ADE, FIK equal.

Diagram. Two similar pentagons, each divided into three triangles by diagonals drawn from an apex.

- Pentagon ABCDE: A (apex, top), B (upper left), E (upper right), C (lower left), D (lower right); diagonals AC, AD
- Pentagon FGHIK: F (apex, top), G (upper left), K (upper right), H (lower left), I (lower right); diagonals FH, FI

[Margin: c. *Elements* VI.4] [Margin: d. *Elements* V.22] [Margin: e. *Elements* VI.6]

Next [Margin: c. *Elements* VI.4], since AC is to CB as FH is to HG, on account of the similarity of the triangles ABC, FGH; and since CB is to CD, by hypothesis, as HG is to HI, on account of the similarity of the polygons: it follows [Margin: d. *Elements* V.22], *ex aequali*, that as AC is to CD, so is FH to HI. And since the angle BCD is supposed equal to the angle GHI, while the angle ACB, subtracted, has been shown equal to the subtracted angle FHG, the remaining angle ACD will also be equal to the remaining angle FHI. Therefore [Margin: e. *Elements* VI.6] the triangles ACD, FHI, since they have the sides about the equal angles ACD, FHI proportional, will be equiangular, and therefore similar. And the same reasoning holds of all the other triangles, if there be more of them.

I say, moreover, that these triangles correspond to the whole polygons — that is, that any triangle in the one polygon is to its corresponding triangle in the other polygon as polygon is to polygon. For since the triangles ABC, FGH are similar, their ratio will be the duplicate of the ratio of the corresponding sides AC, FH. And by the same argument the ratio of the triangles ACD, FHI will be the duplicate of the ratio of the same corresponding sides AC, FH. Therefore, as triangle ABC is to triangle FGH, so will triangle ACD be to triangle FHI, since each of these ratios of triangles is the duplicate of the same ratio of the side AC to the side FH. And by a like reasoning it will be concluded that triangle ADE is to triangle FIK as ACD is to FHI; and so on in turn, if there be more triangles. The triangles of the one polygon are therefore proportional with the triangles of the other, so that the triangles of the one are the antecedents, and the triangles of the other the consequents, of the proportions. But as one antecedent is to one consequent, so [Margin: f. *Elements* V.12] are all the antecedents to all the consequents. Therefore, as any triangle of the one polygon is to its corresponding triangle of the other, so will the whole polygon be to the whole polygon; and therefore the corresponding triangles are to the whole polygons.

Lastly, I say that the polygons have to one another the duplicate of the ratio which the corresponding sides have — that is, if to the corresponding sides, say AB, FG, a third proportional line be found, then the polygon ABCDE is to the polygon FGHIK as the first line AB is to the third line found. And consequently, since the ratio

of AB to that third line is called the duplicate of the ratio of AB to FG, the ratio of polygon to polygon is likewise said to be the duplicate of the ratio of the corresponding sides AB, FG. For since triangle ABC is to triangle FGH as polygon ABCDE is to polygon FGHIK; and since triangle ABC has to triangle FGH [Margin:

g. *Elements* VI.19] the duplicate of the ratio which the corresponding sides AB, FG have — that is, the same ratio which AB has to that third line found — the polygons too will have to one another the duplicate of the ratio of the same corresponding sides AB, FG, that is, the same ratio which AB has to that third line found. And so similar polygons are divided into similar triangles, and so on. Which was to be demonstrated.

COROLLARIVM

From this it is manifest that, if there be three proportional straight lines, then as the first is to the third, so is the polygon described upon the first to the polygon similarly and similarly-situatedly described upon the second; or again, as the polygon described upon the second is to the polygon similarly and similarly-situatedly described upon the third.

This will be demonstrated from the present theorem in no other way than the corollary of the preceding theorem was shown from its theorem — as is plain in the figure here set alongside.

Diagram. Two squares of unequal size, with a short line-segment, illustrating the corollary.

- A (the larger square)
- B (the smaller square)
- C (a short horizontal segment to the right): the third proportional line

SCHOLIVM (to Prop. 20)

When similar polygons are pentagons, as in the figure, it may also be demonstrated in this way that the middle triangles ACD, FHI are similar. Since the angles ACB, FHG are equal, on account of the similarity of the triangles ABC, FGH; if they be taken away from the angles BCD, GHI, which are likewise equal, on account of the similarity of the polygons: the remaining angles ACD, FHI will also be equal. And by the same reasoning the angles ADC, FIH will be shown equal [Margin: h. *Elements* I.32]; and therefore the remaining angles CAD, HFI will be equal as well. Since, then, the triangles ACD [printed ACC], FHI are equiangular, they will have the sides about the equal angles proportional [Margin: i. *Elements* VI.4], and consequently they will be similar.

[Margin: k. *Elements* VI.6]

But when the polygons have more angles, it must be demonstrated that the middle triangles are similar by way of equality of ratios and by Prop. 6 of this

book, as was done in the proposition — because the angles cannot be proved equal, except for one only, namely the one that lies next to the preceding similar triangle, such as the angle ACD and FHI. For the angle ADC cannot be shown equal to the angle FIH, since in that case there still remain further triangles down to the similar triangles AED, FKI, and so on. I have said this so that no one may suppose we spoke to no purpose in the proposition when we said that AC is to CB as FH is to HG, and CB is to CD as HG is to HI, and consequently, by equality of ratios, AC is to CD as FH is to HI; and therefore, the angles ACD, FHI being equal, the triangles ACD, FHI are similar. For this demonstration is altogether necessary, and fits all polygons of however many angles, since all the middle triangles are shown to be similar in this manner, as is evident. For if the triangles ADE, FIK were not the last, they would be demonstrated similar in this way: since AD is to DC as FI is to IH, on account of the similarity of the triangles ACD, FHI; and likewise DC is to DE as IH is to IK, on account of the similarity of the polygons: it follows, *ex aequali*, that AD is to DE as FI is to IK. And since the angle CDE is supposed equal to the angle HIK, while the subtracted angle ADC has been shown equal to the subtracted angle FIH: the remaining angle ADE will also be equal to the remaining angle FIK. Therefore [Margin: k. *Elements* VI.6] the triangles ADE, FIK will be similar, since they have the sides about the equal angles ADE, FIK proportional. And so on in turn, if there be more triangles.

But if the similar polygons are equilateral and equiangular, they too will be divided into similar triangles, equal in number and corresponding to the wholes, when from the centres of the circles that circumscribe them straight lines are drawn to all the angles. For let ABCDEF, GHIKLM be similar, equilateral, and equiangular polygons, circumscribed by circles about the centres N, O, from which let the straight lines NA, NB, and so on be drawn. I say that the triangles NCD, OIK are similar and correspond to the whole polygons. For the angles CND, IOK are equal (because they are equally submultiples of four right angles: for each of the spaces about N and O, being equivalent to four right angles, is divided — by the corollary 2 of Prop. 15 of Book 1 — into angles equal both in number and in magnitude. For since the angles at the centre N stand upon equal arcs [Margin: l. *Elements* III.27], they will be equal to one another; and by the same reasoning the angles at the centre O will be equal). And the sides about them are proportional, since in each case there is a ratio of equality [Margin: m. *Elements* VI.6]. Therefore the triangles NCD, OIK will be similar. And the same reasoning holds of the rest. But that these triangles correspond to the whole polygons will be demonstrated with no trouble: for since the whole polygons are equimultiples of their triangles, as is evident, they will assuredly have the same ratio as the triangles themselves [Margin: n. *Elements* V.15].

Diagram. Two regular hexagons inscribed in circles, each divided into six triangles by radii from the centre.

- Hexagon ABCDEF inscribed in a circle with centre N: A (upper left), B (left), C (lower left), D (lower right), E (right), F (upper right); centre N, with radii to each vertex — triangle NCD is the lower triangle
- Hexagon GHIKLM inscribed in a circle with centre O: G (upper left), H (left), I (lower left), K (lower right), L (right), M (upper right); centre O, with radii to each vertex — triangle OIK is the lower triangle

[Margin: l. *Elements* III.27] [Margin: m. *Elements* VI.6] [Margin: n. *Elements* V.15]

Furthermore, from this theorem we shall very easily demonstrate that theorem which we have already shown by another method in the scholium of Prop. 4 of Book 2. Namely:

If one straight line is double another straight line, the square of the former will be quadruple the square of the latter; and conversely, if a square is quadruple a square, the side of the former will be double the side of the latter.

Clavius's demonstration. First, let the straight line A be double the straight line B. I say that the square of A is quadruple the square of B. For since all squares are similar, as is clear from Definition 1 of this book, the ratio of the square of A to the square of B will, by this present proposition, be the duplicate of the ratio of the corresponding sides A and B; and since these have to one another a double ratio, the ratio of the squares will be quadruple. For a quadruple ratio is the duplicate of a double ratio, as appears here: 1, 2, 4.

(Continuation of the Scholium to Proposition 20 — the converse of the "line double $\Rightarrow$ square quadruple" theorem, begun in the preceding pages.)

Next, let the square of A be quadruple the square of B. I say that the side A is double the side B. For since the ratio of the squares, which is assumed to be quadruple, is the duplicate of the ratio of the corresponding sides (as has been said), the corresponding sides A and B will have a double ratio; for a quadruple ratio is the duplicate of a double ratio, as is clear in the example already brought forward.

And so, if the ratio of side AB to side FG in the polygons we described in the proposition is expressed in numbers — say, as 5 to 4 — we shall also come to know, by this Proposition 20, in numbers, what ratio polygon ABCDE has to polygon FGHIK. For if the ratio 5 to 4 is continued through three numbers, in this

way — 25, 20, 16 — polygon will be to polygon as 25 to 16. This ratio has a denominator [i.e. quotient] of $1 \frac{9}{16}$. For since the ratio 25 to 16 is the duplicate of the ratio 25 to 20, or 5 to 4, which the corresponding sides AB, FG are assumed to have, and since it has been demonstrated in this theorem that the polygons too have the duplicate ratio of that which their corresponding sides have, polygon will be to polygon as 25 to 16. The same denominator of the ratio of the polygons is found if the denominator of the ratio of the corresponding sides — namely $1 \frac{1}{4}$ — is multiplied into itself; for the number produced is $\frac{25}{16}$, that is, $1 \frac{9}{16}$, which will be the denominator of the ratio said to be the duplicate of that which the corresponding sides have (from what we wrote in Definition 10 of Book V). And accordingly the denominator of the ratio of the polygons, since it too is the duplicate of the same ratio of the corresponding sides.

So too, if the corresponding sides had a tenfold ratio, the polygons would have a hundredfold ratio, since this is the duplicate of that, as is clear here: 100, 10, 1. Or again, because the denominator 10 of the tenfold ratio, multiplied into itself, produces the denominator 100 of the hundredfold ratio, which is called the duplicate of the tenfold.

Diagram. Two squares side by side illustrating the numerical example, the larger about twice the linear size of the smaller.

- A: the larger square (left)
- B: the smaller square (right)

THEOR. 15. PROPOS. 21.

Figures that are similar to the same rectilinear figure are also similar to one another.

Let the rectilinear figures ABC and DEF be similar to the rectilinear figure GHI. I say that they are also similar to one another. For since, by reason of their similarity, the angles of figure ABC are equal to the angles of figure GHI, and for the same reason the angles of figure DEF are equal to the angles of the same figure GHI, [Margin: a — Common Notion 1 (I, pron. 1)] the angles of figure ABC will therefore be equal to the angles of figure DEF. Again, since by the same similarity the sides of figure ABC are proportional to the sides of figure GHI — namely those about the equal angles — and likewise, for the same reason, the sides of figure DEF are proportional to the sides of the same figure GHI, [Margin: b — V.11] the sides of figure ABC will also be proportional to the sides of figure DEF, namely those which contain equal angles. And so, by the definition, figures

ABC and DEF are similar. Therefore figures that are similar to the same rectilinear figure are also similar to one another. Which was to be shown.

[Margin: xx.]

Diagram. Three triangles standing on a common baseline: a large triangle on the left, a small triangle in the middle, and a triangle on the right. Figures ABC and DEF are each similar to the middle figure GHI.

- A: apex of the large left triangle
- B: bottom-left vertex of the large triangle
- C: bottom-right vertex of the large triangle
- G: apex of the small middle triangle
- H: bottom-left vertex of the middle triangle
- I: bottom-right vertex of the middle triangle
- D: apex of the right triangle
- E: bottom-left vertex of the right triangle
- F: bottom-right vertex of the right triangle

THEOR. 16. PROPOS. 22.

If four straight lines are proportional, the rectilinear figures similar and similarly described upon them will also be proportional. And if the rectilinear figures similar and similarly described upon the straight lines are proportional, the straight lines themselves will also be proportional.

Let there first be four straight lines AB, CD, EF, GH that are proportional, so that as AB is to CD, so is EF to GH. Upon AB and CD let there be constructed any two similar and similarly described rectilinear figures ABI, CDK; and likewise upon EF, GH two others, any similar and similarly described rectilinear figures EFML, GHON. I say that these figures too are proportional, so that as ABI is to CDK, so is EM [EFML] to GO [GHON].

[Margin: a — VI.11] For let a third proportional P be found to the straight lines AB, CD, and a third proportional Q to the straight lines EF, GH. Then as AB is to P, so is EF to Q. But as AB is to P, so is the rectilinear figure ABI to the rectilinear figure CDK — since they are similar and similarly described — by the corollary of Proposition 20 of this book, or, if they are triangles, by the corollary of Proposition 19. [Margin: b — V.22] And it will be, by equality [*ex aequo*], that as the figure ABI is to the figure CDK, so is the figure EM to the figure GO. By the

same reasoning, as EF is to Q, so is EM to GO. [Margin: c — V.11] Therefore, as ABI is to CDK, so is EM to GO. Which is what was proposed.

Next, let ABI, CDK, EM, GO be proportional rectilinear figures. I say that the four straight lines AB, CD, EF, GH are also proportional, so that as AB is to CD, so is EF to GH. [Margin: d — VI.12] For let a fourth proportional RS be found to the three straight lines AB, CD, EF, and upon it let a rectilinear figure RSVT be described, similar and similarly placed to the figure EM. [Margin: e — VI.21] Since, then, as AB is to CD, so is EF to RS, it will also be — as has already been shown — that as ABI is to CDK, so is EM to RV [RSVT]. But as ABI is to CDK, so too is EM to GO by hypothesis. [Margin: f — V.11] Therefore as EM is to RV, so is EM to GO; and consequently RV and GO are equal. Since these are similar and similarly placed, they will necessarily stand — as we shall show presently — upon equal straight lines RS and GH. [Margin: g — V.9] Hence RV and GO are equal. [Margin: h — V.7] Therefore as EF is to RS, so is EF to GH. But EF was set as to RS in the ratio of AB to CD. [Margin: i — V.11] Therefore it will also be that as AB is to CD, so is EF to GH. Wherefore, if the four straight lines are proportional, etc. Which was to be demonstrated.

[Margin: xxi.]

Diagram. A compound figure illustrating Proposition 22. Four straight line-segments each carry a similar figure: triangles ABI and CDK are described on segments AB and CD; four-sided figures EFML and GHON are described on segments EF and GH. Third proportionals P (to AB, CD) and Q (to EF, GH) are drawn as short segments, and a fourth proportional RS bears the four-sided figure RSVT.

- A, B: endpoints of segment AB (base of triangle ABI)
- I: apex of triangle ABI
- P: third-proportional segment (of AB, CD)
- C, D: endpoints of segment CD (base of triangle CDK)
- K: apex of triangle CDK
- Q: third-proportional segment (of EF, GH)
- E, F: bottom vertices of the four-sided figure on EF
- L, M: top vertices of that figure (L left, M right)
- G, H: bottom vertices of the four-sided figure on GH
- N, O: top vertices of that figure (N left, O right)
- R, S: endpoints of the fourth-proportional segment RS
- T, V: top vertices of the four-sided figure RSVT (T left, V right)

LEMMA.

That equal rectilinear figures, similar and similarly described — such as GO and RV — stand upon equal straight lines is shown thus. For if the straight lines GH, RS are unequal, let GH be the greater. Since, then, on account of the similarity of the figures, GH is to HO as RS is to SV, and since GH is set greater than RS, HO will also be greater than SV; and therefore the rectilinear figure GO will be greater than the rectilinear figure RV, seeing that the latter can be placed within the former — which is absurd, since it is contrary to the hypothesis. Therefore the straight lines GH, RS are not unequal. Which is what was proposed.

Otherwise. Let there be two rectilinear figures ABC, DEF, equal and similar and similarly placed. I say that the corresponding sides — such as the straight lines AB, DE — are equal. For if they be not believed equal, let AB be greater than DE, and let a third proportional G be found to the straight lines AB, DE. Since, then, as AB is to DE, so is DE to G, and AB is greater than DE, DE will also be greater than G, and therefore AB much greater than G. But as AB is to G, so is the rectilinear figure ABC to the rectilinear figure DEF, by the corollary of Proposition 19 or 20 of this book. Since, then, AB is greater than G, the rectilinear figure ABC will also be greater than the rectilinear figure DEF — which is absurd, since it was assumed equal. Therefore AB is not a greater straight line than DE. Neither will it be less, by the same reasoning; for then the rectilinear figure ABC would be shown less than the rectilinear figure DEF, which is contrary to the hypothesis. Therefore the straight lines AB, DE are equal.

Diagram. Two similar house-shaped (pentagonal) rectilinear figures side by side, with a separate horizontal segment G drawn above them (the third proportional).

- G: the separate segment above the figures
- A: bottom-left vertex of the left figure
- B: bottom-right vertex of the left figure
- C: upper vertex of the left figure
- D: bottom-left vertex of the right figure
- E: bottom-right vertex of the right figure
- F: upper vertex of the right figure

SCHOLIVM.

In the same way, if there are three proportional straight lines, the rectilinear figures similar and similarly described upon them will also be proportional, and so on. For if the middle line is taken, and its figure taken twice, there will be four

proportional straight lines, and hence four proportional rectilinear figures, as Euclid here demonstrated. Since, then, what is described upon the second line is equal to what is described upon the third — indeed, it is the very same — what is proposed is manifest.

Diagram. A row of similar triangles illustrating the three-proportionals case: a triangle on the first line, the figure of the middle line drawn twice, and a triangle on the third line.

- A, B: base of the first triangle
- G [?]: apex of the first triangle (apex letter at the resolution floor)
- C, D: base of the two middle triangles (each drawn on line CD)
- N: apex of each of the two middle triangles
- E, F: base of the last triangle
- I: apex of the last triangle

A shorter proof. This whole proposition will be demonstrated more expeditiously in this way. Let it first be posited that as AB is to CD, so is EF to GH. I say that it is also as ABI is to CDK, so is EM to GO. [Margin: a — VI.19 or 20] For since the ratio of figure ABC to CDK is the duplicate of the ratio AB to CD, and likewise the ratio of figure EM to figure GO is the duplicate of the ratio EF to GH, [Margin: d] the ratios ABI to CDK and EM to GO will be equal, seeing that they are the duplicates of equal ratios AB to CD and EF to GH. Which is the first point.

Let it next be posited that as ABI is to CDK, so is EM to GO. I say that it is also as AB is to CD, so is EF to GH. [Margin: b — VI.19 or 20] For since the ratio ABI to CDK is the duplicate of the ratio AB to CD, and likewise the ratio EM to GO the duplicate of the ratio EF to GH, the ratios AB to CD and EF to GH will be equal, seeing that their duplicates — ABI to CDK and EM to GO — are set equal. Which is the second point.

Diagram. The reconstruction of the Proposition-22 figure for the shorter proof: triangles ABI and CDK with a segment P; four-sided figures on EF and GH with top vertices L, M and N, O; the square RSVT with top vertices T, V; and a segment Q.

- I: apex of triangle ABI; A, B: its base
- P: segment
- K: apex of triangle CDK; C, D: its base
- L, M: top vertices of the figure on EF; E, F: its base
- N, O: top vertices of the figure on GH; G, H: its base
- T, V: top vertices of the square RSVT; R, S: its base

- Q: segment

[Margin: xxiii] (?) (roman marginal proposition-number at the resolution floor)]

THEOR. 17. PROPOS. 23.

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.

Let AC and CF be equiangular parallelograms, having the angles BCD, ECG equal. I say that their ratio is compounded of the two ratios which the two sides of the one about the equal angle have to the two sides of the other about the equal angle — the antecedents of the ratios being in the one parallelogram and the consequents in the other. That is, the ratio of parallelogram AC to parallelogram CF is compounded of the ratios of side BC to CG and of side DC to CE; or else of the ratios of side BC to CE and of side DC to CG. That is, if three straight lines I,

K, L are taken, such that I is to K as side BC is to side CG, and K to L as side DC is to side CE, then the parallelogram AC will be to the parallelogram CF as the straight line I is to the straight line L. And accordingly, since the ratio of AC to CF is said to be compounded of the ratios I to K and K to L, that is, of the ratios BC to CG and DC to CE.

For let the parallelograms be joined so as to be about equal angles, so that BC and CG make one straight line. Then, since the angles BCD, ECG are equal, DC and CE will also be one straight line, as we proved above (at Proposition 15 of Book I, from Proclus [Proclus]). Let AD, FG then be produced until they meet at H. Taking any straight line I, [Margin: a — VI.11] let a fourth proportional K be found to the three lines BC, CG, and I; and likewise a fourth proportional L to the three lines DC, CE, and K. Since, then, [Margin: b — VI.1] as BC is to CG, so is AC to CH, and as BC is to CG, so is I set to K, [Margin: c — V.11] it will also be that as AC is to CH, so is I to K. And by the same argument you will show that as HC is to CF, so is K to L; for as DC is to CE, [Margin: d — VI.1] so is HC to CF. Since, then, K has been set to L as DC to CE, HC will also be to CF as K to L. [Margin: e — V.22] Therefore, by equality, it will be that as AC is to CF, so is I to L. But the ratio I to L, by Definition 5 of this book, is compounded of the ratios BC to CG and DC to CE. Out of these same ratios, therefore, will the ratio of parallelogram AC to parallelogram CF also be compounded. By the same reasoning we shall show that the ratio of AC to CF is compounded of the ratios BC to CE and DC to CG — provided the parallelograms are so joined about the equal angles that BC, CE make one straight line, and so on. Therefore equiangular parallelograms have to one another the ratio, etc. Which was to be shown.

Diagram. Two equiangular parallelograms AC and CF joined at the common vertex C, with AD and FG produced to meet at H; three auxiliary segments I, K, L are drawn below.

- A: top-left vertex
- D: top-middle vertex
- H: top-right vertex (meeting-point of produced lines)
- B: mid-left vertex
- C: central common vertex
- G: mid-right vertex
- E: lower vertex (below C)
- F: bottom-right vertex
- I, K, L: the three auxiliary segments below the figure

SCHOLIUM.

The same will be demonstrated more readily in this way. Let the parallelograms be joined as before. [Margin: f — VI.1] Since it is as AC is to CH, so is BC to CG, and as CH is to CF, so is DC to CE, and since the ratio of AC to CF is compounded, by the definition, of the intermediate ratios AC to CH and CH to CF, the same ratio of AC to CF will also be compounded of the ratios BC to CG and DC to CE, which are equal to those intermediates. Which is what was proposed.

Euclid also demonstrates this proposition arithmetically in Book VIII, Proposition 5.

And so, if the ratios of the sides — namely BC to CG and DC to CE — are known in numbers, we shall likewise come, from this Proposition 23, to the knowledge in numbers of the ratio of parallelogram AC to parallelogram CF, in this way.

[Margin: g — VI.23] Let the ratio BC to CG be the same as 11 to 5, and the ratio DC to CE the same as 7 to 11; and let these two ratios be continued in three numbers, 77, 35, 55, so that 77 is to 35 as 11 is to 5, and 35 is to 55 as 7 is to 11. And since the ratio 77 to 35, together with 35 to 55 — that is, the ratio compounded of the ratios of the sides — gives that as 77 is to 55, so is parallelogram AC to parallelogram CF (because the ratio of the parallelograms is likewise compounded of the same ratios of the sides), the ratio of the parallelograms will have a denominator [quotient] of $1 \frac{2}{5}$.

Now the two given ratios will be continued in three numbers either by those which are demonstrated at Proposition 4 of Book VIII, or else in this way. Given the former ratio, 11 to 5, let it be made that, as 7 is to 11 (which is the latter ratio),

so is 5 to another; and there will be found the number $7\frac{6}{7}$ [?]. Thus there will stand three numbers having those two ratios: 11, 5, $7\frac{6}{7}$ [?]. And if the third be reduced to that single [improper] fraction, $55/7$, and the other two numbers be multiplied by the denominator 7, there will be the two products 77, 35 and the numerator 55 — three whole numbers in the same ratios.

So too, if the ratios of the sides be as 4 to 2 and 3 to 3, these two ratios will be continued in these three numbers, 4, 2, 2; or in these least numbers, 2, 1, 1. And since the first is to the third as double, the ratio of the parallelograms will likewise be double. This is evident from the figure. For if the side BC is divided into 4 equal parts, and CG into 2, but DC into 3 and CE into 3, so that BC has to CG a double ratio, but DC to CE a ratio of equality, as was posited — then, if through the points of division lines are drawn parallel to the sides, the parallelogram AC will contain 12 equal rhombi, but CF only 6; and thus the parallelogram AC will have to the parallelogram CF a double ratio, as was said.

Diagram. The joined parallelograms AC and CF (as before), here subdivided by lines drawn parallel to the sides through the points of division: parallelogram AC is cut into 12 equal rhombi, parallelogram CF into 6.

- A: top-left vertex
- D: top-middle vertex
- H: top-right vertex
- B: mid-left vertex
- C: central common vertex
- G: mid-right vertex
- E: lower vertex
- F: bottom-right vertex

It is also evident that one of the ratios that compose the compound ratio can be greater than the compound ratio itself, and also equal to it. For in the former example the ratio of side BC to CG is as 11 to 5 — that is, one whose denominator is $2\frac{1}{5}$ — which is greater than the ratio of the parallelograms, since the denominator of that is $1\frac{2}{5}$. But in the latter example the ratio of side BC to CG is double, just as the ratio of the parallelograms also is. From this, therefore, it can further be established that this composition of ratios is not an addition, as some interpreters would have it; and likewise that a ratio of greater inequality can be compounded both out of a ratio of lesser inequality and out of a ratio of equality — which those same interpreters deny. For in the former example the ratio of

side DC to CE is one of lesser inequality, but in the latter one of equality. But on this matter we shall write more at the end of Book IX.

In this place Federico Commandino [Federico Commandino] demonstrates certain other things which pertain either to the composition of ratios or are demonstrated from it — matters not without use, which we too have resolved to bring forward, though with the demonstrations somewhat altered. They are as follows.

1.

Triangles which have one angle equal to one angle have to one another the ratio compounded of the sides which contain the equal angle.

Let there be triangles ABC and DEF having angle C equal to angle F. I say that the ratio of triangle ABC to triangle DEF is compounded of the sides — that is, of the ratio of BC to EF together with the ratio of AC to DF; or else of the ratio of BC to DF together with the ratio of AC to EF. For when the parallelograms CG and FH are completed, they will be equiangular, [Margin: a — Elements VI.23] and therefore their ratio will be compounded of the sides. Since, then, the triangles ABC and DEF have the same ratio to those [parallelograms] [Margin: b — Elements I.34] of which they are the halves, [Margin: c — Elements V.15] the ratio of triangle ABC to triangle DEF will likewise be compounded of the ratios of the sides BC, AC to the sides EF, DF.

Diagram. Two parallelograms, each with its diagonals drawn, illustrating the completion of the two triangles into parallelograms CG and FH.

- Left parallelogram (CG, completing triangle ABC): G (top-left), A (top-right), B (bottom-left), C (bottom-right); the diagonal from B to A and a further diagonal are drawn.
- Right parallelogram (FH, completing triangle DEF): H (top-left), D (top-right), E (bottom-left), F (bottom-right), with the point I marked near the base at F; diagonals drawn.

II.

To compound one ratio out of two ratios, or out of several.

How this is done is easily gathered from the demonstration of Proposition 23 of this book. For let there be three ratios — A to B, C to D, and E to F — set out in lines. It is now required to compound a single ratio out of them. [Margin: d — Elements VI.12] Let it be arranged that as A is to B, so G is to H; and as C is to D, so

H is to I; and as E is to F, so I is to K. I say that the ratio of G to K is compounded of the three given ratios. For since it is compounded of the ratios G to H, H to I, and I to K — by Definition 5 of this book — it will also be compounded of the ratios A to B, C to D, and E to F, because these are respectively equal to those.

Diagram. A row of ten vertical straight-line segments of varying lengths, representing the magnitudes named in the demonstration, labelled left to right: A, B, C, D, E, F, G, H, I, K.

III.

To subtract a lesser ratio from a greater.

Let A to B be the lesser ratio, to be subtracted from the greater ratio C to D.
 [Margin: e — Elements VI.12] Let it be arranged that as A is to B, so C is to E; and let E be established as a mean term between C and D. I say that the ratio A to B has been taken away from the ratio C to D, and that the remaining ratio is E to D. For since the ratio C to D is compounded of the ratios C to E and E to D, if the ratio C to E — that is, A to B — is taken away, the ratio E to D will be left.

But this compounding and removal of ratios is not, properly speaking, addition and subtraction; for otherwise the whole would be equal to a part, and less, and a greater ratio could be subtracted from a lesser — as we shall show at the end of Book 9.

Diagram. A column of five horizontal straight-line segments of varying lengths, representing the magnitudes of the demonstration, labelled top to bottom: A, B, C, E, D.

IV.

Triangles which have one angle equal to one angle have the same ratio as the rectangles contained by the sides comprehending the equal angle.

Let there be triangles ABC and DEF having angle A equal to angle D. I say that triangle ABC is to triangle DEF as the rectangle contained by AB, AC is to the rectangle contained by DE, DF. For when the perpendiculars BG and EH are drawn to AC and DF, the triangles ABG and DEH will be equiangular, as is clear from the corollary to Proposition 32 of Book 1, since the two angles A and AGB are equal to the two angles D and DHE. [Margin: f — Elements VI.4] Therefore, as GB is to BA, so HE is to ED. But as GB is to BA, [Margin: g — Elements VI.1] so is the rectangle contained by BG, AC to the rectangle contained by AB, AC (for if GB and BA are taken as bases, they will have the same altitude AC). And by the same

reasoning, as HE is to ED, so is the rectangle contained by EH, DF to the rectangle contained by DE, DF. [Margin: h — Elements V.11] The rectangle contained by BG, AC is therefore to the rectangle contained by AB, AC as the rectangle contained by EH, DF is to the rectangle contained by DE, DF; and, alternately, the rectangle contained by BG, AC is to the rectangle contained by EH, DF as the rectangle contained by AB, AC is to the rectangle contained by DE, DF. But as the rectangle contained by BG, AC is to the rectangle contained by EH, DF, [Margin: i — Elements V.15] so is triangle ABC to triangle DEF, [Margin: k — Elements I.41] because these triangles are the halves of those rectangles (for they have the same bases AC, DF as those rectangles, and the same altitudes BG, EH, and hence are set up between the same parallels as they). Therefore triangle ABC will also be to triangle DEF as the rectangle contained by AB, AC is to the rectangle contained by DE, DF. Which was proposed.

ANOTHER WAY, and more easily. From the first theorem of this scholium, the ratio of triangle ABC to triangle DEF is compounded of the ratios AB to DE and AC to DF. [Margin: l — Elements VI.23] But the ratio of the rectangle contained by AB, AC to the rectangle contained by DE, DF is likewise compounded of the same ratios AB to DE and AC to DF. Therefore triangle ABC will be to triangle DEF as the rectangle contained by AB, AC is to the rectangle contained by DE, DF. Which was proposed.

Diagram. Two triangles, each with a cevian drawn from a base vertex to the foot of a perpendicular on the opposite side.

- Left triangle ABC: A (apex), B (bottom-left), C (bottom-right); G marked on side AC (the foot of the perpendicular BG), with the segment from B to G drawn.
- Right triangle DEF: D (apex), E (bottom-left); the bottom-right vertex and the interior foot-point are labelled F and H, with a segment drawn from E. [The printed placement of the labels F and H appears transposed relative to the demonstration, in which H is the foot of the perpendicular EH on DF and F is a vertex.] [?]

V.

Parallelograms equiangular to one another have the same ratio as the rectangles contained by the sides of them comprehending the equal angle.

Let ABCD and EFGH be parallelograms equiangular to one another, whose angles B and F are equal. I say that as parallelogram BD is to parallelogram FH, so is the

rectangle contained by AB, BC to the rectangle contained by EF, FG. For when the diagonals AC and EG are drawn, subtending the equal angles B and F, the triangles ABC and EFG will have angle B equal to angle F. Therefore, as we have already demonstrated, as triangle ABC is to triangle EFG, so is the rectangle contained by AB, BC to the rectangle contained by EF, FG. [Margin: m — Elements V.15] Since, then, the parallelograms BD and FH have the same ratio as the triangles ABC and EFG — [Margin: n — Elements I.34] because these are the halves of them — [Margin: o — Elements V.11] it will also follow that as parallelogram BD is to parallelogram FH, so is the rectangle contained by AB, BC to the rectangle contained by EF, FG. Which was proposed.

Diagram. Two equiangular parallelograms, each with a diagonal drawn to form the triangles ABC and EFG.

- Left parallelogram ABCD: A (top-left), D (top-right), B (bottom-left) [?], C (bottom-right); diagonal AC drawn.
- Right parallelogram EFGH: F (top-left), H (top-right), E (bottom-left) [?], G (bottom-right) [?]; diagonal EG drawn.

VI.

Triangles and parallelograms have to one another a ratio compounded of the ratio of the bases and the ratio of the altitudes.

Let there be triangles ABC and DEF and parallelograms CG and EH, whose altitudes are AI and DK. I say that their ratio is compounded of the ratio of the base BC to the base EF and the ratio of the altitude AI to the altitude DK. First let the altitudes be equal, but the bases either equal or unequal. Let it be arranged that as BC is to EF, so L is to M; and as AI is to DK, so M is to N. This being done, M will be equal to N, since AI too is assumed equal to DK. [Margin: p — Elements V.7 (?)] And accordingly L will be to N as L is to M — that is, as BC is to EF. [Margin: q — Elements VI.1] But as BC is to EF, so is triangle ABC to triangle DEF, and parallelogram CG to parallelogram EH. [Margin: r — Elements V.11] Therefore it will also follow that as L is to N, so is triangle ABC to triangle DEF, and parallelogram CG to parallelogram EH. But the ratio of L to N is compounded of the ratio L to M — that is, of the ratio of the base BC to the base EF — and of the ratio M to N — that is, of the ratio of the altitude AI to the altitude DK. The ratio, therefore, of triangle ABC to triangle DEF, and of parallelogram CG to parallelogram EH, is compounded of the same ratios. Which was proposed.

Diagram. A compound figure for the first case (equal altitudes): the parallelogram CG with its inscribed triangle ABC (altitude AI, foot I on base BC),

and the parallelogram EH with its inscribed triangle DEF (altitude DK, foot K on base EF); to the right, three vertical auxiliary segments labelled L, M, N.

- Left group: G (top-left), A (apex of triangle), B (bottom-left), I (foot of altitude AI), C (bottom-right).
- Right group: D (apex), H (top-right), E (bottom-left), K (foot of altitude DK) [?], F (bottom-right).
- Auxiliary segments: L, M, N.

Now let the altitudes AI and DK be unequal, with AI the greater; and the bases BC, EF either equal or unequal. Let it be arranged that as AI is to DK, so O is to P; and as BC is to EF, so P is to Q. Then, cutting off IL equal to DK, draw through L the line LM parallel to BC, cutting AC in N, and join the straight line BN. Since, then, triangle ABC is to triangle NBC, and parallelogram CG to parallelogram CM, as the altitude AI is to the altitude IL — or to DK, which is equal to IL — that is, as O is to P, by what we showed from Commandino [Federico Commandino] at Proposition 1 of this book (since the triangles and parallelograms have the same base BC); and since triangle NBC is to triangle DEF, and parallelogram CM to parallelogram EH, [Margin: s — Elements VI.1] as the base BC is to the base EF (the altitude being the same) — that is, as P is to Q — [Margin: t — Elements V.22] it will follow *ex aequo* that ABC is to DEF, and CG to EH, as O is to Q. Therefore, since the ratio O to Q is compounded of the ratio O to P — that is, of the ratio of the altitude AI to the altitude DK — and of the ratio P to Q — that is, of the ratio of the base BC to the base EF — the ratio of triangle ABC to triangle DEF, and of parallelogram CG to parallelogram EH, will be compounded of the same ratios. Which was proposed.

In the same way it will be shown that the ratio of triangle DEF, whose altitude is the less, to triangle ABC, and of parallelogram EH to parallelogram CG, is compounded of the ratio of the base EF to the base BC and the ratio of the altitude DK to the altitude AI. For if it be arranged that as EF is to BC, so Q is to P; and as DK is to AI, so P is to O; and the rest be done as before, [Margin: u — Elements VI.1] then triangle DEF will be to triangle NBC, and parallelogram EH to parallelogram CM, as EF is to BC — that is, as Q is to P. Likewise triangle NBC will be to triangle ABC, and parallelogram CM to parallelogram CG, as LI (or DK) is to AI — that is, as P is to O — from what we demonstrated from Commandino at Proposition 1 of this book. [Margin: x — Elements V.22] Therefore *ex aequo* it will follow that as DEF is to ABC, and EH to CG, so Q is to O. Wherefore, since the ratio Q to O is compounded of the ratio Q to P — that is, of the ratio of the base EF to the base CB — and of the ratio P to O — that is, of the ratio of the altitude DK to

the altitude AI — the ratio of DEF to ABC, and of EH to CG, will also be compounded of the same ratios. Which was proposed.

Diagram. A compound figure for the second and third cases (unequal altitudes): the parallelogram CG with triangle ABC and the inscribed line LM (cutting AC in N, with BN joined), and the parallelogram EH with triangle DEF; to the right, three vertical auxiliary segments O, P, Q.

- Left group: G (top-left), A (apex), L, N (on AC), M; B (bottom-left), I, C (base).
- Right group: D (apex), H (top-right); E, K, F along the base.
- Auxiliary segments: O, P, Q.

[Several small vertex-letters in this crowded figure are at the resolution floor]

THEOR. 18. PROPOS. 24.

[Margin: xxii.]

In every parallelogram, the parallelograms about the diameter are similar both to the whole and to one another.

Let ABCD be a parallelogram, in which the diameter AC is drawn, and through any point I of it let two straight lines EF, GH be drawn parallel to the sides of the parallelogram. I say that the parallelograms EG [?] and FH about the diameter are similar both to the whole parallelogram and to one another. That they are equiangular with the whole is easily shown. For angle GAE is the same as angle BAD; [Margin: a — Elements I.29] and the exterior angle AEI is equal to the interior ADC; and the exterior angle AGI to the interior BFI; and this exterior to the interior BCD. Therefore parallelogram EG is equiangular with parallelogram BD; and by the same reasoning FH will be equiangular with BD. That the sides about the equal angles are proportional to the sides of the whole we shall demonstrate as follows. Since triangle AGI is equiangular with triangle

triangle ABC, and triangle AEI with triangle ADC, as is evident from Proposition 29 of Book 1, or also from the corollary to Proposition 4 of this book, [Margin: b — Elements VI.4] it will follow that as AB is to BC, so AG is to GI, and thus the sides about the equal angles B and G are proportional. Again, [Margin: c — Elements VI.4] it will follow that as BC is to CA, so GI is to IA; likewise as CA is to CD, so IA is to IE. [Margin: d — Elements V.22] Therefore *ex aequo*, as BC is to CD, so is GI to IE, and hence the sides about the equal angles BCD, GIE are proportional. In no other way will it be demonstrated that the sides about the remaining equal angles are proportional. Therefore, by the definition, parallelogram EG will be similar to

the whole parallelogram BD. By the same technique you will show that parallelogram FH is similar to the same parallelogram BD; [Margin: e — Elements VI.21] and thus they will also be similar to one another. Therefore in every parallelogram the parallelograms about the diameter are [similar], etc. Which was to be demonstrated.

Diagram. A parallelogram with diameter and two internal lines forming the parallelograms about the diameter.

- A (top-left), E (top-middle), D (top-right).
- G (left-middle), I (centre, on the diameter), H (right-middle).
- B (bottom-left), F (bottom-middle), C (bottom-right).
- The diameter runs from A (top-left) to C (bottom-right); the line EF runs vertically (E to F) and GH horizontally (G to H), crossing at the point I on the diameter.

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It must be understood, however, that the parallelograms about the diameter of the whole are such as have one angle in common with the whole parallelogram, as is manifest from the form of the demonstration.

Now if, about the produced diameter of some parallelogram, another parallelogram stands — such that two of its sides form straight lines with two sides of the other, or at least are parallel to them — it will be shown by nearly the same means that this is similar to that. For let BD be the diameter of parallelogram ABCD, produced to F, and about it let there stand a parallelogram DEFG whose two sides DE, DG form straight lines with AD, DC, the sides of parallelogram AC. I say that parallelogram GE is similar to parallelogram AC. That the two are equiangular with each other is easily shown. [Margin: f — Elements I.29] For since angle A is equal to the alternate angle ADG, and to this the alternate angle G is likewise equal, the angles A and G will be equal. [Margin: g — Elements I.34] Wherefore their opposites C and E will also be equal. [Margin: h — Elements I.15] Again, since the angles ADC, GDE at the vertex are equal, [Margin: i — Elements I.34] their opposites ABC, GFE will also be equal to these. Therefore the parallelograms GE, AC are equiangular. That the sides about the equal angles are proportional will be made clear as follows. Since triangle BAD is equiangular with triangle DGF, and triangle BCD with triangle DEF, [Margin: k — Elements VI.4] it will follow that as BA is to AD, so DG is to GF. Again, as AD is to DB, so GF is to FD; and as DB is to DC, so FD is to FE; [Margin: l — Elements V.22] and therefore *ex aequo*, as AD is to DC, so GF is to FE. The sides about the angles A,

ADC are therefore proportional to the sides about the angles G, GFE, which are equal to them. In the same way you will show that the remaining sides about the equal angles are proportional. Therefore the parallelograms AC, GE are similar.

And if about the same diameter there stands a parallelogram HIKF having its sides parallel to the sides of parallelogram AC, the same will be demonstrated. For when AD, CD are produced until they meet the straight lines FK, FH produced in E and G, HK will be similar to GE, [Margin: m — Elements VI.24] as Euclid demonstrated; but AC too is similar to the same GE, as we have just shown. [Margin: n — Elements VI.21 (?)] Therefore HK and AC are also similar to one another. Which was proposed.

But let us also, following Peletier [Jacques Peletier], dispatch the following problem.

Diagram. The scholium's figure for the two joined parallelograms DEFG standing about the produced diameter BD of ABCD. Readable labels include A, B, C, D (parallelogram ABCD), the produced diameter through the vertex region, and the outer parallelogram vertices E, F, G. [Small vertex-letters at the resolution floor]

Given two equiangular but dissimilar parallelograms: from either of them to cut off one similar to the other.

Let ABCD and CEFG be two equiangular but dissimilar parallelograms, and from ABCD let a parallelogram similar to CEFG be cut off. Let the two be joined at the equal angles BCD, ECG, so that BCG is one straight line, and consequently — as was demonstrated at Proposition 15 of Book 1 — ECD is likewise one straight line. Then, the diameter FC having been drawn, let it be produced until at H it cuts either side AD or side AB (for it will not fall on the point A, since AC would then be similar to CF, as has been demonstrated, which is contrary to the hypothesis); and through H let HI be drawn parallel to AB or to AD. I say that the parallelogram HC cut off is similar to CF. For if the whole parallelogram KL be completed, HC and CF will be about the diameter. Therefore they will be similar to one another, as Euclid demonstrated in this proposition.

By nearly the same technique either of them can also be enlarged so as to become similar to the other. For let ABCD be enlarged so as to become similar to CEFG: let it be joined as before, and let the diameter FC be extended until at H it cuts either side BA produced or side DA produced (for it will not fall on the point A, since AC would then be similar to CF, as has been demonstrated, which is not the case supposed). Then through H let HI be drawn parallel to AD or to AB, until it cuts either side CD produced or CB produced in I. I say that the enlarged

parallelogram HC is similar to parallelogram CF. For if the whole parallelogram KL be completed, HC and CF will stand about the diameter; [Margin: o — Elements VI.24] wherefore they will be similar to one another.

Diagram. Two pairs of parallelogram figures for the Peletier problem — the first pair illustrating the cutting-off (abscission) construction, the second the enlargement (augmentation). Readable labels across the figures include A, B, C, D, E, F, G, H, I, K, L, with the diameter FC drawn and the completed parallelogram KL indicated. [Many small vertex-letters at the resolution floor]

Euclid, *Elements* Book VI (Clavius's commentary) — continued

(Continuing the scholium appended to Proposition 24: the Peletier problem of cutting off or enlarging a parallelogram similar to a given one.)

It is clear, both from Euclid's demonstration of this theorem and from the proof of the theorem we proposed in this scholium, that parallelograms about the same diameter are not only similar but also similarly situated. Therefore, given any parallelogram ABCD, if a larger one is to be described similar to it and similarly situated, one side must be extended — namely BC. Then, from E (any point beyond C), draw EF parallel to CD, cutting the diameter BD (extended) at F; and through F draw FG parallel to AD, meeting the line BA (extended) at G. [Margin: p — *Elements* VI.24] For the parallelogram GE will then be similar and similarly situated to AC, and larger than it. But if a smaller one is to be described, the point E must instead be taken on the near side of C, and the rest carried out as before, as the figure indicates.

Diagram. A parallelogram with one side extended, illustrating both the enlargement and the reduction of a given parallelogram similar to itself. A diagonal runs from the lower-left corner B upward to the outer corner F at the upper right; the original parallelogram lies inside, the enlarged one outside.

- G (upper left): outer corner of the enlarged parallelogram
- F (upper right): outer corner, on the extended diagonal
- A (left, below G): upper-left corner of the original ABCD
- D (upper middle): upper-right corner of the original
- F (interior, on the diagonal), G (interior, lower-left corner): printed duplicate letters (as in the source), marking the reduction case

- B (lower left): base corner
- E (lower, between B and C): the point taken short of C (reduction case)
- C (lower middle): corner of the original
- E (lower right): the point taken beyond C (enlargement case)

PROBL. 7. PROPOS. 25. [Margin: xxv.]

Given a rectilinear figure, to construct another that is similar and similarly situated to it, and equal to a second given figure.

Let two rectilinear figures A and B be given, and let another rectilinear figure be constructed that is similar to A but equal to B. On CD, one side of the figure to which the new one is to be made similar, [Margin: a — *Elements* I.44 or 45] construct a parallelogram CE, at any angle, equal to the rectilinear figure A. And on the straight line DE, at the angle EDG (which is equal to angle DCF), [Margin: b — *Elements* VI.13] construct a parallelogram DH equal to B; then both CDG and FEH will each form a single straight line, as was demonstrated in Proposition 45 of Book I. Now let IK be found as the mean proportional between the straight lines CD and DG, [Margin: c — *Elements* VI.18] and on IK construct a rectilinear figure L similar and similarly situated to A. I say that L is equal to the other figure B.

For since the three straight lines CD, IK, DG are proportional, by the corollary of Proposition 19 or 20 of this book, as CD (the first) is to DG (the third), so is A — the figure described on the first side CD — to L, the figure similarly described on the second, IK. [Margin: d — *Elements* VI.1] But as CD is to DG, so is the parallelogram CE to the parallelogram DH, since they have the same height. [Margin: e — *Elements* V.11] Therefore as CE is to DH, so is A to L. [Margin: f — *Elements* V.7] But as CE is to DH, so is A to B, since the parallelogram CE was constructed equal to the figure A, and the parallelogram DH equal to the figure B. [Margin: g — *Elements* V.11] Therefore as A is to B, so is A to L; [Margin: h — *Elements* V.9] and consequently the figures B and L are equal. But L is also, by construction, similar and similarly situated to A. Therefore, given a rectilinear figure, we have constructed another similar and similarly situated to it and equal to a second given figure. Which was to be done.

Diagram. The construction for Proposition 25. At the upper left is the given pentagon A standing on the base line through C, D and G; to its right is the given triangle B. Below is the pair of parallelograms CE and DH set on the straight line CDG with FEH along their base; at the lower centre is the constructed pentagon L standing on the mean proportional IK.

- A (upper-left pentagon): the given figure to which the new one is made similar
- B (upper-right triangle): the second given figure, to which L is made equal
- C, D, G (along the upper base line): the joined bases of the parallelograms CE, DH
- F, E, H (along the lower base line): the far corners of parallelograms CE and DH
- L (lower-centre pentagon): the constructed figure, similar to A and equal to B
- I, K (base of pentagon L): the segment IK, the mean proportional between CD and DG

THEOR. 19. PROPOS. 26. [Margin: xxiii.]

If from a parallelogram a parallelogram is taken away that is similar to the whole and similarly situated, having a common angle with it, then it lies about the same diameter as the whole.

From the parallelogram BD let a parallelogram EG be cut off, similar to it and similarly situated, having the common angle EAG with it. I say that EG lies about the diameter of the whole BD.

For let the straight lines AF and CF be drawn. If these form a single straight line, it is clear — since AF is the diameter of EG and AC the diameter of BD — that the parallelogram EG lies about the diameter AFC of the whole parallelogram. But if AF and CF are said not to form a straight line, draw AC, the diameter of the whole parallelogram, cutting the side EF at the point H; and through H draw HI parallel to FG. Now since the parallelograms BD and EI lie about the same diameter AHC, [Margin: a — *Elements* VI.24] they are similar and similarly situated. [Margin: b — *Elements* VI, def. 1] Therefore as BA is to AD, so is EA to AI. But as BA is to AD, so also is EA to AG, since the parallelograms BD and EG are likewise assumed to be similar and similarly situated. [Margin: c — *Elements* V.11] Therefore as EA is to AI, so is EA to AG; [Margin: d — *Elements* V.9] and consequently the lines AI and AG are equal — the part and the whole — which is absurd.

But if AHC is said to cut the other side FG, then, when HI is drawn parallel to EF, [Margin: e — *Elements* VI.24] the parallelograms BD and IG are again similar and similarly situated. [Margin: f — *Elements* VI, def. 1] Therefore as DA is to AB, so is GA to AI. But as DA is to AB, so also is GA to AE, because of the similarity of the parallelograms BD and EG. [Margin: g — *Elements* V.11] Therefore as GA is to AI, so is GA to AE; [Margin: h — *Elements* V.9] and consequently the lines AI and AE

are equal — the part and the whole — which is absurd. Therefore the lines AF and FC form a single straight line; that is, the diameter AC, when drawn, passes through the point F, and the diameter AF, when drawn, falls on the point C. And so, if from a parallelogram a parallelogram is taken away, and so on. Which was to be demonstrated.

Diagram. First figure for Proposition 26 — the parallelogram BD with the cut-off parallelogram EG, the diameter AC cutting the side EF at H.

- B (upper left), C (upper right): two corners of the whole parallelogram BD
- A (lower left), D (lower right): the remaining corners; AC is the diameter
- H (interior, on the diameter): where AC meets EF
- F (interior, right of H): corner of the cut-off parallelogram EG
- I (lower, between A and G): foot of HI
- G (lower middle): corner on the base AD

Diagram. Second figure for Proposition 26 — the case in which AHC cuts the side FG.

- B (upper left), C (upper right): corners of the whole parallelogram
- A (lower left), D (lower right): remaining corners; AC is the diameter
- E (left side): corner of the cut-off parallelogram
- I (left side, below E): foot of HI
- F (interior, upper): corner of EG
- H (interior, lower): where the diameter meets FG
- G (lower middle): point on the base

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We shall demonstrate the same theorem in another way, directly, as follows. Let the sides AB and BC be bisected at the points H and I — whether the point H falls on the point E, or above it, or below it — and draw the straight lines HI and AE. Now since, because of the similarity of the parallelograms, as AB is to BC so is AE to EF; [Margin: i — *Elements* V.15] and since, as the whole AB is to the whole BC, so is the half HB to the half BI, [Margin: k — *Elements* V.11] it follows that as HB is to BI, so is AE to EF. Therefore the triangles HBI and AEF — since about the equal angles B and E (which are equal because of the similarity of the parallelograms BD and EG; [Margin: l — *Elements* I.30] or because the one is interior and the other exterior with respect to the parallels BC and EF, which are parallel since

each is parallel to AD) they have proportional sides — [Margin: m — *Elements* VI.6] will be equiangular, and will have the angles BHI and EAF equal, the exterior and the interior between the lines HI and AF. [Margin: n — *Elements* I.28] Therefore the lines HI and AF are parallel. But the straight line conceived as drawn from the point A to the point C [Margin: o — *Elements* VI.2] is also parallel to HI, because the sides AB and BC of the triangle ABC then constructed would be cut proportionally at H and I, being bisected; and it passes through the point F, since from the point A only one line can be drawn parallel to HI, as is evident. Therefore the parallelograms BD and EG, similar and similarly situated, lie about the same diameter AFC. Which is what was proposed.

Diagram. Three small parallelograms (printed rotated 90°) illustrating the three cases of the ostensive proof — the bisection-point H falling on E, above E, or below E. Each shows the whole parallelogram BD with its diagonal, the cut-off parallelogram EG, and the auxiliary lines HI and AE.

- A, B, C, D: the corners of the whole parallelogram (diagonal AC)
- E, F, G: the cut-off parallelogram EG with inner corner F
- H (on AB), I (on BC): the bisection points; HI joins them
- (The three figures repeat these labels for the three positions of H relative to E.) [?]

But if two similar and similarly situated parallelograms do not have a common angle, but one lies outside the other — yet connected to each other along two of their equal angles in such a way that two sides of the one form two straight lines with two sides of the other — we shall demonstrate, by almost the same means, that they lie about the same diameter. Let there be two similar and similarly situated parallelograms BD and EG, joined at the equal angles BCD and GCE so that the lines BC and CG lie in one straight line; and for this reason, by what we showed at Proposition 15 of Book I, the lines DC and CE also form a single straight line. I say that the parallelograms BD and EG lie about the same diameter — that is, that the diameter AC together with the diameter FC forms one straight line.

For if AC and FC do not form one straight line, draw from A to F a straight line cutting BC at the point H, and through H draw HI parallel to CD, meeting the line FE (extended) at K. Now since the parallelograms BI and KG lie about the same diameter AHF (extended), and the two lines BH, HI together with the two lines HG, HK form two straight lines, they are similar and similarly situated, by what we demonstrated at Proposition 24 of this book. Therefore as HB is to BA, so is FK to KH. [Margin: p — *Elements* V.8] But CB, the greater, has to BA a greater ratio than HB, the lesser, has to the same BA; and as CB is to BA, so is FE to EC, since the

parallelograms BD and EG are assumed similar and similarly described. Therefore FE also has to EC — that is, to KH, which is equal to it — a greater ratio than HB has to BA, that is, than FK has to KH. Consequently, if FE has to KH a greater ratio than FK has to the same KH, [Margin: q — *Elements* V.10] then EF will be greater than FK — the part greater than the whole. Which is absurd.

Diagram. The two similar parallelograms BD and EG joined at C, one outside the other, with the lines BC, CG collinear. A tall rectangle with two diagonals drawn from A upward.

- K (upper left), F (upper middle), F (upper right): the top edge, the two F's a printed duplicate (as in the source), the far F at the tip of the extended diagonal AHF
- B (left), H (interior), C (interior right), G (right): the middle line, BC–CG collinear
- A (lower left), I (lower middle), D (lower right): the base

Diagram. The companion case in which the line AHF cuts the side CD. A cross-shaped configuration of two parallelograms meeting at C.

- B (upper left), C (upper centre), D (upper right): tops of the two joined parallelograms
- I (left), H (centre), K (right): the line HI extended to K
- A (lower left), D (lower right): base corners

But if someone says that the line AHF cuts the side CD, then, drawing through H a line parallel to BC — namely HI, meeting the line FG (extended) at K — the parallelograms ID and EK are again similar and similarly situated, by what we showed at Proposition 24 of this book. Therefore as HD is to DA, so is FK to KH. [Margin: r — *Elements* V.8] But CD has to DA a greater ratio than HD has to DA; and as CD is to DA, so is FG to GC, since the parallelograms BD and EG were granted to be similar and similarly situated. Therefore FG also has to GC — that is, to KH, which is equal to it — a greater ratio than FK has to KH; [Margin: s — *Elements* V.10] and therefore FG will be greater than FK — the part greater than the whole — which is absurd.

The same will be shown directly by this reasoning. Since, because of the similarity of the parallelograms BD and EG, the angles B and E are equal, and as AB is to BC so is CE to EF, the triangles ABC and CEF will have proportional sides about the equal angles B and E; [Margin: t — *Elements* VI.6] and therefore they will be equiangular, and will have the angles BCA and EFC equal. Therefore,

adding the common angle BCF, the two angles BCA and BCF will be equal to the two angles EFC and BCF. [Margin: u — *Elements* I.29] But these, lying between the parallels BC and EF, are equal to two right angles. Therefore BCA and BCF too will be equal to two right angles; [Margin: x — *Elements* I.14] and consequently AC and FC form a single straight line. Which is what was proposed.

Diagram. The figure for the ostensive proof of the previous paragraph — two similar parallelograms BD, EG meeting at C, with the diagonal AC through F.

- E (upper left), F (upper right): top of the upper parallelogram
- B (left), C (centre), G (right): the middle line
- A (lower left), D (lower right): base corners; A–C–F lies along the diagonal

Euclid was right, in his theorem, to require that the parallelogram cut off from the whole be not only similar to the whole but also similarly situated, so that it may be shown to lie about the same diameter as the whole. For if from BD, on its longer side, a longer piece EG is cut off, lying about the same diameter as the whole, [Margin: x — *Elements* VI.24] then EG will be similar and similarly situated. But if, in a rectangle IL that is equilateral and equiangular to BD, one takes HM equal to EF and MN equal to AE, and so on, then the rectangle MO will indeed be equal and similar to the rectangle EG — because of the equality of the sides and of the angles, which are right — and for that reason similar to the rectangle IL; yet, because it is not similarly situated, it will not lie about the same diameter as the whole rectangle IL.

The same thing is also seen here in similar rectangles, one of which lies outside the other, yet connected to each other along their angles in such a way that two sides of the one lie in a straight line with two sides of the other — such as the rectangular parallelograms BD and EG, and IL and MO. Of these, EG lies about the same diameter as the rectangle BD, since it is similarly situated; but MO by no means lies about the same diameter as the rectangle IL, because it is not similarly situated, although it is similar to it — being indeed exactly equal to EG. For KM is equal to EF, and MN to FC, and so on.

Diagram. Rectangle BD with the similarly-situated rectangle EG about its diameter.

- B (upper left), C (upper right): top corners
- E (left side), F (interior, on the diagonal): the rectangle EG
- A (lower left), G (lower middle), D (lower right): base

Diagram. Rectangle IL with the rectangle MO, which is equal and similar to EG but not similarly situated.

- I (upper left), K (upper right): top corners
- M (left side), N (interior): the rectangle MO
- H (lower left), O (lower middle), L (lower right): base

Diagram. The rectangles BD and EG connected in line at a common angle, EG being similarly situated and therefore about the same diameter as BD.

- E (upper), F (upper right): top of EG
- C (centre): the common angle
- G (right), D (lower): further corners
- B (left): corner of BD

Diagram. The rectangles IL and MO connected in line, MO not being similarly situated and therefore not about the same diameter as IL.

- M (upper), N (upper right): top of MO
- K (centre): the common angle
- O (right), I (left): further corners
- H (lower), L (lower right): base corners

THEOR. 20. PROPOS. 27. [Margin: xxvi.]

Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar and similarly situated to that described on the half of the line, the greatest is the one applied to the half — the deficient parallelogram being similar to the defect.

Let a straight line AB be given, bisected at C; and on its half BC construct any parallelogram CDEB, whose diameter is BD. Then, if the whole parallelogram ABEH is completed, the parallelogram AD, standing on the half AC, will be applied along AB, deficient by the parallelogram CE, and will be similar to the defect CE. I say that the parallelogram AD, applied to the half AC and deficient by the parallelogram CE, is the greatest of all those applied along AB and deficient by parallelograms similar and similarly situated to CE.

For take any point G on the diameter BD, and through G draw the lines FGI and KG parallel to AB and BE. Then the parallelogram FK, applied along AB, is deficient by the parallelogram KI, [Margin: a — *Elements* VI.24] which is similar to CE and similarly situated, since it lies about the same diameter as CE. [Margin:

b — *Elements* I.43] Now since the complements CG and GE are equal, if the common KI is added, CI and KE will also be equal. [Margin: c — *Elements* I.36] But CI is equal to CF, because the bases AC and CB are equal. Therefore CF and KE are also equal; and when the common CG is added, the parallelogram AG and the gnomon LM will be equal. Therefore, since CE is greater than the gnomon LM — for CE contains, besides the gnomon, the parallelogram DG as well — AD too, being equal to CE because of the equal bases AC and CB, [Margin: d — *Elements* I.36] will be greater than the parallelogram AG by that same parallelogram DG. And in the same way it will be shown that AD is greater than all parallelograms applied along the line AB in such a way that the point G lies between the points B and D — that is, those which occupy a line greater than the half AC and have a smaller height than AD — provided their defects are similar to CE.

The same — that AD is greater than the parallelogram AG — will be demonstrated another way, as follows. [Margin: c — *Elements* I.36] The parallelograms FD and DI are equal, since the bases HD and DE are equal. But DI is greater than GE — that is, than the complement CG [Margin: f — *Elements* I.43] (which is equal to GE) — by the parallelogram DG. Therefore FD too is greater than CG by that same parallelogram DG. And accordingly, when the common CF is added, AD will be greater than AG by that same parallelogram DG.

Diagram. The principal figure for Proposition 27. The parallelogram ABEH stands on AB (bisected at C); the parallelogram AD is applied to the half AC, deficient by CE; the arbitrary parallelogram FK is deficient by KI about the diameter BD. A small arc (semicircle) is drawn about the point G near L.

- H (upper left), D (upper middle), E (upper right): top edge of the whole ABEH
- F (left side), M, G, I (interior middle row): F on the left, then M, the point G with L just above it, and I toward the right
- L (interior, above G): apex of the drawn arc
- A (lower left), C (lower middle), K, B (lower right): the base line AB, bisected at C
- (gnomon LM and parallelograms AG, DG, CE, FK as described in the text)

But if the point G is taken on the diameter BD extended beyond the parallelogram CE, then, drawing through G the line HM parallel to AB (meeting the lines AK and BE, extended, at H and M) and likewise drawing GF parallel to AH, the parallelogram AG will be applied along AB, deficient by the parallelogram FM, [Margin: g — *Elements* VI.4] which is similar and similarly situated to CE, since it

lies about the same diameter as CE. I say that AD is still greater than AG. For, CD being extended to L, the lines HL and LM will be equal, [Margin: h — *Elements* I.34] since they are equal to the equals AC and CB; [Margin: i — *Elements* I.36] and therefore the parallelograms HD and DM are equal. [Margin: k — *Elements* I.43] Since, then, DM is equal to the complement DF, HD too will be equal to DF. But HD is greater than HI by the parallelogram IL. Therefore DF too is greater than HI by that same parallelogram IL. And accordingly, when the common AI is added, AD will be greater than AG by that same parallelogram IL. By the same arguments you will conclude that AD is greater than any parallelogram so applied along the line AB that the point G lies beyond D on the diameter BD extended — that is, one that occupies a line less than the half AC and has a greater height than AD — provided its defect is similar to the parallelogram CE. And so, of all parallelograms applied to the same straight line, and so on. Which was to be demonstrated.

Diagram. The figure for the case in which G falls beyond D on the extended diameter. The applied parallelogram AG is deficient by FM.

- H (upper left), G (upper), L (upper), M (upper right): top edge, with HM parallel to AB
- K (left), I (interior left), D (centre), E (right): middle line
- A (lower left), F (lower), C (lower middle), B (lower right): the base line AB

Euclid, *Elements*, Book VI (Clavius's commentary)

Scholium (to Proposition 27, continued)

Moreover, it is clear that the line to which the deficient parallelogram is applied is either greater than the half AC — as AK is, in the former figure — or less than it — as AF is, in the latter figure — according as the point G is taken either on the diameter BD or on its extension toward D.

PROBL. 8. PROPOS. 28.

[Margin: xxvii.]

To a given straight line to apply a parallelogram equal to a given rectilinear figure and deficient by a parallelogrammic figure that is similar to another given parallelogram. But the given rectilinear figure, to which the parallelogram to be applied is to be equal, must not be greater than the parallelogram that is applied to the half of the line,

when the defects are similar — that is, not greater than both the parallelogram applied to the half and the parallelogram to which the defect must be similar.

Demonstration. Let it be required to apply to the given straight line AB a parallelogram equal to the given rectilinear figure C and deficient by a parallelogram similar to the other given parallelogram D. Bisect AB at E, and [Margin: a — *Elements* VI.18] on the half EB describe the parallelogram EFGB similar to D and similarly placed; then complete the whole parallelogram AHGB. Now if AF is equal to C, then — since it has been applied to AB, deficient by the parallelogram EG similar to D — what is required is done.

But suppose AF is greater than C. (It cannot be less: for since, by the preceding proposition, AF is the greatest of all figures applied to the half — provided the defects are similar — no figure equal to C could be applied to AB, but all would be smaller. This is why Euclid added the clause, "but the given rectilinear figure must not be greater," etc.) Then EG, being equal to AF, will likewise be greater than C. Let it exceed by the rectilinear figure I. (How the excess of two rectilinear figures is to be found, we taught at Proposition 45 of Book 1.) [Margin: b — *Elements* VI.25] Now construct the parallelogram KLMN, similar and similarly placed to D (that is, to EG), but equal to the excess I that was found; so that EG is equal to the rectilinear figure C together with the parallelogram KM, and on that account greater than KM.

Since, then, by the similarity, EF is to FG as NK is to KL, the sides EF, FG will also be greater than the sides NK, KL. For if these were equal to or less than those, EG too would be equal to or less than NL, as is evident. Therefore cut off the straight lines FO, FQ equal to KN, KL, and complete the parallelogram FQPO; this will be equal to LN, and similar and similarly placed to it, and therefore to EG as well. [Margin: c — *Elements* VI.26] And so it will lie about the same diameter as EG — namely BF. Then, when the straight lines QP, OP have been produced, [Margin: d — *Elements* VI.24] the parallelogram AP, applied to the straight line AB and deficient by the parallelogram PB, will be the figure sought — for PB is similar to EG and similarly placed, and therefore similar to D.

I say, then, that AP is equal to the rectilinear figure C. [Margin: e — *Elements* I.43] For since PG is equal to the complement PE, if the common PE is added, BQ will be equal to ER — that is, to ES [?] — [Margin: f — *Elements* I.36] because the bases EA, EB are equal. Therefore, if the common EP is added to the equals AO, BQ, AP will become equal to the gnomon TV. But the gnomon TV is equal to the rectilinear figure C. (For since EG is equal to the figure C together with LN, if the

equals QO, LN are taken away, the gnomon TV will remain equal to C.) Therefore AP too will be equal to C. To the given straight line AB, then, has been applied the parallelogram AP, deficient by the parallelogram PB — which is similar to the given parallelogram D — and equal to the given rectilinear figure C. Which was to be done.

Diagram. The construction for Proposition 28 (application of a parallelogram deficient by one similar to a given figure). At upper left stand two auxiliary triangles — the larger labelled **C** and a smaller one at the right labelled **I** — together with a small quadrilateral labelled **D** (the given parallelogram). Below these is the main figure: a large parallelogram/gnomon construction on the base AB. Readable vertex-labels:

- H, F, Q, G (top edge, left to right)
- K, L (short bracketed segment at the right, marking KL / the constructed parallelogram KLMN)
- S, O, V, P, R (interior/middle line; a semicircular arc is drawn about the region near P–R)
- A, F, B, N, M (base edge, left to right)

The diagonal runs from the upper-left toward B (the common diameter BF); the gnomon TV and complements referred to in the text occupy the interior. Several interior and peripheral labels sit at the resolution floor; the reused letter F at both top and base could not be disambiguated with full confidence. (more than two labels uncertain).

Scholium (to Proposition 28)

Jacobus Peletarius [Jacques Peletier] and Nicolaus Tartalea [Niccolò Tartaglia] raise a certain doubt at this point, which has no place in our construction, since upon EB we set up EG as a parallelogram not only similar to D but also similarly placed — which they did not do. On this matter, if you wish, consult their commentaries.

It is also clear that if a parallelogram deficient by a square is applied to a straight line, the applied figure is equal to the rectangle contained by the segments of the line made by the application. Thus, if AC is applied to AB, deficient by the square CB, the applied figure AC will be the rectangle contained by AD and DC. Since, then, DC is equal to DB — because of the square CB — AC will likewise be contained by the segments AD, DB made by the application. Which is what was proposed.

Diagram. A small rectangle illustrating the note on a parallelogram deficient by a square. The base carries the points **A** (bottom left), **D** (midpoint), **B** (bottom right); a vertical line rises from **D** to the point **C** at the top edge, dividing the rectangle into the applied rectangle **AC** (left) and the square **CB** (right).

PROBL. 9. PROPOS. 29.

[Margin: xxviii.]

To a given straight line to apply a parallelogram equal to a given rectilinear figure and exceeding by a parallelogrammic figure that is similar to another given parallelogram.

Demonstration. Let it be required to apply to the given straight line **AB** a parallelogram equal to the given rectilinear figure **C** and exceeding by a parallelogram similar to the other given parallelogram **D**. Bisect **AB** at **E**; [Margin: a — *Elements* VI.18] on the half **EB** construct the parallelogram **EFGB** similar to **D** and similarly placed. [Margin: b — *Elements* II.14] Next construct a square **H** equal to the rectilinear figure **C** and the parallelogram **EG** together; [Margin: c — *Elements* VI.25] and let there be made a parallelogram **IKLM** equal to that square, but similar to **EG** and similarly placed. **IKLM** will therefore be greater than **EFGB**, since it is equal to the rectilinear figure **C** together with the parallelogram **EG**.

Since, then, on account of the similarity of **MK** and **EG**, **MI** is to **IK** as **EF** is to **FG**, the sides **MI**, **IK** will also be greater than the sides **EF**, **FG**. For if those were equal to or less than these, **MK** would likewise be equal to or less than **EG**, as is plain. Therefore produce **FE**, **FG** so that the straight lines **FO**, **FN** are equal to **IM**, **IK**, and complete the parallelogram **ON**; [Margin: d — *Elements* VI.26] this will be similar and similarly placed to **EG**, since it is equal, similar, and similarly placed to **MK**. Consequently **ON** and **EG** lie about the same diameter — let it be **FP**.

Now produce **AB**, **GB** to **Q** and **R**, and **PO**, until it meets **AS** (which is parallel to **FO**) at **S**; then the parallelogram **AP**, applied to the straight line **AB** and exceeding by the parallelogram **QR**, [Margin: c — *Elements* VI.24] is the figure sought — for **QR** is similar to **EG** and therefore to **D**. I say, then, that **AP** is equal to the rectilinear figure **C**. [Margin: f — *Elements* I.36] For since **AO** and **ER** are equal, [Margin: g — *Elements* I.43] and **ER** is equal to the complement **BN**, **AO** too will be equal to **BN**. Adding, then, the common **OQ**, **AP** becomes equal to the gnomon **EPG**. But the gnomon **EPG** is equal to the rectilinear figure **C**. (For since **MK** — that is, **ON** — is equal to the rectilinear figure **C** together with **EG**, if the common **EG** is taken away, the gnomon **EPG** and the rectilinear figure **C** will remain equal.) Therefore

AP too will be equal to the rectilinear figure C. To the given straight line AB, then, has been applied a parallelogram AP equal to the given rectilinear figure C and exceeding by the parallelogram RQ, which is similar to the other given figure D. Which was to be done.

Diagram. The construction for Proposition 29 (application of a parallelogram exceeding by one similar to a given figure). At the left is a square labelled **H** (the constructed square equal to C plus EG); to its right a large parallelogram/gnomon construction with an inscribed diagonal and a semicircular arc. Below-left is a small quadrilateral labelled **D** (the given parallelogram). Readable vertex-labels:

- K, L (top-left region of the parallelogram IKLM)
- I, M (its lower vertices)
- N, G, Q (upper edge, right portion)
- B, P (with a doubled P near the arc and the point B), F, E, O, A (base and interior)

The gnomon EPG and complements named in the proof occupy the interior. Several peripheral labels are at the resolution floor. (more than two labels uncertain).

PROBL. 10. PROPOS. 30.

[Margin: xxix.]

To cut a given finite straight line in extreme and mean ratio.

Demonstration. Let AB be the straight line to be cut in extreme and mean ratio. Describe the square ABC upon it; [Margin: a — *Elements* VI.29] and to the side DA apply the rectangle DF, equal to the square AC and exceeding by a parallelogram AF similar to the square — so that AF too is a square, since only a square is similar to a square. Let the straight line EF cut the straight line AB at H. I say that AB is cut at H in extreme and mean ratio.

For since DF and AC are equal, if the common AE is taken away, the remainders GH, HC will be equal. And since these have equal angles AHF, BHE — right angles, that is — [Margin: b — *Elements* VI.14] the sides about those angles will be reciprocally proportional; that is, EH — that is, AB [?], equal to EH — will be to HF (that is, to AH, equal to HF) as AH is to HB. Therefore, since the whole AB is to the segment AH as the segment AH is to the segment HB, AB has been cut in extreme and mean ratio, by definition. So the given finite straight line, etc. Which was to be done.

Otherwise. We shall also show that AB is cut at H in extreme and mean ratio thus. Since three lines are given — AB, AH, HB — and the rectangle HC, contained by the first AB and the third HB, is equal to the square on the mean AH, [Margin: c — *Elements* VI.17] these will be proportional: as AB, the first, is to AH, the second, so AH, the second, is to HB, the third. Therefore, by the definition, AB has been cut at H in extreme and mean ratio.

Otherwise. We shall carry out the whole problem in another way. [Margin: d — *Elements* II.11] Let AB be divided at C in such a way that the rectangle contained by the whole AB and the segment CB is equal to the square on the other segment AC. I say that AB is cut at C in extreme and mean ratio. [Margin: e — *Elements* VI.17] For here again the three lines AB, AC, CB will be continuously proportional. So the proposition stands.

Diagram. The figure for Proposition 30 (cutting a line in extreme and mean ratio): a large square standing on the base AB with an applied rectangle. Readable labels along the base are **D** (bottom left), **A** (bottom middle), **C** (bottom right); along the top **G** [?] (top left) and **B** (top right); and in the middle band **E**, **H**, **F** [?] (left to right). The square ABC and the rectangles GH, HC named in the proof are formed by the horizontal EH-line and the vertical through H. Two labels are at the resolution floor.

Diagram. A single straight segment for the second "Otherwise": the line **A**——**C**——**B**, marked with a division point (C) between A and B.

Scholium (to Proposition 30)

The practical construction for dividing a straight line in extreme and mean ratio is to be carried out as we set forth at Proposition 11 of Book 2. Moreover, this section of a line in extreme and mean ratio has remarkable uses and properties, as will be evident in the books of Stereometry — so much so that, not without reason, a line so divided is said by very many mathematicians to have, in a manner of speaking, a "divine" proportion, on account of its wonderful power and nature; while others simply call it a "line divided proportionally."

THEOR. 21. PROPOS. 31.

In right-angled triangles, the figure described on the side subtending the right angle is equal to the figures, similar and similarly placed, described on the sides containing the right angle.

Demonstration. Let ABC be a right-angled triangle having the angle BAC a right angle; and on BC let there be described [Margin: a — *Elements* VI.18] any rectilinear figure BCDE, and on AB, AC let figures ABFG, ACIH be set up similar and similarly placed to it. I say that the figure BD is equal to the two figures AF, AI together.

For drop from A the perpendicular AK to BC; then, by the corollary to Proposition 8 of this book, BC is to CA as CA is to CK. Therefore, as BC is to CK (the first line to the third), so is the figure BD on the first to the figure CH on the second, similar and similarly placed, by the corollary to Proposition 19 or 20 of this book. And, converting, as CK is to BC, so is the figure CH to the figure BD. In the same way it will be shown that, as BK is to BC, so is the figure BG to the figure BD — since the three lines BC, BA, BK are likewise proportional, etc.

Now since CK, the first magnitude, is to BC, the second, as CH, the third, is to BD, the fourth; and likewise BK, the fifth magnitude, is to BC, the second, as BG, the sixth, is to BD, the fourth — [Margin: b — *Elements* V.24] it follows that the first CK together with the fifth BK is to the second BC as the third CH together with the sixth BG is to the fourth BD. But CK and BK, the first and fifth together, are equal to the second BC. Therefore CH and BG, the third and sixth together, will also be equal to the fourth BD. Which is the proposition.

Otherwise. [Margin: c — *Elements* VI.8] Since the triangle KAC is similar to the triangle ABC, and their corresponding sides are BC, CA [Margin: d — *Elements* VI.19] (for as BC is to CA in triangle ABC, so is CA to CK in triangle KAC), the triangle KAC will have to the triangle ABC the duplicate ratio of that which CA has to BC. [Margin: e — *Elements* VI.19 or 20] But the figure CH also has to the figure BD the duplicate ratio of CA to BC. [Margin: f — *Elements* V.11] Therefore, as the triangle KAC is to the triangle ABC, so is the figure CH to the figure BD. By the same reasoning it will be shown that, as the triangle KBA is to the triangle ABC, so is the figure BG to the figure BD.

Since, then, again KAC, the first magnitude, is to ABC, the second, as CH, the third, is to BD, the fourth; and likewise KBA, the fifth, is to ABC, the second, as BG, the sixth, is to BD, the fourth — [Margin: g — *Elements* V.24] the first KAC composed with the fifth KBA will be to the second ABC as the third CH composed with the sixth BG is to the fourth BD. But KAC, KBA, the first and fifth together, are equal to the second ABC. Therefore CH, BG, the third and sixth together, will also be equal to the fourth BD. Which is the proposition.

Otherwise. As the square on the line AC, the first magnitude, is to the square on BC, the second, so is the figure CH, the third magnitude, to the figure BD, the fourth — [Margin: i — *Elements* VI.19 or 20] since each pair is in the duplicate ratio of AC to BC. Similarly, as the square on the line AB is to the square on BC, the second magnitude, so is the figure BG, the sixth, to the figure BD. [Margin: i — *Elements* V.24] Wherefore, as the first magnitude together with the fifth — namely the square on AC together with the square on AB — is to the second, that is, to the square on BC, so is the third magnitude together with the sixth — namely the figure CH together with BG — to the fourth, namely to the figure BD. [Margin: k — *Elements* I.47] But the squares on the lines AC, AB together are equal to the square on the line BC. Therefore the figures CH, BG will also be equal to the figure BD. Which is the proposition. So, in right-angled triangles, any figure, etc. Which was to be demonstrated.

Diagram. The figure for Proposition 31 (the generalized Pythagorean theorem): the right triangle ABC with a figure erected on each side. Readable vertex-labels:

- A (apex of the right angle at the top center)
- H (top, above the figure on AC)
- G (upper left, on the figure over AB)
- F (left), B (lower left) — corners of the figure ABFG on side AB
- K (foot of the perpendicular AK on BC), Q (lower right, at vertex C of the base figure) — [the base vertex read as C/Q]
- E (bottom left), D (bottom right) — the figure BCDE on the hypotenuse BC
- I (right, on the figure over AC)

The perpendicular AK falls from A to BC. (*The same construction appears in partial view at the foot of p. 293, where only the labels F, B, E of its left portion are legible.*)

Scholium (to Proposition 31)

You see, then, that this theorem of Euclid's is far more general — since it extends to all similar and similarly described figures — than that discovery of Pythagoras, which embraces squares alone, as we noted at Proposition 47 of Book 1. Yet that theorem, the one we there demonstrated from Pappus, seems in some respect still more general than this one, since it is propounded for every triangle and for parallelograms even when not similar, whereas this one is propounded only for a right-angled triangle and for figures that are similar and similarly placed.

We shall also convert this theorem, following Campanus, in just the same way as the 47th proposition of Book 1, as follows.

If the figure described on one of the sides of a triangle is equal to those described on the remaining sides of the triangle — the figures being similar and similarly placed — then the angle contained by the remaining two sides of the triangle is a right angle.

Let a triangle ABC be given, and let any figure described on the side BC be equal to two figures, similar to it and similarly described, on the remaining sides AB, AC. I say that the angle BAC is a right angle. For draw AD perpendicular to AC, equal to AB, and join the straight line CD. Now since the angle CAD is a right angle, [Margin: l — Elements VI.31] the figure described on CD (similar to the one on BC and similarly placed) will be equal to the figures described on AD, AC that are similar to it and similarly placed. But the figure on AD is equal to the figure on AB, because the sides are equal. Therefore the figure on CD will be equal to the figures on AB, AC together. And since the figure on BC is assumed equal to those same figures on AB, AC, the figures on CD and BC will be equal to one another; and therefore the straight lines CD, BC will be equal, as is clear from the lemma to Proposition 22 of this book. Since, then, the sides AC, AD of triangle ADC are equal to the sides AB, AC of triangle ABC, and the base DC has been shown equal to the base BC, [Margin: m — Elements I.8] the angles DAC, BAC will be equal. Therefore, since DAC is a right angle by construction, BAC will also be a right angle. Which is the proposition.

Diagram. The figure for the converse (Campanus's conversion): a triangle with vertices **D** (top), **A** (upper left, interior/apex region), **B** (lower left), and **C/Q** (lower right) — the constructed perpendicular AD and the joined line CD forming triangle ADC alongside the original triangle ABC.

THEOR. 22. PROPOS. 32.

[Margin: xxx.]

If two triangles that have two sides proportional to two sides be set together at a single angle so that their corresponding sides are also parallel, then the remaining sides of those triangles will be found to lie in a straight line.

Demonstration. Let the triangles ABC, DCE have the sides AB, AC proportional to the sides DC, DE — namely, as AB is to AC, so DC is to DE — and let them be set together at the angle ACD so that the corresponding sides AB, DC, and likewise AC,

DE, are parallel to one another. I say that the two remaining sides BC, CE form a straight line.

For since AB, DC are parallel, [Margin: a — *Elements* I.29] the angle A will be equal to the alternate angle ACD; and for the same reason the angle D will be equal to that same angle ACD; and therefore A and D will also be equal to one another. Now since the triangles ABC, DCE have the sides about the equal angles A, D proportional, [Margin: b — *Elements* VI.6] they will be equiangular with one another and will have the angles B, DCE equal. Therefore, adding the equals A and ACD, the two angles B, A will be equal to the two angles DCE, ACD — that is, to the angle ACE. Again, adding the common angle ACB, the three angles of the triangle ABC will become equal to the two angles ACE, ACB; [Margin: c — *Elements* I.32] but those three are equal to two right angles. Therefore the two angles ACE, ACB will also be equal to two right angles; [Margin: d — *Elements* I.14] and for that reason BC, CE will constitute one straight line. And so, if two triangles that have two sides proportional to two sides, etc. Which was to be demonstrated.

Diagram. The figure for Proposition 32: two triangles set together along the base line. The larger triangle **ABC** has apex **A** (top), base vertices **B** (bottom left) and **C** (bottom middle); the smaller triangle **DCE** has apex **D** (upper right), sharing the vertex region at **C** and reaching **E** (bottom right). The base runs B — C — E in a single straight line.

SCHOLIUM

(Continuation of the scholium to Prop. 32.)

The two aforesaid triangles must be joined at one angle in such a way that each of the angles contained by the proportional sides is *alternate* to the angle at which the triangles are joined — as you see done in the figure of the theorem. For to the angle ACD, at which the triangles are joined, both angle A and angle D are alternate, and each of these is contained by proportional sides. From this it follows that angles A and D are equal, and therefore that the triangles are equiangular; and thus BC and CE form a single straight line, as is clear from the demonstration.

But if each of the angles contained by the proportional sides is *not* alternate to the angle at which the triangles are joined, then — even though the remaining hypotheses of the theorem are preserved — the conclusion does not necessarily follow. For the two triangles ABC and DBE have two sides AB, AC proportional to the two sides DE, DB (namely, as AB is to AC, so is DE to DB), and they are joined

at the angle CBD, so that the corresponding sides AB, DE, as well as AC, DB, are parallel. Nevertheless the remaining two sides CB, BE do *not* form a single straight line, because neither of the angles A and D is alternate to the angle CBD — indeed, neither of them is, as is evident. For this reason the demonstration of the theorem has no place here.

Diagram. The counterexample figure for the scholium: two triangles joined at the vertex B, showing that the remaining sides do not lie in one straight line when the alternate-angle condition fails.

- C (top): apex of the upper triangle
- D (right, below C): apex of the second triangle
- A (upper left): apex of triangle ABC
- B (lower left): the common joining vertex
- E (lower middle/right): far endpoint of side BE
- Straight lines are drawn from B to A, C, D, and E; the triangles ABC and DBE share the vertex B, and sides CB, BE visibly fail to be collinear.

THEOR. 23. PROPOS. 33.

[Margin: xxxij. — proposition number in the outer margin; the middle glyph is ambiguous between *i* and *l*, read here as *xxxij* (32), lagging the body number 33 by one[?]]

In equal circles, the angles have the same ratio as the arcs on which they stand, whether they are constructed at the centres or at the circumferences. Moreover, the sectors do likewise, since they stand at the centres.

Let there be two equal circles ABC, EFG, whose centres are D and H; and let any two arcs whatever, BC and FG, be taken from the circles, on which stand the angles BDC, FHG at the centres, and the angles BAC, EFG at the circumferences. I say that — by the sense of definition 6 of Book 5 — as the arc BC is to the arc FG, so is the angle BDC to the angle FHG, and the angle BAC to the angle FEG; and moreover that the sector BDC, which is contained by the straight lines BD, DC and the arc BC, is to the sector FHG, which the straight lines FH, HG and the arc FG comprehend, in the same ratio.

For draw the straight lines BC, FG, and [Margin: a — *Elements* IV.1] apply to them, within the circles, equal straight lines: CI equal to BC, and GK, KL equal to FG; and draw the straight lines ID, KH, LH. Since, then, the straight lines BC, CI are equal,

[Margin: b — III.28] the arcs BC, CI will also be equal, [Margin: c — III.27] and therefore the angles BDC, CDI will be equal too. By the same reasoning the arcs FG, GK, KL will be equal, and so will the angles FHG, GHK, KHL.

Therefore, whatever multiple the arc BCI is of the arc BC, the angle BDI — that is, the aggregate of the angles near the centre D standing on the arc BCI — is the same multiple of the angle BDC; and whatever multiple the arc FGKL is of the arc FG, the angle FHL is the same multiple of the angle FHG, since the angles BDI, FHL are divided into just as many equal angles as the arcs BCI, FGKL are cut into equal arcs. And since, if the arc BCI is equal to the arc FGKL, [Margin: d — III.27] the angle BDI is necessarily equal to the angle FHL; if the arc BCI is greater than the arc FGKL, the angle BDI is necessarily greater than the angle FHL; and if less, less: therefore the arc BCI and the angle BDI — equimultiples of the first magnitude BC and the third BDC — will, together with the arc FGKL and the angle FHL — equimultiples of the second magnitude FG and the fourth FHG — either together fall short, or together be equal, or together exceed, if those are taken that correspond to one another. [Margin: e — V def. 6] Hence, as the arc BC, the first magnitude, is to the arc FG, the second magnitude, so will the angle BDC, the third magnitude, be to the angle FHG, the fourth magnitude.

Diagram. Two equal circles for the proof of Prop. 33: a left circle with centre D and a right circle with centre H, in each of which equal arcs are stepped off from a common chord and radii are drawn to the circumference, together with the inscribed (circumference) angles. Several small peripheral vertex-letters sit at the resolution floor.

- D (centre, left circle): centre
- A (top of left circle): vertex of the circumference angle BAC
- B (lower left of left circle): endpoint of the base chord
- C (lower right of left circle): common endpoint of the equal arcs BC, CI
- I (upper right of left circle): far endpoint of the stepped-off equal arc CI
- M (bottom of left circle): peripheral point (role at the resolution floor) [?]
- N (interior, right of D): labelled point on/near a radius [?]
- H (centre, right circle): centre
- Top vertex of right circle (the circumference-angle vertex, expected E) [?]
- F, G (lower left / lower right of right circle): endpoints of the base arc FG
- K (upper right of right circle): stepped-off equal-arc point
- L (right of right circle): far endpoint of the stepped-off equal arcs [?]

- Radii DB, DC, DI and HF, HG, HK, HL are drawn, with chords BC, CI and FG, GK, KL.

And this is likewise true of the *regions* [spatia] about the centres — that is, as the arc BAC is to the arc FEG, so is the region at the centre D standing on the arc BAC to the region at the centre H standing on the arc FEG. For these regions too, if they are equal, stand on equal arcs, and if unequal, on unequal arcs, and so on.

Now, since [Margin: f — V.15] as the angle BDC is to the angle FHG, so is the angle BAC to the angle FEG — [Margin: g — III.20] because the former of each pair is the double of the latter — it is evident [Margin: h — V.11] that the angle BAC is likewise to the angle FEG as the arc BC is to the arc FG. This, however, can be demonstrated by the same arguments we used for the angles constructed at the centres, provided the straight lines IA, KE, LE, and so on, are first drawn.

But if angles of this kind, BAC, FEG, are constructed at the circumferences in such a way that the straight lines drawn BD, CD, FH, GH do not form angles at the centres facing the arcs BMC, FG, then one must take the regions BDC, FHG, which are likewise double the angles at the circumferences, by the scholium to Prop. 20 of Book 3.

Now let there be constructed, in the segments BC, CI, the angles BMC, CNI, [Margin: i — III.27] which will be equal, since they stand on the equal arcs BAC, CBAI. Therefore the segments BMC, CNI will be similar, [Margin: k — III.24] and moreover equal to one another, because they stand on the equal straight lines BC, CI. Adding, then, the triangles BDC, CDI, [Margin: l — I.4] which are likewise equal, the sectors BDC, CDI become equal. Hence the sector BDI is the same multiple of the sector BDC as the arc BCI is of the arc BC. In the same way we shall show that the sector FHL is the same multiple of the sector FHG as the arc FGKL is of the arc FG. And since, if the arc BCI is equal to the arc FGKL, the sector BDI is also equal to the sector FHL — as was shown for the sectors BDC, CDI — and if greater, greater, and if less,

less: therefore the arc BCI and the sector BDI — equimultiples of the first magnitude BC and the third BDC — will, together with the arc FGKL and the sector FHL — equimultiples of the second magnitude FG and the fourth FHG — either together fall short, or together be equal, or together exceed, if those are taken that correspond to one another. [Margin: i — V def. 6] Therefore, as the arc BC, the first magnitude, is to the arc FG, the second magnitude, so will the sector BDC, the third magnitude, be to the sector FHG, the fourth magnitude. Therefore,

in equal circles, the angles have the same ratio as the arcs, and so on. Which was to be demonstrated.

Perhaps the demonstration will be set up more conveniently in this way. Take (at least in thought) any number of circles equal to the circle ABC, and in each of them mark off a single arc equal to the arc BC, on which stand angles both at the centres and at the circumferences. For all these angles will be equal to the angles BDC, BAC, because of the equality of the arcs on which they stand; and consequently their aggregates will be the same multiples of the angles BDC, BAC as the aggregate of all the arcs is of the arc BC. Then take likewise any number of circles equal to the circle EFG, and in each mark off a single arc equal to the arc FG, and so on. For in this way one avoids the confusion of lines and angles within the circles ABC, EFG — a confusion that necessarily arises when the arcs BC, FG are large and their multiples have to be taken, as is evident. For if the arcs were BCI, FGL, on which stand the angles BDI, FHL at the centres and BAI, FEL at the circumferences, their multiples could not be taken without confusion, as is plain. But this confusion is avoided if several equal circles are employed, as we have said.

COROLLARY I

From this it is evident that the sector is to the sector as the angle is to the angle. For each of the two ratios is equal to the ratio of the arc to the arc; [Margin: k — V.11] therefore they will also be equal to one another.

COROLLARY II

It is likewise evident that, as an angle at the centre is to four right angles, so is the arc subtended by that angle to the whole circumference; and conversely, as four right angles are to the angle at the centre, so is the whole circumference to the arc subtended by that angle.

For as an angle at the centre is to a right angle at the centre, [Margin: l — VI.33] so is the arc subtended by that angle to the quadrant subtended by the right angle. Hence, as the angle at the centre is to the quadruple of the right angle — that is, to four right angles — so will the arc subtended by that angle be to the quadruple of the quadrant — namely, to the whole circumference — by what we demonstrated at Prop. 22 of Book 5. This is the first point. Since, then, as the angle at the centre is to four right angles, so is the arc subtended by that angle to the whole circumference, it will also follow, by conversion, that as four right angles are to the angle at the centre, so is the whole circumference to the arc subtended by the angle at the centre. This is the second point.

But this will also be demonstrated in the following way. [Margin: m — VI.33]
Since, as a right angle at the centre is to the angle at the centre, so is the quadrant subtended by the right angle to the arc subtended by that angle, it will also follow — by what we showed at Prop. 22 of Book 5 — that as the quadruple of the right angle, namely four right angles, is to the angle at the centre, so is the quadruple of the quadrant — namely, the whole circumference — to the arc subtended by that angle. Which was proposed.

The same is true of the region [spatium] at the centre: that is, as the region at the centre is to four right angles, so is the arc subtended by that region to the whole circumference, and so on, as is clear from the demonstration.

SCHOLIUM

Moreover, from this theorem it is gathered, more clearly than daylight, that an angle standing on some arc must be referred to the arc that is the *base* of the angle, and not to the arc *in which it lies*. For the ratio of angle to angle is not the same as that of arc to arc if one takes the arcs in which the angles lie, as Euclid intends in the present theorem. For let there be equal circles ABC, DEF, in which the inscribed angles B, E are constructed, B being the greater and E the lesser. On this supposition, the arc AC will be greater than the arc DF, by the scholium to Prop. 26 of Book 3, and therefore the remaining arc ABC will be less than the remaining arc DEF. Hence the ratio of angle B to angle E is one of greater inequality, whereas the ratio of arc ABC to arc DEF is one of lesser inequality. Therefore the ratio of angle to angle is not the same as the ratio of arc to arc. But if we take the arcs on which the angles stand — namely, the arcs AC, DF — then at last angle B will be to angle E as arc AC is to arc DF, as Euclid rightly demonstrated. And so, when we say an angle *is in* a segment, we must understand something different from when we say an angle *stands on* a segment or arc. This we pointed out in the exposition of definition 8 of Book 3.

Diagram. Two equal circles illustrating the scholium: in each, an inscribed angle stands on a chord, the angle at B being greater than the angle at E.

- B (top of left circle): vertex of the inscribed angle, standing on chord AC
- A (lower left of left circle): endpoint of the base chord AC
- C (lower right of left circle): endpoint of the base chord AC
- E (top of right circle): vertex of the inscribed angle, standing on chord DF
- D (lower left of right circle): endpoint of the base chord DF
- F (lower right of right circle): endpoint of the base chord DF

It can also be demonstrated, and not obscurely, from this theorem, that the similarity of segments in similar circles — which Euclid, in definition 10 of Book 3, defined by means of the equal angles existing within those segments — consists in this: that the segments, or the similar arcs, have the same ratio to the whole circumferences of the circles; and therefore, when the segments are commensurable with the whole circles, whatever part one arc is of its whole circumference, such a part likewise is the other similar arc of its whole circumference — as we taught in the exposition of the aforesaid definition. For let there first be two equal circles ABC, DEF, in which equal angles BAC, EDF are constructed at the circumferences. On this supposition, the segments BAC, EDF will be called similar, according to Euclid's aforesaid definition. Now it is evident that their arcs have the same ratio to the whole circumferences of the circles. For since, because of the equality of the circles, [Margin: n — III.28] the arcs BC, EF, on which the equal angles stand, are equal, it follows that the remaining arcs BAC, EDF are also equal. Therefore, in relation to the whole circumferences — which are also assumed equal — [Margin: o — V.7] they will have the same ratio; and accordingly, whatever part the arc BAC is of the whole circumference ABCA, the same part will the arc EDF be of the whole circumference DEFD.

Diagram. Two equal circles, each with an inscribed triangle, illustrating similar segments defined by equal inscribed angles.

- A (top of left circle): vertex of the inscribed angle BAC
- B (lower left of left circle): base vertex
- C (lower right of left circle): base vertex
- D (top of right circle): vertex of the inscribed angle EDF
- E (lower left of right circle): base vertex
- F (lower right of right circle): base vertex

Next, let there be two *unequal* circles ABC, DEF, in which equal angles BAC, EDF are constructed at the circumferences. On this supposition, the segments BAC, EDF will be called similar, according to Euclid's view. I say again that the arcs BAC, EDF have the same ratio to their whole circumferences. For draw to the centres G, H the straight lines BG, CG, EH, FH. Since, then, the angles A and D are assumed equal, the angles G and H at the centres will also be equal, [Margin: p — III.20] since these are the doubles of those; or else the regions [spatia] at the centres will be equal, since these too are double the angles at the circumferences, by the scholium to Prop. 20 of Book 3. [Margin: q — V.7] Therefore four right angles will have the same ratio to the angle (or region) G as to the angle (or region) H. But as four right angles are to the angle or region G, so is the whole

circumference ABCA to the arc BC; and as four right angles are to the angle or region H, so is the whole circumference DEFD to the arc EF, by Corollary 2 of this Proposition 33. [Margin: r — V.11] Therefore, as the whole circumference ABCA is to the arc BC, so will the whole circumference DEFD be to the arc EF. Hence, by conversion of the ratio, it will also follow that as the whole circumference ABCA is to the arc BAC, so is the whole circumference DEFD to the arc EDF. And, inverting, as the arc BAC is to the whole circumference ABCA, so is the arc EDF to the whole circumference DEFD. And so, whatever part the arc BAC is of the whole circumference ABCA, the same part will the arc EDF be of the whole circumference DEFD. Which was proposed.

Diagram. Two *unequal* circles, each with an inscribed triangle and radii drawn to the centre, illustrating similar segments in unequal circles.

- A (top of left circle): vertex of the inscribed angle
- G (centre of left circle): centre, with radii GB, GC drawn
- B (lower left of left circle): base vertex
- C (lower right of left circle): base vertex
- D (top of right circle): vertex of the inscribed angle
- H (centre of right circle): centre, with radii HE, HF drawn
- E (lower left of right circle): base vertex
- F (lower right of right circle): base vertex

It is therefore established that Euclid rightly called those segments of circles *similar* in which the angles existing are equal to one another, since segments of this kind have the same ratio to their whole circles.

From these results that theorem will easily be demonstrated which we showed, without proportions, both at the end of chapter 1 on the sphere and at Prop. 22 of Book 3 — namely:

If two or more circles are described about the same centre, and from the centre two or more straight lines are drawn, the arcs intercepted between any two of the lines will be similar.

Let two circles ABCD, EFGH be described about the same centre I. If, then, two straight lines IB, ID go out from the centre I forming a single straight line BD, it is evident that the arcs BAD, FEH are similar, since they are semicircles. Again, if two straight lines IA, ID go out from I forming the right angle AID, it is likewise clear that the arcs AD, EH are similar, since — by the scholium to Prop. 27 of Book 3 — they are quadrants of their circles. Now let two straight lines ID, IK be drawn

from I, making any non-right angle DIK. I say that even so the arcs DK, HL are similar — that is, that the arc DK is to the whole circumference ABCD as the arc HL is to the whole circumference EFGH. For since, as the angle DIK is to four right angles, so — by the corollary of this proposition — is both the arc DK to the whole circumference ABCD and the arc HL to the whole circumference EFGH, [Margin: s — V.11] it will follow that as the arc DK is to the whole circumference ABCD, so is the arc HL to the whole circumference EFGH; and therefore the arcs DK, HL will be similar. Which was proposed.

Diagram. Two concentric circles about the common centre I: the outer circle ABCD and the inner circle EFGH, with two diameters and the radii ID, IK cutting off the arcs DK (outer) and HL (inner).

- I (centre): common centre
- A (top, outer circle); E (top, inner circle, on radius IA)
- B (left, outer circle); F (left, inner circle, on radius IB)
- C (bottom, outer circle); G (bottom, inner circle)
- D (right, outer circle); H (right, inner circle, on the horizontal diameter)
- K (upper right, outer circle): endpoint of the oblique radius IK
- L (upper right, inner circle, on radius IK): where the oblique radius meets the inner circle

The same will be confirmed more briefly in this way. Since on the arcs DK, HL there stand, at the centre I, equal angles — indeed the very same angle — the arcs DK, HL are similar, by the scholium to Prop. 22 of Book 3; and consequently they will have the same ratio to the whole circumferences, as was shown a little earlier.

By the same reasoning we shall show that the arcs BAK, FEL are similar — that is, that they have the same ratio to the whole circumferences.

But it is fitting to demonstrate here also the lemma that we set forth at Prop. 6 of Book 3 of Theodosius, since it is useful for many other purposes. It runs as follows.

Equal straight lines cut off unequal arcs from unequal circles, and the arc of the smaller circle is greater than would make it similar to the arc of the larger circle.

Let there be unequal circles AB, CD described about the same centre E. Draw from E two arbitrary straight lines EA, EB, cutting the circles at the points A, B and C, D; the arcs AB, CD will be similar, as we demonstrated just now. And since

the straight lines EA, EB are cut proportionally at C, D — because EA, EB are equal to one another — [Margin: t — VI.2] the drawn straight lines AB, CD will be parallel; and therefore the triangles EAB, ECD will be similar, by the corollary to Prop. 4 of this book. [Margin: u — VI.4] Therefore, as EA is to AB, so is EC to CD. [Margin: x — V.14] But EA is greater than EC; therefore AB will also be greater than CD. So fit to CD, within the circle AB, an equal straight line BF; and by the scholium to Prop. 28 of Book 3, the arc AB will be greater than the arc FB. Hence, since the arc CD is similar to the arc AB, the arc CD will be greater than would make it similar to FB itself. Therefore the equal straight lines CD, BF cut off unequal arcs from the unequal circles AB, CD, and the arc of the smaller circle is greater than would make it similar to the arc FB of the larger circle. Which was proposed.

Diagram. Two concentric circles about the centre E: the larger circle (through A, B) and the smaller circle (through C, D), with radii EA, EB cutting the smaller circle at C, D, and the chord BF equal to CD fitted into the larger circle.

- E (centre/lower): common centre, from which the radii EA, EB are drawn
- A (upper left): point where radius EA meets the larger circle
- B (upper right): point where radius EB meets the larger circle
- F (top, between A and B): endpoint of the chord BF (equal to CD) on the larger circle
- C (inner left): point where radius EA cuts the smaller circle
- D (inner right): point where radius EB cuts the smaller circle
- Chords AB, CD (parallel) and BF are drawn.

Hence it is evident that, all the more, a greater line cuts off from the smaller circle an arc greater than would make it similar to the one that a smaller line cuts off from the larger circle. For since the straight line CD, equal to BF, cuts off an arc CD greater than would make it similar to the arc FB, much more will a line greater than CD cut off a greater arc than would make it similar to the arc FB, since that line cuts off a greater arc than CD does, as we showed in the scholium to Prop. 28 of Book 3.

But this demonstration establishes the proposition only when the arcs cut off, BF and CD, are less than a semicircle, as is clear from the demonstration itself. For otherwise no angle would be formed at E, the common centre — which is nevertheless required for the demonstration. Yet it will nonetheless be true that, if an arc of the smaller circle less than a semicircle is greater than would make it similar to an arc of the larger circle less than a semicircle, then an arc of the

smaller circle greater than a semicircle is much greater than would make it similar to an arc of the larger circle less than a semicircle. And if it should ever happen that the straight line CD cuts off a semicircle from the smaller circle — as when it is the diameter of the circle — it is plainly established that the semicircle of the smaller circle is greater than would make it similar to an arc of the larger circle less than a semicircle; and then there will be no need of any other demonstration.

Hence too we shall show, with no trouble, that equal straight lines cut off from unequal circles arcs that are unqualifiedly and absolutely unequal — so that the arc of the smaller circle is simply greater than the arc of the larger circle, and not merely greater than would make it similar. For let the straight lines CD, BF be equal, and let CD cut off the arc CED of the smaller circle, and FB the arc FGB of the larger circle. I say that the arc CED is simply greater than the arc FGB. For when the straight line CD is made to coincide with the straight line FB, the arc CED will necessarily fall

will necessarily fall outside the arc FGB; and therefore arc CED will be greater than arc FGB, since the latter is contained wholly within the former — provided that both arcs are concave toward the same side and have the same endpoints, as Archimedes holds in the postulates prefixed to Book I of *On the Sphere and Cylinder*. Nor indeed will arc CED coincide with arc FGB or fall within it. For if it were said to coincide, then the whole circumference of circle CED would also coincide with the whole circumference of circle FGB, and so the circles would be equal — which is absurd, since they were posited unequal. But if arc CED be said to fall within arc FGB — of which kind is arc CAD — then, since (as we showed just above) arc CED, that is CAD, is too great to be similar to arc FGB, let arc BFH be taken similar to arc CAD, and therefore greater than arc FGB. Now, taking any point A whatever on arc CAD, let the straight lines AF, AB be drawn; and, the straight line FA being produced until it cuts arc FGB at G, let the straight lines GH, GB be drawn. Then, since arcs CAD and HFB are similar, the angles CAD and HGB standing in those segments will be equal. [Margin: Elements I.16] But since angle CAD is greater than angle CGB — an exterior angle greater than an interior one — and angle CGB is likewise greater than angle HGB (the whole greater than the part), angle CAD will be very much greater than angle HGB, which is absurd; for it was shown to be equal. Therefore arc CED will not fall within arc FGB; nor does it coincide with it, as was demonstrated. It will therefore fall outside; and so arc CED will be greater than arc FGB, as was stated. Which was the thing proposed.

From this it is plainly established that a greater line drawn from a smaller circle cuts off an unqualifiedly greater arc than the arc which a smaller line cuts off from a larger circle.

Diagram. Two figures accompany the lemma. The upper figure shows two unequal circular arcs sharing the same endpoints C and D (at left and right), the smaller arc lying inside the larger. Several straight lines (chords and radii) are drawn across them, meeting at interior points, and a lower point H is joined by lines to the upper region. The separate lower figure is a semicircle on the chord CD with apex E.

Upper figure:

- E (top apex): highest point on the outer arc
- G (upper, right of centre): point where the produced line FA meets arc FGB
- A (centre): point taken on arc CAD
- C (far left): left endpoint / left extremity of the chords
- F (left, just below C): point on the arcs near C
- I (centre-low): interior intersection of the drawn lines
- B (right): right endpoint
- D (right, near B): endpoint of the inner arc
- H (lower left): point joined to G and B

Lower figure (semicircle):

- C (left): left end of the chord
- E (top): apex of the semicircle
- D (right): right end of the chord

Since Euclid said much about the finding of proportional lines but nothing about the investigation of proportional surfaces or plane figures, I think it not out of place to append to this sixth book — as a kind of appendix — some problems and theorems, many of which turn on the finding of proportional surfaces and are not unpleasant to know. These I draw partly from skilled geometers, partly from my own discoveries, inasmuch as they are readily deduced from what Euclid has demonstrated. Here, then, we shall make our beginning.

I.

From a given rectilinear figure to cut off a required part, in such a way that both what is cut off and what remains are similar to a given rectilinear figure and similarly placed.

Let the third part be that which is to be cut off from the rectilinear figure A — a part similar and similarly placed to the figure B — and let there remain a rectilinear figure likewise similar to B and similarly placed. [Margin a: Elements VI.25] Construct a rectilinear figure CD equal to A, but similar and similarly placed to B; and on one side of it, CE, [Margin b: Elements VI.6] let a semicircle CFE be described. Then, the third part GE — namely the required part — having been cut off from CE, let GF be drawn perpendicular to CE, and let the straight lines CF, EF be joined; [Margin c: Elements VI.18] upon which let the rectilinear figures FH, FI be constructed, similar and similarly placed to CD. I say, then, that what is required has been done. [Margin d: Elements III.31] For since angle CFE is right — being an angle in a semicircle — [Margin e: Elements VI.31] the figure CD will be equal to the figures HF, FI together; and so, if the figure FI (similar and similarly placed to B) be cut off from the figure CD — that is, from A, its equal — there will remain the figure HF, likewise similar to B and similarly placed. That the figure FI cut off is a third part of the figure CD is shown thus. [Margin f: Elements VI.4] Since CG is to GF as CF is to FE, [Margin g: Elements VI.8] because triangles CGF, CFE are similar, and CG has to GE the ratio duplicate of the ratio of CG to GF (seeing that the three lines CG, GF, GE are proportional, by the corollary to Proposition 8 of this book), [Margin h: Elements VI.19 or 20] the figure HF likewise has to the figure FI the ratio duplicate of the ratio of the homologous sides CF, FE. Therefore, as CG is to GE, so is the figure HF to the figure FI, since these are the duplicates of two equal ratios. Compounding, then, as CE is to GE, so are the two figures HF, FI together — that is, the figure CD, which is equal to them — to the figure FI. But CE is triple GE, by construction. Therefore the figure CD is also triple the figure FI; and consequently FI is a third part of CD. Which was the thing proposed.

And if the required part — namely a third — is to be cut off without further qualification, it will be done most briefly by this method. [Margin i: Elements I.45] Let the given rectilinear figure A be reduced to an equal parallelogram GD; and from its side BD let DE, the required third part, be cut off; and through E let EF be drawn parallel to BG. I say that DF is a third part of DG — that is, of the given rectilinear figure A. [Margin k: Elements VI.1] For since DE is to DB as DF is to DG, and, by construction, DE is a third part of DB, DF too will be a third part of DG. Which was the thing proposed.

II.

Given two rectilinear figures, to find a third proportional.

Let two rectilinear figures A and BCD be given, of which a third proportional is to be found. [Margin l: Elements VI.25] Construct a rectilinear figure EFG equal to A, but similar and similarly placed to BCD. Then, on the homologous sides EF, BC, [Margin m: Elements VI.11] let a third proportional line HI be found; [Margin n: Elements VI.18] and upon HI let a rectilinear figure HIK be constructed, similar and similarly placed to EFG and BCD. I say that HIK is the third proportional. For since [Margin o: Elements VI.22] the lines EF, BC, HI are proportional, the figures EFG, BCD, HIK described upon them (being similar and similarly placed) will also be proportional. And since EFG was constructed equal to A, the figures A, BCD, HIK will also be proportional. Which was the thing proposed.

III.

Given three rectilinear figures, to find a fourth proportional.

Let three rectilinear figures A, BCDE, FGH be given, of which a fourth proportional is to be found. [Margin a: Elements VI.25] Construct a rectilinear figure IKLM equal to A, but similar and similarly placed to BCDE. Then, of the three lines IK, BC, FG, [Margin b: Elements VI.12] let a fourth proportional NO be found; [Margin c: Elements VI.18] and upon NO let a rectilinear figure NOP be constructed, similar and similarly placed to FGH. I say that the figure NOP is the fourth proportional. For since the four lines IK, BC, FG, NO are proportional, [Margin d: Elements VI.22] the similar and similarly placed figures described upon them — IL, BD, FGH, NOP — will also be proportional. And since IL was constructed equal to A, the four figures A, BD, FGH, NOP will also be proportional. Which was the thing proposed.

Diagram. The figures for Proposition III: the three given figures (a triangle A, a quadrilateral BCDE, and a triangle FGH) and the two constructed figures (a parallelogram IKLM and a triangle NOP).

- A: a triangle (the first given figure)
- B (lower left), C (lower right), D (upper right), E (upper left): the quadrilateral BCDE
- H (apex), F (lower left), G (lower right): the triangle FGH
- M (upper left), L (upper right), I (lower left) [?], K (lower right): the constructed parallelogram IKLM (lower-left vertex printed in an I/J form)
- P (apex), N (lower left), O (lower right): the constructed triangle NOP

IV.

Given two rectilinear figures, to find a mean proportional.

Let two rectilinear figures A, BCD be given, of which a mean proportional is to be found. [Margin a: Elements VI.25] Construct a rectilinear figure EFG equal to A, and similar and similarly placed to BCD. Then, of the two lines EF, BC, [Margin b: Elements VI.13] let a mean proportional HI be found; [Margin c: Elements VI.22] and upon HI let a rectilinear figure HIK be constructed, similar and similarly placed to EFG and BCD. I say that HIK is the mean proportional sought. For since the three lines EF, HI, BC are proportional, [Margin d: Elements VI.22] the figures described upon them — EFG, HIK, BCD — will also be proportional, being similar and similarly placed. Since, then, EFG was constructed equal to A, the figures A, HIK, BCD will also be proportional; that is, as A is to HIK, so is HIK to BCD; and consequently HIK is the mean proportional between A and BCD. Which was the thing proposed.

Another method. Let a mean proportional again be sought between the rectilinear figures A and B. [Margin e: Elements I.45] Construct any parallelogram CDEF equal to A, [Margin f: Elements VI.25] and a parallelogram EFGH equal to B but similar and similarly placed to CDEF. Let these parallelograms be joined at equal angles, so that DE, EI make one straight line; and consequently — by what we demonstrated at Proposition 15 of Book I — let FE, EG likewise form one straight line, and let the whole parallelogram KL be completed. I say that either EK or EL is a mean proportional between DF and GI — that is, between A and B. For since DF, GI are similar and similarly placed, DE is to EF as GH is to HI, that is, as EI is to EG, [Margin g: Elements I.34] because the lines EI, EG are equal to the lines GH, HI. Alternating, then, DE is to EI as FE is to EG. [Margin h: Elements VI.1] But DE is to EI as DF is to EK, and FE is to EG as EK is to GI. Therefore DF is to EK as EK is to GI; and consequently EK is a mean proportional between DF and GI — that is, between A and B. Which was the thing proposed.

This can also be confirmed in the following way. Since DE, GI are similar and similarly placed, [Margin i: Elements VI.26] they stand about the same diameter. Hence the complements EK, EL will be equal. But DF is to EK as EL is to GI, [Margin k: Elements VI.1] because each ratio is the same as the ratio of DE to EI. Therefore DF is to EK as EK is to GI. Which was the thing proposed.

Diagram. The figures for the first construction of Proposition IV: a triangle A (the first given figure), a triangle BCD, a triangle EFG, and the constructed triangle HIK.

- A: a right-angled triangle (the given figure A)
- D (apex), B (lower left), C (lower right): triangle BCD
- G (apex), E (lower left), F (lower right): triangle EFG

- K (apex), H (lower left), I (lower right): the constructed triangle HIK

Diagram. The figures for the alternative (parallelogram) construction of Proposition IV: a right triangle A, a triangle B, and the joined parallelograms CDEF and ECHI completed into the parallelogram KL.

- A: a right triangle
- B: a small triangle
- C (upper left), F (upper, centre), K (upper right): upper vertices of the joined parallelograms
- D (middle left), E (middle, centre), I (middle right): the middle row (DE, EI forming one straight line)
- L (lower left), G (lower, centre), H (lower right): lower vertices

V.

On a given rectilinear figure, to construct two rectilinear figures equal to it, which are similar and similarly described to any given rectilinear figure whatever, and which have between themselves any proposed ratio.

Let a rectilinear figure A be given, and a ratio of the line B to C; and let it be required to construct two rectilinear figures which are equal to A, which have the ratio of B to C, and which are similar and similarly placed to any given figure D. [Margin a: Elements VI.25] Construct a rectilinear figure EFGH equal to A, and similar and similarly placed to D. Then, its side EH being divided at I in the ratio of B to C — by the method we demonstrated at Proposition 10 of this book — let a semicircle EKH be described about EH; and from I let the perpendicular IK be drawn to EH, and the straight lines EK, HK be joined. [Margin b: Elements VI.18] Now upon EK, KH let the rectilinear figures EKLM, KHNO be described, similar and similarly placed to EG (or to D); which figures I say are also equal to EG (or to A) and have the given ratio B to C. [Margin c: Elements III.31] For since angle EKH, standing in a semicircle, is right, [Margin d: Elements VI.31] the figures EL, HO will be equal to the figure EG, being similar to each other and similarly described. [Margin e: Elements VI.4] And since EI is to IK as EK is to KH — the triangles EIK, EKH being similar — while EI is to IH in the ratio duplicate of the ratio of EI to IK (because the three lines EI, IK, IH are proportional, by the corollary to Proposition 8 of this book), [Margin f: Elements VI.19 or 20] and likewise EL is to HO in the ratio duplicate of that which the side EK has to the homologous side KH, therefore, as EI is to IH — that is, as B is to C — so is EL to HO. Given, then, the rectilinear figure A, we have exhibited two figures at once

equal to it, EL and HO, which are similar and similarly placed to the given figure D, and which have between themselves the given ratio of B to C. Which was the thing proposed.

Diagram. The figure for Proposition V: a square EFGH with a semicircle EKH described on its upper side EH, the perpendicular IK, and the two rectilinear figures EKLM and KHNO described on EK and KH. Two line-segments B and C, and a rectangle D, are shown separately at the left.

- E (upper left of the square), H (upper right): endpoints of the diameter EH
- F (lower left), G (lower right): lower vertices of the square EFGH
- I (on EH, between E and H): point dividing EH; foot of the perpendicular
- K (apex of the semicircle): point where the perpendicular IK meets the semicircle
- L, M: the further vertices of the figure EKLM on EK
- N, O: the further vertices of the figure KHNO on KH
- B, C: two separate straight-line segments (the given ratio)
- D: a separate rectangle (the figure to which the constructions are similar)

VI.

On a given rectilinear figure, to exhibit two rectilinear figures equal to it, which are similar and similarly described to any given rectilinear figure, and whose homologous sides have a given ratio to one another.

Let a rectilinear figure A be given, and a ratio of the line B to the line C; and let it be required to construct two rectilinear figures equal to A, and similar and similarly placed to a given figure D, whose homologous sides have the ratio of B to C. [Margin a: Elements VI.11] Let a third proportional E be found to B and C; [Margin b: Elements VI.25] and let a rectilinear figure FGH be made equal to A, and similar and similarly placed to D. Then, the side FH being divided at I in the ratio of B to E — as we showed at Proposition 10 of this book — let a semicircle FKH be described about FH; and from I let the perpendicular IK be drawn to FH, and the straight lines FK, HK be joined. [Margin c: Elements VI.18] Now upon FK, HK let the rectilinear figures FKL, HKM be described, similar and similarly placed to FGH (or to D). I say that these figures are equal to FGH (or to A), and that their homologous sides FK, HK have the given ratio of the line B to the line C. [Margin d: Elements III.31] For since angle EKH [recte FKH] is right in a semicircle, [Margin e: Elements VI.31] the figures FKL, HKM will be equal to the figure FGH, and therefore to A. [Margin f: Elements VI.4] And since FI is to IK as FK is to HK, on account of the similarity of the triangles FIK, FKH — while FI is to IH in the

ratio duplicate of the doubling of FI to IK (because the three lines FI, IK, IH are proportional, by the corollary to Proposition 8 of this book) — FI is therefore to IH in the ratio duplicate of the ratio of the homologous sides FK, HK. Likewise the ratio of B to E (equal to the ratio of FI to IH, by construction) is the duplicate of the ratio of B to C, by definition. Therefore the ratio of FK to HK will be the same as that of B to C, since the duplicates of these ratios — that of FI to IH and that of B to E — are equal. The proposition therefore stands established.

Diagram. The figure for Proposition VI: a rectangle A, a triangle D, and the semicircle FKH described on FH, with the perpendicular IK, the joined lines FK, HK, and the figures FKL, HKM described on them. Three line-segments B, C, E are shown at the lower left.

- A: a rectangle (the given figure)
- D: a triangle (the figure to which the constructions are similar)
- F (lower left of the semicircle), H (lower right): endpoints of the diameter FH
- I (on FH between F and H): point dividing FH; foot of the perpendicular
- K (apex): point where the perpendicular meets the semicircle
- L (upper left), M (upper right): the further vertices of the figures FKL and HKM
- G (lower point below FH): apex of the figure FGH
- B, C, E: three separate straight-line segments

VII.

Given two rectilinear figures, to construct a rectilinear figure equal to them, which is similar and similarly placed to any given rectilinear figure.

Let two rectilinear figures A and B be given, to which an equal figure is to be constructed, similar and similarly placed to a figure C. [Margin a: Elements VI.25] Let DEF be made equal to A, but similar and similarly placed to C; and likewise let GFH be constructed equal to B, but similar to the same C and similarly placed. Then let the figures DEF, GFH be so joined together that their homologous sides EF, FH [?] form a right angle EFH, subtended by the straight line EH; [Margin b: Elements VI.18] upon which let a rectilinear figure IHE be described, similar and similarly placed to DEF and GFH — that is, to C. [Margin c: Elements VI.31] It is then evident that EHI is equal to the two figures DEF, GFH together, and therefore to the two figures A and B. Which was the thing proposed.

A shorter method. Construct a parallelogram equal to the two rectilinear figures A and B, by the method we taught at Proposition 45 of Book I. [Margin d: Elements VI.25] For if to this parallelogram we construct an equal rectilinear figure EHI, which is similar and similarly placed to the other given figure C, then the constructed figure EHI will be equal to the two figures A, B, and similar and similarly placed to the figure C. Which was to be done.

Diagram. The figure for Proposition VII: three separate triangles A, B, C, and the joined construction below — triangles DEF and GFH set at the right angle EFH, with the figure IHE described on the subtending line EH.

- A: a triangle (first given figure)
- B: a triangle (second given figure)
- C: a triangle (the figure to which the construction is similar)
- D (upper left), F (upper centre), G (upper right): upper vertices of the joined triangles DEF, GFH
- E (lower left), H (lower right): endpoints of the subtending line EH
- I (lowest point): apex of the figure IHE described on EH

VIII.

If in a circle two straight lines cut one another, the segments of the one will be reciprocal to the segments of the other.

In the circle ABCD let the straight lines AC, BD cut one another at E. I say that the segments AE, EC are reciprocal to the segments BE, ED: that is, that as AE is to BE, so is ED to EC; or as AE is to ED, so is BE to EC. [Margin a: Elements III.25 [?] — expected III.35 (rectangle under segments of intersecting chords); printed digit uncertain] For since the rectangle contained by AE, EC is equal to the rectangle contained by BE, ED, [Margin b: Elements VI.14] the sides about equal angles will be reciprocal. Which was the thing proposed.

Diagram. The figure for Proposition VIII: a circle ABCD in which the two chords AC and BD cross at the interior point E.

- A (top): endpoint of chord AC
- B (left): endpoint of chord BD
- C (bottom): endpoint of chord AC
- D (right): endpoint of chord BD
- E (centre): the point where the chords AC, BD intersect
- M (interior, upper): an additional marked point on the arc

IX.

If some point be taken outside a circle, and from it two straight lines fall upon the circle cutting it: the whole lines, together with the segments outside the circle, will be reciprocal. And if from the same point a line be drawn to touch the circle, this will be a mean proportional between any line that cuts the circle and its exterior segment.

IX. (demonstration)

Outside the circle ABCD let a point E be taken, from which the straight lines EC, ED fall, cutting the circle at A and B. I say that the lines EC, EB are reciprocal to the lines ED, EA — that is, that as EC is to ED, so EA is to EB; and likewise, as EC is to EA, so ED is to EB. For when the line EF is drawn tangent to the circle at F, [Margin: 36. tertij → Elements III.36 (secant/tangent)] the rectangle under EC, EB will be equal to the square on EF; and likewise the rectangle under ED, EA will be equal to that same square on EF. Therefore the rectangles under EC, EB and under ED, EA will be equal. [Margin: 14. sexti → Elements VI.14] Hence their sides about the equal angles are reciprocal. [Margin: 36. tertij → Elements III.36] Now since the square on EF is equal both to the rectangle under EC, EB and to the rectangle under ED, EA, [Margin: 17. sexti → Elements VI.17] the three lines EC, EF, EB will be proportional, and so too the three ED, EF, EA; and thus EF is a mean proportional between any line that cuts the circle and its exterior segment. Which was the thing proposed.

Diagram. A circle (ABCD) with an external point E at the upper right, from which two secants and one tangent are drawn.

- E (upper right, outside the circle): the external point from which the lines are drawn
- F (upper left, on the circle): point of tangency of the tangent EF
- A (right, near intersection on the circle): nearer point where the secant EAC meets the circle
- B (right, near intersection, just below A): nearer point where the secant EBD meets the circle
- C (left, on the circle): farther intersection of the first secant
- D (bottom, on the circle): farther intersection of the second secant

X.

If two straight lines cut one another, and from two of their endpoints perpendiculars are dropped each onto the other line, there will be two

lines — one lying between one of the endpoints and the intersection, the other between the intersection and the perpendicular drawn from the assumed endpoint of the former line — which are reciprocally proportional to two others enclosed in the same way.

Let two straight lines AB, CB cut one another at B, and from the endpoints A, C let perpendiculars be dropped onto them: AD onto CB, and CE onto AB. These perpendiculars will fall on AB, CB extended beyond B if the angle ABC is obtuse, but inward upon the lines themselves if that angle is acute, as the figures show. I say that the lines AB, BE (the former of which lies between the endpoint A and the intersection B, the latter between the intersection B and the perpendicular CE, which of course is drawn to the assumed line AB) are reciprocal to the two lines CB, BD (which are enclosed between corresponding endpoints) — that is, that as AB is to BC, so is BD to BE; and likewise, as AB is to BD, so is BC to BE. For since the angles ABD, ADB of triangle ABD are equal to the angles CEB, of triangle CBE — for ADB and CEB are right angles; [Margin: 15. primi → Elements I.15] and ABD, CBE are equal, being vertical in the first figure, while in the second figure they are one and the same angle — the triangles ABD, CBE will be equiangular. [Margin: 4. sexti → Elements VI.4] Therefore, as AB is to BD, so is CB to BE; and hence, by alternation also, as AB is to BC, so is BD to BE. Which was the thing proposed.

Moreover, by the same reasoning the segments of the perpendiculars AD, CE, cutting one another at F in the second figure, will be reciprocal. For since the triangles AFE, CFD are equiangular, [Margin: 4. sexti → Elements VI.4] as AF is to FE, so will CF be to FD; and by alternation also, as AF is to CF, so is FE to FD. Which was the thing proposed.

Diagram. The first (obtuse-angle) figure for Proposition X: two straight lines crossing in an X, with perpendiculars falling outside.

- A (upper left): endpoint of one line
- C (upper right): endpoint of the other line
- B (center): the intersection of the two lines
- D (lower left): foot of the perpendicular from A
- E (lower right): foot of the perpendicular from C

Diagram. The second (acute-angle) figure for Proposition X, with the perpendiculars falling inward and meeting at F.

- A (upper left): endpoint of one line

- C (right): endpoint of the other line
- F (center): intersection of the two perpendiculars AD, CE
- D (right of F): foot of one perpendicular
- E (lower left): foot of the other perpendicular
- B (bottom): the intersection of the two original lines

XI.

In a parallelogram, two straight lines parallel to the sides and cutting one another divide the parallelogram into four proportional parallelograms.

In the parallelogram ABCD let two straight lines EF, GH, parallel to the sides AD, DC, cut one another at I. I say that the four parallelograms EG, GF, BI, IC are proportional. [Margin: 1. sexti → Elements VI.1] For since, as EI is to IF, so is EG to GF; and likewise, as EI is to IF, so is BI to IC; [Margin: 11. quinti → Elements V.11] it will also be that, as EG is to GF, so is BI to IC. Which was the thing proposed.

Diagram. A parallelogram divided into four smaller parallelograms by two lines parallel to the sides.

- A (top left), G (top center), D (top right): the top side, with G where the line GH meets it
- E (middle left), I (center), F (middle right): the line EF, meeting GH at the center I
- B (bottom left), H (bottom center), C (bottom right): the bottom side, with H where GH meets it

XII.

Every quadrilateral is divided by two diagonals cutting one another into four proportional triangles.

Let the diagonals AC, BD, cutting one another at E, divide the quadrilateral ABCD. I say that the four triangles AED, CED, AEB, CEB are proportional. [Margin: 1. sexti → Elements VI.1] For, as AE is to EC, so is triangle AED to triangle CED, and triangle AEB to triangle CEB. [Margin: 11. quinti → Elements V.11] Hence, as triangle AED is to triangle CED, so is triangle AEB to triangle CEB. By the same reasoning you will show that, as BEA is to DEA, so is BEC to DEC, [Margin: 1. sexti → Elements VI.1] since each ratio is the same as the ratio of BE to ED. The thing proposed is therefore established.

Diagram. A quadrilateral with its two diagonals meeting at an interior point.

- A (top left), D (top right): two vertices of the quadrilateral
- B (bottom left), C (bottom right): the other two vertices
- E (center): the intersection of the diagonals AC and BD

XIII.

In a right triangle in which the perpendicular dropped from the right angle cuts the base in extreme and mean ratio, the three sides are continuously proportional. And if the three sides of a right triangle are continuously proportional, the perpendicular dropped to the base from the right angle cuts the base in extreme and mean ratio.

In the right triangle ABC, let the perpendicular AD dropped from the right angle A cut the base BC at D in extreme and mean ratio, the greater segment being BD; and hence, by the ninth theorem appended to Proposition 47 of Book I, the side AB is greater than the side AC. I say that the three sides BC, BA, AC are continuously proportional. For since by hypothesis, as BC is to BD, so is BD to DC, [Margin: 17. sexti → Elements VI.17] the square on BD will be equal to the rectangle under BC, CD. Again, since by the corollary to Proposition 8 of this book AC is a mean proportional between BC and CD, [Margin: 17. sexti → Elements VI.17] the square on AC will likewise be equal to that same rectangle under BC, CD. Therefore the squares on BD and AC will be equal to one another, and consequently the lines themselves will also be equal. But by the corollary to Proposition 8 of this book, AB is a mean proportional between BC and BD. Therefore that same AB will be a mean proportional between BC and AC — that is, as BC is to AB, so is AB to AC. Which was the thing proposed.

Diagram. A right triangle inscribed in a semicircle, with the perpendicular from the right-angle vertex to the base.

- A (apex, on the semicircle): the right-angle vertex
- B (bottom left): one end of the base and diameter
- D (base, between B and C): foot of the perpendicular AD
- C (bottom right): the other end of the base and diameter
- (a dotted semicircle stands on BC as diameter)

Next, in the right triangle ABC let the three sides BC, AB, AC be continuously proportional — that is, let BC be to AB as AB is to AC — and let the perpendicular AD be dropped to the base BC. I say that BC is cut at D in extreme and mean ratio. For since, as BC is to AB, so is AB to AC; and also, as BC is to AB, so is AB to BD (because AB is a mean proportional between BC and BD, by the corollary to

Proposition 8 of this book), [Margin: 11. quinti → Elements V.11] it will be that, as AB is to AC, so is AB to BD; [Margin: 9. quinti → Elements V.9] and consequently AC and BD will be equal. [Margin: 7. quinti → Elements V.7] Therefore, as AC is to CD, so will BD be to CD. But as AC is to CD, so is BC to AC — since, by the corollary to Proposition 8 of this book, AC is a mean proportional between BC and CD. [Margin: 11. quinti → Elements V.11] Therefore it will also be that, as BC is to AC (that is, to BD), so is AC (that is, BD) to CD; and on that account BC is cut at D in extreme and mean ratio. Which was the thing proposed.

And so, if upon a given straight line BC a right triangle is to be constructed whose three sides are continuously proportional, the given line BC must be cut in extreme and mean ratio at D. Then, a semicircle BAC having been described about this same given line BC, the perpendicular DA must be erected upon BC, and the lines BA, CA joined. For the triangle ABC will be right-angled, [Margin: 31. tertij → Elements III.31] since the angle BAC in the semicircle is a right angle; and hence, since the perpendicular AD cuts the base in extreme and mean ratio, the three sides will be continuously proportional, as was just demonstrated.

XIV.

From a given point on the side of a triangle, to draw a straight line that divides the triangle into two parts according to a given ratio.

Let there be a triangle ABC, and let it be required from a given point D on its side BC to draw a straight line that cuts the triangle into two parts according to the given ratio of E to F. Let BC be divided at G in the ratio of E to F, and let the point G fall first upon the given point D, as in the first figure, and let the line DA be drawn. [Margin: 1. sexti → Elements VI.1] Since, then, as BD is to DC, so is triangle BDA to triangle CDA, the line DA will cut the given triangle in the given ratio of BG to GC — that is, of E to F. Which was to be done.

Diagram. The first figure for Proposition XIV: the dividing point coincides with the point of division of the base.

- A (apex): the top vertex
- H (interior, right of center): auxiliary point
- B (bottom left): base vertex
- D (base): the given point, here coinciding with G
- G (base, near D): point dividing BC in the given ratio
- C (bottom right): base vertex

Next, let the point G fall between C and D, as in the second figure. Let the line DA then be drawn, and through G let GH be drawn parallel to it, and the line DH joined. I say that the line DH cuts the given triangle in the given ratio — that is, that the trapezium BDHA is to the triangle CDH as E is to F, or as BG is to GC. For when the line is drawn, [Margin: 37. primi → Elements I.37] the triangles HGA, GHD will be equal, since they stand on the same base GH and between the same parallels GH, DA. Adding, then, the common triangle CGH, the triangles CGA, CDH become equal; [Margin: 7. quinti → Elements V.7] and hence the triangle ABC will have the same ratio to CGA as to CDH. [Margin: 17. quinti → Elements V.17] Therefore, by separation, as triangle BGA is to triangle CGA, so is the trapezium BDHA to the triangle CDH. [Margin: 1. sexti → Elements VI.1] But BGA is to CGA as BG is to GC. Therefore the trapezium BDHA will also be to the triangle CDH as BG is to GC — that is, as E is to F. Which was the thing proposed.

Diagram. The second figure for Proposition XIV, where G falls between C and D and the line DH is drawn.

- A (apex): the top vertex
- E (upper right): endpoint marking the given ratio
- F (upper right, below E): endpoint marking the given ratio
- B (bottom left): base vertex
- D (base): the given point
- G (base, between D and C): point dividing BC in the given ratio
- C (bottom right): base vertex

Let the point G fall, thirdly, between B and D, and let the line DA be drawn, through which again a parallel GH is drawn through G. I say once more that the line DH, when drawn, cuts the triangle ABC in the given ratio of E to F — that is, that the triangle BDH is to the trapezium CDHA as E is to F. For when the line GA is drawn, [Margin: 37. primi → Elements I.37] the triangles HGA, GHD will, as before, be equal; and, the common triangle BGH being added, BGA and BDH become equal. [Margin: 7. quinti → Elements V.7] Hence ABC is to BGA as it is to BDH. [Margin: 17. quinti → Elements V.17] Therefore, by separation, as CGA is to BGA, so is the trapezium CDHA to the triangle BDH; and by conversion, as BGA is to CGA, so is BDH to the trapezium CDHA. [Margin: 1. sexti → Elements VI.1] But BGA is to CGA as BG is to GC. Therefore BDH will also be to the trapezium CDHA as BG is to GC — that is, as E is to F. Which was the thing proposed.

Diagram. The third figure for Proposition XIV, where G falls between B and D.

- A (apex): the top vertex

- H (on the left side AB): where the parallel GH meets side AB
- B (bottom left): base vertex
- G (base, between B and D): point dividing BC in the given ratio
- D (base, right of G): the given point
- C (bottom right): base vertex

From this it is clear in what way a required part is to be cut off from a triangle by a straight line drawn from any given point on a side. For if, by a straight line drawn from a given point, the triangle is divided in a multiple ratio whose denominator is smaller by a unit than the denominator of the required part — for instance, in the triple ratio if a fourth part is required, and so on — then what is commanded will be done, as is evident. For the whole triangle will then have to the segment cut off a multiple ratio whose denominator is equal to the denominator of the required part, since to the earlier denominator of the assumed multiple ratio is added the unit that was previously lacking.

XV.

Given a rectilinear figure, to describe a rectilinear figure similar and similarly situated to it, greater or smaller according to a given ratio.

Let A be a given rectilinear figure, to which a greater figure, similar and similarly situated, is to be described according to the given ratio of B to C. Given three straight lines B, C, and DE (let DE be any one side of the given rectilinear figure), [Margin: 12. sexti → Elements VI.12] let a fourth proportional F be found. Then, given the two lines DE and F, [Margin: 13. sexti → Elements VI.13] let a mean proportional GH be found; [Margin: 18. sexti → Elements VI.18] upon which, let a rectilinear figure I be described, similar and similarly situated to A. I say that I is greater than A according to the given ratio of B to C. For since the three lines DE, GH, and F are proportional, then by the corollary to Proposition 19 or 20 of this book, as DE, the first, is to F, the third — that is, by construction, as B is to C — so is the rectilinear figure A, upon the first, to the rectilinear figure I, upon the second, which is similar and similarly situated to it. Which was to be done.

In no other way, given the rectilinear figure A, shall we describe a smaller figure I, similar and similarly situated to it, according to the given ratio of B to C. You see it done, just so, in this figure; for the construction and demonstration are altogether the same.

Easily, then, from these we shall double, triple, or quadruple any square or other rectilinear figure, and so on; and we shall likewise construct another figure that

is its half, or its third part, or its fourth, or its fifth, and so on — the similarity of the rectilinear figures being nonetheless always kept the same, just as the similarity of squares remains the same. For if the ratio of B to C is taken as 1 to 2, or 1 to 3, or 1 to 4, and so on; or again as 2 to 1, or 3 to 1, or 4 to 1, and so on; and the rest is carried out as before — then a similar and similarly described figure will be obtained, which will be the double of the proposed rectilinear figure, or its triple, or its quadruple, and so on; or which will be its half, or its third part, or its fourth, and so on, of that which is proposed.

Diagram. The construction for enlarging a figure: three marked segments above a smaller and a larger similar triangle.

- F (top segment, divided into parts): the fourth proportional line
- C (segment below F): one of the given lines
- B (short segment): one of the given lines
- A (smaller triangle) on base D–E: the given figure
- I (larger triangle) on base G–H: the figure to be described, similar to A

Diagram. The construction for reducing a figure: marked segments above a larger and a smaller similar triangle.

- B (top segment, divided): given line
- C (short segment): given line
- A (larger triangle) on base D–E: the given figure
- I (smaller triangle) on base G–H: the figure to be described, similar to A

Nor does it seem right to omit the method of Albrecht Dürer [Alberti Dureri], by which he easily doubles, triples, quadruples, and so on, any given square or parallelogram. For from this we shall likewise construct, for any given rectilinear figure, a similar and similarly described figure, greater or smaller according to any given ratio. And although Dürer offers no reasoning for this method but simply proposes it, still his demonstration does not differ much from the preceding one, as we shall shortly show.

Let the square ABCD, then, be one to be made fivefold. Let the side AB be extended toward B by any amount, and from the point B, beginning there, let five parts equal to AB be taken up to E, so that BE is five times AB. Then, the whole line AE being bisected at F, let a semicircle AGE be described from F with radius FA (or FE), and let the side BC be extended to the circumference at G. I say that the square BGHI, described on BG, is fivefold the square ABCD. For by the corollary to Proposition 13 of this book, BG is a mean proportional between EB

and BA. Therefore, as EB, the first, is to BA, the third, so is BH, the square on the second, to AC, the square on the third, by the corollary to Proposition 20 of this book. But EB, by construction, is five times AB. Therefore the square BH will also be five times the square AC. Which was the thing proposed.

And if the line BE is taken six times the side AB, the square on the line BG will also be sixfold the square ABCD. But if BE is a third part of AB, the square BH will also be a third part of the square AC. In short, in whatever ratio BE is taken to AB, the square BH will have the same ratio to the square AC.

Diagram. Dürer's construction for multiplying a square: a semicircle on AE, with the original square ABCD at the right and the larger square BGHI erected on BG.

- E (left end of the baseline): endpoint of the extended line AE
- I (baseline, left of F): corner of the larger square BGHI
- F (baseline, center): midpoint of AE and center of the semicircle
- B (baseline, right of center): shared corner of both squares
- A (right end of the baseline): corner of the original square
- H (upper left): corner of the larger square
- G (upper right, on the semicircle): where BC extended meets the circle
- C (middle, above B): corner of the original square ABCD
- D (middle, above A): corner of the original square ABCD

Again, let there be a rectangle ABCD, to which a similar and similarly situated figure, its double, is to be found. From the side AB extended let BE be taken, double AB. Then the whole line AE being bisected at F, and a semicircle described from F as before, and CB extended to G, then BG will be one side of the required rectangle. Therefore if AH is cut off equal to BG, and through H a line HI is drawn parallel to BC, meeting the diagonal AC extended at I, and the parallelogram H is completed, [Margin: 24. sexti → Elements VI.24] then HK will be similar and similarly situated to BD; and I also affirm that it is the double of BD. For since the three lines EB, BG, BA are proportional, by the corollary to Proposition 13 of this book, it will be, as before, that as EB, the first, is to BA, the third, so is the rectangle HK, upon AH, the second (for AH was taken equal to BG, the second), to the rectangle BD, upon the third AB, which is similar and similarly described. Which was the thing proposed.

In the same way, if upon AB any rectilinear figure whatever has been constructed, the figure described from BG similar and similarly situated to it will be its double. And in this way the figure from BG will always have to the similar

figure from AB the same ratio that the line EB is set to have to the line BA, by construction.

It is evident that this method, together with its demonstration, does not differ from the one we set forth earlier, except that this one at the same time furnishes the finding of the mean proportional line. For this reason Dürer joins the lines EB, BA into one straight and continuous line, since they have the given ratio, so that at once, by a single operation, he may obtain the mean proportional, and so on.

Diagram. Dürer's construction for doubling a rectangle: a semicircle on the baseline with the rectangle BD at the right and the constructed rectangle at the lower left.

- G (top, on the semicircle): where CB extended meets the circle
- E (left end of the baseline): endpoint of the extended line
- F (baseline, center): midpoint and center of the semicircle
- H (baseline, right of F): point where AH is cut off equal to BG
- B (baseline, right of H): shared corner
- A (right end of the baseline): corner of the original rectangle
- C (middle): corner of the original rectangle ABCD
- D (middle right): corner of the original rectangle ABCD
- I (bottom left): where HI meets the extended diagonal
- K (bottom right): corner of the completed parallelogram

XVI.

In any given triangle, to describe a square:

(Continuation of the construction of appended Proposition XVI, "In any given triangle, to describe [inscribe] a square." The enunciation and catchword "SIT" stood at the end of the preceding page; Clavius's construction and proof follow.)

Let ABC be the given triangle, acute-angled. From any vertex A drop the perpendicular AD to BC; by the scholium to Proposition 13 of Book 2 [Elem. II.13] it falls inside the triangle. Cut this perpendicular at E — by the method we taught in the scholium to Proposition 10 of this book [Elem. VI.10] — so that AE is to ED as AD is to BC. Next, through E draw FG parallel to BC. Finally, from F and G draw FH and GI parallel to DE. I say that the rectilinear figure FGIH, inscribed in triangle ABC, is a square.

For since FG is parallel to BC, then by the scholium to Proposition 4 of this book [Elem. VI.4] BD is to DC as FE is to EG; and, *componendo*, BC is to DC as FG is to EG.

[Margin: a — 4.sex. (Elem. VI.4)] But DC is to AD as EG is to AE, because, by the corollary to Proposition 4 of this book [Elem. VI.4], the triangles ADC and AEG are similar. [Margin: b — 22.quin. (Elem. V.22)] Therefore, *ex aequali*, BC is to AD as FG is to AE. And since by construction AD is to BC as AE is to ED, [Margin: c — 22.quin. (Elem. V.22)] then again, *ex aequali*, BC is to BC as FG is to ED. But BC is equal to BC — indeed it is the very same line. Therefore FG is also equal to ED.

[Margin: d — 34.pri. (Elem. I.34)] Hence, since FG is equal to HI, and ED is equal to EH and GI, the four sides FG, GI, IH, HF will be equal to one another. [Margin: e — 29.pri. (Elem. I.29)] And since the angles EDH and FHD together equal two right angles — and EDH is right by construction — FHD will also be right. Therefore, by what we demonstrated at Definition 1 of Book 2 [Elem. II, def. 1], the remaining angles HFG, FGI, GIH of the parallelogram FGIH are also right; and consequently the figure FGIH [text: "FI"] is a square. Which is what was proposed.

We shall solve the same problem in exactly the same way if the given triangle is right-angled or obtuse-angled, provided we drop the perpendicular from the right or the obtuse angle, as appears in the latter two figures. For in that way the perpendicular AD will always fall inside the triangle, by the scholium to Proposition 13 of Book 2 [Elem. II.13].

Diagram. An acute triangle with an inscribed square, illustrating Proposition XVI.

- A (apex): vertex from which the perpendicular AD is dropped
- B (bottom-left): base vertex
- C (bottom-right): base vertex
- F (upper-left): top-left corner of the inscribed square, on side AB
- E (top-middle): point on FG where the perpendicular AD crosses
- G (upper-right): top-right corner of the inscribed square, on side AC
- H (base): foot of the square's left side, on BC
- D (base): foot of the perpendicular AD, on BC
- I (base): foot of the square's right side, on BC
- The inscribed square is FGIH (with FH and GI its vertical sides)

Diagram. A second triangle with an inscribed square (one of the "latter two triangles" — the right/obtuse cases of Prop. XVI).

- A (apex), B (bottom-left), C (bottom-right): triangle vertices
- F (upper-left), E (top-middle), G (upper-right): top edge of the inscribed square

- H, D, I (on the base BC): feet of the square's sides and of the perpendicular AD

Diagram. A third triangle with an inscribed square, the perpendicular still falling within it. Printed caption beneath: "semper cadet perpendicularis AD, intra triangulum, ex scholio propos. 13. lib. 2." — "the perpendicular AD always falls inside the triangle, by the scholium to Proposition 13 of Book 2."

- A (apex), B (bottom-left), C (bottom-right): triangle vertices
- F (upper-left), E (top-middle), G (upper-right): top edge of the inscribed square
- P (far bottom-left), H, D, I (on the base): the base points, with P marking the extension of the base beyond the triangle

Now if one wishes to describe the square in a right triangle so that two of its sides rest on the two sides of the triangle about the right angle: we divide the perpendicular AB at D so that AD is to DB as AB is to BC; through D we draw DE parallel to BC, and through E another parallel EF to AB.

[Margin: f — 4.sex. (Elem. VI.4)] Since, then, BC is to AB as DE is to AD — because the triangles ABC and ADE are similar, by the corollary to Proposition 4 of this book [Elem. VI.4] — and since by construction AB is to BC as AD is to DB, [Margin: g — 22.quin. (Elem. V.22)] it follows *ex aequali* that BC is to BC as DE is to DB. But BC is equal to itself — indeed it is the same line. Therefore DE is also equal to DB; and hence, as before, DEFB is a square. Which is what was proposed.

But more ingenious than this problem of Commandino's [Federico Commandino] is the one that follows.

Diagram. A right triangle with an inscribed square resting on the two legs about the right angle.

- A (top): apex
- B (bottom-left): the right angle
- C (bottom-right): base vertex
- D (on side AB, left): upper-left corner of the square
- E (on hypotenuse AC, right): interior corner of the square
- F (on base BC): lower corner of the square
- The inscribed square is DEFB

XVII.

Within a given square, to describe another square in a given ratio of greater inequality. But the given ratio must not be greater than double [2:1].

Let ABCD be the square within which another square is to be described, such that the square ABCD has to it the same ratio of greater inequality that the line E has to the line F — whatever that ratio may be, provided it is not greater than double. For if the sides of the square are bisected and the points of division joined by straight lines, a square will be inscribed within the other, having the same center as it, as we demonstrated at the end of Book 4.

And since the circle inscribed in the square ABCD is circumscribed about that inscribed square — as is clear in this figure — the square ABCD will be double the inscribed square, by the scholium to Proposition 9 of Book 4 [Elem. IV.9]. Since, then, no smaller square can be described within the square ABCD than the one whose circumscribed circle touches the sides of ABCD — as is evident — it clearly follows that the square ABCD can have to no inscribed square a ratio greater than double.

Let the given ratio E to F, then, be less than double — namely sesquialterate [3:2]. By Proposition 15 of this scholium find a square to which the given square ABCD has the given ratio E to F. Its semidiameter will be greater than the semidiameter of the square whose circumscribed circle touches the sides of ABCD (which square, as we said, is half of ABCD), but less than the semidiameter of the square ABCD.

Diagram. A small square containing an inscribed circle, and within the circle a second (diagonal) inscribed square, with the diagonals drawn — illustrating that the outer square is double the inner. No vertex letters are printed on this figure.

Therefore, if from the center G of the given square a circle is described equal to the one described about the square just found, it will cut the sides of the square ABCD at the points H, I, K, L, M, N, O, P. Join the straight lines HK, KM, MO, OH, leaving the four points I, L, N, P in the middle. I say that HKMO is a square described in the circle HIKLMNOP, and hence equal to the one that was found — so that the square ABCD has to the square HKMO [text: "HK, MO"] the given ratio E to F.

For, a circle having been described about the given square ABCD from the center G, and perpendiculars GQ, GR drawn from G to the sides AD, AB: [Margin: a —

3.ter. (Elem. III.3)] both the straight lines AD, AB and the straight lines HI, OP will be bisected at Q, R. And since AD, AB are equal, [Margin: b — 14.ter. (Elem. III.14)] they are equally distant from the center G, and therefore HI, OP are likewise equally distant from the same center. [Margin: c — 14.ter. (Elem. III.14)] Hence HI, OP will be equal, and consequently

their halves QH, QI, RO, RP will be equal to one another. Moreover the halves QA, QD, RA, RB of the equal sides are also equal. Therefore, if the former are subtracted from the latter, the remainders HA, ID, PA, OB will be equal. By the same reasoning, if other perpendiculars are drawn to the sides BC, CD, the lines NB, MC, LC, KD will be shown equal both to one another and to those four HA, ID, PA, OB.

Again, since HI, OP are equal, if the equals ID, PA are added, the wholes HD, AO become equal. And for the same reason BM, KC will be equal, both to one another and to those two HD, AO. And so, since the two sides AO, AH are equal to the two sides DH, DK, and contain equal angles — namely right angles — [Margin: d — 4.pri. (Elem. I.4)] the bases HK, HO will also be equal. In the same way KM, MO will be shown equal both to one another and to the two HK, HO. Therefore the figure HKMO is equilateral. I say it is also right-angled.

For since the sides HK, KM, MO, OH are equal, [Margin: e — 28.tertij (Elem. III.28)] the four arcs they subtend will also be equal, and hence will be quadrants. Therefore OHK, HKM, KMO, MOH are semicircles, [Margin: f — 31.ter. (Elem. III.31)] and so the four angles H, K, M, O in those semicircles will be right. Therefore HKMO is a square, and hence equal to the square that was found, since the same circle circumscribes both. For the circle HKMO was described with a radius equal to the semidiameter of the circle circumscribed about the square that was found. Which is what was proposed.

Diagram. Above the circle stand the two given lines: F (the shorter segment) and E (the longer, tick-marked segment) — the terms of the ratio $E : F$. Below them, a circle with two inscribed squares (illustrating Proposition XVII): the axis-aligned square ABCD and the rotated square HKMO, with center G and the feet Q, R of perpendiculars from G.

- A (top-left), D (top-right), C (bottom-right), B (bottom-left): corners of the axis-aligned inscribed square ABCD
- H, K, M, O: corners of the rotated inscribed square HKMO (H top, K right, M bottom, O left)

- I (top, between H and D), L (right, between K and C), N (bottom, between M and B), P (left, between O and A): the four intermediate points left "in the middle"
- Q (upper interior): foot of perpendicular from G to AD
- R (left interior): foot of perpendicular from G to AB
- G (center): center of the square and circle

XVIII.

If at the endpoints of a circle's diameter two perpendiculars are erected, and from those same endpoints, through one and the same point of the circumference, two other straight lines are drawn cutting the circle and meeting the two perpendiculars: then the rectangle contained by either of the secants and its interior segment will be equal to the square on the diameter.

This proposition is Cardano's [Girolamo Cardano], from Book 16 of *De Subtilitate*, chapter 1 — as is the one that follows. Both were demonstrated by Giovanni Battista Benedetti [Io. Baptista Benedictus]. Let ABC be a circle whose diameter is BC. At B and C erect two perpendiculars BD, CE; and through any point A taken on the circumference, let the lines BA, CA — drawn from those same points B, C — cut the perpendiculars at E and D. I say that both the rectangle under EB, BA and the rectangle under DC, CA are equal to the square on the diameter BC.

For since in the right triangle BCE the perpendicular CA has been dropped from the right angle C to the base BE — [Margin: a — 31.ter. (Elem. III.31)] because the angle BAC, being in a semicircle, is right — then by the corollary to Proposition 8 of this book [Elem. VI.8] BC is the mean proportional between EB and BA.

[Margin: b — 17.sexti (Elem. VI.17)] Hence the rectangle under the extremes EB, BA will be equal to the square on the mean BC. By the same reasoning the rectangle under DC, CA will be equal to the same square, because BC is also the mean proportional between DC and CA, by the same corollary to Proposition 8 of this book [Elem. VI.8]. Thus what is proposed is established.

Diagram. A circle on diameter BC with perpendiculars erected at its ends, illustrating Proposition XVIII.

- B (bottom-left): endpoint of the diameter
- C (bottom-right): endpoint of the diameter (the right angle of triangle BCE)
- A (on the upper circumference): the assumed point

- D (upper-left): where line CA meets the perpendicular BD (top of the perpendicular at B)
- E (upper-right, tall): where line BA meets the perpendicular CE (top of the perpendicular at C)

XIX.

If in a circle two diameters cut each other at right angles, and from the endpoint of one of them a straight line is drawn cutting the circumference at will and cutting the other diameter — whether that diameter is produced or not — then the rectangle contained by the two segments of this drawn line (one segment lying between the endpoint of the first diameter and the second diameter, the other between that same endpoint and the circumference) will be equal to the square described within the circle.

In the circle ABCD with center E, let the two diameters AC, BD cut each other at right angles, and from the point D draw a line DF at will, cutting the diameter AC at F and the circumference at G. I say that the rectangle under FD, DG is equal to the square described within the circle.

For when the lines DC, CG are drawn, the triangles DCF and DCG will be equiangular with each other. For the angle CDF is common, and the angle DCF is equal to the angle DGC; [Margin: a — 32.pri. (Elem. I.32)] and hence the remaining angle DFC is equal to the remaining angle DCG. That the angles DCF, DGC are equal we shall show thus. If the point F is inside the circle, [Margin: b — 27.ter. (Elem. III.27)] the angles DCF, DGC, standing on the equal quadrants AD, DC, will be equal. But if the point F is outside the circle, then, the line AD having been drawn, [Margin: c — 27.ter. (Elem. III.27)] the angles DAC, DCA, standing on quadrants, will be equal. [Margin: d — 22.ter. (Elem. III.22)] But both DAC and DGC, [Margin: e — 13.pri. (Elem. I.13)] and likewise DCA and DCF, are equal to two right angles; therefore, when the equals DAC, DCA are taken away, the remainders DGC, DCF will also be equal.

For this reason, since the triangles DCF, DCG are equiangular, [Margin: f — 4.sexti (Elem. VI.4)] DF will be to DC as DC is to DG — that is, the three lines DF, DC, DG will be in continued proportion. [Margin: g — 17.sexti (Elem. VI.17)] Therefore the rectangle under the extremes DF, DG is equal to the square on the mean DC — that is, to the square described within the circle, since DC is the side of the square inscribed in the circle, as is evident from Proposition 6 of Book 4 [Elem. IV.6]. Which is what was proposed.

Alternatively. Draw the line BG. [Margin: h — 31.ter. (Elem. III.31)] The angle BGD, being in a semicircle, will be right; and therefore the two triangles BGD, DEF — since they have the right angles BGD, DEF and share the common angle BDG (or EDF) — [Margin: i — 32.pri. (Elem. I.32)] will be equiangular. [Margin: k — 4.sex. (Elem. VI.4)] Therefore DF is to DE as DB is to DG; [Margin: l — 16.sex. (Elem. VI.16)] so the rectangle under DF, DG will be equal to the rectangle under DB, DE. [Margin: m — 17.sex. (Elem. VI.17)] But equal to the rectangle under DB, DE

is the square on the line AD — that is, the square inscribed in the circle — because, by the corollary to Proposition 8 of this book [Elem. VI.8], AD is the mean proportional between DB and DE, as is evident if the line AB is drawn. [Margin: n — 31.ter. (Elem. III.31)] For a right triangle BAD will be formed, and from the right angle the perpendicular AE will be dropped. Therefore the rectangle under DF, DG will also be equal to that same square on the line AD. Which is what was proposed.

Diagram. A circle with two diameters meeting at right angles at the center, illustrating Proposition XIX and its alternative proof. Because two demonstrations share the figure, the auxiliary points F and G each appear at two positions (one for the main proof, one for the *Aliter*).

- A (top): endpoint of the vertical diameter
- C (bottom): endpoint of the vertical diameter
- B (left): endpoint of the horizontal diameter
- D (right): endpoint of the horizontal diameter (source of line DF)
- E (center): intersection of the diameters
- F (interior, below E; and a second position lower down): where the drawn line cuts diameter AC
- G (left, on the circumference; and a second position lower): where the drawn line cuts the circumference

More universally, this may be stated as follows:

If in a circle a perpendicular is erected to the diameter from any point of the diameter, and to this perpendicular two straight lines are drawn from one of the endpoints of the diameter — one to the perpendicular's intersection with the circumference, the other at will, cutting the circumference at another point — then the rectangle under the segments of this latter line (of which one lies between the said endpoint and the perpendicular, the other between the same endpoint and the

circumference) will be equal to the square on the former line, intercepted between the same endpoint and the circumference (or the perpendicular).

In the circle ABCD, let EF be perpendicular to the diameter BD, cutting the circumference at C; and join the lines DC, DF, the latter of which cuts the perpendicular EF at F and the circumference at G. I say that the rectangle under DF, DG is equal to the square on DC. For when the line BG is drawn, the triangles BGD, FDE will be equiangular, because of the right angles at G and E and the common angle FDE. [Margin: o — 4.sex. (Elem. VI.4)] Therefore FD is to DE as BD is to DG. [Margin: p — 16.sex. (Elem. VI.16)] Hence the rectangle under FD, DG will be equal to the rectangle under DE, BD. [Margin: q — 17.sex. (Elem. VI.17)] But this is equal to the square on DC, since — by the corollary to Proposition 8 of this book [Elem. VI.8] — DC is the mean proportional between BD and DE, because, the line BC being drawn, [Margin: r — 31.tertij (Elem. III.31)] the angle BCD would be right, being in a semicircle. Therefore the rectangle under FD, DG will also be equal to the square on DC. Which is what was proposed.

Diagram. A circle with the diameter BD, a perpendicular EF, and the lines DC, DF, illustrating the universal version of the theorem.

- A (top of circle): endpoint of the vertical diameter
- B (left): endpoint of the diameter BD
- D (right): endpoint of the diameter BD
- E (on BD, interior): foot of the perpendicular on the diameter
- C (lower-left, on the circumference): where the perpendicular meets the circumference
- F (interior, on the perpendicular EF): where line DF cuts the perpendicular
- G (lower, on the circumference): where line DF cuts the circumference

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Hence it follows that the former line DC is the mean proportional between the segments FD, DG of the latter line — one of which lies between the endpoint D and the perpendicular, the other between the same endpoint and the circumference.

XX.

Given a circle and two points — whether outside the circle or inside, provided neither lies on the circumference — to draw through these two

points two straight lines to some one point of the circumference, in such a way that the straight line joining the two points at which those two lines cut the circumference is parallel to the straight line connecting the two given points.

This problem — no less subtle than curious — Pappus [Pappus of Alexandria] distributed among three propositions; reducing it to one, we shall demonstrate it much more clearly. Let ABCD be the circle, and first let the two given points E, F be outside the circle, joined by the straight line EF. If, then, straight lines are to be drawn from E and F to some point of the concave (near) circumference, and so on, we shall proceed thus.

From E draw the line EB tangent to the circle at B, and find a third proportional to the two lines EF, EB; let EG be equal to it. Let the point G fall first between E and F — which happens when EF is greater than EB. Then draw GC tangent to the circle at C, on the side toward the earlier tangent EB; from E, through C, draw a line cutting the concave circumference at A; and join AF, cutting the circumference at D. I say that the drawn line CD is parallel to EF.

For since the three lines EF, EB, EG are in continued proportion, [Margin: a — 17.sex. (Elem. VI.17)] the rectangle under EF, EG will be equal to the square on EB. [Margin: b — 36.ter. (Elem. III.36)] But equal to the same square on EB is the rectangle under EA, EC; therefore the rectangle under EF, EG will be equal to the rectangle under EA, EC. And consequently, by the scholium to Proposition 36 of Book 3 [Elem. III.36], a circle can be described through the four points A, C, G, F. [Margin: c — 22.ter. (Elem. III.22)] Therefore the two opposite angles CAF, CGF of the quadrilateral ACGF, described within that circle, are equal to two right angles; [Margin: d — 13.pri. (Elem. I.13)] and the angles CGF, CGE are likewise equal to two right angles. So, taking away the common angle CGF, the remaining angle CGE will be equal to the remaining angle A [i.e., CAF]. [Margin: e — 32.ter. (Elem. III.32)] Moreover the angle DCG is equal to the angle A in the alternate segment. Therefore the angles DCG, CGE will be equal to each other; and since they are alternate angles, [Margin: f — 27.pri. (Elem. I.27)] the lines CD, EF will be parallel. Which is what was proposed.

Diagram. A circle ABCD with two external points E, F and the tangent/secant construction, illustrating the first case of Proposition XX (G between E and F).

- A (top, on the circumference): where the secant from E cuts the far circumference
- B (left, on the circumference): tangent point of EB

- C (interior-left, on the circumference): tangent point of GC
- D (right, on the circumference): where AF cuts the circumference
- E (bottom-left, external): given point
- G (bottom-center, on line EF): third-proportional point
- F (bottom-right, external): given point

Next, let the point G fall on the point F — which happens when EF is equal to BE. Then again draw GC tangent to the circle at C, on the side toward the earlier tangent EB; from E, through C, draw a line cutting the concave circumference at A; and join AF, cutting the circumference at D. Again I say that the drawn line CD is parallel to EF.

For since EB, EF are equal, [Margin: g — 36.ter. (Elem. III.36)] and the rectangle under AE, EC is equal to the square on EB, this same rectangle will also be equal to the square on EF. Hence [Margin: h — 5.quar. (Elem. IV.5)] if a circle is described about the triangle ACF — since the line EA cuts it, while the line EF is applied to it — [Margin: i — 37.ter. (Elem. III.37)] the line EF will be tangent to that circle. [Margin: k — 32.ter. (Elem. III.32)] Therefore the angle EFC will be equal to the angle A in the alternate segment of that circle; [Margin: l — 32.ter. (Elem. III.32)] and the angle FCD is likewise equal to that same angle A in the alternate segment of the circle ABCD. Therefore the angles EFC, FCD will be equal to each other; and since they are alternate, [Margin: m — 27.pri. (Elem. I.27)] the lines CD, EF will be parallel. Which is what was proposed.

Diagram. A circle ABCD with two external points E, F and the tangent/secant construction, illustrating the further cases of Proposition XX (G coinciding with F, and G beyond F).

- A (top, on the circumference): where the secant from E cuts the far circumference
- B (left, on the circumference): tangent point of EB
- C (interior-center, on the circumference): tangent point of GC
- D (right, on the circumference): where AF cuts the circumference
- E (bottom-left, external): given point
- G (bottom, near F): third-proportional point
- F (bottom-right, external): given point

Thirdly, let the point G fall beyond F — which occurs when EF is less than EB. With the same construction made, since the three lines EF, EB, EG are in continued proportion, [Margin: n — 17.sex. (Elem. VI.17)] the rectangle under EF,

EG will be equal to the square on EB. [Margin: o — 36.ter. (Elem. III.36)] But equal to the same square on EB is the rectangle under AE, EC. Therefore the rectangle under EF, EG — that is, under EG, EF — will be equal to the rectangle under EA, EC; and consequently, by the scholium to Proposition 36 of Book 3 [Elem. III.36], a circle can be described through the four points A, C, F, G. [Margin: p — 21.ter. (Elem. III.21)] Therefore the two angles A, G, in the same segment CAGF of that circle (whose chord would be the drawn line CF), [are equal] to each other...

will be equal to one another. [Margin: q — Elements III.32] And the angle GCD equals the angle A, which lies in the alternate segment of the given circle ABCD. Therefore the angle GCD will also equal the angle G; and since these two are alternate angles, [Margin: r — Elements I.27] CD and EF will be parallel. Which was to be proved.

But if from the very start a line tangent to the circle had been drawn from the point F, and the rest were carried out as before, the very same point A would be found — for no other point can solve the problem. To see this, imagine any other point H near A, between A and D. (If you mark that point H in ink, the demonstration will be easier; and let the same be understood of other points H to be conceived mentally, and likewise of the point I in the last figure.) Then the line FH will cut the circle above D, whereas the line EH will cut it below C; and so the straight line joining the two points of section will cut CD. Produced, therefore, it will also cut the line parallel to CD, namely EF — and hence it cannot be parallel to EF itself. The contrary happens if the point H is taken between A and C: for then the line FH falls below D, and EH above C, and so on.

But suppose that from the same given points E and F two straight lines are to be drawn to some point on the convex circumference, such that, when produced, they cut the concave part of the circumference in two points, and so that the line passing through those two points is parallel to the line joining E and F. We shall proceed as follows. Draw EB tangent to the circle at B, and to the two lines EF and EB find a third proportional EG; let the point G first fall between E and F. Then from G draw GD tangent to the circle at D — but not on the side toward the earlier tangent EB — and draw ED, cutting the circumference at the point C; through C project a line from F, cutting the circumference at A. I say that the line AD so drawn is parallel to EF. For since the three lines EF, EB, EG are in continued proportion, [Margin: s — Elements VI.17] the rectangle under EF, EG equals the square on EB. But the rectangle under ED, EC also equals that same square on EB. [Margin: t — Elements III.36] Therefore the rectangles under EF, EG and under ED, EC are equal to one another; and consequently, by the scholium to III.36, a

circle can be described through the four points C, D, F, G. [Margin: u — Elements III.21] So the two angles CDG and F, lying in the same segment CDFG of that circle (whose chord would be the line GC), are equal to one another. But the angle CDG equals the angle A, in the alternate segment of the given circle ABCD. [Margin: x — Elements III.32] Therefore the angle F too will equal the angle A alternately, [Margin: y — Elements I.27] and so AD and EF are parallel. Which was to be proved.

Next, let the point G fall on F. Again draw GD tangent to the circle at D — not, however, on the side toward the earlier tangent EB — and draw ED, cutting the circumference at the point C; through C let a line drawn from F cut the circumference at A. I say that the line AD so drawn is parallel to EF. For since EB and EF are equal — which comes about because EF is to EB as EB is to EF [that is, to EG] — the squares on them will be equal. But the rectangle under ED, EC equals the square on EB, and therefore also the square on EF. So, since [Margin: z — Elements IV.5] if a circle be circumscribed about the triangle CDF the line ED cuts it, while EF is applied to it, [Margin: a — Elements III.37] EF will be tangent to that circle. [Margin: b — Elements III.32] Therefore the angle EFC equals the angle FDE, in the alternate segment of that circle. But the angle FDE equals the angle A, in the alternate segment of the circle ABCD. [Margin: c — Elements III.32] Therefore the angle EFC too will equal that same angle A alternately; hence AD and EF are parallel. [Margin: d — Elements I.27] Which was to be proved.

Thirdly, let the point G fall beyond F. With the same construction made, since the three lines EF, EB, EG are in continued proportion, [Margin: e — Elements VI.17] the rectangle under EF, EG (or under EG, EF) equals the square on EB. But the rectangle under ED, EC is also equal to that same square. [Margin: f — Elements III.36] Therefore the rectangle under EG, EF equals the rectangle under ED, EC; and so, by the scholium to III.36, a circle can be described through the four points C, D, G, F. [Margin: g — Elements III.22] So the two angles EDG and CFG, opposite in the quadrilateral CDGF inscribed in that circle, together equal two right angles, and hence equal the two angles at F, [Margin: h — Elements I.13] which likewise equal two right angles. Taking away the common angle CFG, the remaining angle CDG equals the remaining angle CFE. [Margin: i — Elements III.32] But the angle GDC equals the angle A, in the alternate segment of the given circle ABCD. Therefore the angle CFE too will equal that same angle CAD alternately, [Margin: k — Elements I.27] and for that reason AD and EF are parallel. Which was to be proved.

But if from the start a line tangent to the circle were drawn from F, and the rest were carried out as before, the very same point C would be found, since no other point can accomplish the problem. For let there be any other point H near C, conceived in the mind between A and C: then the line FH will fall below A, and the line EH above D. So the straight line joining the points of section will cut AD, and consequently, as was said above, it cannot be parallel to EF itself. If the point H were taken between D and C, the line FH would fall above A, and the line EH below D, and so on.

But now let the points E and F be given inside the circle ABCD. Draw AG through E arbitrarily, and to the three lines EF, EA, EG find a fourth proportional, and let EH equal it (H will necessarily fall outside the circle). [Margin: l — Elements VI.16] For since the rectangle under AE, EG — the second and third — equals the rectangle under EF, EH — the first and fourth — a circle can be described about the four points A, F, G, H, by the scholium to III.35, cutting the circle ABCD at A and G. Then, since the arc of that circle through F lies inside this circle, the other arc, through H, will lie outside; [Margin: m — Elements III.10 [?]] for a circle cuts a circle in only two points, and hence the circle drawn through A, F, G, after the points A and G, falls outside the circle; and so the point H, lying on that circle, will fall outside the circle. Now draw HB tangent to the circle at B.

Diagram. A circle with two points marked on it, and three collinear points on a base line below; secant and tangent lines connect them. Readable labels:

- B (on the circle, left): tangent point
- A (on the circle, upper right): point where the projected secant meets the circumference
- C (lower interior, left of center): a point of section
- D (lower interior, right of center): tangent point
- E (base line, far left)
- F (base line, center-right)
- G (base line, right)

[Positions read on the leaf; the internal line-connections are not all separately resolvable.]

Diagram. Circle with a quadrilateral-and-secants construction (case with G between E and F). Readable labels:

- A (on the circle, upper left)
- D (on the circle, upper right)

- C (center interior)
- E (base line, left)
- G (base line, center)
- F (base line, right)

Diagram. Circle with inscribed lines (case with G falling on F). Readable labels:

- B (on the circle, upper left)
- A (on the circle, left)
- D (on the circle, right)
- C (center interior)
- F (base line, left) [the lower-right base labels read as "F" and "G" together and are not cleanly separable [?]]

Diagram. Circle with inscribed lines (case with G beyond F). Readable labels:

- A (on the circle, upper left; a mark, apparently "P", stands just above it [?])
- D (on the circle, upper right)
- C (center interior)
- E (base line, left)
- F (base line, center)
- G (base line, right)

Diagram. Circle with an interior star of secants (the case where E and F lie inside the circle, with the tangent point H). Readable labels:

- A (on the circle, upper left)
- H (top of the figure)
- E (interior, center)
- F (interior, right of E)
- B (lower left)
- G (base, center-bottom)
- C (lower right)

Diagram. A second circle for the interior-points case, with an external point H at the left. Readable labels:

- A and D (near the top, close together)
- H (far left)
- E and F (upper-center interior)

- B (lower left)
- G (just below B)
- C (lower right)

For draw a line from B through E, cutting the circumference at D, and from D through F a line cutting the circumference at C: I say that the line BC so drawn is parallel to EF. For since EF is to EA as EG is to EH, [Margin: n — Elements VI.16] the rectangle under EF, EH equals the rectangle under EA, EG. But the rectangle under EA, EG equals the rectangle under ED, EB. [Margin: o — Elements III.35] Therefore the rectangle under EF, EH equals that same rectangle under ED, EB; and so, by the scholium to III.35, a circle can be described about the four points B, H, D, F. [Margin: p — Elements III.21] Hence the two angles HBD and DFH, lying in the same segment DFBH of that circle (whose chord would be the line DH), are equal. But the angle HBD equals the angle C, in the alternate segment of the given circle ABCD. [Margin: q — Elements III.32] Therefore the angle DFH too will equal that same angle C — the exterior [equal to] the interior. [Margin: r — Elements I.28] So the lines BC and EF are parallel. Which was to be proved.

But if it happens that the tangent HB falls on the point G, there is no need to draw another line BD; rather AG, drawn at the outset, will conclude the proposition, as is clear in the second figure.

Now if the solution of the problem were to begin from the point F rather than from E, the very same point D would be found — for the problem can be accomplished by no other point. For if any other point I be taken near D, between D and C, the line FI will fall below C and EI above B. So the straight line joining the points of section will cut BC, and therefore, as was said above, it will not be parallel to EF. If the point I were taken between D and B, the line FI would fall above C and EI below B, and so on.

On the Marvelous Nature of a Certain Curved Line, by which a Figure of any Number of Equal Sides is Inscribed in a Circle, the Circle is Squared, and Many Other Things Most Delightful to Know are Accomplished

Perhaps last year I came upon Book 4 of Pappus of Alexandria, where he explains a certain curved line which — as he reports Dinostratus, Nicomedes, and some more recent geometers devised for the squaring of the circle — was for that office called by them τετραγωνίζουσα [Greek: tetragōnizousa, "the squaring [line]"]; for the same reason we shall call it the Quadratrix. Now although the authors just

named try to describe a line of this kind by two imaginary motions of two straight lines — in doing which they beg the very principle at issue, so that Pappus rejects it as useless and as one that cannot be described — we shall nonetheless describe it geometrically, without those motions, by finding as many of the points through which it must pass as we please, just as is customarily done in describing the conic sections. We judged that this discovery should be appended, in place of an epilogue, to this sixth book, because by the help of this line geometrical problems most pleasant to know, and sought after to this very day, are accomplished with the greatest ease. This we had undertaken, at the end of Book 4, to do here, while deferring a fuller treatment of the quadrature to our *Practical Geometry*. So, then, the description of the Quadratrix line will run as follows.

In the square ABCD let the quadrant BD be described. Suppose, as the inventors of the Quadratrix wish, that the radius AD is carried uniformly about the center A, while the topmost side CD likewise descends uniformly, in such a way that, in the same time in which the point D traverses the circumference DB with an ever-uniform motion, the line DC — also descending with uniform motion — reaches the side AB, yet always staying parallel to AB and making right angles with the sides AD and BC. Then the radius, swept round in the circumference DB, and the line DC, carried downward, will continually cut one another in points which will trace the Quadratrix line — that is, through which the Quadratrix will pass; such is the curved line DE. But because these two uniform motions — one along the circumference DB, the other along the straight lines DA, CB — cannot be produced unless the ratio of a circular line to a straight line be known, Pappus rightly finds fault with this description; for that ratio is as yet unknown, and is the very thing proposed to be investigated by means of this line. We shall therefore describe the same Quadratrix line geometrically, in this way. Let the arc BD be divided into any number of equal parts, and each side AD, BC into the same number of equal parts. The division will be easiest if both the arc DB and each side AD, BC be first bisected, then each half bisected again, and each single part bisected again, and so on as far as one pleases. The more divisions there are, the more accurately the Quadratrix line will be drawn. To avoid confusion, we have divided both the arc DB and the two sides AD, BC into 8 equal parts.

Next, let pairs of points on the sides AD, BC that are equally distant from the side DC (or AB) be joined by hidden straight lines, and from the center A let other hidden straight lines be extended to each of the division-points of the quadrant DB. For where these lines cut the former ones — the first the first, the second the second, and so on — through those points the Quadratrix line is to be drawn

smoothly, so that it is not wavy but always advances evenly, making no hump or angle anywhere: such is the curved line DE, cutting the radius AB at E.

But since the point E on the side AB cannot be found geometrically — because there every intersection of straight lines ceases — in order to find it without notable error (one, that is, that would fall under the senses) we shall use this device. We shall repeatedly bisect the lowest part AF of the side AD, if it is not small enough, until the lowest little part is very small; and in the same way we shall repeatedly bisect the lowest part BI of the arc DB, until as many subdivisions are made as were made in the part AF, so that the little part BK is the same fraction of the whole arc DB as AG is of the whole side AD. Then, equal to the little part AG, we shall cut off BL, BN, AM, and we shall draw the hidden lines GL, MN. Drawing from the center A the hidden line AK, which cuts GL at the point H (to be marked most exactly), we shall take MP equal to GH. For if we continue the Quadratrix, drawn up to H, with an even and uniform exten-

Diagram. The construction of the Quadratrix in the square ABCD with the quadrant BD centered at A. The interior shows the family of radial lines from A and horizontal lines descending from side DC, whose intersections trace the curve; several auxiliary division-points cluster near the base for the "device" described in the text. Readable labels:

- D (top left corner) and C (top right corner)
- A (bottom-left corner / center of the quadrant)
- B (bottom-right corner)
- Down the left side toward the base: F, G, then A, and M (lowest)
- Along the base near the center: H, E, P
- Down the right side toward the base: I, K, L, then B, and N

[The dotted radial lines from A and the horizontal dashed lines are the construction lines; individual division-points beyond those lettered are not separately labelled.]

extension as far as P, the Quadratrix line will cut the side AB at E, the point sought. For, because of the small distance between the lines GH, AE, MP, they turn out to be nearly equal — although, geometrically speaking, the line AE is always somewhat greater, however little those lines are apart; but that excess cannot be detected with compasses. The same happens on the circumference of the circle: the lines GL, AB, MN, if they lie little apart, will be judged wholly equal within the circle, although in truth AB is somewhat greater. And so, if those three lines GH, AE, MP have a very small distance between them, there can be no doubt that the

point E — where the Quadratrix line cuts the radius AB — does not differ from the one that truly lies on the Quadratrix there, provided the points H and P have been found exactly and with the greatest diligence.

Furthermore, we shall call the line AD the *side* of the Quadratrix, the line AE its *base*, and finally the point A its *center*.

But we shall find the points of the Quadratrix near the base more reliably, without intersections of lines, in this way. Draw the chord BQ in the quadrant, and bisect it at D. This will be done if from A to C, the midpoint of the quadrant, the line AC be drawn; for by the scholium to III.26 this will bisect the line BQ. Then let AD be taken equal to AE, and joining DE, let it be bisected at F — which will also be done if from A to I, the midpoint of the arc BC, the line AI be drawn; for this (if drawn) would bisect the chord BC, by the scholium to III.27, and consequently also the line DE, by the scholium to Proposition 4 of this [book], [Margin: a — Elements VI.2] since it is parallel to the chord BC — because the sides AB, AC are cut proportionally at D, E, seeing that AB and AC are equal, and so are AE and AD. Again, let AG be taken equal to AF, and joining FG, let it be bisected at H — which will also be done by the line AK drawn to K, the midpoint of the arc BI. And in this way, if AH be taken equal [to AI], and the rest done as before, another point will be found between H and G. And so, one after another, we shall find as many other points as we please nearer to AB, by perpendicular lines and not by oblique sections. [Margin: b — Elements III.3] For AD is perpendicular to QB, AF to DE, AH to FG, and so on. Now all these points found — D, F, H, and so on — lie on the Quadratrix, which I show as follows. Draw DL, FM, HN parallel to AB, cutting DB at O, P. [Margin: c — Elements VI.2] Then, as DB is to DQ, so [will AL be to LQ] [?]; and therefore AQ too will be bisected at L. But the arc BQ is also bisected at C. Therefore, as has been shown, the point D lies on the Quadratrix. Again, since DE is bisected at F, [Margin: d — Elements VI.2] DB too will be bisected at O; [Margin: e — Elements VI.2] and so, as FG is to FD, so is BO to OD. But as BO is to OD, so is AM to ML; and as EF is to FD, so is the arc BI to IC — since in both cases the ratio is one of equality. Therefore also, as AM is to ML, so is BI to IC. So, as we have shown, AI and MO cut one another at a point of the Quadratrix. And the same reasoning holds for the other points found by this art.

Diagram. A quadrant with center A and the auxiliary construction for finding points of the Quadratrix near the base by successive chords and bisections.

Readable labels:

- Q (top of the arc)
- C (on the arc, upper right)

- D (interior, below C; foot of the first chord-bisection)
- I (on the arc, right)
- Down the left side: L, M, N
- Interior, center-low: F, O
- Along the base: A (bottom-left corner), G, E, B (bottom-right); H and P near the center of the base; K to the right, near B

[The dashed lines are the hidden auxiliary lines (chords, radii, and parallels) named in the text.]

It is clear that this curved line DE, described by us point-by-point, is geometrically the very same as the one that Dinostratus and Nicomedes conceived as described by those two imaginary motions. For if the radius AD, about the center A, be moved through the arc DB with uniform motion, in the same time in which the side DC is carried downward, likewise with uniform motion, then when the radius AD has passed over any part whatever of the arc DB, the side DC will have traversed a like part of the sides DA, CB — otherwise either those two motions would not be uniform, or the radius AD and the side DC would not reach the side AB in the same time. Since, then, the lines drawn from the center A through the parts of the arc DB always cut off, from the arc DB and from the sides DA, CB, like parts, by construction, it plainly follows that the points of the curved line DE, found by us geometrically, do not differ from the points that would be found by those two motions. This, then, is the description of the Quadratrix line, which may in a sense be called geometrical — just as the descriptions of the conic sections, which are also made point-by-point, as handed down by Apollonius, are called geometrical, even though they are more liable to error than our description, on account of the finding of the many mean proportional lines needed for those descriptions, of which there is no need in describing the Quadratrix. Therefore, unless one is willing to reject as useless and non-geometrical the whole doctrine of the conic sections — which Apollonius of Perga pursued with such keenness of intellect that for it he was called the Great Geometer (a thing I judge no one will do, since the most excellent geometers have employed the conic sections for their demonstrations; for Menaechmus used the hyperbola and the parabola in finding two mean proportional lines between any two given straight lines, Archimedes himself proved many splendid things about these same conic sections, and finally such sections have a notable use in gnomonics, as appears from our *Gnomonica*) — one will be wholly compelled to admit that this our description of the Quadratrix line is in a sense geometrical. Add to this that the conchoid, by which Nicomedes most acutely investigates two mean proportional lines, is likewise described point-by-point, as we shall say in

our book *On Measurements*. But let us now set out the use of the Quadratrix line in a few propositions.

I.

If from the center, through any points of the Quadratrix line, straight lines be drawn as far as the circumference of the quadrant described from the same center, and from the same points perpendiculars be dropped to the base, and other straight lines be drawn parallel to that base: then the arcs of the quadrant intercepted between the radii will be proportional to the perpendiculars, or to the segments of the radius lying between the parallels.

From the center A, through points F, K taken anywhere on the Quadratrix, draw straight lines meeting the arc of the quadrant at H, L; drop the perpendiculars FG, KM to the base AE; and draw FI, KN parallel to that base AE. I say that as the whole arc DB is to the arc HB, so is the whole side DA to the perpendicular FG, or to the segment IA, and so on. For since the arc DH is the same fraction of the whole arc DB as the line DI is of the whole side DA — as is clear from the description of the Quadratrix line, whether it be thought of as described by those two proportional motions of Dinostratus and Nicomedes, or by the method that we have pre-

that we have prescribed. For in that curve the arc DH always comprehends as many small parts of the whole arc DB as the line DI contains parts of the whole side DA — because the straight lines AH and IF intersect at the point F of the Quadratrix. Nor is this proportionality impaired even if the arc DH is incommensurable with the whole arc DB, and the line DI incommensurable with the whole side DA, since the Quadratrix advances with perpetual uniformity through all its points. For suppose the line DI were *not* such a part of the whole side DA — whether commensurable or incommensurable — as the arc DH is of the whole arc DB. Then, if one imagined such a part, smaller than DI or larger, the parallel drawn through its endpoint would cut the straight line AH either above F or below it, at a point through which the Quadratrix would have to be described; and so the curve would not pass through F — which is absurd and contrary to the hypothesis. Since, I say, the arc DH is the same part of the whole arc DB as the line DI is of the whole side DA, the remaining arc HB will likewise be the same part of the whole arc DB as the remaining line IA is of the whole side DA. Hence it will follow that, as the whole arc DB is to the arc HB, so the whole side DA is to the line IA — that is, to the line

FG, [Margin: a — *Elements* I.34] which is equal to IA. And therefore, both by *dividendo* (separation): as the arc DH is to the arc HB, so is the line DI to the line IA, or FG; and by *convertendo* (conversion): as the arc HB is to the arc DB, so is the line FG, or IA, to the line DA, and so on.

By the same reasoning it will be: as the arc DB is to the arc LB, so is the line DA to the line NA, or to KM, [Margin: b — *Elements* I.34] which is equal to NA. And: as the arc DL is to the arc LB, so is the line DN to the line NA, or KM. And: as the arc LB is to the arc DB, so is the line KM, or NA, to the line DA. Since therefore, as LB is to DB, so KM is to DA; and as DB is to HB, so DA is to FG; it will follow, *ex aequo*: as LB is to HB, so KM is to FG. And by conversion: as HB is to LB, so FG is to KM — that is, so IA is to NA. And by separation: as HL is to LB, so IN is to NA, or FQ is to QG.

Again, since as DH is to HB, so DI is to IA; and as HB is to LB, so IA is to NA; it will follow, *ex aequo*: as DH is to LB, so DI is to NA, or to KM. Therefore the arcs intercepted between the radii are always proportional to the perpendiculars, or to the segments of the radius placed between the parallels. Which was to be proved.

Diagram. The Quadratrix figure: a square with corners D (top left), C (top right), B (bottom right), and A (bottom left), the last being the center. A quadrant arc runs from D to B, and the Quadratrix curve runs from D down to E on the base. Two radii from A meet the arc at H (upper) and L (lower), crossing the Quadratrix at F and K respectively; from F and K the perpendiculars FG and KM drop to the base, and the parallels FI and KN run to the side DA.

- D (top-left corner): upper end of quadratrix and of side DA
- C (top-right corner)
- B (bottom-right corner): far end of quadrant arc
- A (bottom-left corner): center of the quadrant
- H (on the arc, upper): endpoint of radius AH
- F (on the quadratrix, on radius AH): FI parallel to base, FG perpendicular to base
- I (on side DA): foot of parallel FI
- L (on the arc, lower): endpoint of radius AL
- K (on the quadratrix, on radius AL): KN parallel to base, KM perpendicular to base
- N (on side DA): foot of parallel KN

- Q (interior, intersection near center): intersection of FG with the parallel KN
- G (on the base): foot of perpendicular FG
- M (on the base): foot of perpendicular KM
- E (on the base): foot of the quadratrix

II.

To divide a given arc of a circle in a given ratio.

First, let the arc HB in the preceding figure be the one to be divided, in the given ratio of the line O to the line P. Through the point H draw to the center A the straight line HA, cutting the Quadratrix at F, and draw FI parallel to the base AE. Then let the line IA be cut at N — by the scholium of Proposition 10 of this book — so that IN is to NA as O is to P; and, drawing NK parallel to the base AE, cutting the Quadratrix at K, let a straight line be sent out from the center A through K, cutting the arc HB at L. I say that the arc HB has been cut at L in the given ratio of O to P. For since, as we showed in Proposition 1, HL is to LB as IN is to NA; and IN is to NA, by construction, as O is to P; [Margin: a — *Elements* V.11] HL will also be to LB as O is to P.

The three given ratios are represented by these drawn line-segments, with their assigned lengths:

- O ——— 3
- P ——— 2
- R ——— 3
- S ——— 5
- T ——— 8
- V ——— 2

Next, let the whole quadrant DB be cut in the ratio R to S. The radius DA being divided at I so that DI is to IA as R is to S, draw IF parallel to AE, cutting the Quadratrix at F. Then, the straight line AFH being drawn, it will be, by Proposition 1, that DH is to HB as DI is to IA — that is, as R is to S.

Thirdly, let the same quadrant DB be cut into two arcs, so that the whole quadrant has to one of them the given ratio T to V. With the three lines T, V, DA, let a fourth proportional be found, and let AN be cut off equal to it. Then, drawing NK parallel to AE, cutting the Quadratrix at K, draw from A through K the straight

line AKL. I say that the quadrant has been cut at L as proposed. For it is, by Proposition 1, that DB is to LB as DA is to NA — that is, as T is to V.

Fourthly, let the semicircle ABC be divided in a given ratio, namely a triple one. Let each quadrant AB, BC be cut according to the given triple ratio at E and F, and let the arc EG be cut off equal to the arc AE, or BF. Then if from the common arc ABC equal parts are taken away — namely the two arcs AE, BF together, and the arc AG — the remainders will become equal, namely the two arcs EB, FC together, and the arc GC. And since, as AE is to EB, so BF is to FC, [Margin: b — *Elements* V.12] the arcs AE, BF together will be to the arcs EB, FC together — that is, the arc AG to the arc GC — as AE is to EB. But AE has to EB the given triple ratio. Therefore the arc AG will also have to the arc GC the same given ratio.

Fifthly, let the arc ACD, containing three quadrants, be cut in the same ratio. The three quadrants AB, BC, CD being divided in that ratio at E, F, H, let the arc EG be taken equal to BF, and the arc GI equal to CH. Then if from the common arc ACD equal parts are taken away — namely the three arcs AE, BF, CH together, and the arc ABI — the remainders will become equal, namely the three arcs EB, FC, HD together, and the arc ID. And since the three arcs AE, BF, CH have to the three arcs EB, FC, HD the same ratio, [Margin: c — *Elements* V.12] all three together will have to all three together — that is, the arc ABI to the arc ID — the same ratio that the single arc AE has to the single arc EB, namely the given one.

Lastly, let the arc ACK be divided in the same way. The three quadrants being again divided, together with the arc DK, as proposed, at E, F, H, L, let the arc EG be taken equal to BF, the arc GI equal to CH, and the arc IM equal to DL. Then if from the common arc ACK equal parts are taken away — namely the four arcs AE, BF, CH, DL together, and the arc ABM — the remainders will become equal, namely the four arcs EB, FC, HD, LK together, and the arc MK. And since the four arcs AE, BF, CH, DL have to the four arcs EB, FC, HD, LK

the same ratio, [Margin: d — *Elements* V.12] all four together will have to all four together — that is, the arc ABM to the arc MK — the same ratio that the single arc AE has to the single arc EB, namely the one that was given. Therefore what is proposed is established.

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From these results we easily divide any rectilinear angle into two angles having a given ratio, and moreover any arc, and any angle, into any number of equal parts.

For, once an arc has been described from a given angle, and that arc divided as required, if a straight line be drawn from the angle to the point of division, the angle will be divided in the same way as the arc — [Margin: c — *Elements* VI.33] since, as arc is to arc, so is angle to angle at the center.

But if the arc, or the angle, is to be cut into any number of equal parts, it must be divided into two parts having a multiple ratio denominated by the number that is one less than the number of proposed parts. For example, if the arc BC, or the angle BAC, is to be divided into five equal parts, the arc BC must be cut at G in the quadruple ratio. For GC will be the fifth part of the arc BC; and if from the rest four equal parts GF, FE, ED, DB be cut off, the arc BC will be divided into five equal parts. And if from A the straight lines AG, AF, AE, AD be drawn, the angle BAC will likewise be cut into five equal angles. Which was proposed.

Diagram. A fan-shaped figure: from the apex A a bundle of straight lines fans down to an arc at the bottom. The arc runs from B (left end) through the division-points D, E, F, G to C (right end); rays AD, AE, AF, AG divide the angle at A into equal parts.

- A (apex): vertex of the divided angle
- B (left end of arc)
- D, E, F, G (division points along the arc, left to right)
- C (right end of arc)

III.

To construct an isosceles triangle each of whose equal angles has a given ratio to the remaining angle.

Let the given ratio be that of the line AB to the line BC. Let BC be bisected at D, and, a circle FGH of any size being described from a center E with a diameter FG drawn in it, let the semicircumference FHG be cut at H so that the ratio of the arc FH to the arc HG is the same as that of the line AB to the line BD; and let the lines HE, HG, HF be joined. I say that each of the equal angles of the isosceles triangle EGH at the base GH has to the remaining angle GEH the given ratio AB to BC. For since, as AB is to BD, so is the arc FH to the arc HG; and, as the arc FH is to the arc HG, [Margin: a — *Elements* VI.33] so is the angle EGH [?] to the angle GFH; [Margin: b — *Elements* V.11] it will be: as AB is to BD, so the angle FGH to the angle GFH. But as BD is to BC — the double of BD — so is the angle GFH to the angle GEH, [Margin: c — *Elements* III.20 [?]] which [GEH] is double the angle GFH

[?]. Therefore, *ex aequo*, it will be: as AB is to BC, so is the angle EGH to the angle GEH. Which was proposed.

Diagram. A circle with center E and vertical diameter FG (F at top, G at bottom). On the upper diameter, below F, are the tick-marks C, D, B (marking the reference segments AB, BC, BD). The point H lies on the circle at the lower right; the chords HF and HG, together with the diameter, form the isosceles triangle. A is marked near the bottom, below E toward G.

- F (top of circle): top of diameter
- C, D, B (tick-marks on the diameter below F): reference segments [?]
- E (center of circle)
- A (near the bottom, below the center)
- G (bottom of diameter)
- H (on the circle, lower right): apex of the constructed triangle

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Therefore, if isosceles triangles are constructed in which the angles equal at the base have to the remaining angle ratios that are both multiple-sesquialter and multiple in order, then by the former all equilateral figures with an even number of sides will be described, and by the latter all equilateral figures with an odd number of sides, inscribed in a circle, as we showed at the end of Book 4.

We shall accomplish the same thing without triangles of this kind, if we cut the whole circumference into as many equal parts — as we taught in the immediately preceding corollary — as the figure to be inscribed ought to have sides or angles, and so on.

IV.

If the quadrant and the quadratrix have the same center, then the arc of the quadrant, the radius, and the base of the quadratrix are in continued proportion.

Let there be a quadrant, and a quadratrix described from it, as above. I say that the arc BD, the radius DA, and the base AE of the quadratrix are in continued proportion — that is, that BD is to DA as DA is to AE. If not, let it be that BD is to DA as DA is to AF, [where AF is] greater than AE itself or less; and first let AF be greater than AE. A quadrant FG being described from the center A through F, cutting the quadratrix at H, draw through H the radius AHK, and drop the perpendicular HI. Since, then, the arc BD is posited to be to the line DA as DA —

that is, as AB — is to AF; and since, as AB, the radius, is to the radius AF, so is the arc BD to the arc FG (for since, by the demonstrations of Pappus, as we too shall show in our book *On Measurements*, the diameter is to the diameter as the circumference of a circle is to the circumference of a circle, [Margin: a — *Elements* V.15] the radius AB will also be to the radius AF as the same circumference is to the same circumference, [Margin: b — *Elements* V.15] and hence also as the quarter of the circumference to the quarter of the circumference — that is, as the arc BD to the arc FG): [Margin: c — *Elements* V.11] the arc BD will also be to the line DA as the same arc BD is to the arc FG. [Margin: d — *Elements* V.9] And therefore the line DA and the arc FG will be equal. But since, by Theorem 1, as the arc BD is to the arc BK, so is the line DA to the line HI; and as the arc BD is to the arc BK, so is the arc FG to the arc FH — because the arcs BD, BK are similar to the arcs FG, FH, as was shown above in this scholium: [Margin: e — *Elements* V.11] the line DA will also be to the line HI as the arc FG is to the arc FH. Since, then, the line DA has been shown equal to the arc FG, [Margin: f — *Elements* V.14] the line HI will also be equal to the arc FH — which is absurd. For the line HI is less than the arc FH, since it is the half of the chord subtending an arc double of FH [Margin: g — *Elements* III.3] (for the line AF bisects that chord, and hence also the arc, by the scholium of Proposition 27 of Book 3); and a chord is always less than its arc. Therefore the arc BD is not to the radius DA as DA is to a line greater than the base AE of the quadratrix.

Diagram. The Quadratrix figure (first case of Proposition IV): the square D (top left), C (top right), B (bottom right), A (bottom left), with the quadrant arc from D to B and the quadratrix from D to E on the base. A smaller quadrant arc through F is drawn; the radius AHK cuts the quadratrix at H, with the perpendicular from H falling on the base. G lies on the side DA.

- D (top-left corner)
- C (top-right corner)
- G (on side DA): endpoint of the inner arc
- K (on the quadratrix / arc, upper right): end of radius AHK
- H (interior, intersection): where radius crosses the quadratrix
- A (bottom-left corner): center
- I (on the base): foot of the perpendicular HI
- E (on the base): foot of the quadratrix
- F (on the base): foot of the inner quadrant
- B (bottom-right corner)

Next, if it can be done, let AF be less than AE. A quadrant FG being described from the center A through F, erect from F, to AE, the perpendicular FL, cutting the quadratrix at L — a point through which let the radius ALK be drawn, cutting the arc FG at H. We shall then show, as before, that the arc FG is equal to the line DA; and likewise that the arc BD is to the arc BK — that is, the arc FG to the arc FH — as the line DA is to the line LF. Therefore, since the arc FG has been shown equal to the line DA, the arc FH will also be equal to the line LF. Which is absurd. For the line LF is greater than the arc FH. For if from L there were drawn toward G another straight line tangent to the circle FG, just as LF touches the same circle at F, these tangents would be equal, by the scholium of Proposition 36 of Book 3; and the arc intercepted between them would be bisected at H, because, by the scholium of Proposition 37 of Book 3, the angle contained by them would be bisected, [Margin: h — *Elements* I.4 or I.8] and hence also the angle at the center A, if a straight line were joined to the other point of contact; [Margin: i — *Elements* III.26] and therefore also the arcs on which they stand would be equal. Since, then, by Archimedes at the beginning of *On the Sphere and Cylinder*, those two tangents together are greater than the arc contained by them, their half LF will also be greater than the half FH of that arc. Therefore the arc BD is not to the radius DA as DA is to a line less than the base AE of the quadratrix; nor, as was shown, as DA to a greater line. Therefore it is as DA to the base AE itself. Which was proposed.

Diagram. The Quadratrix figure (second case of Proposition IV): the square D (top left), C (top right), B (bottom right), A (bottom left), with the quadrant arc from D to B and the quadratrix. An inner quadrant through F is drawn; the perpendicular FL rises from the base and cuts the quadratrix at L, through which the radius ALK passes, cutting the inner arc at H. G lies on the side DA.

- D (top-left corner)
- C (top-right corner)
- K (on the quadratrix / arc, upper right): end of radius ALK
- G (on side DA): endpoint of the inner arc
- L (interior): where the perpendicular FL cuts the quadratrix
- H (interior, lower): where radius ALK cuts the inner arc
- A (bottom-left corner): center
- F (on the base): foot of the perpendicular FL and of the inner quadrant
- E (on the base): foot of the quadratrix
- B (bottom-right corner)

COROLLARIVM I.

Hence we shall easily find a straight line equal to any arc of a circle whose quadrant is BD, from which the Quadratrix has been described.

For since the arc BD is to DA as DA is to AE, it will be, by conversion, that AE is to DA as DA is to the arc of the quadrant BD. If, then, for the two lines AE, AD a third proportional be found, [Margin: k — *Elements* V.11] AD will be to that third [line] as it is to the arc BD, since each ratio is the same, namely that of AE to AD.

[Margin: l — *Elements* V.9] Therefore that third proportional will be equal to the arc of the quadrant BD; and if it be doubled, a line equal to the semicircumference of the same circle will result; if it be quadrupled, a line equal to the whole circumference. But if a line equal to the arc BK, which is less than a quadrant, is to be found: let it be arranged that, as DA is to the perpendicular LF dropped from L to AB, so that third proportional is to another line; and the fourth line found will be equal to the arc BK. For since, by Theorem 1, as DA is to the perpendicular LF, so is the arc DB to the arc BK, [Margin: m — *Elements* V.11] that third proportional will also be to the fourth line found as the arc DB is to the arc BK. But that third proportional is equal to the arc of the quadrant DB. [Margin: n — *Elements* V.14] Therefore the fourth line found will also be equal to the arc BK. But if a line equal to an arc greater than a quadrant is to be found, one must first find a line equal to a quadrant, or a semicircle, or three quadrants, according as the given arc includes one, or two, or three quadrants; then another line equal to the remaining arc, which is less than a quadrant. For these two lines joined together will be equal to the whole proposed arc.

COROLLARIVM II.

It also follows from these things that, if the base AE of the Quadratrix is set as the radius of some circle, its side AD is equal to the quadrant of that same circle, and the line double the side is equal to the circumference of that circle, and the line quadruple [the side is equal] to the whole circumference.

For since, as we said above from Pappus, the radii are proportional to the circumferences of circles, and hence also to their quadrants, it will be: as DA is to AE — that is, as that third proportional is to DA — so is the quadrant BD to the quadrant of the radius AE. But that third proportional is equal to the quadrant BD, as was shown. [Margin: o — *Elements* V.14] Therefore the line DA is also equal to the quadrant of the radius AE. Therefore the line double DA will be equal to the semicircumference of the circle whose radius is AE, and the quadruple to the whole circumference.

By the same reasoning, if two lines have the same ratio as DA and AE, and the lesser is set as the radius of some circle, the greater will be shown equal to the quadrant of that circle, and so on. For let it be, as DA is to AE, so some line B to a line C. Then, by permutation, as DA is to B, so AE is to C. But as AE is to C, so is the quadrant of the radius AE to the quadrant of the radius C, as was said above from Pappus. Therefore it will also be: as DA is to B, so the quadrant of the radius B to the quadrant of the radius C. Since, then, DA is equal, as was shown, to the quadrant of the radius AE, [Margin: p — *Elements* V.14] B will also be equal to the quadrant of the radius C. Which was proposed.

- B —————
- C —————

But if one wishes to explore by numbers what ratio, more or less, the circumference of a circle has to its diameter from this excellent discovery — or (what is the same) the semicircumference to the radius, [Margin: q — *Elements* V.15] since, as circumference is to diameter, so is semicircumference to radius — we shall do it in this way. Let us conceive the quadrant DB, in the figure in which we described the Quadratrix, to be divided into 90 degrees, and each degree into 60 minutes, so that the whole arc DB comprehends 5400 equal parts. If, then, we conceive the side DA to be divided into just as many equal parts, the parallel drawn from the last part up to the Quadratrix will be almost equal to the base AE, on account of the tiny distance between that parallel and the base AE — since in that figure the parallel GH also, cutting off AG, the sixteenth part, which is $1/16$, much greater than $1/5400$, is scarcely exceeded by the base AE: so that without notable error we may take that parallel for the base. And since that last part of the side DA is the sine of an arc of 1 minute, and that parallel is the sine of the complement, namely of an arc of 89 degrees 59 minutes — as is evident, if AG be set as the last part, and the arc BK as the last minute, and the straight line AH be drawn. For GH will be the sine of the angle GAH, 89 degrees 59 minutes, and hence AG the sine of the angle AHG, 1 minute, the whole sine being set as AH, as we showed in the treatment of Sines. If, then, the whole sine AH be set as 10000000 (for thus the desired ratio will be found more exactly than if the whole sine be set as only 100000), since in this case the sine of 89 degrees 59 minutes...

Diagram. The Quadratrix figure prepared for the numerical computation: the square with corners D (top left), C (top right), B (bottom right), A (bottom left). The side DA is divided into many equal parts, from each of which a dashed parallel runs across toward the quadratrix; dashed radial lines also fan from A to the quadrant arc, so the figure is filled with a fine grid of dashed lines. Near the base,

the last narrow strip is marked by the clustered points. This is a dense figure at the resolution floor; several of the tightly-packed lower labels are uncertain.

- D (top-left corner)
- C (top-right corner)
- G (upper on side DA): endpoint of a parallel GH near the base
- K (on the quadratrix / arc)
- I (on the right side, near B): [?]
- H (interior, low center): on the quadratrix
- F, G, A, M (clustered at the lower left, top to bottom): [?]
- E, B (on the right, near the base)
- N (lower right corner region): [?]
- P (on the lowest parallel, center): [?]

(Continuing Proposition IV, Corollary II — the numerical computation of the ratio of circumference to diameter.)

59), so that it does not differ from the whole sine, AG will be 2909 and GH 9999999. But the whole side DA, in those same parts, will be 15708600, as is established from what we said, namely that AG is 2909. The double of the side DA, therefore, will be 31417200. And since, by Corollary 2 of Proposition IV of this book, the double of the side DA is equal to the semicircumference [of the circle] whose semidiameter is AE, the ratio of that semicircumference to the semidiameter AE will be nearly the same as that of the number 31417200 to 9999999 — a ratio denominated by 3 and $\frac{1417203}{9999999}$, that is, in lowest terms, by 3 and $\frac{157467}{1111111}$. This ratio is smaller than the triple-and-a-seventh ratio [tripla sesquiseptima, $3\frac{1}{7}$], but larger than the triple-with-ten-seventy-first-parts ratio [tripla superdecupartiens septuagesimas primas, $3\frac{10}{71}$]; and the true ratio of the circumference to the diameter lies between these two, as was demonstrated by Archimedes.

Therefore, in order that a straight line equal to the circumference of the circle may be found, it is enough to find the point E, even if the whole Quadratrix line has not been drawn. But in order that an arc, or an angle, may be cut in a given ratio, we do not need the point E, as is plain.

V.

Given a circle, to construct a square equal to it.

Since Archimedes demonstrated that any circle whatever is equal to a right triangle one of whose sides about the right angle is equal to the semidiameter of the circle, while the other is equal to the circle's circumference — and since from this, in Proposition 4 of the *Isoperimetric Figures* in our commentary on the *Sphere*, we showed that the same circle is equal to the rectangle contained by its semidiameter and its semicircumference — it follows that if, by the Corollary of the preceding proposition, a straight line equal to the semicircumference of the given circle is found, and on that line together with the semidiameter [Margin: a — 14. secun. (Elements II.14)] a square is constructed equal [to the rectangle], this same square will be equal to the given circle.

Now a straight line equal to the semicircumference, or to the quadrant, or to the whole circumference, will be found [Margin: b — 12. sexti (Elements VI.12)] if it is made so that as the base of the drawn Quadratrix is to its side, so is the semidiameter of the given circle to a fourth line. For the fourth line so found will be equal to the quadrant of the circle, as we said in Corollary 2 of the preceding proposition; and doubled it will yield a line equal to the semicircumference, while quadrupled it will exhibit a line equal to the whole circumference.

But so that any circle may be squared more expeditiously, two figures most suited to squaring circles are to be constructed in this way. Let a right angle DAE be set up, and let AD be equal to the semidiameter of the Quadrant from which the Quadratrix was drawn, and AE equal to the base of that same Quadratrix; and from center A let a new Quadratrix DE be described, whose side is AD and whose base is AE. When the line DE has been drawn, one instrument most suited to squaring circles will have been constructed. For if the semidiameter of the circle to be squared is equal to the line AE, then the line AD will be equal to the circumference of the Quadrant of that same circle, as is clear from Corollary 2 of the preceding proposition. But if the semidiameter is smaller than AE, we shall cut off AB equal to it, and draw BC parallel to DE. If, finally, the semidiameter is larger than AE, we shall cut off AF equal to it on AE produced, and through F draw FG parallel to DE. For, by Corollary 2 of the preceding proposition, the line AC will be equal to the circumference of the Quadrant whose semidiameter is AB; while the line AG will be equal to the circumference of the Quadrant whose semidiameter is AF [Margin: c — 2. vel 4. sexti (Elements VI.2 or VI.4)], because as AE is to AD, so is both AB to AC and AF to AG.

Next, when a straight line HI of any length whatever has been drawn, let a perpendicular KL of any magnitude be raised upon it; and a second instrument, fitted for squaring circles, will have been prepared.

Diagram. The first instrument: a right triangle with its right angle at A (lower left). From A a vertical leg rises and a horizontal base runs to the right. On the vertical leg, read from the bottom: A, I [?], K [?], C, D, G (G at the top). On the base, from left to right: A, H, L, B, E, F. The line DE is the new Quadratrix, running from D on the vertical leg to E on the base; BC is drawn from B on the base up to C, parallel to DE; FG is drawn from F on the base up to G, parallel to DE. A construction circle (the circle to be squared) is drawn about the interior. Several of the small vertex letters sit at the resolution floor.

- A (lower-left, right-angle vertex): apex of the triangle
- G (top of vertical leg): meets parallel FG
- D (vertical leg): upper terminus of the Quadratrix DE
- C (vertical leg): meets parallel BC
- K [?], I [?] (lower vertical leg): auxiliary points
- H, L, B, E, F (base, left→right): division points; E is the base-terminus of the Quadratrix

Diagram. The second instrument, with the semicircle used in the demonstration. A long horizontal base line carries, from left to right, the points H, S, M, N, K, O [?], R, I. From K a perpendicular KL rises to the point L. A large semicircle, described from center N through H and I, cuts the perpendicular KL at L. Straight lines run from L down to points on the base. Above and to the left is a separate short horizontal segment labelled P (a given line). Point Q lies on one of the lines descending from L. The peripheral labels are small and partly at the resolution floor.

- H, S, M, N (base, left half)
- K, O [?], R, I (base, right half); N is the midpoint of HI
- L (apex, on the perpendicular and on the semicircle)
- Q (on a line from L)
- P (separate horizontal segment, upper left)

Diagram. The result of the quadrature: a square with an inscribed circle touching its four sides, the center marked with radii drawn to the circumference. This illustrates the square constructed equal to the given circle. No vertex letters are clearly legible on the square itself at this resolution.

Through these two instruments we shall square any circle whatever without difficulty, in this manner. Let there be a circle to be squared, whose semidiameter is equal, in the first instrument, to the line AB; and through B let BC be drawn

parallel to DE. But in the second instrument let two lines KM, MH be taken equal to AC, and the line KI equal to the semidiameter AB. Then, the whole line HI having been bisected at N, let a semicircle be described from N with radius NH (or NI), cutting the perpendicular KL at L. I say that the square on KL is equal to the given circle — whose semidiameter is AB in the first instrument, or KI in the second. For since KL is the mean proportional between KH and KI, by the scholium to Proposition 13 of this book, the square on KL will be equal to the rectangle under KH, KI [Margin: d — 17. septimi (?) (likely Elements VI.17, the mean-proportional theorem)]. And since this rectangle is equal to the given circle — as we said at the beginning of this proposition, KH being equal to the semicircumference and KI to the semidiameter — the same square on KL will also be equal to the given circle whose semidiameter is AB (or KI). If, therefore, the square on the line KL is described, and from its center, with radius AB (in the first instrument) or KI (in the second), a circle is drawn, we shall have a square equal to the circle, as appears in the figure set alongside.

Moreover, the division of the line HI into two equal parts, in the second instrument, will be easily made thus. Let the semidiameter KI be bisected at O — for since it is smaller than HI, it will easily be bisected — and let a line KO be taken equal to

MN. For N will be the midpoint of the line HI. For since MN, OI are equal, the common NO being added, NI will be equal to MO. But MO is equal to HN (for since HM, MK are equal, the equal parts MN, KO being added, the wholes HN, MO become equal). Therefore NI too will be equal to HN.

VI.

Given a square, to describe a circle equal to it.

Let a square be given whose side is P. From the perpendicular KL of the second instrument of the preceding proposition, cut off KQ equal to P. Then, for any circle proposed whose semidiameter is KI, let a square equal to it be found by the preceding proposition, whose side is KL. Next, when the line LI has been drawn, let QR be drawn parallel to it. I say that the circle whose semidiameter is KR is equal to the square whose side is KQ (that is, P). For, by the preceding proposition, a line KH has been found equal to the semicircumference of the circle whose semidiameter is KI; and when the line LH has been drawn, let QS be drawn parallel to it. Since therefore, because of the similarity of the triangles, as HK is to KL, so is SK to KQ; and as KL is to KI, so is KQ to KR — it will follow, *ex aequo*, that as HK is to KI, so is SK to KR. Since then HK is equal to the

semidiameter KI [recte: HK is equal to the semicircumference and KI to the semidiameter], it follows also, by Corollary 2 of Proposition 4, that SK is equal to the semicircumference of the circle whose semidiameter is KR. And since KQ, by the Corollary to Proposition 8 of this book, is the mean proportional between SK and KR — because the angle RQS is right [Margin: a — 29. primi (Elements I.29)], seeing that its parts RQK, SQK are equal to the parts ILK, HLK of the right angle HLI (for [Margin: b — 31. tertij (Elements III.31)] the angle HLI is right, being in a semicircle, as is evident from the construction of the preceding proposition, by which KL is found to be the mean proportional between HK and KI) — it follows [Margin: c — 17. sexti (Elements VI.17)] that the square on KQ is equal to the rectangle under SK, KR; that is, to the circle whose semidiameter is KR. That is, the circle of semidiameter KR is equal to the square on the side KQ (that is, P). Which was to be shown.

COROLLARIVM.

From what has been demonstrated we shall construct a circle equal to any rectilinear figure whatever; and conversely, we shall construct, equal to any circle whatever, a rectilinear figure similar to any other given rectilinear figure. For if to a given rectilinear figure [Margin: d — 14. secun. (Elements II.14)] we describe an equal square, and to this square construct an equal circle, then by Proposition 6 this same circle will be equal to the given rectilinear figure.

Again, if by Proposition 5 we construct a square equal to a given circle, and to this square [Margin: e — 25. sexti (Elements VI.25)] construct a rectilinear figure equal [to it] and similar to another given rectilinear figure, then this same rectilinear figure so constructed will be equal to the given circle. Which was to be shown.

VII.

Given a straight line, to find a circumference of a circle equal to it.

By what method a straight line may be found equal to the circumference of a given circle, we taught in Proposition 5; now, so that conversely a circumference of a circle equal to a given straight line may be found, we must proceed thus. Let a straight line A be given, to which a circumference is to be found equal. Any circle BC having been described about center D, let a line E equal to it [that is, to A] be found, which will easily be done thus. In the first instrument of Proposition 5, let AH be taken equal to the semidiameter BD of the described circle, and through H let HI be drawn parallel to DE. For AI, quadrupled, will give a line E equal to the circumference of the circle BC, as we said in Proposition 5. Then, to

the three lines E, A, BD, let a fourth proportional FG be found, and from G, with radius GF, let a circle FH be described. I say that its circumference is equal to the given line A. For since, as E is to A, so is BD to FG — that is [Margin: a — 15. quint. (Elements V.15)], so is the whole diameter of circle BC to the whole diameter of circle FH; and since [Margin: b — 11. quint. (Elements V.11)] as diameter is to diameter, so is the circumference BC to the circumference FH (as Pappus demonstrated) — it follows also that as E is to A, so is the circumference BC to the circumference FH. Since then E has been made equal to the circumference BC, it follows [Margin: c — 14. quint. (Elements V.14)] that the line A too is equal to the circumference FH; that is, the circumference of the circle FH has been found equal to the given line A. Which was to be shown.

Otherwise, and more easily. Take AK equal to the fourth part of the given line A, in the first instrument of Proposition 5, and through K draw KL parallel to DE. For the circumference of the circle whose semidiameter is AL will be equal to the given line A; because the quadrant of that circumference is equal to the line AK, the fourth part of the given line A, as we showed in Proposition 5.

VIII.

Given two unequal circles, and given an arc in one of them, to cut off from the other an arc equal to it. But the arc given in the larger circle must not be greater than the circumference of the smaller circle.

Let there be two unequal circles AB, CD, whose centers are E, F and semidiameters AE, CF, and let AB be the larger and CD the smaller. First let an arc CD be given in the smaller circle, to which an equal arc is to be cut off from the larger. Since the circle AB is larger than the circle CD, the semidiameter AE will also be larger than the semidiameter CF. Let AG then be cut off equal to CF, and let the arc CD be cut at H, by Proposition 2, so that the ratio of arc CH to arc HD is the same as that of the line AG to the line GE; and let there be cut off from AB an arc similar to CH — which will be done easily if, when the line FH has been drawn, the angle AEB is made equal to the angle CFH. I say that the arc AB is equal to the given arc CD. For since, as AG is to GE, so is arc CH to arc HD; and, by inversion, as EG is to AG, so is arc DH to arc CH — it follows, by composition, that as AE is to AG (that is, to CF, which is equal to AG), so is arc CD to arc CH. But as AE, the semidiameter, is to the semidiameter CF [Margin: a — 15. quint. (Elements V.15)], that is, as the whole diameter is to the whole diameter, so — from what Pappus demonstrated — is the circumference of circle AB to the circumference of circle CD, that is, so is arc AB to the similar arc CH. Therefore [Margin: b — 11. quint. (Elements V.11)] it will also follow that as arc CD is to arc CH, so is arc AB to

that same arc CH; and consequently the arcs CD, AB [Margin: c — 9. quinti (Elements V.9)] will be equal to one another. Which was to be shown.

Diagram. The figure for Proposition VII. At the top are two horizontal segments: the upper one, A, divided by tick-marks into four equal parts (the given line); and below it the shorter segment E. Beneath them are two circles: on the left a smaller circle with horizontal diameter B–D (B at the left, D the center) and C marked at the top — this is the circle BC; on the right a larger circle with diameter F–G (F at the left, G the center) and H at the upper right — this is the circle FH.

- A (upper segment): the given line, divided into four parts
- E (lower segment): line found equal to the circumference of circle BC
- B, C, D (left circle): B and C on the circle, D the center
- F, G, H (right circle): F and H on the circle, G the center

Diagram. The figure for Proposition VIII: two unequal circles. On the left, the larger circle AB with center E: A at the left end of a horizontal diameter, G near the center, and B, K, L at the bottom marking the arc AB and its construction. On the right, the smaller circle CD with center F: C at the left, I near the center, and H, D at the bottom marking the arc CD.

- A, G, E (left circle): A on the circle, E the center, G on the diameter AE (with $AG = CF$)
- L, B, K (left circle, bottom): division/construction points of arc AB
- C, F, I (right circle): C on the circle, F the center, I on the diameter
- H, D (right circle, bottom): H the point of section of arc CD, D on the circle

Next let there be given, in the larger circle, an arc AB not greater than the circumference of the smaller circle CD, from which an arc equal to the arc AB is to be cut off. Since the circle AB is larger than the circle CD, the semidiameter AE will also be larger than the semidiameter CF. Let CF then be produced to I, so that CI is equal to AE; let the arc AB be cut at L, by Proposition 2, so that the ratio of AB to BL is the same as that of CF to FI; and to the arc BL let there be taken an equal arc BK, and from the whole arc AK let a similar arc be cut off, so that the angle CFD is made equal to the angle AEK. I say that the arc CD is equal to the given arc AB. For since, as CF is to FI, so is arc AB to arc BL (that is, to arc BK, which is equal to BL), it will follow, by composition, that as CI is to FI, so is arc AK to arc BK; and, by conversion of ratio, that as CI is to CF — that is, as AE is to CF — so is arc AK to [arc AB]. But as AE is to the semidiameter CF [Margin: d — 15. quin. (Elements V.15)], that is, as the whole diameter is to the whole diameter, so — from what Pappus demonstrated — is the circumference of circle AB to the circumference of

circle CD, that is, so is arc AB to the arc CD similar to it. Therefore [Margin: e — 11. quinti (Elements V.11)] it will also follow that as arc AK is to arc CD, so is that same arc AK to arc AB; and consequently the arcs AB, CD [Margin: f — 9. quinti (Elements V.9)] will be equal to one another. Which was to be shown.

More briefly. Let a straight line equal to the arc CD be found, by Corollary 1 of Proposition 4 on the Quadratrix; and to this line let an arc AB be found equal, as will be shown in the following Corollary 1. In the same way, let a straight line equal to the arc AB be found, so that to this line an arc CD may be found equal: and what is required will be done.

COROLLARIVM I.

From what has been demonstrated it is manifest that from any circle whatever whose circumference is not smaller than a given straight line, a circumference can be cut off equal to any given straight line — just as it was shown above, in Corollary 1 of Proposition 4, that a straight line equal to any given circumference can be found — in this way. By the means which we showed a little before in Corollary 1 of Proposition 4, let a straight line be found equal to the quadrant of the given circle. And if indeed this line so found is equal to the given straight line, then the circumference of the quadrant will be equal to the given straight line.

But if that line found equal to the quadrant of the circle is larger than the given straight line, let the circumference of the quadrant be cut into two parts, by Proposition 2, so that the quadrant has the same ratio to one part as the line found has to the given straight line. For that part of the quadrant will be equal to the given straight line. For since, as the line found is to the given line, so is the quadrant to that arc cut off; and since the line found is equal to the quadrant [Margin: g — 14. quin. (Elements V.14)], it follows also that the given straight line is equal to the arc cut off.

But if, finally, that line found equal to the quadrant is smaller than the given straight line, let its double be taken — which of course will be equal to the circumference of the semicircle. For if this is equal to the given straight line, the circumference of the semicircle will be equal to the same given straight line.

But if that double line is larger than the given straight line, let the circumference of the semicircle be cut into two arcs, so that the semicircumference has the same ratio to one of them as that double line has to the given straight line. This indeed will easily be done if from that double line one equal to the given line is cut off, and the semicircumference is cut, by Proposition 2, in the same way as that double line has been cut. For it will follow, by composition, that as the whole

double line is to the part cut off — that is, to the given line — so is the whole semicircumference to the arc cut off. For that arc will be equal to the given line. For since, as the double line is to the given line, so is the semicircumference to that arc; and since that double line is equal to the semicircumference [Margin: h — 14. quin. (Elements V.14)], it follows also that the given line is equal to that arc.

But if, finally, that double line is smaller than the given straight line, let the earlier line found — which of course is equal to the quadrant of the circle — be added to it, so that the whole composite line is equal to the circumference of three quadrants, and triple the earlier line found. For if this triple line is equal to the given straight line, the arc of three quadrants will be equal to the same given straight line.

But if that triple line is larger than the given straight line, let the circumference of three quadrants be cut into two arcs, by Proposition 2, so that it has the same ratio to one of them as that triple line has to the given line — which will easily be done if from that triple line a line equal to the given line is cut off, and the circumference of three quadrants is cut, by Proposition 2, in the same way as that triple line has been cut. For, by composition, it will follow that as the whole triple line is to the part cut off — that is, to the given line — so is the whole circumference of three quadrants to the arc cut off. For that arc will be equal to the given straight line. For since, as that triple line is to the given line, so is the circumference of three quadrants to the arc cut off; and since that triple line is equal to the circumference of three quadrants [Margin: i — 14. quint. (Elements V.14)], it follows also that the given straight line is equal to that arc cut off.

Finally, if that triple line is smaller than the given straight line, let there be added to it again the earlier line found equal to the quadrant, so that the whole composite line becomes equal to the whole circumference of the circle, and quadruple the earlier line found. For if this quadruple line is equal to the given straight line, the whole circumference of the given circle will be equal to the given straight line.

But if that quadruple line is larger than the given straight line (it cannot be smaller: otherwise the given straight line would be larger than the circumference of the given circle, which is contrary to the hypothesis), let the whole circumference be cut into two arcs, by Proposition 2, so that it has the same ratio to one of them as that quadruple line has to the given line — which will easily be done if from that quadruple line a line equal to the given line is taken away, and the whole circumference of the circle is cut in the same way as that quadruple line has been cut. For, by composition, it will follow that as the whole quadruple

line is to the part cut off — that is, to the given line — so is the whole circumference of the circle to the arc cut off. For that arc will be equal to the given line. For since, as that quadruple line is to the given straight line, so is [the whole circumference] to the arc cut off; and since that quadruple line is equal to the whole circumference [Margin: k — 14. quin. (Elements V.14)], it follows also that the given straight line is equal to the arc cut off.

Otherwise, and more briefly. When, by Corollary 1 of Proposition 4 on the Quadratrix (or by Proposition 5 of the same), a straight line equal to the circumference of the given circle has been found, let there be cut off from it a line equal to the given straight line, and let the whole circumference be cut, by Proposition 2, into two parts in the same way as that line found has been cut. For, by composition, it will follow that as the whole line is to the part removed from it — that is, to the given line — so is the whole circumference to the arc cut off. Since then the whole line is equal to the whole circumference [Margin: l — 14. quint. (Elements V.14)], it follows also that the given straight line is equal to the arc cut off. Which was to be shown.

COROLLARY II.

From what has been said it can also be inferred that both arcs and rectilinear angles may be found that are incommensurable with one another. For let the arc HB be given, together with the rectilinear angle HAB. I say that the arc HB can be divided into two arcs incommensurable with one another, and likewise the angle HAB into two angles incommensurable with one another.

For, having drawn the quadratrix DE, which cuts the straight line HA at F, draw through F the line FI parallel to the base AE. Then, having found two straight lines incommensurable with one another (as is taught in Book 10), divide the straight line IA at N — by the scholium to proposition 10 of this book — in such a way that IN has to NA the same ratio that the two lines found have to each other; and so IN and NA will likewise be incommensurable. Now draw the straight line NK parallel to AE, cutting the quadratrix at K, and from the centre A draw through K a straight line cutting the arc HB at L. I say that the arcs HL and LB, as well as the angles HAL and LAB, are incommensurable with one another.

For, by proposition 1, as IN is to NA, so is the arc HL to the arc LB; but IN and NA are incommensurable; therefore the arcs HL and LB will also be incommensurable. [Margin: d — VI.33.] Since then it also holds that, as the arc HL is to the arc LB, so is the angle HAL to the angle LAB, the angles HAL and LAB will

likewise be incommensurable with one another. Which was what was to be proved.

Diagram. A square with corners A (bottom-left), B (bottom-right), C (top-right), and D (top-left), with A the centre of a quarter-circle arc running from D (top-left) down to B (bottom-right). The quadratrix curve runs from D down to E on the base AB. From the centre A two radial lines are drawn to the arc, one through F to H, the other through K to L. Two horizontal lines (FI and NK) are drawn parallel to the base. The left side AD and the right side BC carry equal tick-mark divisions.

- A (bottom-left corner): centre of the arc and of the square
- B (bottom-right corner): endpoint of the arc on the base line
- C (top-right corner)
- D (top-left corner): endpoint of the arc; start of the quadratrix
- H (on the arc, upper): endpoint of the radius through F
- F (on radius AH): where the quadratrix cuts the line HA; left end of the horizontal FI
- I (on the left side AD): right — level with F; endpoint of the horizontal FI
- L (on the arc, lower right): where the radius through K cuts the arc HB
- K (on the quadratrix): where the horizontal NK cuts the quadratrix
- O (near the centre): interior intersection point
- N (on the left side AD, lower): left end of the horizontal NK; the division point of IA
- G (on the base AB)
- M (on the base AB)
- E (on the base AB): foot of the quadratrix

FINIS ELEMENTI SEXTI

[The End of the Sixth Element.]

EUCLID'S ELEMENTS, BOOK SEVEN

[EVCLIDIS ELEMENTVM SEPTIMVM.]

DEFINITIONS

I.

Unity is that according to which each of the things that exist is said to be one.

Up to this point Euclid has dealt with the earlier part of geometry — that, namely, which concerns plane figures; there remained the other part, which concerns solids. But it was first necessary for him to treat of commensurable and incommensurable lines, since for demonstrating the properties of very many bodies — and most of all those called regular — and for explaining them as one should, a knowledge of these lines is required. So much so, indeed, that without them the treatment of solids would be incomplete and not fully worked out in its numbers. To this is added that, without these same lines, very many sides both of plane figures and of solids can be neither expressed nor understood, if the theory of geometry is put to work and use. For not seldom the sides of figures are those lines which by the Greeks are called ἄλογοι [Greek: ἄλογοι (alogoi, "irrational," "inexpressible")] and by the Latins "irrational"; or, if they are not irrational, they are at least incommensurable with one another in length, and so do not fall under the measure of numbers. And because the explanation and understanding of these same lines is bound up and joined with numbers, so that without numbers they cannot be known at all, it was necessary — in order that the doctrine might keep its proper order and reasoning — that the account of numbers be set before those lines. For this reason, in this seventh book and in the two that follow it, he is occupied with the properties and affections of numbers, so far as they serve the geometrical subject, so that thereafter, in the tenth book, the demonstrations concerning commensurable and incommensurable lines may be carried through more easily and more fully.

Beginning therefore, in his usual manner, from first principles, he defines at the outset "unity," and teaches that it is that according to which each of the things that exist is said to be one. For it is in respect of unity that we are accustomed to speak of one stone, one animal, one body, and so on. Moreover, unity admits of no division among numbers, just as the point admits of none among magnitudes, as we taught in the first book.

II.

Number, moreover, is a multitude composed of units.

Since number is a certain multitude composed of units, it is clear that any number whatever has as many parts as there are units constituting it — so that the unit is a part of any number, named after the number itself of which it is a part. Thus the number 8, composed of eight units, is divided into just so many parts, namely into 8 units, each of which is called an eighth part of the number eight. In the same way the number 100, composed of a hundred units, is distributed into just as many parts, each of which is a hundredth part of it, and so on.

From this it follows that all numbers, however many there be, are commensurable with one another, since one and the same measure — namely the unit, as has been said — measures them. This can in no way hold for all magnitudes, since very many of them have no common measure, but are utterly incommensurable, as will be shown most clearly in Book 10.

III.

A part is a number of a number — the lesser of the greater — when the lesser measures the greater.

This definition is not unlike the one by which Euclid, in Book 5, set out the "part" of a continuous quantity. For just as there, so here too, he defines only an aliquot part, since only this is properly said to measure the whole, as we explained more fully in that place. And so the number 6 will be called a part of all these numbers: 12, 18, 24, 30, 60, 630, and so on, because it measures each of them. Likewise the parts of the number 576 will be all these numbers: 3, 4, 6, 8, since each of these measures it, as is evident.

[Signature]

Now every part takes its name from that number by which it measures the number of which it is a part: thus 6, a part of the number 42, draws its name from 7, since 6 measures 42 by 7. And so 6 will be a seventh part of the number 42, and it is the same with the rest.

IV.

"Parts," however, when it does not measure.

Euclid intends that a number less than a greater number, which it does not measure, be called not a "part" but "parts": such is the number 5 when compared with the number 18. For although — since it does not measure the greater except through its own units — it ought not to be called a part of it, nevertheless it can

fittingly and appropriately be called "parts," because it contains five units, each of which is an eighteenth part of the number 18. Hence we shall call the number 5 "five eighteenth-parts" of the number 18. From this it plainly follows that Euclid understood by the name "part" only an aliquot part, and not also an aliquant part, as some would have it; otherwise this fourth definition, which comprehends the aliquant part, would be superfluous.

Furthermore, "parts," whatever they may be, take their name from those two numbers by which the common measure of the two numbers measures each of them — namely both the one which is called "parts" and the one of which it is called the parts. So that if the common measure of two numbers measures the lesser by 3 and the greater by 5, the lesser is said to be "three fifths" of the greater. Such parts are: 6 of the number 10. For their common measure, 2, measures 6 by 3 and 10 by 5. By the same reasoning we shall call the same number 6 "six tenths" of the same number 10, since the unit, their common measure, measures the one by 6 and the other by 10. Hold the same judgement for the rest.

But if you should ask why Euclid in this place defines not only the lesser number that is a part of the greater, but also the one that is "parts," whereas in the fifth book, dealing with magnitudes, he did not do the same — for he did not there call "parts" that lesser magnitude which does not measure the greater, but only the one which measures the greater by a part — we reply that the reason is this: every lesser number is either a part or parts of any greater number, as will be shown in proposition 4 of this book — a part, when it measures it; parts, when it does not measure it. But with magnitudes the matter stands far otherwise. For, given two unequal magnitudes, the lesser is not necessarily either a part or parts of the greater, since they are very often incommensurable, as the tenth book will openly demonstrate, and so the lesser may in no way contain several parts of the greater. For only among commensurable magnitudes does the lesser comprehend several parts of the greater, if it does not measure it. Rightly, therefore, did Euclid in the fifth book explain only the "part" among magnitudes, but here, among numbers, both "part" and "parts."

V.

A multiple, on the other hand, is the greater of the lesser, when the lesser measures the greater.

Just as only that lesser number is called a part of the greater which measures the greater, so too only that greater number is called a multiple of the lesser which

the lesser measures — so much so that the greater number, of which the lesser is a part, is in turn a multiple of the lesser. Thus the number 6 is a part of the number 30, and the latter is a multiple of the former, and so on. But if the lesser does not measure the greater, the greater will in no way be a multiple of the lesser. For if the greater were a multiple of the lesser, the lesser would measure the greater, by this definition. And conversely, if the greater is not a multiple of the lesser, the lesser will not measure the greater; for if the lesser did measure the greater, then by this definition the greater would be a multiple of the lesser.

VI.

An even number is one that is divided into two [equal parts].

Thus all these numbers — 4, 10, 40, 100, 1000 — are called even, because they are divided into two, that is, into two equal parts, since their halves are 2, 5, 20, 50, 500.

VII.

An odd number, on the other hand, is one that is not divided into two [equal parts]. Or, one that differs from an even number by a unit.

All these numbers — 5, 11, 15, 37, 101, 1001 — are called odd, because they cannot be divided into two equal parts. Or certainly because they differ by a unit from these even numbers — 4, 10, 14, 36, 100, 1000 — or also from these — 6, 12, 16, 38, 102, 1002. From this passage it can be plainly gathered that the unit in numbers is utterly indivisible. For if it were divided, every odd number would have a half, and so could be divided into two equal parts. For the half of this odd number, 11, would be five units and a half — the contrary of which Euclid taught by this definition.

VIII.

An even-times-even number is one which an even number measures by an even number.

Since an even number is one that is divided into two equal parts, it comes about that some even number, at least two, measures any even number. That even number, then, which an even number measures by an even number, is called "even-times-even." Such is this even number, 32; for the even number 8 measures it by the even number 4. In the same way the even number 24 will be called even-times-even, since the even number 4 measures it by the even number 6, and so on.

IX.

An even-times-odd number, on the other hand, is one which an even number measures by an odd number.

But if an even number measures an even number by an odd number, it will be called "even-times-odd." Such is the even number 30; for the even number 2 measures it by the odd number 15. In the same way the even number 6 measures the same [30] by the odd number 5, and so on.

Furthermore, if the two immediately preceding definitions are rightly weighed, it will be evident that it can happen that one and the same even number is both even-times-even and even-times-odd. For the even number 24, since the even number 6 measures it by the even number 4, will be even-times-even. Again, because the even number 8 measures the same [24] by the odd number 3, it will also be called even-times-odd. Hence some interpreters, thinking this to be absurd, in order to exclude even numbers of this kind — which appear to be both even-times-even and even-times-odd — added to each definition the qualifying word "only," so that an even-times-even number, according to them, is one which an even number

measures it by an even number only; whereas an even-times-odd number is one that an even number measures by an odd number only. For it turns out that the given even number 24 is neither even-times-even — since it is measured not only by 6 taken an even number of times (four), but also by 8 taken an odd number of times (three) — nor even-times-odd, since it is measured not only, as was said, by 8 through the odd number 3, but also by 6 through the even number 4. Rather it can fittingly be called both even-times-even and even-times-odd, for it shares in a way the nature of both, as is clear. And so three genera of even number, most distinct among themselves, are established: even-times-even; even-times-odd; and the even-times-even-and-odd, which some call "even-times-even and odd-wise" [*pariter par & impariter*].

Yet although all this is true, and correctly explained on the view of the Pythagoreans, Nicomachus, Boethius [Boethius], and others, it is nonetheless wholly foreign to Euclid's [Euclid's] approach — as is clear both from the definitions handed down, in which that word "only" [*tantum*] which they add is not found, and from Propositions 32, 33, and 34 of Book 9 (*Elements* IX.32–34). There Euclid plainly calls every even number that some even number measures by an even number "even-times-even," but the one that some even number measures by an odd number "even-times-odd"; and finally the one that an even

number measures both by an even and by an odd number he calls both even-times-even and even-times-odd. And he demonstrates that all numbers that are successive doubles of two — such as 2, 4, 8, 16, 32, 64, 128, etc. — are even-times-even only; that is, that even numbers measure them by even numbers only. But numbers that have odd halves are even-times-odd only — that is, even numbers measure them by odd numbers only — of which sort are 6, 10, 14, 18, 22, etc. And finally, numbers that are neither doubles of two nor have odd halves are both even-times-even and even-times-odd, such as 12, 20, 24, 28, 36, etc. And so Euclid holds, in the demonstrations of those propositions, that these last numbers, and others like them, are truly, according to the definitions handed down, even-times-even; and again are rightly called even-times-odd — although they are neither even-times-even only nor even-times-odd only. But these matters will be understood more clearly from Book 9.

X.

An odd-times-odd number [*impariter impar*], by contrast, is one which an odd number measures by an odd number.

As this odd number 15 is called odd-times-odd, since the odd number 3 measures it by 5, an odd number. In the same way these odd numbers — 9, 21, 25, 27, 33, 35, 39, 135, 2025, and infinitely many others — are named odd-times-odd.

XI.

A prime [first] number [*primus*] is one which unity alone measures.

If no number, but unity alone, measures some number — so that it can be called neither even-times-even, nor even-times-odd, nor odd-times-odd — it will be called a prime number; such are all these: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, etc. For unity alone measures them.

XII.

Numbers prime to one another [*primi inter se*] are those which unity alone, as a common measure, measures.

Just as the number that unity alone measures is called prime, so too two, three, four, or even more numbers — which no number besides unity measures as a common measure, although each of them individually has numbers that measure it besides unity — are called prime to one another. Thus 15 and 8 are numbers prime to one another, because unity alone, as a common measure, measures them: for although the numbers 3 and 5 measure the former, and 2 and 4 the

latter, yet none of these measures both, but unity alone is the common measure of both. So too these numbers 7, 10, 15 are said to be prime to one another, because besides unity they have no number as a common measure — even though the last two [10 and 15] have this number 5 as a common measure. Finally, unity and any number can be said, though improperly, to be numbers prime to one another, because unity alone measures both unity and any number, as a common measure.

XIII.

A composite number [*compositus*] is one which some number measures.

A number which, besides unity, some other number measures, the geometers call composite — such as this 15. For each of these numbers, 3 and 5, measures it. And it is clear that all even numbers except 2 are composite, since 2 measures them all. From which it follows that all prime numbers except 2 are odd, since among the even numbers 2 alone is prime, as we said.

XIII.

Numbers composite to one another [*compositi inter se*] are those which some number, as a common measure, measures.

Two or more numbers which, besides unity, some other number measures as a common measure, are said to be composite to one another — even though not every one of them is itself a composite number; just as the number that, besides unity, some number measures is called composite. Thus these numbers 15 and 24 are composite to one another, because this number 3 measures them as a common measure. In the same way these numbers 7, 21, 35 will be composite to one another. For the first of them measures both itself and the other two — although, taken by itself, it is called prime.

XV.

A number is said to multiply a number when the one that is multiplied is composed as many times as there are units in the multiplier itself, and some number is produced.

Thus 6 is said to multiply 8 when 8 is composed six times — namely, as many times as there are units in the multiplier 6 — and the number 48 is produced. And likewise, conversely, 8 is said to multiply 6 if we take 6 eight times — that is, as many times as there are units contained in the multiplier 8 — and produce the same number 48. By the same reasoning, finally, these numbers 100, 1000, 20, etc.

are said to multiply 456 when it is taken a hundred times, a thousand times, or twenty times, etc., and these numbers are generated: 45600, 456000, 9120, etc. And so a number is said to be produced, generated, or procreated from two numbers when, one multiplying the other, a product results. Thus 63 is said to be generated from these numbers 7 and 9, because it is procreated from the multiplication of 7 into 9, or the reverse. And so with the rest.

It follows from this that the number produced by the multiplication has the same ratio to either of the multipliers as the other multiplier has to unity. For since, by Euclid's definition, either multiplier must be composed as many times, to make the product, as there are units in the other multiplier, the product will contain either multiplier as many times as the other multiplier contains unity; and therefore the product will have the same ratio to either multiplier as the other multiplier has to unity. And so multiplication — that is, the drawing [*ductus*] of one number into another — can also be described thus:

The multiplication of a number into a number is the finding of a number that has the same ratio to either multiplier as the other multiplier has to unity.

Thus you see that from the multiplication, or drawing, of 6 into 8, the number 48 is generated, which stands to 6 as 8 to 1, or to 8 as 6 to 1.

To this definition must be added another, teaching what it is for a number to divide a number — since it is altogether necessary for the things we are going to demonstrate in what follows.

A number is said to divide a number when a number is taken which, by its units, indicates how many times the dividing number is contained in the divided one.

Thus 6 is said to divide 48 when the number 8 is taken, which by its eight units indicates that the dividing number 6 is contained eight times in the divided number 48. And likewise, conversely, 8 is said to divide the same 48 if 6 is taken, which by its six units indicates that the dividing number 8 is contained six times in the divided number 48.

Hence it follows that the number produced by the division has the same ratio to unity as the divided number has to the divisor. For since, as we said in the definition, the produced number must indicate by its units how many times the dividing number is contained in the divided one, the product will contain unity as many times as the divided number contains the divisor; and therefore the

number generated by the division will have the same ratio to unity as the divided number has to the divisor. And so the division of one number by another can also be described thus:

The division of a number by a number is the finding of a number that has the same ratio to unity as the divided number has to the divisor.

Thus you see that from the division of 48 by 6 there is produced 8, which stands to 1 as 48 to 6; and likewise from the division of the same number 48 by 8 there is produced 6, which stands to 1 as 48 to 8.

From these it is also established that, when a number is divided by a number, the divided number is generated by the multiplication of the number found through the division into the dividing number. For let a number A be divided by B, and let the number C be produced; I say that A is generated from the drawing of the number C into the number B. For since, by the definition of multiplication, the number produced from C into B stands to B as C to unity D; and by the definition of division, A likewise stands to B as C to unity D; it is clear that the number generated from C into B is the number A, since both that generated number and A have the same ratio to B as C to D, unity.

A, 48. B, 8. C, 6. D, 1.

All this holds also for fractions, and for integers together with fractions: that is, a fraction is said to multiply a fraction — or an integer a fraction, or a fraction an integer (whether the fractions are attached to integers or not) — when the one that is multiplied is composed as many times as there are units in the multiplier itself, and some number is produced; and to divide, when a number is taken which indicates how many times the divisor is contained in the divided one. So that in multiplication there is likewise found a number having the same ratio to either multiplier as the other multiplier has to unity, while in division there is produced a number having the same ratio to unity as the divided number has to the divisor. Thus $1/2$ is said to multiply 20 when 20 is composed as many times as there are units in $1/2$, and the number 10 is produced. For since unity is in $1/2$ only by its half, 20 too must be taken by its half, which is 10. And likewise, conversely, 20 is said to multiply $1/2$ if $1/2$ is taken twenty times — namely, as many times as unity is in 20 — and the same number 10 is produced. Here you see that the produced number 10 has the same ratio to $1/2$ as the other multiplying number 20 has to 1; or that 10 is to 20 as $1/2$ to 1. So too $1/2$ and $1/3$ are said to multiply each other, when $1/2$ is taken by its third part — inasmuch as $1/3$ contains only a third part of unity — or when $1/3$ is taken by its half, because

$1/2$ too contains only a half of unity. In either way $1/6$ is produced, which is a third part of the number $1/2$, or $3/6$, and half of the number $1/3$, or $2/6$. In what manner multiplication is to be carried out with fractions we have taught in the *Arithmetic*, and will demonstrate at the end of Book 9.

Again, $1/2$ is said to divide 10 when the number 20 is taken, indicating that the dividing number $1/2$ is contained twenty times in the divided number 10 — so that the same ratio holds of the produced number 20 to 1 as of the divided number 10 to the divisor,

$1/2$. So too $1/2$ is said to divide $1/6$ when the number $1/3$ is taken, indicating that the dividing number $1/2$ is not contained wholly in the divided number $1/6$, but only a third part of it. For since the number $1/2$ is the same as $3/6$, it clearly follows that a third part of it, namely $1/6$, is contained in $1/6$. As to how division is to be carried out with fractions, we have handed it down in the *Arithmetic*, and will demonstrate at the end of Book 9, where all these things we have said about the multiplication and division of fractions will become clearer.

XVI.

But when two numbers multiplying each other make some number, the number so made will be called plane [*planus*]. And the numbers that have multiplied each other will be called its sides.

Every number produced from the mutual multiplication of two numbers is called plane, because, by its units arranged in length and breadth, it represents a rectangular parallelogram whose sides are the two multiplying numbers — which are therefore called the sides of the produced number, because they contain it, just as straight lines enclosing a right angle are said to contain a rectangular parallelogram, as we explained more fully in Book 2. Thus 24, produced from the mutual multiplication of 4 and 6, is called plane, and its sides are 4 and 6, because its units, arranged in length and breadth as the sides require, represent a rectangular parallelogram, one side of which comprises six units and the other 4. In the same way 64, generated from the mutual multiplication of 8 and 8, is called plane, and its sides are 8 and 8.

However, although there are infinitely many kinds of plane numbers among the arithmeticians — just as there are plane figures among the geometers — Euclid defines only the rectangular quadrilateral plane number, namely the one contained under two numbers from whose mutual multiplication it is generated, since in these books he discusses only this kind of number: one which in every respect is similar and equal to a geometrical square, and to a figure longer on one

side [an oblong], whether you regard the perimeter, the area, or the capacity. He says nothing about triangular, pentagonal, hexagonal, and other such numbers, because these, although they agree with a geometrical triangle, pentagon, hexagon, etc. as regards the perimeter, nevertheless, if you consider the area or capacity, differ greatly from them. This is clear to anyone who reads carefully through these books of Euclid and the *Arithmetic* of Jordanus [Jordanus Nemorarius].

Moreover, one and the same plane number can have several sides, since it can be produced from several multiplications. Thus 24 has not only the sides 4 and 6, but also 3 and 8, and moreover 2 and 12, because it is generated as much from the multiplication of 4 into 6 [as from the others]. Likewise the plane number 100 has the sides 5 and 20; 4 and 25; 2 and 50; 10 and 10 — because it is generated from all these numbers, if they are multiplied two by two.

And since two numbers multiplying each other measure every plane number — because each of them, taken as many times as there are units in the other, produces it — it clearly follows that every plane number is composite. This can also be said of a solid number, which is being defined just now. It is true that unity can sometimes be called a plane number, though improperly, because its sides are two unities, which, multiplied together, produce unity itself.

XVII.

But when three numbers multiplying each other make some number, the one procreated will be called solid [*solidus*]. And the numbers that have multiplied each other will be called its sides.

Thus, because these three numbers 2, 3, 4, multiplying one another, produce 24. For from 2 into 3 the number 6 is procreated, and from 6 into 4, 24 results; or from 2 into 4 the number 8 is generated, and from 8 into 3, 24 is effected; or, finally, from 3 into 4 the number 12 is produced, and from 12 into 2, 24 is generated — the number 24 will be called solid, and the numbers 2, 3, 4 will be called its sides, because its units, arranged in length, breadth, and depth, represent a certain solid figure, which is called a parallelepiped, as we shall explain in Book 11. Its three dimensions are all expressed by the three numbers multiplying one another: one gives the length, another the breadth, and the remaining one the height. For if the first is multiplied by the third — that is, if 2 by 4 — the number 8 results, the base of the solid number, four units long and two broad; and if this base is multiplied by 3 — that is, taken three times — the whole solid number 24 arises, having a height of three units. But if 2 is multiplied

by 3, they will have a base of 6 units, which multiplied by 4 makes the whole solid number 24, having 4 units in height. If, finally, 3 is multiplied by 4, the number 12 is produced as a base, which taken twice makes the same solid number 24, whose height contains two units. All of which is evident in the figure set out.

Diagram. A rectangular parallelepiped (an oblong box) drawn in oblique projection and ruled into a grid of unit cubes, illustrating the solid number 24 (4 long $\times$ 2 deep $\times$ 3 high). Eight vertices are lettered :

- E (top face, back-left corner)
- H (top face, back-right corner)
- A (top face, front-left corner)
- D (top face, front-right corner)
- B (bottom face, front-left corner)
- C (bottom face, front-right corner)
- F (bottom face, back-left corner)
- G (bottom face, back-right corner)

The base BCGF has length BC (four units) and breadth BF (two units); the height BA (equivalently AE) runs three units.

For if the base is BCGF, of eight units — whose length BC comprises four units, and BF, the breadth, two — two other bases, similar and equal, will be superimposed upon it, so that the whole solid number contains 24 units, and its height BA is three units. Similarly, if the base is ABFE, of six units — whose length AB contains three and breadth BF two units — three other similar and equal bases will be superimposed upon it, and the whole solid number will again become 24, having a height BC of four units. If, finally, the base is ABCD, of 12 units — whose length BC contains four and breadth AB three — another base similar and equal, EFGH, will be superimposed upon it, and the whole solid will consist of 24 units, two of which will give the height AE, or BF. This same solid number 24 has these sides 6, 2, 2, since it is produced from these multiplied together. And in the same way one must judge of other solid numbers.

Finally, the unit too will sometimes be called a solid number, though not properly so, since its sides are three units, which produce the unit itself by multiplying one another.

Here too Euclid defines only the *right-angled* solid number — whose opposite bases are parallel, and which is contained under three numbers — passing over the infinitely many other kinds treated by Jordanus [Nemorarius], for the reason

given in the preceding definition: namely, because these are exactly equal and similar to the cubes and parallelepipeds of geometry.

XXVIII. [recte: XVIII]

A **square number** is one that is equal times equal; or one that is contained under two equal numbers.

Diagram. A square array of 25 dots (five rows of five), illustrating the square number 25 arranged as a plane figure five units on each side. No labels; the array is uncaptioned.

He calls a *square* that plane number which is equal times equal — that is, one which, when its units are laid out lengthwise and breadthwise, represents a rectangular parallelogram whose length equals its breadth, so that all its sides are equal; or one produced by the mutual multiplication of two equal numbers, and so contained under them. Of this kind is the number 25, contained under the equal numbers 5 and 5 — that is, generated by their mutual multiplication. For if its units are reduced to a plane figure, it represents a perfect geometrical square with five units on each side, and so it is equal times equal.

Either one of the equal numbers under which a square number is contained, or from whose multiplication it is produced, is called by geometers the *side* of the square, but by most arithmeticians the *square root*.

XIX.

A **cube**, again, is that which is equal times equal equally; or one that is contained under three equal numbers.

Diagram. A cube drawn in oblique projection, its edges subdivided into three parts with dots at the grid intersections, illustrating the cube number 27 as a solid three units in every dimension. No vertex letters are printed.

He likewise calls a *cube* that solid number which is equal times equal equally — that is, one whose units, arranged in length, breadth, and depth, represent a geometrical cube, so that all its dimensions (length, breadth, and height or depth) are equal; or one produced by the mutual multiplication of three equal numbers among themselves. Such is the number 27, contained under the equal numbers 3, 3, 3, or generated by their mutual multiplication: for 3 times 3 makes 9, and 9 times 3 yields the cube number 27. All its units, reduced to solid form, represent a perfect geometrical cube, and there are three units in length, in breadth, and in depth alike. Therefore the number 27 is equal times equal equally.

Any one of the three equal numbers under which a cube is contained, or from whose mutual multiplication it is produced, is called by geometers the *side* of the cube, but by most arithmeticians the *cube root*.

XX.

Numbers are **proportional** when the first is the same multiple, or the same part, or the same parts, of the second as the third is of the fourth; or else when the first equally contains the second, and the third the fourth, together with the same part of it or the same parts.

In order to take in proportional numbers in every kind of rational proportion of inequality (which is what all unequal numbers compared with one another are), we have added to this definition the words: *or else when the first equally contains the second, and the third the fourth, together with the same part of it or the same parts*. For Euclid's received definition — which I judge to be corrupt, since it is defective and incomplete — takes in only proportional numbers in multiple or submultiple proportion, and in the remaining proportions of lesser inequality. In multiple proportion any four numbers are proportional when the first is the same multiple of the second as the third is of the fourth; in submultiple, when the first is the same part of the second as the third is of the fourth; and in the remaining proportions of lesser inequality, when the first is the same parts of the second as the third is of the fourth, as Euclid's definition requires. But from it we can by no means detect which numbers are proportional in superparticular proportion, in the superpartient of a multiple-superparticular, and in the multiple-superpartient. For in all of these the first number is neither the same multiple of the second, nor the same part, nor the same parts, as the third is of the fourth; rather the first equally contains the second, and the third the fourth — say either once or several times — together with the same part of it, or the same parts, as is clear from what we taught on definition 4 of book 5, where we set out all the rational proportions fully.

And so these numbers 12, 4, 9, 3 are proportional, since the first is the same multiple of the second as the third is of the fourth, namely triple. Likewise these, 4, 12, 3, 9: for the first is the same part of the second, and the third of the fourth, namely a third. Again, these numbers 6, 8, 9, 12 are proportional, because the first is the same parts of the second, and the third of the fourth, namely three-fourths. Finally, 7, 6, 14, 12 and 7, 4, 14, 8 and 11, 5, 22, 10 and 12, 5, 24, 10 are proportional numbers. For in the first example the first number contains the second, and the third the fourth, once and moreover the same part of it, namely a sixth; in the second, once, and the same parts, namely three-fourths; in the third,

twice, and still the same part, namely a fifth; in the last, twice, and besides the same parts, say two-fifths. But if the first number [is not the same multiple] of the second, and the third of the fourth,

is not the same multiple, nor the same part, nor the same parts; or, finally, if the first does not equally contain the second, and the third the fourth, together with the same part of it or the same parts — then the given numbers must in no way be called proportional.

Therefore, whenever four numbers are set down as proportional, it must necessarily be granted — if the greater are compared with the less — that the first is the same multiple of the second as the third is of the fourth, or else that the first equally contains the second, and the third the fourth, together with the same part or the same parts. And conversely, if it is granted that the first is the same multiple of the second as the third is of the fourth, or that the first is said to contain the second equally, and the third the fourth, together with the same part or the same parts, then the numbers are concluded to be proportional. But if the less are referred to the greater and are said to have the same proportion, it must be admitted that the first is the same part of the second, and the third the same part of the fourth, or the same parts; and conversely, if it is granted that the first is the same part of the second, and the third of the fourth, or the same parts, it is concluded that these numbers have the same proportion.

Euclid defines as proportional only those numbers that have the same proportion of *inequality*. For if we speak of a proportion of equality, it is clear that the first must be equal to the second, and the third to the fourth, for them to be called proportional numbers.

From this definition it is plainly gathered that equal numbers have the same proportion to the same number, and conversely that the same number has the same proportion to equal numbers; and likewise that numbers having the same proportion to the same number, or to which the same number has the same proportion, are equal. For since equal numbers are of the same number either the same multiple, or the same part, or the same parts; or else equally contain it together with the same part of it or parts; and likewise since the same number is of equal numbers either the same multiple, or the same part, or the same parts, or else equally contains them together with the same part of them — it is clear, by this definition, that equal numbers have the same proportion to the same number, and the same number to equal numbers.

Again, since numbers having the same proportion to the same number are of it either the same multiples, or the same part, or the same parts, or else equally contain it together with the same part of it or parts; and likewise since the same number is of them either the same multiple, or the same part, or the same parts, or else equally contains them together with the same part of them or parts — by this same definition it is manifest that numbers which have the same proportion to the same number, or to which the same number has the same proportion, are equal.

By parallel reasoning it is inferred that a greater number has a greater proportion to the same number than a lesser one does; and conversely, that the same number has a greater proportion to a lesser number than to a greater. Likewise, that of two numbers the one which has the greater proportion to the same number is the greater, while the one to which the same number has the greater proportion is the lesser. All of which are clear, provided this definition is rightly understood.

This definition holds good even for fractional numbers, whether whole numbers are entirely absent or not. For these four numbers are proportional: $\frac{2}{3}$, $\frac{1}{3}$, 1, $\frac{1}{2}$ — because the first is the same multiple of the second as the third is of the fourth, namely double, as is plain, if the first two are brought to the same denomination, as to [?], and the latter two likewise to the same denomination, as to [?]. (How fractional numbers are reduced to the same denomination we taught in the *Arithmetic*, and we shall demonstrate it at the end of book 9.) By the same reasoning these four numbers are proportional: 2, 4, $1\frac{1}{2}$, 3 — since the first is the same part of the second, and the third of the fourth, namely a half, as is evident, if the first two are brought to these of the same denomination, [?], and the latter two to these, [?]; and so of the rest.

XXI.

Similar plane and solid numbers are those that have proportional sides.

For one plane number to be similar to another, it is not necessary that any two sides of the one be proportional to any two sides of the other; it is enough that the one have some sides which are proportional to some two sides of the other. For by this reasoning their breadths will be proportional to their lengths, if they are reduced to a plane figure according to their units, as the sides taken are joined. Thus the plane numbers 24 and 6 are similar, since the sides of the former, 6 and 4, are proportional to the sides of the latter, 3 and 2 — although the other sides of the former, namely 8 and 3, or 12 and 2, are not proportional to these same sides.

In the same way, for two solid numbers to be similar, it is not required that any three sides of the one be proportional to any three sides of the other; it suffices that three sides of the one be found proportional to three of the other, because by this reasoning, if they are reduced to solid form according to their units, their breadths will be proportional to their lengths, and their lengths to their heights. Thus the solid numbers 192 and 24 are similar, because the sides of the former, 8, 6, 4, are proportional to the sides of the latter, 4, 3, 2 — even though other sides of the former, namely 12, 8, 2, or 16, 4, 3, are by no means proportional to these same sides.

And so two plane or solid numbers can be similar even though, when some sides of the one are taken, no proportional sides of the other can be found in any way. For the plane numbers 24 and 6 are similar, as has been said; and yet if the sides of the former are taken as 3 and 8, none proportional to them will be found in the latter. So too the solid numbers 192 and 24 are similar, even though when the sides of the former are taken as 3, 4, 16, none proportional to them are found in the latter.

Moreover, this similarity of plane [and solid] numbers is also found among fractional numbers, or among whole numbers together with fractional ones. For if four non-whole proportional numbers are set down, and both the two former and the two latter are multiplied by one another, there will usually be produced two similar non-whole plane numbers, and so on. I said "usually," because it can happen that sometimes whole numbers are generated. For if two sides are $\frac{2}{3}$ and 6, and two others $1\frac{1}{3}$ [?] and 12, which have the same ninefold proportion, the former two will produce the plane number 4, and the latter the number 16.

XXII.

A **perfect number** is one that is equal to its own parts.

That number to which all its parts taken together are equal (we are speaking of *aliquot* parts, according to the definition in this book) is called by mathematicians *perfect*. Of this kind are these numbers: 6, 28, 496. For the first contains only these aliquot parts, 1, 2, 3, which taken together make up the number 6. The aliquot parts of the second number are all these: 1, 2, 4, 7, 14, whose sum constitutes the number 28 itself. The third number has these aliquot parts: 1, 2, 4, 8, 16, 31, 62, 124, 248 —

— all of which, collected into one sum, evidently make up the number 496. Which numbers are perfect, and by what method and principle they are generated (for

besides the three mentioned there are innumerable others), will be demonstrated by Euclid in the last proposition of book 9.

But if all the aliquot parts of some number taken together are greater than the number itself, that number is usually called *abundant*; if less, *deficient*.

It is gathered from this passage, more clearly than daylight, that "part" is taken by Euclid only in the sense of an aliquot part. For otherwise every number would be perfect, being equal to its own parts, if any lesser number could be called a part of a greater number whether it measures it or not.

To these definitions set out by Euclid, it seems to me that some others should be added from Campanus and other writers; and immediately after these will follow postulates and common notions [axioms], especially those which both Euclid and his interpreters seem to use in demonstrating the properties of numbers.

XXIII.

A **number** is said to *measure* a number by that number which — multiplying it, or being multiplied by it — produces it.

For example, the number 4 is said to measure the number 12 by 3, because when multiplied by 3 it produces 12. And by the same reasoning the number 3 measures the same number 12 by 4, since from the multiplication of 4 by 3 the number 12 is produced. That this is so will become clear by the following reasoning: since the number 4 measures 12 by 3, 4 makes up 12 itself, taken as many times as there are units in 3. Therefore, by definition 15, the number 3 multiplying the number 4 will produce 12. But since, as we shall show in proposition 16 of this book, the same number is produced from multiplying 4 by 3 as from 3 by 4, it remains manifest that the same number 12 is generated if the number 4 multiplies the number 3.

This definition also fits non-whole numbers. For the number 2[?] is said to measure the number 13[?] by 5[?], because when multiplied by 5[?] it produces 13[?].

XXIV.

Proportion of numbers is a certain relation of one number to another, according to which it is a multiple of it, or a part or parts of it; or else contains it once or several times, together with some part of it or parts.

If the number 20 is compared with the number 4 in respect of that by which it is a multiple of it — namely fivefold — this relational comparison will be called a *proportion*. So too a proportion is the relation by which the same number 20 is compared with the number 60, according to which it is a third part of it; and so of the rest.

Since this is so, it is clear that four numbers are called proportional only when the first is the same multiple of the second as the third is of the fourth, or the same part, or the same parts; or else when the first equally contains the second, and the third the fourth, together with the same part of it or the same parts, as we taught above at definition 20.

XXV.

The **terms**, or *roots*, of a proportion are the two numbers than which no smaller can be taken in the same proportion.

XXVI.

When three numbers are proportional, the first is said to have to the third the **duplicate ratio** of that which it has to the second. And when four numbers are proportional, the first is said to have to the fourth the **triplicate ratio** of that which it has to the second; and so on continuously, one more each time, as long as the proportion continues.

This definition, as it applies to magnitudes, was fully explained at definition 10 of book 5. Since everything we wrote there can easily be transferred to numbers here, there is no reason to repeat it again.

XXVII.

Given any numbers set in order, the proportion of the first to the last is said to be **compounded** of the proportions of the first to the second, of the second to the third, of the third to the fourth, and so on, as long as the proportion continues.

That this is so we have demonstrated at length at definition 5 of book 6. Here too may be referred those definitions handed down in the fifth book concerning alternate ratio and inverse ratio; concerning the composition, division, and conversion of a ratio; concerning ratio *ex aequali* [ratio of equality]; and concerning ordered and perturbed proportion. For all these modes of argument that deal with proportions will be shown to apply also to the proportions of numbers in this seventh book.

Euclid, Book VII (Arithmetic) — Postulates, Axioms, and Propositions 1–2

(*Clavius's commentary on the Elements**. Enunciations of the postulates, axioms, and propositions are set as blockquotes; Clavius's demonstrations and scholia follow in ordinary prose. Lettered marginal references keyed to the superscript letters of a demonstration are given as [Margin: ...]; "pron." = pronuntiatum/axiom, "septi." = Euclid,* *Elements VII.*)

POSTULATES, OR REQUESTS [POSTVLATA, SIVE PETITIONES]

I.

Let it be postulated that, for any number, any number of numbers equal to it, or multiples of it, can be taken.

II.

That a greater number than any given number can be taken.

Clavius's note: For although a number cannot be diminished infinitely — rather, the diminution necessarily arrives at the indivisible unit — nevertheless it can be increased infinitely by the continual addition of a unit. Therefore, for any proposed number a greater one can be exhibited, namely the one that arises from the addition of a single unit, or even of several.

AXIOMS, OR PRONOUNCEMENTS [AXIOMATA, SIVE PRONVNTIATA]

I.

Numbers that are equal multiples of equal numbers, or of the same number, are equal to one another.

II.

Numbers of which the same number is an equal multiple, or of which equal numbers are equal multiples, are equal to one another.

III.

Numbers that are the same part, or the same parts, of equal numbers, or of the same number, are equal to one another.

IV.

Numbers of which the same number, or equal numbers, are the same part or the same parts, are equal to one another.

V.

The unit measures every number by the units that are in it — that is, by the very number itself.

Clavius's note: For the unit, taken as many times as there are units in the proposed number, constitutes that number. It therefore measures the number by the units contained in it — that is, by the very number made up of its own units.

VI.

Every number measures itself by the unit.

Clavius's note: Since any number taken once is equal to itself, it is clear that every number measures itself by the unit.

VII.

If a number, multiplying a number, produces some number, the multiplier measures the product by the multiplicand, and the multiplicand measures the same product by the multiplier.

Clavius's note: For let the number A, multiplying the number B, produce the number C. I say that A measures C by B, and that B measures the same C by A. For since, by definition 15, the number B — repeated as many times as there are units in A — constitutes C, it is clear that B measures C by A; and by the same reasoning A measures that same C by B, since B too, multiplying A, produces the number C, as will be demonstrated in Proposition 16 of this book.

Diagram. A number-line figure of dotted segments standing for numbers, illustrating Axiom VII.

- Upper line: labelled **A** at the left, then **B** partway along (two dotted numbers, A the multiplier and B the multiplicand).
- Lower line: labelled **C** at the left, a longer dotted segment (the product).

VIII.

If a number measures a number, then the number by which it measures will also measure that same number by the units that are in the measuring number — that is, by the measuring number itself.

Clavius's note: For example, since the number 6 measures the number 18 by 3, the number 3 will also measure that same number 18 by 6 — that is, by the units found in the measuring number 6. That this is so we shall confirm in the following way. Since 6 measures 18 by 3, the number 18 will result from the multiplication of 6 by 3, or of 3 by 6, by definition 23. Therefore, by the preceding axiom, the number 3 will measure the number 18 by 6.

IX.

If a number measuring a number multiplies the number by which it measures, or is multiplied by it, it will produce the number it measures.

Clavius's note: For let the number A measure the number C by B. I say that A, multiplying B — or being multiplied by B — produces C. For a number A is said to measure a number C by that number which, when A multiplies it (or by which A is multiplied), produces C, by definition 23. Since, then, A is supposed to measure C by B, it is clear that A, multiplying B (or being multiplied by it), produces C.

Diagram. A dotted-segment figure illustrating Axiom IX.

- Upper line: **A** at the left, then **B** partway along.
- Lower line: **C** at the left, a longer dotted segment.

X.

A number measuring any number of numbers also measures the number composed of them.

Clavius's note: Let the number A measure the numbers BC and CD. I say that the same A also measures BD, which is composed of them. For since A measures BC and CD, each of BC and CD will be a multiple of A. Therefore, when BC is divided into the parts BE and EC [recte: EC], equal to A, and CD into the parts CF, FG, GD, equal to the same A, the number BD — composed of all the parts BE, EC, CF, FG, GD, equal to A — will be a multiple of A. Therefore A measures BD. Which is the thing proposed.

Diagram. A dotted-segment figure illustrating Axiom X.

- Upper line: **A** (the measuring number), a short dotted segment.

- Lower line: the composite segment marked, in order, **B ... E ... C ... F ... G ... D** [final endpoint engraved ambiguously; the text's composition "BE, EC, CF, FG, GD" of the whole BD requires **D**]. BC is divided at E; CD is divided at F and G.

XI.

A number measuring any number also measures every number that the latter measures.

Clavius's note: Let the number A measure the number B, and B measure the number CD. I say that the same A also measures the number CD, which B measures. For since B measures CD, CD will be a multiple of B. Therefore, when CD is divided into the parts CE and ED, equal to B, A will measure the numbers CE and ED, since it is supposed to measure the number B, which is equal both to CE and to ED. Therefore the same A, by pronuntiatum 10, also measures the number CD, composed of CE and ED. Which is the thing proposed.

Diagram. A dotted-segment figure illustrating Axiom XI.

- First line: **A**, a short dotted segment.
- Second line: **B**, a longer dotted segment.
- Third line: **C ... E ... D**, with E dividing CD into CE and ED.

XII.

A number measuring the whole and the part taken away also measures the remainder.

Clavius's note: Let the number A measure the whole BC and the part taken away BD. I say that the same A also measures the remainder DC. For since A measures both BC and BD, each of BC and BD will be a multiple of A, or else ED [recte: BD] will be equal to A itself. Therefore, when both BC and BD are divided into parts equal to A, the remaining number DC will be either one part of the number BC equal to A, or several — and so DC will be either equal to A or a multiple of it. Therefore A measures DC. Which is the thing proposed.

Diagram. A dotted-segment figure illustrating Axiom XII, showing three cases of the segment BC with the point D at different positions.

- Top line: **A**, a short dotted segment.
- Line 1: **B ... D ... C** (D near C).
- Line 2: **B ... D ... C** (D closer to the middle).

- Line 3: **B ... D ... C** (D near B).

THEOREM 1. PROPOSITION 1. [THEOR. 1. PROPOS. 1]

If, two unequal numbers being set out, the lesser is continually subtracted from the greater in a certain alternating subtraction, and the remainder never measures the number preceding it until a unit is reached, then the numbers set out at the beginning are prime to one another.

Let two unequal numbers AB and CD be set out, of which the lesser, CD, is subtracted from the greater, AB, as many times as possible; and let the remainder EB be subtracted from CD as many times as possible; and the remainder FD from EB. And suppose that in this alternating subtraction the remainder never measures the preceding number, from which it was subtracted, until the subtraction arrives at the unit GB — which indeed measures the preceding number FD. I say that the numbers AB and CD are prime to one another; that is, that only the unit measures them as a common measure.

For if they are not said to be prime to one another, some number, other than the unit, will measure them as a common measure; let it be H. Now since H measures the number CD, and CD measures the number AE (because CD is either a part of AE or equal to it, seeing that when it was subtracted from AB it left the number EB), [Margin: (a) 11.pron.] H will also measure AE. But H measures the whole AB as well. [Margin: (b) 12.pron.] Therefore it also measures the remainder EB. Now EB measures CF. [Margin: (c) 11.pro.] Therefore H will measure CF as well; and consequently, since it also measures the whole CD, [Margin: (d) 12.pron.] it will also measure the remainder FD. Since, then, FD measures EG, [Margin: (e) 11.pron.] H will also measure FG. But H was also measuring the whole EB. [Margin: (f) 12.pron.] Therefore the number H will also measure the remaining unit GB — the part [measuring] the whole. Which is absurd. Therefore no number other than the unit will measure the numbers AB and CD; and so they are prime to one another. And so, if, two unequal numbers being set out, etc. Which was to be demonstrated.

Diagram. A dotted-segment figure for Proposition 1 (anthyphairesis).

- Line 1: **A ... E ... G ... B** (the greater number AB, with the successive remainder-points E and G, and G · B the terminal unit).
- Line 2: **C ... F ... D** (the lesser number CD, with F).
- Line 3: **H** — (a short solid segment: the supposed common measure).

SCHOLIUM

We shall convert this proposition, together with Campanus, in the following way:

If, two numbers prime to one another being set out, the lesser is always subtracted from the greater in a certain alternating subtraction, the remainder will never measure the preceding number until a unit is reached.

Let AB and CD be two numbers prime to one another, of which the lesser, CD, is subtracted from the greater, AB, as many times as possible; and the remainder EB from CD, as many times as possible; and the remainder FD from EB, leaving GB. I say that in this alternating subtraction the remainder never measures the preceding number, until a unit is reached.

For, if it were possible, let the remainder GB — before the subtraction reaches the unit — measure the preceding number FD taken away from EB. Since, then, the number GB measures the number FD, and FD measures EG, [Margin: (g) 11.pron.] GB will also measure EG. And since GB measures itself as well, [Margin: (h) 10.pron.] it will also measure the number EB, composed of EG and GB. But EB measures CF; [Margin: (i) 11.pron.] therefore GB will also measure that same CF. And further, since it is supposed to measure FD as well, [Margin: (k) 10.pro.] it will also measure CD, composed of CF and FD. But CD measures AE; [Margin: (l) 11.pron.] therefore it will also measure that same AE, as has been shown; [Margin: (m) 10.pron.] and it will also measure AB, composed of both AE and EB. Therefore, since the number GB measures both numbers AB and CD, they will be composite to one another. Which is absurd, since they are supposed to be prime to one another. Therefore no remaining number will ever measure the preceding one, until a unit is reached. Which is the thing proposed.

In the same way this too is true:

If, two numbers composite to one another being set out, the lesser is always subtracted from the greater in a certain alternating subtraction, the subtraction will not reach the unit, but [will reach] a number that measures the preceding subtracted number.

Clavius's note: For if a subtraction of this kind reached all the way to the unit, the numbers set out would be prime to one another, [Margin: (n) 1.septi. — Euclid, *Elements* VII.1] as Euclid demonstrated. Which is absurd, since they are supposed to be composite to one another.

From these things we shall easily discern whether any two given numbers are prime to one another or not. For, when the lesser is always subtracted from the greater in a certain alternating subtraction, if the remainder never measures the preceding number until a unit is reached, they will be prime to one another, [Margin: (o) 1.septi. — Euclid, *Elements* VII.1] as Euclid demonstrated. But if some remaining number does measure the preceding one, they will be composite to one another, since that same remaining number measures both of the given numbers, as is clear from the earlier demonstration of this scholium. For from the fact that the remaining number GB was said to measure the preceding number FD, it was shown that this same remainder GB measures both AB and CD.

PROBLEM 1. PROPOSITION 2. [PROBL. 1. PROPOS. 2] [Margin: ii.]

Given two numbers not prime to one another, to find their greatest common measure.

Let two numbers AB and CD, not prime to one another, be given, whose greatest common measure is required. Let the lesser, CD, be subtracted from the greater, AB, as many times as possible, leaving the number EB; let EB, subtracted from CD, leave FD; and so on in turn, the lesser always subtracted from the greater in a certain alternating subtraction. In this process one will arrive at a number that measures the one preceding it. For if one arrived at the unit, [Margin: (a) 1.septi. — Euclid, *Elements* VII.1] the numbers AB and CD would be prime to one another, which is contrary to the hypothesis. Suppose, then, that we have now arrived at the remaining number FD, which, subtracted from EB, leaves nothing but measures it. I say that FD is the greatest common measure of the numbers AB and CD.

That it measures both, we shall show as follows. Since FD measures EB, and EB measures CF, [Margin: (b) 11.pro.] FD will also measure CF; and since it also measures itself, [Margin: (c) 10.pro.] it will also measure the whole CD, composed of CF and FD. But CD measures AE. [Margin: (d) 11.pron.] Therefore FD will also measure that same AE. And since FD measures EB as well, [Margin: (e) 10.pron.] it will also measure the whole AB, composed of both AE and EB. Therefore FD measures both numbers AB and CD.

That FD is their greatest common measure, we shall prove thus. If it were possible, let there be a common measure G greater than FD. Since, then, G measures both AB and CD, and CD measures AE, [Margin: (f) 11.pron.] G will also measure AE, [Margin: (g) 12.pron.] and therefore the remainder EB as well. But

EB measures CF; [Margin: (h) 11.pron.] therefore G will also measure that same CF, [Margin: (i) 12.pron.] and therefore the remainder FD as well — the greater measuring the smaller. Which is absurd. Therefore no number greater than FD measures the numbers AB and CD; and so FD is the greatest measure of the numbers AB and CD.

But if the lesser number CD measures the greater AB — so that, subtracted from AB, it leaves nothing — then CD itself will be the greatest common measure of both, since it also measures itself, as appears in this figure. Therefore, given two numbers not prime to one another, etc. Which was to be done.

Diagram. The principal dotted-segment figure for Proposition 2.

- Line 1: A ... E ... B (the greater number AB, with the remainder-point E).
- Line 2: C ... F ... D (the lesser number CD, with F).
- Line 3: G — (a short solid segment: the supposed greater common measure).

Diagram. A second dotted-segment figure for Proposition 2, for the case in which the lesser number measures the greater directly.

- Line 1: A ... B (a dotted segment).
- Line 2: C ... D (a shorter dotted segment measuring the first).

COROLLARY [COROLLARIUM]

From this it is evident that a number measuring two numbers also measures their greatest common measure.

Clavius's note: This is drawn from that part of the demonstration in which it was shown that FD is the greatest measure of AB and CD. For it was demonstrated there that the number G, if it measures the numbers AB and CD, also measures the number FD, their greatest common measure. And the same reasoning holds for the rest.

SCHOLIUM

From what has been said, we can easily test, together with Campanus, whether any given numbers are prime to one another or not. For let there be three numbers A, B, C. First, then, I test — by the method we taught at Proposition 1 — whether the two A and B are prime to one another. If they are prime to one another, the three numbers A, B, C will not be composite among themselves, since

they can have no common measure other than the unit, on account of the numbers A and B being prime to one another.

Diagram. A dotted-segment figure of three numbers for the scholium.

- Line 1: **A**, a dotted segment.
- Line 2: **B**, a shorter dotted segment.
- Line 3: **C**, a shorter dotted segment still.

*[Continues Euclid **Book VII** (arithmetic). This opening continues the scholium begun on the previous page — the extension of the "prime-to-one-another" test to three numbers.]*

but if A and B are composite to one another, [Margin a: 2.septi. — Elements VII.2] let their greatest common measure D have already been found. If this D also measures the number C, it is clear that the three numbers A, B, C are composite to one another, since they have the number D as a common measure.

A
 B
 C
 D ...

But if D, the greatest measure of A and B, does not measure C, then C and D are either prime to one another or not. If they are prime to one another, the three numbers A, B, C will not be composite to one another, but prime to one another. For suppose they were said to be composite to one another, so as to have a common measure: that common measure would then measure the number D, the greatest measure of A and B, by the corollary of this proposition; and therefore, since that same measure also measures C, the numbers C and D would not be prime to one another — which is contrary to the hypothesis.

A
 B
 C
 D ...

But if C and D are not prime to one another, then the three numbers A, B, C are composite to one another. [Margin b: 2.septi. — Elements VII.2] For once the greatest common measure of C and D — call it E — has been found: since E measures D, and D in turn measures A and B, [Margin c: 11.pron. — Axiom 11 of this book] E too will measure those same A and B. Therefore, since this same E likewise measures C, E measures the three numbers A, B, C; and so they are composite to one another. Which is what was proposed.

A
 B
 C
 D E ..

By a similar method we shall test whether more than three numbers are prime to one another, or rather composite to one another. For if the given numbers are four, we must first carry out the test on three of them; if five, on four; and so on. The remaining steps are to be completed just as we described for three given numbers.

PROBL. 2. PROPOS. 3.

[Margin: iii.]

Given three numbers not prime to one another, to find their greatest common measure.

Let three numbers A, B, C be given, not prime to one another, whose greatest common measure is to be found. Let D be the greatest measure of A and B. If, then, D also measures C, it is clear that D is the greatest measure of the given numbers A, B, C. For if some greater number were said to measure A, B, C, that same number would — by the corollary of Proposition 2 of this book — measure D, the greatest measure of A and B: a greater number measuring a lesser, which is absurd.

But if D does not measure C, then at least D and C are composite to one another. For since A, B, C are composite to one another, any common measure of theirs will — by the corollary of Proposition 2 of this book — measure D, the greatest measure of A and B; and since that same measure also measures C, D and C are composite to one another. [Margin a: 2.septi. — Elements VII.2] Let their greatest measure be E. I say that E is the greatest common measure of the given numbers A, B, C. That it is a common measure of theirs is shown as follows. Since E measures the numbers C and D, and D measures A and B, [Margin b: 11.pro. — Axiom 11 of this book] E too will measure A and B; and so it will measure the three numbers A, B, C.

A
 B D
 C E .. F—

That E is their *greatest* common measure is also clear. For, if it were possible, let F be a common measure of theirs greater than E. Since, then, F measures A and B, it will also — by the corollary of Proposition 2 of this book — measure D, their

greatest common measure; and it also measures C. Therefore F, measuring D and C, will — by the same corollary — measure E, their greatest measure: a greater number measuring a lesser, which is absurd. So no number greater than E measures A, B, C; and therefore E is their greatest measure. And so, given three numbers not prime to one another, and so forth. Which was to be done.

COROLLARIVM.

From this it is clear that a number which measures three numbers also measures their greatest common measure.

This is also gathered from the last part of the demonstration. For it was shown there that the number F, if it measures A, B, C, also measures E, their greatest common measure; and the same reasoning holds in all other cases.

By like reasoning, when more than three numbers are given, not prime to one another, their greatest common measure will be found, and this same corollary will hold. For if the given numbers are four, the greatest common measure of three of them must first be found; if five, the greatest common measure of four must first be taken; and so on. All the rest is to be carried out as was said for three numbers.

THEOR. 2. PROPOS. 4.

[Margin: iiii.]

Every number is, of every number — the less of the greater — either a part or parts.

Let there be two numbers, A the less and B the greater. I say that A is either a part or parts of B. First, let A and B be prime to one another. Since, then, every unit of the number A is a part of the number B, it is clear that the number A is parts of the number B — namely, as many parts as there are units in A.

A
B

Next, let A and B not be prime to one another, but composite to one another, and let A measure B. On this supposition it is manifest that A is a part of the number B, by Definition 3 of this book.

A
B

**

But now let A not measure B. [Margin a: 2.septi. — Elements VII.2] Let their greatest common measure be C, and let the number A be divided into the parts AD, DE, EF, each of them equal to C. [Margin b: 3.defin. — Definition 3 of this book] Since, then, C is a part of B (because it measures it), AD too will be a part of that same B; and likewise DE and EF. And therefore the whole number A is parts of the number B — namely, as many as the number of times C is contained in AF. Every number, then, is of every number — the less of the greater — either a part or parts. Which was to be demonstrated.

A .. D .. E .. F
 B
 C ..

THEOR. 3. PROPOS. 5.

[Margin: v.]

If a number be a part of a number, and another the same part of another: then also the two together will be the same part of the two together as the one is of the one.

Let the number A be the same part of the number BC as the number D is of the number EF. I say that both numbers A and D together are the same part of both numbers BC and EF together as A is of BC, or D is of EF. For let the numbers BC, EF be divided into the parts BG, GC; EH, HF, equal to A and D. Then the multitude of the parts of the number BC will be equal to the multitude of the parts of the number EF, because A is the same part of BC as D is of EF. Since, then, A and BG are equal, if there be added to them the equals D and EH, A and D together will be equal to BG and EH together; and by the same argument A and D together will be equal to GC and HF together. And so on in turn: if there be more parts in BC, EF, the sum of the numbers A and D will be equal to as many aggregates of the parts of the numbers BC and EF as the number of times A is contained in BC, or D in EF. And therefore both A and D together are the same part of both BC and EF together as A is of BC, or D is of EF. If, then, a number be a part of a number, and so forth. Which was to be shown.

A D
 B G C E H F

SCHOLIVM.

The same holds true in fractional numbers, and precisely the same demonstration will apply. This appears in the following example, where the

number A is the same part of the number BC, and the number D of the number EF; and accordingly both A and D together will be shown to be the same part of both BC and EF together as A is of BC — provided, of course, that BC and EF are divided into parts BG, GC, EH, HF equal to A and D, and so on.

A, $\frac{2}{7}$ D, $\frac{2}{7}$

B, $\frac{2}{7}$ G, $\frac{2}{7}$ C E, $\frac{2}{7}$ H, $\frac{2}{7}$ F

Now if it should ever happen that a fractional number cannot be divided into the proposed equal parts, because its numerator cannot be split into those parts, then both numerator and denominator must be multiplied by the number of parts. For by this method there will be produced another fraction equivalent to the first, and one whose numerator can be divided into the proposed parts. Thus, if the fraction $\frac{5}{9}$ is to be cut into two equal parts, each of its numbers is to be multiplied by 2; if into three, by 3; if into four, by 4; and so on. In this way there are generated the fractions $\frac{10}{18}$, $\frac{15}{27}$ [?], $\frac{20}{36}$ — of which the first will be cut into these two equal parts, $\frac{5}{18}$, $\frac{5}{18}$; the second into these three, $\frac{5}{27}$, $\frac{5}{27}$, $\frac{5}{27}$ [?]; and the third into these four, $\frac{5}{36}$, $\frac{5}{36}$, $\frac{5}{36}$, $\frac{5}{36}$.

Again, when there are whole numbers present, they must first be reduced together with the fraction to a single fraction; then, in the same way, both numerator and denominator must be multiplied by the number of parts, and so on. Thus, if the number $4\frac{2}{7}$ is to be cut into three parts, we first reduce it to this single fraction, $\frac{30}{7}$; then we multiply each of its numbers by 3, so that there results the fraction $\frac{90}{21}$, of which the three equal parts are $\frac{30}{21}$, $\frac{30}{21}$, $\frac{30}{21}$. And these same rules are to be observed in the following propositions too, whenever they are applied to fractions; and by the name "fractions" whole numbers combined with fractions are always to be understood as well, and likewise the cases where some are fractional and others whole.

In the same way we shall demonstrate what follows.

If a unit be a part of a number, and another unit, or a number, the same part of another number: then also both the units together, or the unit and the number together, will be the same part of both numbers together as the unit is of the number.

This appears clearly in the examples set out here, for the demonstration is exactly the same.

A. D. A. D...

B. G. C E. H. F B. G. C E... H... F

We can also extend this proposition to any number of numbers, in this way.

If there be any number of numbers, equal in multitude to any number of others, each of each the same part: then also all of them together will be the same part of all together as the one is of the one.

Let the numbers A, B, C be, each of each, the same part of the numbers DE, FG, HI. I say that all the numbers A, B, C together are the same part of all the numbers DE, FG, HI together as A is of DE. For let the numbers DE, FG, HI be divided into parts equal to A, B, C.

A B . . . C . .
D . . . K . . . E F . . . L . . . G H . . M . . I

Then the multitude of the parts of the number DE will be equal to the multitude of the parts both of the number FG and of the number HI. Since, then, A and DK are equal, if there be added to them the equals B and FL, A and B together will be equal to DK and FL together; and if to these there be again added the equals C and HM, then A, B, C together will also be equal to DK, FL, HM together. By like reasoning A, B, C will be equal to KE, LG, MI together; and so on in turn. If there be more parts in DE, FG, HI, the sum of the numbers A, B, C will be equal to as many aggregates of the parts of the numbers DE, FG, HI as the number of times A is contained in DE. Therefore A, B, C together are the same part of DE, FG, HI together as A is of DE.

**

The same follows if, in place of one of the numbers A, B, C, a unit be taken, or in place of several, or even of all of them, several units, as was said in the case of two. This is what the following figures indicate.

A. B. . C. . .
D.K.E. F. . L. . G H. . . M. . . I

A. B. C.
D.K.E F.L.G H.M.I

These things hold also for fractional numbers. For if any number of fractional numbers be, each of each, the same part of any number of fractional numbers equal in multitude, then all together will be the same part of all together as the one is of the one. This will be demonstrated in the same way, even if some of the numbers are whole numbers or units, as may be seen here.

A, 1 B, $\frac{1}{2}$ C, $\frac{1}{3}$ [?]
D 1 K 1 E F $\frac{1}{2}$ L $\frac{1}{2}$ G H $\frac{1}{3}$ M $\frac{1}{3}$ I

We shall also propose something similar to the first theorem of the fifth book, in this manner.

If there be any number of numbers, equal in multitude to any number of others, each of each equimultiples: as one number is a multiple of one, so will all together be of all together.

The demonstration is the same here as in Book 5. Nevertheless we shall also demonstrate it otherwise, thus. Let A, B, C be numbers that are equimultiples, each of each, of the numbers D, E, F. I say that all of A, B, C together are the same multiple of all of D, E, F together as A is of D. For since A is the same multiple of D as B is of E, and C of F, then conversely D will be the same part of A as E is of B, and F of C. Therefore, by what we have just demonstrated, D, E, F together will be the same part of A, B, C together as D is of A; and accordingly, conversely, all of A, B, C together are the same multiple of all of D, E, F together as A is of D.

A B C
D E ... F ..

If the numbers A, B, C are fractional, and equimultiples of their fractional numbers D, E, F, then, since the multiple of one is one, all together are also multiples of all together — as is clear from the demonstration.

But if, in place of one of the numbers D, E, F, a unit be assumed — or, in place of several or of all, several units — the theorem will be demonstrated in the same way, as appears from the following figures.

A .. B C A .. B .. C ..
D · E .. F ... D · E · F ·

THEOR. 4. PROPOS. 6.

[Margin: vi.]

If a number be parts of a number, and another the same parts of another: then also the two together will be the same parts of the two together as the one is of the one.

Let the number AB be the same parts of the number C as the number DE is of the number F. I say that both numbers AB and DE together are the same parts of both numbers C and F together as AB is of C, or DE is of F. For let the numbers AB, DE be divided into parts AG, GB; DH, HE, that are parts of the numbers C and F. Then the multitude of parts in the number AB will be equal to the multitude of parts in the number DE, because AB is the same parts of the number C as DE is of the

number F. Since, then, the part that AG is of the number C, that same part is DH of the number F, [Margin a: 5.septi. — Elements VII.5] both AG and DH together will be the same part of both C and F together as AG is of C, or DH is of F. By the same reasoning, both GB and HE together will be the same part of both C and F together as GB is of C, or HE is of F. And so on in turn: if there be more parts in AB, DE, there will be as many aggregates of parts contained in the numbers AB, DE, each of which is the same part of the numbers C, F together as AG is of C, as there are equal parts that the number AB is of C, or DE is of F. And accordingly both numbers AB and DE together are the same parts of both numbers C and F together as AB is of C, or DE is of F. If, then, a number be parts of a number, and so forth. Which was to be demonstrated.

A ... G ... B D H E
C F

SCHOLIVM.

This same proposition holds in fractional numbers, together with its demonstration, as this example shows.

A, $\frac{1}{5}$ G, $\frac{1}{5}$ B D, $\frac{2}{7}$ H, $\frac{2}{7}$ E
C, $\frac{3}{5}$ F, $\frac{6}{7}$

But we shall also amplify this proposition, extending it to any number of numbers, whether whole or fractional, in this way.

If there be any number of numbers, of any number of numbers, each of each the same parts: then also all together will be the same parts of all together as the one is of the one.

For the demonstration is the same, if only, in place of the fifth proposition, we assume what we demonstrated in the second place in the preceding scholium; and here it is clear.

A .. K .. B C ... L ... D E M F
G G I

THEOR. 5. PROPOS. 7.

[Margin: vii.]

If a number be a part of a number, such as a subtracted number is of a subtracted number: then the remainder will be the same part of the remainder as the whole is of the whole.

Book VII

THEOR. 5. PROPOS. 7. (continued) [Margin: vii.]

Demonstration. Let the number AB be the same part of the number CD that the subtracted AE is of the subtracted CF. I say that the remainder EB is the same part of the remainder FD that the whole AB is of the whole CD.

For let EB be made the same part of the number GC that AE is of CF — that is, the same part that the whole AB is of the whole CD. Now since AE is the same part of CF that EB is of GC, [Margin: a — Elements VII.15] the two together, AE and EB, are the same part of the two together, CF and GC, that AE is of CF — that is, that the whole AB is of the whole CD. And therefore, since AB is the same part of each of the whole segment FG (that is, of CF together with GC) and of CD, the segments FG and CD are equal to one another. So when the common CF is removed, the remainders GC and FD are left equal. Therefore EB is the same part of FD that it is of GC — that is, that the whole AB is of the whole CD. Therefore, if a number is a part of a number, and so on. Which was to be shown.

A.... E... B G.... C..... F.... [Margin: a — Elements VII.15]

SCHOLIUM

This proposition, together with its demonstration, holds also for fractional numbers, as is clear here:

A, $\frac{3}{8}$. E, $\frac{2}{5}$ [?]. B, G, $\frac{4}{5}$. C, $\frac{6}{8}$. F, $\frac{4}{5}$ [?]. D,

This same theorem is true both in whole numbers and in fractions, even when the subtracted amount AE is a unit, or when the remainder EB is a unit, or finally when both the subtracted amount and the remainder in whole numbers are units — as is clear from these examples:

A. E... B | A... E. B | A. F. B G.... C.. F..... D | G.. C..... F.. D | G.. C..
F. D

Moreover, from this we shall demonstrate something similar to the fifth theorem of Book V, both in whole numbers and in fractions:

If a number is the same multiple of a number as a subtracted number is of a subtracted number, then the remainder also will be the same multiple of the remainder that the whole is of the whole.

This we shall show in exactly the same way as that theorem of Book V. But we shall confirm the same result from what has already been demonstrated, as

follows. Let the whole AB be the same multiple of the whole CD as the subtracted AE is of the subtracted CF. I say that the remainder EB is the same multiple of the remainder FD that the whole AB is of the whole CD. For since AB and AE are, conversely, the same part of CF that the subtracted CF is of the subtracted AE — [Margin: b — Elements VII.7] wherefore the remainder FD is also the same part of the remainder EB that the whole CD is of the whole AB. And accordingly, conversely, EB is the same multiple of FD that AB is of CD, which is a multiple.

A..... E.... B C... F.. D [Margin: b — Elements VII.7]

And if from CD a unit CF is subtracted, or if the remainder FD is a unit, or finally if both in whole numbers the subtracted amount CF is a unit and the other remainder FD is a unit, the same is demonstrated, as these examples show together:

A... E..... | A.... E.. B | A.... E.... B C. F.. D | C.. F. D | C. F. D

THEOR. 6. PROPOS. 8. [Margin: viij.]

If a number is the same parts of a number as a subtracted number is of a subtracted number, then the remainder will be the same parts of the remainder that the whole is of the whole.

Demonstration. Let the number AB be the same parts of the number CD that the subtracted AE is of the subtracted CF. I say that the remainder EB is the same parts of the remainder FD that the whole AB is of the whole CD. For let a number GH be taken equal to AB; then GH will be the same parts of CD that AE is of CF. So let GH be divided into the parts GI, IH that measure CD, and AE into the parts AK, KE that measure CF; then the multitude of the parts GI, IH will be equal to the multitude of the parts AK, KE, and each of GI and IH will be the same part of CD that each of AK and KE is of CF.

Now since CD is a greater number than CF, each of GI and IH — being a part of CD — will be greater than each of AK and KE, which is a part of CF. So let numbers GL, IM be taken equal to AK, KE; then GL will be the same part of CF that AK is of the same CF — that is, the part that GI is of CD. And accordingly, since the whole GI is the same part of the whole CD that the subtracted GL is of the subtracted CF, [Margin: a — Elements VII.7] the remainder LI will also be the same part of the remainder FD that the whole GI is of the whole CD. By the same argument we shall show that MH is the same part of FD that the whole IH is of CD.

A..... K..... E.... B C..... F..... D G..... L.. I..... M.. H
[Margin: a — Elements VII.7]

Since, therefore, each of GI and IH is the same part of FD, both GI and IH together will be the same parts of FD that both LI and MH are; the same parts of CD, that is, that AB is of the same CD, on account of the equality of the numbers AB and GH. Therefore both LI and MH together will also be the same parts of FD that AB is of CD. But since, if from the equals AB and GH the equals AK, KE and GL, IM are taken away, the remainders EB and LI, MH together are equal, the remainder EB will also be the same parts of the remainder FD that the whole AB is of the whole CD — namely, the same parts that LI and MH together were of the same FD. Therefore, if a number is the same parts of a number, and so on. Which was to be shown.

SCHOLIUM

In fractional numbers this same proposition will be demonstrated in exactly the same way, as is here evident:

A, $\frac{2}{19}$. K, $\frac{2}{19}$. E, $\frac{6}{19}$. B C, $\frac{6}{19}$. F, $\frac{9}{19}$. D G, $\frac{2}{19}$. L, $\frac{3}{19}$. I, $\frac{2}{19}$ [?].
M, $\frac{3}{19}$. H

Euclid did not demonstrate this proposition in the way he did the preceding one — as some interpreters do — because here it was not established that the remaining number EB is the same parts of some number that the number AE is of the number CF, [p. 320] whereas there it was evident that the remainder EB is the same part of some number that AE is of CF. For one may take the double of EB, or the triple, quadruple, and so on, until EB becomes a submultiple of the assumed number GC in the same way that AE is a submultiple of CF.

THEOR. 7. PROPOS. 9. [Margin: ix.]

If a number is a part of a number, and another the same part of another; then, conversely, whatever part or parts the first is of the third, the second will be the same part or parts of the fourth.

Demonstration. Let the number A be the same part of the number D that the number BC is of the number EF, and let A, BC be smaller than D, EF, each than each — for the proposition is to be understood of these. I say that, conversely, A is the same part, or the same parts, of D that BC is of EF. For when the numbers BC, EF are divided into the parts BG, GC and EH, HF equal to A and D respectively, the multitude of the parts of BC will be equal to the multitude of the parts of EF. And since BG, GC are equal to one another and smaller than EH, HF (which are likewise equal to one another) — because the whole BC is supposed smaller than the whole EF — BG will be the same part or parts of EH that GC is of HF. [Margin: a — Elements VII.5 or 6] Wherefore both BG, GC together, namely BC, will be the

same part or parts, second of the two, that EH, HF together, namely EF, are; that is, the same that A, the first, is of D, the third. Therefore, if a number is a part of a number, and so on. Which was to be demonstrated.

A.... B.... G.... C D..... E..... H..... F [Margin: a — Elements VII.5 or 6]

SCHOLIUM

In fractional numbers this proposition, together with its demonstration, also holds good, as is here evident:

A, $\frac{3}{11}$. B, $\frac{3}{11}$. G, $\frac{3}{11}$. C D, $\frac{14}{22}$. E, $\frac{14}{22}$. H, $\frac{14}{22}$. F

And if in place of the first number a unit is taken, which is a part of some number, of which another number is the other part: then, conversely, the unit will be the same part of the third number that the second number is of the fourth. This is confirmed by the same argument, if in place of the parts we assume a part in the demonstration, as is clear from this example:

A. B. G. C D..... E..... H..... F

THEOR. 8. PROPOS. 10. [Margin: x.]

If a number is the same parts of a number, and another the same parts of another; then, conversely, whatever parts or part the first is of the third, the second also will be the same parts or part of the fourth.

Demonstration. Let the number AB be the same parts of the number C that the number DE is of the number F, and let AB, C be smaller than DE, F, each than each — for this proposition too is to be understood of these, just as the preceding one. I say that, conversely, the number AB is the same parts, or the same part, of the number DE that the number C is of the number F. For when the numbers AB, DE are divided into the parts AG, GB and DH, HE, equal to the numbers C and F, the multitude of the parts in AB will be equal to the multitude of the parts in DE, and each of AG and GB will be the same part of C that each of DH and HE is of F. [Margin: a — Elements VII.9] Conversely, therefore, AG will be the same part, or the same parts, of DH that GB is of HE. [Margin: b — Elements VII.5 or 6]

Therefore both AG, GB together, namely the whole AB, will be the same part or parts of both DH, HE together, namely DE, the third; that is, the same that C, the second, is of F, the fourth. Therefore, if a number is the same parts of a number, and the other the same parts of the other, and so on. Which was to be demonstrated.

A... G... B C..... D..... H..... E F..... [Margin: a — Elements VII.9;
b — Elements VII.5 or 6]

SCHOLIUM

The demonstration of this proposition is suited to fractional numbers as well, as appears in this example:

A, $\frac{2}{15}$. G, $\frac{2}{15}$. B C, $\frac{6}{15}$. D, $\frac{3}{10}$. H, $\frac{3}{10}$. E F, $\frac{9}{10}$.

THEOR. 9. PROPOS. 11. [Margin: xii. — see note (a)]

If the whole is to the whole as the subtracted is to the subtracted, then the remainder will be to the remainder as the whole is to the whole.

Demonstration. Let the whole number AB be to the whole CD as the subtracted AE is to the subtracted CF. I say that the remainder EB is to the remainder FD as the whole AB is to the whole CD. For since AB is to CD as AE is to CF, [Margin: a — Elements VII, def. 20] AB will be the same multiple of CD that AE is of CF, or the same part, or the same parts; or else AB will measure CD, and AE will measure CF, an equal number of times, and moreover will contain the same part or the same parts of it in addition.

Let AB first be a multiple of CD. On this supposition, conversely, CD will be the same part of the whole AB that the subtracted CF is of the subtracted AE, on account of AB, AE being equal multiples of CD, CF. [Margin: b — Elements VII.7] Therefore the remainder FD also is the same part of the remainder EB that the whole CD is of the whole AB; and accordingly, conversely, AB is to CD as EB is to FD, an equal multiple. [Margin: c — Elements VII, def. 20] Therefore the whole AB is to the whole CD as EB, the remainder, is to the remainder FD.

A..... E..... B C..... F... D [Margin: a — Elements VII, def. 20]

[p. 321]

Next, let AB be the same part, or the same parts, of CD that AE is of CF. On this supposition [Margin: d — Elements VII.7 or 8] the remainder EB also will be the same part, or the same parts, of the remainder FD that the whole AB is of the whole CD. And accordingly, it will be that as the whole AB is to the whole CD, so is EB, the remainder, to the remainder FD. [Margin: c — Elements VII, def. 20]

A..... E... B | A.... E.. B C..... F... D | C..... F... D [Margin: d — Elements VII.7 or 8; c — Elements VII, def. 20]

Finally, let AB contain CD a third time, and AE contain CF an equal number of times, and moreover the same part or the same parts of it. On this supposition, conversely, CD will contain the whole AB the same number of times, and the same parts, that the subtracted CF contains the subtracted AE — as we shall show presently. [Margin: f — Elements VII.8] Therefore the remainder FD also will be the same parts of the remainder EB that the whole CD is of the whole AB; and accordingly, conversely, AB will contain CD, and EB will contain FD, an equal number of times, together with the same part or parts of it in addition. [Margin: g — Elements VII.8]

A..... E..... B C..... F..... D

Therefore FD, the remainder, will be the same parts of the remainder EB that the whole CD is of the whole AB; and accordingly, it will be that as the whole AB is to the whole CD, so is EB, the remainder, to the remainder FD.

And if AB is equal to the whole CD, and the subtracted AE to the subtracted CF, it is evident that the remainder EB is equal to the remainder FD.

A..... E... B | C..... F... D

For if from equals equals are taken away, the remainders are equal. And so, if it is that as the whole is to the whole, so the subtracted is to the subtracted, and so on. Which was to be demonstrated.

SCHOLIUM

This is demonstrated in the same way in fractional numbers. This is plainly established in these examples, which correspond to the examples of the third case of the demonstration:

A, $\frac{6}{8}$. E, $\frac{1}{10}$ [?]. B | A, $\frac{14}{21}$ [?]. E, $\frac{28}{29}$ [?]. B C, $\frac{4}{9}$. F, $\frac{2}{10}$ [?]. C | C, $\frac{7}{21}$ [?]. F, $\frac{10}{29}$ [?]. D

LEMMA

But that, if AB contains CD, and AE contains CF, an equal number of times, together with the same part or parts of it in addition, then conversely CD is the same parts of AB, and CF the same parts of AE; and that, if CD is the same parts of AB, and CF the same parts of AE, then conversely AB contains CD, and AE contains CF, an equal number of times, together with the same part or parts of it in addition — this, both in fractional numbers and in whole numbers, we shall demonstrate as follows.

Let AB first contain CD, and AE contain CF, an equal number of times — namely once, or twice, or three times, and so on — and moreover the same part in addition, namely GB of CD, and HE of CF; so that the remaining numbers AG, AH are either equal to CD, CF themselves, or equal multiples of them. So when the numbers CD, CF are divided into the parts CI, IK, KD and CL, LM, MF equal to GB, HE, the multitude of the parts of the number CD will be equal to the multitude of the parts of the number CF (because GB is the same part of CD that HE is of CF). Similarly, when the numbers AG, AH are divided into the parts AN, NO, OG and AP, PQ, QH equal to the same GB, HE, the multitude of the parts of the number AG will also be equal to the multitude of the parts of the number AH.

A.. N.. O.. G.. B C.. I.. K.. D A.. P.. Q.. H.. E C.. L.. M.. F

For since AG, AH are either equal to CD, CF themselves, or equal multiples of them, there will be either just as many parts in AG, AH as in CD, CF; or else the number of the parts of the number CD will be contained in AG as many times as the number of the parts of the number CF is contained in AH; and on that account the multitude of the parts of the number AG will be equal to the multitude of the parts of the number AH. If, therefore, the parts GB, HE are added to these, the multitude of the parts of the number AB will also be equal to the multitude of the parts of the number AE. And so one part of the number CD is the same part of the number AB that one part of the number CF is of the number AE. Therefore, since the multitude of the parts of the number CD is equal to the multitude of the parts of the number CF, CD will be the same parts of the number AB that CF is of the same AE.

Next, let AB contain CD, and AE contain CF, an equal number of times — namely once, twice, three times, four times, and so on — and moreover the same parts of it in addition: the number AB the parts GB of the number CD, and the number AE the parts HE of the number CF; so that the remaining numbers AG, AH are, again, either equal to CD, CF themselves, or equal multiples of them. So when the numbers GB, HE are divided into the parts GI, IB and HK, KE, of the numbers CD, CF, the multitude of the parts of GB will be equal to the multitude of the parts of HE. In like manner, when the numbers CD, CF are divided into the parts CL, LM, MD and CN, NO, OF, equal to the parts GI, IB and HK, KE, [it will follow that these multitudes too are equal,] because each of the parts of the number GB is the same part of the number CD that each of the parts of the number HE is of the number CF. Finally, when the numbers AG, AH are divided into the parts AP, PQ, QG and AR, RS, SH, equal to the same parts GI, IB and HK, KE, the multitude of the parts of the number AG will also be equal to the multitude of the parts of the number AH.

A.. P.. Q.. G.. I.. B C.. L.. M.. D A.. R.. S.. H.. K.. E C.. N.. O.. F

For since AG, AH are either equal to CD, CF themselves, or equal multiples of them, there will be either just as many parts in AG, AH as in CD, CF; or else the number of the parts of the number CD will be contained in AG as many times as the number of the parts of the number CF is contained in AH; and on that account the multitude of the parts of the number AG will be equal to the multitude of the parts of the number AH. If to these are added the equal multitudes of the parts of the numbers GB, HE, there will also be a multi-[tude]

(Continuation of the LEMMA begun in the previous section — the second case, on the equivalence of "parts" with "containing an equal number of times, plus a part or parts," in whole and fractional numbers.)

[multi]tude of the parts of the number AB is equal to the multitude of the parts of the number AE. And thus one part of the number CD is the same part of the number AB as one part of the number CF is of the number AE. Therefore, since the multitude of the parts of the number CD is equal to the multitude of the parts of the number CF, CD will be the same parts of the number AB as CF is of AE. Which was the first thing proposed.

Now let CD be the same parts of AB, and CF the same parts of AE. I say, conversely, that AB contains CD, and AE contains CF, an equal number of times, and moreover the same part or parts of it. For when the numbers CD, CF are divided into parts equal to the numbers AB, AE, these parts will be equal to one another in multitude. Likewise, when the numbers AB, AE are divided into parts equal to the parts of the numbers CD, CF, these too will be equal to one another in multitude. Therefore all the parts of the number CD will be contained just as many times in AB — with the same part or parts of CD left over — as all the parts of the number CF are contained in AE, with whatever part or parts of CF remaining, because of the equal multitudes of the parts of the numbers CD, CF and AB, AE. For it so happens that the equal multitudes of the parts of the numbers AB, AE contain an equal number of times the equal multitudes of the parts of the numbers CD, CF, and moreover that in those two numbers there remain over parts of the numbers CD, CF equal in multitude. For this reason AB will contain CD, and AE will contain CF, an equal number of times, and moreover the same part or parts of it. Which was the second thing proposed.

[Margin: xiiij.]

THEOR. 10. PROPOS. 12.

If there be any number of proportional numbers, then as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents.

Let there be any number of proportional numbers A, B; C, D; E, F — that is, as A is to B, so C is to D, and E to F. I say that all of A, C, E together are also to all of B, D, F together as A is to B.

First let A, C, E be less than B, D, F. Since, then, by the same proportion, A is the same part (or parts) of B as C is of D and E is of F [Margin: a — Elem. VII.def.20], both A and C together will also be the same part (or parts) of both B and D together as A is of B, or as E is of F [Margin: b — Elem. VII.9 or 10]. Again, since both A and C together, taken as one, are the same part (or parts) of both B and D together, taken as one, as E is of F [Margin: c — Elem. VII.5], both A and C taken as one, together with E, will be the same part (or parts) of B and D taken as one, together with F, as A is of B [Margin: d — Elem. VII.def.20]. Therefore the ratio of all A, C, E together to all B, D, F together is the same as that of A to B.

Number-segment figure (first case — A, C, E less than B, D, F):

A.... C.. E...
B..... D.... F.....

Next let A, C, E be greater than B, D, F, and equal multiples of them. On this supposition, conversely B will be the same part of A as D is of C and F is of E; and so, as before [Margin: e — Elem. VII.5], all B, D, F together will be the same part of all A, C, E together as B is of A; and therefore, conversely, all A, C, E together are an equal multiple of all B, D, F together as A is of B [Margin: f — Elem. VII.def.20]. Hence the ratio of all A, C, E together to all B, D, F together is the same as that of A to B. This same thing holds even if some of the ratios — or indeed all of them — are ratios of numbers to a unit; for the demonstration is always the same, as appears here, with the help, however, of the scholium to proposition 5 of this book.

Number-segment figure (second case — A, C, E greater, and equal multiples):

A..... C.... E.....
B.... D.. F...

Paired number-segment figure (the two configurations of the foregoing conclusion):

A... C..... E..... | A... C... E...
 B. D.. F.... | B. D. F.

Third, let A, C, E be greater than B, D, F, but not multiples of them [Margin: g — Elem. VII.def.20]. Since, then, by the same proportion, A contains B, and C contains D, and E contains F an equal number of times, and moreover the same part or parts, then by the lemma before the preceding proposition [Margin: h — Elem. VII.6] B, D, F together will be the same parts of A, C, E together as B is of A; and so, as before, conversely A, C, E together will contain B, D, F together — and A will contain B — an equal number of times, and moreover the same part or parts [Margin: i — Elem. VII.def.20]. Therefore the ratio of all A, C, E together to all B, D, F together is the same as that of A to B.

Number-segment figure (third case — A, C, E greater, not multiples):

A..... C.... E.....
 B..... D... F.....

Fourth and last, let A, C, E be equal to B, D, F. Since, if to the equals A and B the equals C and D are added, A and C together become equal to B and D together; and if to these in turn the equals E and F are added, all A, C, E together become equal to all B, D, F together — they will be, as A is to B, so all A, C, E together to all B, D, F together, since in both cases the ratio is one of equality. Therefore, if there be any number of proportional numbers, then &c. Which was to be demonstrated.

Number-segment figure (fourth case — A, C, E equal to B, D, F):

A.... C... E....
 B... D... F....

SCHOLIVM.

In fractional numbers too this proposition will be proved true in the same way, as is plain, if fractions be assumed in place of whole numbers.

[Margin: xiv.]

THEOR. 11. PROPOS. 13.

If four numbers be proportional, they will also be proportional alternately.

For let A be to B as C to D. I say that, alternately, A is to C as B to D. First let A and C be less than B and D, and A also less than C. On this supposition, by the same

proportion, A will be the same part (or parts) of B as C is of D [Margin: a — Elem. VII.9 or 10]. Alternately, then, A will also be the same part (or parts) of C as B is of D [Margin: b — Elem. VII.def.20].

Next let A and C be less than B and D, but A greater than C. On this supposition, by the same proportion, C will be the same part (or parts) of D as A is of B. Alternately, then, C will be the same part (or parts) of A as D is of B [Margin: c — Elem. VII.9 or 10]; and so, conversely, either A will be an equal multiple of C as B is of D, or else — by the lemma to proposition 11 of this book — A will contain C, and B will contain D, an equal number of times, and moreover the same part or parts [Margin: d — Elem. VII.def.20]. Therefore, as A is to C, so B to D.

Number-segment figure (second case):

B..... D....
A..... C...

Third, let A and C be greater than B and D, but A less than C. On this supposition, by the same proportion, either A is an equal multiple of B as C of D, or else A contains B, and C contains D, an equal number of times, and moreover the same part or parts. And so, conversely, B will be either the same part of A (and D of C), or else — by the lemma to proposition 11 of this book — the same parts [Margin: e — Elem. VII.9 or 10]. Alternately, then, B will be the same part (or parts) of D as A is of C [Margin: f — Elem. VII.def.20]. And hence the ratio of B to D is the same as that of A to C — that is, as A is to C, so B to D.

Number-segment figure (third case):

A.... C.....
B... D.....

Fourth, let A and C be greater than B and D, and A also greater than C. On this supposition, by the same proportion, C will be either an equal multiple of D as A of B, or else C will contain D, and A will contain B, an equal number of times, and moreover the same part or parts. And so, conversely, D will be either the same part of C (and B of A), or else — by the lemma to proposition 11 of this book — the same parts [Margin: g — Elem. VII.9 or 10]. Alternately, then, D will be the same part (or parts) of B as C of A; and for that reason, conversely, either B will be an equal multiple of D as A of C, or else — by the lemma to proposition 11 of this book — B will contain D, and A will contain C, an equal number of times, and moreover the same part or parts [Margin: h — Elem. VII.def.20]. Therefore the ratio of B to D is the same as that of A to C — that is, as A is to C, so B to D.

Number-segment figure (fourth case):

A..... C....
B..... D...

Fifth, let A and C be equal to B and D, and A less than C. The equal numbers A and B are either the same part, or the same parts, of the equal numbers C and D [Margin: i — Elem. VII.def.20]. Therefore, as A is to C, so B to D.

Number-segment figure (fifth case):

A... C....
B... D....

Sixth, let A and C be equal to B and D, but A greater than C. Since, then, the equal numbers A and E [recte: B] are either equal multiples of the equal numbers C and D, or else those contain these an equal number of times, and moreover the same part or parts [Margin: k — Elem. VII.def.20], it will be, as A is to C, so B to D.

Number-segment figure (sixth case):

A..... C...
B..... D...

Seventh and last, let A and C be equal to one another — whether they be greater than B and D, or less, or equal. Since, then, by the same proportion, A is an equal multiple of B as C of D, or the same part, or the same parts; or else A contains B, and C contains D, an equal number of times, and moreover the same part or parts; and since A and C are equal, B and D will also be equal [Margin: l — Elem. VII.def.20]. And so it will be, as A is to its equal C, so B to its equal D. Wherefore, if four numbers be proportional, they will also be proportional alternately. Which was to be demonstrated.

Number-segment figure (seventh case):

A..... C.....
B... D...

SCHOLIVM.

If you substitute fractional numbers for whole numbers, you will prove in the same way that this proposition is true in fractional numbers.

It is also evident that this proposition does not vary even if one of the numbers be taken as a unit.

We have indeed been compelled, in this proposition and the two preceding ones, to review so many cases and to confirm them by the most certain demonstrations, together with the lemma to proposition 11, so that their truth might appear in every kind of rational ratio. For Theon [Theon of Alexandria] and some other commentators demonstrate them only for rational ratios of lesser inequality — those, namely, in which the antecedent numbers are parts of the consequents — as is plainly clear from the demonstrations of these same authors; unless we should say that a greater number is "parts" of a lesser number, as some concede — among whom is even (to my astonishment) Federico Commandino [Federico Commandino], an outstanding geometer — which is absurd, and foreign to Euclid's practice, since he would then have called a number "parts" of a number, a lesser of a greater, whereas the lesser does not measure the greater. This is also established, clearer than sunlight, from Definition 20, where he taught that numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second as the third is of the fourth, and so on. For if he had thought a greater number to be "parts" of a lesser, it would have been enough to say "when the first is the same part, or the same parts, of the second as the third is of the fourth." For thus he would have comprehended proportional numbers in every kind of ratio, as is manifest; so that all the remaining words of the definition would be superfluous.

[Margin: xv.]

THEOR. 12. PROPOS. 14.

If there be any number of numbers, and others equal to them in multitude, taken two and two in the same ratio, they will also be in the same ratio *ex aequali*.

Let there be any number of numbers A, B, C, and as many others D, E, F, such that as A is to B, so D to E, and as B is to C, so E to F. I say that *ex aequali* also, as A is to C, so D to F. For since as A is to B, so D to E, alternately A will be to D as B to E [Margin: a — Elem. VII.13]. And similarly, for the same reason, since as B is to C, so E to F, alternately B will be to E as C to F. Therefore, as A is to D, so C to F. (For since each ratio — A to D and C to F — is the same as the ratio B to E, as has been demonstrated, they will also be the same as one another, as will presently be shown.) [Margin: b — Elem. VII.13] And hence, alternately, as A is to C, so D to F.

But if there be more numbers than three, so that also as C is to G, so F to H, I say that still, as A is to G, so D to H. For since it has now been shown, for three numbers, that as A is to C, so D to F, and it is supposed that as C is to G, so F to H,

there will be three numbers A, C, G and three others D, F, H, taken two and two in the same ratio. Therefore, by what was shown *ex aequali* for three numbers, it will again be that as A is to G, so D to H. And in the same way we shall show the same for five numbers by means of four, just as it was demonstrated for four numbers by means of three; and so on for more. Therefore, if there be any number of numbers, and others equal to them in multitude, &c. Which was to be demonstrated.

Number-segment figure (Prop. 14):

A..... D.....
 B..... E.....
 C.... F...
 G..... H.....

SCHOLIVM.

It is evident, moreover, that this proposition can be demonstrated in the same way for fractional numbers, if fractions be assumed in place of whole numbers.

Number-segment figure (fractional example):

A. D...
 B.. E.....
 C... F.....
 G. G...

The same is true if in place of a single number a unit be taken, or even, in place of several numbers, several units, as is clear in this example.

LEMMA.

That two ratios of numbers which are the same as the same ratio are also the same as one another — such as, in the demonstration, are the ratios A to D and C to F, which were shown to be the same as the ratio B to E — whether the numbers be whole or fractional, will be demonstrated thus. By the same proportion,

Number-segment figure (Lemma):

A..... B..... C....
 D..... E..... F...

B will be an equal multiple of E, and both A of D and C of F, or the same part, or the same parts; or else B will contain E — and both A will contain D and C will contain F — an equal number of times, and moreover the same part or parts

[Margin: c — Elem. VII.def.20]. Therefore the numbers A, D, C, F are proportional, namely, as A is to D, so C to F. Which was proposed.

This same thing was demonstrated by Euclid in Book 5, on the proportions of magnitudes, proposition 11 (*Elements* V.11).

[Margin: xvi.]

THEOR. 13. PROPOS. 15.

If a unit measure some number as many times as another number measures another number, then alternately the unit too will measure the third number equally many times, and the second the fourth.

Let the unit A measure the number BC as many times as the number D measures the number EF. I say that, alternately, the unit A measures the number D, and the number BC measures the number EF, equally many times. For when the number BC is divided into the units BG, GH, HC, and the number EF into the parts EI, IK, KF equal to D, the multitude of the units of the number BC will be equal to the multitude of the parts of the number EF; and the unit A will measure the number D as often as the unit BG measures EI, the unit GH measures IK, and the unit HC measures KF; and therefore the unit A is the same part of the number D as the unit BG is of the number EI, as the unit GH is of the number IK, and as the unit HC is of the number KF. Wherefore, by what we showed at proposition 5 of this book, the units BG, GH, HC together will be the same part as the unit BG is of the number EI — that is, as the unit A is of the number D. And hence the unit A will measure the number D, and the number BC (composed of the units BG, GH, HC) will measure the number EF (composed of the numbers EI, IK, KF), equally many times. If, therefore, a unit measure some number, &c. Which was to be demonstrated.

Number-segment figure (Prop. 15):

A. D. .
B. G. H. C. E. . I. . K. . F

SCHOLIVM.

That very thing which Euclid demonstrated concerning numbers at proposition 13, he here shows separately concerning the unit and three numbers, because the unit is not a number. This, indeed, we shall demonstrate more briefly thus. Since the unit A measures the number BC as often as the number D measures the number EF, the unit A will be the same part of the number BC as the number D is

of the number EF. Therefore, by what we demonstrated at proposition 9, alternately the unit A will be the same part of the number D as the number BC is of the number EF; and for that reason the unit A will measure the number D, and the number BC the number EF, equally many times.

This proposition cannot hold for fractional numbers. For if a unit measure some number, and another fractional number measure another fractional number equally many times, the unit will not, alternately, measure equally the third number (which is taken as fractional), nor the second the fourth fractional number [Margin: a — Elem. VII.13]; but the unit will only have the same ratio to the third number as the second has to the fourth.

[Margin: xvij.]

PROBL. 14. BROPOS. 16.

If two numbers, multiplying one another, make certain numbers, the products of them will be equal to one another.

Let two numbers A and B, multiplying one another, make the numbers C and D, such that A multiplying B makes C, and B multiplying A makes D. I say that the numbers C and D are equal to one another. For, taking the unit E: since A multiplying B makes C [Margin: a — Elem. VII.def.15], C will be composed of B as many times as there are units in A; and so the unit E will measure the number A, and the number B will measure the number C, equally many times [Margin: b — Elem. VII.15]. Alternately, then, the unit E will measure the number B, and the number A will measure the number C, equally. Again, in the same way, since B multiplying A makes D [Margin: c — Elem. VII.def.15], D will be composed of A as many times as there are units in B; and so the unit E will measure the number B, and the same number A will measure the number D, equally. But the same unit E was also measuring the same number B, and the same number A was measuring C, equally. Therefore the number A measures both numbers C and D equally; and hence C and D will be equal to one another. If, therefore, two numbers, multiplying one another, make certain numbers, &c. Which was to be demonstrated.

Number-segment figure (Prop. 16):

E.
A... B....
D..... C.....

SCHOLIUM

In fractional numbers this proposition must be demonstrated as follows. Since A, multiplying B, makes C, then by the definition of multiplication C will be to B as A is to E, the unit. Therefore, alternately, C is to A as B is to E, the unit. But by the same definition, as B is to E, the unit, so too is D to A — D being what B produces by multiplying A. Consequently, by the lemma of proposition 14, C is to A as D is to that same A; and therefore C and D are equal to one another. Which was to be demonstrated.

E.

A, $\frac{2}{3}$. B, $4\frac{5}{7}$.

D, $3\frac{1}{7}$. C, $3\frac{1}{7}$.

This same proposition can also be stated, with Campanus [Campanus], in the following way:

If two numbers multiply one another, one and the same number will be produced.

For let A multiply B and make C. I say that the same C is generated if B multiplies A. As before, when A multiplying B makes C, we shall show that the unit E measures the number B, and that the number A measures the number C, in the same way. And likewise, if the number B multiplies A, the unit E also measures the number B, and the number A measures the number produced, in the same way. Therefore the same number C is generated from the multiplication of B into A, since it too measures the number A just as the unit E measures B.

E.

A... B....

C.....

[Margin: a — Def. VII.15]

But if the numbers A and E are both fractional, or one of them is, we shall demonstrate the same in this way. Since A, multiplying B, makes C, then by definition 15, C is to B as A is to E, the unit. And alternately, C is to A as B is to E, the unit. But if B multiplies A, then by the same definition 15, B is to E, the unit, as the produced number is to A. Therefore C is to A as this produced number is to the same A; and consequently this produced number will be the same as C. Which is what was proposed.

THEOR. 15. PROPOS. 17.

[Margin: xvii.]

If a number, multiplying two numbers, produces certain products, the numbers generated from them will have the same ratio as the numbers multiplied.

For let the number A, multiplying the numbers B and C, make D and E. I say that D is to E as B is to C. Take the unit F. Then D is composed [Margin: a — Def. VII.15] out of B as many times as the unit F is contained in A; and likewise E is composed out of C as many times as that same unit F is contained in the same A. And therefore B measures D, and C measures E, in the same way. Hence B is the same part of D that C is of E. Consequently B is to D as C is to E [Margin: b — Def. VII.20]; and, alternately, B is to C as D is to E [Margin: c — VII.13]. So if a number, multiplying two numbers, produces certain products, and so on. Which was to be shown.

F.
A...
B.. C....
D..... E.....

SCHOLIUM

If the numbers A, B, C are fractional — one of them, or two — we shall show the same in this way. Since A, multiplying B and C, makes D and E, then by definition 15 both D is to B and E is to C as A is to F, the unit; and consequently, by the lemma of proposition 14, D is to B as E is to C. Therefore, alternately, D is to E as B is to C. Which is what was proposed.

THEOR. 16. PROPOS. 18.

[Margin: xix.]

If two numbers, multiplying some number, produce certain products, the numbers generated from them will have the same ratio as the multipliers.

For let the numbers A and B, multiplying the number C, make D and E. I say that A is to B as D is to E. For since D arises from A into C, the same D will also arise [Margin: a — VII.16] from C into A. And by the same reasoning, since E arises from B into C, the same E will arise from C into B. Since, then, the same C, multiplying A and B, makes D and E, it follows that A is to B as D is to E [Margin: b

— VII.17]. So if two numbers, multiplying some number, produce certain products, and so on. Which was to be demonstrated.

A.... B.....

C...

D..... E.....

SCHOLIUM

It is perfectly clear that this same proposition is demonstrated in the same way if the numbers A, B, C are fractional — one of them, or two.

Both this proposition and the preceding one we shall extend, with Campanus [Campanus], to any number of numbers, whether the numbers are all whole or not, by the following reasoning:

If one number multiplies any number of numbers, or any number of numbers multiply some one number, the products will have the same ratios as the numbers multiplied or the multipliers.

For let E, F, G arise from A into B, C, D, or from B, C, D into A. I say that the products E, F, G have the same ratios that the multiplying — or multiplied — numbers B, C, D have: that is, that B is to C as E is to F, and C is to D as F is to G. For since EF arise from A into BC, or from BC into A, B will be to C as E is to F [Margin: a — VII.17 or 18]. Likewise, because F, G arise from A into C, D, or from C, D into A, C will be to D as F is to G. And so for the rest.

A...

B.. C... D....

E..... F..... G.....

THEOR. 17. PROPOS. 19.

[Margin: xix.]

If four numbers are proportional, the number produced from the first and fourth will be equal to the one produced from the second and fourth [recte: third]. And if the number produced from the first and fourth is equal to the one produced from the second and third, the four numbers will be proportional.

Let there be four proportional numbers A, B, C, D, such that A is to B as C is to D; and let E arise from A, the first, into D, the fourth, and F from B, the second, into C, the third. I say that the numbers E and F are equal to one another. Let G in turn arise from A into C. Then, since G, E arise from A into C, D, C will be to D — that is,

A to B — as G is to E [Margin: a — VII.17]. Again, since G, F arise from A and B into C, it will likewise hold that A is to B as G is to F [Margin: b — VII.18]. Therefore, by the lemma of proposition 14, G has the same proportion to both E and F — namely the one that A has to B; and consequently the numbers E and F are equal, by what we wrote at definition 20.

Now let E, generated from A, the first, into D, the fourth, be equal to F, produced from B, the second, into C, the third. I say that the four numbers A, B, C, D are proportional, so that A is to B as C is to D. Let G again arise from A into C. Since, then, G, E arise from A into C, D, C will be to D as G is to E [Margin: c — VII.17], or as it is to F, which equals E — for G has the same ratio to the equal numbers E, F, as we taught at definition 20. Again, since G, F arise from A and B into C, it will likewise hold that A is to B as the same G is to the same F [Margin: d — VII.18]. Therefore the proportions of A to B and of C to D, since they are the same as the proportion of G to F, are the same as one another, by the lemma of proposition 14. And consequently A is to B as C is to D. So if four numbers are proportional, and so on. Which was to be shown.

A...
 B..
 C.....
 D....
 E.....
 F.....
 G.....

SCHOLIUM

It is clearer than daylight that the same demonstration of this proposition holds in fractional numbers, whether they are all fractional or not.

The first part of this theorem could also be stated in the following manner, in fractional numbers as well as in whole ones:

If two numbers multiply two numbers having the same ratio as they do — the antecedent of the former multiplying the consequent of the latter, and the consequent the antecedent — the products will be equal to one another.

For it has been shown in just this way that the number E, which arises from A, the antecedent, into D, the consequent, is equal to the number F, which is produced from B, the consequent, into C, the antecedent.

Moreover, the following theorem, in fractional numbers as well as in whole ones, will easily be demonstrated from this proposition 16.

If the ratio of the first to the second is greater than that of the third to the fourth, the number produced from the first and fourth will be greater than the one produced from the second and third. And if the number produced from the first and fourth is greater than the one produced from the second and third, the ratio of the first to the second will be greater than that of the third to the fourth.

First, let the ratio of A, the first, to B, the second, be greater than that of C, the third, to D, the fourth. I say that the number produced from A into D is greater than the one produced from B into C. For suppose E to be to B as C is to D — whether E be a whole number, or fractional, or a whole with a fraction; and such an E will be found, as we shall show at proposition 19 of book 9, if the number produced from B into C is divided by D. Then the ratio of A to B will also be greater than that of E to B; and consequently A is greater than E. Hence a greater number will arise from A into D than from E into the same D. But the number arising from E into D is equal to the one arising from B into C [Margin: a — VII.19]. Therefore the number generated from A into D will be greater than the number produced from B into C. Which is what was proposed.

A..... E.....
B....
C.....,
D.....

Next, let a greater number arise from A into D than from B into C. I say that the ratio of A, the first, to B, the second, is greater than that of C, the third, to D, the fourth. For if the number E is understood to be such that from it into D the same number arises as from B into C — whether E be a whole number, or fractional, or a whole with a fraction — a greater number will also arise from A into D than from E into the same D; and consequently the ratio of A to B will be greater than that of E to B [Margin: b — VII.19]. But E is to B as C is to D. Therefore the ratio of A to B will also be greater than that of C to D. Which is what was proposed.

But if the ratio of the first to the second is less than that of the third to the fourth, the number produced from the first and fourth will be less than the one produced from the second and third; and if the number produced from the first and fourth is less than the one produced from the second and third, the ratio of the first to the second will be less than that of the third to the fourth. For the demonstration will be exactly the same, provided you change the word "greater" into the word

"less," as appears in this appended example. Nevertheless, it can also be shown in this way. Since the ratio of A to B is less than that of C to D — that is, the ratio of C, the first, to D, the second, is greater than that of A, the third, to B, the fourth — a greater number will arise from C, the first, into B, the fourth, or from B into C, than from D, the second, into A, the third, or from A into D, as has already been demonstrated: that is, a smaller number will arise from A into D, or from D into A; which is what was proposed. Then, if a smaller number arises from A into D than from B into C — that is, a greater one from B into C, or from C into B, than from A into D, or from D into A — the ratio of C, the first, to D, the second, will be greater than that of A, the third, to B, the fourth, as has already been demonstrated: that is, the ratio of A to B will be less than that of C to D. Which is what was proposed.

A..... E.....
 B....
 C...
 D..

THEOR. 18. PROPOS. 20.

[Margin: xx.]

If three numbers are proportional, the number contained by the extremes is equal to the one produced from the mean; and if the number contained by the extremes is equal to the one described from the mean, the three numbers will be proportional.

Let there be three proportional numbers A, B, C, such that A is to B as B is to C. I say that the number arising from A, the first, into C, the third, is equal to the one produced from B, the mean. For take D equal to B: then B will be to C — that is, A to B — as D is to C; and the number generated from B into D will be equal to the one produced from B into itself. Since, then, the four numbers A, B, D, C are proportional, the number made from A into C will be equal [Margin: a — VII.19] to the one arising from B into D — that is, to the one produced from B into itself.

Now let the number produced from A, the first, into C, the third, be equal to the one arising from B, the mean, into C. I say that the three numbers A, B, C are proportional. For take D again equal to B: then B will be to C as D is to C, and the number arising from B into D will be equal to the one arising from B into itself — that is, to the one arising from A into C. Since, then, the number arising from A, the first, into C, the fourth, is equal to the one arising from B, the second, into D, the third, the four numbers A, B, C, D will be proportional [Margin: b — VII.16], so

that A is to B as D is to C, or as B is to C. So if three numbers are proportional, and so on. Which was to be demonstrated.

A.....
 B..... D.....
 C.....

SCHOLIUM

This demonstration will not vary even if the numbers A, B, C are fractional, or partly whole and partly fractional.

But if the ratio of the first to the second is greater than that of the second to the third, a greater number will arise from the first into the third than from the second into itself; and if a greater number arises from the first into the third than from the second into itself, the ratio of the first to the second will be greater than that of the second to the third. Likewise, if the ratio of the first to the second is less than that of the second to the third, a smaller number will arise from the first into the third than from the second into itself; and if a smaller number arises from the first into the third than from the second into itself, the ratio of the first to the second will be less than that of the second to the third. This is clear from the scholium of the preceding proposition, if another number equal to the second is taken, so that four numbers are had. For then the ratio of the first to the second will be greater than that of the third to the fourth, or less, and so on — as appears in the appended examples, even if the numbers are fractional, or partly fractional and partly whole.

A..... A....
 B..... D..... B..... D
 C..... C.....

THEOR. 19. PROPOS. 21.

[Margin: xxj.]

The least numbers among all those having the same ratio with them measure equally the numbers that have the same ratio with them — the greater indeed measuring the greater, the less the less.

Let AB, CD be the least numbers in the ratio in which two other, greater, numbers E, F stand, so that AB is to CD as E is to F. I say that AB, CD measure E, F equally — the greater, AB, measuring the greater, E, and the less, CD, the less, F: that is, the antecedent measuring the antecedent, and the consequent the consequent. For since AB is to CD as E is to F, it will hold, alternately, that AB is to E as CD is to F

[Margin: a — VII.13]. And so, since AB, CD are less than E, F, AB will be the same part, or the same parts, of E that CD is of F [Margin: b — Def. VII.20]. Now they are certainly not "parts": for if AB, CD could be divided into parts AG, GB and CH, HD of the numbers E, F, the number of parts AG, GB would be equal to the number of parts CH, HD; and therefore AG would be the same part of E that CH is of F. Then AG would be to E as CH is to F [Margin: c — Def. VII.20]; and, alternately, AG would be to CH as E is to F, or as AB is to CD [Margin: d — VII.13]. And consequently the numbers AG, CH, less than AB, CD, would have the same proportion as AB to CD. Which is absurd, since AB, CD are supposed to be the least in their proportion. So AB, CD are not the same parts of E, F; they are therefore the same part. And on that account AB will measure E, and CD measure F, equally. Therefore the least numbers among all those having the same ratio with them, and so on. Which was to be demonstrated.

A... G.. B
 C.. H. D
 E.....
 F.....

SCHOLIUM

The same holds true if one says that three numbers are in continued proportion and that the first two are the least in their proportion. For, this being posited, it will be shown in the same way that the first measures the second, and the second the third — as appears in this appended example, where the third number is equal to the second. And although three numbers in continued proportion, of which the first two are the least in their proportion, cannot be given unless the first is a unit, the same is nevertheless demonstrated in three numbers, even if the first is not conceded by the opponent to be a unit, as we said. I have said this so that proposition 12 of book 9 may be demonstrated — in which the opponent is forced to concede that three numbers are in continued proportion, and that the first two are the least in their proportion, and consequently that the first measures the second, by this proposition, which he previously denied. But this will be made plainer at proposition 12 of book 9.

By like reasoning, that too is true which Campanus [Campanus] teaches.

A.. G.. B
 C... H... D
 E.....
 F.....

Any number of least numbers in the continuation of their ratios — whether the ratios are the same or different — measure equally as many other numbers that have the same ratios with them: the first measuring the first, the second the second, the third the third, and so on.

For let there be more than two numbers AB, CD, EF, the least in the continuation of their ratios — whether the ratio of AB to CD is the same as that of CD to EF, or not — such that no other numbers less than AB, CD, EF can be found whose first is to the second ...

A... K... B C... L. D E.... M.... F
G..... H..... I.....

Euclid, *Elements*, Book VII (Clavius's commentary) — continued

(Global = printed folios 328–330. Running headers: verso "EVCLID. GEOM.", recto "LIBER VII.")

[continuation of PROPOS. 21 — the least numbers in a continued series of ratios]

such that the first is to the second as AB is to CD, and the second to the third as CD to EF, and so on. (Such ratios may indeed be found separately in smaller numbers not standing in continued proportion: namely the ratio of AB to CD in the numbers 4 to 2, or 2 to 1, which are smaller than AB and CD themselves; just as the ratios of the numbers 16, 20, 25 — which are the least in the continuation of two sesquiquartan ratios [5:4], since they cannot be continued in smaller numbers — are found separately in smaller numbers: the ratio 16 to 20 indeed in 8 and 10, but the ratio 20 to 25 in 4 and 5, or in 12 and 15.)

Let there then be as many other numbers G, H, I, not the least, continued in the same ratios — namely G to H as AB to CD, and H to I as CD to EF. I say that AB measures G, CD measures H, and EF measures I, equally.

For since AB is to CD as G is to H, [Margin: a — VII.13] it will alternately be, as AB is to G, so CD to H. In the same way, since CD is to EF as H is to I, [Margin: b — VII.13] it will likewise be, alternately, as CD to H, so EF to I. [Margin: c — VII, Def. 20] Therefore AB will be either the same part, or the same parts, of G that CD is of H and EF is of I. But "the same parts" is impossible. For if AB, CD, EF be divided (were it possible) into AK, KB; CL, LD; EM, MF — parts of the numbers G, H, I —

there will be as many parts in AB as in CD and in EF, and thus AK will be the same part of G that CL is of H [Margin: d — VII, Def. 20] and EM of I. It will therefore be, as AK is to G, so CL to H [recte: ita CL, ad H]; [Margin: e — VII.13] and alternately, as AK is to CL, so G to H, or AB to CD. In the same way it will be as CL to FN [?] as CD to EF. Hence the numbers AK, CL, EM are continued in the ratios of the numbers AB, CD, EF, and are smaller than AB, CD, EF. This is absurd, since these last were assumed to be the least in the continuation of their ratios. Therefore AB, CD, EF are not the same parts of the numbers G, H, I. Hence each is the same single part of its own; and consequently AB will measure G, CD will measure H, and EF will measure I, equally. Which is what was proposed.

And if three given numbers A, B, C be the least in the continuation of their ratios, in such a way that also any two of them are the least, the same will be demonstrated still more easily in this manner. Let there be three others, D, E, F, not the least, having the same ratios as A, B, C. I say that A, B, C measure D, E, F equally. For, by this proposition 21, since A, B are the least in the ratio of A to B, they measure D, E equally; and by the same reasoning B, C measure E, F. Therefore, since A measures D and

D.... E..... F.....
A.. B..... C...

C measures F equally, and B measures E, all of A, B, C will measure D, E, F equally.

[Clavius's remark.] For the rest, this proposition, together with its scholium, can in no way apply to fractional numbers, because among fractions no least numbers can be given in a given ratio: whatever be given, still smaller ones can be given, and so on to infinity. And the same is to be understood in all the other propositions in which mention is made of least numbers, for they are all to be understood of whole numbers only. So too, whenever numbers prime to one another are treated, fractions must be excluded, since these cannot be prime to one another — any fraction whatever can measure them as a common measure. For if they be reduced to the same denomination, it is evident that they have as a common measure one small part, or several, of the same denomination. But all the other propositions about numbers, in which least numbers in their ratio, or numbers prime to one another, are not employed, apply equally to whole and to fractional numbers — which it is enough to have pointed out this once, here.

THEOR. 20. PROPOS. 22.

[Margin: xix.]

If there be three numbers, and others equal to them in multitude which are taken two and two and in the same ratio, but their proportion be perturbed, then ex aequali they too will be in the same ratio.

Let there be three numbers A, B, C, and as many others D, E, F, taken two and two and in the same ratio, and let their proportion be perturbed — namely as A is to B, so E to F; and as B is to C, so D to E. I say that ex aequali also it is, as A is to C, so D to F.

A..... H...
B... D....
C..... E.....
G..... F..

For since A is to B as E is to F, [Margin: a — VII.19] the number produced from A into F will be equal to the number that arises from B into E. By like reasoning, since also B is to C as D is to E, [Margin: b — VII.19] there will be, equal to the same number that arises from B into E, the number that arises from C into D. And thus the number produced from A, the first, into F, the fourth, will be equal to the number produced from C, the second, into D, the third. [Margin: c — VII.19] Therefore it will be, as A is to C, so D to F.

And if there be more numbers than three, in such a way that also C is to G as H is to D, I say that still it is, as A is to G, so H to F. For since it has now been shown, in the case of three numbers, that as A is to C, so D to F, and it is assumed that as C is to G, so H to D, there will be three numbers A, C, G and three others H, D, F, taken two and two in the same ratio, and their proportion is perturbed. By the ex aequali case already shown for three numbers, it will again be, as A is to G, so H to F. In the same way we shall show it for five numbers by means of four, just as it was demonstrated for four numbers by means of three; and so for more. Accordingly, if there be three numbers, and others equal to them in multitude taken two and two, and so on. Which was to be shown.

SCHOLIVM.

The proposition will be demonstrated in the same way for fractional numbers, as is evident.

Now since Euclid, out of those six modes of arguing in proportions which he set forth and confirmed by demonstrations for magnitudes in Book 5, has here exhibited for numbers only two — namely the one drawn from alternated proportion (permutando), in proposition 13, and the one called ex aequali, in propositions 14 and 22 of this book — it will not be foreign to our purpose to set

forth briefly the remaining four modes in numbers, and certain others among the following theorems of the fifth book, all of which apply to fractional numbers as well as to whole ones.

I.

If four numbers be proportional, they will also be proportional inversely, that is, by inverting.

Let it be, as A is to B, so C to D. I say that also, by inverting (that is, by inverse ratio), it is as B to A, so D to C.

A C . . .
B D . .

For since A is to B as C is to D, [Margin: a — VII.13] it will alternately be, A to C as B to D. Again, since B is to D as A is to C, [Margin: b — VII.13] it will alternately be, B to A as D to C. Which is what was proposed.

II.

If compounded numbers be proportional, these too, separated, will be proportional.

[LIBER VII. — printed folio 329]

Let it be, as AB is to CB, so DE to FE. I say that, by separation (dividendo), it is as AC to CB, so DF to FE. For since AB is to CB as DE is to FE, [Margin: a — VII.13] alternately the whole AB will be to the whole DE as the taken-away CB to the taken-away FE; [Margin: b — VII.11] and consequently, as the whole AB is to the whole DE, so the remainder AC to the remainder DF; that is, AC is to DF as CB is to FE. [Margin: c — VII.13] Alternately, therefore, it will also be AC to CB as DF to FE. Which is what was proposed.

A C . . . B
D F . . E

By the same reasoning we shall demonstrate the converse and the contrary of separation of ratio, as in Book 5. First, let it be as AB to CB, so DE to FE. I say that, by the converse of separation of ratio, it is also as CB to AC, so FE to DF. For since AB is to CB as DE is to EF, by separation it will also be as AC to CB, so DF to FE; and by conversion, as CB to AC, so FE to DF. Which is what was proposed.

Next, let it be as AC to AB, so DF to DE. I say that, by the contrary of separation of ratio, it is also as AC to CB, so DF to FE. For since AC is to AB as DF is to DE, by

conversion it will be as AB to AC, so DE to DF. Therefore by separation, as CB to AC, so FE to DF; and by conversion, as AC to CB, so DF to FE. Which is what was proposed.

III.

If separated numbers be proportional, these too, compounded, will be proportional.

Let it be, as AB is to BC, so DE to EF. I say that, by compounding (componendo), it is also as AC to BC, so DF to EF. For since AB is to BC as DE is to EF, [Margin: a — VII.13] it will alternately be as AB to DE, so BC to EF; [Margin: b — VII.12] and consequently AB, BC together are to DE, EF together as BC is to EF; [Margin: c — VII.13] and alternately AB, BC together — that is, the whole AC — is to BC as DE, EF together — that is, the whole DF — is to EF. Which is what was proposed.

A B . . . C
D E . . F

Likewise the converse and the contrary of compounding of ratio will be shown in this place, as in Book 5. First, let it be as AB to BC, so DE to EF. I say that, by the converse of compounding of ratio, it is also as AC to AB, so DF to DE. For since AB is to BC as DE is to EF, it will be by conversion as BC to AB, so EF to DE; and by compounding, as AC to AB, so DF to DE. Which is what was proposed.

Next, let it be again as AB to BC, so DE to EF. I say that, by the contrary of compounding of ratio, it is also as AB to AC, so DE to DF. For since AB is to BC as DE is to EF, it will be by conversion as BC to AB, so EF to DE. Therefore by compounding, as AC to AB, so DF to DE; and by conversion, as AB to AC, so DE to DF. Which is what was proposed.

IV.

If compounded numbers be proportional, these too will be proportional by conversion of ratio.

Let it be, as AB is to CB, so DE to FE. I say that, by conversion of ratio, it is also as AB to AC, so DE to DF. For since AB is to CB as DE is to FE, [Margin: a — VII.13] alternately the whole AB will be to the whole DE as the taken-away CB to the taken-away FE; [Margin: b — VII.11] and consequently, as the whole AB to the whole DE, so will the remainder AC be to the remainder DF. [Margin: c — VII.13] Alternately, therefore, as AB to AC, so DE to DF. Which is what was proposed.

A C . . . B
D F . . E

Again, from these we shall easily demonstrate in numbers that theorem which Euclid shows for magnitudes in proposition 24 of Book 5. Namely:

V.

If a first have to a second the same ratio as a third to a fourth, and a fifth also have to the second the same ratio as a sixth to the fourth, then the first compounded with the fifth will have to the second the same ratio as the third compounded with the sixth has to the fourth.

Let it be, as AB, the first, is to C, the second, so DE, the third, to F, the fourth; likewise as BG, the fifth, is to C, the second, so EH, the sixth, to F, the fourth. I say that AG, compounded of the first and fifth, is to C, the second, as DH, compounded of the third and sixth, is to F, the fourth.

A..... B.. G | D..... E... H
C.... F.....

For since BG is to C as EH is to F, it will be by conversion, as C to BG, so F to EH. Since, then, AB is to C as DE is to F, and C is to BG as F is to EH, it will be ex aequali as AB to BG, so DE to EH. By compounding, therefore, as AG to BG, so DH to EH. And so, since again AG is to BG as DH is to EH, and BG is to C as EH is to F, ex aequali it will be as AG to C, so DH to F. Which is what was proposed.

In the same way we shall show this theorem too, which we demonstrated for magnitudes in proposition 24 of Book 5.

VI.

If two numbers have to two numbers the same ratio, and certain subtracted numbers have to the same the same ratio, then the remainders too will have to the same the same ratio.

Let it be, as the whole AB is to C, so the whole DE to F; likewise as the subtracted AG is to C, so the subtracted DH to F. I say that the remainder GB too is to C as the remainder HE is to F. For since AG is to C as DH is to F, it will be by conversion as C to AG, so F to DH. Since, then, AB is to C as DE is to F, and C is to AG as F is to DH, it will be ex aequali as AB to AG, so DE to DH. By separation, therefore, as GB to AG, so HE to DH. And so, since again GB is to AG as HE is to DH, and AG is to C as DH is to F, ex aequali it will be as GB to C, so HE to F. Which is what was proposed.

A..... G., B D..... H... E
C.... F.....

This too we shall demonstrate.

VII.

[EVCLID. GEOM. — printed folio 330]

If a first have to a second the same ratio as a third to a fourth, and the first also have to a fifth the same ratio as the third to a sixth, then the first will have to the second and fifth compounded the same ratio as the third has to the fourth and sixth compounded.

Let the first, A, be to the second, BC, as the third, D, is to the fourth, EF; and let the first, A, be to the fifth, CG, as the third, D, is to the sixth, FH. I say that A, the first, is to BG (compounded of the second and fifth) as D, the third, is to EH (compounded of the fourth and sixth).

A..... D.....
B... C.. E..... F.... H

For since A is to BC as D is to EF, it will be by conversion as BC to A, so EF to D. Since, then, BC is to A as EF is to D, and A is to CG as D is to FH, it will be ex aequali as BC to CG, so EF to FH; and by compounding, as BG to CG, so EH to FH. Therefore, since A is to CG as D is to FH, [and] as CG is to BG, so FH [is to EH; ex aequali, as A is to BG, so D is to EH] [?]. Which is what was proposed. [recte: propositum; printed "rpropositum"]

And finally, from all these we shall infer this theorem.

VIII.

If any numbers whatever have to the same number the ratios that others equal to them in multitude have to some other same number, then all these together will also have to the same number the ratio that all those others together have to that other same number. And if the same number have to any numbers whatever the ratios that the same other number has to others equal to them in multitude, then the same number will also have to all of them together the ratio that the same other number has to all these together.

Let any numbers whatever AB, BC, CD have ratios to the same number K — that is, let it be as AB to E, so FG to K; and as BC to E, so GH to K; and as CD to E, so HI to K. I say that all of them together — that is, AD itself — has to E the same ratio that all these together have, that is, that FI itself has to K. For since it is, as AB, the first, is to E, the second, so FG, the third, to K, the fourth; likewise as BC, the fifth,

is to E, the second, so GH, the sixth, to K, [the fourth,] it will also be, as AC, the first with the fifth, is to E, the second, so FH, the third with the sixth, to K, the fourth.

A.... B..... C..... D | F. G... H.... I
E..... K.....

Again, since AC, the first, is to E, the second, as FH, the third, is to K, the fourth; likewise as CD, the fifth, is to E, the second, so HI, the sixth, to K, the fourth; it will also be, as AD, the first with the fifth, is to E, the second, so FI, the third with the sixth, to K, the fourth. And so of the rest, if there be more.

But now let the same number E have to any numbers whatever AB, BC, CD the same ratios that the same other number K has to as many others FG, GH, HI — that is, let it be as E to AB, so K to FG; and as E to BC, so K to GH; and as E to CD, so K to HI. I say that it is, as E to all of them together, namely to AD, so K to all of them together, namely to FI. For since it is, as E, the first, is to AB, the second, so K, the third, to FG, the fourth; likewise as E, the first, is to BC, the fifth, so K, the third, to GH, the sixth; it will also be, as E, the first, is to AC, the second with the fifth, so K, the third, to FH, the fourth with the sixth. Again, since it is, as E, the first, is to AC, the second, so K, the third, to FH, the fourth; likewise as E, the first, is to CD, the fifth, so K, the third, to HI, the sixth; it will also be, as E, the first, is to AD, the second with the fifth, so K, the third, to FI, the fourth with the sixth. And so of the rest, if there be more.

Now, these things being demonstrated, the nine last propositions of Book 5, added by Campanus, will be shown for disproportional numbers in the same way in which they were demonstrated for magnitudes: if, in place of magnitudes, numbers be understood, whether whole or fractional, and, in place of the modes of arguing in proportions demonstrated in Book 5, the same modes demonstrated in this book be assumed — so that there is no need to repeat them here. For it is enough, as I have said, that those propositions of the fifth book be taken in hand, and the magnitudes be understood to be numbers, and the very same demonstrations be applied.

THEOR. 21. PROPOS. 23.

[Margin: xxj.]

Numbers prime to one another are the least of all having the same ratio with them.

Let there be numbers A, B prime to one another. I say that they are the least of all that have the same ratio as A and B. For if they are not the least, there will be

certain others smaller than they, being the least, that have the ratio which A and B have. Let there be, then, if it be possible, C, D, the least in the ratio of A and B, and therefore smaller than A and B.

A..... B....
 C—— D—
 E——

Since, then, C, D are the least in the ratio of A and B, [Margin: a — VII.21] C will measure A, and D will measure B, equally; and thus according to the same number, which let be E, so that C measures A as many times, and D measures B, as there are units in E. Accordingly, since the unit measures the number E, and the number C measures the number A, equally, [Margin: b — VII.15] alternately too the unit will equally measure the number C, and the number E measure the number A. Again, since the unit measures the number E, and the number D measures the number B, equally, [Margin: c — VII.15] alternately too the unit will equally measure the number D, and the number E measure the number B. And thus, since the same number E measures both A and B, the number E will be their common measure. Therefore A and B are not prime to one another, but composite. This is absurd, and contrary to the hypothesis. There are, then, no other numbers smaller than A and B that are the least in the ratio of A to B; and consequently A and B are the least. Therefore numbers prime to one another are the least, and so on. Which was to be demonstrated.

THEOR. 22. PROPOS. 24. [Margin: xxij.]

The least of all numbers having the same ratio with them are prime to one another.

Let A and B be the least of all numbers having the same ratio with them. I say they are prime to one another — that is, that no number except unity measures them by a common measure. For if they are not prime to one another but have some number as a common measure, let the number C be their common measure, and let C measure A as many times as there are units in the number D, and measure B as many times as there are units in E. Now since C, taken as many times as there are units in D, produces A, and the same C, taken as many times as there are units in E, produces B, then D and E, multiplying C, produce A and B. [Margin: a — Axiom 9 [?]] Therefore the ratio of D to E will be the same as that of A to B. [Margin: b — VII.18] But since D and E, being parts of A and B, are smaller than A and B, then A and B will not be the least of all numbers having the same ratio with them — which is absurd. Therefore the numbers A and B are prime to

one another; and consequently the least of all numbers having the same ratio with them are prime to one another. Which was to be demonstrated.

A B
 C —————
 D ——— E —

SCHOLIVM

We shall extend this proposition, and the preceding one, following Campanus, to more numbers, in this way.

Any numbers whatever prime to one another are the least in the continuation of their proportions. And any numbers whatever that are the least in the continuation of their proportions are prime to one another.

Let A, B, C be any numbers prime to one another. I say they are the least in the continuation of their proportions, so that they cannot be continued in smaller numbers — even though the ratio of two of them may be found in smaller numbers. For if they are not the least, there will be some others smaller than them, say D, E, F, the least in the continuation of those proportions. Now since D, E, F are the least in the ratio of the numbers A, B, C, then D will measure A, E will measure B, and F will measure C equally, by what we demonstrated at proposition 21 of this book [Margin: VII.21] — and indeed by one and the same number, which let be G, so that D measures A, E measures B, and F measures C, as many times as there are units in G. Since, then, unity measures the number G just as the number D measures A, [Margin: a — VII.15] likewise unity measures the number D just as the number G measures A. And by the same reasoning the same G measures both B and C, just as unity measures E and F. Therefore A, B, C, since they have the number G as a common measure, are composite to one another. Which is absurd, and contrary to the hypothesis. Therefore there are no other numbers smaller than A, B, C that are the least in the continuation of the proportions of A to B and B to C; but A, B, C themselves are the least.

Now let A, B, C be the least in the continuation of their proportions. I say they are prime to one another. For if they are not prime to one another, let the number G measure them by a common measure, so that G measures A as many times as there are units in D, measures B as many times as there are units in E, and measures C as many times as there are units in F. Since, then, A, B, C are measured as many times as there are units in D, E, F, it follows that D, E, F, multiplying G, produce A, B, C and have the same ratios which A, B, C have, by

what we showed at proposition 18 of this book. [Margin: VII.18] And so, since D, E, F are smaller than A, B, C, then A, B, C will not be the least in the continuation of their proportions — which is absurd. Therefore A, B, C are prime to one another. Which is what was proposed.

A B C
D — E — F —
G —

THEOR. 23. PROPOS. 25. [Margin: xxiv.]

If two numbers be prime to one another, the number that measures one of them will be prime to the other.

Let A and B be prime to one another, and let the number C measure A. I say that C is prime to the other, B. For if B and C are not prime to one another, let the number D measure them by a common measure, if that be possible. Now since D measures C, and C measures A, [Margin: a — Axiom 11] D will also measure A. But it also measures B. Therefore A and B are not prime to one another, since they have the number D as a common measure — which is absurd, and contrary to the hypothesis. Therefore C is prime to B. In the same way, if some number measures B, it will be prime to A. Accordingly, if two numbers be prime to one another, etc. Which was to be demonstrated.

A B
C
D

THEOR. 24. PROPOS. 26. [Margin: xxv.]

If two numbers be prime to some number, then the number produced from them [their product] will also be prime to that same number.

Let each of the numbers A and B be prime to C, and let the number D be produced from A into B, or from B into A. I say that D too is prime to that same C. For if C and D are not prime to one another, let the number E be their common measure, measuring D as many times as there are units in the number F. Now since E, taken as many times as there are units in F, makes D, [Margin: a — Axiom 9] it follows that F, multiplying E, generates D; [Margin: b — VII.16] and conversely E, multiplying F, produces the same D. But this same D was generated from A into B. Therefore, since from E (first) into F (fourth) there arises the same number as from A (second) into B (third), [Margin: c — VII.19] it will be: as E (first) is to A (second), so B (third) is to F (fourth). But since A and C are prime to

one another, and E is supposed to measure C, [Margin: d — VII.25] E will be prime to A;

and so E and A will be the least in their ratio. [Margin: e — VII.23] Therefore they will measure B and F equally, since these have the same ratio — namely E measures B, as A measures F. [Margin: f — VII.21] And so, since E measures both B and C, then B and C will not be prime to one another — which is absurd, and contrary to the hypothesis. Therefore D will be prime to that same C; and consequently, if two numbers be prime to some number, etc. Which was to be demonstrated.

A B
 E
 C D
 F

THEOR. 25. PROPOS. 27. [Margin: xxvj.]

If two numbers be prime to one another, then the number produced from one of them [i.e., its square] will also be prime to the other.

Let A and B be prime to one another, and let C be produced from A into itself. I say that C is prime to the other, B. For if D be taken equal to A, then D too will be prime to B. Now since A and D are prime to B, [Margin: a — VII.26] the number produced from A into D — that is, produced from A into itself, namely C — will be prime to that same B. And by the same technique we shall show that the number produced from B into itself is prime to A. Therefore, if two numbers be prime to one another, etc. Which was to be demonstrated.

A ... B
 C
 D

THEOR. 26. PROPOS. 28. [Margin: xxvij.]

If two numbers be prime to two numbers, each to each, then the numbers produced from them [their products] will also be prime to one another.

Let each of A and B be prime to each of C and D, and let E be produced from A into B, and F from C into D. I say that E and F are prime to one another. For since each of A and B is prime to C, [Margin: a — VII.26] E, produced from them, will also be prime to that same C. Again, since each of A and B is prime to D, in the same way E, produced from them, will be prime to that same D. Now since each

of C and D is prime to E, [Margin: b — VII.26] F, produced from them, will also be prime to E. Therefore, if two numbers be prime to two numbers, each to each, etc. Which was to be shown.

A B ...
 E
 C ... D ..
 F

THEOR. 27. PROPOS. 29. [Margin: xxvii.]

If two numbers be prime to one another, and each multiplying itself makes some number [its square], then the numbers produced from them will also be prime to one another. And if the original numbers, multiplying those products, make some numbers [the cubes], these too will be prime to one another. And this will always come about with respect to the extremes.

Let A and B be prime to one another; and from A into itself let C be made, and from B into itself let D be made. I say that C and D too are prime to one another. And if again E be made from A into C, and F from B into D, I say that E and F are likewise prime to one another. For since A and B are prime to one another, [Margin: a — VII.27] C, made from A into itself, will be prime to the other, B; and in the same way, since B and C are prime to one another, D, made from B into itself, will be prime to C; and so the two products C and D will be prime to one another.

AGAIN, since A and B are prime to one another, [Margin: b — VII.27] C, made from A into itself, will be prime to B, and D, produced from B into itself, will be prime to A. But C has also been shown prime to D. Therefore each is prime to each; [Margin: c — VII.28] and consequently E, made from A into C, will be prime to F, made from B into D. And if further G be made from A into E, and H from B into F — since A and C are prime to B, [Margin: d — VII.26] E, produced from them, will be prime to B; and by the same reasoning F will be prime to A. Now since each of A and E is prime to each of B and F, [Margin: e — VII.28] G, made from A into E, will be prime to H, made from B into F. And so on in succession, if there be more. For in the same way, since A and E are prime to B, [Margin: f — VII.26] G, made from them, will be prime to B, and likewise H to A. [Margin: g — VII.28] Therefore I, produced from A into G, will be prime to K, produced from B into H, since each of A and G is prime to each of B and H. And so, if two numbers be prime to one another, etc. Which was to be shown.

A ... B ..
 C D
 E F
 G, 81. H, 16.
 I, 243. K, 32.

THEOR. 28. PROPOS. 30. [Margin: xxix.]

If two numbers be prime to one another, then both together [their sum] will also be prime to each of them; and if both together be prime to some one of them, then the original numbers will also be prime to one another.

Let AB and BC be prime to one another. I say that both together, AC, is also prime both to AB and to BC. For if AC and AB are not prime to one another, let the number D measure them by a common measure, if that be possible. Now since D measures the whole AC and the subtracted part AB, [Margin: a — Axiom 12] it will also measure the remainder BC. Therefore AB and BC are not prime to one another, since the number D measures them — which is absurd, and contrary to the hypothesis. Therefore AC will be prime to AB. And in the same way we shall show it is prime to BC.

But now let both AB and BC together be prime to some one of them, namely to AB. I say that AB and BC are also prime to one another. For if they are not prime to one another, let the number D measure them, if that be possible. Now since D measures AB and BC, [Margin: b — Axiom 10] D will also measure the number AC, compounded of AB and BC; and consequently AC and AB are not prime to one another, since the number D measures them. Which is absurd, and contrary to the hypothesis. Therefore AB and BC are prime to one another.

A B C
 D ———

And by the same argument we shall show that AB and BC are prime to one another if AC be supposed prime to BC. Therefore, if two numbers be prime to one another, etc. Which was to be demonstrated.

COROLLARIVM

From this it follows that a number compounded of two numbers, which is prime to one of them, is also prime to the other. For if AC is prime to AB, then AB and BC will be prime to one another, by the second part of this proposition. Therefore AC will also be prime to BC, by the first part of the same. Which is what was proposed.

THEOR. 29. PROPOS. 31. [Margin: xxxij. [?]] — see anomaly (a)]

Every prime number is prime to every number which it does not measure.

Let the prime number A not measure the number B. I say that A is prime to B — that is, that A and B are prime to one another, even though B be composite. For if A and B are not prime to one another, let some number C, other than unity, measure them by a common measure, if that be possible. Then C will not be the same as A, since A is supposed not to measure B. Now since another number, C, measures the number A, A will not be prime — which is absurd, and contrary to the hypothesis. Therefore A is prime to B; and accordingly every prime number is prime to every number, etc. Which was to be demonstrated.

A B
C

THEOR. 30. PROPOS. 32. [Margin: xxxij. [?]] — see anomaly (a)]

If two numbers, multiplying one another, make some number, and some prime number measures the number produced from them, then it will also measure one of the original numbers.

Let the two numbers A and B, multiplying one another, make C, and let the prime number D measure C. I say that D measures at least one of A and B, if it does not measure both. For let D not measure A, but let it measure C as many times as there are units in the number E, so that C is made from D into E — the same C that was made from A into B. Now since the number produced from D (first) into E (fourth) is equal to the number produced from A (second) into B (third), [Margin: a — VII.19] it will be: as D (first) is to A (second), so B (third) is to E (fourth). But since the prime D is prime to A, because it does not measure it, [Margin: b — VII.31] [Margin: c — VII.23] D and A will be the least in their ratio. [Margin: d — VII.21] Therefore they will measure B and E equally — namely D measures B, and A measures E. And consequently, if D does not measure A, it will measure at least B; and in the same way, if D does not measure B, it will measure at least A. Therefore, if two numbers, multiplying one another, make some number, etc. Which was to be demonstrated.

A B
 C
 D E

SCHOLIVM

In the same way we shall demonstrate the following theorem too.

If two numbers, multiplying one another, make some number, and some non-prime number measures the number produced from them — or at any rate is composite with respect to it — then it will also be composite with respect to one of the original numbers.

For from A into B let C be made, which some non-prime number D either measures, or at any rate is composite with respect to it — that is, D and C are composite to one another. I say that D too is composite with respect to one of A and B — that is, that either D and A, or D and B, are composite to one another. For if D is composite with respect to neither of them, then each of A and B will be prime to D. [Margin: a — VII.26] Therefore C, produced from them, will also be prime to that same D — which is absurd, since D was supposed either to measure the number G [recte: C] or at any rate to be composite with respect to it. Therefore D is composite with respect to A or to B, since it is not prime to both.

A . . . B
 C
 D

PROBL. 31. PROPOS. 33. [Margin: xxx.]

Every composite number is measured by some prime number.

Let A be a composite number. I say that some prime number measures it. For let the number B measure it; if B is prime, we have what was proposed. But if it is composite, let the number C measure it, which will be either prime or composite. If it is prime, then since it measures B, and B measures A, [Margin: a — Axiom 11] C, being prime, will also measure A. But if C is composite, another number will measure it. And since a number is not diminished infinitely, we shall at last arrive at some number which no other measures — that is, at a prime — which, since it measures all the preceding, [Margin: b — Axiom 11] will also measure the composite A. Which is what was proposed.

A
 B C . . .

ANOTHER WAY. Since the number A is composite, some number — or even several — will measure it. Let B be the least of all those measuring it, which I say is prime. For if B is not prime, let the number C measure it, if that be possible. Then since C measures B, and B measures A, [Margin: c — Axiom 11] C, which is smaller than B, will also measure A — which is absurd, since B was supposed the least of all those measuring it. Therefore the number B is prime. Therefore every composite number is measured by some prime number. Which was to be demonstrated.

A
 B ... C —

THEOR. 32. PROPOS. 34. [Margin: xxxj.]

Every number is either prime, or is measured by some prime number.

[Demonstration of Proposition 34, continued from the preceding page. The enunciation — "Every number is either prime, or is measured by some prime number" — was given at the end of the previous page.]

Let A be any number whatever.

A.....

I say that it is either prime, or else some prime number measures it. For every number is either prime or composite: if A is prime, we have what was proposed; but if it is composite, then some prime number measures it [Margin: a | 33.sept — i.e. *Elements* VII.33]. Therefore every number is either prime, or else some prime number measures it. Which was to be shown.

PROBL. 3. PROPOS. 35. [Margin: xxxiiij.]

Given any quantity of numbers whatever, to find the least of all the numbers that are in the same ratio with them.

Let A, B, C be any number of numbers whatever, holding any ratios—whether the ratio of A to B is the same as that of B to C, or not—and let it be required to find just as many numbers that are the least in the same ratios.

D..
 A..... B.... C.....
 E... F.. G....
 H — I — K —
 L —.

Now A, B, C are either prime to one another or composite. If they are prime to one another, then they themselves are the least in the sequence of their ratios, by what we demonstrated at proposition 24 of this book. But if they are not prime to one another, let the greatest common measure of them be found—the number D, which measures A, B, C by the numbers E, F, G [Margin: a | 3.sept — *Elements* VII.3]. I say that E, F, G are the least in the ratios of the numbers A, B, C.

That they have the same ratios as the numbers A, B, C is shown thus. Since D measures A, B, C by E, F, G, it follows that D, multiplying E, F, G, produces A, B, C [Margin: b | 9.pron — Axiom 9]. Therefore, by what we showed at proposition 18 of this book, E, F, G will have the same ratios as the numbers A, B, C.

That E, F, G are the least of all the numbers having the same ratio with them will become clear in this way. If they are not the least, there will be some other numbers, smaller than they, that are the least in the same ratios. So let there be, if possible, the least numbers H, I, K; and since these measure A, B, C, let them measure them by the number L. On this supposition, it follows that L, multiplying the numbers H, I, K, produces the numbers A, B, C [Margin: c | 9.pron — Axiom 9]; and conversely L measures A, B, C by H, I, K [Margin: d | 8.pron — Axiom 8]. Now since E, as the first term multiplying D, makes the fourth term A, while H, as the second term multiplying L, makes that same A as the third product, it follows that as E, the first, is to H, the second, so is L, the third, to D, the fourth [Margin: e | 19.sept — *Elements* VII.19]. But E is greater than H. Therefore L too will be greater than D; and so, since L measures A, B, C, D will not be the greatest common measure of the numbers A, B, C. This is absurd and contrary to the hypothesis. Therefore there will be no other numbers, smaller than E, F, G, that are the least in the sequence of the ratios of A to B and B to C; rather E, F, G themselves are the least. Hence, given any quantity of numbers, we find the least, and so on. Which was to be done.

COROLLARIVM

From this it follows that the greatest measure of any set of numbers measures them by numbers that are the least of all those having the same ratio with them. For it has been shown that the numbers E, F, G—by which D, the greatest measure of A, B, C, measures A, B, C—are the least in the sequence of the ratios of A to B and B to C. And the same holds in the remaining cases.

SCHOLIVM

From these results we shall easily find, by a simple route, the two least numbers that have the same ratio as any given set of continuously proportional numbers.

For example, suppose the continuously proportional numbers A, B, C, D, E are proposed—whether or not they are the least in the sequence of the ratio of A to B: we shall find the two least numbers in the same ratio if, by means of this problem, we take F and G, the least in the ratio of the two numbers A and B, namely those by which I, their greatest common measure, measures them.

A, 16. B, 24. C, 36. D, 54. E, 81.
 I, 8.
 F, 2. G, 3.

It sometimes happens, however, that one of the numbers E, F, G found by this proposition is a unit—namely when D, the greatest measure, is equal to one of the numbers A, B, C, as is clear from these examples:

A... B..... C..... A.... B.. C.....
 D... D..
 E. F.. G... E.. F. G...

And it is then evident that the numbers E, F, G so found are the least in the sequence of their ratios, since no number smaller than a unit can be given.

PROBL. 4. PROPOS. 36. [Margin: xxxvj.]

Given two numbers, to find the least number that they measure.

Let it be required to find the least of all the numbers that the given numbers A and B measure. First let the given numbers A and B be prime to one another, and let them, multiplying each other, make C.

A.... B.....
 C.....
 D ————.
 E — F —

I say that C is the least number that A and B measure. That they measure it is evident. For since C is produced from A into B, or from B into A, A measures C by B, and B measures that same C by A [Margin: a | 7.pron — Axiom 7]. Both A and B, therefore, measure C. That C is the least of all the numbers that A and B measure we shall demonstrate as follows. If C is not the least, let A and B measure, if possible, another number D smaller than C; and let A measure D by E, and B measure that same D by F. On this supposition, D is produced both from A into E and from B into F, and conversely [Margin: b | 9.pron — Axiom 9]. Now since the same number D is made from A (the first term) into E (the fourth), and from B (the second) into F (the third), it follows that as A, the first, is to B, the second, so is F, the third, to E, the fourth [Margin: c | 19.sept — *Elements* VII.19].

Therefore A and B—since they are supposed prime to one another, and for that reason the least in their ratio [Margin: d | 23.sept — *Elements* VII.23]—measure F and E equally [Margin: e | 21.sept — *Elements* VII.21], namely A measures F and B measures E.

For since A, multiplying B and E, makes C and D, it follows that C is to D as B is to E [Margin: f | 17.sept — *Elements* VII.17]. And accordingly, since B measures E, as was shown, C, the greater number, will also measure D, the smaller. Which is absurd. Therefore A and B will not measure any number smaller than C; and so C is the least of all the numbers they measure.

Next, let the given numbers A and B not be prime to one another. Let C and D be found, the least in the same ratio, so that there are four proportional numbers, namely A is to B as C is to D [Margin: g | 35.sept — *Elements* VII.35]. On this supposition, the same number is made from A (the first term) into D (the fourth), and from B (the second) into C (the third) [Margin: h | 19.sept — *Elements* VII.19]. So let it be made, and call it E. I say that E, produced in this way, is the least number that A and B measure. That they measure it is manifest. For since E is produced both by A multiplied by D and by B multiplied by C, both A and B measure E [Margin: i | 7.pron — Axiom 7]. That E is the least of all the numbers that A and B measure will be proved thus. If E is not the least, let A and B measure, if possible, another number F smaller than E; and let A measure that F by G, and B measure that same F by H. On this supposition, F is made both from A into G and from B into H [Margin: k | 9.pron — Axiom 9]. Now since the same number F is made from A (the first) into G (the fourth) and from B (the second) into H (the third), it follows that as A, the first, is to B, the second, so is H, the third, to G, the fourth [Margin: l | 19.sept — *Elements* VII.19]. Therefore C and D, since they are the least in the ratio of A to B, or of H to G, measure H and G equally [Margin: m | 21.sept — *Elements* VII.21], namely C measures H and D measures G. And since A, multiplying D and G, makes E and F, it follows that E is to F as D is to G [Margin: n | 17.sept — *Elements* VII.17]. And accordingly, since D measures G, as was shown, E, the greater number, will also measure F, the smaller. Which is absurd. Therefore A and B will not measure any number smaller than E; and so E is the least of all the numbers they measure. Hence, given two numbers, we find the least number that they measure. Which was to be done.

COROLLARIVM

From this it follows that if two numbers multiply the least numbers having the same ratio—the greater multiplying the smaller and the smaller the greater—the

number produced is the least that they measure. For, given C and D as the least in the ratio of A to B, it was demonstrated that the number E—formed from A (the smaller) into D (the greater), and from B (the greater) into C (the smaller)—is the least number that A and B measure.

SCHOLIVM

This corollary, in Campanus, is proposition 35 of this seventh book. And the following proposition, in the same author, is the corollary to proposition 35.

THEOR. 33. PROPOS. 37. [Margin: xxxvi.]

If two numbers measure some number, then the least number that they measure will also measure that same number.

Let two numbers A and B measure any number CD, and let E be another number, the least that these same A and B measure.

A... B...
C..... F — D
E.....

For if E does not measure CD, then when E is subtracted from CD as many times as it can be taken away, some number smaller than E will be left over. So let E, subtracted from CD as often as possible, leave—if it can—a number FD smaller than itself, such that E measures the part taken away, CF. Now since both A and B measure E, and E measures CF, both A and B measure CF as well [Margin: a | 11.pron — Axiom 11]. Therefore, since A and B measure the whole CD and the part taken away, CF, they also measure the remainder FD [Margin: b | 22.pron — Axiom 22]. But FD is smaller than E. So E is not the least number that A and B measure—which is absurd and contrary to the hypothesis. Therefore E measures CD. So if two numbers measure some number, and so on. Which was to be demonstrated.

PROBL. 5. PROPOS. 38. [Margin: xxxvj.]

Given three numbers, to find the least number that they measure.

Let it be required to find the least number that three given numbers A, B, C measure. When D, the least number that the two numbers A and B measure, has been found [Margin: a | 36.sept — *Elements* VII.36], the remaining number C either measures that same D or does not. First let C measure D, so that all three numbers A, B, C measure D.

A... B.... C.....
D.....
E _____

I say that the number D, found as the least that A and B measure, is also the least that the three numbers A, B, C measure. For if D is not the least, let A, B, C measure, if possible, another number E smaller than D. Then, since A and B measure E, which is smaller than D, D will not be the least number that A and B measure—which is absurd and contrary to the hypothesis. Indeed, since A and B measure E, and D is the least that these same A and B measure, D, the greater, will also measure E, the smaller [Margin: b | 31.sept — *Elements* VII.31]. Which is absurd.

But now let C not measure D as found. Then, when E, the least number that C and D measure, has been found [Margin: c | 36.sept — *Elements* VII.36], I say that E is the least number that A, B, C measure.

A.. B... C....
D.....
E.....
F _____

That they measure it is shown thus. Since A and B measure D, and D measures E, A and B also measure E [Margin: d | 11.pron — Axiom 11]. But C also measures that same E. Therefore all three numbers A, B, C measure E. That E is the least that A, B, C measure will be proved in this way. If E is not the least, let A, B, C measure, if possible, another number F smaller than E. Then, since A and B measure F, the number D—namely the least found that the numbers A and B measure—also measures that same F [Margin: e | 37.sept — *Elements* VII.37]. And so, since C and D measure F, which is smaller than E, E will not be the least that C and D measure—which is absurd and contrary to the hypothesis. Indeed, since C and D measure F, the number E—the least that these same C and D measure—will also measure F, the greater measuring the smaller [Margin: f | 37.sept — *Elements* VII.37]. Which is absurd. Therefore A, B, C will not measure any number smaller than E; rather E itself is the least. Hence, given three numbers, we find the least number that they measure. Which was to be done.

COROLLARIVM

From these results it follows that if three numbers measure some number, then the least number that they measure also measures it. For in the last part of this demonstration, from the fact that A, B, C were supposed to measure F, it was shown that E—the least that A, B, C measure—also measures that same F.

SCHOLIUM

We can show this corollary in another way, exactly as in proposition 37 of this book. For let A, B, C measure any number DE, and let F be the least that these same A, B, C measure.

A. B... C....

D..... G — E

F.....

I say that F also measures DE. For if it does not, let it measure a part DG of it and leave a remainder GE smaller than itself. Now since A, B, C measure F, and F measures DG, A, B, C also measure that same DG [Margin: g | 11.pron — Axiom 11]. And accordingly, since they are supposed to measure the whole DE as well, they will also measure the remainder GE, which is smaller than F [Margin: h | 12.pron — Axiom 12]. Therefore F will not be the least that A, B, C measure—which is absurd and contrary to the hypothesis. So F measures DE.

By the same reasoning, given more numbers than three, we shall find the least number that they measure, and this same corollary will hold. For if the given numbers are four, one must first find the least that three of them measure; if five, the least that four measure, and so on. Everything else is to be carried out as was said for three numbers.

THEOR. 34. PROPOS. 39. [Margin: xxxvij.]

If some number measures a number, then the number it measures will have a part named after the measuring number.

Let the number B measure the number A. I say that A has some part named after B.

A.....

B.... C...

For let B measure A as many times as there are units in the number C. Since, then, the unit measures C just as B measures A [Margin: a | 15.sept — *Elements* VII.15], the unit is accordingly the same part of B that C is of A. But the unit is a part of B named after B, as we taught at definition 2 of this book. Therefore C too is a part of A, named after that same B. So if some number measures a number, and so on. Which was to be shown.

THEOR. 35. PROPOS. 40. [Margin: xxxvii.]

If a number has any part whatever, then the number named after that part will measure it.

Let the number A have a part B, after which the number C is named. I say that C measures A.

A.....
B... C.....

For since the part B is named after C, and the unit is likewise a part of C named after that same C, the unit measures C just as B measures A. Conversely, then, the unit measures B just as C measures A [Margin: a | 15.sept — *Elements* VII.15]. So if a number has any part whatever, and so on. Which was to be demonstrated.

PROBL. 6. PROPOS. 41. [Margin: xxxix.]

To find a number which, being the least, has given parts.

Let the given parts be A, B, C, and let it be required to find the least number having the given parts. Let the numbers named after the parts A, B, C—that is, the numbers that name them—be D, E, F; and let G be the least number that D, E, F measure [Margin: a | 38.sept — *Elements* VII.38].

D.. A, Secunda [a half]
E... B, Tertia [a third]
F.... C, Quarta [a fourth]
G.....
H —————

I say that G is the least number having the given parts A, B, C. That it has such parts is shown thus. Since D, E, F measure G, G has parts named after D, E, F [Margin: b | 39.sept — *Elements* VII.39]—that is, the parts A, B, C, since these are named after D, E, F. And that G is the least having those parts is clear. For if it is not the least, let H, smaller than G, have, if possible, the same parts A, B, C. Then, since H has the parts A, B, C, the numbers D, E, F—named after the parts A, B, C—measure it [Margin: c | 40.sept — *Elements* VII.40]. And so, since H is smaller than G, G will not be the least that D, E, F measure—which is absurd and contrary to the hypothesis. Therefore no number smaller than G will have the given parts A, B, C; rather G itself is the least. Hence we have found a number which, being the least, has the given parts. Which was to be done.

SCHOLIUM (to Proposition 41, continued)

[Clavius's commentary, continuing the scholium begun on the previous page.]

If we take the numbers I, K, L by which the numbers D, E, F measure G, then I, K, L will be the given parts A, B, C of G — that is, the parts named after D, E, F. For since D, E, F measure G by way of I, K, L, the unit measures I, K, L in the same way that the numbers D, E, F measure G. [Margin: VII.15] Conversely, then, the unit measures D, E, F just as the numbers I, K, L measure G; and so the unit is the same part of D, E, F as the numbers I, K, L are of G. Since, therefore, the unit is a part of D, E, F named after them, I, K, L will likewise be parts of G named after D, E, F.

A, Second D.. I.....

B, Third E... K.... [Margin: VII.15]

C, Fourth F.... L...

G.....

From this it follows that the least number which any given numbers measure is the least having parts named after the measuring numbers. For it has been shown that G — the least number that D, E, F measure — is the least that has the parts A, B, C, which are I, K, L, named after the measuring numbers D, E, F.

Now, as Campanus says, if the least number found having the given parts is doubled, tripled, and so on, we obtain the second number after the least, then the third, the fourth, and so on, each containing the same parts. For once G, the least, has been found having the parts A, B, C named after D, E, F, let its double be the number H, its triple I, and so on. I say that H is the second number that has these same parts A, B, C named after the numbers D, E, F, that I is the third, and so on — so that neither between G, the least, and its double H, nor between H, the double, and I, the triple, and so on, does any other number fall having the same parts; rather, H, I alone and the remaining multiples of G contain the said parts. That H and I, and so on, have the parts A, B, C named after D, E, F, we show as follows. Since D, E, F measure G, by construction, and G measures H, I, and the remaining multiples of G, [Margin: Axiom 11] D, E, F will also measure those same numbers H, I and the remaining multiples of G. [Margin: VII.39] Therefore H, I and the remaining multiples of G will have parts named after the measuring numbers D, E, F — namely the parts posited as A, B, C.

A, D..

B, E...

C, F....

G.....

H.....

I.....

K----- M----- L [Margin: Axiom 11]
 N----- P----- O [Margin: VII.39]

That H, the double of the least G, is the second number having those parts, we shall demonstrate in this way. If H is not the second, let there be, if possible, another number KL prior to it — that is, one greater than the least G and less than H, the double of G. When G is subtracted from KL, let the remainder ML be less than G. Since, then, KL has the parts A, B, C, [Margin: VII.40] the numbers D, E, F, named after those parts, will measure it. And therefore G also, the least that D, E, F measure, will measure that same KL, by the corollary of Proposition 38 of this book. But G also measures the piece KM cut off, which is equal to itself. [Margin: Axiom 12] Therefore G will measure the remainder ML as well — the greater measuring the less. This is absurd. Therefore no other number falling between G and H has the parts A, B, C; and consequently H is the second number having parts of this kind.

In just the same way we shall show that I, the triple of G, is the third number having the said parts. For if it is not the third, let there be, if possible, another — say NO — prior to it, greater than H the double, but less than I the triple. When H, the double, is subtracted from NO, let the remainder PO be less than G. Since, then, NO has the parts A, B, C, [Margin: VII.40] the numbers D, E, F, named after those parts, will measure it. And for that reason G also, the least that D, E, F measure, will measure that same NO, by the corollary of Proposition 38 of this book. But G also measures NP, the piece cut off, which is equal to H the double. [Margin: Axiom 12] Therefore G will measure the remainder PO as well — the greater measuring the less. This is absurd. Therefore no other number falling between H and I has the given parts A, B, C; and consequently I is the third number having those parts. By the same reasoning the quadruple of G will be the fourth, the quintuple the fifth, and so on.

To this place may also be referred the following problem.

[Appended problem] To find a number which, being the least, has given parts under this condition: that each part contains the following part.

Let the given parts be A, B, C, and let it be required to find the least number that has them in this order, so that part A contains part B, and part B contains part C. Let D, E, F be the numbers named after the parts A, B, C. Let G be made from E multiplied into F, and H from D multiplied into G. I say that H is the least number sought.

A, Second B, Third C, Fourth
 D... E... F....

G.....
H.....

. Unit

I----- - - - -
K-----
L-----
M---

That it has the given parts in the aforesaid order is easily demonstrated. For since H is made from D multiplied into G, G will be contained in H as many times as the unit is in D. But the unit is a part of D, named after D itself. Therefore G too is a part of H, named after the same D; and so H has the part A — namely the number G — named after D. Next, since G is made from E multiplied into F, by the same reasoning F will be a part of G, named after E; and so A, the part of H (that is, the number G), has the part B — namely the number F — named after E. Finally, since F has the unit as a part named after F itself, it is clear that B, the part of G (namely the number F), also has the part C — named after F, namely the unit. Hence the number found, H, has the part A; part A has part B; and part B has part C.

That H is the least containing those parts in this order will be shown by the following reasoning. For if it is not the least, let a smaller number I, if possible, have the same parts in the same order, so that K is the part A of I, named after the number D; L the part B of K, named after E; and M the part C of L, named after F. Since, then, K is a part of I named after D, K will be contained in I as many times as the unit is in D; [Margin: Def. 15] and thus I will be produced from D multiplied into K. By the same reasoning K will be produced from E multiplied into L, and L from F multiplied into M. So, since D, multiplying G and K, produces H and I, [Margin: VII.17] it will follow that as H is to I, so is G to K. By the same reasoning, since G and K are produced from E multiplied into F and into L, as G is to K, so is F to L. And since F and L are produced from F multiplied into the unit and into M, as F is to L, so is the unit to M. Since, then, as H is to I, so G is to K; and as G is to K, so F is to L; and as F is to L, so the unit is to M — it will follow, by the lemma to Proposition 14 of this book, that as H is to I, so is the unit to M.

H. G. F. Unit
I. K. L. M

But H is assumed to be a number greater than the number I. Therefore the unit too will be greater than the number M — the part greater than the whole. This is

absurd. Therefore no number smaller than H has the parts A, B, C in the aforesaid order, but H itself is the least. Which was what was proposed.

But if there are more than three parts, exactly the same procedure and demonstration are to be followed. For example, if the numbers 2, 3, 4, 5, 6 are the denominators of the parts, then 30 will be produced from 5 into 6; 120 from 4 into 30; 360 from 3 into 120; and finally 720 from 2 into 360. For the number 720 will have a part named after 2; that part another named after 3; that another named after 4; that another named after 5; and that, finally, a part named after 6 — as is evident.

If we double, triple, and so on, the number H once it has been found, we shall obtain other numbers — the second, third, fourth, and so on — which have the same parts in the same order, though doubled or tripled, and so on. For G doubled or tripled, and so on, will be the half of H doubled or tripled, and so on — just as G is the half of H. And the same reasoning holds for the other parts.

FINIS ELEMENTI SEPTIMI

[End of the Seventh Element.]

EUCLID'S EIGHTH ELEMENT

[EVCLIDIS ELEMENTVM OCTAVVM]

THEOREM 1. PROPOSITION 1.

[Margin: i.]

If there are any number of numbers in continued proportion, and the extremes of them are prime to one another, then they are the least of all having the same ratio as they.

[Clavius's commentary.]

Let there be any numbers in continued proportion whatsoever — A, B, C, D — and let the extremes A and D be prime to one another. I say that A, B, C, D are the least of all having the same ratio as they. For if they are not the least, there will be others smaller than they, in the same ratio. Let there be, then, if possible, others smaller than they in the same ratio: E, F, G, H. Since, therefore, there are the numbers A, B, C, D, and others equal to them in multitude — E, F, G, H — taken two by two and in the same ratio, [Margin: VII.14] it will follow, *ex aequali* (by

equality of ratios), that as A is to D, so E is to H. [Margin: VII.23] But A and D are the least in their ratio, since they are assumed to be prime to one another. [Margin: VII.21] Therefore A will measure E, and D will measure H alike — the greater measuring the less. This is absurd. Therefore there are no numbers smaller than A, B, C, D in the same ratio as they, but they themselves are the least. And consequently, if there are any number of numbers in continued proportion, etc. Which was to be demonstrated.

A..... E--
 B..... F----
 C..... G---- [Margin: VII.14]
 D..... H----- [Margin: VII.23]
 [Margin: VII.21]

SCHOLIUM.

It is clear from the demonstration that the numbers A, B, C, D are the least in the continuation of their ratios — whether the extremes A and D are prime to one another, and whether A, B, C, D are in continued proportion (as Euclid intends) or not. This is shown by the following example, in which all the ratios are different, whereas in the previous one the same ratio is always found.

A.... E---
 B..... F----
 C..... G----
 D..... H----

PROBLEM 1. PROPOSITION 2.

[Margin: ij.]

To find the least numbers in continued proportion, as many as anyone may require, in a given ratio.

[Clavius's commentary.]

Let A and B be the least numbers of a given ratio, and let it be required first to find the three least numbers in the given ratio of A to B. Let A, multiplying itself and the number B, produce C and D. Then let B, multiplying itself, produce E. I say that C, D, E are the three least in the ratio of A to B. That they are proportional in the given ratio of A to B — that is, that as A is to B, so C is to D, and D to E — we show as follows. Since C and D are produced from A multiplied into A and into B, [Margin: VII.17] it will follow that as A is to B, so C is to D...

A.. B...
 C.... D..... E.....

F, 8. G, 12. H, 18. I, 27.
K, 16. L, 24. M, 36. N, 54. O, 81.

Euclid, *Elements* Book VIII — continued

(Continuation of the demonstration of Prop. 2: to find the least numbers in continued proportion, as many as you wish, in a given ratio. The worked instance uses $A = 2$, $B = 3$ (the given ratio $2 : 3$), from which come the three numbers $C = 4$, $D = 6$, $E = 9$; the four $F = 8$, $G = 12$, $H = 18$, $I = 27$; and the five $K = 16$, $L = 24$, $M = 36$, $N = 54$, $O = 81$.)

A .. B ...
C D E
F, 8. G, 12. H, 18. I, 27.
K, 16. L, 24. M, 36. N, 54. O, 81.

so that C is to D. Again, since D and E are produced when B is multiplied into A and into B, it likewise follows that as A is to B, so D is to E. [Margin: b, *Elements* VII.17] The three numbers C, D, E are therefore in continued proportion in the given ratio A to B. That they are also the least in that same ratio is shown as follows. The extremes C and E have been produced from A and B multiplied into themselves; and A and B are prime to one another, [Margin: c, *Elements* VII.24] because they are the least in their ratio. Hence the extremes C and E too will be prime to one another. [Margin: d, *Elements* VII.29 [?]] Therefore C, D, E are the least numbers in the ratio A to B. [Margin: e, *Elements* VIII.1]

Now let F, G, H be produced from A multiplied into the three numbers already found, C, D, E, and let I be produced from B multiplied into the last of them, E. I say that the four numbers F, G, H, I are the least in the same given ratio A to B. That they are proportional in the ratio A to B is shown thus. Since A, by multiplying C, D, E, produced F, G, H, these numbers F, G, H will have — by what we established at Prop. 18 of Book 7 — the same ratios as C, D, E, that is, the ratio that A has to B. [Margin: f, *Elements* VII.18] Again, since A and B, by multiplying E, produced H and I, it likewise follows that as A is to B, so H is to I. F, G, H, I are therefore in continued proportion in the given ratio A to B. [Margin: g, *Elements* VII.24] That they are the least in the given ratio becomes plain as follows. Since A and B, being the least in their ratio, are prime to one another, and since C and E were produced from A and B multiplied into themselves, and since F and I in turn were produced from A and B into C and E — F, that is, from A into C, and I from B into E — the extremes F and I too will be prime to one another. [Margin: h, *Elements* VII.29 [?]] Hence F, G, H, I are the least in their ratio, which is A to B. [Margin: i, *Elements* VIII.1]

In just the same way, if from A multiplied into the four numbers found, F, G, H, I, there arise K, L, M, N, and from B into the last, I, there arises O, then K, L, M, N, O will be five least numbers in the given ratio A to B. By the same procedure we shall find six, seven, eight, and so on. In this way we obtain the least numbers in continued proportion, and so forth. Which was to be done.

Scholium

(A second route to the same numbers, working downward from B rather than upward from A.)

A .. B ...
 C D E
 F, 8. G, 12. H, 18. I, 27.
 K, 16. L, 24. M, 36. N, 54. O, 81.

By the same method the four least proportional numbers will be produced from the multiplication of B into the three numbers found, E, D, C, and from the multiplication of A into C. For let I, H, G be produced from B into E, D, C, and F from A into C. Once this is done, [Margin: k, *Elements* VII.18] the numbers F and G will have the same ratio as A and B, since A and B, by multiplying C, produced F and G. Likewise G, H, I will — by what we demonstrated at Prop. 18 of this book — stand in the same ratio as C, D, E, that is, in the ratio of A to B, since B, by multiplying C, D, E, produced G, H, I. Therefore F, G, H, I are in the same ratio as A to B. That they are the least will be shown as before.

In just the same way, if from B multiplied into the four numbers found, I, H, G, F, there arise O, N, M, L, and from A into F there arises K, then K, L, M, N, O will be five least numbers in the given ratio A to B. By the same procedure we shall find six, seven, eight, and so on. Thus, whether A is multiplied into all the numbers found and B into the last, or B into all those found and A into the first, there will always be produced more numbers, and least numbers, in the same given ratio A to B. For Euclid demonstrated that the four numbers F, G, H, I are produced from A into C, D, E and from B into E; whereas we have shown that these same numbers are produced from B into E, D, C and from A into C.

Corollary I

From this it follows that, if three least numbers are in continued proportion, their extremes are squares; and if there are four, they are cubes. For of the three least numbers C, D, E the extremes C and E were made from A and B multiplied into themselves, and are therefore equal-times-equal; hence by Definition 18 they are squares. In the same way, of the four least numbers F, G, H, I the extremes F and I

are generated from A and B — the sides — multiplied into C and E, the squares of A and B respectively; that is, the number F is produced from the mutual multiplication of the three equal numbers A, A, A, and the number I from the mutual multiplication of the three equal numbers B, B, B, so that each is equal-times-equal-times-equal. By Definition 19, therefore, they are cubes.

Likewise, if there are five least numbers in continued proportion, their extremes will be squares of squares; and so, if there are still more, their extremes will be numbers under yet other names, of the kind usually explained in algebra. All this is evident from the multiplications by which the proportional numbers are generated in accordance with this proposition.

Nor can anyone object that this corollary holds only for three or four least numbers in continued proportion found by the technique taught in this proposition, and not for any three or four least numbers in continued proportion however proposed. No one can say this, because any given three or four least numbers in continued proportion are exactly equal to those found by this proposition. For if the given numbers were smaller than those found, the ones found would not be the least, which is contrary to this proposition; and if the given numbers were larger than those found, then the given ones would not be the least, which is contrary to the hypothesis. This will be made perfectly clear as follows.

A — B — C —
D, 2. E, 3.
F, 4. G, 6. H, 9.

Let three least numbers A, B, C in continued proportion be given. Taking the two least numbers D, E in the ratio A to B or B to C, from the scholium to Prop. 35 of Book 7, let there be found by this proposition, from these two least numbers D, E, the three least numbers F, G, H in the same ratio. I say that the three given numbers A, B, C do not differ from the three found, F, G, H. For A will necessarily be equal to F; and so, since as A is to B, so F is to G, it will also follow, by alternation, that as A is to F, so B is to G. Hence, if A is equal to F, B too will be equal to G; and by the same reasoning we shall show that C is equal to H. Therefore the three numbers A, B, C are equal, each to each, to the three numbers F, G, H. The same reasoning holds for four or more. But if A were said to be smaller than F, we should show by the same argument that B and C too are smaller than G and H; and then F, G, H would not be the least — which is absurd and contrary to this proposition. And if A were granted larger than F, then by the same argument B and C too would be larger than G and H, and so A, B, C would not be the least — which is contrary to the hypothesis. Since, then, A is neither

smaller nor larger than F, it is altogether equal to it; and therefore, as has been shown, B and C too will be equal to G and H. Which is what was proposed.

Corollary II

It is likewise plain that the extreme numbers of any set of proportionals found, according to this proposition, as least numbers in a given ratio, are prime to one another. This we shall easily show from the multiplications by which the proportional numbers are produced, and from Prop. 29 of Book 7 — just as it was demonstrated of the extreme numbers C, E and F, I, and so on.

Corollary III

It is settled, moreover, that the two least numbers in a given ratio measure all the middle numbers, however many, of the least set in that same ratio; for these middle numbers are produced by the multiplication of the two least into certain other numbers. Thus, in the given example, the middle number D is produced from A into B; the middle numbers G, H from A into D and E, or from B into C and D; likewise the middle numbers L, M, N from A into G, H, I, or from B into F, G, H — as is apparent both from Euclid's demonstration and from the one we set out in the scholium.

Scholium

Since both the numbers A, C, F, K and the numbers B, E, I, O are, by construction, in continued proportion from unity — the ratios of the former being reckoned by the number A, those of the latter by B, as will be shown more clearly at Prop. 9 of this book — it comes about that the extreme numbers of any set of least numbers in continued proportion are the last terms of as many numbers in continued proportion from unity, reckoned by the least numbers of the given ratio, as there are least proportionals proposed. Thus in the example above you see that C and E, the extremes of the three numbers in continued proportion, are the last of three numbers in continued proportion from unity, whose ratios are reckoned by A and B; while F and I, the extremes of the four in continued proportion, are the last of four numbers in continued proportion from unity, and so on. Hence, if anyone wishes to find any number of least whole numbers in a given ratio in continued proportion — the task being an easy one, since it begins from unity — he will readily accomplish it by this route. Once the two least numbers in the given ratio have been found, let there be taken, from unity, as many numbers in continued proportion, in the ratio reckoned by the smaller of those two, as there are least numbers proposed to be found. For the last of those numbers in

continued proportion from unity will be the first of the proportionals to be found; and if this is multiplied by the denominator of the given ratio, the second will be generated; and if this in turn is multiplied by the same denominator, the third will be generated; and so on. For example, if one wants eight least numbers in the sesquialter ratio: having found the two least numbers in the sesquialter ratio, 3 and 2, one must take eight numbers from unity in continued proportion in the double ratio, whose denominator is 2, in this manner: 1. 2. 4. 8. 16. 32. 64. 128. For the last of these, 128, will be the first of the eight numbers to be found, which, when multiplied by the denominator, [gives] 128. 192. 288. 432. 648. 972. 1458. 2187 — if the greater numbers are referred to the smaller. And the same holds for the rest.

Boethius [Boethius] and others explain this rule in the same way, as we too set it out in Book 5, in our treatment of proportions. Let there be taken, from unity, as many continued multiples according to the denomination of the aliquot part mentioned in the given ratio as there are continued proportionals to be found in the given ratio; for the last number will be the first of those to be found. Thus in the given example, since eight least numbers in the sesquialter ratio are wanted, one must take eight double numbers from unity, as before, which are reckoned by the denominator of the second, or half, part mentioned in [the name] "sesquialter" — that is, by 2. And if five least numbers in the sesquiquintan ratio are wanted, one must take five quintuple numbers from unity, as here: 1. 5. 25. 125. 625. For 625 is the first of the five numbers in the sesquiquintan ratio, as appears here: 625. 750. 900. 1080. 1296. And so for the rest. Furthermore, by this technique it is impossible to find — surprising as it may seem — more continued proportionals than were proposed. For after 1296 in the last example no other number will be found that has to 1296 the sesquiquintan ratio; just as in the earlier example no other number can be found after 2187 that has to 2187 the sesquialter ratio. The reason for this is that the extremes found by this technique are prime to one another, as is clear from the demonstration. For it thus comes about that the last term cannot stand to any other number as the first stands to the second — as will be demonstrated at Prop. 17 of Book 9.

THEOR. 2. PROPOS. 3.

[Margin: iij.]

If there are any number of least numbers in continued proportion among all that have the same ratio with them, their extremes are prime to one another.

A, 8. B, 12. C, 18. D, 27.
E, 2. F, 3.
G, 4. H, 6. I, 9.
K, 8. L, 12. M, 18. N, 17. [recte: 27]

Let A, B, C, D be least numbers in continued proportion. I say that the extremes A and D are prime to one another. For let two least numbers E, F be found in the ratio A to B, or B to C, or C to D. Then, by the route of the preceding problem, let three least numbers G, H, I be found in the same ratio, and also four, K, L, M, N, and so on, until the multitude K, L, M, N equals the multitude A, B, C, D. Now since A, B, C, D are the least in their ratio, and K, L, M, N have likewise been taken as the least in the same ratio, equal in multitude to them, they will be equal each to each — lest numbers smaller than the least be assigned in the same ratio — namely A to K and D to N. This we demonstrated geometrically at the end of Corollary 1 of the preceding proposition. But by Corollary 2 of the preceding proposition K and N are prime to one another. Therefore A and D too are prime to one another. And so, if there are any number of least numbers in continued proportion among all that have the same ratio with them, and so forth. Which was to be shown.

Scholium

This proposition is to be understood of least numbers in *continued* proportion, as can be seen from its demonstration and from Euclid's words. For if A, B, C, D were not in continued proportion, it would be impossible, by the preceding problem, to find K, L, M, N, an equal number of least terms in the same ratio. Indeed, if the ratios are diverse, there will sometimes be given least numbers successively proportional — that is, continued in their several diverse ratios — whose extremes are not prime to one another. For these numbers, 6. 10. 13 [?]. 18, are the least in the continuation of their ratios, and yet the extremes 6 and 18 are composite with respect to one another. This theorem, then, is the converse of the first theorem, so far as it concerns numbers in continued proportion. For any number of continued proportionals whose extremes are prime to one another are the least in their ratio, as is clear from the first theorem; and conversely, the extreme numbers of any set of least continued proportionals are prime to one another, as has here been demonstrated. But the first theorem, in so far as it regards numbers *not* in continued proportion, cannot be converted. For although any number of numbers not in continued proportion,

whose extremes are prime to one another, may be the least in the continuation of their ratios — as is clear from the scholium to Prop. 1 of this book — nevertheless it is not always true, conversely, that the extreme numbers of any set of least

numbers not in continued proportion are prime to one another, as is evident in the example just brought forward, 6. 10. 13 [?]. 18.

PROBL. 2. PROPOS. 4.

[Margin: iv.]

Given any number of ratios in least numbers, to find the least numbers in continued proportion in the given ratios.

Let there first be given two ratios in least numbers, A to B and C to D; and let it be required to find three least numbers in continued proportion in the given ratios.

A, 6. B, 5. C, 4. D, 3.
H, 4. F, 24. E, 20. G, 15.
I — K — L —

Having found the least number E, which B and C — the second and third terms — measure, [Margin: a, *Elements* VII.36] let A measure F as many times as B measures E; and let G be measured by D as many times as C measures that same E. I say that F, E, G are the least in the given ratios. That they have the given ratios in succession we show thus. Since A and B measure F and E equally, that is, by the same number, let them measure by H. This being posited, A and B, multiplying H, will produce F and E. [Margin: b, Axiom 9] Hence, as A is to B, so F is to E.

[Margin: c, *Elements* VII.18] By the same reasoning, since C and D measure E and G equally, it will be that as C is to D, so E is to G. Therefore F, E, G are in continued proportion in the ratios A to B and C to D. That they are the least will be plain as follows. If they are not the least, there will be some numbers smaller than they, each to each, in the same ratios: let these be, if possible, I, K, L. Since A and B are the least in their ratio, they will measure I and K equally, these being in the same ratio — namely B measuring K, the consequent measuring the consequent.

[Margin: d, *Elements* VII.21] And by the same reasoning C and D will measure K and L equally — namely C measuring K, the antecedent measuring the antecedent. Therefore, since B and C both measure K, the least number E, which B and C measure, will also measure that same K — the greater measuring the smaller. [Margin: e, *Elements* VII.37] Which is absurd. There will therefore be no other numbers smaller than F, E, G in the ratios A to B and C to D; and consequently F, E, G are the least in the given ratios.

Let there next be given three ratios in least numbers, A to B, C to D, and E to F; and let four least numbers in continued proportion in the given ratios be required.

A, 6. B, 5. C, 4. D, 3. E, 5. F, 7.
H, 24. G, 20. I, 15. K, 21.
L — M — N — O —

Having again found the least number G, which B — the second term — and C — the third — measure, [Margin: f, *Elements* VII.36] let A measure H as many times as B measures G; and let D measure I as many times as C measures that same G. This done, either E measures I or it does not. Let it first measure it; and let F measure K as many times as E measures I. I say that H, G, I, K are the least in the given ratios. That they are in continued proportion in the given ratios is settled. For since A and B measure H and G equally, it will be, as before, that as A is to B, so H is to G. By the same reasoning it will be that as C is to D, so G is to I; and as E is to F, so I is to K — since C and D measure G and I, and E and F measure I and K, equally. Therefore H, G, I, K are in continued proportion in the given ratios. That they are the least we show thus. If they are not the least, let there be, in the same ratios, numbers smaller than they, each to each, if possible: L, M, N, O. Since A and B are the least in their ratio, they will measure L and M equally, these having the same ratio — namely B, the consequent, measuring the consequent M. [Margin: g, *Elements* VII.21] In the same way C and D will measure M and N equally — namely C, the antecedent, measuring the antecedent M. And so, since B and C both measure M, the least number G, which B and C measure, will also measure that same M — the greater measuring the smaller. [Margin: h, *Elements* VII.37] Which is absurd. There will therefore be no other numbers in continued proportion in the given ratios smaller than G, H, I, K; and hence H, G, I, K are the least in the given ratios.

But now let E *not* measure I.

A, 6. B, 5. C, 4. D, 3. E, 2. F, 7.
H, 24. G, 20. I, 15.
M, 48. L, 40. K, 30. N, 105.
O — P — Q — R —

Having then found the least number K, which E and I measure, [Margin: i, *Elements* VII.36] let G measure L, and H measure M, as many times as I measures K; and let F measure N as many times as E measures K. I say that M, L, K, N are the least in the given ratios. For since H, G, I measure M, L, K equally, it will be, as above, that as H is to G, so M is to L; and as G is to I, so L is to K. But it is by the same reasoning that as H is to G, so A is to B, and as G is to I, so C is to D — since A, B measure H, G, and C, D measure G, I, equally. Therefore as A is to B, so M is to L; and as C is to D, so L is to K. And as E is to F, so too is K to N, since E and F measure K and N equally. M, L, K, N are therefore in continued proportion in the

given ratios. I say further that they are the least in the same ratios. For if they are not the least, let there be assigned smaller numbers than they in the same ratios, O, P, Q, R, if possible. Since A, B are the least in their ratio, they will measure O, P equally, these being in the same ratio — namely B, the consequent, measuring the consequent P. [Margin: k, *Elements* VII.21] By the same argument C, D will measure P, Q equally, namely the antecedent C measuring the antecedent P; and so, since B and C both measure P, [Margin: l, *Elements* VII.37] G, the least number that B and C measure, will also measure that same P. Now since it has been shown that as G is to I, so L is to K, that is, so P is to Q, it will be, by alternation, that as G is to P, so I is to Q; and therefore, since G measures P, I too will measure Q. But E also measures that same Q, [Margin: m, *Elements* VII.21] because E and F, the least in their ratio, measure Q, R — of the same ratio — equally, namely the antecedent E measuring the antecedent Q. Therefore, since I and E measure Q, K too — which I and E measure as their least — will measure that same Q, [Margin: n, *Elements* VII.37] the greater measuring the smaller. Which is absurd. There will therefore be given no other numbers in continued proportion in the given ratios smaller than M, L, K, N; and consequently M, L, K, N themselves are the least.

And if four ratios are given, one must first find numbers in continued proportion in the three former ratios, and then proceed with the fourth, later, ratio as we have just done with the third given ratio.

A, 6. B, 5. C, 4. D, 3. E, 2. F, 7. S, 3. T, 2.
M, 48. L, 50. [recte: 40] K, 30. N, 105. O, 70.

For example, if the fourth given ratio is in the least numbers S to T, we shall first find M, L, K, N, the least numbers in continued proportion in the three ratios A to B, C to D, and E to F. Then, if S measures N, we shall take O, which T measures as many times as S measures N; for M, L, K, N, O will then be in continued proportion...

[Continuation of Book VIII, Prop. 4 (PROBL.), from the preceding page — its final clauses, then the Scholium.]

are the least in continued proportion in the given ratios. But if S does not measure N, [Margin: c — *Elem.* VII.36] we take O, the least number that both S and N measure. Then, as many times as N measures O, let K measure P, and L measure Q, and M measure R the same number of times; and

A, 6. B, 5. C, 4. D, 3. E, 2. F, 7. S, 9. T, 2.
M, 48. L, 40. K, 30. N, 105.
R, 144. Q, 120. P, 90. O, 315. V, 70.

as many times as S measures that same O, let T measure V. Once this is done, R, Q, P, O, V will be the least numbers in continued proportion in the ratios A to B, C to D, E to F, and S to T. All of this we will prove by the same argument as before. We will proceed in the same way if five or more ratios are given in least numbers. And so, given any number of ratios whatever in least numbers, etc. Which was to be done.

SCHOLIVM

If all the given ratios, or some of them, are given in numbers that are not the least, we will nonetheless solve the proposed problem; but the given ratios must first be exhibited in their least numbers before we undertake to find the least numbers in continued proportion, as is clear from the demonstration.

This problem differs, moreover, from the one demonstrated in Proposition 35 of Book 7, in that here the numbers are not given in continued proportion — such that each intermediate one is both an antecedent and a consequent — even though the ratios are different, exactly as they are there.

Furthermore, once the least numbers in continued proportion in the given ratios have been found, if they are multiplied by any one and the same number, others will be produced in continued proportion in the same ratios, as is established from what we demonstrated at Proposition 18 of Book 7.

THEOR. 3. PROPOS. 5.

[Margin: v.]

Plane numbers have to one another a ratio compounded of their sides.

Let A and B be two plane numbers, the sides of the first being any C, D, and those of the second E, F. I say that the ratio of A to B is compounded of the ratios of the sides — namely of the ratios C to E and D to F, or of the ratios C to F and D to E — in such a way that the sides of the one are antecedents and those of the other consequents, just as we taught in Proposition 23 of Book 6. Let D and E, multiplying each other, make the number G. Since, then, D multiplying C and E made A and G, it follows [Margin: a — Elem. VII.17] that A is to G as C is to E. And in the same way, since E multiplying D and F made G and B, G is to B as D is to F. Therefore A, G, B are in continued proportion in the ratios of the sides C to E and D to F. Now the ratio of A to B is compounded of the ratios A to G and G to B [Margin: b — Elem. def. 27 [?]]. Therefore the same ratio of A to B is compounded of the ratios of the sides C to E and D to F.

A, 24. B, 48.

G, 18.

C, 4. D, 6. E, 3. F, 16.

By the same argument we will show that the ratio of A to B is compounded of the ratios C to F and D to E, if the sides E and F change places and G is made by the mutual multiplication of D and F. For since D, multiplying C and F, makes A and G, it follows [Margin: c — Elem. VII.17] that A is to G as C is to F. Similarly, since F, multiplying D and E, makes G and B, G is to B as D is to E. And so again A, G, B are in continued proportion in the ratios of the sides C to F and D to E. Since, then, the ratio of A to B is compounded of the ratios A to G and G to B, [Margin: d — Elem. def. 27 [?]] it will likewise be compounded of the ratios C to F and D to E.

Therefore plane numbers have to one another a ratio compounded of their sides. Which was to be demonstrated.

A, 24. B, 48.

G, 96.

C, 4. D, 6. F, 16. E, 3.

THEOR. 4. PROPOS. 6.

[Margin: vi.]

If there be any number of numbers in continued proportion, and the first does not measure the second, then neither will any other measure any other.

Let A, B, C, D, E be in continued proportion, and let A, the first, not measure B, the second. I say that no one of them measures any other. That none measures the one immediately following it is evident. For since A is to B as B is to C, C to D, and D to E, while A does not measure B, neither will B measure C, nor C measure D, nor D measure E. That no other of them measures any other we will show as follows. Take the three least numbers F, G, H in the ratio of A to B; [Margin: a — Elem. VIII.2] then, *ex aequali*, A will be to C as F is to H. And since A is to B as F is to G, while A does not measure B, neither does F measure G; and for this reason F will not be a unit — for otherwise F would measure G, since a unit measures every number. Therefore, since F and H are prime to one another [Margin: b — Elem. VIII.3] and F is not a unit, F will not measure H; and on that account neither will A measure C — for it was shown that A is to C as F is to H. By the same reasoning we will show that neither does B measure the third number from it, D, nor C the third, E. And if four least numbers be taken in the ratio of A to B, we will show in like manner that neither does A measure the fourth, D, nor B the

fourth, E. And so on for the rest. If, therefore, there be any number of numbers in continued proportion, etc. Which was to be shown.

A, 16. B, 24. C, 36. D, 54. E, 81.
F, 4. G, 6. G, 9. [recte: H, 9]

SCHOLIVM

If greater numbers be referred to smaller, this proposition may be stated in this manner:

If there be any number of numbers in continued proportion, and the first is not a multiple of the second, then neither will any other be a multiple of any other.

For let A, B, C, D, E be in continued proportion, and let A, the first, not be a multiple of the second, B. I say that no one of them is a multiple of any other. That none is a multiple of the one immediately following it is established. For since A is to B as B is to C, C to D, and D to E, while A is not a multiple of B, neither will B be a multiple of C, nor C of D, nor D of E. And that none of them is a multiple of any other will be confirmed thus. Since D is not a multiple of E, conversely E will not measure D, by what we wrote in Definition 5 of Book 7. Since, then, E, D, C, B, A are numbers in continued proportion — (for since D is to E as C is to D, B to C, and A to B, it follows by conversion that E is to D as D is to C, C to B, and B to A) — and E, the first, does not measure D, the second, [Margin: c — Elem. VIII.6] neither will any other of them measure any other. Therefore, conversely, if greater numbers be referred to smaller, none of them will be a multiple of any other, by what we wrote in the definitions of Book 7. But from these considerations we will also demonstrate the following theorem.

A, 81. B, 54. C, 36. D, 24. E, 16.

If there be any number of numbers in continued proportion, and the first measures the second, then any one whatever will measure any of those following it. And if the first is a multiple of the second, then any one whatever will be a multiple of any of those following it.

Let A, B, C, D, E be in continued proportion, and let the first, A, measure the second, B. I say that each of them measures any of those following it. That each measures the one immediately following it is clear. For since A is to B as B is to C, and D to E, and A measures B, then B will measure C, C measure D, and D measure E. And that any of them measures any of those following it — namely that B measures D and E — will be shown thus. Since B measures C, and C measures D, [Margin: d — 11. pro.

[?]] B will also measure D. Again, since D measures E, [Margin: e — 11. pro. [?]] B (which measures D) will also measure that same E. And so for the rest.

A, 3. B, 6. C, 12. D, 24. E, 48.

But now let the first, A, be a multiple of the second, B. I say that each of them is a multiple of any of those following it. For since A is to B as B is to C, C to D, and D to E, while A is a multiple of B, then B will be a multiple of C, C of D, and D of E.

Therefore each is a multiple of the one immediately following it. Again, since D is a multiple of E, conversely E will measure D; but by the inverse ratio E, D, C, B, A are in continued proportion. Therefore each of them will measure any of those following it, as was demonstrated; and on that account, conversely, if greater be referred to smaller, each will be a multiple of any of those following it.

A, 48. B, 24. C, 12. D, 6. E, 3.

We will also convert this sixth proposition, and the theorem which we demonstrated in the first place in the scholium, in this way:

If there be any number of numbers in continued proportion, and the first — or any other — measures none of those reckoned from the second onward, then neither will the first measure the second. And if the first, or any other, is a multiple of none of those from the second onward, then neither will the first be a multiple of the second.

Let A, B, C, D, E be in continued proportion, and let the first, A, or any other, measure none of the numbers reckoned from the second, B. I say that A, the first, does not measure B, the second. For if A, the first, be said to measure the second, B, then it would also measure — by what we showed in the second place in this scholium — any one whatever of those following. Which is absurd, since neither the first nor any other is supposed to measure any number from the second. Therefore A does not measure B.

A, 16. B, 24. C, 36. D, 54. E, 81.

Now let A, the first, or any other, not be a multiple of any number reckoned from the second, B. I say that A, the first, is not a multiple of B, the second. For if A, the first, be said to be a multiple of B, the second, then it would also be a multiple of any one whatever of those following, from what we demonstrated in the second theorem of this scholium. Which is absurd. For neither the first nor any other is supposed to be a multiple of any number from the second. Therefore A will not be a multiple of B.

A, 81. B, 45. [recte: 54] C, 36. D, 24. E, 16.

THEOR. 5. PROPOS. 7.

[Margin: vii.]

If there be any number of numbers in continued proportion, and the first measures the last, it will also measure the second.

Let A, B, C, D, E be in continued proportion, and let A, the first, measure the last, E. I say that A, the first, also measures B, the second. For if A be said not to measure B, [Margin: a — Elem. VIII.6] then neither will any other measure any; and so A will not measure E either. Which is absurd, for A is supposed to measure E. Therefore A, the first, will measure B, the second; and accordingly, if there be any number of numbers in continued proportion, etc. Which was to be demonstrated.

A, 3. B, 6. C, 12. D, 24. E, 48.

SCHOLIVM

In order to convert the theorem which we demonstrated in the second place in the scholium of the preceding proposition, we will amplify this proposition in the following way:

If there be any number of numbers in continued proportion, and the first — or any other — measures any number reckoned from the second, then the first will also measure the second. And if the first, or any other, is a multiple of any number from the second, then the first will also be a multiple of the second.

Which we will demonstrate in the same way. For if the first be said not to measure the second, or not to be a multiple of it, then neither will any other measure any, nor be a multiple of it, by Proposition 9 and the first theorem of the preceding scholium. Which is contrary to the hypothesis.

THEOR. 6. PROPOS. 8.

[Margin: viii.]

If between two numbers there fall numbers as means in continued proportion, then as many numbers as fall between them as means in continued proportion, so many will also fall as means in continued proportion between others having the same ratio with them.

Let there fall between two numbers A and B the mean proportionals C, D in continued proportion, and let E be to F as A is to B. I say that as many mean proportionals fall in continued proportion between E and F as fall between A and B. For, taking as many least numbers G, H, I, K in the ratio of A to C as there are numbers A, C, D, B, then *ex aequali* A will be to B — and therefore E to F — as G is to K.

A, 24. C, 36. D, 54. B, 81.
 G, 8. H, 12. I, 18. K, 27.
 E, 32. L, 48. M, 72. F, 108.

A, 40. C, 20. D, 10. B, 5.
 G, 8. H, 4. I, 2. K, 1.
 E, 8. L, 4. M, 2. F, 1.

A, 27. C, 9. D, 3. B, 1.
 G, 27. H, 9. I, 3. K, 1.
 E, 108. L, 36. M, 12. F, 4.

Therefore, since G and K are prime to one another, being the extremes of the least numbers, [Margin: a — Elem. VIII.3] and on that account the least in their ratio, [Margin: b — Elem. VII.23] G will equally measure E, and K will equally measure F. [Margin: c — Elem. VII.21] As many times, then, as G and K measure E and F, so many times let H and I measure the other numbers L and M, so that the numbers G, H, I, K measure the numbers E, L, M, F equally, each its own. [Margin: d — 9. pro. [?]] Since, then, G, H, I, K, multiplying the number by which they measure the numbers E, L, M, F, produce E, L, M, F themselves, E, L, M, F will be in the same ratios as G, H, I, K, as we demonstrated at Proposition 18 of Book 7. But G, H, I, K are in continued proportion. Therefore E, L, M, F are also in continued proportion; and accordingly, since the multitude of E, L, M, F equals the multitude of A, C, D, B, as many mean proportionals fall between E and F as between A and B. If, therefore, between two numbers there fall means in continued proportion, etc. Which was to be demonstrated.

SCHOLIVM

It is established, not only from the demonstration, that as many mean proportionals fall between E and F as between A and B, but also that the ratio of the numbers E, L, M, F is the same as that of the numbers A, C, D, B. For it was shown that E, L, M, F are in the same ratio as G, H, I, K; but these have, by construction, the same ratio as A, C, D, B. Therefore E, L, M, F also have the same ratio as A, C, D, B.

This same proposition is true if, A and B being numbers, either E or F be taken as a unit; likewise if either A or B be a unit while E and F are numbers, as appears in the examples.

It is also established from this theorem that between numbers in double ratio, or in any superparticular ratio, or in a superbipartient ratio, no mean proportional number can fall. For since the double ratio is found, in least numbers, between two and a unit; the superparticular between numbers differing by a unit only; and the superbipartient, finally, between numbers whose difference is two — if a mean proportional number could fall between two numbers in double, or superparticular, or superbipartient ratio, then, by this theorem, some mean proportional number would also fall between two and a unit, or between numbers differing by a unit only, or by two — namely between the least numbers that have the same ratios as those — which can in no way happen. For between two and a unit no number is interposed at all — far from their admitting any mean proportional; and likewise between two numbers distant from one another by two, only one number is inserted, which differs from each by a unit. That this can in no way be a mean proportional between them we will demonstrate in the following manner. Let the numbers AB, CD (whether CD be a unit and AB three, or not) differ by two, and let there fall between them a number EF, less than AB by a unit and greater than CD, also by a unit. I say that EF is not a mean proportional between AB and CD. For if it be said that AB is to FE as FE is to CD, then, when there is taken away from AB a number AG equal to EF itself, so that the remainder is the unit GB; and

[Dotted lettered number-segments (the magnitudes AB, EF, CD with the points G, H marked), printed in the margin beside this demonstration — two schematics side by side:]

A . . G . B A G . B
 E . H . F E H . F
 C . D C D

[Book VIII — continued]

Scholium to Proposition 8 (continued)

The demonstration of the preceding proposition concludes. The numbers under discussion are shown as lettered dotted number-segments, at two scales:

A . . G . B A G . B
 E . H . F E H . F
 C . D C D

GB; and from EF let the number EH, equal to CD, be taken away, so that the unit HF is likewise left over. Then, as the whole AB is to the whole EF, so too will the part removed, AG, be to the part removed, EH — since AG and EH are, by construction, equal to EF and CD respectively. Therefore the remainder GB will also stand to the remainder HF — that is, a unit to a unit — as the whole AB stands to the whole EF, the greater to the less. But this is absurd. Therefore EF is not a mean proportional between AB and CD.

And so no mean proportional can fall between numbers in triple ratio. Otherwise, by this same theorem, a mean proportional would also fall between 3 and 1, the least numbers of the triple ratio, which differ from one another by two — and we have just now shown that this cannot happen.

By the same reasoning, no mean proportional can fall between numbers in quintuple ratio. For suppose one is said to fall; then, by this same theorem, a mean proportional will also fall between 5 and 1, the least numbers of the quintuple ratio. That this is impossible we shall demonstrate as follows. First, let there fall — if it is possible — between the number five, AB, and the unit C, a mean proportional, either the number two or the number three, D, such that AB is to D as D is to C; and, conversely, C is to D as D is to AB.

A B . E
D . . D . . .
C .

[Margin: a — 11.pron. (Axiom 11)]

Now since the unit C measures D, D will in turn measure AB. But two or three, D, also measures six, AE. Therefore D measures the whole AE and the part removed AB; and consequently D, a number, will measure the remaining unit BE. Which is absurd.

Next, let there fall — if it is possible — between the number five, AB, and the unit C, the mean proportional four, D. We shall show, as before, that D measures the number five, AB. But four, D, also measures four, AE. Therefore D measures the whole AB and the part removed AE; and consequently D, a number, will measure the remaining unit EB. Which is absurd.

A E . B
D
C .

[Margin: b — 11.pron. (Axiom 11)]

From these facts it follows that neither the musical interval consisting in the double ratio — the *Diapason* [Greek: διαπασῶν (*diapasōn*, the octave)], or as it is commonly called, the eighth — nor the interval found in the sesquioctave ratio [9:8] — namely the *Tonus* (the whole tone, or in common speech the second) — can be bisected, that is, divided into two equal ratios. For a ratio is said to be bisected (so far as our purpose requires) when a mean proportional falls between its terms. Thus, because between the terms 24 and 6, which are in quadruple ratio, there falls the mean proportional number 12 — in this manner, 24. 12. 6 — the quadruple ratio is said to be divided into two double ratios. Since, then, it has been shown that no mean proportional number falls between numbers in double ratio or in a superparticular ratio [*ratio superparticularis*] such as the sesquioctave, it is plain that the octave and the whole tone cannot be bisected. Hence the octave, in its first division into the *Diapente* [Greek: διαπέντε (*diapente*, the fifth)] and the *Diatessaron* [Greek: διατεσσάρων (*diatessarōn*, the fourth)] — of which intervals the former consists in the sesquialterate ratio [3:2], the latter in the sesquitercian [4:3] — is divided by the musicians, as appears here, 2. 3. 4. For here the double ratio is divided into the sesquialterate and the sesquitercian, as into its greatest parts unequal to one another. In the same way the whole tone is customarily divided into two semitones, of which one is called the greater, the other the less. But you will find more on these matters among the writers on music.

Again, from these results it can be demonstrated that the ratio of the diagonal of any square to its side cannot be expressed in numbers, but is irrational. For since (as we demonstrated at Prop. 47 of Book I) the square on the diagonal is double the square described on the side, while the ratio of the squares is the duplicate of the ratio of the sides, it follows that the ratio of the square described on the diagonal to the square on the side is the duplicate of the ratio of the diagonal to the side. Since, then, that ratio whose duplicate is the double ratio cannot be expressed in numbers — because, as we have shown, no mean proportional falls between numbers in double ratio to bisect it — it is manifest that the ratio of the diagonal to the side cannot be expressed in numbers either, but is irrational; or, what is the same thing, that the diagonal is incommensurable with the side. On this matter we shall write more at Prop. 9 and the last proposition of Book X.

[Margin: ix.]

THEOR. 7. PROPOS. 9.

If two numbers are prime to one another, and between them numbers fall in continued proportion, then however many numbers fall in

continued proportion between them, just so many will fall in continued proportion between each of them and a unit.

Let the mean proportionals C, D fall in continued proportion between the numbers A and B, which are prime to one another. I say that just as many continued proportionals fall between the unit and A, and likewise between the unit and B, as fall between A and B. For let two numbers E and F be found, the least in the ratio of A to C [Margin: a — 2.oct. = Elements VIII.2]; and in the same proportion let there be taken the least three, G, H, I, and the least four, K, L, M, N, and so on, until the multitude of those taken equals the multitude of A, C, D, B.

A, 8. C, 12. D, 18. B, 27.

Vnitas.

1.

E, 2. F, 3.

G, 4. H, 6. I, 9.

K, 8. L, 12. M, 18. N, 27.

Now since A and B, the extremes, are prime to one another [Margin: b — 1.oct. = Elements VIII.1], A, C, D, B will be the least numbers in the ratio of E to F. But K, L, M, N are likewise just as many and, by construction, the least in the same ratio. Therefore K, L, M, N are severally equal to A, C, D, B — K to A and N to B [Margin: c — 1.pron. = Axiom 1] — lest numbers less than the least be assigned. And since, as is clear from the demonstration of Prop. 2 of this book, E multiplied by itself produced G, and multiplying G made K, E measures G by E [Margin: d — 5.pron. = Axiom 5]; but the unit also measures E by E. Equally, then, the unit measures E, E measures G, and G measures K; and therefore the unit is the same part of E that E is of G, and G is of K. So the unit and the numbers E, G, K are, one after another, proportional [Margin: e — 20.def. = Elements VII, def. 20]. By the same reasoning — since it is likewise manifest from the demonstration of Prop. 2 of this book that F multiplied by itself produces I, and multiplying I makes N — F measures I by F, and I measures N by the same F [Margin: f — 7.pron. = Axiom 7]; but the unit also measures F by F [Margin: g — 5.pron. = Axiom 5]. Equally, then, the unit measures F, F measures I, and I measures N; and therefore the unit is the same part of F that F is of I, and I is of N.

So the unit and the numbers F, I, N are, one after another, proportional [Margin: c — 20.defin. = Elements VII, def. 20]. And accordingly, since the multitude E, G, K, as well as F, I, N, together with the unit, equals the multitude K, L, M, N (or A, C, D, B) — as is clear from Prop. 2 of this book, for E and F are the second terms from the unit, G, H, I the third, K, L, M, N the fourth, and so on —

A, 8. C, 12. D, 18. B, 27.

Vnitas.

1.

E, 2. F, 3.

G, 4. H, 6. I, 9.

K, 8. L, 12. M, 18. M, 27.

[recte: the final term should read N, 27; the compositor here prints "M, 27." On the identical array of the previous page it correctly reads N, 27.]

just so many mean proportional numbers in continued proportion will fall between the unit and K (that is, A, its equal), and between the unit and the number N (that is, B, its equal), as fall between the numbers A and B. Therefore, if two numbers are prime to one another and mean proportionals fall between them, and so on. Which was to be shown.

SCHOLIUM

It is evident that the numbers which fall in continued proportion between the numbers A and B and the unit have their ratios named after the least numbers of the ratio of A to C — that is, the ratios which the numbers E and F bear to the unit — as we also asserted in the scholium to Prop. 2 of this book.

It is likewise established from this demonstration that, if a number multiplying itself makes some number and then multiplies the product, and so on in turn, all the products are in continued proportion from the unit. For it has been shown, from the demonstration of Prop. 2 of this book, that E, G, K and F, I, N, generated in this way, are in continued proportion from the unit.

[Margin: x.]

THEOR. 8. PROPOS. 10.

If between two numbers and a unit numbers fall in continued proportion, then however many numbers fall in continued proportion between each of the two and the unit, just so many will fall in continued proportion between the two numbers themselves.

Let there fall, between each of the numbers A, B and the unit, any number of continued proportionals: between A and the unit the numbers C, D, E; and between B and the unit the numbers F, G, H, equal in multitude to the former. I say that just as many mean proportionals in continued proportion fall between A and B as fall between each of A, B and the unit. For let C and F, multiplied into one another, make I. Then let C, multiplying I and G, make K and L. Again let the

same C, multiplying K, L, and H, make M, N, and O — just as many between A and B as there are between each of A, B and the unit.

A, 81. M, 162. N, 324. O, 648. B, 1296.

E, 27. K, 54. L, 108. H, 216.

D, 9. I, 18. G, 36.

C, 3. F, 6.

1.

vnitas.

Now since, as the unit is to C, so is C to D, D to E, and E to A, the unit measures C just as C measures D, D measures E, and E measures A. But the unit measures C by C [Margin: a — 5.pron. = Axiom 5]; therefore C also measures D, D measures E, and E measures A, all by C [Margin: b — 9.pron. = Axiom 9]. Hence C, multiplying itself, made D; multiplying D, made E; and multiplying E, made A. By the same reasoning F, multiplying itself, made G; multiplying G, made H; and multiplying H, made B. And so, since C, multiplying C and F, made D and I, then as C is to F, so is D to I [Margin: c — 17.septi. = Elements VII.17]. Similarly, since F, multiplying C and F, made I and G, then as C is to F, so is I to G [Margin: d — 17.septi. = Elements VII.17 (printed "17.sexti")]. Therefore the three numbers D, I, G are in continued proportion in the ratio of C to F. Again, since C, multiplying D, I, G, made E, K, L, the numbers E, K, L will have (by what we showed at Prop. 18 of Book VII) the same continued proportion as D, I, G — that is, the ratio of C to F. In the same way, since C and F, multiplying G, made L and H, as C is to F, so is L to H [Margin: e — 18.septi. = Elements VII.18]. So the four numbers E, K, L, H are in continued proportion in the ratio of C to F. Finally, since C, multiplying E, K, L, H, made A, M, N, O, these — by what we demonstrated at Prop. 18 of Book VII — A, M, N, O will have the same continued proportion as E, K, L, H, that is, the ratio of C to F. By the same reasoning, since C and F, multiplying H, made O and B, as C is to F, so is O to B [Margin: f — 18.septi. = Elements VII.18]. Therefore the five numbers A, M, N, O, B are in continued proportion in the ratio of C to F; and consequently just as many mean proportionals fall between A and B as fall between H (or B) and the unit, since the multitude M, N, O has been made equal to the multitude C, D, E (or F, G, H). Therefore, if between two numbers and the unit continued proportionals fall, and so on. Which was to be demonstrated.

SCHOLIUM

From these things it is established that the numbers which fall as mean proportionals between A and B have the ratio which the two numbers nearer to the unit — namely C and F — have.

It is also plain from this demonstration that, given any numbers in continued proportion from the unit, the second from the unit multiplied into itself produces the third; from the same second multiplied into this third arises the fourth; from the same into this the fifth; and so on. For it has been demonstrated, from the fact that C, D, E, A are in continued proportion from the unit, that D, the third, arises from C, the second, into itself; E from C into D; and A from the same C into E, and so on.

It is fitting here to demonstrate some other theorems pertaining to numbers in continued proportion, which will be useful both for what follows and for many other things; and we take our beginning here.

I.

If from a unit there be two series of numbers in continued proportion, equal in multitude, then the third from the unit will have to the third the duplicate of the ratio which the second from the unit has to the second; the fourth the triplicate; the fifth the quadruplicate; and always so on, each by one more.

Let A, B, C, D be in continued proportion from the unit, and let there be as many others, E, F, G, H, from the same unit. I say that the ratio of B, the third from the unit, to F, the third from the same unit, is the duplicate of the ratio which A, the second, has to E, the second, and so on. For since between each of the numbers B, F and the unit there falls a single mean proportional — between B and the unit the number A, and between F and the unit the number E [Margin: a — 10.oct. = Elements VIII.10] — one mean proportional will also fall between B and F, and indeed (by the scholium to this proposition) in the ratio of A to E.

D, 81. M, 108. N, 144. O, 192. H, 256.

C, 27. K, 36. L, 48. G, 64.

B, 9. I, 12. F, 16.

A, 3. E, 4.

1.

vnitas.

Let it fall, then, and be I. In the same way two means will fall between C and G in the same ratio of A to E, namely K and L — just as many as fall between each of C, G and the unit. And likewise three, M, N, O, in the same ratio of A to E, will fall between D and H. Since, then, by definition B has to F the duplicate of the ratio which B has to I, and as B is to I so is A to E, B will also have to F the duplicate of the ratio which A has to E. In no other way, C will have to G the triplicate of the ratio of A to E; for the ratio of C to G is the triplicate of the ratio of C to K, and as C

is to K so is A to E. So too D will have to H the quadruplicate of the ratio of A to E; and so with the rest.

We shall demonstrate this same thing if there be two series of numbers in continued proportion beginning from some one and the same number. Hence we shall state this same theorem as follows.

II.

If from some one and the same number there be two series of numbers in continued proportion, equal in multitude, then the third from that number will have to the third the duplicate of the ratio which the second from it has to the second; the fourth the triplicate; the fifth the quadruplicate; and always so on, each by one more.

Let B, C, D, E be in continued proportion from the number A, and let there be as many others, F, G, H, I, from the same A. I say again that the ratio of B to F is the duplicate of the ratio of C to G; that of D to H the triplicate; and that of E to I the quadruplicate. From A into itself let K be made; from A into K, L; and from A into L, M. Similarly from B into itself let N be made; from B into N, O; and from B into O, P. Finally from F into itself let Q be made; from F into Q, R; and from F into R, S. When this is done, A, K, L, M will be in continued proportion from the unit, as is established from the scholium to Prop. 9 of this book. By the same reasoning B, N, O, P will be proportional from the unit; and likewise F, Q, R, S.

E, 162. I, 512.

D, 54. Vnitas. H, 128.

C, 18. 1. G, 32.

B, 6. F, 8.

A, 2.

N, 36. K, 4. Q, 64.

O, 216. L, 8. R, 512.

P, 1296. M, 16. S, 4096.

Since, then, the ratio of N to K is the duplicate of the ratio of B to A, by Theorem 1 of this scholium — and, by definition, the ratio of C to A is likewise the duplicate of the same ratio of B to A — C will be to A as N to K. In the same way A will be to G as K to Q; for by the same Theorem 1, and by definition, the ratio of A to G is the duplicate. Therefore, since C is to A as N to K, and A is to G as K to Q, *ex aequali* C will be to G as N to Q. But the ratio of N to Q is the duplicate of the ratio of B to F, by Theorem 1 of this scholium — because both B, N and F, Q are in continued proportion from the unit, by the scholium to Prop. 9 of this book. Therefore the ratio of C to G is also the duplicate of the ratio of B to F. By the same arguments the ratio of D to H will be the triplicate of the ratio of B to F. For the ratio of O to L

is, by Theorem 1, the triplicate of the ratio of B to A — because both B, N, O and A, K, L are in continued proportion from the unit, by the scholium to Prop. 9 of this book — and the ratio of D to A is likewise the triplicate, by definition. Therefore D is to A as O to L.

E, 81. I, 1.
D, 54. Vnitas. H, 2.
C, 36. 1. G, 4.
B, 24. F, 8.
A, 16.
N, 576. K, 256. Q, 64.
O, 13824. L, 4096. R, 512.
P, 331776. M, 65536. S, 4096.

In the same way A is to H as L to R, since each ratio is the triplicate of the ratio of A to F: the ratio of A to H by definition, and the ratio of L to R by Theorem 1 of this scholium, because both A, K, L and F, Q, R are in continued proportion from the unit, by the scholium to Prop. 9 of this book. Therefore *ex aequali* D is to H as O to R. But the ratio of O to R is the triplicate of the ratio of B to F, by Theorem 1, because both B, N, O and F, Q, R are in continued proportion from the unit, by the scholium to Prop. 9 of this book. Therefore the ratio of D to H is also the triplicate of the same ratio of B to F. In no different way shall we demonstrate that the ratio of E to I is the quadruplicate of the ratio of B to F. And so on.

Euclid, *Elements* — Book VIII (Clavius's commentary)

[Continuation of the scholium on duplicate, triplicate, and quadruplicate ratios begun in the preceding pages.]

The very same thing will be demonstrated if the numbers of one series terminate in a unit, as is clear from the second figure, in which I is the unit.

Moreover, we shall demonstrate this theorem also for lines proportional from one and the same line (Book 14, prop. 28), though not for any number of lines whatever. Hence it holds far more generally for numbers than it does for lines. And from what has thus been demonstrated we shall make this proposition 10 of Euclid more universal, in the following way.

III.

If between two numbers and some other assumed number there fall numbers in continued proportion, then as many means in continued

proportion as fall between each of the two and the assumed number, just so many will also fall as means in continued proportion between the two numbers themselves.

Between each of the numbers A and B and the assumed number C let any quantity of continued proportionals fall: between A and C the numbers D, E, F; and between B and C the same quantity of terms, G, H, I. I say that as many means in continued proportion fall between A and B as fall between A and C and between B and C; likewise, that as many fall between F and I as fall between F and C and between I and C; and as many between E and H as between E and C and between H and C. For take three numbers K, L, M, the least in the ratio of D to G; then by definition:

A, 162. Q, 216. R, 288. S, 384. B, 512.
F, 54. O, 72. P, 96. I, 128.
E, 18. N, 24. H, 32.
D, 6. G, 8.
C, 2.
K, 9. L, 12. M, 16.

the ratio of K to M is the duplicate of the ratio K to L, that is, of D to G. But by theorem 2 of this scholium the ratio of E to H is likewise the duplicate of that same ratio, D to G. Therefore as K is to M, so is E to H; and so, since one mean falls between K and M — namely N — one mean also falls between E and H, just as one falls between E and C and between H and C.[Margin: (a) Elements VIII.8] In the same way, if four least numbers are taken in the same ratio, D to G, we shall show that two means fall between F and I, just as two means fall between F and C and between I and C. And by the same reasoning three means will fall between A and B, if five least numbers are taken in the same ratio, D to G.

This holds also if, in place of one of the numbers — namely B — a unit is assumed, as is plain in this second figure, in which the number B is a unit, and there fall between A and C, as well as between B and C, three numbers in continued proportion — that is, as many as fall between A and B, and so on.

A, 1048576. Q, 32768. R, 1024. S, 32. B, 1.
F, 65536. O, 2048. P, 64. I, 2.
E, 4096. N, 128. H, 4.
D, 256. G, 8.
C, 16.
K, 1024. L, 32. M, 1.

From these things it is also apparent that the numbers which fall as mean proportionals between A and B — and likewise between F and I, and between E

and H — bear a ratio nearer to the assumed number C than the two numbers D and G do.

THEOR. 9. PROPOS. 11. [Margin: xj.]

Between two square numbers there is one mean proportional number;
and a square has to a square the duplicate ratio of side to side.

Let A and B be square numbers, whose sides are C and D. I say that one mean proportional number falls between A and B, and that the ratio of A to B, square to square, is the duplicate of the ratio C to D, side to side. For let C and D, multiplied by one another, make E. Since, then, C multiplying itself made the square A (by the definition of a square), and multiplying D made E (by the construction), [Margin: (a) Elements VII.17] it will be as C is to D, so A to E. Again, since D multiplying C made E (by the construction), and multiplying itself made the square B (by the definition of a square), [Margin: (b) Elements VII.17] it will likewise be as C is to D, so E to B. Therefore A, E, B are continued proportionals in the ratio of the sides C and D; and accordingly the mean E falls in continued proportion between the squares A and B.

A, 9. E, 21. B, 49. C, 3. D, 7.

Now since, from the fact that A, E, B are continued proportionals, the number A has to B the duplicate of the ratio it has to E, the square A will also have to the square B the duplicate of the ratio which the side C has to the side D — since this ratio is the same as that of A to E. Therefore between two square numbers there is one mean proportional, and so on. Which was to be demonstrated.

SCHOLIVM.

It is clear from what has been said that between two square numbers there falls a mean proportional number, in the continued proportion of side to side. For the ratio of the square A to the mean proportional E is the same as that of the side C to the side D, as has been demonstrated.

EVCLID. GEOM. — Book VIII (continued)

By the same reasoning it is evident that the mean proportional E and each of the squares A, B are composite with respect to one another [i.e. share a common measure]. For since A, E, B have been shown to be proportional in the ratio C to D — that is, since A is to E, and E to B, as C to D — and since the side D measures its own square B, the consequent measuring the consequent (if we speak of the ratios E to B and C to D), C will also measure E, the antecedent measuring the

antecedent, seeing that E is to B as C is to D. But the side C also measures its own square A. Therefore E and A have C as a common measure, and so are composite with respect to one another. Similarly, since A is to E as C to D, and the side C measures its own square A, the antecedent measuring the antecedent, D will also measure E, the consequent measuring the consequent. And since the side D also measures its own square B, E and B will have D as a common measure, and so will be composite with respect to one another. It is moreover clear from this demonstration that the common measure of the prior numbers A, E is C, the side of the prior square A; while the common measure of the posterior numbers E, B is D, the side of the posterior square B.

THEOR. 10. PROPOS. 12. [Margin: xj.]

Between two cube numbers there are two mean proportional numbers;
and a cube has to a cube the triplicate ratio of side to side.

Let A and B be cube numbers, whose sides are C and D. I say that two mean numbers fall between A and B in continued proportion, and that the ratio of A to B, cube to cube, is the triplicate of the ratio C to D, side to side. For let C multiplying itself make E; let D multiplying itself make F; let C and D multiplying one another make G; and let G, multiplied [by C and by D], make H and I. Since, then, C multiplying itself and D made E and G, [Margin: (a) Elements VII.17] it will be as C is to D, so E to G. By the same reasoning, since D multiplying C and itself made G and F, [Margin: (b) Elements VII.17] it will likewise be as C is to D, so G to F; and therefore E, G, F are continued proportionals in the ratio C to D. Again, since (by the definition of a cube) C multiplying E made A, and (by the construction) multiplying G made H, [Margin: (c) Elements VII.17] it will be as E is to G — that is, as C to D — so A to H. Similarly, since D (by the construction) multiplying G made the number I, and multiplying F (by the definition of a cube) made the cube B, [Margin: (d) Elements VII.17] it will likewise be as G is to F — that is, as C to D — so I to B. And H is to I as C is to D, [Margin: (e) Elements VII.18] because C and D, multiplying G, made H and I by the construction. Therefore A, H, I, B are continued proportionals in the ratio C to D; and accordingly the two means H and I fall between the cubes A and B in continued proportion. Now since, from the fact that A, H, I, B are continued proportionals, the number A has to B the triplicate of the ratio it has to H, the cube A will also have to the cube B the triplicate of the ratio which the side C has to the side D — since C is to D as A is to H. For this reason, between two cube numbers there are two mean proportionals, and so on. Which was to be shown.

A, 27. H, 36. I, 48. B, 64.
E, 9. G, 12. F, 16.
C, 3. D, 4.

SCHOLIVM.

Here too it is manifest that between two cubes there fall two mean proportional numbers in the continued proportion of side to side. For it has been shown that A is to H, and H to I, and I to B, as C is to D.

In the same way it is established that the two mean proportionals H, I and each of the cubes A, B are composite with respect to one another. For since A, H, I, B have been shown to be proportional in the ratio C to D — that is, since A is to H, and I to B, as C to D — and since the side D measures its own cube B, the consequent measuring the consequent (if we speak of the ratios I to B and C to D), C will also measure I, the antecedent measuring the antecedent, seeing that I is to B as C is to D. But the side C measures its own cube A as well; indeed it also measures H, since H was made by the multiplication of C into G, by the construction. Therefore A, H, I have C as a common measure, and so are composite with respect to one another. Similarly, since A is to H as C to D, and the side C measures its own cube A, the antecedent measuring the antecedent, D will also measure H, the consequent measuring the consequent. And since the side D measures its own cube B — indeed also I, since I was made by the multiplication of D into G, by the construction — H, I, B have D as a common measure; and accordingly they are composite with respect to one another. It is moreover manifest here too that the common measure of the prior three numbers A, H, I is C, the side of the prior cube A; while the common measure of the posterior three, H, I, B, is D, the side of the posterior cube E [recte: B?].

A, 27. H, 36. I, 48. B, 64.
E, 9. G, 12. F, 16.
C, 3. D, 4.

THEOR. 11. PROPOS. 13. [Margin: xij.]

If there be any number of numbers in continued proportion, and each, multiplying itself, produces certain numbers, those produced from them will be proportional; and if the numbers first set down, multiplying those already made, produce certain numbers, these too will be proportional; and this always happens with respect to the extremes.

Let A, B, C be continued proportionals, which multiplying themselves make D, E, F; and let these multiplying D, E, F make G, H, I; and again, multiplying G, H, I,

make K, L, M, and so on in succession. I say that D, E, F, and G, H, I, and K, L, M are likewise continued proportionals. For let A and B, multiplying one another, make N; and let B and C, multiplying one another, make O. Then let A, multiplying N and E, make P and Q. Likewise let B, multiplying O and F, make R and S. In the same way let A, multiplying P, Q, H, make T, V, X; and let B, multiplying R, S, I, make Y, Z, a. [Margin: (a) Elements VII.17] Since, then, A multiplying itself and B made D and N, it will be as A is to B, so D to N. In the same way, since B multiplying A and itself made

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N and E, it will also be as A is to B, so N to E. Therefore D, N, E are continued proportionals in the ratio A to B. Again, since B multiplying itself and the number C made E and O, [Margin: (b) Elements VII.17] it will be as B is to C, so E to O. And by the same

A, 2. B, 4. C, 8.

D, 4. N, 8. E, 16. O, 32. F, 64.

G, 8. P, 16. Q, 32. H, 64. R, 128.

K, 16. T, 32. V, 64. X, 128. L, 256. Y, 512. Z, 1024. a, 2048. M, 4096.

reasoning, since C multiplying B and itself made O and F, it will be as B is to C, so O to F; and therefore E, O, F too are continued proportionals in the ratio B to C, or A to B. And so, since D, N, E are continued proportionals in the same ratio as E, O, F, it will be, *ex aequali*, that D is to E as E is to F; and accordingly D, E, F are continued proportionals.

Next, since A multiplying D, N, E made G, P, Q, then — from what we showed at proposition 18 of Book 7 [Margin: (c) Elements VII.18] — G, P, Q will be proportional in the same ratio as D, N, E, that is, in the ratio A to B. Likewise, since A and B multiplying E made Q and H, it will also be as A is to B, so Q to H. Therefore G, P, Q, H are proportional in the ratio A to B. Similarly, since B multiplying E, O, F made H, R, S, then — from what was demonstrated at proposition 18 of Book 7 — H, R, S will be proportional in the ratio E, O, F, that is, B to C, or A to B. And in the same way, since B and C multiplying F made S and I, [Margin: (d) Elements VII.18] it will be as B is to C — that is, as A to B — so S to I; and therefore H, R, S, I too are proportional in the ratio A to B. For which reason, since G, P, Q, H have the same continued proportion as H, R, S, I, it will be, *ex aequali*, that G is to H as H is to I; and accordingly G, H, I are continued proportionals.

By an argument not unlike this we shall demonstrate that K, L, M too are continued proportionals; and we shall proceed in the same way in the remaining cases. Therefore, if there be any number of numbers in continued proportion, and each multiplying itself, and so on. Which was to be shown.

SCHOLIUM.

Now it is clear from the demonstration that the numbers produced in the first place, D, E, F, are proportional in the duplicate ratio of the given numbers A, B, C. For D has to E the duplicate of the ratio it has to N, that is, of A to B, and so on.

Likewise the numbers begotten in the second place, G, H, I, will be in the triplicate ratio of A to B; and those produced in the third place in the quadruplicate ratio; and so onward, always one degree higher.

We shall demonstrate the whole proposition more briefly in this way. From A, B, C, continued proportionals, multiplied into themselves, let the numbers D, E, F be made; and into these, from the same, the numbers G, H, I; and again, from the same into these, the numbers K, L, M. I say that D, E, F, and G, H, I, and K, L, M are likewise continued proportionals. For, by the scholium of proposition 9 of this book, both A, D, G, K and B, E, H, I, and C, F, I, M, are proportional from unity in continued proportion. Therefore, by theorem 1 of the scholium of proposition 10 of this book, D will be to E in the duplicate ratio of A to B, or of B to C. But of that same ratio, B to C, the ratio E to F is the duplicate, by the same theorem. Therefore D, E, F are continued proportionals in the duplicate ratio of A to B and of B to C. In the same way we shall show that G, H, I are continued proportionals in the triplicate ratio of A to B and of B to C; and K, L, M in the quadruplicate; and so with the rest.

A, 2. B, 4. C, 8.
D, 4. E, 16 [?]. F, 64.
G, 8. H, 64. I, 512.
K, 16. L, 256. M, 4096.

From this demonstration in turn it appears that the numbers produced in the first place are in the duplicate ratio of the two [given] numbers, while those begotten in the second place are in the triplicate, and so on. Which is drawn out more clearly than daylight from theorem 1 of the scholium of proposition 10 of this book.

But although in each demonstration of this proposition 13 only three continued proportionals A, B, C have been given, we shall nevertheless demonstrate the same proposition even if there be more than three. For let there be more than

three numbers in continued proportion, A, B, C, D; which multiplying themselves make E, F, G, H; and then, multiplying E, F, G, H, make I, K, L, M, and so on. Since, then, E, F, G are proportional from what has been demonstrated;[Margin: (e) Elements VIII.13] and likewise F, G, H — since the former were produced from A, B, C, proportionals, into themselves, and the latter from B, C, D, likewise proportionals, into themselves — E, F, G, H will be continued proportionals. In no other way will I, K, L, M, and the others in succession, produced in the same manner, be proportional.

A, 2. B, 4. C, 8. D, 16.

E, 4. F, 16. G, 64. H, 256.

I, 8. K, 64. L, 512. M, 4096.

THEOR. 12. PROPOS. 14. [Margin: xii.]

If a square number measures a square number, the side of the one will also measure the side of the other; and if the side of one square measures the side of the other, the square too will measure the square.

Let the square A, whose side is C, measure the square B, whose side is D. I say that the side C also measures the side D. For let C and D, multiplying one another, make E. Since, then — as is clear from the demonstration of proposition 11 of this book — A, E, B are continued proportionals in the ratio C to D, and A, the first, measures B, the last extreme,[Margin: (a) Elements VIII.7] A the first will also measure E the second. Therefore, since A is to E as C is to D, C will also measure D.

A, 4. E, 12. B, 36. C, 2. D, 6.

Now let C, the side, measure the side D. I say that the square A also measures the square B. For in the same way it will be, as C is to D, so A to E, because A, E, B are continued proportionals in the ratio C [to D], as was demonstrated at proposition 11 of this book. Therefore, since C measures D, A the first will also measure E the second, and consequently the extreme B as well, by theorem 2 of the scholium of proposition 6 of this book. Therefore, if a square number measures a square number, and so on. Which was to be shown.

THEOR. 13. PROPOS. 15.

[Margin: xiv.]

If a cube number measures a cube number, then the side of the one measures the side of the other. And if the side of one cube measures the side of the other, then the cube measures the cube.

Let cube A, whose side is C, measure cube B, whose side is D. I say that side C also measures side D. For let each of C and D, multiplying itself, make E and F respectively; let C and D, multiplying one another, make G; and finally, multiplying G, let them make H and I. Since, then — as is clear from the demonstration of Prop. 12 of this book — both E, G, F and A, H, I, B are continued proportionals in the ratio of C to D, and since A, the first term, measures B, the last, [Margin: a — 7. oct. (*Elements* VIII.7)] the same A, the first, will also measure H, the second. And since, as A is to H, so is C to D (because A, H, I, B are continued proportionals in the ratio C to D, by Prop. 12 of this book), and since A measures H, side C will also measure side D.

A, 8. H, 24. I, 72. B, 216.
 E, 4. G, 12. F, 36.
 C, 2. D, 6.

Now let side C measure side D. I say that cube A also measures cube B. For by the same argument, as C is to D, so will A be to H, because A, H, I, B are continued proportionals in the ratio C to D, as was shown at Prop. 12 of this book. Therefore, since side C measures side D, A will also measure H; and hence it will also measure the last term B — cube measuring cube — from what we demonstrated at Prop. 6 of this book. And so, if a cube number measures a cube number, and so on. Which was to be demonstrated.

THEOR. 14. PROPOS. 16.

[Margin: xv.]

If a square number does not measure a square number, then neither will the side of the one measure the side of the other. And if the side of one square does not measure the side of the other, then neither will the square measure the square.

Let A and B be squares whose sides are C and D, and let A not measure B. I say that neither does side C measure side D. For, if it be possible, let C measure D. Since, then, C, the side of square A, measures D, the side of square B, [Margin: a — 14. octa. (*Elements* VIII.14)] square A will also measure square B. But this is absurd, for it was assumed not to measure. Therefore side C does not measure side D.

A, 16. B, 81. C, 4. D, 9.

But now let side C not measure side D. I say that neither will square A measure square B. For if A be said to measure B, [Margin: b — 14. octa. (*Elements* VIII.14)]

C, the side of the former, would also measure D, the side of the latter. But this is absurd, for it was assumed not to measure. Therefore square A will not measure square B. Therefore, if a square number does not measure a square number, and so on. Which was to be shown.

THEOR. 15. PROPOS. 17.

[Margin: xv. [?] — reads xv; under the lag-1 convention xvi would be expected; see anomaly note]

If a cube number does not measure a cube number, then neither will the side of the one measure the side of the other. And if the side of one cube does not measure the side of the other, then neither will the cube measure the cube.

Let A and B be cubes whose sides are C and D, and let A not measure B. I say that neither does side C measure side D. For if C be said to measure D, [Margin: a — 15. oct. (*Elements* VIII.15)] cube A would also measure cube B. But this is absurd, since it was assumed not to measure. Therefore side C does not measure side D.

A, 8. B, 27. C, 2. D, 3.

But now let side C not measure side D. I say that neither will cube A measure cube B. For if it be said to measure, [Margin: b — 15. oct. (*Elements* VIII.15)] C, the side, would also measure the side D. But this is absurd, for it was assumed not to measure. Therefore cube A will not measure cube B. Therefore, if a cube number does not measure a cube number, and so on. Which was to be demonstrated.

SCHOLIVM.

The four immediately preceding propositions can also be stated in this way, if the greater numbers are referred to the smaller.

If a square number is a multiple of a square number, and a cube of a cube, then the side of the one will also be a multiple of the side of the other. And if the side of one is a multiple of the side of the other, then the square will be a multiple of the square, and the cube of the cube. But if the square is not a multiple of the square, nor the cube of the cube, then neither will the side be a multiple of the side. And if the side is not a multiple of the side, then neither will the square be a multiple of the square, nor the cube of the cube.

For if square A is a multiple of square B, and cube A of cube B, then B will measure A — the square measuring the square, and cube B the cube A. [Margin: c

— 14. & 15. oct. (*Elements* VIII.14 & 15)] Therefore side D will also measure the side; and consequently side C will be a multiple of side D.

A, 36. B, 9. C, 6. D, 3.

But if side C is a multiple of side D, then side D measures side C. [Margin: d — 14. & 15. oct. (*Elements* VIII.14 & 15)] Therefore B will also measure A — the square measuring the square, and the cube the cube; and consequently A will be a multiple of B — the square of the square, and the cube of the cube.

But if A is not a multiple of B — neither the square of the square, nor the cube of the cube — then B will not measure A, neither the square measuring the square nor the cube the cube. [Margin: g — 16. & 17. oct. (*Elements* VIII.16 & 17)] Therefore neither will side D measure side C. Hence C will not be a multiple of side D.

A, 64. B, 8. [B prints as "S"; read 8 — see anomaly note]
C, 4. D, 2.

A, 49. B, 9.
C, 7. D, 3.

A, 125. B, 8.
C, 5. D, 2.

Likewise, if side C is not a multiple of side D, then D will not measure the side C. [Margin: h — 16. & 17. oct. (*Elements* VIII.16 & 17)] Therefore neither will B measure A — neither the square the square, nor the cube the cube; and so A will not be a multiple of B — neither the square of the square, nor the cube of the cube.

THEOR. 16. PROPOS. 18.

[Margin: xvi. [?]]

Between two similar plane numbers there is one mean proportional number; and a plane number has to a plane the duplicate ratio of that which the corresponding side has to the corresponding side.

Let A and B be similar plane numbers; let the sides of the former be C and D, and those of the latter, proportional to them, E and F — so that C and D, multiplying one another, make A, and E and F, multiplying one another, make B, and as C is to D, so is E to F. I say that between A and B there falls one mean proportional number, and that the ratio of A to B is the duplicate of the ratio of the

corresponding sides C to E (or D to F). For let D and E, multiplying one another, make G. Since, then, C is to D as E is to F, then, alternately, C will be to E as D is to F. And because D, multiplying C and E, made A and G, [Margin: a — 17. septi. (*Elements* VII.17)] A will be to G as C is to E — that is, as D is to F. Similarly, because E, multiplying D and F, made G and B, [Margin: b — 17. septi. (*Elements* VII.17)] G will also be to B as D is to F.

A, 12. G, 18. B, 27.
C, 6. D, 2. E, 9. F, 3.

Therefore A, G, B are proportional in the ratio of C to E (or D to F); and thus G falls as a mean proportional between A and B.

Now, since A, G, B are continued proportionals, A has to B the duplicate ratio of that which A has to G — that is, of C to E (or D to F). It is therefore clear that the ratio of the plane number A to the plane B is the duplicate of that which C has to E (or D to F), corresponding side to corresponding side. Therefore, between two similar plane numbers, and so on. Which was to be demonstrated.

SCHOLIVM.

Since it has been shown that A, G, B are proportional in the ratio of C to E (or D to F), it is evident that between the two similar plane numbers A and B there falls a mean proportional G, in the ratio of the corresponding sides C, E (or D, F) that were taken.

It is also established from what has been said that the mean proportional G and either of the plane numbers A, B have a common measure. For since A, G, B have been shown proportional in the ratio C to E (or D to F) — that is, A is to G, and G to B, as C is to E (or D to F) — and since E, F, the sides, measure the number B, of which they are the sides (consequents measuring a consequent, if we speak of the ratios G to B and C to E, or D to F), then C, D will also measure G (antecedents measuring an antecedent), since G is to B as C is to E (or D to F). But C, D, the sides, also measure the number A, of which they are sides. Therefore G and A have a common measure, both C and D; and hence they have a common measure with one another. Again, since A is to G as C is to E (or D to F), and since C, D, the sides, measure their own plane A (antecedents measuring an antecedent), then E, F will also measure G (consequents measuring a consequent). Therefore, since E, F, the sides, also measure their own plane B, G and B will have a common measure, both E and F; and so they have a common measure with one another.

From this demonstration it is also left manifest that the common measures of the two prior numbers A, G are C, D, the sides of the prior plane A; while the common measures of the latter pair G, B are E, F, the sides of the latter plane B.

THEOR. 17. PROPOS. 19.

[Margin: xviii.]

Between two similar solid numbers there are two mean proportional numbers; and a solid number has to a solid the triplicate ratio of that which the corresponding side has to the corresponding side.

Let A and B be similar solid numbers; let the sides of the former be C, D, E, and those of the latter, proportional to them, F, G, H — so that C is to D as F is to G, and D is to E as G is to H. I say that between A and B there fall two mean proportionals, and that the ratio of A to B is the triplicate of that which the corresponding sides C to F (or D to G, or E to H) have. For let C and D, multiplying one another, make I; and let F and G, multiplying one another, make K. Likewise let D and F, multiplying one another, make L. Finally let E and H, multiplying one another, make M and N. Since, then, C, D, E are proportional to F, G, H, they will also be proportional alternately — namely, as C is to F, so D is to G, and E to H. Now since D, multiplying C and F, made I and L, [Margin: a — 17. septi. (*Elements* VII.17)] as C is to F, so will I be to L. In like manner, since F, multiplying D and G, made L and K, [Margin: b — 17. septi. (*Elements* VII.17)] as D is to G, so will L be to K. Therefore I, L, K are continued proportionals in the ratio of C to F (or D to G, or E to H). And since the solid A is made from the mutual multiplication of the sides C, D, E, and I was made from the mutual multiplication of C, D, it follows that E, multiplying I, makes A. In the same way, since the solid B is made from the mutual multiplication of the sides F, G, H, and K was made from the mutual multiplication of F, G, the number H, multiplying K, will make B. Therefore, since E, multiplying I and L, made A and M, [Margin: c — 17. septi. (*Elements* VII.17)] A will be to M as I is to L — that is, as C is to F (or D to G, or E to H). By the same reasoning, since H, multiplying L and K, made N and B, [Margin: d — 17. septi. (*Elements* VII.17)] N will be to B as L is to K — that is, as C is to F (or D to G, or E to H). [Margin: e — 18. septi. (*Elements* VII.18)] Moreover, M is to N as E is to H, since E and H, multiplying L, made M and N. Therefore A, M, N, B are continued proportionals in the ratio of C to F (or D to G, or E to H); and hence between the similar solids A, B there fall two mean proportionals M, N in continued proportion.

A, 30. M, 60. N, 120. B, 240.
I, 6. L, 12. K, 24.
C, 2. D, 3. E, 5. F, 4. G, 6. H, 10.

Now since A, M, N, B are continued proportionals, the ratio of A to B is the triplicate of that which A has to M. But A is to M as C is to F (or D to G, or E to H). Therefore A to B — solid to solid — will also have the triplicate ratio of that which C has to F (or D to G, or E to H), side to corresponding side. Therefore, between two similar solid numbers there fall two means, and so on. Which was to be shown.

SCHOLIVM.

Since it has been demonstrated that A, M, N, B are proportional in the ratio of C to F (or D to G, or E to H), it is clear that between the two similar solids A and B there fall two mean proportionals M and N, in the ratio of the corresponding sides C, F (or D, G, or E, H) that were taken.

From what has been said, it is also evident that the two mean proportionals M, N and either of the solids A, B have a common measure with one another. For since A, M, N, B have been shown proportional in the ratio C to F (or D to G, or E to H) — that is, A is to M, and M to N, and N to B, as C is to F (or D to G, or E to H) — and since F, G, H, the sides, measure their own solid B (consequents measuring a consequent, if we speak of the ratios N to B and C to F, or D to G, or E to H), then C, D, E will also measure N (antecedents measuring an antecedent), since N is to B as C is to F (or D to G, or E to H). But C, D, E also measure M. (For since M was made, by construction, from E into L, E measures M. Next, since M, N, B are proportional to I, L, K, and K measures B — because B was made from K into H — I will also measure M. But C, D, the sides, measure their own plane I. [Margin: c — 11. pro. [?]] (Prop. 11 [?])] Therefore C, D will also measure M; and so C, D, E will measure M.) And likewise they measure A too — namely, the sides measuring their own solid. Therefore A, M, N have a common measure — C, D, and E alike; and so they have a common measure with one another.

Again, since A is to M as C is to F (or D to G, or E to H), and since C, D, E, the sides, measure their own solid A (antecedents measuring an antecedent), then F, G, H will also measure M (consequents measuring a consequent). But F, G, H also measure N. (For since N was made from H into L, by construction, H measures N. Next, since A, M, N are proportional to I, L, K, and I measures M, K will also measure N. Since, then, F, G, the sides, measure their own plane K, F, G will also measure N; and so F, G, H will measure N.) And likewise they measure B too — namely, the sides measuring their own solid. Therefore M, N, B have a common

measure — F, G, and H alike. Hence they have a common measure with one another.

A, 30. M, 60. N, 120. B, 240.

I, 6. L, 12. K, 24.

C, 2. D, 3. E, 5. F, 4. G, 6. H, 10.

From all this it plainly follows that the common measures of the three prior numbers A, M, N are C, D, E, the sides of the prior solid A; while F, G, H, the sides of the latter solid B, are the common measures of the three latter numbers M, N, B.

Now, since a solid number has three sides, from whose mutual multiplication it is generated, let us here demonstrate that the same solid number is produced whether the first side is multiplied into the second and the number so produced into the third; or the first into the third and the number generated into the second; or, finally, the second into the third and the resulting number into the first. Indeed, let us show that, given any numbers whatever, the same number is always generated from their mutual multiplication, in whatever order they may be permuted among themselves. This matter is of great use in arithmetical questions. Let, then, such a theorem be proposed:

Given any number of numbers whatever, the same number will always be produced from their mutual multiplication, in whatever order they permute their arrangement among themselves.

First, let three numbers A, B, C be given; and from A into B let D be made, and from D into C let E be made. I say that the same E is produced if from A into C, F is made and F is multiplied into B; likewise if from B into C, G is made and G is multiplied into A. For since C, multiplying D, makes E, [Margin: f — 15. def. (*Elements* VII, Def. 15)] as E is to D, so is C to unity. But as C is to unity, [Margin: g — 15. def. (*Elements* VII, Def. 15)] so also is F to A, since A, multiplying C, makes F. Therefore, as E is to D, so is F to A. And as D is to B, [Margin: h — 15. def. (*Elements* VII, Def. 15)] so is A to unity, because A, multiplying B, makes D. Therefore, from equality, as E is to B, so will F be to unity; [Margin: i — 19. septi. (*Elements* VII.19)] and hence the number made from F into unity — that is, F itself — will be equal to the number made from B into F, F being the product from A into C. Therefore, just as the number E is made from A into B and the product into C, so the same E is made from A into C and the product into B. Which is the first case.

A, 3. B, 5. C, 8.

D, 15. E, 120. F, 24. G, 40.

E, F,
D, A,
B, unity

Again, since E is made from C into D, [Margin: k — 15. def. (*Elements* VII, Def. 15)] as E is to D, so is C to unity. But as C is to unity, [Margin: l — 15. def. (*Elements* VII, Def. 15)] so also is G to B, because G is made from B into C. Therefore, as E is to D, so is G to B. And as D is to A, [Margin: m — 25. def. [?] (*Elements* VII, Def. 25 [?] — likely misprint for 15. def.)] so is B to unity, because D is made from A into B. Therefore, from equality, as E is to A, so will G be to unity; [Margin: n — 19. sept. (*Elements* VII.19)] and hence the number made from E into unity — that is, E itself — will be equal to the number generated from A into G, G being the product from B into C. Therefore, just as E is generated from A into B and the product into C, so the same E is produced from B into C and the product into A. Which is the second case.

E, G,
D, B,
A, unity

Next, let there be four numbers A, B, C, D; and from A into B, and the product into C, and the product into D, let E be made. I say that the same E is generated:

- from A into B, and the product into D, and the product into C;
- from A into C, and the product into B, and the product into D;
- from A into C, and the product into D, and the product into B;
- from A into D, and the product into B, and the product into C;
- from A into D, and the product into C, and the product into B;
- from B into C, and the product into A, and the product into D;
- from B into C, and the product into D, and the product into A;
- from B into D, and the product into A, and the product into C;
- from B into D, and the product into C, and the product into A;
- from C into D, and the product into A, and the product into B;
- from C into D, and the product into B, and the product into A.

A, 2. B, 3. C, 4. D, 5. E, 120.

—so that, besides the first multiplication, by which all four numbers are multiplied together in order, there arise eleven distinct multiplications, all producing the same number every time.

For since, as was shown in the case of three numbers, the same number results whether the number generated from A into B is multiplied first into C and then this product into D, or first into D and then this product into C, what was proposed in the first place is established. Next, since, as was shown in the case of three, the same number that arises from A into B and the product into C is the one that arises from A into C and the product into B, it is clear that the same number is generated both from A into B, the product into C, and the product into D, and from A into C, the product into B, and the product into D — which is the second case.

(Continuation of Clavius's inserted, un-numbered theorem on the order-independence of a continued product. The running example throughout is $A = 2$, $B = 3$, $C = 4$, $D = 5$, whose full product is $E = 120$. Cases one and two were given in the preceding pages; the enumeration of the eleven distinct multiplications, besides the first, resumes here at the third.)

Thirdly. Since it was shown for three numbers that the same number is produced — whether the number generated from A into C is multiplied first into B and then this product into D, or first into D and then this product into B — it is clear that one and the same number is generated from A into C, then the product into D, then that product into B. Since, then, that number is equal to the first product just shown, this one too will be equal to the same. This is the third case.

Fourthly. Since it was shown for three numbers that the same number arises from A into B, then the product into D, as arises from A into D, then the product into B, there will likewise be generated the same number from A into B, then the product into D, then the product into C. But that number was shown in the first place to be equal to the one made from A into B, then the product into C, then the product into D. Therefore this one also is equal. This is the fourth case.

Fifthly. Since it was demonstrated for three numbers that the same number is generated from A into C, then the product into D, as is generated from A into D, then the product into C, there will be produced the same number from A into C, then the product into D, then the product into B, as is produced from A into D, then the product into C, then the product into B. Since, then, that number was shown in the third place to be equal to the one made from A into B, then the product into C, then the product into D, this one too will be equal to the same. This is the fifth case.

Sixthly. Since we demonstrated for three numbers that the same number arises from A into B, then the product into C, as arises from B into C, then the product

into A, it is evident that one and the same number is procreated both from A into B, then the product into C, then the product into D, and from B into C, then the product into A, then the product into D. This is the sixth case.

Seventhly. Since we showed for three numbers that the same number is generated whether the product [from B into C] is multiplied first into D, or first into D and then this product into A, it is established that the same number arises from B into C, then the product into D, then the product into A, as arises from B into C, then the product into A, then the product into D. But this latter was shown just above to be equal to the one made from A into B, then the product into C, then the product into D. Therefore the former also. This is the seventh case.

Eighthly. Since it was shown for three numbers that the same number is procreated from A into B, then the product into D, as from B into D, then the product into A, there will likewise be generated the same number from A into B, then the product into D, then the product into C, as from B into D, then the product into C, then the product into A. But that number was shown in the first place to be equal to the one made from A into B, then the product into C, then the product into D. Therefore this one also. This is the eighth case.

Ninthly. Since it was demonstrated for three numbers that the same number arises from B into C, then the product into D, as from B into D, then the product into C, there will be procreated the same number from B into C, then the product into D, then the product into A, as from B into D, then the product into C, then the product into A. But that number was shown in the seventh place to be equal to the one made from A into B, then the product into C, then the product into D. Therefore this one also. This is the ninth case.

A, 2. B, 3. C, 4. D, 5.
E, 120.

Tenthly. Since it was demonstrated for three numbers that the same number is generated from A into C, then the product into D, as from C into D, then the product into A, there will also be produced the same number from A into C, then the product into D, then the product into B, as from C into D, then the product into A, then the product into B. But that number, as was shown in the third place, is equal to the one made from A into B, then the product into C, then the product into D. Therefore this one also. This is the tenth case.

Lastly. Since it was shown for three numbers that the same number is generated from B into C, then the product into D, as from C into D, then the product into B, there will also be procreated the same number from B into C, then the product

into D, then the product into A, as from C into D, then the product into B, then the product into A. But that number was shown in the seventh place to be equal to the one made from A into B, then the product into C, then the product into D. Therefore this one too will be equal to the same. This is the eleventh case.

The same manner of arguing must be used with five or more numbers, provided only that with five one assumes what has already been demonstrated for four, and with six what was demonstrated for five, and so on. Thus with the five numbers A, B, C, D, E it is plain — where we have altered the order three times, and yet always the same number is procreated from their continued multiplication among themselves. For the product of A into B, together with C, D, E, constitutes four numbers; therefore, in whatever way they interchange their order among themselves, they will always generate the same number. And so both the second order and the third will produce the same number as the first. Again, since the three numbers A, B, C, however they change their order, make the same number, it follows that the fourth order produces exactly the same number as the first — and so with the rest.

A. B. C. D. E.
A. B. C. E. D.
A. B. E. D. C.
B. C. A. D. E.

THEOR. 18. PROPOS. 20.

[Margin: xvij.]

If between two numbers one mean proportional number falls, those numbers will be similar plane numbers.

Let a mean proportional C fall between A and B. I say that A and B are similar plane numbers. Take D and E, the least numbers in the ratio A, C, B. Since, then, D and E are the least in the ratio of A to C, [Margin: a — Elements VII.21] they will measure A and C equally; let them measure by F. Similarly they will measure C and B equally; let them measure by G. [Margin: b — Axiom 9] And so F, multiplying D and E, will produce A and C; [Margin: c — Axiom 9] likewise G, multiplying the same D and E, will produce C and B. Since, therefore, E, multiplying F and G, made C and B, [Margin: d — Elements VII.17] it follows that as C is to B, so is F to G. But as C was to B, so was D to E. Therefore as D is to E, so is F to G; and, by permutation, as D is to F, so is E to G. Now since F, multiplying D, made A, A will be a plane number whose sides are D and F. In the same way, since G, multiplying E, made B, B too will be a plane number whose sides are E

and G. Since, then, these sides have been shown to be proportional — that is, D is to F as E is to G — A and B will, by definition, be similar plane numbers. Therefore, if between two numbers one mean proportional falls, and so on. Which was to be demonstrated.

A, 18. C, 24. B, 32.
D, 3. E, 4. F, 6. G, 8.

THEOR. 19. PROPOS. 21.

[Margin: xix.]

If between two numbers two mean proportional numbers fall, those numbers are similar solids.

Let two mean proportionals C, D fall between A and B. I say that A and B are similar solids. Take the three least numbers E, F, G in the ratio A, C, D, B. [Margin: a — Elements VIII.20] Since, then, a mean proportional F falls between E and G, E and G will be similar plane numbers: let the sides of E be H, I, and the sides of G be K, L, proportional to those. And since E, F, G, being the least in the ratio of A, C, D, B, [Margin: b — Elements VII.21] measure A, C, D — which have the same ratio as they — equally, let them measure by M. [Margin: c — Elements VII.21] In the same way, since E, F, G are also the least, they equally measure C, D, B, which have the same ratio as they; let them measure by N. [Margin: d — Axiom 9] Thus M, multiplying E, F, G, produces A, C, D; and N, multiplying the same E, F, G, produces C, D, B. Now E was produced from the mutual multiplication of its sides H and I; likewise G, from the mutual multiplication of its sides K and L. Therefore A is produced from the mutual multiplication of H, I, M; and B from the mutual multiplication of K, L, N. And so A, by definition, is a solid number having as its sides H, I, M; and B too is a solid having as its sides K, L, N. But since M and N, multiplying F, produce C and D, as we showed, [Margin: e — Elements VII.18] it follows that C is to D as M is to N. [Margin: f — Elements VII.17] And C and D are in the same ratio as E and F, because N, multiplying E and F, made C and D; they therefore have the same ratio as HK, or as I, L. For between the similar planes E, G falls the mean proportional F, by the corollary to Proposition 19 of this book, in the ratio of the homologous sides H, K, or I, L. Therefore H will also be to K, and I to L, as M is to N; and, by permutation, H is to I as K is to L, and I is to M as L is to N. The sides H, I, M are therefore proportional to the sides K, L, N; and for that reason the solid numbers A and B are similar. Wherefore, if between two numbers two mean proportionals fall, and so on. Which was to be demonstrated.

A, 24. C, 72. D, 216. B, 648.
E, 1. F, 3. G, 9.
H, 1. I, 1. M, 24. K, 3. L, 3. N, 72.

A, 8. C, 12. D, 18. B, 27.
E, 4. F, 6. G, 9.
H, 2. I, 3. M, 2. K, 3. L, 3. N, 3.

SCHOLIVM.

We have set down two examples. In the first of them it plainly appears that the unit E is a plane number, and the units H, I its sides — though improperly so, as we noted in the definition of a plane number. For if we exclude the unit from among plane numbers, we shall not be able, by this argument of Euclid's, to demonstrate that the numbers A and B are similar solids. The same difficulty will arise with all other numbers that contain two mean proportionals in a multiple ratio — of which kind are A and B in this same first example. But this is absurd, since Euclid states the proposition in general terms and demonstrates it without any exception.

THEOR. 20. PROPOS. 22.

[Margin: xx.]

If three numbers be in continued proportion, and the first is a square, then the third will also be a square.

Let there be three numbers A, B, C in continued proportion, and let the first of them, A, be a square. I say that the third, C, is also a square. For since a mean proportional falls between A and C, [Margin: a — Elements VIII.20] A and C will be similar plane numbers. Therefore, A being a square, C, which is similar to it, will also be a square. If, then, three numbers be in continued proportion, and so on. Which was to be demonstrated.

A, 9. B, 54. C, 324.

THEOR. 21. PROPOS. 23.

[Margin: xxi.]

If four numbers be in continued proportion, and the first is a cube, then the fourth will also be a cube.

Let there be four numbers A, B, C, D in continued proportion, and let the first, A, be a cube. I say that the fourth, D, is also a cube. For since two mean

proportionals B, C fall between A and D, [Margin: a — Elements VIII.21] A and D will be similar solids. Therefore, A being a cube, D, which is similar to it, will also be a cube. Wherefore, if four numbers be in continued proportion, and so on. Which was to be demonstrated.

A, 27. B, 45. C, 75. D, 125.

SCHOLIUM.

This is the received demonstration of the two preceding propositions, found in all the interpreters of Euclid whom I have seen; but anyone who cares to examine it closely will find it decidedly defective and incomplete. For since, at Definition 21 of Book 7, we handed down that two plane or solid numbers can be similar even though not for *any* sides whatever of the one can other sides of the other be exhibited proportional — it being enough that for *some* sides of the one certain sides of the other are proportional — anyone could, with perfect right, deny that, if one of two similar plane numbers be a square, the other is also a square, and that, if one of two similar solids be a cube, the other too is a cube; yet in the demonstration this was assumed as granted and as evident to all. For, given the two similar plane numbers 36 and 64, it is enough that for two sides of the former, such as 3 and 12, two other sides of the latter be exhibited proportional, namely 4 and 16. Hence, even though the former be a square, having two equal sides 6 and 6, one may rightly doubt — indeed altogether deny — that the latter has two other sides proportional to these, that is, equal to one another, and so is a square: just as it is likewise manifest that to these sides of the former, 4 and 9, or 2 and 18, no other sides in the latter correspond proportional. If, then, to these sides of the former number no other sides are found in the latter proportional, although the numbers themselves are similar, why should we grant, without any demonstration, that to these equal sides of the former, 6 and 6, two others can be assigned in the latter proportional — that is, equal to one another as well? The same must be said of similar solids. For this reason we shall demonstrate both propositions of Euclid otherwise, in the following manner.

Let there be three numbers A, B, C in continued proportion, and let the first, A, be a square. I say that C, the third, is also a square. For, taking D, the side of the square A — since D, multiplied by itself, makes A — D and A will, by the scholium to Proposition 9 of this book, be in continued proportion from the unit;

and so, by conversion, A will be to D as D is to the unit. And thus, since between the number A itself — and both the number C and the unit — mean proportionals fall equal in multitude, namely one, there will fall, by Theorem 3 of the scholium

to Proposition 10 of this book, just as many between the unit and the number C, namely one, which let be E. Since, therefore, the unit and E, C are in continued proportion, C will be produced from E multiplied into itself, by the scholium to Proposition 10 of this book. And so C will, by definition, be a square, whose side is E.

A, 36. B, 48. C, 64.
D, 6. E, 8.
unitas.

Otherwise. Taking three numbers D, E, F, the least in the ratio of A to B and of B to C, D will measure A, and F will measure C, equally, by what we showed at Proposition 21 of Book 7. And since the extremes D, F of the three least numbers are, by Corollary 1 of Proposition 2 of this book, squares — and A too is a square, by hypothesis — [Margin: b — Elements VIII.11] one mean proportional, say G, will fall between the squares A and D. And since, by equality, as A is to C, so is D to F, and, by permutation, as A is to D, so is C to F, [Margin: c — Elements VIII.8] one mean proportional, namely H, will also fall between C and F. Since, then, F, the first, measures C, the last, [Margin: d — Elements VIII.7] it will measure the second, H, as well. Now, taking I, the side of the square F, let I measure the number K as many times as F measures H itself, so that H is to F, and K to I, in the same multiple proportion. Let L also be the square of K. [Margin: e — Elements VIII.11] Thus, since the ratio of L to F, square to square, is the duplicate of the ratio of K to I, side to side — that is, of H to F — while the ratio of C to F is likewise the duplicate of the same ratio H to F, by the definition, so that by conversion C, H, F are also proportional, it follows that L is to F as C is to F. And therefore L and C are equal. Since, then, L is a square by construction, C too will be a square.

A, 36. B, 48. C, 64.
G, 18. H, 32.
D, 9. E, 12. F, 16.
L, 64. K, 8. I, 4.

Otherwise. [Margin: f — Elements VIII.20] Since A and C are similar planes, let their sides be proportional: D, E those of A, and F, G those of C, so that D is to E as F is to G; and let H be taken as the side of the square A. Since, then, the same number A is produced both from D into E and from H into itself, [Margin: g — Elements VII.20] D, H, E will be in continued proportion. Since, therefore, F is to G as D is to E, and a mean proportional H falls between D and E, [Margin: h — Elements VIII.8] one mean proportional, say I, will also fall between F and G. Wherefore, since F, I, G are in continued proportion, [Margin: i — Elements VII.20] the same number will arise from F into G as from I into itself. But C arises

from F into G. Therefore C too will arise from I into itself; and so C will, by definition, be a square, whose side is I.

A, 36. B, 48. C, 64.

H, 6. I, 8.

D, 3. E, 12. F, 4. G, 16.

A, 36. B, 48. C, 64.

H, 6. I, 8.

D, 6. E, 6. F, 8. G, 8.

Again, let there be four numbers A, B, C, D in continued proportion, and let the first, A, be a cube. I say that the fourth, D, is also a cube. For, taking E, the side of the cube A, let the number F be made from E into itself; and so, by definition, the cube A is made from E into F. Since, then, E multiplied by itself makes F, and multiplied by F makes A, E, F, A will, by the scholium to Proposition 9 of this book, be in continued proportion from the unit; and therefore, by conversion, the numbers A, F, E, and the unit will likewise be proportional. And so, since between both the number D and the unit — and the same number A — mean proportionals fall equal in multitude, namely two, there will fall, by Theorem 3 of the scholium to Proposition 10 of this book, just as many means between the unit and the number D, which let be G, H. And so, since the unit and the numbers G, H, D are in continued proportion, H will be made from G into itself, and D from G into the product H, as was demonstrated by us at Proposition 10 of this book. Wherefore D is a cube, by definition, whose side is G.

A, 64. B, 160. C, 400. D, 1000.

F, 16. H, 100.

E, 4. G, 10.

unitas.

Otherwise. Having found the four least numbers E, F, G, H in the ratio of A to B, of B to C, and of C to D, E will measure A, and H will measure D, equally, by what we demonstrated at Proposition 21 of Book 7. And since E, H, the extremes of the four least numbers, are cubes, by Corollary 1 of Proposition 2 of this book — while A too is a cube, by hypothesis — [Margin: k — Elements VIII.12] two mean proportionals, say I, K, will fall between the cubes E and A. And since, by equality, as A is to D, so is E to H, and, by permutation, as A is to E, so is D to H, [Margin: l — Elements VIII.8] two mean proportionals, say L, M, will also fall between H and D. Since, then, H, the first, measures D, the extreme, [Margin: m — Elements VIII.7] it will measure the second, L, as well. Then, taking N, the side of the cube H, let N measure the number O as many times as H measures L, so that L is to H, and O to N, in the same multiple proportion; and let P also be the cube of O.

[Margin: n — Elements VIII.12] Thus, since the ratio of P to H, cube to cube, is the triplicate of the ratio of O to N, side to side — that is, of L to H — while the ratio of D to H is likewise the triplicate of the same ratio L to H, by the definition, so that by conversion D, M, L, H are also proportional, P will be to H as D is to H. And therefore P and D will be equal. Wherefore, since P is a cube by construction, D too will be a cube.

A, 64. B, 160. C, 400. D, 1000.
K, 32. M, 500.
I, 16. L, 250.
E, 8. F, 20. G, 50. H, 125.
P, 1000. O, 10. N, 5.

THEOR. 22. PROPOS. 24.

[Margin: xxij.]

If two numbers have to one another the ratio which a square number has to a square number, and the first is a square, then the second will also be a square.

For let A be to B as C, a square, is to the square D, and let A be a square. I say that B is also a square. For since A is to B as C is to D, [Margin: a — Elements VIII.11] and one mean proportional, namely E, falls between the squares C and D, [Margin: b — Elements VIII.8] one mean, say F, will also fall between A and B. Since, therefore, the three numbers A, F, B are in continued proportion, and A, the first, is a square, B, the third, will also be a square. [Margin: c — Elements VIII.22] Wherefore, if two numbers have to one another the ratio which a square number has to a square number, and so on. Which was to be shown.

A, 36. F, 48. B, 64.
C, 9. E, 12. D, 16.

COROLLARIVM.

It is clear from these things that the ratio of any square number to any non-square number can in no way be exhibited between two square numbers. For if it could be exhibited, [Margin: d — Elements VIII.24] then both the former numbers — having the ratio which the exhibited squares have — would also be squares, since the first is posited to be a square. But this is absurd; for the second is posited to be non-square. Hence numbers in a double ratio do not have the propor-

[Continuation of the Corollary begun on the preceding page — that a square-to-non-square ratio cannot be exhibited between two squares.]

do not have to one another the ratio that a square has to a square. For if they did, all these doubling numbers — 4, 8, 16, 32, 64, 128, and so on — would be squares. Since 4 is a square to begin with, 8 too would have to be a square [Margin: f — *Elements* VIII.24]; and then 16 and 32 as well, and so on [Margin: g — *Elements* VIII.11] — which is absurd [reading uncertain: the sequence is printed "16.&32.&0."; evidently "&c."]. For a mean proportional would fall between 4 and 8, between 8 and 16, between 16 and 32, and so on, if they were squares; yet it has been shown in the scholium to Prop. 8 of this book that no mean proportional whatever can fall between numbers standing in a double ratio.

In the same way, numbers in a quintuple ratio will not have to one another the ratio that a square number has to a square number. For if they did, a mean proportional would fall between them [Margin: h — *Elements* VIII.11]; and hence one would fall between 5 and 1 as well [Margin: i — *Elements* VIII.8], the least numbers in quintuple ratio — which we have demonstrated to be impossible in the scholium to Prop. 8 of this book.

THEOR. 23. PROPOS. 25.

[Margin: xxiiij.]

If two numbers have to one another the ratio that a cube number has to a cube number, and the first is a cube, then the second will also be a cube.

Let A be to B as the cube C is to the cube D, and let A be a cube. I say that B too is a cube. For since A is to B as C is to D, and since two cube mean proportionals — call them E and F — fall between the cubes C and D [Margin: a — *Elements* VIII.12], two means will likewise fall between A and B — call them G and H [Margin: b — *Elements* VIII.8].

A, 8. G, 12. H, 18. B, 27.
C, 64. E, 96. F, 144. D, 216.

Now since the four numbers A, G, H, B are in continued proportion, and A, the first, is a cube, B, the fourth, will also be a cube [Margin: c — *Elements* VIII.23]. And so, if two numbers have to one another the ratio a cube number has to a cube number, etc. Which was to be demonstrated.

COROLLARIUM.

[Margin: d — *Elements* VIII.25]

It is also clear from these results that the ratio of any cube number to any non-cube number cannot be exhibited between two cube numbers. For if it could be, then both of the former numbers — those having the ratio found between the two cubes — would themselves be cubes as well, since the first is assumed to be a cube. And that is absurd, for the second is assumed to be non-cube.

THEOR. 24. PROPOS. 26.

[Margin: xxiv.]

Similar plane numbers have to one another the ratio that a square number has to a square number.

Let A and B be similar plane numbers. I say that A is to B as some square number is to some other square number. For since A and B are similar planes, a single mean proportional — namely C — falls between them [Margin: a — *Elements* VIII.18].

A, 20. C, 30. B, 45.
D, 4. E, 6. F, 9.

Take then the three least numbers D, E, F in continued ratio with A, C, B; the extremes D and F will be squares, by Corollary 1 of Prop. 2 of this book. Since, therefore, *ex aequali*, A is to B as D is to F, it is clear that A is to B as one square, namely D, is to another square, namely F. Therefore similar plane numbers have to one another the ratio, etc. Which was to be shown.

[The lettered magnitude-figure D, 4 · E, 6 · F, 9 with A, 20 · C, 30 · B, 45 is printed twice in the side column, once against each half of the demonstration; the values are identical.]

SCHOLIUM.

We shall also easily demonstrate the converse of this, namely:

Numbers that have to one another the ratio a square number has to a square number are similar planes.

A, 8. B, 18.
C, 16. D, 36.

Let the numbers A, B have to one another the ratio that the squares C, D have. I say that A, B are similar planes. For since a single mean proportional falls between the squares C, D [Margin: b — *Elements* VIII.11], and since C is to D as A is to B, a mean proportional will also fall between A and B [Margin: c — *Elements* VIII.8]. Therefore A and B are similar planes [Margin: d — *Elements* VIII.20].

From this it is evident that plane numbers which are not similar do not have the ratio a square has to a square. For if they did, they would be similar planes, as was just demonstrated — which is absurd, since they are assumed not to be similar.

THEOR. 25. PROPOS. 27.

[Margin: xxv.]

Similar solid numbers have to one another the ratio that a cube number has to a cube number.

Let A, B be similar solids. I say that A is to B as some cube number is to some other cube. For since A, B are similar solids, two mean proportionals — call them C, D — fall between them [Margin: a — *Elements* VIII.19].

A, 240. C, 360. D, 540. B, 810.
E, 8. F, 12. G, 18. H, 27.

Take the four least numbers E, F, G, H in the continued proportion A, C, D, B; the extremes E, H will be cubes, by Corollary 1 of Prop. 2 of this book. Since, therefore, *ex aequali*, A is to B as E is to H, it follows that A is to B as one cube, namely E, is to another cube, namely H. Similar solid numbers, therefore, have to one another the ratio [a cube has to a cube], etc. Which was to be demonstrated.

SCHOLIVM.

We shall also demonstrate the converse of this, namely:

Numbers that have to one another the ratio a cube number has to a cube number are similar solids.

A, 16. B, 54.
C, 64. D, 216.

Let the numbers A, B have to one another the ratio that the cubes C, D have. I say that A, B are similar solids. For since A is to B as C is to D, and since two mean proportionals fall between the cubes C, D [Margin: b — *Elements* VIII.12], two

means will also fall between A, B [Margin: c — *Elements* VIII.8]. Therefore A and B are similar solids [Margin: d — *Elements* VIII.21].

From all this it is clearly inferred that no numbers standing in a double, superparticular, or superbipartient ratio are similar planes or similar solids [Margin: e — *Elements* VIII.18]. For if they were similar planes, a single mean proportional would fall between them — which, it was long ago shown in the scholium to Prop. 8 of this book, is impossible. By the same reasoning they will not be similar solids, since two means cannot fall between them; for otherwise two means would also fall between the least numbers of those same ratios [Margin: f — *Elements* VIII.8]. But that cannot happen, since these differ from one another by only a single unit, or at most by two, as was set out in the aforesaid scholium; and it is certain that between such numbers two numbers cannot be interposed, let alone two mean proportionals.

Likewise, no two prime numbers can be similar planes or similar solids, since they cannot have proportional sides. For any plane number that is prime has as its sides only unity and itself (though the latter improperly). Thus the plane numbers 19 and 23 have as sides 1 and 19, and 1 and 23; and since they are produced by the mutual multiplication of these — which are plainly not proportional [they are not similar]. And any solid number that is likewise prime has as its sides only two units and itself: thus the solid numbers 29 and 47 have as sides $1 \cdot 1 \cdot 29$ and $1 \cdot 1 \cdot 47$, since they are produced by the mutual multiplication of these. And it is evident that these cannot be proportional.

Again, no two numbers prime to one another which are not squares or cubes (even though neither of them is prime) can be similar planes or similar solids. For if two such numbers prime to one another were said to be similar planes or solids, one mean proportional would fall between them, or two [Margin: g — *Elements* VIII.18 & 19]. Since, then, the two extremes are assumed to be prime to one another, all three (or all four) will be prime to one another, because no number can measure them all as a common measure, on account of the two extremes being prime to one another. Hence the least numbers [in that ratio] will be in continued proportion, as we demonstrated in the scholium to Prop. 24 of Book 7. And therefore, by Corollary 1 of Prop. 2 of this book, the two extremes will be squares or cubes — which is absurd, for they are assumed to be neither squares nor cubes.

From this it follows that if two numbers are similar planes or similar solids, of which the smaller is prime, the smaller measures the larger. For if it did not measure it, the two given numbers would be prime to one another [Margin: h —

Elements VII.31]; and hence, as we have just shown, they could not be similar planes or solids — which is absurd and contrary to the hypothesis. But we shall demonstrate this affirmatively as well, thus. First let A, B be two similar plane numbers, with A the smaller and prime. I say that A measures B. For since A and B are assumed to be similar planes, they will have proportional sides.

A, 3. B, 48.

C, 1. D, 3. E, 4. F, 12.

Since, then, the sides of the prime number A are C (unity) and the number D equal to A — for no number other than C, unity, and the number D equal to A measures the prime number A — let E, F be the sides of B, proportional to the sides C, D, so that C is to E as D is to F. Then, by alternation, C will be to D as E is to F [Margin: i — *Elements* VII.19]. Therefore the number produced from C, the first, into F, the fourth, is equal to the one produced from D, the second, into E, the third. But from C (unity) into F comes F itself. So the same number F comes from D into E. Hence D measures F [Margin: k — *Elements* VII.4]. But F measures B [Margin: l — *Elements* VII, Axiom 4], since B is generated from E into F. Therefore D — that is, A, which is equal to it — also measures B [Margin: m — *Elements* VII, Axiom 11], which is what was proposed.

From this it is evident that two numbers prime to one another, of which the smaller is prime, cannot be similar planes. For if they were similar, the smaller would measure the larger, as we just showed. And since it also measures itself, they would not be prime to one another but composite — which is contrary to the hypothesis.

Next let A, B be two similar solids, with A the smaller and prime. I say that A measures B. For since A and B are similar solids, they will have proportional sides.

A, 3. B, 24.

B, 1. C, 1. D, 3. E, 2. F, 2. G, 6.

Since, then, the sides of the prime number A are B, C (units) and the number D equal to A — for no number other than unity and the number D equal to A measures the prime number A — let E, F, G be the sides of the number B, proportional to the sides B, C, D, so that B is to E as C is to F and D is to G. Now since C is to F as D is to G, by alternation C will be to D as F is to G [Margin: n — *Elements* VII.19]. Therefore the number produced from C, the first, into G, the fourth, is equal to the one produced from D, the second, into F, the third. But from C (unity) into G comes G itself. So the same number G comes from D into F. Hence D measures G [Margin: o — *Elements* VII.4]. But G measures B [Margin: p —

Elements VII, Axiom 4], since G is a side of B. Therefore D — that is, A, which is equal to it — also measures B [Margin: q — *Elements* VII, Axiom 11]. Which is what was proposed.

And from this it is likewise established that two numbers prime to one another, of which the smaller is prime, cannot be similar solids. For if they were similar, the smaller would measure the larger, as we just showed; and since it also measures itself, they would not be prime to one another but composite — which is contrary to the hypothesis.

And so it is quite easy to find two plane, or two solid, numbers that are not similar. For if we take two numbers standing in a double, superparticular, or superbipartient ratio; or else two prime numbers, or two numbers prime to one another, neither of which is a square or a cube; these very numbers will, by what has been set out, be plane or solid numbers that are not similar.

Again, any two numbers of which one is a square and the other not a square are plane numbers that are not similar — such as 16 and 20. For if they were similar planes, they would have the ratio a square has to a square [Margin: r — *Elements* VIII.28]; and since 16 is a square, 20 would also be a square [Margin: s — *Elements* VIII.24] — which is absurd, for it is assumed not to be a square.

By the same reasoning, any two numbers of which one is a cube and the other not a cube are solid numbers that are not similar — such as 27 and 40. For if they were similar solids, they would have the ratio a cube has to a cube [Margin: t — *Elements* VIII.27]; and since 27 is a cube, 40 would also be a cube [Margin: u — *Elements* VIII.25] — which is absurd, for it is assumed not to be a cube.

FINIS ELEMENTI OCTAVI. — End of the Eighth Book [of the *Elements*].

EVCLIDIS ELEMENTVM NONVM.

Euclid's Ninth Book of the *Elements*.

THEOR. 1. PROPOS. 1.

[Margin: i.]

If two similar plane numbers multiplied into each other make some number, the product will be a square.

Let the two similar plane numbers A, B, multiplied into each other, make C. I say that C is a square. For let A, multiplied into itself, make D, a square. Since, then, A

multiplied into A [makes D], and A [multiplied] into B [makes C], A will be to B as D is to C [Margin: a — *Elements* VII.17]. But between A and B, being similar planes, a single mean proportional falls [Margin: b — *Elements* VIII.18]; therefore between D and C a single mean proportional will also fall [Margin: c — *Elements* VIII.8]. Let E, then, fall, so that D, E, C are in continued proportion.

A, 6. B, 54.

D, 36. E, 108. C, 324.

And so, since the three numbers D, E, C are in continued proportion, and the first, D, is a square by construction, C, the third, will also be a square [Margin: d — *Elements* VIII.22]. Therefore, if two similar plane numbers are multiplied into each other, etc. Which was to be demonstrated.

THEOR. 2. PROPOS. 2.

[Margin: ii.]

If two numbers multiplied into each other make a square, they will be similar planes.

Let the two numbers A, B, multiplied into each other, make C, a square. I say that A, B are similar planes. For let A, multiplied into itself, make D, a square. Since, then, A multiplied into A made D, and A [multiplied] into B made C, A will be to B as D is to C [Margin: a — *Elements* VII.17]. But between D and C, being squares, a single mean proportional falls [Margin: b — *Elements* VIII.11]; therefore between A and B a single mean proportional will also fall [Margin: c — *Elements* VIII.8]. And on that account A and B are similar planes [Margin: d — *Elements* VIII.20]. Therefore, if two numbers multiplied into each other [make a square], etc. Which was to be demonstrated.

A, 6. B, 54.

D, 36. C, 324.

SCHOLIVM.

From these we shall demonstrate four following theorems, not without their use.

I.

If two square numbers multiplied into each other make some number, the product will be a square.

Let the two squares A, B, multiplied into each other, make C. I say that C is a square. For since A and B are similar planes — being, of course, squares — C, produced from them, will be a square [Margin: a — *Elements* IX.1].

A, 4. B, 25.
C, 100.

II.

If two numbers multiplied into each other make a square, and one of them is a square, then the other will be a square as well.

Let the two numbers A, B, multiplied into each other, make C, a square; and let A be a square. I say that B too is a square. For since A and B, multiplied into each other, make C, a square, A and B will be similar planes [Margin: a — *Elements* IX.2]; and so a single mean proportional, namely D, will fall between them [Margin: b — *Elements* VIII.18]. Therefore, since A is a square, B will also be a square [Margin: c — *Elements* VIII.22].

A, 9. D, 12. B, 16.
C, 144.

III.

If two numbers multiplied into each other make a non-square, and one of them is a square, then the other will be a non-square.

Let the two numbers A, B, multiplied into each other, make C, a non-square; and let A be a square. I say that B is a non-square. For if B were said to be a square, then C — the product of the squares A, B — would also be a square, by Theorem 1 of this scholium; which is absurd and contrary to the hypothesis. Therefore B is not a square number.

A, 4. B, 20.
C, 80.

IV.

If two numbers, a square and a non-square, multiplied into each other make some number, the product will be a non-square.

EUCLID'S ELEMENTS, BOOK IX

(Continuation of the Scholium to Proposition 2 — demonstration of sub-theorem IV, "a square multiplied by a non-square makes a non-square." Its enunciation stood at the close of the preceding page.)

Let two numbers, A (a square) and B (a non-square), multiplying each other, make C. I say that C is not a square. For if C were a square, then — since it is produced from A into B, and A is a square — B too would be a square, by Theorem 1 of this scholium; which is absurd and contrary to the hypothesis. Therefore C will not be a square.

A, 9. B, 20.
C, 180.

(9, a square, into 20, a non-square, makes 180, a non-square.)

THEOR. 3. PROPOS. 3. [Margin: iij.]

If a cube number, multiplying itself, produces some number, the product will be a cube.

Let the cube A, multiplying itself, make the number B. I say that B is a cube. For let C be the side of the cube A; from C into itself let D be made, and accordingly let the cube A itself be generated from C into D. Since, then, C multiplying itself made D, [Margin: a — Ax. 7] C measures D by C; [Margin: b — Ax. 5] but the unit also measures C by C. Therefore the unit is the same part of C that C is of D — namely, one named after C. And accordingly [Margin: c — Def. 20] as the unit is to C, so is C to D. Again, since C multiplying D made A, [Margin: d — Ax. 7] D measures A by C; but C also was measuring D by C. Therefore C is the same part of D that D is of A, [Margin: e — Def. 20] and so as C is to D, so is D to A. But as C is to D, so was the unit to C. Therefore as the unit is to C, so is C to D, and D to A; and thus between the unit and the number A two mean proportional numbers, C and D, fall.

But since A measures B by A — because A multiplying itself made B — [Margin: f — Ax. 7] and [Margin: g — Ax. 5] the unit also measures A by A, the unit will be the same part of A that A is of B; [Margin: h — Def. 10] and therefore as the unit is to A, so is A to B. Wherefore, since between the unit and the number A two mean proportionals C and D fall, [Margin: i — Elem. VIII.8] just as many will fall between A and B — namely E and F. Since therefore the four numbers A, E, F, B are in continued proportion, and A, the first, is a cube, [Margin: k — Elem. VIII.23] E [recte: B] too, the fourth, will be a cube. If therefore a cube number multiplying itself, etc. Which was to be demonstrated.

A, 8.
E, 16. D, 4.
F, 32. C, 2.
B, 64. Unit.

THEOR. 4. PROPOS. 4. [Margin: iv.]

If a cube number, multiplying a cube number, makes some number, the product will be a cube.

From the cube A into the cube B let the number C be made. I say that C is a cube. For let A, multiplying itself, make D; [Margin: a — Elem. IX.3] and D will be a cube. Now since A, multiplying A and B, made D and C, [Margin: b — Elem. VII.17] as A is to B, so will D be to C. But [Margin: c — Elem. VIII.12] between A and B, which are cubes, two mean proportionals fall. Therefore [Margin: d — Elem. VIII.8] between D and C also two mean proportionals will fall; and accordingly, [Margin: e — Elem. VIII.23] since D is a cube, C too will be a cube. Wherefore, if a cube number multiplying a cube number, etc. Which was to be shown.

A, 8. B, 27.
D, 64. C, 216.

THEOR. 5. PROPOS. 5. [Margin: v.]

If a cube number, multiplying a certain number, makes a cube, the number multiplied will also be a cube.

From the cube A into the number B let the cube number C be made. I say that B too is a cube. For let A, multiplying itself, make D; [Margin: a — Elem. IX.3] and it will be a cube. Since therefore A, multiplying A and B, made D and C, [Margin: b — Elem. VII.17] as A is to B, so will D be to C. But [Margin: c — Elem. VIII.12] between D and C, which are cubes, two mean proportionals fall. Therefore [Margin: d — Elem. VIII.8] just as many will fall between A and B; and accordingly, [Margin: e — Elem. VIII.22] since A is a cube, B too will be a cube. Wherefore, if a cube number multiplying a certain number, etc. Which was to be shown.

A, 8. B, 27.
D, 64. C, 216.

SCHOLIUM

From these, the following results too will be easily demonstrated, namely:

I.

If a cube number, multiplying a non-cube number, makes some number, the product will be a non-cube.

From A, a cube, into B, a non-cube, let the number C be made. I say that C is not a cube. For if C were a cube, [Margin: a — Elem. IX.5] then the number multiplied, B, would also be a cube — which is not posited.

A, 8. B, 20.
C, 160.

II.

If a cube number, multiplying a certain number, makes a non-cube, the number multiplied is also a non-cube.

From the cube A into the number B let the non-cube C be made. I say that B too is not a cube. For if B were a cube, [Margin: a — Elem. IX.4] then the product C would be a cube — which is contrary to the hypothesis.

A, 8. B, 17.
C, 136.

THEOR. 6. PROPOS. 6. [Margin: vj.]

If a number, multiplying itself, makes a cube, it too will be a cube.

Let the number A, multiplying itself, make B, a cube. I say that A too is a cube. For from A into B let the number C be made, which will be a cube — since A into itself makes B, and C itself is generated from A into B, and so it is "equal, equally equal," as is clear from what we wrote on the definition of a cube. Since, then, B — a cube — multiplying a certain number A makes the cube C, [Margin: a — Elem. IX.5] A too will be a cube. Wherefore, if a number multiplying itself makes a cube, etc. Which was to be demonstrated.

A, 8. B, 64. C, 512.

SCHOLIUM

The commentators on Euclid bring another demonstration in this place, and indeed a longer one. For after they have shown that C, produced from A into B, is a cube, they infer: [Margin: b — Elem. VIII.12] therefore between B and C, which are cubes, two mean proportionals fall. And since [Margin: c — Elem. VII.17] as B is to C, so is A to B — because A, multiplying A and B, made B and C themselves — [Margin: d — Elem. VIII.8] two mean proportionals will also fall between A and B. And so, [Margin: e — Elem. VIII.23] since B is a cube, A too will be a cube. But our demonstration is shorter and clearer, as is evident.

A, 8. B, 64. C, 512.

THEOR. 7. PROPOS. 7. [Margin: vij.]

If a composite number, multiplying some number, makes another, the product will be a solid number.

Let the composite number A, multiplying any number B, make C. I say that C is a solid number. For since A is composite, [Margin: a — Def. 13] some number besides the unit measures it; let D measure A by E. This being posited, [Margin: b — Ax. 9] with D multiplying E, A will be produced. Since therefore B, multiplying A, made C, C will be generated from the mutual multiplication of the three numbers D, E, B. And accordingly it will be a solid number by the definition, its sides being D, E, B. Wherefore, if a composite number, multiplying some number, etc. Which was to be shown.

A, 6. B, 11. C, 66.
D, 2. E, 3.

THEOR. 8. PROPOS. 8. [Margin: viij.]

If from the unit there are any number of continued proportionals, then the third from the unit is a square, and all that skip one; the fourth is a cube, and all that skip two; and the seventh is at once both a cube and a square, and all that skip five.

Let there be, from the unit, any number of continued proportionals A, B, C, D, E, F, G, H, I, K, L, M. I say that the third, B, from the unit is a square, together with all that skip one — which are D, F, H, K, M:

Unit. A, 3. B, 9. C, 27. D, 81. E, 243. F, 729. G, 2187.
H, 6561. I, 19683. K, 39049. L, 177147. M, 531441.

(K printed 39049; the $\times 3$ progression requires 59049. See numerical impossibles.)

The fourth, C, is a cube, together with all that skip two — of which sort are F, I, M; and the seventh, F, is at once a cube and a square, together with all that skip five — namely M.

For since the unit is to A as A is to B, [Margin: a — Ax. 5] the unit measures A equally as A measures B; but the unit measures A by A. [Margin: b — Ax. 9] And so A, multiplying A, will produce B. Therefore B is a square. And since B, C, D are three successive proportionals, and B is a square, [Margin: c — Elem. VIII.22 [?]] D too will be a square. By the same reasoning, taking the three successive proportionals D, E, F, since D is a square, [Margin: d — Elem. VIII.22] F too will be a square — and likewise all that skip one.

Again, since the unit is to A as B is to C, the unit measures A equally as B measures C. [Margin: e — Ax. 5] But the unit measures A by A. Therefore B will also measure C by A, [Margin: f — Ax. 9] and so B, multiplying A, will produce the number C. Wherefore, since A multiplying itself makes B, and multiplying B makes C — as has been shown — C is a cube. And since C, D, E, F are four successive proportionals, and C is a cube, [Margin: g — Elem. VIII.23] F too will be a cube. By the same argument, taking the four successive proportionals F, G, H, I, since F is a cube, [Margin: h — Elem. VIII.23] I too will be a cube — and so all that skip two.

Finally, since F, the seventh from the unit, has been shown to be both a square and a cube, it will be at once cube and square. And in the same way M, the seventh from F — five being skipped, namely G, H, I, K, L — will be at once cube and square; and likewise all that skip five. If therefore from the unit there are any number of continued proportionals, etc. Which was to be demonstrated.

SCHOLIUM

This theorem can also be proposed in this way:

If there are any number of continued proportionals from the unit, then all those set in odd places — namely in the third, fifth, seventh, ninth, eleventh, and so on — are squares. All those, however, that immediately follow the numbers set in the places third, sixth, ninth, twelfth, fifteenth, eighteenth, and the others that three measures — of which sort are the fourth, seventh, tenth, thirteenth, sixteenth, nineteenth, and so on — are cubes. And all those that immediately follow the numbers set in the places sixth, twelfth, eighteenth, twenty-fourth, and the others that six measures — such as the seventh, thirteenth, nineteenth, twenty-fifth, and so on — are at once cubes and squares.

For the numbers of the first order always skip one after the third from the unit; the numbers of the second order always omit two numbers after the fourth from the unit; and finally the numbers of the third order always interpose five after the seventh from the unit, just as Euclid proposes in the theorem.

THEOR. 9. PROPOS. 9. [Margin: ix.]

If from the unit there are any number of continued proportionals, and the one right after the unit is a square, then all the rest will be squares too. But if the one after the unit is a cube, then all the rest will be cubes too.

Let there be, from the unit, any number of continued proportionals A, B, C, D, E, F, and let the first, A, next to the unit, be a square. I say that all the rest too are squares. That the third from the unit, and all that skip one — which are B, D, F — are squares, [Margin: a — Elem. IX.8] has already been demonstrated. That the remaining intermediate ones, C and E, are also squares will become clear thus. Since A, B, C are three continued proportionals, and A is a square, [Margin: b — Elem. VIII.22] C too will be a square. And in the same way, taking the three continued proportionals C, D, E, since C is a square, [Margin: c — Elem. VIII.22] E too will be a square. Therefore all of A, B, C, D, E, F are squares. We shall show the same if more numbers are in continued proportion.

Unit. A, 4. B, 16. C, 64. D, 256. E, 1024. F, 4096.

Now let A, next to the unit, be a cube. I say that the rest too are cubes. That the fourth from the unit, and all that skip two — of which sort are C and F — are cubes, [Margin: d — Elem. IX.8] has already been demonstrated. That the remaining ones B, D, E are cubes we shall show thus.

Unit. A, 8. B, 64. C, 512. D, 4096. E, 32768. F, 262144.

Since the unit is to A as A is to B, the unit measures A equally as A measures B. [Margin: e — Ax. 5] But the unit measures A by A itself. Therefore A too will measure B by itself; [Margin: f — Ax. 9] and so the cube A, multiplying itself, will produce the number B. [Margin: g — Elem. IX.3] Wherefore B is a cube. And since A, B, C, D are four continued proportionals, and A is a cube, [Margin: h — Elem. VIII.23] D too will be a cube. And by the same reasoning, taking the four successive proportionals B, C, D, E, since B is a cube, [Margin: i — Elem. VIII.23] E too will be a cube. Therefore all of A, B, C, D, E, F are cubes. In no other way shall we demonstrate that all are cubes, if more are in continued proportion. And so, if from the unit there are any number of continued proportionals, etc. Which was to be demonstrated.

SCHOLIUM

The Geometer [Euclid] did not demonstrate that, if the number after the unit is at once a cube and a square, the rest too are all at once cubes and squares — because this very thing is plainly gathered from what has been demonstrated. For if the one next to the unit is at once a cube and a square, then in the respect in which it is a cube, all the rest will by that account be cubes; and in the respect in which it is a square, all the rest too will by that account be squares, as has been demonstrated. Therefore all will be at once cubes and squares.

But the whole proposition can be more easily demonstrated in this way. Let A, next to the unit, be first a square. I say that all are squares. For since, by the scholium to Proposition 10 of Book VIII, A multiplying itself makes B, B will by definition be a square. Likewise,

Unit. A, 4. B, 16. C, 64. D, 256. E, 1024. F, 4096.

since, by the same scholium, A multiplying B makes C, and A and B are squares, C too will be a square, by the scholium to Proposition 2 of this book. Again, since by the same scholium to Proposition 10 of Book VIII, A multiplying C makes D, and A and C are squares, D too will be a square, by the same scholium to Proposition 2 of this book. And so on in succession: since by that scholium to Proposition 10 of Book VIII, A multiplying D produces E, and multiplying E makes F, and so forth.

Let A, next to the unit, then be a cube. I say that all are cubes. For since, by the scholium to Proposition 10 of Book VIII, the cube A multiplying itself makes B, [Margin: k — Elem. IX.3] B too will be a cube. Likewise, since by the same scholium A multiplying B makes C, and A and B are cubes, [Margin: l — Elem. IX.4] C too will be a cube. Again, since by the same scholium A multiplying C makes D, and A and C are cubes, [Margin: m — Elem. IX.4] D too will be a cube. And so with the rest, since by that scholium A multiplying D generates E, and multiplying E produces F, and so forth.

Unit. A, 8. B, 64. C, 512. D, 4096. E, 32768. F, 262144.

THEOR. 10. PROPOS. 10. [Margin: x.]

If from the unit there are any number of continued proportionals, and the one after the unit is not a square, then no other will be a square either, except the third from the unit and all that skip one. But if the one after the unit is not a cube, then no other will be a cube either, except the fourth from the unit and all that skip two.

Let there be, from the unit, any number of continued proportionals A, B, C, D, E, F, G, H, I, and so on, and let the first, A, next to the unit, not be a square. I say that no other is a square either, except the third from the unit and all that skip one — that is, except B, D, F, H, and so on. For if, besides these, some other is a square, let it be a square:

Unit. A, 2. B, 4. C, 8. D, 16. E, 32. F, 64. G, 128. H, 256.
I, 512. K, 1024. L, 2048. M, 4096.

[Margin: a — Elem. IX.8] Since, then, D too is a square, and since as D is to E, or E is to F, so is A to B — and, conversely, as E is to D, or F is to E, so is B to A — B will

have to A the ratio that the square number E has to the square number D, or the square F to the square E. [Margin: b — Elem. IX.8] But B, the third from the unit, is a square. [Margin: c — Elem. VIII.24] Therefore A too, the first from the unit, will be a square; which is absurd, for A is posited as not a square. Therefore E is not a square. By the same reasoning we shall show that no other besides those named is a square.

But now let A, next to the unit, not be a cube. I say that no other is a cube either, except the fourth from the unit and all that skip two — namely, except C, F, I, and so on. For if, besides these, some other can be a cube, let it be D, ...let it be D, a cube. [Margin: d — IX.8] Since, then, F too is a cube, and since, *ex aequali*, as D is to F, so A is to C (because the three numbers D, E, F have the same ratio as the three A, B, C), then by inversion, as F is to D, so C is to A; and so C will bear to A the ratio that the cube F bears to the cube D. [Margin: e — IX.8] But C is a cube, since it is the fourth from the unit. [Margin: f — VIII.25] Therefore A also, the first from the unit, will be a cube. And this contradicts the hypothesis. So D is not a cube.

And should E be believed to be a cube: [Margin: g — IX.8] since C too is a cube, and, *ex aequali*, as the cube C is to the cube E, so A is to C, then by inversion, as the cube E is to the cube C, so C is to A; and since C is a cube, [Margin: h — VIII.25] A will be a cube as well — which is not supposed. So E is not a cube.

And just as it was shown that the two numbers D and E lying between the two cubes C and F are not cubes, so by the same reasoning we shall show that neither G and H, between the cubes F and I, nor any other two numbers lying between two cubes, are cubes. Therefore, if from a unit there be any number of continued proportionals, and so on. Which was to be demonstrated.

SCHOLIUM

It is not necessary that, if the number after the unit is not both a cube and a square, no other should be both a cube and a square except the seventh from the unit and all that skip five; for sometimes the third from the unit, sometimes the fourth, can be both a cube and a square, even though the first from the unit is not both a cube and a square.

For let the first number after the unit, A, be a cube but not also a square. I say that B, the third, is both a cube and a square: a cube, [Margin: i — IX.9] because, the first being a cube, all are cubes; a square, [Margin: k — IX.8] because it is the third from the unit.

Unit. A, 8. B, 64.

Next, let A after the unit be a square but not also a cube. I say that the fourth, C, is both a cube and a square: [Margin: l — IX.8] a cube, because it is the fourth from the unit; [Margin: m — IX.9] a square, because, the first being a square, all are squares.

Unit. A, 4. B, 16. C, 64.

Finally, let A after the unit be neither a cube nor a square. I say that no other is both a cube and a square except the seventh, F, and all that skip five. For, the first not being a square, [Margin: n — IX.10] no other is a square except the third from the unit and all that skip one — of the kind we taught in the scholium to Proposition 8 of this book, namely the third, fifth, seventh, ninth, eleventh, thirteenth, fifteenth, seventeenth, nineteenth, and so on. And the first not being a cube, [Margin: o — IX.10] no other is a cube except the fourth from the unit and all that skip two — such as, as appears from the same scholium, the fourth, seventh, tenth, thirteenth, sixteenth, nineteenth, and so on. It is therefore clear that only the seventh, the thirteenth, and the nineteenth — all of which skip five — are at once squares and cubes, and so on for the rest. For only into these places do a square and a cube fall together, as is plainly evident to anyone who has rightly weighed this proposition and the eighth of this book.

Unit. A, 6. B, 36. C, 216. D, 1296. E, 7776. F, 46656.

THEOR. 11. PROPOS. 11.

[Margin: xj.]

If from a unit there be any number of continued proportionals, the less measures the greater by one of the numbers that are among the proportionals.

Let there be from a unit any number of continued proportionals, A, B, C, D, E, F, G. I say that any of the smaller ones, say C, measures any of the greater, say G, by one of the numbers A, B, C, D, E, F, G. For since, *ex aequali*, as C is to G, so the unit is to D

Unit. A, 3. B, 9. C, 27. D, 81. E, 243. F, 729. G, 2187.

(because the five numbers C, D, E, F, G have the same continuous ratio as the unit and the four numbers A, B, C, D), [Margin: a — Def. 20] the unit measures the number D just as the number C measures the number G. [Margin: b — Axiom 5] But the unit measures the number D by D itself. Therefore C too, the number, will

measure the number G by D. By the same reasoning E will measure F by A, since as E is to F, so is the unit to A, and so on. So likewise A will measure G by F, since, *ex aequali*, as A is to G, so the unit is to F, and so on. And so with the rest in the same way. Therefore, if from a unit there be any number of continued proportionals, and so on. Which was to be shown.

SCHOLIUM

It is moreover plain from this that, by however much the greater number is distant from the greater measuring number, by so much is the number distant from the unit by which the less measures the greater; because the ratio of the less to the greater ought to be the same as that

Unit. A, 3. B, 9. C, 27. D, 81. E, 243. F, 729. G, 2187.

of the unit to the number by which the less measures the greater, as appears from the demonstration. And so B will measure F by D; and C will measure F by C, that is, by itself, and so on — because between B and F, just as between the unit and D, three numbers are interposed; and between C and F, just as between the unit and C, two numbers lie between.

[Margin: c — IX.9] From this, again, it results that any number that multiplies itself produces a number which, among the proportionals, is as far distant from it as it is itself from the unit; but if some smaller number multiplies a greater one, there is produced a number which is as far distant from the greater as the smaller is from the unit — because, if a number measuring a number multiplies the one it measures, the number that is measured is produced. So C, multiplying itself, will beget F, since F is as far from C as C is from the unit; and therefore C measures F by C, as we said. So likewise B, multiplying D, will make the same F; so that D measures F by B, and so on with the rest.

THEOR. 12. PROPOS. 12.

[Margin: xij.]

If from a unit there be any number of continued proportionals, then whatever prime numbers measure the last, these also measure the one that is nearest to the unit.

Let there be from a unit any number of continued proportionals, A, B, C, D. I say that whatever prime numbers measure the last, D, measure also A, the one nearest the unit. For let the prime number E measure D. I say that this same E, a prime, also measures the number A, nearest the unit. For if it does not measure A,

[Margin: a — VII.31] then E, being prime, will be prime to A: that is, A and E will be prime to one another. Let E measure D by F, [Margin: b — IX.9] so that E, multiplying F, makes D. But since A too measures D by C — and so A, multiplying C, makes D, as we showed in the scholium to the preceding proposition — the same number D will be produced from A, the first, into C, the fourth, and from E, the second, into F, the third. [Margin: c — VII.19] Therefore, as A, the first, is to E, the second, so F, the third, is to C, the fourth. [Margin: d — VII.23] Since, then, A and E are prime to one another, [Margin: e — VII.21] and hence least in their ratio, [Margin: f — IX.9] A will measure F just as E measures C. Let E measure C by G, so that E, multiplying G, makes C. But since A too, by the same scholium of the preceding proposition, measures C by B, [Margin: g — IX.9] so that A, multiplying B, makes C, the same number C will be procreated from A, the first, into B, the fourth, and from E, the second, into G, the third. [Margin: h — VII.19] Therefore, as A, the first, is to E, the second,

Unit. A, 10. B, 100. C, 1000. D, 10000. E, 5. H, 20. G, 200. F, 2000.

so is G, the third, to B, the fourth. Wherefore, since A and E are prime to one another, [Margin: i — VII.23] and therefore least in their ratio, [Margin: k — VII.21] A will measure G just as E measures B. Let E measure B by H, [Margin: l — Axiom 6] so that E, multiplying H, makes B. But A too measures B by A, and therefore A, multiplying itself, makes B, as appears from the same scholium of the preceding proposition. So the same number B is begotten from E, the first, into H, the third, and from A, the middle, into itself. [Margin: m — Def. 20] Therefore, as E, the first, is to A, the second, so A, the second, is to H, the third. Since, then, E and A are prime to one another, [Margin: n — VII.23] and hence least in their ratio, [Margin: o — VII.21] E will measure A just as A measures H. But this contradicts the hypothesis, for E was supposed not to measure A. Therefore it is false that E does not measure A. For from the very supposition that E is said not to measure A, it will always be demonstrated that E does measure A — which is absurd. Therefore E measures A. By the same argument we shall show that whatever other prime numbers measure D also measure A; and the reasoning will be the same if C or B be set as the last number.

Another way. [Margin: p — VII.31] Let the prime number E measure the last, D. I say that E measures A as well. For if it does not measure it, E will be prime to A. Since, then, A and E are prime to one another, and A, multiplying itself, makes B, as is established from the scholium of the preceding proposition, [Margin: q — VII.27] B will be prime to E. And since A and B are prime to E, while C is made from A into B, by the same scholium of the preceding proposition, [Margin: r — VII.26] C will be prime to E. Again in the same way, since A and C are prime to E,

and D is made from A into C, by the same scholium of the preceding proposition, [Margin: s — VII.26] D will be prime to E. Therefore E does not measure D — which contradicts the hypothesis. Therefore E measures A.

Unit. A, 6. B, 36. C, 216. D, 1296. E, 3.

Therefore, if from a unit there be any number of continued proportionals, and so on. Which was to be shown.

COROLLARIUM

And so neither any prime number greater than the one nearest the unit, nor any smaller one that does not measure the one nearest the unit, can measure the last. For if it measured the last, it would also measure the one nearest the unit, as has been demonstrated — which is not supposed.

SCHOLIUM

The demonstration of this first proposition is admirable. For in it Euclid, from the fact that E is said *not* to measure A, shows by an affirmative demonstration that E *does* measure A — which seems impossible to be done. For if someone were to set out to demonstrate that Socrates is white from the fact that he is not white, he would seem to bring forward something paradoxical and unexpected. Yet something not unlike this is what Euclid has done here with numbers, and in several other propositions that follow. Cardano [Cardanus] too effected something similar with magnitudes, in Book 5 of his *De proportionibus*, Proposition 201, and boasts that he was the first of all to discover this manner of demonstrating — which I think he would not have said, had he weighed more carefully the force of this demonstration, or at least, if he did weigh it, had he recalled it to memory; since Euclid, long before him, used this very mode of demonstrating, as is manifest from this twelfth theorem. Theodosius too used the same kind of demonstration in Book 1 of the *Spherics*, Proposition 12, as we noted there.

[Margin: t — IX.12] Furthermore, from what has been demonstrated it is clear that whatever prime number measures the last also measures all the others before it. For since it measures the one nearest the unit, and this one measures all that follow, [Margin: u — IX.11] because the less always measures the greater by one of the numbers that are among the proportionals, [Margin: x — Axiom 11] it is manifest that it measures them all.

THEOR. 13. PROPOS. 13.

[Margin: xiiij.]

If from a unit there be any number of continued proportionals, and the one after the unit is prime, then no other number will measure the greatest except those that are among the proportionals.

Let there be from a unit any number of continued proportionals, A, B, C, D, of which A, the one nearest the unit, is prime. I say that no other number except A, B, C themselves measures the greatest, D. For if it be possible, let another number E, different from A, B, C, measure D. Now it is manifest that E is not a prime number. For if it were prime and measured the last, D, [Margin: a — IX.12] it would also measure A, nearest the unit, which is supposed prime — which is absurd. Therefore E is not prime, but composite; [Margin: b — VII.33] and so some prime number will measure it. And I say this can be no other than the prime A. For if another prime

Unit. A, 5. B, 25. C, 125. D, 624. [recte: D, 625] E — H — G — F —

than A were to measure E, then, since E measures the last, D, [Margin: c — IX.11] that other prime would also measure the same D, [Margin: d — IX.12] and hence A itself as well, the first, namely the one nearest the unit. Which is absurd. Therefore no prime other than A measures E.

Now let E measure D by F. I say that F is different from A, B, C. For if F were the same as some one of A, B, C, and E measured D by F, then E would measure D by one of A, B, C; [Margin: e — VII.8] and conversely one of A, B, C (namely the one by which E measures D) would measure D by E. [Margin: f — IX.11] Since, then, each of A, B, C measures D by one of A, B, C, E would be the same as some one of A, B, C — which contradicts the hypothesis, for E was supposed *not* the same as any of A, B, C. Therefore F is not the same as any of A, B, C, but different. And since E measures D by F, [Margin: g — VII.8] conversely F measures D by E. But F will not be prime: for if it were prime and measured the last, D, [Margin: h — IX.12] it would also measure A, nearest the unit, which is prime — which is absurd. Therefore F is not prime, but composite; [Margin: i — VII.33] and so some prime number will measure it. And this again, I say, can be no other than the prime A.

Unit. A, 5. B, 25. C, 125. D, 625. E — H — G — F —

For if a prime other than A were to measure F, then, since F measures the last, D, [Margin: k — IX.11] that other prime would also measure the one nearest the

unit, which is prime — which is absurd. Therefore no prime other than A measures F.

Now since E measures D by F, [Margin: m — IX.9] D is made from E into F; but D is also made from A into C, as we taught in the scholium to Proposition 11 of this book. So the same number D is made from A, the first, into C, the fourth, and from E, the second, into F, the third; [Margin: n — VII.19] and hence, as A, the first, is to E, the second, so F, the third, is to C, the fourth. But A measures E, as was shown. Therefore F too measures C. Let it measure it by G. Since, then, F, different from A and B, measures the last, C (for we now leave aside the number D), by G, we shall show similarly that G is different from A and B, and is not prime but composite, and that only the prime A measures it — just as it was shown of the number F, by which E, different from A, B, C, measured the last, D.

[Margin: o — VII.8] For if G were the same as some one of A, B, and F measured C by G, then F would measure C by one of A, B; and conversely one of A, B (namely the one by which F measures C) would measure C by F. Since, then, each of A, B measures C by one of A, B, F would be the same as some one of A, B — which is absurd, for it was shown that F is not the same as any one of A, B. [Margin: p — VII.11] Therefore G is not the same as any of A, B, but different. And since F measures C by G, [Margin: q — VII.8] conversely G measures C by F. But G will not be prime. For if it were prime and measured the last, C, [Margin: r — IX.12] it would also measure the prime number A, nearest the unit — which is absurd. Therefore G is not prime, but composite; [Margin: s — VII.33] and hence some prime will measure it, and I say that this can be no other than A. For if a prime other than A were to measure G, then, since G measures the last, C, [Margin: t — IX.11] that other prime would also measure the same C, [Margin: u — IX.12] and so A itself as well, nearest the unit, which is prime — which is absurd. Therefore no prime other than A measures G. [Margin: x — IX.9] Now since C is made from F into G — because F measures C by G — while the same C is made from A into B, as we taught in the scholium to Proposition 11 of this book, the same number C is made from A, the first, into B, the fourth, and from F, the second, into G, the third. [Margin: y — VII.19] Therefore, as A, the first, is to F, the second, so G, the third, is to B, the fourth. But A measures F, as was demonstrated. Therefore G too measures B. Let it measure it by H. Since, then, G, different from A, measures the last, B (for we now leave aside the numbers C and D), by H, we shall show similarly that H is different from A, just as it was shown of the numbers F and G.

[Margin: z — IX.9] For if H were the same as A, and G measured B by H, then G would measure B by A; and conversely A would measure B by G. Since, then, A

measures B by A, as is established from the scholium to Proposition 11 of this book, G would be the same as A — which is absurd. For it was shown that G is not the same as A. Therefore H is not the same as A, but different. [Margin: a — IX.9] Now since B is made from G into H — because G measures B by H — while the same B is also made from A into itself, as we taught in the scholium to Proposition 11 of this book, the same number B is made from H, the first, into G, the third, and from A, the middle, into itself. [Margin: b — VII.20] Therefore, as H, the first, is to A, the second, so A, the second, is to G, the third. But A measures G, as has been demonstrated. Therefore H too measures A, a prime number. Which is absurd. Therefore, if A is prime, no number measures the greatest, D, except A, B, C themselves. And hence, if from a unit there be any number of continued proportionals, and so on. Which was to be demonstrated.

THEOR. 14. PROPOS. 14.

[Margin: xiv.]

If prime numbers measure the least number, no other prime number will measure it except those that measured it from the beginning.

Let A be the least number that the prime numbers B, C, D measure. I say that no other prime except B, C, D measures A. For, if it be possible, let another prime number E, different from B, C, D, measure A by the number F.

A B .. C ... D E — F —

Since, then, E measures A by F, [Margin: a — IX.9] E, multiplying F, will make A. [Margin: b — VII.32] Wherefore each of B, C, D will measure one or other of E, F — not indeed E, which is prime and different from them: therefore F, which is smaller than A. Which is absurd. For A was set as the least that the primes B, C, D measure. Therefore, if prime numbers measure the least number, and so on. Which was to be demonstrated.

SCHOLIUM

Campanus adds in this place the following theorem, not useless for what follows. Namely:

If any number of continued proportionals are the least of all having the same ratio with them, then a number measuring one of them will measure also one or other of the two numbers that are taken as least in the same ratio, or else it will be composite with respect to one of them.

SCHOLIUM (Campanus) — demonstration (*continued*)

(The theorem enunciated at the close of the preceding page: "If any number of continued proportionals are the least of all having the same ratio, then a number measuring one of them will measure also one or other of the two numbers taken as least in the same ratio, or else it will be composite with respect to one of them." Its demonstration now follows.)

Let there be any number of continued proportionals A, B, C, D, E, the least in their ratio, in which two least numbers F, G are likewise taken. Let H first measure the first of them, namely A. I say that H also measures either F or G, or at least is composite with respect to one of them.

For suppose H is not prime but composite. [c] Let there be taken, in the same ratio, three least numbers I, K, L, and four M, N, O, P, and so on, until there are as many, less one, as A, B, C, D, E themselves are. [Margin: c — *Elements* VIII.2]

A, 16. B, 24. C, 36. D, 54. E, 81.

M, 8. N, 12. O, 18. P, 27.

I, 4. K, 6. L, 9.

H — F, 2. G, 3.

Now (as is established from the method of producing any number of least terms given in Book VIII, proposition 2) the number A is produced from F into M, and H measures A. Therefore, by the scholium to proposition 32 of Book VII, H will be composite with respect to F or to M. If with respect to F, we have what was proposed. But if with respect to M — which is produced from F into I, as is established from the demonstration of proposition 2 of Book VIII — then, by the scholium to that same proposition 32 of Book VII, H will again be composite with respect to F or to I. If with respect to F, we have what was proposed; but if with respect to I, then, since I is generated from F into itself, H will be composite with respect to F, by the same scholium to proposition 32 of Book VII. And if H should be prime, we shall show in the same way, from proposition 32 of Book VII (but not from its scholium), that it measures F itself.

Next, let H, not being prime, measure the second, B. Since B is produced from F into N, it will be, as before, that H is composite with respect to F or to N. If with respect to F, we have what was proposed; but if with respect to N — which is produced from F into K — then H will in the same way be composite with respect to F or to K. If with respect to F, we have what was proposed; but if with respect to K, then, since K is generated from F into G, H will also be composite with

respect to F or to G. And if H should be prime, we shall demonstrate by the same reasoning, from proposition 32 of Book VII, that it measures either F or G.

Again, let H, not being prime, measure the third, C. Since it is produced from F into O, H will be composite with respect to F or to O. If with respect to F, we have what was proposed; but if with respect to O — which is produced from F into L — then H will be composite with respect to F or to L. If with respect to F, we have what was proposed; but if with respect to L, then, since L is generated from G into itself, H will likewise be composite with respect to G. And if H should be prime, we shall prove in the same way, from proposition 32 of Book VII, that it measures G itself.

Moreover, let H, not being prime, measure the fourth, D. Since it is produced from F into P, H will be composite with respect to F or to P. If with respect to F, we have what was proposed; but if with respect to P — which is produced from G into L — then H will be composite with respect to G or to L. If with respect to G, we have what was proposed; but if with respect to L, since it is produced from G into itself, H will likewise be composite with respect to G. And if H should be prime, we shall show in the same way, from proposition 32 of Book VII, that it measures G itself.

In the same way, if H, not being prime, measures the last, E, we shall show that H is composite with respect to G; and if H should be prime, that it measures G itself. And so, in the remaining cases, the demonstration will always be the same.

Therefore, if H measures the first number A, it will measure F itself, or will be composite with respect to it. If H measures the last, E, it will measure G itself, or will be composite with respect to it. And finally, if H measures any of the others besides the first and the last, it will measure either F itself or G itself, or will be composite with respect to one or the other. This is clear from the demonstration itself.

DEMONSTRATION IN NUMBERS OF THOSE THINGS WHICH EUCLID DEMONSTRATED IN LINES IN THE SECOND BOOK, IN THE FIRST TEN THEOREMS

Since, in order to demonstrate the theorem that follows, Theon [Theon of Alexandria] assumes certain things concerning numbers which have been demonstrated concerning lines in the second book — just as if the same had been shown of numbers — we have thought it not foreign to our purpose to demonstrate here, of numbers, some of those things which Euclid demonstrated

geometrically of lines in Book II. The same, it is reported by some, was done also by the monk Barlaam [Barlaam of Seminara]. We shall follow the same order which we observe Euclid to have kept in the second book.

I.

If there be two numbers, and one of them be cut into any number of parts: the plane number contained by those two numbers is equal to the numbers contained by the undivided number and each of the parts of the divided number.

Let there be two numbers AB and C, of which AB is divided into AD, DE, EB. Let F be produced from C into AB; likewise GH from C into AD; HI from C into DE; and IK from C into EB.

A . . . D . . E . B C . .
 F
 G H . . . I . . K

I say that F is equal to the numbers GH, HI, IK — that is, to the whole number GK composed of GH, HI, IK. For since C, multiplying AB, produced F, [a] AB measures F by C; that is, AB is a part of F, denominated by C. [Margin: a — 2. pronunc. (Axiom 2)] By the same reasoning, AD is a part of GH, DE of HI, and EB of IK, a part denominated by C — namely, the same part that AB is of F. But since, by what we demonstrated at proposition 5 of Book VII, the whole AB is the same part of the whole GK as AD is of GH, the whole AB will also be the same part of the whole GK as it is of F. [b] And therefore F and GK will be equal to one another. [Margin: b — 4. pronunc. (Axiom 4)] Which was what was proposed.

II.

If a number be divided into two parts: the plane numbers contained by the whole and each of the parts are together equal to the square number produced from the whole.

Let the number AB be divided into AC, CB. I say that the numbers produced from the whole AB into the parts AC, CB are together equal to the square number produced from the whole AB.

D
 A C . . B

For, taking a number D equal to AB, then by theorem 1 the number produced from D — that is, from AB into AB, namely the square of AB — is equal to the

numbers produced from D, that is, from AB, into AC and into CB. Which was what was proposed.

The same will be demonstrated if AB is cut into more parts than two, as can be seen from the second figure appended. For by the same reasoning the number produced from E — that is, from AB, namely the square of AB — is equal to the numbers generated from E, that is, from AB, into the several parts AC, CD, DB.

E
A . . . C . . D . B

III.

If a number be divided into two parts: the plane number contained by the whole and one of the parts is equal both to the number contained by the two parts, and to the square of the said part.

Let the number AB be divided into AC, CB. I say that the number produced from the whole AB into the part AC is equal to that contained by the parts AC, BC, together with the square of the said part AC.

D
A C B

For, taking a number D equal to the said part AC, then by theorem 1 the number produced from D — that is, from AC into AB, or (what is the same) from AB into AC — is equal to the numbers produced from D, that is, from AC into CB, and from D, that is, from AC into AC, namely the square of AC. Which was what was proposed.

IV.

If a number be divided into two parts: the square produced from the whole is equal to the squares produced from the parts, together with the plane number contained twice by the parts.

Let the number AB be divided into AC, CB. I say that the square number produced from AB is equal to the squares of the parts AC, CB, together with the number produced twice from the part AC into the part CB.

A C B

For by theorem 2 the square number of AB is equal to the numbers produced from AB into AC and into CB. But the number produced from AB into AC is, by theorem 3, equal to the number generated from AC into CB, together with the square of AC. Likewise the number produced from AB into CB is, in the same way,

equal to the number produced from AC into CB, together with the square of CB. Therefore the square number produced from AB is also equal to the square numbers of the parts AC, CB, together with the number produced twice from AC into CB. Which was what was proposed.

V.

If a number be cut into two equal and into two unequal parts: the plane number contained by the unequal parts, together with the square number of the number lying between the two sections, is equal to the square produced from the half.

Let the number AB be cut into equal parts AC, CB, and into unequal parts AD, DB. I say that the number contained by the unequal parts AD, DB, together with the square of the intermediate number CD — which is indeed the excess between the half AC (or CB) and one or other of the unequal parts AD, DB — is equal to the square produced from the half number CB.

A C . . . D . . B

For, since the square number of CB is, by theorem 4, equal to the squares of the parts CD, DB, together with the plane number contained twice by CD, DB; and since the number contained once by CD, DB, together with the square of DB, is equal (by theorem 3) to the number produced from CB into DB — let there be taken, in place of the square of DB together with the number produced from CD into DB, the equivalent: then the square number of CB is also equal to the remaining square of CD, together with the number produced from CD into DB. But by theorem 1 the number produced from the whole AD into DB is equal to the numbers produced from CD into DB and from AC into DB. Therefore the square of CB will be equal to the square of CD together with the number produced from AD into DB; that is, the number produced from AD into DB, together with the square of CD, will be equal to the square of CB. Which was what was proposed.

VI.

If a number be divided into two equal parts, and some other number be added to it: the number produced from the whole-with-the-addition into the added part, together with the square of the half number, is equal to the square of the number composed of the half and the added part.

Let the number AB be cut into equal parts AC, CB, and let the number BD be added to it. I say that the number produced from the whole AB together with the added BD, taken as one — namely from AD — into the added part BD, together

with the square of the half number CB, is equal to the square produced from CB, the half, together with the added BD, taken as one; that is, to the square of CD.

A C B . . . D

For, since the square of CD is, by theorem 4, equal to the squares of the parts CB, BD, together with the number contained twice by them, CB, BD — that is, equal to the squares of the parts CB, BD, together with the numbers contained by CB, BD and by AC, BD; and since the number contained by AD into BD is (by theorem 1) equal to the numbers contained by CB, BD, by AC, BD, and by BD, BD — that is, together with the square of BD — it follows that, if this number produced from AD into BD be taken in place of the square of the part BD together with the numbers produced from CB into BD and from AC into BD, the square of CD is also equal to the remaining square of CB, together with the number produced from AD into BD; that is, the number produced from AD into BD, together with the square of CB, is equal to the square of CD. Which was what was proposed.

VII.

If a number be divided into two parts: the square of the whole, together with the square of one of the parts, is equal to twice the number produced from the whole into that part, together with the square of the remaining part.

Let the number AB be divided into the parts AC, CB. I say that the square of the whole AB, together with the square of the part CB, is equal to twice the number produced from the whole AB into the said part CB, together with the square of the remaining part AC.

A C B

For since, by theorem 4, the square of the whole AB is equal to the squares of the parts AC, CB, together with the number produced twice from AC into CB: if the square of CB be added in common, the squares of the numbers AB, CB together will be equal to the squares of the numbers AC, CB, CB, together with the number produced twice from AC into CB.

But to that which is produced twice from AC into CB, together with the square of CB taken twice, there is equal (by theorem 3) the number produced twice from AB into CB. Therefore, if in place of the number produced twice from AC into CB together with twice the square of CB, there be taken the number produced twice from AB into CB, the squares of the numbers AB, CB together will be equal to the

number produced twice from AB into CB, together with the square of the remaining part AC. Which was what was proposed.

VIII.

If a number be divided into two parts: four times the number produced from the whole into one of the parts, together with the square of the remaining part, is equal to the square of the number composed of the whole and the aforesaid part.

Let the number AB be cut into the parts AC, CB. I say that the number produced four times from the whole AB into the part CB, together with the square of the remaining part AC, is equal to the square of the number composed of the whole AB and the aforesaid part CB.

A C . . . B . . . D

For, adding the number BD, which is equal to the said part CB: since the square of the whole AD, composed of AB and BD (that is, the said part CB), is equal (by theorem 4) to the squares of the numbers AB, BD, together with the number produced twice from AB into BD — that is, that which is produced twice from AB into CB; and since (by theorem 7) the squares of the numbers AB, CB together are equal to the number contained twice by AB, CB, together with the square of the number AC — it follows that, if in place of the squares of the numbers AB, CB together with the square of AC there be taken the number produced twice from AB into CB, then the square produced from AD is equal to the number produced four times from AB into CB, together with the square of the remaining part AC. Which was what was proposed.

IX.

If a number be cut into two equal and into two unequal parts: the squares produced from the unequal parts are double the squares produced from the half number and from the intermediate number.

Let the number AB be divided into equal parts AC, CB, and into unequal parts AD, DB. I say that the squares of the unequal parts AD, DB are double the squares produced from AC, the half, and from CD, the intermediate number — which CD is indeed the excess between the half AC (or CB) and one or other of the unequal parts AD, DB.

A C . . . D . . B

For, since the square of the number AD is equal (by theorem 4) to the squares of the numbers AC, CD, together with the number produced twice from AC into CD: if the square of the part DB be added in common, the squares of the parts AD, DB will be equal to the squares of the parts AC, CD, DB, together with the number produced twice from AC — that is, from CB — into CD. But to the square of DB, together with the number produced twice from CB into CD, there are equal (by theorem 7) the squares produced from CB, that is, from AC and CD. Therefore, if in place of the square of DB together with that which is produced twice from CB into CD there be taken the squares of AC, CD themselves, the squares of the parts AD, DB will be equal to the squares of the parts AC, CD taken twice; and therefore the squares of the parts AD, DB are double the squares produced from the parts AC, CD. Which was what was proposed.

X.

If a number be divided into two equal parts, and some other number be added to it: the square of the number composed of the whole and the added part, and the square of the added number, together are double both the square produced from the half, and the square of the number composed of the half and the added part.

Let the number AB be divided into two equal parts AC, DB [recte: CB], and let the number BD be added to it. I say that the square of the number AD, composed of the whole AB and the added BD, and the square of the added number BD, both together, are double the squares produced from AC, the half, and from CD, composed of CB, the half, and the added BD.

A C B . . D

For, since the square of AD is equal (by theorem 4) to the squares of the parts AC, CD, together with the number produced twice from AC — that is, from CB — into CD: if the square of the part BD be added in common, the squares of the numbers AD, BD will be equal to the squares of the parts AC, CD, BD, together with the number produced twice from CB into CD. But (by theorem 7) to the square of BD, together with the number produced twice from CB into CD, there are equal the squares of the numbers CB, CD; and to the square of CB there is equal the square of AC. Therefore, if in place of the square of BD together with the number produced twice from CB into CD there be taken the squares of the numbers CB, CD — that is, of the numbers AC, CD — the squares of the numbers AD, BD will be equal to the squares of the numbers AC, CD taken twice; and therefore the squares of the numbers AD, BD are double the squares produced from AC, CD. Which was what was proposed.

Now we shall demonstrate that theorem 11 of Book II cannot be accommodated to numbers — that is, that it is in no way possible for any number to be divided into two parts such that the plane number produced from the whole into one of the parts is equal to the square of the remaining part — as we noted there.

Let the number AB, if it be possible, be divided at C, so that the number produced from the whole AB into the part CB is equal to the square of the remaining part AC.

A D E C B

Now, since the number contained by the extremes AB, CB is equal to the square of the mean number AC, [a] it will be, as AB is to AC, so AC to CB. [Margin: a — *Elements* VII.20] But AB is greater than AC; therefore AC too is greater than CB. Now let CD, equal to CB, be taken away: then it will also be, as AB is to AC, so AC to CD. Since therefore the whole AB is to the whole AC as AC — taken away from the whole AB — is to CD, taken away from the whole AC, [b] it will also be, as the whole AB is to the whole AC [Margin: b — *Elements* VII.11], or as the subtracted AC is to the subtracted CD, so CB (the remainder from AB, that is, CD, equal to CB) is to AD, the remainder from AC. And since it is, as AC to CD, so CD to AD, and AC is greater than CD, CD too will be greater than AD. Let DE, equal to AD, be taken away: then it will also be, as AC to CD, so CD to DE. And so, since it is, as the whole AC to the whole CD, so CD — taken away from the whole AC — to DE, taken away from the whole CD, [c] it will also be, as the whole AC is to the whole CD [Margin: c — *Elements* VII.11], or as the subtracted CD is to the subtracted DE, so AD (the remainder from AC, that is, DE, equal to AD) is to EC, the remainder from CD. Therefore, since it is, as CD to DE, so DE to EC, and CD is greater than DE, DE too will be greater than EC. And accordingly a number equal to EC can be taken away from ED; and so on, without any end ever coming to this subtraction.

— which is absurd, since a number cannot be divided to infinity. Therefore the number AB will not be divided in such a way that the plane number produced from the whole multiplied into one of its parts is equal to the square of the remaining part. Which is what was proposed.

Alternatively [Aliter]. The number AB is divided at C in such a way that the product of AB into CB is equal to the square of the remaining part AC. Hence four times the product of AB into CB will be four times the square of AC; and consequently four times AB into CB, together with the square of AC, will be five times the square of the part AC. But the number contained four times under AB and CB, together with the square of AC, is itself a square number, since it is equal

to the square of the number compounded of AB and CB, by the third theorem [of the preceding scholium]. Therefore two square numbers — namely the one contained four times under AB, CB together with the square of AC, and the square of AC itself — have to one another the ratio of 5 to 1, or of 25 to 5. This is absurd, as is clear from the corollary to VIII.24 [coroll. propos. 24. lib. 8]. Therefore the number AB is not divided at C in such a way that the product of AB into CB is equal to the square of AC. Which is what was proposed.

The line divided at C:

A—————C——B

But we shall demonstrate this same thing more clearly at Proposition 29 of this book [ad propos. 29. huius lib.].

THEOR. 15. PROPOS. 15.

[Margin: xvj.]

If three numbers in continued proportion are the least of all those having the same ratio as they, then any two of them added together will be prime to the remaining one.

Demonstration. Let A, B, C be three numbers, the least of all having the same ratio as they. I say that any two of them added together are prime to the remaining one — namely A and B together to C; B and C together to A; and A and C together to B.

Worked values:

A, 9. B, 12.

C, 16.

D, 3. E, 4.

Take two numbers D, E, the least in the same ratio as A, B, C, by the scholium to VII.35 [scholio propos. 35. lib. 7]. Then, from the demonstration of VIII.2 [propos. 2. lib. 8], it is clear that D multiplied into itself makes A; D multiplied into E makes B; and E multiplied into itself makes C. [Margin: a — Elements VII.24] Now since D and E, being the least in their ratio, are prime to one another, [Margin: b — Elements VII.30] each of D, E — and so their sum too — will be prime to the other. And so, since both the sum of D and E, and D itself, are prime to E, [Margin: c — Elements VII.26] the number produced by the sum of D and E multiplied into D is likewise prime to that same E. But the number produced by (D+E) into D is equal, by the third theorem of the preceding scholium, to A (the square of D) together

with B (the product of D and E). Therefore A and B together are prime to E; [Margin: d — Elements VII.27] and consequently they are also prime to C, which is the square of E. So A and B added together are prime to C.

Next, as before, since each of D, E — and so their sum — is prime to the other, [Margin: e — Elements VII.30] it follows (both the sum of D and E, and E itself, being prime to D) [Margin: f — Elements VII.26] that the number produced by the sum of D and E multiplied into E is prime to D. But the number produced by (D+E) into E is equal, by the third theorem of the preceding scholium, to C (the square of E) together with B (the product of D and E). Therefore B and C together are prime to D; [Margin: g — Elements VII.27] and hence also to A itself, which is the square of D. So B and C added together are prime to A.

Second setup (same values, rearranged in the source):

B, 12. C, 16.
A, 9.
D, 3. E, 4.

Finally, as before, since each of D, E — and so their sum — is prime to the other, [Margin: h — Elements VII.30] and conversely each of D, E is prime to the sum of D and E, [Margin: i — Elements VII.26] the product of D into E will also be prime to the sum of D and E; [Margin: k — Elements VII.27] and consequently that same product of D into E will be prime to the sum of D and E multiplied into itself. But the sum of D and E multiplied into itself is equal, by the fourth theorem of the preceding scholium, to the numbers made from D and E each into themselves, together with twice the product of D into E. Therefore the product of D into E is prime to the numbers made from D and E each into themselves and to twice the product of D into E.

Third setup:

A, 9. C, 16.
B, 12.
D, 3. E, 4.

Now, since two numbers taken together — namely the sum of the squares of D and E, and the product of D into E — are prime to one of them, namely to the product of D into E, as has been shown, [Margin: l — Elements VII.20] those two, the sum of the squares of D and E and the product of D into E, will also be prime to one another. Again, since two numbers taken together — the sum of the squares of D and E, and the product of D into E — are prime to one of them, namely to the product of D into E, [Margin: m — Elements VII.30] as has been

shown, those two, the sum of the squares of D and E and the product of D into E, will likewise be prime to one another. Since, then, the squares of D and E make A and C, and the product of D into E makes B, A and C together will be prime to B. And therefore, if three numbers in continued proportion, etc. Which was to be shown.

SCHOLIVM.

Campanus demonstrates this theorem otherwise, and indeed more easily, for any number of numbers whatever in continued proportion, proposing it in the following way:

If any number of numbers in continued proportion are the least of all having the same ratio as they, then all the rest added together are prime to each one of them.

Let there be any number of numbers A, B, C, D in continued proportion, the least [in their ratio]. I say that all the rest added together are prime to each one of them — namely A, B, C together to D; A, B, D together to C; A, C, D together to B; and B, C, D together to A.

Worked values:

A, 8. B, 12. C, 18. D, 27.
E, — F, 2. G, 3. H, —

For if A, B, C together are not prime to D, then A, B, C together and D will be composite to one another, and so some number will measure them as a common measure. Let the number E measure them, if that is possible; and let F and G be taken, the two least in the ratio of the numbers A, B, C, D. Since, then, A, B, C, D are in continued proportion and least, while E measures one of them, namely D, E will also measure one of F, G — or else it will be composite to one of F, G — by the scholium to Proposition 14 of this book [scholio propos. 14. huius libri]. And therefore some number will measure both E and one of F, G.

Let H measure them. And so, since H measures E, and E is supposed to measure the sum of A, B, C and D itself, [Margin: n — Axiom 11] H will also measure that same sum of A, B, C and D itself. But since H measures one of F, G as well — and both F and G measure the middle numbers B, C, by the third corollary to VIII.2 [coroll. 3. propos. 2. lib. 8] — [Margin: o — Axiom 11] H will also measure the middle numbers B, C; [Margin: p — Axiom 10] and on that account the sum of B and C as well. And so, since H measures the whole sum of A, B, C and, when the sum of B and C is subtracted, [Margin: q — Axiom 12] it will also measure the

remainder A. But it was measuring D too. Therefore A and D, the extremes of the least numbers A, B, C, D, are composite to one another. [Margin: r — Elements VIII.3] Yet they are also prime to one another — which cannot be. Therefore the sum of A, B, C and the number D are not composite to one another. So they are prime to one another.

Again, if A, B, D together are not prime to C, then, as before, some number will measure them. Let E measure them, which again, by the scholium to IX.14 [scholio propos. 14. huius lib.], either will measure one of F, G, or will be composite to one of F, G. Let H then measure both E and one of F, G. And so, since H measures E, and E measures the sum of A, B, D, [Margin: s — Axiom 11] H will also measure that same sum of A, B, D. But since H measures one of F, G, which (by the third corollary to VIII.2) measure the middle numbers B, C, [Margin: t — Axiom 11] H will also measure B, C themselves. And so, since H measures the whole sum of A, B, D and, when B is subtracted, [Margin: u — Axiom 12] it will also measure the remaining sum of A, D. And since H measures one of F, G — of which F measures A and G measures D, as is clear from the demonstration of VIII.2 — [Margin: x — Axiom 11] H will also measure one of A, D. [Margin: y — Axiom 12] Therefore it will measure the other too. So A and D, the extremes, are composite to one another. [Margin: z — Elements VIII.3] Yet they are also prime to one another — which cannot be. Therefore A, B, D together are not composite to C. So they are prime to it. Which is what was proposed.

We shall show in no other way that A, C, D together are prime to B, and likewise B, C, D together to A. And the same reasoning holds for any number of numbers whatever in continued proportion.

THEOR. 16. PROPOS. 16.

[Margin: xvij.]

If two numbers are prime to one another, it will not be the case that as the first is to the second, so is the second to any other whatever.

Demonstration. Let A and B be prime to one another. I say that it is not the case that as A is to B, so is B to some other number. For if it were possible, let it be that as A is to B, so is B to another, namely to C. Since, then, A and B are prime to one another, [Margin: a — Elements VII.23] and therefore the least in their ratio, [Margin: b — Elements VII.21] they will equally measure the numbers B and C, which stand in the same ratio — namely A will measure B, and B will measure C. But A also measures itself. Therefore A measures A and B, which are prime to one

another. This is absurd. Therefore it is not the case that as A is to B, so is B to another number C. And by the same reasoning it will not be the case that as B is to A, so is A to another. And so, if two numbers are prime to one another, etc. Which was to be demonstrated.

The magnitudes:

A
 B
 C —————

THEOR. 17. PROPOS. 17.

[Margin: xvii.]

If there are any number of numbers in continued proportion, and their extremes are prime to one another, it will not be the case that as the first is to the second, so is the last to any other whatever.

Demonstration. Let there be any number of numbers A, B, C, D in continued proportion, whose extremes A, D are prime to one another. I say that it is not the case that as A is to B, so is D to some other number. For if it were possible, let it be that as A is to B, so is D to another, namely to E. Then, alternately, it will be that as A is to D, so is B to E. But A and D, being prime to one another, [Margin: a — Elements VII.23] are the least in their ratio. [Margin: b — Elements VII.21] Therefore they will equally measure B and E — namely A will measure B, and D will measure E. But it is the case that as A is to B, so is B to C. So, since A measures B, [Margin: c — Axiom 11] A will on that account also measure C. And since it is the case that as B is to C, so is C to D, and B measures C, C too will measure D. And so, since A measures C, [Margin: d — Axiom 11] it will also measure D. But it measures itself as well. Therefore A measures A and D, which are prime to one another. This is absurd. Therefore it is not the case that as A is to B, so is D to another number E. And by the same argument it will not be the case that as D is to C, so is A to another. And so, if there are any number of numbers in continued proportion, etc. Which was to be shown.

Worked values:

A, 8. B, 12. C, 18. D, 27. E, —

PROBL. 1. PROPOS. 18.

[Margin: xix.]

Given two numbers, to consider whether a third proportional to them can be found.

Demonstration. Let A and B be the two given numbers, and let it be required to consider whether a third proportional to them can be found — that is, whether there is some other number to which B stands as A stands to B. Now either A and B are prime to one another, or they are not. If they are prime to one another, [Margin: a — Elements IX.16] it is clear from what has been demonstrated that a third proportional to them cannot be found.

A, 4. B, 7.

But if A and B are not prime to one another, let B multiplied into itself make C. Then either A measures C, or it does not. First let A measure C, by D. I say that a third proportional to A and B can be found — indeed, that D itself is the third proportional. For since A measures C by D, [Margin: b — Axiom 9] C will be produced from A into D. But by construction the same C is produced from B into itself. Therefore the number contained under the extremes A and D is equal to the number described from the mean B; [Margin: c — Elements VII.20] and consequently it will be that as A is to B, so is B to D. And so D has been found as the third proportional to A and B.

A, 4. B, 6. D, 9. C, 36.

But now let A not measure C. I say that a third proportional to A and B cannot be found. For if it could be found, let D be found, such that A is to B as B is to D. Since, then, it is the case that as A is to B, so is B to D, [Margin: d — Elements VII.20] the number contained under the extremes A, D will be equal to the number produced from the mean B into itself — that is, to C. And so, since C is produced from A into D, [Margin: e — Axiom 7] A will measure C, by D. But A was supposed not to measure it. This is absurd. Therefore

a third proportional to A, B cannot be found when A, the first, does not measure C, the product of B, the second, into itself. By the same method we shall consider whether some other third number can be found for B and A, such that A stands to it as B stands to A. And so, given two numbers, we have considered, etc. Which was to be done.

PROBL. 2. PROPOS. 19.

[Margin: xx.]

Given three numbers, to consider whether a fourth proportional to them can be found.

Demonstration. Let A, B, C be three given numbers, whether in continued proportion or not; and let it be required to consider whether a fourth proportional to them can be found — that is, whether there is some other number to which C stands as A stands to B. Let B multiplied into C make D. Then either A measures D, or it does not. First let A measure D, by E. I say that a fourth proportional to A, B, C can be found — indeed, that E itself is the fourth proportional. For since A measures D by E, [Margin: a — Axiom 9] D will be produced from A into E. But the same D is produced from B into C by construction. Therefore the number contained under the extremes A, E is equal to the number contained under B, C, the second and third; [Margin: b — Elements VII.19] and on that account it will be that as A is to B, so is C to E. And so E has been found as the fourth proportional to A, B, C.

A, 8. B, 12. C, 18. E, 27. D, 216.

A, 4. B, 8. C, 9. E, 18. D, 72.

But now let A not measure D. I say that a fourth proportional to A, B, C cannot be found. For let the fourth proportional E be found, if that is possible, such that as A is to B, so is C to E. Since, then, the four numbers A, B, C, E are proportional, [Margin: c — Elements VII.19] the number contained under the extremes A, E will be equal to the number produced from B, the second, into C, the third — that is, to D. And so, since D is produced from A into E, [Margin: d — Axiom 7] A will measure D, by E. But it is supposed not to measure it. This is absurd. Therefore a fourth proportional to A, B, C cannot be found when A, the first, does not measure D, the product of B, the second, into C, the third. By the same method we shall consider whether some fourth number can be found for C, B, A, such that A stands to it as C stands to B. And so, given three numbers, we have considered whether it can be done, etc. Which was to be done.

A, 4. B, 6. C, 9. E, — D, 54.

A, 3. B, 4. C, 10. E, — D, 40.

SCHOLIVM.

What has been demonstrated in the last four propositions — namely 16, 17, 18, and 19 — must be understood of whole numbers. For if we speak of fractions, a further proportional can be adjoined to any two or three numbers, even though their extremes are prime to one another. For, given two numbers, if the square of the second is divided by the first, a third proportional will be produced. Thus,

given the two numbers 3 and 8: if the square of the second, 8 — namely 64 — is divided by 3, the third proportional $21\frac{1}{3}$ will be found, in this way: 3, 8, $21\frac{1}{3}$. For since, when 64 is divided by 3, the number $21\frac{1}{3}$ results, the number divided, 64, will be reproduced by the multiplication of the quotient $21\frac{1}{3}$ by the divisor 3, as we set out at VII, Definition 15 [ad defin. 15. lib. 7]. Since, then, this same number 64 is made from the mean number 8 [Margin: e — Elements VII.20] multiplied into itself, the numbers 3, 8, $21\frac{1}{3}$ will be in continued proportion.

Given three numbers, if the number generated from the second into the third is divided by the first, a fourth proportional will be produced. Thus, given the three numbers 9, 6, 20: if the number 120, generated from the second into the third, is divided by the first, 9, the fourth proportional $13\frac{1}{3}$ will be found, in this way: 9, 6, 20, $13\frac{1}{3}$. For since, when 120 is divided by 9, $13\frac{1}{3}$ results, the number divided, 120, will be reproduced by the multiplication of the quotient $13\frac{1}{3}$ by the divisor 9, from what we wrote at VII, Definition 15. But this same divided number is made from the second into the third. [Margin: f — Elements VII.19] Therefore the four numbers 9, 6, 20, $13\frac{1}{3}$ are proportional.

From these considerations we shall easily recognize, given any number of numbers in continued proportion, whether another proportional can be adjoined to them. For if, taking the last three, another can be found for them, that same one will be proportional to all of them.

Theon, and those who follow him — among whom is also Federico Commandino [Federicus Commandinus] — dispose of this problem in four cases. For (they say) either the three given numbers are in continued proportion and their extremes are prime to one another; or they are not indeed in continued proportion, but their extremes are prime to one another; or they are in continued proportion, but their extremes are not prime to one another; or, finally, they are neither in continued proportion nor are their extremes prime to one another. Here I cannot sufficiently wonder how it is that these most careful interpreters of Euclid — most vigilant in their other demonstrations — nodded off in demonstrating the second case of this problem, and, as it were forgetting themselves and slackening in their diligence, fell into an error, and no light one. For they say that if three given numbers are not in continued proportion, but their extremes are prime to one another, no fourth proportional can be found for them. This is both false, and the demonstration by which they try to establish it is faulty. For the falsity of the matter appears plainly in these numbers, not in continued proportion, 4, 8, 9, whose extremes 4 and 9 are prime to one another, and to which nonetheless a fourth proportional, 18, can be adjoined. For who does not see that as 4 is to 8, so

is 9 to 18, since in each case the ratio is one-half? The same may be seen in countless other examples, as here: 2, 4, 7, 14; likewise 3, 9, 4, 12; likewise 2, 10, 3, 15; likewise 5, 10, 6, 12; and so on. For in all these numbers the extremes of the first three are prime to one another, and yet a further, fourth proportional is adjoined to them. It is false, therefore, that no fourth proportional can be given for them. And that their demonstration, by which they strive to prove this, is faulty will become clear if we bring the demonstration itself into the open. They reason, then, as follows.

Let there be three numbers A, B, C, not in continued proportion, whose extremes A, C are prime to one another. I say that in no way can a fourth proportional be found for them. For if one can be found, let it be D, such that as A is to B, so is C to D, and as B is to C, so is D to E.

A, 3. B, 6. C, 8. D, 16. E, —

It will then follow, from equality of ratios, that as A is to C, so is C to E. But A and C are prime to one another, [Margin: g — Elements VII.23] and on that account the least [in their ratio]. [Margin: h — Elements VII.21] Therefore they will equally measure C and E — namely A will measure C, and C will measure E. But A also measures itself. Therefore A measures the two numbers A, C, which are prime to one another. This cannot be. Therefore a fourth proportional to A, B, C cannot be found.

This is their demonstration, in which they assume that as B is to C, so is D to some other number, namely to E — a thing they have nowhere demonstrated can be done; indeed, this very point is called into doubt in this problem. For it is no less asked whether a

Euclid, Book IX (Clavius's commentary) — Propositions 20–29, with the *Ex Campano* addendum

(Marginal citations follow Clavius's abbreviations: *N. noni* = Elements Book IX, prop. N; *N. septi*/*N. septim*/*N. sept* = Book VII, prop. N; *N. pron* = the Nth axiom [pronuntiatur]. The outer-margin lowercase-Roman numeral beside each heading is Clavius's proposition tally, which here runs consistently one ahead of the Arabic PROPOS. number.)

[Conclusion of the Scholium to Prop. 19]

whether in the three numbers B, C, D a fourth proportional can be found, than in the three A, B, C — since this holds no more in the one case than in the other. Therefore, since in that place they assume as granted the very thing which here they undertake to prove impossible, it is plainly a case of begging the question (to speak with the logicians) in their demonstration. What is more, whenever four numbers are given proportional but not consecutive, of which the first and third are prime to one another — as A, B, C, D are supposed to be — there is no way that another number can be given to which the fourth stands as the second stands to the third. Yet this is precisely what they assume as granted and proved, since without any proof they demand that a number E be given, to which the fourth D stands as the second B stands to the third C. But that this cannot be done we shall easily demonstrate by the very argument they themselves use.

For let there be four numbers A, B, C, D, proportional but not consecutive, and let A and C — the first and third — be prime to one another. I say it is impossible that D should stand to some other number as B stands to C. For suppose, if it can be done, that as B is to C, so D is to E:

A, 4. B, 12. C, 3. D, 9. E —

Then, *ex aequali*, as A is to C, so is C to E. And so, exactly as they themselves showed, A would measure the two numbers A and C, which are prime to one another. This is absurd. Therefore it cannot be that D is to E as B is to C. Hence, in the demonstration of the second member, Theon and the other interpreters of Euclid who follow him assumed something manifestly false.

The first member, however, they show clearly and openly. For if three numbers are consecutively proportional and their extremes are prime to one another [Margin: d — IX.17], it is clear, from what has been demonstrated, that no fourth proportional to them can be found. The last two members, finally, they dispatch in just the way we ourselves resolved the whole problem.

THEOR. 18. PROPOS. 20. [Margin: xxi.]

Prime numbers are more than any assigned multitude of prime numbers.

Let any number of prime numbers A, B, C be proposed. I say there are more primes than A, B, C. For take the least number DE which A, B, C measure, and add to it a unit EF [Margin: a — VII.38]. Then the whole DF is either prime or not

prime. Let it first be prime. Then A, B, C, and DF are prime numbers, more than the assigned multitude A, B, C.

A, 2. B, 3. C, 5.

30.

D ————— E. F

But now let DF not be prime. Then some prime number will measure it — say G [Margin: b — VII.33]. I say that this prime G is the same as none of A, B, C. For if G were the same as one of A, B, C, it would measure the same DE which they measure. Hence G, measuring the whole DF as well as the subtracted DE, would also measure the remainder EF — that is, the unit [Margin: c — Axiom 11]. This is absurd. Therefore the prime G is not the same as any of A, B, C; and so prime numbers A, B, C, G have been found, more than the assigned multitude of primes A, B, C. And by the same route more will be found than A, B, C, G, if the least number they measure be taken, and so on. Therefore prime numbers are more, etc. Which was to be demonstrated.

A, 3. B, 5. C, 7.

105.

D ————— E. F

53.

G —

SCHOLIVM.

This same theorem could have been proposed in the manner of a problem, thus:

Given any number of prime numbers, to find another prime number different from them.

For if any number of primes A, B, C be proposed, we shall find in the same way another prime different from them — namely DF, if it is prime, or else G, which measures DF. And in the same way we shall find, for four primes, a fifth; for five primes, a sixth; and so on we shall find as many prime numbers as anyone wishes.

THEOR. 19. PROPOS. 21. [Margin: xxii.]

If any number of even numbers be added together, the whole will be even.

Let any number of even numbers AB, BC, CD be added together. I say the whole compound AD is also even. For since AB, BC, CD are even, each will have its own half, by definition. Let E be the half of AB, F of BC, and G of CD. Since, then, as AB is to E, so BC is to F and CD is to G (each being a double ratio), it will also be that

as AB is to E, so is AD to E, F, G together [Margin: a — VII.12]. But AB is the double of E. Therefore the whole AD is likewise the double of the compound of E, F, G; and so AD, having as its half the number compounded of E, F, G, is even, by definition. If, then, any number of even numbers, etc. Which was to be demonstrated.

A..... B.... C..... D
E... F.. G....

THEOR. 20. PROPOS. 22. [Margin: xxij.]

If any number of odd numbers be added together, and their multitude be even, the whole will be even.

Let any number of odd numbers, whose multitude is even, AB, BC, CD, DE, be added together. I say the whole compound AE is even. For since AB, BC, CD, DE are odd, each differs from an even number by a unit, by definition.

A.... B.... C..... D..... E

Hence, a unit being subtracted from each, each of the remainders will be even [Margin: a — IX.21]; and so the compound of them will be even. But the multitude of the subtracted units is also even. Therefore the whole AE, so compounded, will be even [Margin: b — IX.21]. If, then, any number of odd numbers, etc. Which was to be shown.

THEOR. 21. PROPOS. 23. [Margin: xxiv.]

If any number of odd numbers be added together, and their multitude be odd, then the whole too will be odd.

Let any number of odd numbers, whose multitude is odd — AB, BC, CD — be added together. I say the whole compound AD is odd.

A..... B... C..... E. D

For since an odd number differs from an even by a unit, by definition, then, the unit ED being taken from the odd CD, the remainder CE will be even. But AC, compounded of the odds AB, BC — even in multitude — is even [Margin: a — IX.22 [?: printed digit reads 12 or 22; the argument requires IX.22]]. Therefore AE too, compounded of the evens AC, CE, will be even [Margin: b — IX.21]. Hence, the unit ED being added, the whole AD will be odd, since an odd differs from an even by a unit, by definition. If, then, any number of odd numbers, etc. Which was to be shown.

SCHOLIVM.

By the same reasoning, an odd number added to an even number, or to several even numbers, will make an odd number. For, a unit being taken from the odd, the remainder will be even — since by definition an odd differs from an even by a unit. Hence that even remainder, together with another given even, or with several given evens, will make an even [Margin: c — IX.21]; and so, that unit being added back again, the whole will become odd, since it differs from an even by a unit alone. Which is what was proposed.

THEOR. 22. PROPOS. 24. [Margin: xxv.]

If an even number be subtracted from an even number, then the remainder too will be even.

From the even number AB let the even CB be subtracted. I say the remainder AC is also even. For the subtracted even CB is either the half of AB, or greater than half, or less. Let it first be the half.

A C B

Since, then, AC — the other half — is equal to the even CB, AC too will be an even number.

But now let the subtracted even CB be greater or less than the half of AB. Since AB, CB are even numbers, they have their halves, by definition; let DB be the half of AB, and EB the half of CB.

A C . . D . . . E B

Since, then, as AB is to its half DB, so is CB to its half EB, by alternation, as AB is to CB, so is DB to EB; and by division, as AC is to CB, so is DE to EB; and again by alternation, as AC is to DE, so is CB to EB. But CB is the double of EB. Therefore AC too is the double of DE; and hence AC, being bisected (in that DE is its half), is even. If, then, an even number be subtracted from an even number, etc. Which was to be shown.

SCHOLIVM.

We shall demonstrate this proposition more easily by reduction to the impossible, thus. If the remainder AC be called not even but odd, then, the unit AD being subtracted, the remainder DC will be even.

A. D . . . C B

Therefore the compound DB will also be even [Margin: a — IX.21]; and hence, the unit AD being added back, the whole AB will become odd — which is absurd, since it is posited even.

THEOR. 23. PROPOS. 25. [Margin: xxvj.]

If an odd number be subtracted from an even number, then the remainder will be odd.

From the even number AB let the odd CB be subtracted. I say the remainder AC is odd.

A C . D B

For, the unit CD being taken from the odd CB, the remaining number DB becomes even. Since, then, the whole AB is likewise posited even, the remainder AD will also be even [Margin: a — IX.24]. Therefore, the unit CD being removed, the remainder AC will be odd. If, then, an odd number be subtracted from an even number, etc. Which was to be demonstrated.

THEOR. 24. PROPOS. 26. [Margin: xxvij.]

If an odd number be subtracted from an odd number, the remainder will be even.

From the odd number AB let the odd CB be subtracted. I say the remainder AC is even.

A C D . B

For, the unit DB being taken from the odds AB, CB, the remainders AD, CD will be even. Since, then, the even CD is taken from the even AD, the remainder AC will also be even [Margin: b — IX.24]. Hence, if an odd number be subtracted from an odd number, etc. Which was to be demonstrated.

THEOR. 25. PROPOS. 27. [Margin: xxvii.]

If an even number be subtracted from an odd number, the remainder will be odd.

From the odd AB let the even CB be subtracted. I say the remainder AC is odd.

A . D C B

For, the unit AD being taken from the odd AB, the remainder DB will be even; and when the even CB is subtracted from it, the remainder DC too will be even

[Margin: a — IX.24]. And so, the unit AD being added back, AC becomes odd. If, then, an even number be subtracted from an odd number, etc. Which was to be demonstrated.

THEOR. 26. PROPOS. 28. [Margin: xxix.]

If an odd number, multiplying an even number, makes some number, the product will be even.

Let the even number C be made from the odd A into the even B. I say C is even. For since C is made from A into B, C will be composed of as many numbers equal to B as there are units in A. Hence, since B is even, C will be composed of as many evens equal to B as there are units in A; and so it will be even [Margin: a — IX.21]. If, then, an odd number multiplying an even makes some number, etc. Which was to be shown.

A... B....
C.....

SCHOLIVM.

By the same demonstration we shall show this too:

If an even number, multiplying an even number, makes some number, the product will be even.

For again C will be composed of as many evens equal to B as there are units contained in A, etc.

A.. B.....
C.....

COROLLARIVM.

It is also clear from these that an even number multiplied into itself produces an even number — as is evident if the evens A and B be supposed equal.

A... B...
C.....

THEOR. 27. PROPOS. 29. [Margin: xxx.]

If an odd number, multiplying an odd number, makes some number, the product will be odd.

From the odd A into the odd B let C be made. I say C is odd. For since C is made from A into B, C will be composed of as many numbers equal to B as there are units in A. Hence, since both A and B are odd, C will be composed of odds equal to B, whose multitude is odd — namely equal to the multitude of units contained in the odd A. Therefore C is odd [Margin: a — IX.23]. Accordingly, if an odd number multiplying an odd makes some number, etc. Which was to be shown.

A... B...
C.....

COROLLARIVM.

From this it follows that an odd number multiplied into itself begets an odd number — as is clear if the odds A and B be supposed equal.

A... B...
C.....

EX CAMPANO.

An odd number measuring an even number measures it by an even number.

Let the odd A measure the even B by C. I say C is even. For if it were odd, then B, made from the odd A into the odd C, would be odd [Margin: b — IX.29]. This is absurd, since B is posited even. Therefore C is even.

A... C...
B.....

An odd number measuring an odd number measures it by an odd number.

Let the odd A measure the odd B by C. I say C is odd. For if it were even, then B, made from the odd A into the even C, would be even [Margin: b — IX.28]. This is absurd, since B is posited odd. Therefore C is odd.

A... C...
B.....

SCHOLIVM.

From these results we shall also most clearly demonstrate that the eleventh theorem of Book 2 [*Elements* II.11] cannot be adapted to numbers — which we also showed by another route in the scholium to Proposition 14 of this book, having undertaken to demonstrate it more clearly in this place. For let the

number AB be cut, if it can be done, into the parts AC, CB, in such a way that the number made from AB into CB is equal to the square of the remaining part AC. Then either both parts AC, CB are odd; or the greater AC is odd and the lesser CB even; or, conversely, the greater AC is even and the lesser CB odd; or, finally, both parts AC, CB are even.

A ——— C ——— B
D — E —

First, let both parts AC, CB be odd, so that the whole number AB is even [Margin: d — IX.22]. Then the number made from AB into CB is even [Margin: c — IX.28], whereas the number made from AC into itself is odd (by the corollary to Proposition 29 of this book). Since, then, the number generated from AB into CB is posited equal to the one made from AC into itself, an even number will be equal to an odd — which cannot be.

Next, let AC be odd and CB even, so that by the scholium to Proposition 32 [?: reads 32; the parallel case below cites the scholium to Prop. 23] of this book the whole AB is odd. Since, then, the number made from AB into CB is even [Margin: f — IX.28], while the number generated from the odd AC into itself is odd (by the corollary to Proposition 29 of this book), that even number will again be equal to this odd one — which is absurd.

Thirdly, let AC be even and CB odd, so that by the scholium to Proposition 23 of this book the whole AB is odd. Since, then, the number made from the odd AB into the odd CB is odd [Margin: g — IX.29], while the number generated from the even AC into itself is even (by the corollary to Proposition 28 of this book), that odd number will again be equal to this even one — which is absurd.

Fourthly, let both parts AC, CB be even. Then, having found (by the scholium to Proposition 35 of Book 7) the two least numbers D, E in the ratio of AC to CB, D and E will be prime to one another [Margin: h — VII.24]; and hence they will not both be even, but either both odd, or one odd and the other even.

A ——— C ——— B
D — E —

And since, as AC is to CB, so is D to E, then by conversion, as CB is to AC, so is E to D; and by composition, as AB is to AC, so is the whole DE to D [Margin: i — VII.20]. But as AB is to AC, so is AC to CB — since the number made from AB into CB is posited equal to the square of the mean AC. Therefore also, as the whole DE is to D, so is D to E; and hence the number made from the whole DE into E will be equal to the square of D [Margin: k — VII.20]. So the whole number D, E has been

cut into the parts D, E — of which either both are odd, or one is odd and the other even — in such a way that the number made from the whole DE into the part E is equal to the square of the other part D. But this we have just shown cannot happen in the number AB, when both parts AC, CB are either odd, or one odd and the other even. Therefore the proposition is established.

THEOR. 28. PROPOS. 30.

[Margin: xxxiii.]

If an odd number measures an even number, it will also measure its half.

Let the odd A measure the even B. I say that A measures the half of B as well. Let A measure B by C. Then, from what we demonstrated after [Campanus] in the preceding proposition, the number C is even, and so it has a half part. Since A measures B by C, C likewise measures B by A [Margin: a. 8. pro.]; and consequently C is a part of B named after A, as we taught at the 3rd definition of Book 7. Now, as C is to its own half, so is B to its own half; and, alternately, as C is to B, so is the half of C to the half of B. But C is a part of B named after A, as was shown; therefore the half of C will likewise be a part of the half of B, named after the same A. And for that reason A measures the half of B [Margin: b. 40. septi. — Elements VII.40]. So, if an odd number measures an even number, and so on. Which was to be shown.

The lettered segments:

A . . . C
B

THEOR. 29. PROPOS. 31.

[Margin: xxxiv.]

If an odd number is prime to some number, it will also be prime to its double.

Let the odd number A be prime to the number B, whose double is C. I say that A is prime to C as well. For if A and C are not prime to one another, some number will measure them—say D—which will necessarily be odd. For if it were even, then, since it measures the odd A, A would be the product of D (an even number) into the number by which it measures A, and so even [Margin: a. 28. noni — Elements IX.28]. But that is absurd, since A is assumed odd. Therefore D, odd and measuring the even C (for C is even, since it has B as its half), will also measure the number B, its half [Margin: b. 30. noni — Elements IX.30]. But it measures A as

well. Therefore D measures A and B, which are prime to one another. That is absurd. It is not the case, then, that A is not prime to C. Hence it is prime. Consequently, if an odd number is prime to some number, and so on. Which was to be shown.

The lettered segments:

A B
C D –

COROLLARIVM.

It follows from this that an odd number which is prime to some number is prime not only to its double—as Euclid proved—but also to its quadruple, its octuple, its sixteen-fold, and so on, proceeding by continual doubling. For if it is prime to the double, it will also be prime to the quadruple, which is the double of the double [Margin: c. 31. noni — Elements IX.31]; and likewise to the octuple, which is the double of the quadruple; and so forth.

THEOR. 30. PROPOS. 32.

[Margin: xxxv.]

Of the numbers doubled from the binary, each one is even-times-even only.

Let there be any numbers B, C, D, E doubled from the binary A. I say that B, C, D, E are even-times-even only. That they are even-times-even is evident. For, a unit being set out, since A is the binary and B, C, D, E are doubles from the binary, the numbers A, B, C, D, E will be successively proportional from the unit—namely in double proportion. Therefore A measures each of B, C, D, E, and every smaller measures the greater one following it, through some one of A, B, C, D, E [Margin: a. 11. noni — Elements IX.11]; and since these are all even, being doubles from the binary, each of B, C, D, E is measured by an even number through an even number. And for that reason each is even-times-even by definition.

That these same numbers are even-times-even *only* is likewise clearly established. For since A, B, C, D, E are continually proportional from the unit, and A, the number next to the unit, is prime—namely the binary—no other number measures any of them except A, B, C, D, E themselves [Margin: b. 13. noni — Elements IX.13]; and since these are all even, each of them is measured by an even number through an even number only. And for that reason each is even-

times-even only. Therefore, of the numbers doubled from the binary, and so on. Which was to be demonstrated.

Vnitas. A, 2. B, 4. C, 8. D, 16. E, 32.

SCHOLIVM.

Hence, together with the arithmeticians, we shall gather the art by which we may find all numbers that are even-times-even only. For let A, B, C, D, E, F be doubles from the binary, which, by what has here been demonstrated, are all even-times-even only, besides those contained in this series; and so, if the series of numbers doubled from the binary be increased to infinity, all even-times-even numbers are found, with none left out. For if it be possible, let there be some other number G, outside the series of numbers doubled from the binary, even-times-even only, whose half is H. Then H will be an even number. (For if it be called odd, then, since it is the half of G, the binary would measure G by H, an odd number; and so G would be even-times-odd. That is absurd, for it is assumed even-times-even only.) Let its half, again, be I. Then again I will be an even number. (For if it be called odd, since it is the fourth part of G, the even number 4 would measure G by I, an odd number; and on that account G would be even-times-odd. That is absurd, for it is assumed even-times-even only.) And so on, the half will always be an even number, until we come to the unit. Let, then, the half of I be a unit, so that I is the binary. Therefore H and G are doubles from the binary I; and consequently G will be some one of the series A, B, C, D, E, F—but it is assumed not to be. That is absurd. No other number, therefore, is even-times-even only, besides those that are doubles from the binary.

A, 4. B, 8. C, 16. D, 32. E, 64. F, 128.

I — — — — H — — — — G — — — —

THEOR. 31. PROPOS. 33.

[Margin: xxxvj.]

If a number has an odd half, it is even-times-odd only.

Let the number A have an odd half. I say that A is even-times-odd only. That it is even-times-odd becomes evident thus. Since A has an odd half, the binary, an even number, measures A by that odd half. Therefore A, by definition, is even-times-odd. That the same A is even-times-odd *only* we shall demonstrate in this way. Let B be the half of A, and C the binary. If, then, A is not even-times-even only, it will also be even-times-even. Therefore some even number will measure it by an even number. Let D, even, measure it by the even E. Then A is the

product of D into E [Margin: a. 9. pro.]; but the same A is the product of C, the binary, into B, its half. Therefore the number produced from C, the first, into B—the fourth—is equal to that produced from D, the second, into E—the third [Margin: b. 19. sept. — Elements VII.19]; and for that reason, as C is to D, so E is to B. But C, the binary, measures the even D. Therefore E too will measure B—an even number an odd one. That is absurd. Therefore A is not even-times-even. Hence it is even-times-odd only. Wherefore, if a number has an odd half, and so on. Which was to be demonstrated.

The lettered segments:

A

A

B C . .

D — — — — E — — —

SCHOLIVM.

Thus, if the doubles of all odd numbers be taken, there will be found all the numbers that are even-times-odd only. That any of them is even-times-odd only is established by this proposition, since each has an odd half. And that these alone are even-times-odd only will be clear from the following theorem, in which it is demonstrated that all even numbers which are neither doubles from the binary nor have odd halves are both even-times-even and even-times-odd. Since, then, no other even number has an odd half except the doubles of odd numbers, it is manifest that no other besides these is even-times-odd only; but each is either even-times-even only, or certainly both even-times-even and even-times-odd.

Impares 3. 5. 7. 9. 11. 13. 15.

Pariter impares tantum. 6. 10. 14. 18. 22. 26. 30.

THEOR. 32. PROPOS. 34.

[Margin: xxxvij.]

If an even number is neither a double from the binary nor has an odd half: it is both even-times-even and even-times-odd.

Let the even number A be neither a double from the binary nor have an odd half, but an even one. I say that A is both even-times-even and even-times-odd. That it is even-times-even is clear. For, since it has an even half, the binary—an even number—measures it by an even number, namely by its half. Therefore, by definition, it is even-times-even.

That the same A is also even-times-odd we shall show thus. When A is divided in two, and its half again in two, and so on, we shall at length fall upon some odd number. For if we were to fall upon the binary, A would be a double from the binary; but it is assumed not a double. That is absurd. Therefore, since we fall upon an odd number, that odd number will measure A by an even number. For otherwise, if it measured A by an odd number, then—since an odd multiplying an odd makes an odd—A would be the product of that odd into this odd, and so odd [Margin: a. 29. noni — Elements IX.29]. That is absurd, for it is assumed even. So, since an odd number measures A by an even, and hence, conversely, an even by an odd, A will, by definition, be even-times-odd. But it was also even-times-even. Therefore it is both even-times-even and even-times-odd. If, then, an even number is neither a double from the binary, and so on. Which was to be shown.

A

SCHOLIVM I.

Easily, from these, we shall find all numbers that are both even-times-even and even-times-odd. For, setting aside all those that are doubles from the binary, and all those that have odd halves, all the other even numbers will, by this proposition, be both even-times-even and even-times-odd.

Furthermore, from the last three theorems, what we wrote at the 9th definition of Book 7 is plainly understood.

THEOR. 33. PROPOS. 35.

[Margin: xxxviii.]

If there are any numbers in continued proportion, and from the second and the last there be subtracted [amounts] equal to the first: then, as the excess of the second is to the first, so will the excess of the last be to all those preceding it.

Let there be any numbers in continued proportion A, B, C, D, E, F; and from B C (the second) and E F (the last) let there be subtracted C G and F H, equal to the first, A. I say that as B G is to A, so is E H to A, B, C, and D together. Let F I be subtracted equal to B C, and F K equal to D. Since, then, F I is equal to B C, and the subtracted F H equal to the subtracted C G, the remainder I H will also be equal to the remainder B G. And since as A is to B C, so B C is to D, and D to E F; and, by conversion, as E F is to D, so D is to B C, and B C to A; and since K F is equal to D, and I F to B C, and H F to A: then also, as E F is to K F, so is K F to I F, and I F to H F. Therefore, dividing, as E K is to K F, so is K I to I F, and I H to H F; and

consequently, so also will all E K, K I, I H together be to all K F, I F, H F together [Margin: a. 12. septi. — Elements VII.12]—that is, the whole E H to D, B C, A together (to which K F, I F, H F were taken equal)—as I H is to H F, that is, as B G is to A; for B G and A are equal to I H and H F. Wherefore, if there are any numbers in continued proportion, and so on. Which was to be demonstrated.

A
 B G C
 D
 E K I H F

A .
 B . . G . C
 D
 E K I . . H . F

SCHOLIUM.

From this theorem we shall come to know the sum of any number of continually proportional numbers, provided the first, the second, and the last be known. For since, as the excess of the second is to the first, so is the excess of the last to all those preceding it: if the first be taken from the second and from the last, and it be arranged that, as the remainder of the second is to the first, so the remainder of the last is to another number—this other number will be the sum of all the proportional numbers that precede the last. If, then, the last be added, the whole sum will be had. And that fourth proportional is found if the first be multiplied into the excess of the last, and the product be divided by the excess of the second: as is clear from the scholium of proposition 19 of this book; since the three given numbers, to which the fourth proportional is to be found, are the excess of the second, the first, and the excess of the last. As in the example set alongside: if the first number—namely the unit—be subtracted from 3, the second, and from 2187, the last, the remaining excesses will be 2 and 2186. Since, then, as 2 is to 1, so must 2186 be to the fourth: therefore, since the first number 1 is multiplied by the third, 2186, the number 2186 is produced; which, when divided so as to give the sum of all except the last—the second number 1 being multiplied into the first, 2—yields the number 1093. That is, the sum of all, the last, 2187, excluded; and if this be added to the found sum 1003 [recte: 1093], the sum of all, 3280, is produced.

1. 3. 9. 27. 81. 243. 729. 2187.

THEOR. 34. PROPOS. 36.

[Margin: xxxix.]

If from a unit any number of numbers are set out successively in double proportion, until the whole composite becomes prime; and this whole, multiplied into the last, makes some number: the product will be perfect.

Let there be, from a unit, any numbers A, B, C, D, doubles, such that E, composed of them, is prime; and let E, multiplying D, the last, make F. I say that F is a perfect number. For as many as are the numbers A, B, C, D, let just as many doubles be taken from E—namely E, G, H, I. Since, then, A, B, C, D have the same ratio as E, G, H, I, there will be, *ex aequali*, as A is to D, so E to I [Margin: a. 19. sept. — Elements VII.19]; and for that reason the number produced from A, the first, into I—the fourth—is equal to that produced from D, the second, into E—the third. But F is produced from D into E. Therefore the same F will be produced from A into I [Margin: b. 7. pron. — Axiom 7]; and hence I measures F by A, the binary; and on that account F will be the double of I. Therefore E, G, H, I, F are successively doubles. Let there be subtracted from G (the second) and from F (the last) the numbers K, L, equal to the first, E, and let the remaining excesses be M, N. Then, as M is to E, so is N to E, G, H, I together [Margin: c. 35. noni — Elements IX.35]. But M is equal to E (for, since G is the double of E, and K, taken away, is equal to E, the remainder M will also be equal to E). Therefore N too is equal to E, G, H, I together. Adding, then, equals—namely the number L (which is equal to E, taken away) and the number composed of the unit and the numbers A, B, C, D (which composite is, by construction, equal to E)—the composite of L and N, namely F itself, will be equal to the unit and the numbers A, B, C, D, E, G, H, I together. Wherefore, since the unit and all the numbers A, B, C, D, E, G, H, I measure F (for, since F is produced from E into D, D measures F [Margin: d. 7. pro.]; and so the unit and the numbers A, B, C [Margin: e. 11. pron. — Axiom 11] which measure D measure the same F [Margin: f. 11. pron. — Axiom 11]; again, since I measures F, as has been shown, the numbers E, G, H [Margin: g. 11. pron. — Axiom 11], which measure I on account of the double proportion [Margin: h. 11. noni — Elements IX.11], also measure the same F), and no other number measures F, as we shall presently show—the unit and the numbers A, B, C, D, E, G, H, I will be all the parts that F can have. And since F has been shown equal to them, F will, by definition, be perfect.

Vnitas. A, 2. B, 4. C, 8. D, 16.
E, 31. G – 62. H, 124. I, 248. F – 496.
K, 31. M, 31. L, 31. N, 465.
O – P –

That no other number, besides A, B, C, D, E, G, H, I, measures F, we shall demonstrate thus. Let, if it be possible, another number O, besides those, measure

F by P [Margin: i. 9. pron. — Axiom 9]; and so F is produced from O into P. But the same F is produced from E, the first, into D—the fourth—and from P, the second, into O—the third; and for that reason, as E is to P, so is O to D [Margin: k. 19. septi — Elements VII.19]. Now, since A, B, C, D are proportional from the unit, and A, next to the unit, is prime, no other number besides A, B, C measures the last, D [Margin: l. 13. oct. — Elements VIII.13]; and O is assumed not the same as any one of A, B, C; therefore O will not measure D. But as O was to D, so was E to P; therefore neither will E measure P. Since, then, E is prime, E and P will be prime to one another [Margin: m. 31. septi — Elements VII.31], and therefore least in their proportion [Margin: n. 23. septi — Elements VII.23]. And consequently E measures O just as P measures D [Margin: o. 21. sexti — Elements VI.21]. But no other number besides A, B, C measures D [Margin: p. 23. octau — Elements VIII.23]. Therefore P will be the same as some one of A, B, C. Let it, then, be the same as B; and as many as are B, C, D, let just so many doubles be taken from E: E, G, H. There will then be, *ex aequali*, as B is to D, so E to H [Margin: q. 18. septi — Elements VII.18]; and consequently the same number will be produced from B, the first, into H—the fourth—as from D, the second, into E—the third. But what is produced from D into E was equal to what is produced from P into O. Therefore the same will be produced from P into O as from B into H, and so, as P is to B, so is H to O [Margin: r. 29. septi — Elements VII.29]. But P was the same as B; therefore H too is the same as O. That is absurd, for O is assumed different from all of A, B, C, D, E, G, H, I. No other number O, therefore, measures F; but A, B, C, D, E, G, H, I alone measure it. For which reason, if from a unit any number of numbers be set out successively in double proportion, and so on. Which was to be demonstrated.

SCHOLIVM.

From this theorem is drawn the method of finding all perfect numbers, and their aliquot parts, which, taken together, are—according to the definition of a perfect number—equal to the number itself. For if any number of numbers, doubled from a unit, be gathered into one sum until the composite is a prime number, the number produced from that prime into the last of the doubles will be perfect. And if, from that prime which is composed of those doubles, there be taken just as many continued doubles (the prime itself also being reckoned) as there are of those numbers doubled from the unit, the unit excluded—then these numbers taken, and those doubled from the unit, together with the unit itself, will be all the aliquot parts that the perfect number found can have. All of which are evident from the demonstration of the theorem, and easily from the examples set below—

can be grasped from the examples. For since the number 3, composed of 1 and 2, is prime, the number 6 — produced from 3 into 2, the last of the doubles — will consequently be perfect, and its aliquot parts are 1, 2, 3. Again, since the number composed of these, 1, 2, 4, namely 7, is prime, therefore 28, produced from 7 into 4, is perfect, having these aliquot parts: 1, 2, 4, 7, 14. But since the number composed of these, 1, 2, 4, 8 — that is, 15 — is not prime, 120, produced from 15 into 8, will not be perfect.

Finally, since the number composed of these, 1, 2, 4, 8, 16, namely 31, is prime, 496, produced from 31 into 16, will be a perfect number, whose aliquot parts are 1, 2, 4, 8, 16, 31, 62, 124, 248. In the same way we shall find the remaining perfect numbers as well.

It is worth demonstrating here certain properties of geometrical proportionality, on which rests that remarkable property we set out in Book 5, in the seventh place, when we were treating of geometrical proportionality; and we begin from here.

I.

When any number is divided by any two numbers, these two numbers are proportional, in reverse order, to the quotients.

For let the number A, divided by B and C, yield the quotients D and E. I say that as B is to C, so E is to D. For since, when A is divided by B, the quotient D results, A will be produced from B into D — that is, $A = B \times D$ — as we stated at Definition 15 of Book 7. By the same reasoning, the same A will be produced from C into E. Since, therefore, the same number A is produced from B (the first) into D (the fourth), and from C (the second) into E (the third), it follows [Margin: a — VII.19] that as B (the first) is to C (the second), so E (the third) is to D (the fourth). Which was to be proved.

A, 48.
B, 8. C, 3.
D, 6. E, 16.

II.

When any number is divided by any number of numbers, each pair of dividing numbers is proportional, in reverse order, to the corresponding pair of quotients.

For let the number A be divided by B, C, D, E, and let the quotients be F, G, H, I. I say that as B is to C, so G is to F; and as C is to D, so H is to G; and as D is to E, so I

is to H. This is clear from the first theorem, just demonstrated. For since the two divisors B, C of A yield F, G, it follows that as B is to C, so G is to F. Likewise, since the two divisors C, D of A yield G, H, it follows that as C is to D, so H is to G. Likewise, since the two divisors D, E of A yield H, I, it follows that as D is to E, so I is to H — and so on in turn, if there are more. Which was to be proved.

A, 48.

B, 3. C, 4. D, 8. E, 6.

F, 16. G, 12. H, 6. I, 8.

III.

When any number is divided by any number of numbers in continued proportion, the quotients, in reverse order, are in continued proportion in the same ratio.

For let the number A be divided by B, C, D, E, which are in continued proportion, and let the quotients be F, G, H, I. I say that I, H, G, F are in continued proportion in the same ratio as B is to C. For, by the second theorem, as D is to E, so I is to H; and as C is to D — that is, as D is to E — so H is to G; and as B is to C — that is, as C is to D, and as D is to E — so G is to F. Therefore I, H, G, F are in continued proportion in the ratio of D to E, or of B to C. Which was to be proved.

A, 15.

B, 1. C, 2. D, 4. E, 8.

F, 15. G, $7\frac{1}{2}$. H, $3\frac{3}{4}$ [?] I, $1\frac{1}{8}$.

IV.

If there are any numbers whatever, and just as many others in the same ratios, then as the sum of the first is to any one of them, so the sum of the second is to the one that corresponds to it among the latter.

Let there be any numbers A, B, C, whose sum is D; and just as many others E, F, G, whose sum is H; and let A be to B as E is to F, and B be to C as F is to G. I say that as D is to A, B, C severally, so is H to E, F, G severally. For since A is to B as E is to F, then by composition [componendo] A, B together will be to B as E, F together are to F; and B is to C as F is to G. Therefore, by equality [ex aequo], A, B together will be to C as E, F together are to G; and by composition, A, B, C together — that is, D — is to C as E, F, G together — that is, H — is to G.

Since, therefore, D is to C as H is to G, and, by conversion [convertendo], C is to B as G is to F, it follows by equality that D is to B as H is to F. Finally, since D is to B as H is to F, and, by conversion, B is to A as F is to E, it follows by equality that D is

to A as H is to E. Which was to be proved. And the same reasoning holds for more terms.

A, 3. B, 6. C, 20.

D, 29.

E, 6. F, 12. G, 40.

H, 58.

V.

If there are any numbers whatever, and just as many others in the same ratios, then the sum of the first, divided by them severally, will yield exactly the same quotients as the sum of the second divided by them.

For, as was shown in the fourth theorem, the ratio of the sum of the first terms to the first term is the same as the ratio of the sum of the second terms to the first; and therefore both ratios will have the same denominator. Now, since the denominator of a ratio is generated by dividing the antecedent term by the consequent, the very same quotient will be produced when each sum is divided separately by the first numbers, each by its own. And the same reasoning holds if the division is made by the second, third, fourth terms, and so on. So the proposition is established.

VI.

If the sum of any numbers in continued proportion is divided by them severally; and the sum of the quotients by those same quotients; and the sum of these second quotients by those same second quotients; and so on to infinity: the same first quotients will always be produced alternately, in reverse order.

Let there be any numbers A, B, C, D in continued proportion, whose sum E, divided by them, yields the quotients F, G, H, I; and let the sum K of these, divided by them, yield the quotients L, M, N, O. I say that these quotients and those are the same alternately, in reverse order — that is, that F and O are the same; likewise G and N; H and M; I and L; and so on.

A, 1. B, 2. C, 4. D, 8. E, 15.

F, 15. G, $7\frac{1}{2}$. H, $3\frac{3}{4}$. I, $1\frac{1}{8}$. K, $28\frac{1}{8}$.

L, $1\frac{1}{8}$. M, $3\frac{3}{4}$. N, $7\frac{1}{2}$. O, 15.

For, by the third theorem, the quotients F, G, H, I have, in reverse order, the same continued proportion as the numbers A, B, C, D; and the quotients L, M, N, O have, in reverse order, the same proportion as the quotients F, G, H, I — that is, the

same, in the same order, as the numbers A, B, C, D. Therefore, by the fourth theorem, E is to A as K is to I. Since the denominator of a ratio is produced by dividing the antecedent term by the consequent, the same quotient will result from dividing E by A as from dividing K by I. Since, then, these quotients are F and O, F and O will be equal. By the same reasoning G and N will be equal; H and M; I and L; and so on. Which was to be proved.

VII.

If the sum of any numbers in continued proportion is divided by them severally, the sum of all the quotients equals the number generated by multiplying the first into the last, or by the mutual multiplication of any two means equally distant from the extremes, or finally (if the number of quotients is odd) by multiplying the middle quotient into itself.

First let there be any numbers, even in count, in continued proportion — A, B, C, D, E, F — whose sum G, divided by them severally, yields the quotients H, I, K, L, M, N, whose sum is O. I say that the number O equals the number produced from H into N, or from I into M, or from K into L. For, by the fourth theorem, the quotients H, I, K, L, M, N are in continued proportion

A, 1. B, 2. C, 4. D, 8. E, 16. F, 32. G, 63.
H, 63. I, 31½. K, 15¾. L, 7¾. M, $3^{15}/_{16}$. N, $1^{31}/_{32}$. O, 124½ [?].

[Margin: a — VII.19] in reverse order in the ratio of A to B, so that N is to M as I is to H. Therefore the same number is generated from N into H as from M into I; that is, the same from H into N as from I into M; and by the same reasoning the same from I into M as from K into L. And since, by the sixth theorem, the sum O, divided by H, I, K, L, M, N, produces the same quotients in reverse order — that is, the sum O divided by H yields the quotient N — the sum O will be produced from H into N, by the definition; and accordingly also from I into M, and from K into L.

Next let there be any numbers, odd in count, in continued proportion — A, B, C, D, E — whose sum F, divided by them severally, yields the quotients G, H, I, K, L, whose sum is M. I say that M equals the number generated from G into

A, 1. B, 2. C, 4. D, 8. E, 16. F, 31.
G, 31. H, 15½. I, 7¾. K, 3¾. L, $1^{15}/_{16}$. M, $60^{1}/_{16}$.

L, and from H into K, and from I into itself. For that it is equal to the number made from G into L, and from H into K, we shall demonstrate as before. [Margin: b — VII.20] Then, since the three numbers H, I, K are in continued proportion, the

number generated from H into K — that is, the sum M — will be equal to the number generated by multiplying I into itself. Which was to be proved.

From this it follows that, when the number of terms is odd, the sum of the quotients is a square number whose side, or root, is the middle term.

Accordingly, if any number of terms in continued proportion in a given ratio are to be found whose sum equals the number generated from the first into the last, or from any two equally distant from the extremes, one must take, in the given ratio, as many numbers in continued proportion as are sought. For if their sum, divided by them severally,

is divided, the quotients will be the numbers sought, as is clear from this seventh theorem. But if any number of terms, odd in count, is desired whose sum is a square number whose side or root is the middle term, one must take in the given ratio the same number of terms in continued proportion, and their sum must be divided by them severally. For the quotients will be the numbers sought, as is clear from this same seventh theorem. And it is truly remarkable that one can find any number of terms in continued proportion in a given ratio — even a thousand or more — the sum of all of which equals the product of the first into the last, and the product of any two equally distant from the extremes multiplied together, and finally, if the number of terms is odd, the product of the middle term into itself.

Likewise, if one wishes for two numbers in a given ratio whose sum equals the number produced by multiplying one into the other, let him take any two numbers in the given ratio and divide their sum by each. The quotients will yield the numbers sought, as has been shown — namely, because their sum equals the number produced from the first into the last, that is, into the second. For example: let two numbers be found in sesquiquintan ratio [proportio sesquiquinta, i.e. $6 : 5$] whose sum equals the number produced by multiplying one into the other. Take any two numbers in the given sesquiquintan ratio, namely 6 and 5, and divide their sum, 11, by each. For the quotients $1\frac{5}{6}$ and $2\frac{1}{5}$ are the numbers sought. For they have the given sesquiquintan ratio, and both their sum and the number produced by multiplying them equal $4\frac{1}{30}$.

OF FRACTIONS, OR FRACTIONAL NUMBERS

DEMONSTRATIONS.

I.

Two fractions [minutiae] having the same denominator, one of which has unity for its numerator, have the same ratio as their numerators.

Let there be two fractions AB, CB having the same denominator, and let the numerator C be unity. I say that the fraction AB is to the fraction CB as the numerator A is to the numerator C. For as often as the unit C is contained in A, so often is the fraction CB included in AB; because the fraction AB is divided into as many fractions equal to CB as there are units in A, so that each of those fractions has C for its numerator and B for its denominator. [Margin: a — Def. 29]

Therefore the unit C is the same part of the number A as the fraction CB is of the fraction AB; and accordingly, as the unit C is to A, so the fraction CB is to the fraction AB. And, by conversion, as A is to C — numerator to numerator — so is the fraction AB to the fraction CB. Which was to be proved.

A, 6. C, 1.

B, 11. B, 11.

(that is, $AB = 6/11$, $CB = 1/11$)

Now, since fractions are portions, or small parts, either of unity or of some other number — for I cannot approve the opinion of those who hold that they are to be taken only with respect to unity, since they wrongly suppose that their demonstrations cannot all be carried out in whole numbers, but only in unity; whereas we, in this treatise, shall accommodate all the demonstrations to whole numbers as well — we shall endeavor to explain all the demonstrations of fractions in whole numbers, so that the diligent reader may learn even by examples that the demonstrations we bring forward are true.

Let there be given, then, the fractions $\frac{6}{11}$ and $\frac{1}{11}$, portions, for example, of the whole number 44. Since its $\frac{1}{11}$ is 4 and its $\frac{6}{11}$ is 24, it is clear that 4 is to 24 as the numerator 1 is to the numerator 6; or, put otherwise, 24 is to 4 as 6 is to 1. This proposition we shall demonstrate in general, at Proposition 4, for any two fractions of the same denomination, even if neither has unity for its numerator.

II.

The numerator of any fraction has the same ratio to the denominator as the fraction has to the whole of which it is a fraction.

Let AB be any fraction. I say that the numerator A is to the denominator B as the fraction AB is to its whole. Take the fraction CB, whose numerator C is unity and whose denominator is the same B; likewise take another fraction whose numerator D is equal to that same denominator B, so that it equals its whole. For when the numerator equals the denominator, the fraction is equivalent to its whole: since it contains as many parts of the whole as those into which the whole itself is divided, and accordingly contains the whole. [Margin: b — Prop. 1 of this treatise] Since, therefore, A is to C as the fraction AB is to the fraction CB, and C is to D as the fraction CB is to the fraction DB, it follows by equality that A is to D — that is, the numerator A is to the denominator B (since D equals B) — as the fraction AB is to the fraction DB, that is, to the whole. Which was to be proved.

A, 3. C, 1. D, 5.
B, 5. B, 5. B, 5.

(that is, $AB = 3/5$, $CB = 1/5$, $DB = 5/5$)

Otherwise: Let it first be granted that in the given fraction AB the numerator A is unity, so that the fraction is an aliquot part of the whole C — whether C be unity or a number — an aliquot part, I say, denominated by B; because as many fractions equal to AB make up the whole C as there are units in the denominator B. Since, therefore, the unit A is a part of the number B, being derived from that same B,

A, 1. C, 1.
B, 5. C, 30. [?]

(that is, $AB = 1/5$; the second array reads $1/30$, but both labels appear as C — uncertain)

Note on the reference letters in the margins: each lowercase letter (a, b, c...) keys a superscript marker in the demonstration to a cited authority. In the citations, "**huius**" = a proposition of this treatise on fractions; "**sept.**" / "**septi.**" = a proposition of Euclid's *Elements*, Book VII; "**def.**" = a definition. These are rendered below as [Margin: <letter> · <citation>].

The lettered number- and fraction-schematics are set inline as monospace panels; a $\times$ between two fractions marks the cross-multiplication of numerators into denominators.

[Continuation of the demonstration of Proposition II, from the preceding page — the numerator being unity.]

as we wrote at definition 2 of Book VII; the unit A will be the same part of the number B that the fraction AB is of the whole C. [Margin: c · 20. def.] Therefore the numerator A will be to the denominator B as the fraction AB is to the whole C — which was to be proved.

Next, let the numerator A **not** be unity. Take the fraction DB, whose numerator is D — unity — and whose denominator is the same B as that of the fraction AB. [Margin: d · 1. huius] Since, then, A is to D as numerator to numerator, so is the fraction AB to the fraction DB; and since D is to B as the fraction DB is to the whole C (as we proved just above), it follows *ex aequo* that A is to B, numerator to denominator, as the fraction AB is to the whole C. Which was to be proved.

A, $\frac{3}{5}$ D, $\frac{1}{5}$ C, 1
C, 30

And so, if the given fraction $\frac{3}{5}$ be a fraction of the whole number 30, its $\frac{1}{5}$ will be the number 6, and accordingly its $\frac{3}{5}$ the number 18 — where you see that 18 is to 30 as 3 is to 5, numerator to denominator.

III.

Every fraction is a part of its numerator, named by its denominator.

[Margin: c · 2. huius] Let AB be any fraction. I say that it is a part of the numerator A, named by the denominator B. For since A is to B, numerator to denominator, as the fraction AB is to its own whole C, then by alternation A is to the fraction AB as B is to the whole; and by inversion, the fraction AB is to A as the whole is to B (that is, the whole taken as many times as there are units in the denominator B), named by that same denominator B. (For since the unit measures the denominator, and likewise measures the whole itself — the whole being taken as many times as there are units in the denominator; and since the unit is a part of the denominator named by the denominator, as we wrote at definition 2 of Book VII — the whole too will be the same part of the whole-taken-repeatedly as there are units in the denominator.) Therefore the fraction AB likewise will be that same part of the numerator A (that is, of the whole taken as many times as there are units in the numerator A), named by the denominator B. Which was to be proved.

Let the fraction $\frac{3}{5}$ be supposed to be a fraction of this whole, 30; then its $\frac{3}{5}$ is worth 90, and its denominator is worth 150. (Just as, when the same fraction $\frac{3}{5}$

is taken with respect to unity, the numerator contains three units and the denominator five.) It is then plainly established that 18 is the fifth part of the numerator — that is, of three units, each of which stands for the whole number 30, and which three units make 90 — the part, I say, named by the denominator 5; just as 30, the given whole number, is the fifth part of the denominator — that is, of five units, each of which stands for the whole number 30, and which five units make up 150 — the part, I say, named by the denominator 5.

IV.

Given fractions that have the same denominator, neither of which has unity for its numerator, are to one another in the same ratio as their numerators.

[Margin: f · 2. huius] What was shown in the first proposition of two fractions, one of which has unity for its numerator, is here demonstrated of any fractions whatever. So let there be two fractions AB, CB having the same denominator B. I say that the fraction AB is to the fraction CB as A is to C, numerator to numerator. [Margin: f] For since A is to B as the fraction AB is to its own whole, and B is to C as the whole is to the fraction CB [Margin: g · 2. huius] (for [6] since C is to B as the fraction CB is to the whole, then by inversion B is to C as the whole is to the fraction CB), it follows *ex aequo* that A, the numerator, is to C, the numerator, as the fraction AB is to the fraction CB. Which was to be proved.

A, $\frac{6}{7}$ C, $\frac{3}{7}$

Let the given fractions $\frac{6}{7}$ and $\frac{3}{7}$ be fractions of this whole, 28; then its $\frac{6}{7}$ is 24 and its $\frac{3}{7}$ is 12 — where it is clear that 24 is to 12 as the numerator 6 is to the numerator 3.

V.

If the numerator of the first of two fractions is multiplied into the denominator of the second, and the numerator of the second into the denominator of the first, then the first fraction will be to the second as the first product is to the second.

Let there be two fractions AB, CD; and from A into D let F be made, and from C into B let G be made. I say that the fraction AB is to the fraction CD as F is to G. [Margin: h · 18. septi.] Let H be made from B into D. Since, then, A and B, multiplying D, make F and H, it follows that A is to B as F is to H. But as A is to B, [Margin: i · 2. huius] so is the fraction AB to the whole. Therefore also the fraction AB is to the whole as F is to H. Again, since C and D, multiplying B, make G and H,

[Margin: k · 18. sept.] it follows that C is to D as G is to H; and by inversion, D is to C as H is to G. But as D is to C, so is the whole to the fraction CD; [Margin: l · 2. huius] for since C is to D as the fraction CD is to the whole, then by inversion D is to C as the whole is to the fraction CD. Therefore also H is to G as the whole is to the fraction CD. Hence *ex aequo* it follows that F is to G as the fraction AB is to the fraction CD. Which was to be proved.

F, 10 G, 12
A, $\frac{2}{3} \times C$, $\frac{4}{5}$
H, 15

[p. 383 — running header "LIBER IX."; the demonstration of Proposition V continues, with the correspondence displayed:]

F, AB
H, Integrum (the whole)
G, CD

If two fractions $\frac{2}{3}$ and $\frac{4}{5}$ are fractions of the number 30, then two-thirds of it are 20, and four-fifths of the same are 24. It is clear that 20 is to 24 as 10 is to 12 — the two numbers begotten from the numerators into the denominators, in swapped order.

And so, if the product F be equal to the product G, then the fraction AB will also be equal to the fraction CD; if less, less; if greater, greater — on account of the same ratio, F to G and fraction to fraction. And this is the demonstration of the rule we gave in the *Arithmetic* for discerning which of two fractions is the greater.

VI.

The whole has the same ratio to the sum of two fractions as the number produced by the mutual multiplication of the denominators has to the sum of the two products — of which one is begotten from the numerator of the first fraction into the denominator of the second, the other from the numerator of the second into the denominator of the first.

Let there be two fractions AB, CD. From B into D let H be made; from A into D let F be made, and from C into B let G be made. I say that the whole is to the sum of the fractions AB, CD as H is to the sum of the numbers F, G. [Margin: m · 5. huius] For since the fraction AB is to the fraction CD as F is to G, then by composition the sum of the fractions AB, CD is to the fraction CD as the sum of the numbers F, G is to G. But as the fraction CD is to the whole, so is G to H (for [Margin: n · 2. huius] as the fraction CD is to the whole, so is C to D; [Margin: o · 18. septi.] and as C is to

D, so is G to H, because G and H were made by G and D multiplying B). Therefore *ex aequo* the sum of the fractions AB, CD is to the whole as the sum of the numbers F, G is to H; and by inversion, the whole is to the sum of the fractions AB, CD as H is to the sum of the numbers F, G. Which was to be proved.

F, 10 G, 12
A, $\frac{2}{3} \times C$, $\frac{4}{5}$
H, 15

AB, CD, F, G
CD, G
Integrum, H

Again, let there be two fractions $\frac{2}{3}$ and $\frac{4}{5}$, fractions of the number 30; two-thirds of it are 20, and four-fifths of the same 24, so that the sum of the fractions is 44. Here you see that the whole 30 is to this sum 44 as 15 (the product of the denominators) is to 22 [? — small glyph at the resolution floor; confirmed by the arithmetic and by the stated "sum of the numbers 10, 12"], the sum of the numbers 10 and 12, produced from the numerators into the denominators in swapped order.

VII.

If two fractions have the same numerator, the first fraction will be to the second as the denominator of the second is to the denominator of the first.

Let two fractions AB, AC have the same numerator A. I say that the fraction AB is to the fraction AC as the denominator C is to the denominator B. Let D be made from A into C, and E from A into B. Since, then, A multiplying C and B makes D and E, [Margin: $p \cdot 17$. sept.] it follows that D is to E as C is to B. But as D is to E, [Margin: $q \cdot 5$. huius] so is the fraction AB to the fraction AC. Therefore also the fraction AB is to the fraction AC as C is to B. Which was to be proved.

D, 105 E, 35
A, $\frac{5}{7} \times A$, $\frac{5}{21}$
B, 7 C, 21

Let the two fractions $\frac{5}{7}$ and $\frac{5}{21}$ be fractions of the whole number 42; then its $\frac{5}{7}$ are 30, and its $\frac{5}{21}$ are 10. And it is manifest that 30 is to 10 as the latter denominator 21 is to the former, 7.

VIII.

Fractions whose numerators bear the same ratio to their denominators are equal; and the numerators of equal fractions bear the same ratio to

their denominators. But the fraction whose numerator bears the greater ratio to its denominator is the greater; and the greater fraction is the one whose numerator bears the greater ratio to its denominator.

Let there be two fractions AB, CD, and let the ratio of A to B (numerator to denominator) be the same as that of C to D. I say that these fractions are equal. [Margin: $r \cdot 2$. huius] For since A is to B as the fraction AB is to the whole, and C is to D as the fraction CD is to the same whole, and A is to B (by hypothesis) as C is to D, then also the fraction AB is to the whole as the fraction CD is to the same whole. Therefore the fractions AB, CD are equal. Which was to be proved.

A, $\frac{3}{4}$ C, $\frac{9}{12}$

But now let the fractions AB, CD be equal. I say that A is to B as C is to D. For since the fractions are equal, the fraction AB will be to the whole as the fraction CD is to the same whole. [Margin: $s \cdot 2$. huius] But the fraction AB is to the whole as A is to B, and the fraction CD is to the whole as C is to D. Therefore also A is to B as C is to D. Which was to be proved.

[p. 384 — running header "EVCLID. GEOM."; Proposition VIII continues:]

Next, let the ratio of A to B be greater than that of C to D. I say that the fraction AB is greater than CD. [Margin: $a \cdot 2$. huius] For since A is to B as the fraction AB is to the whole, and C is to D as the fraction CD is to the same whole, and the ratio of A to B is supposed greater than that of C to D, then the ratio of the fraction AC [recte: AB] to the whole will also be greater than that of the fraction CD to the same whole. Therefore the fraction AB is greater than the fraction CD. Which was to be proved.

A, $\frac{3}{4}$ C, $\frac{2}{3}$

But now let the fraction AB be greater than the fraction CD. I say that the ratio of A to B is greater than that of C to D. [Margin: $b \cdot 2$. huius] For the ratio of the fraction AB (the greater) to the whole will be greater than the ratio of the fraction CD (the lesser) to the same whole. Since, then, the fraction AB is to the whole as A is to B, and the fraction CD is to the whole as C is to D, the ratio of A to B will also be greater than that of C to D. Which was to be proved.

Let 24 be the whole number of the fractions $\frac{3}{4}$ and $\frac{12}{16}$ [? — the second, small stacked glyph sits at the resolution floor; it is a fraction equal to $\frac{3}{4}$ (both give 18 of 24) and could also read $\frac{9}{12}$]; both $\frac{3}{4}$ and $\frac{12}{16}$ of it are 18, and so the fractions $\frac{3}{4}$ and $\frac{12}{16}$ are equal. But $\frac{2}{3}$ of the same number 24 are 16 — where

it is manifest that the number 18 is greater than the number 16, and therefore that the fraction $\frac{3}{4}$ is greater than the fraction $\frac{2}{3}$.

IX.

Fractions whose numerators bear the same ratio [to each other] as their denominators are equal; and the numerators of equal fractions bear the same ratio as their denominators. But that fraction whose numerator bears the greater ratio is the greater; and the numerator of the greater fraction bears a greater ratio than does its denominator.

Let there be two fractions AB, CD, and let the ratio of A to C be the same as that of B to D. I say that the fractions themselves are equal. For by alternation A will be to B as C is to D; [Margin: $c \cdot 8$. huius] and accordingly the fractions will be equal. Which was to be proved.

A, $\frac{6}{9}$ C, $\frac{2}{3}$

But now let the fractions AB, CD be equal. I say that A is to C as B is to D. [Margin: $d \cdot 8$. huius] For A will be to B as C is to D. Therefore by alternation A is to C as B is to D. Which was to be proved.

Next, let the ratio of A to C be greater than that of B to D. I say that the fraction AB is greater than the fraction CD. For by alternation the ratio of A to B will also be greater than that of C to D, [Margin: $e \cdot 8$. huius (printed "c")] and on that account the fraction AB will be greater than CD. Which was to be proved.

A, $\frac{6}{9}$ C, $\frac{3}{5}$

But now let the fraction AB be greater than CD. I say that the ratio of A to C is greater than that of B to D. [Margin: $f \cdot 8$. huius] For the ratio of A to B will be greater than that of C to D, and therefore by alternation the ratio of A to C will be greater than that of B to D. Which was to be proved.

Let the two fractions $\frac{6}{9}$ [? — the numerator glyph reads 5 or 6 at the floor; the adjacent printed "60" pins it at $\frac{6}{9}$] and $\frac{2}{3}$ be fractions of the whole number 90. Here you see that the fraction $\frac{6}{9}$, that is 60, is equal to the fraction $\frac{2}{3}$, that is to the number 60; but that same fraction $\frac{6}{9}$, that is 60, is greater than the fraction $\frac{3}{5}$, that is the number 54.

X.

To reduce two fractions of different denominations to two other fractions of the same denomination equal to them.

Let two fractions AB, CD have unlike denominators B, D. Let E be made from A into D, F from C into B, and G from B into D. I say that the fractions EG, FG — whose numerators are E, F and whose common denominator is G — are equal to the fractions AB, CD. [Margin: g · 18. sept.] For since A and B, multiplying D, make E and G, it follows that A is to B as E is to G. [Margin: h · 8. huius] Hence the fractions AB, EG will be equal. In the same way, since C and D, multiplying B, make F and G, [Margin: i · 18. septi.] it follows that C is to D as F is to G. [Margin: k · 9. huius] Therefore the fractions CD, FG will be equal. Which was to be proved.

E, 10 F, 9
A, $\frac{2}{3} \times C$, $\frac{3}{5}$
G, 15

XI.

To reduce any whole number to a fraction of a given denominator.

Let first A be the whole number 1, to be reduced to a fraction whose denominator is C. Take a numerator B equal to the denominator C. I say that the fraction BC is equal to the unit A. [Margin: l · 2. huius] For since B is to C as the fraction BC is to the whole A, and B is equal to C, then the fraction BC will also be equal to the whole A — that is, to unity. Which was to be proved.

A, 1 B, $\frac{6}{6}$

Next, let A be a whole number to be brought back to a fraction whose denominator is C. From A into C let B, the numerator, be made. I say that the fraction BC is equal to the whole A. For let the unit D be supposed under the whole A, so that the fraction AD is made equal to as many units as there are units in A. Since, then, B is made from A into C, it follows from the definition of multiplication that B is to C as A is to D, the unit. [Margin: m · 8. huius] Since, then, the fractions AD, BC have their numerators A, B in the same ratio to their denominators D, C, they will be equal to one another. And since the fraction AD is equal to the whole A (on account of its denominator, which is unity), the fraction BC too will be equal to the same whole A. Which was to be proved.

A, 7 B [recte for printed "C"], 42
D, 1 C, 6

XII.

To double and to halve a given fraction.

Let the fraction A/B first be the one to be doubled. Either double the numerator A to make a numerator C, keeping the same denominator B; or, when it is even,

halve the denominator B to make a denominator D, keeping the same numerator A. I say that each of the two fractions, C/B and A/D , is double the fraction A/B . For since AB and CB have the same denominator, [Margin: a — 4. huius] C will be to A, numerator to numerator, as CB is to AB, fraction to fraction. But C is double A, by construction. Therefore the fraction CB is also double the fraction AB. Which was to be proved.

Again, since the fractions AB and AD have the same numerator, [Margin: b — 7. huius] B will be to D, denominator to denominator, as AD is to AB, fraction to fraction. But B is double D, by construction. Therefore the fraction AD too is double the fraction AB. Which was to be proved.

$A/B = 3/8 \rightarrow C/B = 6/8$ (double the numerator) or $A/D = 3/4$ (halve the denominator)

Next, let the fraction A/B be the one to be halved. Either double the denominator B to make a denominator C, keeping the same numerator A; or, when it is even, halve the numerator A to make a numerator D, keeping the same denominator B. I say that each of the two fractions, A/C and D/B , is half the fraction A/B . For since AB and AC have the same numerator, [Margin: c — 7. huius] B will be to C, denominator to denominator, as AC is to AB, fraction to fraction. Since, then, B is by construction half of C, the fraction AC will also be half the fraction AB. Which was to be proved.

Again, since the fractions AB and DB have the same denominator, [Margin: d — 4. huius] D will be to A, numerator to numerator, as DB is to AB, fraction to fraction. Since, then, D is half of A, by construction, the fraction DB too will be half the fraction AB. Which was to be proved.

$A/B = 6/11 \rightarrow A/C = 6/22$ (double the denominator) or $D/B = 3/11$ (halve the numerator)

In the same way a given fraction will be tripled, quadrupled, and so on, either if the numerator is tripled, quadrupled, and so on, the same denominator remaining; or (when it can be done) if a third, a fourth, and so on, of the denominator is taken, the same numerator remaining.

And likewise a third part, a fourth part, and so on, of a given fraction will be taken, either if the denominator is tripled, quadrupled, and so on, the same numerator remaining; or (when it can be done) if a third part, a fourth part, and so on, of the numerator is taken. Which is proved in the same way.

XIII.

A fraction of a fraction is equal to a simple fraction whose numerator is produced by multiplying the two numerators together, and whose denominator by multiplying the two denominators together.

For let AB be a fraction of the fraction CD with respect to the whole number E, that is, of the unit E, and let a simple fraction K express the value of this fraction. Let F be made from A into C, and G from B into D. I say that the given fraction-of-a-fraction is equal to the simple fraction FG — with respect to the whole E — whose numerator is F and denominator G. For let H in turn be made from C into B. [Margin: e — 2. huius] And since, as A is to B, so is the fraction AB to its own whole, namely to the fraction CD, whose whole is E; [Margin: f — 18. septi] and likewise, as A is to B, so is F to H (because A and B, multiplying C itself, make F and H): therefore, as the fraction AB is to the fraction CD, so also will F be to H. Again, [Margin: g — 2. huius] since, as C is to D, so is the fraction CD to its own whole E; [Margin: h — 17. septi] and likewise, as C is to D, so is H to G (because B, multiplying C and D themselves, has made H and G): therefore, as the fraction CD is to its own whole E, so also will H be to G. Since, then, as the fraction AB is to the fraction CD, so is F to H; and as the fraction CD is to the whole E, so is H to G; it will hold, *ex aequo*, that as AB — a fraction of the fraction CD — is to the whole E, so is F to G. But as AB, a fraction of the fraction CD, is to the whole E, so also is the simple fraction K to that same whole E, since K was set equal to the given fraction AB of the fraction CD (for it expresses its value with respect to the whole E). Therefore it will also hold that, as F is to G, so is the simple fraction K to the whole E. [Margin: i — 2. huius] Since, then, as F is to G, so also is the fraction FG to that same whole E of its own, the simple fractions K and FG will be equal. And accordingly, since K is equal to the given fraction-of-a-fraction, the simple fraction FG too will be equal to the given fraction AB of the fraction CD, whose whole is E. Which was to be proved.

$A/B = 2/3$ (= AB); $C/D = 3/4$ (= CD); $F = A \times C = 6$; $G = B \times D = 12$; $H = C \times B = 9$
simple $K = F/G = 6/12 = 1/2$; whole $E = 24$ (unit $E = 1$) segments listed: A B ·
C D · E || F · H · G

Let $\frac{2}{3}$ be a fraction of the fraction $\frac{3}{4}$ with respect to the integer 24. Three-fourths of this number is 18, and two-thirds of that is 12, which is $\frac{1}{2}$ of the given whole number 24. And it is manifest that if the numerators 2 and 3 are multiplied together, and likewise the denominators 3 and 4, the fraction $\frac{6}{12}$ is produced, which is equivalent to $\frac{1}{2}$ — just such a part, indeed, as the number 12 was with respect to the whole number 24.

XIII.

To reduce a fraction of a fraction, or a fraction of fractions, to a simple fraction.

Let AB first be a fraction of the fraction CD. From A into C let E be made, and from B into D let F be made. [Margin: k — 13. huius] And since the fraction EF expresses

the value of the fraction AB with respect to the fraction CD, the given fraction-of-a-fraction will have been reduced to the simple fraction EF. Which was to be proved.

$A/B = 2/3 (= AB)$; $C/D = 3/4 (= CD)$; $E = A \times C = 6$; $F = B \times D = 12$; simple EF = $6/12$

Next, let AB be a fraction of the fractions CD and EF, such that CD is a fraction of the fraction EF, and AB a fraction of the fraction CD. Let G be made from A into C, and H from B into D; then let I be made from G into E, and K from H into F; and so on in succession, the last-produced numbers I and K being multiplied into the numbers of the next fraction. I say that the fraction IK, whose numerator is I and denominator K, is equal to the given fraction-of-fractions. [Margin: a — 13. huius] For the fraction GH is equal to AB — a fraction of the fraction CD — taken with respect to the fraction EF as a kind of whole. [Margin: b — 13. huius] Likewise the fraction IK is equal to the fraction GH (that is, to AB, a fraction of the fraction CD), a fraction of EF, with respect to the whole of which EF is itself a fraction; and so on in succession, if there are more fractions. Therefore the fraction IK, constituted from the numbers produced in the last place, is equal to the given fraction-of-fractions. Which was to be proved.

$A/B = 2/3$; $C/D = 3/4$; $E/F = 1/2$; $G = A \times C = 6$; $H = B \times D = 12$ $I = G \times E = 6$; $K = H \times F = 24$; $IK = 6/24$

Let $\frac{2}{3}$ be a fraction of the fractions $\frac{3}{4}$ and $\frac{1}{2}$ with respect to the integer 24. Half of this number is 12; three-fourths of that is 9; and finally two-thirds of that is 6, which is $\frac{1}{4}$ of the whole integer 24. And it clearly stands that, if the numerators 2, 3, 1 are multiplied together, and likewise the denominators 3, 4, 2, the fraction $\frac{6}{24}$ is produced, which is equal to $\frac{1}{4}$ — just such a part, indeed, as the number 6 was with respect to the whole 24.

XV.

Given two fractions, to reduce one of them to another equal to it, in such a way that the numbers of the one measure the numbers of this newly

found fraction — namely, by the numbers produced from the numerator of the given fraction being multiplied into the denominator of the other given fraction, and from the numerator of the other given fraction into the denominator of the [first] reduced fraction.

Let there be two fractions AB and CD, of which AB is to be reduced to another equal to it, such that its numerator is measured by the numerator C through the number produced from A into D, and its denominator is measured by the denominator D through the number produced from C into B. Let F be made from C into D; and from F into A and B let G and H be made. Likewise, from A into D let K be made, and L from C into B. I say that the fraction GH is equal to the fraction AB; and that C measures G through K (produced from A into D), and D measures H through L (produced from C into B). For since F, multiplying AB, makes G and H, [Margin: c — 17. sept] as A is to B, so will G be to H; [Margin: d — 7. huius] and accordingly the fractions AB and GH will be equal. Next, since C, multiplying D, makes F, [Margin: e — 9. pro. [?]] C measures F itself through D; [Margin: f — 5. pro. [?]] and the unit likewise measures D itself through D. The same part, therefore, is C of F as the unit is of D. Since, then, the unit is a part of D denominated by D, C too will be a part of F denominated by D. By the same reasoning F will be a part of G denominated by A, because G is made from F into A. Since, then, C is a fraction of the fraction F with respect to the whole G, and the numerators are units, and the denominators are the numbers D and A by which the parts C and F are denominated, [Margin: g — 13. huius] C — a fraction of the fraction F with respect to the whole G — will constitute a simple fraction M (the fractions C, F, M being those we wrote out separately) with respect to that same whole G, whose numerator, that is, is produced from the numerators of the fractions C and F, and whose denominator from their denominators. Therefore, since the numerators produce the unit and the denominators the number K (for the numerators are units, while the denominators are the numbers D and A, by which the parts C and F are denominated, and from whose multiplication the number K was made), the fraction C of the fraction F, with respect to the whole G, will constitute a simple fraction of that same whole G whose numerator is the unit and denominator the number K, equal to the fraction M. And accordingly C, the numerator of the given fraction CD, which constitutes C — a fraction of the whole G denominated by the number K. Wherefore the number C will measure G itself through K, taken as many times, that is, as the unit is contained in the number K. By exactly the same reasoning we shall show that D measures H itself through the number L. Which was to be proved.

$A/B = 2/3$; $C/D = 3/4$; $F = C \times D = 12$; $G = F \times A = 24$; $H = F \times B = 36$ $K = A \times D = 8$; $L = C \times B = 9$; $GH = 24/36 = 2/3 (= AB)$ auxiliary simple fractions: $C = 1/4$; $F = 1/2$; $M = 1/8$

XVI.

Two fractions whose numerators, multiplied crosswise into the denominators, produce equal numbers, are equal; and the numerators of equal fractions, multiplied crosswise into the denominators, produce equal numbers. But that fraction whose numerator, multiplied into the denominator of the other, produces the greater number, is the greater; and the numerator of the greater fraction, multiplied into the denominator of the other, produces the greater number.

Let there be two fractions AB and CD; and from A into D let E be made, and from C into B let F be made; and let the numbers E and F first be equal. I say that the fractions AB and CD are equal. For since the number E, made from A (the first) into D (the fourth), is equal to the number F, made from C (the second) into B (the third), [Margin: h — 19. septi] A, the first, will be to C, the second — numerator to numerator — as B, the third, is to D, the fourth — denominator to denominator. [Margin: i — 9. huius] Therefore the fractions AB and CD are equal. Or thus: [Margin: k — 5. huius] since, as E is to F, so is the fraction AB to the fraction CD, and E is equal to F, the fraction AB will also be equal to the fraction CD. Which was to be proved.

$$A/B = 2/3 \rightarrow$$

$$E/D = 8/12; E = A \times D = 24; F = C \times B = 12$$

But now let the fractions AB and CD be equal. I say that the number E is also equal to the number F. For, on account of the equality of the fractions, [Margin: a — 9. huius] as A is to C, numerator to numerator, so will B be to D, denominator to denominator. [Margin: b — 19. septi] Wherefore the same number will be made from A (the first) into D (the fourth) as from C (the second) into B (the third); and accordingly the numbers E and F will be equal. Or thus: [Margin: c — 5. huius] since, as the fraction AB is to the fraction CD, so is E to F, and the fractions are set equal, the numbers E and F will also be equal. Which was to be proved.

Next, let E be a number greater than the number F. I say that the fraction AB too is greater than the fraction CD. For since, from A into D, the number E is made greater than F, F will be generated from a number smaller than A — into D — namely from G into D. Since, then, the same number F is made both from G (the first) into D (the fourth) and from C (the second) into B (the third), [Margin: d — 19. septi] G, the first, will be to C, the second, as B, the third, is to D, the fourth. But

the ratio of A to C is greater than that of G to C (since A is greater than G).

Therefore [Margin: e — 9. huius] the ratio of A to C will also be greater than that of B to D; and accordingly the fraction AB will be greater than the fraction CD. Or thus: [Margin: f — 5. huius] since, as E is to F, so is the fraction AB to the fraction CD, and E is greater than F, the fraction AB will also be greater than the fraction CD. Which was to be proved.

$A/B = 2/3 \times C/D = 7/12$; $E = A \times D = 24$; $F = C \times B = 21 \rightarrow E > F$, so $AB > CD$ $G = 1\frac{3}{4}$ [?] (the number such that $F = G \times D$)

Finally, let the fraction AB now be greater than the fraction CD. I say that the number E is also greater than the number F. For since the fraction AB is greater than the fraction CD, [Margin: g — 9. huius] the ratio of A to C will be greater than that of B to D. A number smaller than A, therefore, will have to C the same ratio that B has to D; let this be G. [Margin: h — 19. septi] The same number, then, will be made from G (the first) into D (the fourth) as from C (the second) into B (the third). Since, then, from A into D a greater number is made than from G into D — because A is greater than G — E, made from A into D, will also be greater than F, generated from C into B. Or thus: [Margin: i — 5. huius] since, as the fraction AB is to the fraction CD, so is E to F, and the fraction AB is greater than the fraction CD, E will also be greater than F. Which was to be proved.

And this is also the demonstration of that rule which we laid down in the Arithmetic for determining which of two given fractions is the greater. This we noted at the 5th proposition as well.

XVII.

To reduce a given fraction to its lowest terms.

Let AB be the fraction to be brought back to its lowest terms. If, then, A and B are prime to one another, the fraction AB cannot be brought back to smaller terms, but is already constituted in its lowest terms. For let it be reduced, if it can be, to smaller terms C, D, such that the fraction CD is equal to the fraction AB. [Margin: k — 8. huius] Since, then, as A is to B, so is C to D, and C, D are numbers smaller than the numbers A, B, then A and B will not be the least terms in their ratio; [Margin: l — 23. septi] yet they are the least, since they are prime to one another — which is absurd.

$A/B = 11/12$; $C/D = -$ (no smaller equal terms exist, A and B being coprime)

But if A, B are not prime to one another, [Margin: m — 2. septi] let C be their greatest common measure, which measures A through D and B through E. I say

that the fraction DE, whose numerator is D and denominator E, is equal to the given fraction AB, and is constituted in its lowest terms. For since C measures A and B through D and E, [Margin: n — 9. pro. [?]] A and B will be produced from C into D and E. [Margin: o — 17. sept] Wherefore, as A is to B, so will D be to E; [Margin: p — 8. huius] and on that account the fractions AB and DE will be equal. Next, since C, the greatest measure of the numbers A and B, measures them through D and E, then — by the corollary of proposition 35 of Book 7 — D and E are the least in the ratio of A to B. Which was to be proved.

$A/B = 36/48$; $C = 12$ (greatest common measure); $D = 3$; $E = 4$; $DE = 3/4$

XVIII.

To reduce a given fraction to another equal to it of a given denomination, when this can be done.

Let AB be a given fraction, to be brought back to another equal to it whose denominator is a given C. From A, the numerator, into the given denominator C, let D be made, which B — the denominator of the given fraction — measures through E. I say that the fraction EC, whose numerator is E and denominator the given C, is equal to the given fraction AB. For since B measures D through E, [Margin: q — 9. pro. [?]] D will be made from B into E; but the same D was made from A into C. [Margin: r — 19. septi] Therefore, as A, the first, is to B, the second, so will E, the third, be to C, the fourth; [Margin: s — 8. huius] and hence the fractions AB and EC will be equal. Which was to be proved.

$A/B = 3/4$; given denominator $C = 12$; $D = A \times C = 36$; $E = D/B = 9$; $EC = 9/12$

But if B does not measure D — produced from A into C — the given fraction AB cannot be reduced to the given denomination C. For let it be reduced, if it can be, to the fraction EC. Since, then, the fractions AB and EC are equal, [Margin: t — 8. huius] as A is to B, so will E be to C. [Margin: u — 19. septi] The same number, therefore, will be made from A (the first) into C (the fourth) as from B (the second) into E (the third) — namely the number D, made from A into C. [Margin: x — 7. pro. [?]] Therefore B will measure D through E. Which is absurd; for B was assumed not to measure D itself.

$A/B = 3/4$; given denominator $C = 10$; $D = A \times C = 30$; $E = -$ ($B = 4$ does not measure 30, so EC cannot be formed)

XIX.

When a smaller number is divided by a larger, the quotient is a fraction whose numerator is the smaller number divided, and whose denominator is the larger dividing number.

This opening completes the demonstration of Proposition XIX of the fractions treatise (whose enunciation stood at the end of the previous stretch), then runs through the propositions on adding, subtracting, multiplying, dividing, and grafting fractions. The running head remains **LIBER IX** — the *Elements* Book X has not yet begun.

Demonstration of Proposition XIX (a smaller number divided by a larger yields a fraction):

Let the smaller number A be divided by the larger number B. I say that the quotient is the fraction A/B, whose numerator is A and whose denominator is B. [Margin: a. Prop. VIII of this treatise (8. huius)] For A is to B as the divided number is to the divisor — that is, as the numerator is to the denominator — so the fraction A/B is to the whole, that is, to unity. Therefore, by the definition of division, the fraction A/B is the quotient of the division of the number A by the number B, which was the point to be proved.

$$AB = 6/11$$

It follows from this that, when a larger whole number is divided by a smaller whole number and something is left over, one must still add to the whole-number quotient already obtained a fraction whose numerator is the number left over from the division and whose denominator is the dividing number. For that remainder, which is necessarily smaller than the divisor, must still be divided by that same larger divisor. For example: dividing the number 23 by 4, the whole-number quotient is 5, and 3 remains, still to be divided by 4. [Margin: b. Prop. XIX of this treatise (19. huius)] There results, then, the fraction 3/4, so that the whole quotient is $5 \frac{3}{4}$.

XX.

To collect several fractions into a single sum.

Let there first be two fractions AB and CB to be added, of the same denomination. From the numerators A and C form the sum D, and give it the same denominator B. I say that the fraction DB is the sum collected from the addition of the fractions AB and CB. For since the fractions have the same denominator, [Margin: c. Prop.

IV (4. huius)] A is to C as the fraction AB is to the fraction CB; and by composition, A and C together are to C as the fractions AB and CB together are to the fraction CB. [Margin: d. Prop. IV (4. huius)] And as C is to D, so too is the fraction CB to the fraction DB. Therefore, *ex aequo*, A and C together are to D as the fractions AB and CB together are to the fraction DB. Since, then, A and C together are equal to D itself by construction, the fractions AB and CB together will likewise be equal to the fraction DB — which was to be proved.

$$AB = 3/10 \quad CB = 4/10 \quad DB = 7/10$$

Next, let there be two fractions AB and CD to be added, of different denominations. Multiply A, the numerator of the first, into D, the denominator of the second; and C, the numerator of the second, into B, the denominator of the first; and let the sum of these two products be E, to which is assigned the denominator F, produced by the mutual multiplication of the denominators B and D. I say that the fraction EF is the sum collected from the two given fractions AB and CD. [Margin: e. Prop. VI (6. huius)] For the whole is to the sum of the two fractions AB and CD as F, produced from the denominators, is to E, the sum of the products of the numerators multiplied crosswise into the denominators; therefore, by conversion, the sum of the fractions AB and CD is to the whole as E is to F. [Margin: f. Prop. II (2. huius)] And as E is to F, so too is the fraction EF to that same whole. Therefore the sum of the fractions AB and CD is to the whole as the fraction EF is to that same whole. Hence the sum of the fractions AB and CD is equal to the fraction EF, and so the fraction EF is the sum collected from the fractions AB and CD — which was to be proved.

$$AB = 2/3 \times CD = 3/4 \rightarrow EF = 17/12$$

Or thus. From the multiplication of the numerators into the denominators crosswise, and from the multiplication of the denominators by one another, there arise two fractions of the same denomination equal to the fractions AB and CD, as is established by the demonstration of Proposition X. So if to the sum E of the products of the numerators into the denominators one assigns F, the common denominator produced from the denominators, then the fraction EF becomes the sum of the fractions AB and CD — that is, of the two fractions of the same denomination to which AB and CD have been reduced, as was demonstrated at the beginning of this proposition.

Now, when several fractions are given, the first two are to be added; then this sum is to be joined with the third, and this sum with the fourth, and so on in turn, until all are dealt with.

But if whole numbers are given together with fractions, the whole numbers are to be added separately and the fractions separately.

For example, let $\frac{2}{3}$ and $\frac{3}{4}$ be fractions of this whole number 60. Its $\frac{2}{3}$ is 40, and its $\frac{3}{4}$ is 45. From 40 and 45 the sum 85 arises — that is, the whole integer 60, and 25 units besides, which make $\frac{5}{12}$ of the integer 60 — just as from $\frac{2}{3}$ and $\frac{3}{4}$ the sum $\frac{17}{12}$ is collected, that is, one whole 1 and $\frac{5}{12}$ besides.

Likewise, the sum of the fractions $\frac{1}{3}$ and $\frac{4}{9}$ is $\frac{21}{27}$, that is, in lowest terms $\frac{7}{9}$. So if the whole of those fractions is 36, then $\frac{1}{3}$ of it is 12 and $\frac{4}{9}$ of it is 16; and here you see that 12 and 16 make 28, that is, $\frac{7}{9}$ of the whole 36.

XXI.

To subtract a smaller fraction from a larger.

From the larger fraction AB let the smaller CB be subtracted, and let these fractions first be of the same denomination. When the numerator C of the smaller fraction is subtracted from A, the numerator of the larger, let the remainder be the number D, to which the same denominator B is assigned. I say that the fraction DB is the remainder after the subtraction of the fraction CB from the fraction AB. For since C subtracted from A leaves D, A is composed of C and D. Therefore the fraction AB, whose numerator A is collected from the numerators C and D, is the sum of the two fractions CB and DB, as was shown in the first part of the preceding proposition. Hence, when the fraction CB is subtracted from the fraction AB, the remainder becomes the fraction DB — which was to be proved.

$$AB = 7/10 \quad CB = 3/10 \quad DB = 4/10$$

Next, let the fractions AB and CD have different denominators, and let the smaller CD be subtracted from the larger AB. From C, the numerator of the smaller, into B, the denominator of the larger, form E; and from A, the numerator of the larger, into D, the denominator of the smaller, form F; and let E, taken away from F, leave G, to which is assigned the denominator produced by the mutual multiplication of the denominators. I say that the fraction GH is the remainder after the subtraction of the fraction CD from the fraction AB.

$$AB = 3/4 \times CD = 2/5 \rightarrow E = 8, \quad F = 15, \quad GH = 7/20$$

[Margin: g. Elements VII.17 (17. sept.)] For since D, multiplying A and B, makes F and H, [Margin: h. Prop. II (2. huius)] it follows that A is to B as F is to H. And as A is to B, so is the fraction AB to the whole; therefore also F is to H as the fraction AB is to the whole. Again, since B, multiplying C and D, makes E and H, [Margin: i. Elements VII.17 (17. septi.)] C is to D as E is to H; [Margin: k. Prop. II (2. hui.)] and

as C is to D, so is the fraction CD to the whole; therefore also E is to H as the fraction CD is to the whole. Since, then, E, the first, is to H, the second, as the fraction CD, the third number, is to the whole, the fourth number, [Margin: a. Prop. II (2. huius)] and G, the fifth, is to that same second H as the fraction GH, the sixth number, is to that same whole, the fourth — then, by the fifth theorem of the scholium to Proposition 22 of Book 7 (*Elements* VII.22, schol.), E and G together, the first and fifth, are to H, the second, as the fractions CD and GH together, the third and sixth numbers, are to the whole. And as E and G together are to H, so is F — equal to E and G themselves (for E subtracted from F left G) — to that same H. Therefore also F is to H as the fractions CD and GH together are to the whole. Since, then, it has been shown that as F is to H, so is the fraction AB to the whole, it follows that the fraction AB is to the whole as the fractions CD and GH together are to that same whole; and therefore the fraction AB is equal to the fractions CD and GH together. Hence, when the fraction CD is subtracted from the fraction AB, the remainder is the fraction GH — which was to be proved.

Or thus. Since from the multiplication of the numerators into the denominators crosswise, and from the multiplication of the denominators by one another, there arise two fractions of the same denomination equal to the given fractions — as is established from the demonstration of Proposition 10 — if E, produced from C into B, is subtracted from F, produced from A into D, and to the remaining number G one assigns the common denominator H produced from the denominators, then the fraction GH is produced, the remainder after the subtraction of the fraction CD from the fraction AB, just as was shown at the beginning of this proposition. [Margin: b. Prop. XI (11. huius)]

Now, if whole numbers are given together with fractions, the whole numbers are to be subtracted separately from the whole numbers, and the fraction from the fraction. But if the fraction to be subtracted is larger than the one from which the subtraction ought to be made, one must first reduce one unit of the whole number from which the subtraction is made into a fraction of the denominator of the adhering fraction. For example, if $6\frac{2}{5}$ is to be subtracted from $10\frac{3}{4}$: subtracting 6 from 10 leaves 4, and subtracting $\frac{2}{5}$ from $\frac{3}{4}$ leaves the fraction $\frac{7}{20}$; so the whole remaining number is $4\frac{7}{20}$. But if $4\frac{6}{7}$ is to be subtracted from $20\frac{3}{4}$, we shall make from one unit of the number 20 the fraction $\frac{4}{4}$, which together with $\frac{3}{4}$ makes $\frac{7}{4}$, and 19 whole units remain. Subtracting, then, the whole number 4 from 19 leaves 15; and subtracting the fraction $\frac{6}{7}$ from the fraction $\frac{7}{4}$ leaves the fraction $\frac{25}{28}$. So the whole remaining number is $15\frac{25}{28}$.

For example, let $\frac{3}{4}$ and $\frac{2}{5}$ be fractions of this whole number 40. So if $\frac{2}{5}$ — that is, 16 — is subtracted from $\frac{3}{4}$ — that is, from 30 — then 14 remains, namely $\frac{7}{20}$ of the whole number 40, as the calculation above has shown.

XXII.

To multiply a fraction by a fraction.

Let the fraction AB be multiplied by the fraction CD. From the multiplication of the numerators A and C by one another form E, to which is assigned F, produced from the denominators B and D. I say that the fraction EF is generated from the multiplication of the fractions AB and CD by one another. For let G be formed from E into D, and H from C into F; [Margin: c. Prop. V (5. huius)] then the fraction EF is to the fraction CD as G is to H. And since C, multiplying A and F, makes E and H, [Margin: d. Elements VII.17 (17. septi.)] A is to F as E is to H; and by alternation, A is to E as F is to H. Again, since D, multiplying E and B, makes G and F, then by the perturbed proportion, as appears here, A is to B as G is to H. And as G is to H, so we have shown the fraction EF is to the fraction CD; [Margin: e. Elements VII.17 (17. septi.)] and as A is to B, so is the fraction AB to the whole, that is, to 1. Therefore also the fraction EF is to the fraction CD as the fraction AB is to 1. Hence, by the definition of multiplication, the fraction EF is produced from the multiplication of the fraction AB into the fraction CD [Margin: f. Prop. II (2. huius)] — which was to be proved.

$$H = 60 \quad G = 40 \quad AB = \frac{2}{3} \quad CD = \frac{4}{5} \times EF = \frac{8}{15}$$

Proportion terms: $A \cdot G / E \cdot F / B \cdot H$

But if a fraction is to be multiplied by a whole number, one must assign a unit to the whole number, so that it becomes as it were a fraction whose numerator is the given whole number and whose denominator is unity. From this it follows that it is enough if the numerator of the proposed fraction is then multiplied by the given whole number and the denominator of that same fraction is assigned to the resulting number; because the unit assigned to the whole number, multiplied into the denominator of the given fraction, does not increase that denominator. Thus, if the fraction $\frac{3}{4}$ is to be multiplied by 5, the result is $\frac{15}{4}$, as appears here: $\frac{3}{4} \cdot \frac{5}{1}$. For the fraction $\frac{15}{4}$ is generated.

But if fractions adhere to whole numbers, [Margin: g. Prop. XI (11. huius)] the whole numbers must be brought back to a fraction of the same denomination as the adhering fraction, and from the numerator of this fraction and the numerator

of the adhering fraction a single numerator is formed. Thus, if $7 \frac{2}{3}$ is to be multiplied by $4 \frac{1}{2}$, these become the two fractions to be multiplied: $\frac{23}{3} \cdot \frac{9}{2}$.

Now, to adapt the multiplication of fractions to whole numbers, the fraction produced is not to be compared with the whole number assumed — as was done in addition and subtraction — but with the square of the whole number assumed. For since the comparison must be made between like things, and since from the multiplication of two numbers by one another a plane number is generated, the number produced must be compared with the plane of the whole number, that is, with its square. So let $\frac{2}{3}$ and $\frac{4}{5}$ be fractions of this whole number 60; from their multiplication the fraction $\frac{8}{15}$ is generated. It is necessary, then, that $\frac{2}{3}$ of the number 60, multiplied into $\frac{4}{5}$ of the same, make a number that constitutes $\frac{8}{15}$ of the square produced from 60 into itself. And this is plainly established: for $\frac{2}{3}$ of the number 60 is 40, and $\frac{4}{5}$ of the same is 48; and from 40 into 48 arises the number 1920, which makes $\frac{8}{15}$ of the number 3600, that is, of the square of 60 itself.

XXIII.

To divide a fraction by a fraction.

Let the fraction AB be divided by the fraction CD, and first let the numbers C and D of the latter measure the numbers A and B of the former, by E and F, so that when A is divided by C the quotient is E, and when B is divided by D the quotient is F. I say that the fraction EF is the quotient of the division of the fraction AB by the fraction CD. For since C measures A by E, and D measures B by F, [Margin: h. 9. pron. [?] (degraded abbreviation)] A is generated from C into E, and B from D into F. [Margin: i. Prop. XXII (22. huius)] Therefore the fraction AB is produced from the multiplication of the fraction EF by the fraction CD; and hence, by the definition of multiplication, the fraction AB is to the fraction CD as the fraction EF is to 1. Wherefore, since AB, the divided fraction, is to CD, the dividing fraction, as the fraction EF is to 1, then, by the definition of division, the fraction EF is the quotient of the division of the fraction AB by the fraction CD — which was to be proved.

$$\begin{aligned} AB = 12/16 \div CD = 3/8 \rightarrow EF = 4/2 \quad AB = 6/8 \div CD = 3/8 \rightarrow EF = 2/1 \quad AB = 3/8 \div CD \\ = 3/4 \rightarrow EF = 1/2 \end{aligned}$$

Next, let the numbers C and D of the fraction CD not measure the numbers A and B of the fraction AB. [Margin: k. Prop. XV (15. huius)] Let the fraction AB be reduced to another equal fraction EF, whose numbers E and F the numbers C and D of the fraction CD measure by the numbers G and H, produced from A into D

and from C into B. This will happen if K, produced from C into D, is multiplied into A and B, so that E and F are generated, as was shown in Proposition 15. Since, then, C and D measure E and F by G and H, the fraction GH is the quotient of the division of the fraction EF — or of AB, equal to it — by the fraction CD, as was demonstrated at the beginning of this proposition. Which was to be proved.

$$EF = 24/36 \quad AB = 2/3 \quad K = 12 \quad CD = 3/4 \quad GH = 8/9 \quad EF = 90/135 \quad AB = 6/9 \quad K = 15 \quad CD = 3/5 \quad GH = 30/27$$

But since G and H, the numbers of the quotient fraction GH, are generated from A into D and from C into B, as was said, it is enough — for the division of any fraction by any fraction — that the numerator A of the fraction to be divided be multiplied into D, the denominator of the dividing fraction, and C, the numerator of the dividing fraction, into B, the denominator of the fraction to be divided. For the first number produced, G, will give the numerator, and the second, H, the denominator of the quotient fraction GH. And this is true for all fractions, whether or not the numbers of the fraction to be divided measure the numbers of the dividing fraction. For always, from the numerator of the fraction to be divided into the denominator of the dividing fraction, and from the numerator of the dividing fraction into the denominator of the fraction to be divided, two numbers are produced, by which the numbers of the dividing fraction measure the numbers of the fraction to which the fraction to be divided is brought back according to the doctrine of Proposition 15 — as is clear from the demonstration of that proposition, and as appears in this adjoined example, where the numbers C and D measure the numbers A and B by G and H; and the numbers E and F, produced from A and B into K (the product of C into D), those same numbers C and D measure by L and M. And so both the fraction GH and the fraction LM are the quotient of the fraction AB divided by the fraction CD.

$$EF = 48/54 \quad AB = 8/9 \quad K = 6 \quad CD = 2/3 \quad GH = 4/3 \quad LM = 24/18$$

Now, if a whole number is to be divided by a fraction, or a fraction is to be divided by a whole number, a unit must be assigned to it, so that it becomes as it were a fraction whose numerator is the given whole number and whose denominator is 1. Thus, if the number 8 is proposed to be divided by $\frac{1}{2}$, the quotient is $16/1$, as is clear in the former adjoined example. But if the fraction $\frac{1}{2}$ is to be divided by the whole number 8, the quotient is $1/16$, as appears in the latter example.

$$8/1 \div 1/2 \rightarrow 16/1 \quad 1/2 \div 8/1 \rightarrow 1/16$$

But if fractions adhere to whole numbers, the whole numbers must be reduced to a fraction of the same denomination as the adhering fraction, just as we said at the end of the preceding proposition.

But since it is not easy to keep in memory that the numerator of the quotient fraction is generated from the numerator of the fraction to be divided into the denominator of the dividing fraction, and the denominator from the numerator of the dividing fraction into the denominator of the fraction to be divided, it is better, as an aid to memory, to interchange the places of the numbers of the dividing fraction — as we taught in the *Arithmetic* — and then to apply the rule of multiplication. Thus, if the fraction $\frac{3}{4}$ is proposed to be divided by $\frac{2}{7}$, the numbers of the dividing fraction must be interchanged in this way: $\frac{7}{2}$. Whence the quotient is $\frac{21}{8}$, as you see here. For by this method the numerator of the fraction to be divided is always multiplied into the denominator of the dividing fraction, according to the rule of multiplication, which prescribes that the numerators be multiplied by one another to form the numerator, and so on.

$$\frac{3}{4} \div \frac{2}{7} \rightarrow (\text{interchanged}) \frac{7}{2} \rightarrow \frac{21}{8}$$

Let the whole number be 700, whose fraction $\frac{3}{4}$ is to be divided by $\frac{2}{7}$ of the same — that is, the number 525 by 200. It is plain that, dividing 525 by 200, the quotient is $2 \frac{5}{8}$, that is, the fraction $\frac{21}{8}$, which was just found. So too, if the whole number 10 taken eight times is divided by its half — namely, by 5 — the quotient is 16, which was found above from the division of 8 by $\frac{1}{2}$.

XXIV.

To graft one fraction into another.

Let the fraction AB be grafted into the fraction CD, and let the sense first be that the fraction AB of one part of the fraction CD is to be added to the fraction CD — that is, that $\frac{3}{4}$ of one seventh is to be added to $\frac{6}{7}$. From C, the numerator of the latter fraction, into B, the denominator of the former fraction, form F, to which is added A, the numerator of the former; and to the composite number G assign the denominator H, produced from the multiplication of the denominators B and D. I say that the fraction GH is produced from the grafting of the fraction AB into the fraction CD, in the sense we have described — that is, that the fraction GH is equal to the fraction AB with respect to one part of the fraction CD, added to the fraction CD. [Margin: l. Prop. XIV (14. huius)] For since AB is the fraction of one part of the fraction CD, it is reduced to the simple fraction AH, whose numerator is the same as A (produced, namely, from unity into A), while its denominator is the number H, produced from the denominators B and D, as appears in this

adjoined example. Likewise, the fraction FH is equal to the fraction CD, having the same denominator H produced from the denominators B and D, and the numerator F produced from C, the numerator of the latter fraction, into B, the denominator of the former: (for since B, into C and D, makes F and H, [Margin: m. Elements VII.17 (17. septi.)] C is to D as F is to H, [Margin: n. Prop. VIII (8. huius)] and therefore the fractions CD and FH are equal). Thus it is brought about that the fraction GH, whose numerator G is composed of the numerators F and A of the fractions FH and AH (namely, from the number produced from C into B, and from A,)...

$$F = 24 \quad AB = 3/4 \quad CD = 6/7 \quad GH = 27/28$$

$$AB = 3/4 \text{ (one seventh)} \quad 1/7 \quad AH = 3/28$$

(from A). Let the denominator be H — the same denominator as those fractions FH and AH. Then, I say, the fraction GH is the sum of the fractions FH and AH, as was shown at the beginning of Prop. 20. That is, it is the sum gathered from the fraction AB (with respect to one part of the fraction CD), which was shown equal to the fraction AH, and from the fraction CD, which was shown equal to the fraction FH. The fraction GH is therefore produced by grafting the fraction AB into the fraction CD; that is, it equals the fraction AB of one part of the fraction CD, taken together with the fraction CD itself. Which is what was proposed.

Now suppose that several fractions AB, CD, EF, GH are to be inserted — that is, that the fraction AB (of one part of CD, of one part of EF, of one part of GH), and the fraction CD (of one part of EF, of one part of GH), and the fraction EF (of one part of GH) are to be added to the fraction GH. Then one must proceed as follows. From G, the numerator of the last fraction GH, multiplied into F, the denominator of the second-to-last fraction EF, form a number, and to it add E, the numerator of that same second-to-last fraction EF, to make the number K. Next, to the product of K into D, the denominator of the third-from-last fraction CD, add C, the numerator of that same fraction CD, to make M; and to the product of M into B, the denominator of the immediately preceding fraction AB, add A, the numerator of that same fraction AB, to make P. And so on continually: always multiply the number last formed into the denominator of the immediately preceding fraction, and to the product add the numerator of that same immediately preceding fraction, until no fraction remains. Then, under the number P — the one formed at the last step — set the number Q, produced by multiplying the denominators together: first multiply H into F to make L, then L into D to make N, then N into B to make Q, and so on. I say that the fraction PQ is produced by grafting the given fractions, in the sense we have explained.

$K = 14$ $M = 59$ $P = 119$ $A/B = 1/2$ $C/D = 3/4$ $E/F = 2/3$ $G/H = 4/5$ $L = 15$ $N = 60$ $Q = 120$

For, as was shown in the case of two fractions, the fraction KL is produced by grafting the fraction EF into the fraction GH ; that is, it equals the sum gathered from the fraction EF (of one part of GH) and from the fraction GH . Next, the fraction CD of one part of EF , of one part of GH ,

[Margin: d — 14. huius (Prop. XIV of this treatise)]

makes a simple fraction whose numerator is C (namely, the product of C into unity taken twice, thus: 3.1.1.), and whose denominator is N , produced from D into L — that is, from the multiplication of the denominators H , F , D .

[Margin: e — 8. huius (Prop. VIII of this treatise)]

Likewise the fraction whose numerator is the number produced from D into K , and whose denominator is the same number N , produced from D into L , equals the fraction KL ,

[Margin: f — 17. septi. (Elements VII.17)]

because the numerators of these fractions bear the same ratio to the denominators; for D multiplied into K and into L generates the numbers of one and of the other fraction. Therefore, since — by what we demonstrated at the beginning of Prop. 20 — the fraction MN (whose numerator M is formed from the numerator C and from the number produced by D into K) equals the sum gathered from the fraction whose numerator is C and whose denominator is produced from D into K , and from the fraction whose denominator is N ; the same fraction MN will also equal the sum gathered from the fraction GH , from the fraction EF (of one part of GH), and from the fraction CD (of one part of EF , of one part of GH).

Again, the fraction AB of one part of CD , of one part of EF , of one part of GH ,

[Margin: g — 14. huius (Prop. XIV of this treatise)]

makes a simple fraction whose numerator is A , and whose denominator is Q , produced from B into N — that is, from the multiplication of the denominators H , F , D , B .

[Margin: h — 8. huius (Prop. VIII of this treatise)]

Likewise the fraction whose numerator is the number produced from B into M, and whose denominator is that same number Q, produced from B into N, equals the fraction MN,

[Margin: i — 17. septi. (Elements VII.17)]

because the numerators of these fractions bear the same ratio to the denominators; for B multiplied into M and into N generates the numbers of one and of the other fraction. Therefore, since the fraction PQ (whose numerator P is formed from the numerator A and from the number produced by B into M) equals — by what was demonstrated at the beginning of Prop. 20 — the sum gathered from the fraction whose numerator is A and whose denominator is Q, and from the fraction whose numerator is generated from B into M and whose denominator is the same Q; the same fraction PQ will also equal the sum gathered from the fraction GH, from the fraction EF (of one part of GH), from the fraction CD (of one part of EF, of one part of GH), and from the fraction AB (of one part of CD, of one part of EF, of one part of GH). Which is what was proposed. And the reasoning is the same for still more fractions.

Now suppose the sense is that the fraction AB *of the whole* fraction CD is to be added to the fraction CD.

[Margin: k — 14. hui. (Prop. XIV of this treatise)]

From C, the numerator of the latter fraction, multiplied into B, the denominator of the former, make F; to it add E, produced from the numerators A and C; and with the number G thus formed, set H, produced from the denominators B and D. I say that the fraction GH is produced by grafting the fraction AB into the fraction CD, in the sense we mentioned — that is, that the fraction GH equals the sum gathered from the fraction CD and from the fraction AB of the fraction CD.

[Margin: l — 8. huius (Prop. VIII of this treatise)] [Margin: m — 17. sept. (Elements VII.17)]

$$E = 18 \quad F = 24 \quad A/B = 3/4 \quad C/D = 6/7 \quad G = 42 \quad H = 28$$

For since the fraction AB (of the) fraction CD reduces to the simple fraction EH, whose numerator is E, produced from the numerators A and C — as the appended example shows — the fraction GH (which, by what we showed at the beginning of Prop. 20, equals the sum of the fractions EH and FH) is equal to the sum gathered from the fraction CD (that is, from the fraction FH) and from the fraction AB of the fraction CD (that is, from the fraction EH). And therefore the

fraction GH is produced by grafting the fraction AB into the fraction CD, in this latter sense. Which is what was proposed.

$$A/B = 3/4 \quad C/D = 6/7 \quad E/H = 18/28 \quad F/H = 24/28$$

But if several fractions AB, CD, EF, GH are to be inserted — that is, if the fraction AB (of the fraction CD, of the fraction EF, of the fraction GH), and the fraction CD (of the fraction EF, of the fraction GH), and the fraction EF (of the fraction GH) are to be added to the fraction GH — then one must proceed as follows. Multiply G into F, and to the product add the number produced from E into G, to make K. Next multiply K into D, and to the number produced add the number made from C, E, G, to make M. Then, to the product of M into B, add the number produced from A, C, E, G, to make P. And so on continually: always multiply the number last generated into the denominator of the preceding fraction, and to the number produced add the number formed from all the numerators of the fractions taken up to that point, until no fraction remains. Then, under the number P — the one produced at the last step — set the number Q, produced from all the denominators, as in the former sense. I say that the fraction PQ is produced by grafting the said fractions, provided the grafting is understood as stated. For, as was demonstrated in the case of two fractions, the fraction KL is produced by grafting the fractions EF, GH — that is, it equals the sum gathered from the fraction EF (of the whole fraction GH) and from the fraction GH.

$$P = 232 \quad M = 104 \quad K = 20 \quad A/B = 1/2 \quad C/D = 3/4 \quad E/F = 2/3 \quad G/H = 4/5 \quad Q = 120 \quad N = 60 \quad L = 15$$

from the fraction GH. Next, the fraction CD of the whole fraction EF, of the whole fraction GH,

[Margin: m — 14. hui. (Prop. XIV of this treatise)]

makes a simple fraction whose numerator is produced from the numerators C, E, G, and whose denominator is N, produced from the denominators H, F, D.

Likewise the fraction whose numerator is the same N, produced from D into L — that is, from the denominators H, F, D —

[Margin: n — 8. huius (Prop. VIII of this treatise)]

equals the fraction KL,

[Margin: o — 17. sept. (Elements VII.17)]

because the numerators of these fractions bear the same ratio to the denominators; for D multiplied into K and into L makes the numbers of one and

the other of these fractions. Therefore, since the fraction MN (whose numerator M is formed from the number that the numerators C, E, G generate, and from the number that D into K produces) equals — by what was shown at the beginning of Prop. 20 — the sum gathered from the fraction whose numerator is produced from the numerators C, E, G and whose denominator is N (produced from the denominators H, F, D), and from the fraction whose numerator is generated from D into K and whose denominator is the same N; the same fraction MN will also equal the sum gathered from the fraction GH, from the fraction EF (of the fraction GH), and from the fraction CD (of the fraction EF, of the fraction GH).

Again, the fraction AB of the fraction CD

[Margin: p — 14. huius (Prop. XIV of this treatise)]

makes a simple fraction whose numerator is produced from the numerators A, C, E, G, and whose denominator is produced from the denominators H, F, D, B. Likewise the fraction whose numerator is produced from B into M

[Margin: q — 8. huius (Prop. VIII of this treatise)]

equals the fraction MN,

[Margin: r — 17. septi. (Elements VII.17)]

because the numerators of these fractions bear the same ratio to the denominators; for B multiplied into M and into N makes the numbers of one and the other of this fraction. Therefore the fraction PQ (whose numerator P is formed from the number that the numerators A, C, E, G generate, and from the number that B into M makes) — since it equals, by what was demonstrated at the beginning of Prop. 20, the sum gathered from the fraction whose numerator is produced from the numerators A, C, E, G and whose denominator is Q, and from the fraction whose numerator is generated from B into M and whose denominator is the same Q — will also equal the sum gathered from the fraction GH, from the fraction EF (of the fraction GH), from the fraction CD (of the fraction EF, of the fraction GH), and from the fraction AB (of the fraction CD, of the fraction EF, of the fraction GH). Which is what was proposed. And the reasoning is the same for more fractions.

P = 232 M = 104 K = 20 A/B = 1/2 C/D = 3/4 E/F = 2/3 G/H = 4/5 Q = 130 [?] N = 60
L = 15

On the Composition of Proportions [De proportionum compositione]

Having dispatched what pertains to the demonstrations concerning fractional numbers, it remains for us to explain the composition of proportions — which we undertook, at this place, to treat in connection with Def. 10 of Book 5, Def. 5 of Book 6, and Prop. 23 of Book 6. First, then, that "composition" of proportions with which Euclid dealt in Def. 10 of Book 5 and Def. 5 of Book 6 — in those propositions where he demonstrates a doubled, tripled, or compounded proportion of magnitudes or of numbers — is not a true addition of proportions, such that the doubled or tripled proportion would be twice or three times greater than the proportion of which it is called the double or triple. Likewise, that a proportion compounded of several proportions is truly some whole whose parts are the proportions of which it is said to be composed — as most interpreters of Euclid suppose — can be plainly disproved from this: that a part would then be either equal to, or greater than, the whole. For since this composition must be understood whether a greater quantity is compared with a lesser or a lesser with a greater (as we showed at Def. 10 of Book 5), suppose that, given the three continually proportional terms 1, 10, 100, the proportion $1 : 100$ were not only called the double of the proportion $1 : 10$ (as Def. 10 of Book 5 intends), but were truly twice as great — so that it were heaped up out of the two equal proportions $1 : 10$ and $10 : 100$, as Euclid's interpreters suppose. Then who does not see that the part would be greater than the whole? — namely, that the proportion $1 : 10$, which is one part, would be greater than the proportion $1 : 100$. And here, [in the case] 4.4.4 [?], that the part would be equal to the whole? So too, if — given the three terms 4, 2, 8, which are not continually proportional — the proportion $4 : 8$ were truly heaped up out of the proportions $4 : 2$ and $2 : 8$ as out of parts (as those same authors contend), then the part would again be greater than the whole, since the double proportion $4 : 2$ is greater than the sub-double proportion $4 : 8$. And here, [given] 10, 5, 1, 2, 5, the part would be equal to the whole — namely, the proportion $10 : 5$.

24. | 48. 12.2. | 3.16.

to concede in any way, willing or unwilling, from Prop. 5 of Book 8. For let there be two plane numbers, 24 and 48. Let the sides of the former be 12 and 2; of the latter, 3 and 16. Since, then, Euclid there demonstrates that the proportion $24 : 48$ is compounded of the proportions of the sides — that is, of the proportion $12 : 3$ and of the proportion $2 : 16$ — it will be a component part (if this composition is a true addition), namely the sub-double proportion $24 : 48$. By like reasoning, if the

proportion $24 : 48$ is truly a whole, and its component parts are the quadruple proportion $12 : 3$ and the sub-octuple proportion $2 : 16$, then if either of these be subtracted from it, the other will remain. Therefore the greater can also be subtracted from the lesser — namely, the quadruple proportion $12 : 3$ subtracted from the sub-double proportion $24 : 48$, from the proportion $2 : 16$ — and the remainder will be the quadruple proportion $12 : 3$, greater than that from which the subtraction was made. And if, of the latter plane number 48, the sides be taken as 12 and 4, one of the components — namely the proportion $2 : 4$ — will be equal to the whole proportion, that is, to $24 : 48$. And with the proportion $2 : 4$ subtracted, there would still remain the proportion of equality $12 : 12$, which is greater than the whole, that is, than the proportion $24 : 48$. Again it would follow that the whole is not increased even if infinite parts were added to it. For, given these four terms 12, 4, 2, 1, if the proportion $12 : 1$ is truly heaped up out of the proportions $12 : 4$, $4 : 2$, and $2 : 1$ as out of parts, so that these proportions taken together equal the proportion $12 : 1$: if with those terms there are contained a further three terms in this way, 12, 4, 2, 1, 6 — of which the first three are the same as those out of which the proportion was previously composed — then the three proportions $1 : 6$, $6 : 3$, and $3 : 1$, added to the three proportions $12 : 4$, $4 : 2$, and $2 : 1$, do not increase the whole heaped up out of them.

All these consequences are absurd and contrary to the principles of mathematicians. Add to this that two triple proportions [compounded] constitute a sextuple proportion, not a nine-fold one — as the said authors would have it, when a triple proportion is called "tripled doubled," as appears in these numbers, 9.3.1. For who does not see that, if some agent be as 3, that same one is made twice as powerful as 6, not as 9? So too the common understanding grasps that, from a triple and a quintuple, an octuple proportion is produced, because two motive powers — as 3 and as 5 — moving together move as 8, and do not constitute out of themselves the proportion $15 : 1$, [p. 393] even though this [p. 393]

is said to be composed of them, from Def. 5 of Book 6, as is clear in these numbers, 15.5.1. Add to this that the powers of many agents cannot be halved, because between two numbers that express the proportion of the agent to the patient, no proportional mean falls. For example: an agent as 6 could not have an agent half as powerful, because between 6 and 1 no proportional mean number falls; whereas who would not at once understand that an agent as 3 is less, in double proportion, than an agent as 6?

It is clear, then, that a composition of this kind of proportions is not addition, but a *continuation* of proportions, brought about by the multiplication of the denominators, as we demonstrated at Def. 5 of Book 6 — where we also explained why the proportion of the extremes is said to be composed of the mean proportions, on account of the continuation of the terms; just as at Def. 10 of Book 5 we explained why, when three, four, or more continually proportional terms are set out, the proportion of the extremes is called the double, triple, or quadruple (and so on) of the proportion which the first two terms hold. These matters will be perceived more plainly from what we are now about to say concerning the *true* composition of proportions.

Accordingly, all the arithmetical operations on proportions — namely addition, subtraction, multiplication, and division — must be carried out by means of the denominators of the proportions, as Voluminius Rodulphus [?] rightly teaches in his disputation on the proportion of proportions. And since the denominators of proportions are fractions, or *minutiae* (as we taught in Book 5) — for under the denominator of any multiple proportion, which is a whole number, a unit can be supplied, so that it becomes, as it were, a fraction denominated from unity — each of which equals the fraction that any two numbers constitute in the proportion of which that fraction is the denominator (if the numerator is made from the antecedent and the denominator from the consequent): it plainly follows that the arithmetical operations on proportions do not differ from the operations on fractional numbers, whether they are set up by means of the denominators of the proportions or by means of any numbers whatever of those same proportions — provided the consequent terms are placed under the antecedents, in the manner of fractions, as Cardanus [Cardano] excellently prescribed in his *Arithmetic*, as will now appear.

First, let two proportions be gathered into one sum — the triple and the quintuple, 3 and 5 — from whose addition arises the sum 8; an octuple proportion will be heaped up out of them.... supposing first a unit under each of them, let [the numerators] be added in the manner of fractions; or let the numbers between which the given proportions are found be gathered into one sum, just as if they were fractional numbers, with the consequent terms placed under the antecedents.

$3/1 \times 5/1 \rightarrow 8/1$ $12/4 \times 40/8 \rightarrow 256/32$ $18/7 \times 3/2 \rightarrow 57/14$ $54/21 \times 6/4 \rightarrow 342/84$

For by the former method the proportion 8 : 1 is gathered, and by the latter 256 : 32, each of which is octuple. So too, from the double-superquadrupartient-sevenths proportion and the sesquialter — whose denominators are $2\frac{4}{7}$ and 1

1/2, and which are found between the numbers 54, 21 and 6, 4 — arises the quadruple-sesquifourteenth proportion, such as both the numbers 57, 14 and 342, 84 hold. But when the operation is set up by means of the denominators, the denominators must first be brought back to single fractions, as was done here; for we brought $2\frac{4}{7}$ back to $\frac{18}{7}$, and $1\frac{1}{2}$ to $\frac{3}{2}$. And this must be understood in the other operations as well.

Next, if from the octuple proportion the triple proportion be subtracted, and likewise from the quadruple-sesquifourteenth proportion the sesquialter proportion be deducted, the remainders will be the quintuple proportion and the double-superquadripartient-sevenths proportion, as is here evident:

$$8/1 \times 3/1 \rightarrow 5/1 \quad 256/32 \times 12/4 \rightarrow 640/128 \quad 57/14 \times 3/2 \rightarrow 72/28 \quad 342/84 \times 6/4 \rightarrow 864/336$$

Third, if the triple proportion is to be multiplied by 2 — or, what is the same, by the double proportion — and likewise the sesquialter proportion by the supertripartient-fifths, the operation must be set up as you see here; there the sextuple proportion will be produced, but here the double-superbipartient-fifths:

$$3-2 / 1-1 \rightarrow 6/1 \quad 12-10 / 4-5 \rightarrow 120/20 \quad 3-8 / 2-5 \rightarrow 24/10 \quad 9-16 / 6-10 \rightarrow 144/60$$

Fourth and last, if the sextuple proportion be proposed to be divided by the triple proportion, and likewise the double-superbipartient-fifths proportion by the sesquialter proportion: if the terms of the divisor be transposed (as we taught in the division of fractions), the operation will be set up thus:

$$6-1 / 1-3 \rightarrow 6/3 \quad 120-4 / 20-12 \rightarrow 480/240 \quad 24-2 / 10-3 \rightarrow 48/30 \quad 144-6 / 16[?]-9 \rightarrow 864/540$$

You see, then, that the arithmetical operations on proportions differ in nothing from the operations on fractions, except that in proportions it is not necessary to interpose a small line between the numerator and the denominator, as it is with fractions.

Now we shall demonstrate, in the following way, that the multiplication of proportions we have prescribed — which the authors falsely call "addition" — agrees with the composition of proportions treated by Euclid in Def. 10 of Book 5 and Def. 5 of Book 6, and that consequently that Euclidean composition is truly a multiplication and not an addition, as we said. Let there be two proportions, $A : B$ and $C : D$, whether equal or unequal. Multiply A , the antecedent, into the antecedent C , and let E result; and from B , the consequent, into the consequent D , let F result. I say...

[The demonstration continues from the previous page — that Euclid's composition of proportions is in truth multiplication (of the quotients), not addition.]

I say that the proportion $E:F$, which is produced by that multiplication, is composed of the two proportions $A:B$ and $C:D$; so that, if these proportions $A:B$ and $C:D$ are equal, the proportion $E:F$ is called the duplicate [i.e. squared] ratio of either of them, as def. 10 of Book V requires:

A, 6. C, 12. B, 2. D, 4. | A, 3. C, 6. B, 2. D, 5.
E, 72. F, 8. G, 24. | E, 18. F, 10. G, 12.

But if they are unequal, it is said to be composed of them, according to the sense of def. 5 of Book VI. For let G be made from B into C . And since from A and B into C there are produced E and G [that is, $E = A \times C$, $G = B \times C$], [Margin: a — Elem. VII.18] it follows that as $A:B$, so $E:G$. Again, since from B into C and D there are produced G and F , [Margin: b — Elem. VII.17] it follows that as $C:D$, so $G:F$. Now the proportion $E:F$ is composed of the proportions $E:G$ and $G:F$, by Euclid's definition. Therefore the same proportion $E:F$ is composed of the proportions $A:B$ and $C:D$, since these are the same as $E:G$ and $G:F$. This is also evident from the demonstration which we brought forward, at the definition in Book VI, from Eutocius and Witelo [Vitellio]. For since the composition of proportions described by Euclid answers to the multiplication of the denominators [quotients] among themselves, as we there demonstrated, while our own multiplication does not differ from the multiplication of the denominators, it is plainly established that that composition of proportions is the multiplication which we have set forth.

By the same reasoning we shall show that the subtraction of proportions, which these same authors teach, is nothing other than the division we have explained, and that it is the contrary of that composition of Euclid. For let the proportion $C:D$ be subtracted from the proportion $A:B$, as they will have it. Let E be made from A into D , and F from B into C — which is the same as if the terms of the proportion $C:D$ were to exchange places and the rule of multiplication were applied. I say that the proportion $E:F$, which is produced by the division of the proportion $A:B$ by the proportion $C:D$, as we have taught,

A, 9. C, 3. B, 1. D, 1. | A, 4. C, 6. B, 2. D, 2.
E, 9. F, 3. G, 27. | E, 8. F, 12. G, 24.

is that which is left over when the proportion $C:D$ is taken away from the proportion $A:B$: that is, that the proportion $A:B$ is composed of the proportions $C:D$ and $E:F$, in accordance with Euclid's definition. For let G be made from A into C . Since, then, from A and B into C there are produced G and F [$G = A \times C$, $F = B \times C$],

[Margin: c — Elem. VII.18 (printed "1."; apparently a misprint for 18, parallel to note a)] it follows that as $A:B$, so $G:F$. Again, since from A into C and D there are produced G and E , [Margin: d — Elem. VII.17] it follows that as $C:D$, so $G:E$. Since therefore [$G:F$ is composed] of $G:E$ and $E:F$, as Euclid will have it, the proportion $A:B$ also (which is the same as $G:F$) will be composed of the same proportions $G:E$ — that is, $C:D$ — and $E:F$; and accordingly, when the proportion $C:D$ is subtracted from the proportion $A:B$, the proportion $E:F$ will remain: just as, if from the proportion $G:F$ (with the term E placed in the middle) the proportion $G:E$ is taken away, the proportion $E:F$ is left. But who does not see that this is rather to *divide* the proportion $A:B$ by the proportion $C:D$, than to *subtract* the proportion $C:D$ from the proportion $A:B$ — lest one be said to take a greater away from a lesser, as happens in the latter example? And so much for the composition of proportions. Whoever desires more, let him read the most learned treatise written on this subject by Volumnio Rodolfo of Spoleto [Volumnius Rodolphus Spoletanus].

FINIS ELEMENTI NONL [recte: NONI — "End of the Ninth Book"]

Euclid's Tenth Book [EVCLIDIS ELEMENTVM DECIMVM]

Definitions

I.

Commensurable magnitudes are said to be those which the same measure measures.

Euclid, in the three preceding books, has finished with the matters that concern numbers, so far as seemed enough for the understanding of geometrical things. Now, in this tenth book, he proceeds to the discussion of commensurable and incommensurable lines — for whose sake, as we said above, he undertook the treatment of numbers. For without a knowledge of these lines, very many magnitudes, both solid and plane, can neither be perfectly understood nor, when occasion demands, be applied to practical work and use; because for the most part their sides are incommensurable. The same may be said of planes and solids themselves, since these too are very often incommensurable, as we shall demonstrate toward the end of this book. But since this book is beset with many difficulties, on account of the obscurity of the lines it treats of, I shall bend every sinew of my industry to this end: that, out of what has hitherto been demonstrated by Euclid, it may be made so plain and easy that it can be grasped

without much labor by anyone who has rightly understood the demonstrations of the preceding books. For I cannot side with the opinion of those who think that, for understanding it, that part of arithmetic is necessary which treats of the roots of numbers, both rational and irrational, as they call them. On the contrary, I am wholly persuaded that a perfect knowledge of that part of arithmetic depends upon this tenth book — so far am I from thinking that the treatment of roots is required in order that this book may be more easily understood. Yet I would not deny that one who has mastered the theory of roots and their calculation will grasp this book with greater pleasure than one wholly ignorant of them, because the former can turn the demonstrations to practical use, the latter by no means. It was for this reason, indeed, that we too fitted the first ten theorems of the second book to numbers, so that the studious reader might take from it the greater delight of mind, and profit — not, however, that we supposed the things demonstrated in that book could be more easily understood from numbers. Why then (someone will say) did you not, as in that book, likewise here make use of the numerical examples that Algebra employs, so that the matter might be both more pleasant and more convenient for readers? The reason we thought it should be omitted is ready to hand. For since very few at this time are acquainted with that most celebrated art of Algebra, those numbers, if they were brought in, seemed likely to spread darkness rather than to yield any greater light and clarity; so that the minds of the studious, instead of the help and light which we labor to bring them by these our commentaries, would meet with more inconvenience, and would perceive the demonstrations less well. This could not happen in the second book, since scarcely anyone is ignorant of the addition and multiplication of whole numbers, which alone were needful there. To this is added that those versed in Algebra can, with little trouble, fit the demonstrations of this book to numbers by themselves, whenever they wish — especially since this very thing was accomplished, not long before, carefully and accurately, by Michael Stifel [Michael Stifelius], the noble arithmetician, in the second book of the work which he entitled *Arithmetica Integra*.

And so, to return to our purpose: Euclid, beginning the subject itself in his usual manner from the explanations of the terms he will need, declares in the first place what magnitudes are called commensurable, defining them to be those which the same measure measures. Thus two magnitudes A, B, which the same measure C measures (for C measures A when repeated four times, and B when taken three times), are called commensurable:

Diagram. In the margin stand three horizontal straight lines, one above another, illustrating the definition.

- A (top line): the longest of the three; the common measure c is contained in it four times ("repeated four times"), so that $A = 4 \times c$. Marked off internally by short cross-strokes.
- B (middle line): shorter than A; the common measure c is contained in it three times ("taken three times"), so that $B = 3 \times c$. Also marked off internally.
- c (bottom line): the shortest, serving as the common measure of both A and B.

In the same way, a line of 20 palms and a line of 13 palms are commensurable, because a line of one palm, or of half a palm, or of a third part of a palm, and so on, measures them. Similarly, those surfaces are called commensurable which one and the same surface measures; and likewise those bodies, or solids, are commensurable which the same body or solid measures.

II.

But incommensurable [magnitudes] are those for which no common measure can be found.

Such magnitudes are the diameter of any square and its side; for they have no common measure, as will be demonstrated by Euclid in the last proposition of this book. There are also very many other incommensurable lines, for which no common measure can in any way be given; many of which the Geometer explains in this book, teaching by what method they can be found. Again, those surfaces will be called incommensurable, and those solids incommensurable, which admit no common measure.

III.

Straight lines are commensurable in square [potentia] when the same space measures their squares.

Of straight lines, some are commensurable in length — those, namely, which another line, serving as a common measure, measures; such as those which, in the exposition of def. 1, we called simply commensurable. But some are incommensurable in length — those, namely, which no line, serving as a common measure, measures. Again, of the lines that are incommensurable in length, some are such that their squares are commensurable, while others are so disposed that their squares too are incommensurable. Therefore those lines, incommensurable in length, whose squares are commensurable, are here called by Euclid commensurable in square [power]; because, namely, according to their powers —

that is, according to their squares (for the power of any line is, as we said at prop. 47 of Book I, the square described upon it) — they are commensurable, although in respect of length they are altogether incommensurable.

But if anyone should ask why Euclid defined separately the lines commensurable in square, but not those commensurable in length, the answer is that lines commensurable in length have been more than sufficiently explained in the definition of commensurable magnitudes, since one common measure measures lines of this kind, as has been said. But since no common measure can measure lines commensurable in square, but only their squares measure the same space, it was therefore altogether necessary that these should be set out by their own proper definition.

IV.

But [straight lines are] incommensurable [in square] when no space can be found for their squares that would be their common measure.

The remaining lines, incommensurable in length, whose squares also are incommensurable, he calls incommensurable in square [power]. Now lines commensurable both in length and in square have this, as it were, agreement and connection among themselves: that lines commensurable in length are also commensurable in square, so that no lines can be given commensurable in length without their squares being commensurable also; because the square described from their common measure measures, as a common measure, their squares — as appears in the figure set below. For just as the straight line AB measures the straight lines CD, EF, so also the square AG measures the squares CH, EI. But it is not [conversely the case] that all lines commensurable in square are also commensurable in length; for many lines are commensurable in square — that is, have commensurable squares — which nevertheless are altogether incommensurable in length, as will be demonstrated in this book. Again, among lines incommensurable both in length and in square, this connection is found: that all lines incommensurable in square are also incommensurable in length; but not, conversely, that all lines incommensurable in length are also incommensurable in square, since many lines are found incommensurable in length among themselves while nevertheless in square — that is, according to their squares — they are commensurable. All of which will be clear from the corollary of prop. 9 of this book.

Diagram. Three squares stand side by side on three collinear base-segments, illustrating that lines commensurable in length have squares commensurable in area.

- E–F (leftmost, largest base-segment): the square erected upon it has its far upper corner at I; this is the square EI. It is subdivided internally by rules into an integer number of copies of the unit square AG.
- C–D (middle base-segment): the square erected upon it has its far upper corner at H; this is the square CH, likewise subdivided into copies of the unit square.
- A–B (rightmost, smallest base-segment): the square erected upon it has its far upper corner at G; this is the square AG, the unit square, which serves as the common measure of the squares CH and EI.
- The base-segment AB is the shortest and serves as the common measure of the segments CD and EF.

V.

These things being laid down, it is shown that, for any proposed straight line, there are straight lines infinite in multitude both commensurable and incommensurable with it — some in length and in square, others in square only. And let the proposed straight line be called Rational.

If any straight line of known magnitude is proposed, then of all the others that are compared with it, some will be commensurable with it in length and in power, some in power only; and likewise some will be incommensurable with it in length only, others in both length and power — as will be demonstrated in various places in this tenth book. For Euclid does not intend that it be shown, or gathered, from what has been said (although his words seem to sound this way) that these things are so; but he only indicates and hints to us, in a certain manner, that some lines can be said to be commensurable with the proposed straight line in length and in power, others in power only, and so on. Which indeed in no way follows from what has been said, but will be demonstrated in what follows, as we have said.

Now that proposed line, by reference to which the others are commensurable or incommensurable, is called by the Greeks ῥητή [Greek: ῥητή (rhētē, "rational, expressible")], and by the Latins Rationalis [rational]; because it is always posited as fixed and known, whereas the others, when compared with it, are not always known — although each, taken separately, is likewise fixed and known — since any of them can be divided into however many equal parts you please.

VI.

And those commensurable with it, whether in length and in power, or in power only, [are] Rational.

He teaches that the other lines, commensurable in whatever way with that one which is called Rational, are likewise called Rational — not indeed by position, as that one is, but because, being compared with it, they are found commensurable with it either in length and in power, or in power only.

Therefore, in Euclid's view, the square root of this number 20 or 1000 and the like — or, what comes to the same, a straight line whose square is 20 or 1000 and so on — is called Rational, because in power it is commensurable with the Rational line (for both 20 and 1000 are commensurable with any square number, such as 16, 100, and the rest), even though in length it is incommensurable with it. And so not a few arithmeticians are deceived, who for this reason call such a root irrational, on the ground that it cannot be expressed by a number.

VII.

But those [lines] incommensurable with it, let them be called Irrational.

Since in the preceding definition Euclid called Rational those lines that are commensurable with the assigned Rational straight line either in length and in power, or in power only, it is clear that here he calls Irrational those that are incommensurable with the assigned Rational straight line in both ways — that is, both in length and in power — and not in length only. From this it is also evident that the square root of this number 20 or 1000 and the like — or, what is the same, the line whose square is 20 or 1000 and so on — cannot be called Irrational, since it is incommensurable with the assigned Rational line not in length and in power, but only in length.

VIII.

And let the square that is formed on the assigned straight line be called Rational.

Just as that line which is laid down as fixed and known is called Rational, so too the square described upon it is called Rational, because it likewise is fixed and known; and it is by comparison with it that other areas are called commensurable or incommensurable.

IX.

And those areas commensurable with it, indeed, are Rational.

All plane areas commensurable with the square of the assigned Rational line are called Rational — not by position, as that square is, but because, when compared with it, they are found to be either commensurable or incommensurable with it; just as lines commensurable in any way with the Rational line are likewise called Rational. And since lines whose squares are these areas commensurable with the square of the assigned Rational line [*lineae potentes ipsa spatia* — the sides of squares equal to those areas] are at least commensurable in power with the Rational line — by Definition 3 — it is clear that they too are called Rational, according to Definition 6.

X.

But those [areas] incommensurable with it, let them be called Irrational.

In just the same way, plane areas incommensurable with the square described on the Rational line are called Irrational, exactly as the lines that are wholly incommensurable with the assigned Rational line have been called Irrational.

XI.

And [let] the straight lines that can produce them [be called] Irrational: if those [areas] are squares, [their] very sides; but if any other rectilinear figures, [then] the straight lines that describe squares equal to those incommensurable areas.

He calls Irrational the lines which can produce areas incommensurable with the square of the Rational line, just as he called those areas themselves Irrational: so that, if the incommensurable areas are squares, their sides are called Irrational; but if they are not squares but any other rectilinear figures, then the straight lines that describe squares equal to those incommensurable areas are called Irrational.

It is inferred, moreover, from Definition 7, that the lines which can produce areas incommensurable with the square of the Rational line must be called Irrational — just as above, from Definition 6, we inferred that the lines which can produce areas commensurable with the square of the Rational line must be called Rational. For the lines which can produce those areas incommensurable with the same square of the Rational line are, by Definition 4, incommensurable in power with the Rational line. Therefore, since all lines incommensurable in power are

also incommensurable in length — as we said at Definition 4, and as will be demonstrated in the corollary to Proposition 9 — it is evident, from Definition 7, that those lines which can produce areas incommensurable with the square of the Rational line are called Irrational.

It must also not be passed over that Euclid and the ancient geometers almost always attach the phrases "in length and in power" and "in power only" to the words "commensurable" and "incommensurable," but scarcely ever — very rarely — to the words "Rational" and "Irrational." For lines are rightly said to be commensurable in length and in power, or in power only; and likewise incommensurable in length and in power, or in length only. But it is less correct to speak of them as Rational in length and in power, or in power only, or as Irrational in length and in power, or in length only. And because Campanus did not notice this, he gave many an occasion to speak variously and obscurely, in this tenth book, about commensurable and incommensurable lines, as well as about Rational and Irrational ones.

To these definitions we shall add one postulate, and a few axioms whose use is found in this book.

POSTULATE, OR REQUEST

Let it be postulated that any magnitude can be multiplied so many times over that it exceeds any [given] magnitude of the same kind.

For if two unequal magnitudes of the same kind are given, then — since the greater is not infinite, while the lesser can be increased without limit — it is clear that the lesser can be multiplied so many times that it surpasses the greater. This is also established from what we wrote

at Definition 5 of Book 5. For there we said that magnitudes are to be reckoned of the same kind only when either one of them can be so multiplied that it exceeds the other. And it was from the failure of this condition that we showed a rectilinear angle and an angle of contact [angulus contactus] to be of different kinds.

AXIOMS, OR COMMON NOTIONS

I.

A magnitude that measures any number of magnitudes also measures the magnitude composed of them.

II.

A magnitude that measures any magnitude also measures every magnitude which that one measures.

III.

A magnitude measuring a whole magnitude and a part taken from it also measures the remainder.

These axioms, insofar as they concern numbers, were demonstrated by us in the last three common notions of Book 7. Since the same reasoning holds for magnitudes, there is no point in pointlessly repeating their demonstrations here — especially since they can be carried over word for word, changing only the word "number" into the word "magnitude."

Moreover, since in this book we had to depart from the numbering that Theon follows in the propositions of Euclid — for we have everywhere set aside Campanus's sequence, as we explained at the beginning of Book 1 — we have taken care that a double number be placed in the margin opposite the propositions, the upper showing Theon's order and the lower Campanus's sequence. And if only a single number is found in the margin, you should understand that there is then no discrepancy between Theon and Campanus as far as the numbering of the propositions is concerned.

THEOREM 1. PROPOSITION 1.

[Margin: j.]

Two unequal magnitudes being set out: if from the greater more than its half is taken away, and from the remainder again more than its half, and this is done continually, there will at length be left a certain magnitude that will be smaller than the assigned lesser magnitude.

Let two unequal magnitudes AB and C be set out, of which AB is the greater. I say that if from AB more than its half is taken away, and from the remainder again more than its half, and this is done continually, there will at length be left a certain magnitude smaller than C.

For multiply C as many times as needed, until the resulting magnitude DE is greater than AB — so that DE is the multiple of C next greater than AB. Divide DE into the parts DF, FG, GE, each equal to C. Then from AB take away more than its half, namely AH, and from the remainder HB more than its half, namely HI; and

let this be done continually, until the parts of AB are equal in number to the parts of DE. Let the parts AH, HI, IB, then, be as many as DF, FG, GE.

Now since DE is greater than AB, while from DE the part DF taken away is less than half — or exactly half, if DE is double C — whereas from AB the part AH taken away is more than half, the remainder FE will be greater than the remainder HB. (For since DE is greater than AB, if half were taken from DE and likewise half from AB, the remainder of DE would still be greater than the remainder of AB. Therefore, if from DE less than half is taken and from AB more than half, the remainder of DE will be greater than the remainder of AB.) Again, since FE is greater than HB, and from FE the half FG is taken — or at most less than half, if FE is more than double C — while from HB more than half, namely HI, is taken, the remainder GE will in the same way be greater than the remainder IB. Proceeding thus, we shall at length show that the last part of DE, here GE, is greater than the last part of AB, here IB. And since GE, the last part of DE, is equal to C, C too will be greater than IB, the last part of AB. And so IB is left, a magnitude smaller than C. Which was to be demonstrated.

Diagram. The principal figure for Proposition 1: three vertical straight segments drawn side by side.

- Left segment AB — labelled top to bottom B, I, H, A; the greater magnitude, divided at H and I into the parts AH, HI, IB.
- Middle segment C — a short vertical segment (the lesser magnitude C).
- Right segment DE — labelled top to bottom E, G, F, D; the multiple of C, divided at F and G into the parts DF, FG, GE (each equal to C).

The same will be demonstrated if from AB the half AH is taken, and from the remainder HB again the half HI, and so on. For by the same reasoning the remainder FE will be greater than the remainder HB, and likewise the remainder GE greater than the remainder IB — since from the greater DE the part DF taken is less than half, or at most half, while from the lesser AB the half AH is taken, and so on.

Otherwise. With the construction made as before, take KL to be such a multiple of IB as DE is of C. Divide KL, then, into the parts KM, MN, NL, each equal to IB; there will be as many parts in KL as in DE. Now since AH is greater than HB — or at least equal to it, if a half was taken from AB — and HB is greater than IB, AH too will be greater than IB, that is, greater than KM. Again, since HI is greater than IB, or at least equal to it, and IB is equal to MN, HI too will be greater than MN, or equal to it; and thus the whole AI is greater than the whole KN.

Diagram. The figure for the "same"/"otherwise" demonstration of Proposition 1: again three vertical segments.

- Left segment AB — labelled top to bottom B, I, H, A, divided at H and I (parts AH, HI, IB).
- Middle segment C — a short vertical segment.
- Right segment DE — labelled top to bottom E, G, F, D, divided at F and G (parts DF, FG, GE).

than the whole KN. Adding, then, the equals IB and NL, the whole AB will be greater than the whole KL. But DE is greater still than AB; therefore DE is far greater than KL. Hence, since as DE is to C, so is KL to IB (for KL was taken to be as great a multiple of IB as DE is of C), [Margin: a — 14. quint. (Elements V.14)] C too will be greater than IB. And so IB is left, smaller than C. Which was to be demonstrated.

Diagram. The figure for the "otherwise" demonstration of Proposition 1, now including the auxiliary segment KL. Four vertical straight segments are drawn side by side.

- Segment AB — labelled top to bottom B, I, H, A; divided at H and I into AH, HI, IB.
- Segment KL — labelled top to bottom L, N, M, K; divided at M and N into the parts KM, MN, NL (each equal to IB).
- Segment DE — labelled top to bottom E, G, F, D; divided at F and G into DF, FG, GE.
- Segment C — a short vertical segment.

SCHOLIUM.

It is clearer than daylight, from the demonstration of this theorem, that two unequal quantities set out must be such that the lesser, when multiplied, can at length surpass the greater. This we pointed out at Proposition 16 of Book 3, when we were treating the angle of contact against Jacques Peletier [Iacobum Peletarium].

THEOREM 2. PROPOSITION 2.

[Margin: ij.]

If, two unequal magnitudes being set out, the lesser is continually subtracted from the greater by a certain alternating subtraction, and the

remainder never measures the one preceding it: then those magnitudes will be incommensurable.

Two unequal magnitudes AB, CD being set out, of which AB is the lesser: let AB be subtracted from the greater CD, and the remainder of this from AB, and so on in turn — by a certain alternating subtraction let the lesser always be taken from the greater — and let the remainder never measure the preceding magnitude. I say that the magnitudes AB, CD are incommensurable.

For if they are not incommensurable, some common measure will measure them. Let the magnitude E measure them, if it can — E being either equal to AB or smaller. Now let AB, subtracted from CD as often as possible, leave FD smaller than itself, so that AB measures CF. Again let FD, subtracted from AB, leave GB smaller than itself, so that FD measures AG; and let this be done continually. [Margin: a — 1. dec. (Elements X.1)] There will therefore at length be left, from CD or AB, some magnitude smaller than E. (For since AB, taken from CD, leaves FD smaller than itself, the part CF taken away is greater than half of CD; otherwise, if it were the half, or less than the half, AB could still be subtracted from CD. In the same way AG, taken from AB, is greater than half of AB; and since more than the half is always taken away, [Margin: b — 1. dec. (Elements X.1)] there will at length be left some magnitude smaller than E.) Let GB, now left, be smaller than E.

Since therefore E measures AB, and AB measures CF, [Margin: c — 2. pron. (Axiom II)] E will also measure CF. But E measures the whole CD as well. [Margin: d — 3. pron. (Axiom III)] Therefore E will also measure the remainder FD. But FD measures AG; [Margin: e — 2. pron. (Axiom II)] hence E will also measure AG. But E measures the whole AB as well. [Margin: f — 3. pron. (Axiom III)] Therefore E will also measure the remainder GB — a greater measuring a lesser. Which is absurd. Therefore no magnitude will measure the magnitudes AB, CD. Hence they are incommensurable. If, then, two unequal magnitudes being set out, [the lesser is continually subtracted from the greater, and the remainder never measures the preceding one, they are incommensurable,] and so forth. Which was to be shown.

Diagram. The figure for Proposition 2: three vertical segments.

- Segment AB — labelled top to bottom B, G, A (the lesser magnitude), with the division point G, giving the parts AG and GB.
- Segment CD — labelled top to bottom D, F, C (the greater magnitude), with the division point F, giving the parts CF and FD.
- Segment E — a medium vertical segment (the supposed common measure).

SCHOLIUM.

We shall convert this theorem in the following way.

If, two incommensurable magnitudes being set out, the lesser is always subtracted from the greater by a certain alternating subtraction: the remainder will never measure the preceding one.

Let AB, CD be incommensurable magnitudes, and let the lesser AB be subtracted from CD, the remainder being ED; likewise let ED be taken from AB, and let FB be left, and so on. I say that in this alternating subtraction the remainder never measures the preceding magnitude. For, if it be possible, let FB measure the preceding magnitude ED. Since, then, FB measures ED, and ED measures AF, [Margin: g — 2. pron. (Axiom II)] FB will also measure AF; and it measures itself as well. [Margin: h — 1. pron. (Axiom I)] Therefore FB will measure the whole AB too. But the whole AB also measures CE. [Margin: i — 2. pron. (Axiom II)] Therefore FB will measure CE as well. But FB is assumed to measure ED too; [Margin: k — 1. pron. (Axiom I)] therefore FB will also measure the whole CD. Yet it has been shown that FB measures AB as well. Hence FB measures both AB and CD — which is absurd, for AB, CD are assumed incommensurable. Therefore no remainder magnitude ever measures the preceding magnitude. Which is what was proposed.

We shall demonstrate this too in like manner.

If, two commensurable magnitudes being set out, the lesser is always subtracted from the greater by a certain alternating subtraction: some remainder will measure the preceding one.

For if no remainder ever measured the preceding one, the given magnitudes would be incommensurable, as Euclid demonstrated in this theorem. But that is absurd, for they are assumed to be commensurable.

Diagram. The figure for the Scholium of Proposition 2 (the converse): two vertical segments.

- Segment AB — labelled top to bottom B, F, A, with the division point F (parts AF and FB).
- Segment CD — labelled top to bottom D, E, C, with the division point E (parts CE and ED).

(Continuation of the Scholium to Proposition 2 — Clavius's converse test for commensurability.)

So from all this we shall easily recognize whether any two given magnitudes are commensurable or not. For suppose the smaller is always subtracted from the greater in a kind of alternating subtraction: if at some point a remainder measures the term before it, the given magnitudes will be commensurable, since that same remainder then measures both of the given magnitudes — as is clear from the demonstration of the first theorem of this scholium. For from the fact that the remaining magnitude FB was said to measure the preceding term ED, it was shown that this same remainder FB measures both magnitudes AB and CD. But if the remainder never measures the term before it, the given magnitudes will be incommensurable, as Euclid has demonstrated here.

PROBL. 1. PROPOS. 3.

Given two commensurable magnitudes, to find their greatest common measure.

Let AB and CD be the two given commensurable magnitudes, whose greatest common measure we are required to find. Since AB and CD are commensurable, it comes about that if the smaller, AB, subtracted from the greater, CD, as many times as possible, leaves ED; and ED, subtracted from AB as many times as possible, leaves FB; and this proceeds continually by a kind of alternating subtraction, then at last some remainder measures the term before it, by the second theorem of the preceding scholium. Suppose that FB, the remainder, now measures the preceding term ED. I say that FB is the greatest common measure of the magnitudes AB and CD.

That it measures both, we shall show as follows. Since FB measures ED, and ED measures AF, FB will also measure AF itself [Margin: a — 2.pron. (i.e. axiom 2 of Book X)]. But FB measures itself as well. Therefore FB [text prints *EB*; recte FB] will measure the whole of AB [Margin: b — 1.pron.]; and thus also CE, which AB measures [Margin: c — 2.pron.]. Since therefore it measures ED as well, FB will also measure the whole of CD [Margin: d — 1.pron.]; therefore FB measures both AB and CD. So FB is the common measure of both AB and CD.

That FB is their *greatest* common measure, we shall prove thus. If it is not the greatest, there will be some other common measure of them greater than FB. Let G, then, if it be possible, be their common measure greater than FB. Now since G measures both AB and CD, and AB measures CE, G will also measure CE [Margin: e — 2.pron.]. But it also measures the whole of CD; therefore G will also measure the remainder ED [Margin: f — 3.pron.]. But ED measures AF; therefore G too will measure AF [Margin: g — 2.pron.]. But this same G measures the remainder FB —

a greater measuring a smaller [Margin: h — 3.pron.]. This is absurd. Therefore no magnitude greater than FB is a common measure of AB and CD. So FB is their greatest common measure.

Diagram. Two tall vertical line-segments set side by side, with a short segment between them at the foot.

- Left segment AB: A (bottom endpoint), B (top endpoint), with F marked between them (dividing it into the lower part AF and the upper part FB); interior tick-marks show the alternating subtraction.
- Right segment CD: C (bottom endpoint), D (top endpoint), with E marked between them (dividing it into the lower part CE and the upper part ED).
- G: a short vertical segment at the foot, between A and C — the assumed common measure greater than FB used in the reductio.

But if the smaller magnitude AB measures the greater CD, so that when subtracted from CD it leaves nothing, then AB itself will be the greatest common measure of both, since it measures itself as well. Therefore, given two commensurable magnitudes, and so on. Which was to be done.

Diagram. Two vertical line-segments side by side, the short one measuring the long one exactly.

- Left segment AB (short): A (bottom endpoint), B (top endpoint).
- Right segment CD (long): C (bottom endpoint), D (top endpoint); interior tick-marks show AB laid off along CD a whole number of times.

COROLLARIVM.

From this it is clear that a magnitude measuring two magnitudes also measures their greatest common measure.

This is drawn from that part of the demonstration where FB was shown to be the greatest common measure of AB and CD. For it was there demonstrated that the magnitude G, if it measures the magnitudes AB and CD, also measures the greatest measure FB. And the reasoning is the same for the rest.

SCHOLIVM.

From what has been said, it will not be difficult to determine whether any number of given magnitudes are commensurable or not. Let there be three magnitudes A, B, C. First I test, by the method we taught at proposition 2 of this book, whether the two A and B are commensurable or not. If they are

incommensurable, it is plain that all three A, B, C are incommensurable, since they can have no common measure, on account of the incommensurable magnitudes A and B.

But if A and B are commensurable, and their greatest common measure D has been found, then if D also measures the magnitude C, it is clear that the three magnitudes A, B, C are commensurable, since they have D as a common measure.

But if D, the greatest measure of the magnitudes A and B, does not measure C, then C and D will be either commensurable or not. If they are incommensurable, then all three A, B, C will likewise be incommensurable. For if they were supposed commensurable, their greatest common measure would measure D, the greatest measure of magnitudes A and B, by the corollary of this proposition. Since therefore that same measure would also measure C, C and D would not be incommensurable — which contradicts the hypothesis.

Diagram. A column of illustrative horizontal line-segments (with interior tick-division marks), each labelled at its left end with a capital letter, accompanying the scholium: an upper group illustrating the three magnitudes and lower groups illustrating three and four magnitudes with their common measures.

- Segments labelled (top to bottom, left ends) A [?], B, C, and in the lower groups A, B, C, D — the rotated single-letter capitals are at the orientation/resolution floor and several cannot be resolved with certainty.

But if C and D are commensurable, then the three magnitudes A, B, C will also be commensurable [Margin: i — 3.decimi (Elements X.3)]. For, E having been found as the greatest measure of C and D, since E measures D, and D measures A and B, E will also measure A and B [Margin: k — 2.pron.]. Therefore, since the same E also measures C, E will measure the three magnitudes A, B, C; and for that reason they are commensurable. Which was proposed.

In the same way we shall investigate whether more than three given magnitudes are commensurable or not. For if four are given, we shall first test the matter on three; if five, on four; and so on. The rest we shall carry through as was said of three magnitudes.

Diagram. Four horizontal line-segments with interior tick-division marks, illustrating the three magnitudes A, B, C together with their common measure.

- Segments labelled at their left ends A, B, C, and a short segment (the common measure) whose rotated label reads as H [?] — reading uncertain.

PROBL. 2. PROPOS. 4.

[Margin: iv.]

Given three commensurable magnitudes, to find their greatest common measure.

Let A, B, C be the three given commensurable magnitudes, whose greatest measure we must find. Let D be found, the greatest measure of A and B [Margin: a — 3.decimi (X.3)]. If then D also measures C, it is evident that D is the greatest measure of all three A, B, C. For if a magnitude greater than D were said to measure the magnitudes A, B, C, that same magnitude would — by the corollary of proposition 3 of this book — measure the greatest measure of A and B, namely D itself: a greater magnitude measuring a smaller, which cannot happen. But if D does not measure C, then at least D and C will be commensurable. For since A, B, C are commensurable, any common measure of them will measure D, the greatest measure of A and B, by the corollary of the preceding proposition. Since therefore that same measure also measures C, D and C will be commensurable. Let E, then, be found, their greatest measure [Margin: b — 3.decimi (X.3)]. I say that E is the greatest measure of the given magnitudes A, B, C.

That it is their common measure will become plain thus. Since E measures D and C, and D measures A and B, E will also measure A and B [Margin: c — 2.pron.]. But it measures C as well; therefore E is the common measure of A, B, C, as we asserted.

That E is their greatest measure can be confirmed in this way. If it is not the greatest, let F, if it be possible, be a common measure of them greater than E. Since F measures A and B, it will also measure their greatest measure D, by the corollary of the preceding proposition. But it measures C too. Therefore F, measuring D and C, will also measure their greatest measure E, by the same corollary — a greater magnitude measuring a smaller. This is absurd. Therefore no magnitude greater than E measures A, B, C; and for that reason E is their greatest common measure. Wherefore, given three commensurable magnitudes, and so on. Which was to be done.

Diagram. Vertical line-segments illustrating the construction of Proposition 4.

- Segments A, B, C: the three given commensurable magnitudes (labelled A, B, C at the foot).
- D: the greatest measure of A and B (labelled to the right, upper).

- E: the greatest measure of D and C — the greatest common measure of all three (labelled to the right, upper); interior tick-marks show how D and E divide the segments.

COROLLARIVM.

It is also plainly gathered from this that a magnitude measuring three magnitudes also measures their greatest common measure.

This is inferred from the last part of the demonstration. For it was there shown that the magnitude F, if it measures A, B, C, also measures E, their greatest common measure. And the reasoning is the same for the rest.

In like manner, given more than three commensurable magnitudes, we shall find their greatest common measure; and this same corollary will hold. For if the given magnitudes are four, one must first find the greatest common measure of three of them; if five, one must take the greatest measure of four magnitudes; and so on. All the rest is to be carried through as was said of three magnitudes.

LEMMA.

If there are any number of magnitudes, and just as many numbers taken two by two in the same ratio as the corresponding pairs of magnitudes, then *ex aequali* [*ex æquo*] the magnitudes and the numbers will also be in the same ratio.

Let there be any number of magnitudes A, B, C, D, in the same ratios as just as many numbers E, F, G, H — namely, as A is to B, so E to F; and as B to C, so F to G; and as C to D, so G to H. I say that *ex aequali*, as A is to D, so E is to H. For, from what we demonstrated on definition 5 of book 6, the ratio of A to D is compounded of the ratios A to B, B to C, and C to D, just as the ratio of E to H is compounded of the ratios E to F, F to G, G to H. Since, then, the ratios that compound the ratio A to D are set equal to the ratios that compound the ratio E to H, it is clear that the compounded ratios too — namely A to D, and E to H — are equal; that is, as A is to D, so E is to H. Which was proposed.

Diagram. Four horizontal line-segments with interior tick-division marks, representing the magnitudes, each paired with its number in the right margin.

- A (divided into 3 parts) — paired with number E, 3.
- B (divided into 4 parts) — paired with number F, 4.
- C (undivided, short) — paired with number G, 1.
- D (divided into 2 parts) — paired with number H, 2.

The same follows if the proportion of magnitudes and numbers is perturbed [ex aequali in perturbed proportion], as appears in this example, in which as A is to B, so G to H; and as B to C, so F to G; and as C to D, so E to F. For the demonstration is entirely the same.

Diagram. Four vertical line-segments with interior tick-division marks, representing the magnitudes A, B, C, D, each paired with its number in the left margin — the perturbed-proportion example.

- Segments labelled at the foot A, B, C, D.
- Marginal numbers: E, 3; F, 12; G, 4; H, 6.

SCHOLIVM.

Since we shall need to use the preceding lemma both in the proposition that follows and in many others, it seemed good to demonstrate it here briefly, for it does not readily appear to follow from what has been demonstrated. Magnitude and number constitute different kinds of quantity; but the proportion *ex aequali*, in the preceding books, was shown only for quantities of the same kind — for numbers in book 7, and for magnitudes in book 5. Yet the demonstration made in the fifth book could be transferred to this lemma as well, since all the demonstrations of that book apply to numbers too. Or, at all events, the demonstration made in book 7 will fit this same lemma, since the magnitudes A, B, C, D, which have the same ratios as the numbers E, F, G, H, take on the character of numbers.

THEOR. 3. PROPOS. 5.

[Margin: v.]

Commensurable magnitudes have to one another the ratio which a number has to a number.

Let A, B be commensurable magnitudes. I say that they have to one another the ratio which some number has to another number. Let C be found, their greatest measure [Margin: a — 3.decimi (X.3)]; and as often as C, repeated, measures A, so often let a unit, repeated, measure the number D; and as often as the same C, repeated, measures B, so often let a unit, repeated, measure the number E. Since, then, the magnitude C measures the magnitude A, and the unit measures the number D, an equal number of times, the magnitude A will contain the magnitude C as many times as the number D contains the unit. Therefore, as A is to C, so is the number D to the unit. And likewise, as C is to B, so is the unit to the

number E; because C measures B, and the unit measures the number E, an equal number of times. Therefore, by the preceding lemma, *ex aequali*, as the magnitude A is to the magnitude B, so is the number D to the number E. Therefore commensurable magnitudes have to one another the ratio, and so on. Which was to be demonstrated.

Diagram. Three vertical line-segments with interior tick-division marks, illustrating Proposition 5.

- A (measured into 3 parts by C) — paired with number D, 3.
- C: their greatest common measure — paired with the label Vni. tas (the unit).
- B (measured into 2 parts by C) — paired with number E, 2.
- Segments labelled at the foot A, C, B.

SCHOLIUM.

Since Euclid, for the demonstration of this theorem, assumes the greatest common measure of the given commensurable magnitudes — such as C was, of A and B — it is clear that the lines must be commensurable in length, and for that reason commensurable in power [square] as well, so that they may have to one another the ratio of a number to a number. For these alone admit a common measure, and so it is in these that the demonstration has place.

But if the lines are commensurable only in power, their squares will indeed have to one another the ratio of a number to a number — because they are commensurable, and therefore have a common measure, and the demonstration of this theorem fits them. The lines themselves, however, by no means do so; for since they are incommensurable in length, they lack a common measure, and for that reason the demonstration of this theorem can in no way fit them.

Further, from the demonstration of this same theorem it is plainly established that the ratio of commensurable magnitudes is that which the numbers have by which their greatest common measure measures them. For it was shown that the magnitudes A and B have the ratio which the numbers D and E have — the numbers, that is, by which their greatest common measure C measures them.

Hence, if for two given commensurable magnitudes A and B one wishes to find in what ratio they stand as numbers, one must take two numbers D and E which the unit measures in just the way that C, the greatest common measure of the commensurable magnitudes A and B, measures those magnitudes. For, as was

demonstrated, the magnitudes A and B have the ratio which the numbers D and E have.

But if, instead of the greatest measure C, we take any other common measure of them, we shall nonetheless demonstrate this theorem by the same argument. For although the numbers D and E come out greater in this latter demonstration than in the former, they still have the same ratio to one another, as appears from the adjoining figure. But since, in the former demonstration, the numbers found are the least of all having the same ratio, it was perhaps for this reason that Euclid assumed in his demonstration the greatest common measure, and not just any measure — though this is not necessary.

Diagram. Three vertical line-segments with interior tick-division marks, illustrating the scholium's alternative with a smaller common measure.

- A (measured into 6 parts) — paired with number D, 6.
- C: the (non-greatest) common measure taken — paired with the label Vnitas (the unit).
- B (measured into 4 parts) — paired with number E, 4.
- Segments labelled at the foot A, C, B.

THEOR. 4. PROPOS. 6.

[Margin: vj.]

If two magnitudes have to one another the ratio which a number has to a number, the magnitudes will be commensurable.

Let two magnitudes A, B have the ratio which the numbers C, D have. I say that the magnitudes A, B are commensurable. For, as often as the unit is contained in the number C, let the magnitude A be understood as divided into that many equal parts, of which parts let the magnitude E be equal to one. Then, as often as the unit, repeated, measures the number D, [the demonstration continues]

PROP. 6 (demonstration, continued) [Margin: vj.]

If two magnitudes have to one another the ratio which a number has to a number, the magnitudes will be commensurable.

(Continuing the demonstration begun in the previous pages: A and B have the ratio of the numbers C, D; A has been divided into as many equal parts as there are units in C, of which E is one.) Then let the magnitude E, repeated as many times as the unit is contained in D, measure off some other magnitude F. Since,

then, the magnitude E is contained in A just as many times as the unit is contained in C, A will contain E exactly as many times as the number C contains the unit. Therefore A is to E as C is to the unit. And likewise, as E is to F, so is the unit to D — because E measures F, and the unit measures D, the same number of times. So, by the lemma to proposition 4 of this book, *ex aequali*, A will be to F as C is to D. But it was also given that as C is to D, so is A to B. Therefore A is to B as A is to F [Margin: a — 9.quin (Elem. V.9)]; and consequently the magnitudes B and F are equal to one another. Hence, since E measures F, the same E will also measure B; but it measured A as well. Therefore the magnitude E measures both A and B [Margin: b — 1.def (Elem. X, def. 1)], and accordingly A and B are magnitudes commensurable with one another. If, then, two magnitudes have to one another the ratio which a number has to a number, and so on. Which was to be demonstrated.

ALITER (an alternative proof). Let the magnitude A be conceived as divided into as many equal parts as there are units in C, one of which parts is equal to the magnitude E. Since, then, E is to A as the unit is to C — because E measures A, and the unit measures C [recte C; the text prints "D"], the same number of times — and since it is laid down that A is to B as C is to D, then by the same lemma above, *ex aequali*, E will be to B as the unit is to D. But the unit measures the number D. Therefore the magnitude E also measures the magnitude B [recte B; the text prints "E"]; and E measures A as well. Therefore A and B, having the same common measure E, are commensurable [Margin: c — 1.def (Elem. X, def. 1)]. Which is what was proposed.

Diagram. Vertical line-segments illustrating the magnitudes and numbers of Prop. 6. Reading the bottom labels left to right, three segments are lettered B, E, F (a fourth segment, the whole magnitude A on the far left, is largely cut off at the margin [?]). To the right, the corresponding quantities are shown: the number C = 5 (upper), the unit (Vni.tas = unitas, middle), and the number D = 3 (lower).

- B (bottom): magnitude B
- E (bottom): the part E of A
- F (bottom): magnitude F
- C, 5 (right): the number C = 5
- Vni.tas (right): the unit
- D, 3 (right): the number D = 3

Diagram. The line-figure for the alternative proof of Prop. 6. Bottom labels, left to right: E, A, B. To the right, the quantities: the unit (Vni-tas), the number $C = 5$, and the number $D = 3$.

- E (bottom): the part E of A
- A (bottom): magnitude A
- B (bottom): magnitude B
- Vni-tas (right): the unit
- C, 5 (right): the number $C = 5$
- D, 3 (right): the number $D = 3$

COROLLARIVM

From the earlier demonstration of the theorem a method is now open to us for finding a straight line to which any other given straight line has the ratio of a number to a number. For (repeating the earlier figure of this proposition) if a line is to be found to which the given line A has the ratio which the number C has to the number D, then A must be divided into as many equal parts as there are units in C, and another line F must be taken consisting of as many of those same parts as there are units in D. If this is done, A will be to F as C is to D, as has been demonstrated.

From this it further appears by what method one can find a straight line whose square has to the square of another given straight line the ratio of a number to a number. For if a line is to be found whose square has to the square of the given line A the ratio which the number C has to the number D, then first the line F must be found — by the method just described — to which A has the ratio of C to D. Next, the mean proportional G is to be taken between A and F [Margin: d — 13.sex (Elem. VI.13)]. For then the square on A will be to the square on G as the number C is to the number D. Since the three straight lines A, G, F are continually proportional, then by the corollary to proposition 20 of book 6, the square on A will be to the square on G as A is to F — and hence as the number C is to the number D (since A is to F as C is to D) — because all squares are similar figures similarly described.

Diagram. The corollary's line-figure (the earlier figure repeated). Bottom labels, left to right: A, G, F, where G is the mean proportional between A and F. Beside them, the numbers $C = 5$ and $D = 3$.

- A (bottom): the given line A
- G (bottom): the mean proportional between A and F

- F (bottom): the line F, to which A has the ratio $C : D$
- C, 5 (right): the number $C = 5$
- D, 3 (right): the number $D = 3$

SCHOLIVM

In this theorem too it is evident that lines having the ratio of a number to a number are commensurable both in length and in power (square), and not merely in power — since, as is clear from the demonstration, they have a common measure, just as the magnitudes A and B were shown to have the common measure E.

THEOR. 5. PROPOS. 7. [Margin: vij.]

Incommensurable magnitudes do not have to one another the ratio which a number has to a number.

Let A and B be incommensurable magnitudes. I say that they do not have the ratio which a number has to a number. For if A and B are said to have the ratio of a number to a number, they will be commensurable [Margin: a — 6.deci (Elem. X.6)]. But this is absurd, since they are assumed to be incommensurable. Therefore A and B do not have the ratio which a number has to a number. Hence incommensurable magnitudes, and so on. Which was to be shown.

Diagram. Two vertical line-segments labelled A and B (bottom), the two incommensurable magnitudes of Prop. 7.

- A (bottom): magnitude A
- B (bottom): magnitude B

SCHOLIVM

In whatever way lines are incommensurable — whether in length and in power, or in length only — we shall always conclude that they do not have the ratio of a number to a number, as the theorem requires. For otherwise, as has been demonstrated, they would be commensurable in length. But that is not assumed.

THEOR. 6. PROPOS. 8. [Margin: viij.]

If two magnitudes do not have to one another the ratio which a number has to a number, the magnitudes will be incommensurable.

Let the magnitudes A and B not have to one another the ratio which a number has to a number. I say that they are incommensurable. For if A and B are believed to be commensurable, their ratio will be that of a number to a number [Margin: a — 5.deci (Elem. X.5)]. But this is absurd, since they are assumed not to have the ratio of a number to a number. Therefore A and B are not commensurable. Accordingly, if two magnitudes do not have to one another the ratio of a number to a number, and so on. Which was to be shown.

Diagram. Two vertical line-segments labelled A and B (bottom), the two magnitudes of Prop. 8.

- A (bottom): magnitude A
- B (bottom): magnitude B

SCHOLIUM (of Prop. 8)

If lines do not have the ratio which a number has to a number, they will necessarily, by this theorem, be incommensurable in length; otherwise they would have the ratio of a number to a number, as we demonstrated at proposition 5 of this book — which is contrary to the hypothesis. It must not, however, be inferred from this theorem that lines which do not have the ratio of a number to a number are necessarily incommensurable in power, as is evident from the demonstration. For it does not follow, if they are commensurable in power only, that they have the ratio of a number to a number, as is assumed in the demonstration of the theorem; on the contrary, they could in no way have such a ratio, as we taught in the scholium to proposition 5 of this book. The only conclusion drawn from the demonstration of this theorem, therefore, is that lines not having the ratio of a number to a number are incommensurable in length.

LEMMA

If there are three quantities continually proportional, and three other quantities also continually proportional, and if the first of the one set is to its third as the first of the other set is to its third, then also the first of the one set will be to its second as the first of the other set is to its second.

Let there be quantities A, B, C continually proportional, and likewise D, E, F — whether the former are in the same kind as the latter or not — and let A be to C as D is to F. I say also that A is to B as D is to E. For since the ratio of A to C is the duplicate of the ratio of A to B, and the ratio of D to F is the duplicate of the ratio of D to E, and since the ratios A : C and D : F are laid down as equal, the ratios A : B and D : E will also be equal — inasmuch as their duplicate ratios are equal.

The same follows if there are more quantities than three: provided that, for four, we take the triplicate ratio in place of the duplicate for the demonstration; and for five, the quadruplicate; and so on.

Diagram. Three horizontal line-segments, one above another, illustrating the Lemma. Each is doubly labelled — by a letter of the first triple (A, B, C, set sideways at the left ends) and by a letter and value of the second triple (at the right ends): D = 49, E = 42, F = 36. The values 49 : 42 : 36 are continually proportional in the ratio 7 : 6.

- A / D, 49 (top line): first quantity; D = 49
- B / E, 42 (middle line): second quantity; E = 42
- C / F, 36 (bottom line): third quantity; F = 36

THEOR. 1. PROPOS. 9. [Margin: ix.]

Squares formed on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares that have to one another the ratio which a square number has to a square number will also have sides commensurable in length. But squares formed on straight lines incommensurable in length do not have to one another the ratio which a square number has to a square number; and squares that do not have to one another the ratio which a square number has to a square number will not have sides commensurable in length either.

[Part 1.] Let AB, CD be first commensurable in length. I say that the square on AB has to the square on CD the ratio which a square number has to a square number. For since the straight lines AB, CD are commensurable in length, AB will be to CD as a number is to a number [Margin: a — 5.deci (Elem. X.5)]; let it be as the number E is to the number F, and let the squares of E and F themselves be G and H. Since, then, AB is to CD as the number E is to the number F, and since the square on AB has to the square on CD the duplicate ratio of the side AB to the side CD [Margin: b — 20.sex (Elem. VI.20)], and likewise the square number G has to the square number H the duplicate ratio of the side E to the side F [Margin: c — 11.oct (Elem. VIII.11)], the ratio of the square on AB to the square on CD will be the same as that of the square number G to the square number H — inasmuch as both these ratios are the duplicates of equal ratios. Which is what was proposed.

Diagram. Two squares side by side. The left square stands on the base A—B and is divided by a 2×2 grid into 4 cells (the square number G = 4, side E = 2); the right

square stands on the base C—D and is divided by a 3×3 grid into 9 cells (the square number H = 9, side F = 3). Below: the numbers E = 2, F = 3, G = 4, H = 9.

- A, B (base of left square): the line AB
- C, D (base of right square): the line CD
- E, 2 : side of the first square-number
- F, 3 : side of the second square-number
- G, 4 : the square number on E
- H, 9 : the square number on F

ALITER (an alternative proof). With the same construction laid down, let the rectangle BD be contained by the straight lines AB, CD; and let E and F, multiplying one another, produce I, which will be the mean proportional between the squares G and H — as is clear from the demonstration of proposition 11 of book 8 — and hence stands in the ratio of the side E to the side F, as we said in the scholium to that same proposition. Since, then, AB is to CD as E is to F, and since AB is to CD as the square on AB is to the rectangle BD (because of the common height AB, if CD is taken as the base of the rectangle) [Margin: d — 1.sexti (Elem. VI.1)], and likewise E is to F as G is to I, the square on AB will also be to the rectangle BD as the square number G is to the number I. Again, since AB is to CD as E is to F, and since AB is to CD as the rectangle BD is to the square on CD (because of the common height CD) [Margin: e — 1.sexti (Elem. VI.1)], if AB...

Diagram. The figure for the alternative proof: three rectilinear figures in continued proportion. The first is the square on AB (2×2 grid, 4 cells = G); the second is the rectangle BD contained by AB and CD (2×3 grid, 6 cells = the mean I); the third is the square on CD (3×3 grid, 9 cells = H). Their corner labels are set sideways in the print and are only partly legible.

- A, B : corners of the square on AB
- C, D : corners of the square on CD
- I (upper corner): the mean proportional, the rectangle BD [?]
- (two further sideways corner labels are not securely legible [?] [?])

LIBER X (continued)

[Part 2 — ALITER, continued.] ...if AB is taken as the base of the rectangle; and likewise E is to F as I is to H — the rectangle BD will also be to the square on CD as the number I is to the square number H. Therefore three magnitudes — the square on AB, the rectangle BD, and the square on CD — are in the same continued ratio in which the three numbers G, I, H stand. So, *ex aequali*, by the

lemma to proposition 4 of this book, the square on AB will be to the square on CD as the square number G is to the square number H. Which is what was proposed.

[Part 3, converse.] Secondly, let the square on AB be to the square on CD as the square number G is to the square number H. I say that the straight lines AB, CD are commensurable in length. For let the numbers E and F be the sides of the square numbers G and H, as in the earlier figure. Since, then, the square on AB is to the square on CD [recte CD; the text appears to read "BD"] as the square number G is to the square number H, and since the square on AB has to the square on CD the duplicate ratio of the side AB to the side CD [Margin: f — 20.sex (Elem. VI.20)], and likewise the square number G has to the square number H the duplicate ratio of the side E to the side F [Margin: g — 11.oct (Elem. VIII.11)], the ratio of the side AB to the side CD will be that of the side E to the side F — inasmuch as the duplicate ratios of these ratios are equal. Therefore, since the straight lines AB, CD have the ratio which the numbers E, F have, they will be commensurable in length. Which is what was proposed.

ALITER (of Part 3). Let the construction of the later figure be repeated. Since AB is to CD as the square on AB is to the rectangle BD [Margin: h — 6.dec (Elem. X.6)], and likewise AB is again to CD as the rectangle BD is to the square on CD, as we said above, the square on AB, the rectangle BD, and the square on CD will be continually proportional in the ratio of the line AB to the line CD. But the numbers G, I, H are also continually proportional in the ratio of the number E to the number F, as we showed above; and the square on AB is laid down to be to the square on CD as the square number G is to the square number H. Therefore, by the preceding lemma [Margin: i — 1.sextri (Elem. VI.1)], AB will be to CD as G is to I, that is, as E is to F; and therefore AB, CD, being straight lines having the ratio which the numbers E, F have, are commensurable in length [Margin: k — 6.dec (Elem. X.6)]. Which is what was proposed.

[Part 4.] Thirdly, let the straight lines A, B be incommensurable in length. I say that their squares do not have the ratio which a square number has to a square number. For if the square on A is said to have to the square on B the ratio which a square number has to a square number, then A, B will be commensurable in length, as has already been demonstrated. But that is absurd, since they are assumed incommensurable in length. Therefore the square on A does not have to the square on B the ratio which a square number has to a square number. Which is what was proposed.

[Part 5.] Fourthly and lastly, let the square on A not have to the square on B the ratio which a square number has to a square number. I say that they are

incommensurable in length. For if they are believed to be commensurable in length, their squares — as has been demonstrated — will have the ratio which a square number has to a square number. But that is absurd, since they are assumed not to have it. Therefore A and B are not commensurable in length. Which is what was proposed. Therefore squares formed on straight lines commensurable in length, and so on. Which was to be demonstrated.

Diagram. Two vertical line-segments labelled A and B (bottom), the two straight lines incommensurable in length treated in the third and fourth parts of the proposition.

- A (bottom): line A
- B (bottom): line B

COROLLARIVM

From these things it is evident that straight lines which are commensurable in length are altogether commensurable in power (square) as well; but those commensurable in power are not altogether commensurable in length too. And lines that are incommensurable in length are not altogether incommensurable in power; but those incommensurable in power are altogether incommensurable in length as well.

For since the squares of lines commensurable in length have the ratio which a square number has to a square number — as was demonstrated in this theorem — that is, simply the ratio of a number to a number, they will be commensurable [Margin: l — 6.dec (Elem. X.6)]; and therefore their sides too will be commensurable in power [Margin: m — 3.defin (Elem. X, def. 3)]. Hence lines commensurable in length are commensurable in power as well.

Next, since lines whose squares do not have the ratio which a square number has to a square number, but nevertheless the ratio simply of a number to a number, are indeed commensurable in power [Margin: n — 3.defi (Elem. X, def. 3)] — because their squares are commensurable [Margin: o — 6.dec (Elem. X.6)] — but by no means commensurable in length, as was shown in this theorem: it is clear that lines commensurable in power are not altogether commensurable in length. For only those lines commensurable in power whose squares have the ratio which a square number has to a square number are also commensurable in length, as is established from the second part of this theorem.

Again, since lines whose squares do not have the ratio which a square number has to a square number, but nevertheless the ratio of a number to a number, are

indeed incommensurable in length yet commensurable in power, as we have just said, it plainly follows that lines incommensurable in length are not altogether incommensurable in power. For only those lines incommensurable in length whose squares do not have the ratio of a number to a number are incommensurable in power as well [Margin: p — 4.defi (Elem. X, def. 4)], since their squares are incommensurable [Margin: q — 8.deci (Elem. X.8)].

Finally, that lines incommensurable in power are altogether incommensurable in length as well is evident. For if they were commensurable in length, they would be commensurable in power too, as is clear from the first part of the corollary. But that is absurd, since they are assumed incommensurable in power. For this reason lines incommensurable in power are altogether incommensurable in length as well.

SCHOLIVM

In order that the aim of the first two parts of this theorem may be grasped more clearly and completely, the following points are to be carefully considered. First, that straight lines commensurable in length — whose squares, by the first part of the theorem, have the ratio which a square number has to a square number — are not only those which can be expressed by numbers, if with... ..are compared with the proposed Rational line — such as those set out in the demonstration (for line AB is 2 and CD is 3) — but also those which can be expressed by no numbers at all, when compared with the proposed Rational line.

Then, conversely: squares that have to one another the ratio of a square number to a square number — whose lines, that is, sides, by the second part of the theorem, are commensurable in length — must be understood to include not only those that contain just as many square measures, equal to those into which the square of the Rational line is resolved, as there are units in the square numbers that have the same ratio to one another. Of this kind are the squares drawn in the demonstration (for the square on AB contains four square measures, and the square on CD nine — exactly as many units as there are in the square numbers G and H). They must also include those squares that embrace either fewer or more square measures than there are units in those square numbers, when they are compared with the square of the Rational line. For very often given lines are commensurable with one another in length, yet cannot be expressed in numbers, because they are incommensurable in length with the proposed Rational line. And therefore their squares do indeed have the ratio of a square number to a square number, as has been shown in this theorem, yet each contains either

fewer or more square measures than there are units in those square numbers. All this we shall make clear by the following demonstration.

[Margin: 26.oct. — Elements VIII.26] [Margin: 14.sec. — Elements II.14]

Let the Rational line AB be set out, expressed by the number 4; its square AC will contain 16 square measures. Let there also be numbers D and E, similar plane numbers but not squares, which nevertheless have [Margin: r — 26.oct., Elements VIII.26] — as is proved in the arithmetical books — the ratio that the square number F has to the square number G, namely 4 to 16. Next, take a rectangle HI made up of six square measures of the square AC — exactly as many units as are contained in the number D; and [Margin: s — 14.sec., Elements II.14] construct a square KL equal to this HI, whose side is KM. Finally, by what we showed in the corollary to Proposition 6 of this book (*Elements* X.6, Cor.), let a line NO be found such that KL, the square of the line KM, is to its square NP as the number D is to the number E, or as the square F is to the square G.

Since, then, as the square number F is to the square number G — or as the number D is to the number E, namely 6 to 24 — so is the square KL to the square NP, by construction; and since the square KL contains 6 square measures, of which the Rational square AC contains 16 (for the square KL was constructed equal to the rectangle HI, which is made up of 6 such square measures), the square NP will contain 24 of these same square measures. And therefore KM and NO, the sides of the squares KL and NP, will be $\sqrt{6}$ and $\sqrt{24}$, which cannot be expressed in numbers, since 6 and 24 are not square numbers. They will be incommensurable in length with the Rational line AB, but commensurable in length with one another — by this theorem — since their squares have the ratio that the square number F has to the square number G.

Diagram. A group of geometric figures illustrating the worked example.

- Rectangle **HI**: a tall narrow column (H at top-left, I at bottom-left) divided by cross-lines into six equal cells — the rectangle HI, "made up of six square measures" of the Rational square.
- Square **AC**: a square with A at bottom-left, B at bottom-right, C at top-right, divided into a 4×4 grid; the number 4 is marked along the base AB — the square of the Rational line AB (side 4, sixteen square measures).
- A block of numeral labels between the two upper figures: **D.6 E.24 / F.4 G.16 / Q.9 R.36** (the numbers D = 6, E = 24; F = 4, G = 16; Q = 9, R = 36).

- Small square **KL**: with K at bottom-left, L at top (side KM at the base, M at bottom-right), the numeral **6** inside and **R.6** ($= \sqrt{6}$) marked below the base — the square KL of 6 measures.
- Large square **NP**: with N at bottom-left, P at top-right, O at bottom-right, the numeral **24** inside and **R.24** ($= \sqrt{24}$) marked along the base NO — the square NP of 24 measures.

This can also be established as follows. Since the squares KL and NP have [Margin: t — 6.deci., Elements X.6] the ratio which is the duplicate of that of their sides KM and NO (by the corollary to Proposition 20 of Book VI, *Elements* VI.20, Cor.), and since the ratio of the squares is one-fourth (subquadruple), namely 1 to 4, the ratio of the sides will be one-half (subduple), namely 1 to 2 — for the former is the duplicate of the latter, as appears here: 1, 2, 4. Therefore the lines KM and NO are commensurable in length, since they have the ratio of a number to a number. So in this theorem we must include, among lines commensurable in length, also those that cannot be expressed in numbers when compared with the Rational line. But if these same lines KM and NO are considered simply and absolutely, without any regard to the Rational line AB, they can be expressed in numbers. For since they stand in a one-to-two (subduple) ratio, if KM is divided into two equal parts, NO will be divided into four parts of the same size; if KM into three, NO will be cut into six, and so on. But these parts are in no way equal to the parts of the Rational line AB; indeed they are altogether incommensurable in length with them, as we have said.

Again: since the squares KL and NP have the ratio that the square number F has to the square number G, they contain more square measures — equal to those into which AC, the square of the Rational line AB, is resolved — than there are units in the said square numbers F and G (for KL is equal to six such measures, and NP contains 24, whereas the square number F is made up of four units and G of sixteen). And if larger squares Q and R are taken in the same ratio, the squares KL and NP embrace fewer such measures than there are units in the square numbers Q and R, since one is made up of 9 units and the other of 36. It is therefore clear that in this theorem we must also include squares having the ratio of a square number to a square number which comprehend either fewer or more square measures than the units found in those square numbers, when they are compared with the square of the proposed Rational line. But if the same squares KL and NP are considered absolutely and simply, without any regard to the square AC described on the Rational line AB, they can be resolved into just as many square measures as there are units contained in the square numbers F, G and Q, R. For if the line KM is divided into two parts and NO into four, as was

said, the square KL will contain four square measures and the square NP sixteen — exactly as many units as are found in the square numbers F and G. Likewise, if the same lines are cut into three and six parts, their squares will have 9 and 36 square measures — exactly as many units as are comprehended in the square numbers Q and R, and so on. But measures of this kind are in no way equal to the square measures into which AC, the square of the Rational line AB, is resolved.

We shall say the same of all other squares and their sides whose areas are not expressed by square numbers, provided they have to one another the ratio of square numbers to square numbers — such as squares whose areas are 12 and 3, having the ratio that the square number 16 has to the square number 4, and so on.

Accordingly this theorem is to be understood of lines commensurable in length with one another, even though at times they are incommensurable with the Rational line — so that the sense is this: squares described on straight lines commensurable in length with one another — even though they are sometimes incommensurable in length with the proposed Rational line — have to one another the ratio of a square number to a square number, although the squares themselves, when compared with the square of the Rational line, very often cannot be expressed in square numbers at all. And squares having to one another the ratio of a square number to a square

number — whether or not these squares can be expressed in square numbers when compared with the square of the Rational line — will have sides commensurable in length with one another, even if those lines are incommensurable in length with the Rational line, and so on.

[Campanus] infers from this theorem that in square figures the diagonal is incommensurable in length with the side, in the following way. Since the square on the diagonal is double the square on the side, as we proved at Proposition 47 of Book I (*Elements* I.47); and since no double ratio can be the same as that of a square number to a square number — because between numbers in double ratio no mean proportional falls, as we showed in the scholium to Proposition 8 of Book VIII (*Elements* VIII.8, schol.) — the square on the diagonal and the square on the side will not have the ratio of a square number to a square number. Therefore, by the last part of this theorem, their sides — that is, the diagonal and the side in one and the same square — are incommensurable in length with one another. We too demonstrated this in the scholium to Proposition 7 of Book VIII (*Elements* VIII.7, schol.). [Euclid] will show it more clearly, however, in the last proposition of this book.

THEOR. 8. PROPOS. 10.

[Margin: x.] [Margin: xj.]

If four magnitudes be proportional, and the first be commensurable with the second, the third also will be commensurable with the fourth. And if the first be incommensurable with the second, the third also will be incommensurable with the fourth.

Let there be four proportional magnitudes A, B, C, D, and let A, the first, be commensurable with B, the second. I say that the third, C, is also commensurable with the fourth, D. For since A and B are commensurable with one another, [Margin: a — 5.deci., Elements X.5] A will have to B the ratio of a number to a number, namely that of E to F. But it is laid down that as A is to B, so is C to D. Therefore C also has to D the ratio that the number E has to the number F. [Margin: b — 6.dec., Elements X.6] Hence C and D are commensurable with one another.

But now let A be incommensurable with B. I say that C is also incommensurable with D. For since A and B are incommensurable, [Margin: c — 7.dec., Elements X.7] A will not have to B the ratio of a number to a number. But it is laid down that as A is to B, so is C to D. Therefore neither does C have to D the ratio of a number to a number. [Margin: d — 8.deci., Elements X.8] Hence C and D are incommensurable with one another. If, therefore, four magnitudes be proportional, and so on. Which was to be shown.

Diagram. Two short vertical line-segments at top, labelled **E.4** and **F.2** (the numbers $E = 4$ and $F = 2$), and below them four vertical line-segments of differing heights labelled, left to right, **A, B, C, D** — the four proportional magnitudes.

SCHOLIVM.

If the four given magnitudes are lines, and the first is commensurable in length with the second, the third also will be commensurable in length with the fourth, as appears from the demonstration of this theorem. For in the same way we shall show that the latter two have the ratio [recte: proportionem] of a number to a number. Therefore, as we said in the scholium to Proposition 5 of this book, they are commensurable in length.

But if the first is commensurable with the second in square only, the third also will be commensurable with the fourth in square only. For let the same figure be repeated. Since, then, A is commensurable with B, [Margin: e — 3.defin., Def. 3]

their squares will be commensurable, [Margin: f — 5.deci., Elements X.5] and therefore will have the ratio of a number to a number. [Margin: g — 22.sex., Elements VI.22] But as the square on A is to the square on B, so is the square on C to the square on D (because the four given lines A, B, C, D are proportional, and their squares are similar figures, similarly described). Therefore the squares on C and D also will have the ratio of a number to a number. [Margin: h — 6.deci., Elements X.6] Hence they will be commensurable; [Margin: i — 3.def., Def. 3] and consequently the lines C and D are commensurable in square. That they are commensurable in square only will be made clear thus. Since the lines A and B, being commensurable in square only, are incommensurable in length, their ratio will not be that of a number to a number, as we taught in the scholium to Proposition 7 of this book. And therefore neither will C and D have the ratio of a number to a number. So C and D are incommensurable in length, by the scholium to Proposition 8 of this book. But they are commensurable in square, as we showed. Therefore they are commensurable in square only. Which was proposed.

By the same reasoning, if the first is incommensurable with the second in length only, the third also will be incommensurable with the fourth in length only. For if A and B are incommensurable in length only, their squares will be commensurable [in square] only; and therefore they themselves will be commensurable in square only. Therefore, as we have just shown, the lines C and D also will be commensurable in square only; and for that reason they will be incommensurable in length only. Which was proposed.

Again, if of four magnitudes the first two are lines and the remaining two are surfaces or solids, it will nonetheless follow that, if the lines are commensurable or incommensurable in length, the remaining two magnitudes are commensurable or incommensurable as well. For if the lines are commensurable in length, their ratio will be that of a number to a number, from what we taught in the scholium to Proposition 5 of this book. Since, then, the remaining two magnitudes have the same ratio as the lines, their ratio also will be that of a number to a number; [Margin: k — 6.deci., Elements X.6] and so they will be commensurable. But if the lines are incommensurable in length — whether or not they are commensurable in square — their ratio will not be that of a number to a number, as is clear from the scholium to Proposition 7 of this book. Therefore neither will the ratio of the remaining two magnitudes, which have the same ratio as those lines, be that of a number to a number. [Margin: l — 8.dec., Elements X.8] Hence they will be incommensurable. Which was proposed.

In the same way, if the first two magnitudes are surfaces or solids and the latter two are lines, we shall prove that, whether the planes or solids are commensurable or incommensurable, the lines are commensurable or incommensurable in length. For if the planes or solids are commensurable, [Margin: m — 5.deci., Elements X.5] they themselves will have the ratio of a number to a number. Therefore the lines too, having the same ratio as they, will have the ratio of a number to a number; and consequently they will be commensurable in length, as we said in the scholium to Proposition 6 of this book. But if the planes or solids are incommensurable, they will not have the ratio of a number to a number. Therefore neither will the lines, having the same ratio as they, have the ratio of a number to a number. So they will be incommensurable in length, by what we taught in the scholium to Proposition 8 of this book. Which was proposed.

LEMMA.

To find two plane numbers that do not have the ratio of a square number to a square number.

Let two dissimilar plane numbers be found, by what we taught toward the end of Book VIII. For numbers of this kind will not have the ratio of a square number to a square number, as we demonstrated in the scholium to Proposition 26 of Book VIII (*Elements* VIII.26, schol.). Which was proposed.

But if we wish to find several numbers, no two of which have the ratio of a square number to a square number, we shall take as many prime numbers as we please — by what we set forth in the scholium to Proposition 20 of Book IX —

A, 3. B, 5. C, 7. D, 11. E, 13.

namely A, B, C, D, E. For no two of these accepted primes have to one another — by the scholium to Proposition 26 of Book VIII — the ratio of square numbers to square numbers, since they are not similar plane numbers, as we taught toward the end of Book VIII.

SCHOLIUM.

Now, the method of finding dissimilar plane numbers — that is, numbers which do not have the ratio of a square number to a square number — some have handed down incorrectly at this place, among them [Federico Commandino]. To show this, we must set out the method by which they try to find the said numbers. They proceed thus:

A, 2. C, 3. B, 6. D, 16. E, 12. F, 48.

Let four numbers A, B, C, D be set out such that A is not to C as B is to D; and let the number E be made from A and B [$E = A \times B$], and the number F from C and D [$F = C \times D$]. It is then clear that E and F are plane numbers, but dissimilar plane numbers, since their sides are not proportional. Which was required.

This is their method of finding dissimilar plane numbers. But such interpreters are mistaken, because very often the numbers found by their method are similar plane numbers. For the numbers found by their demonstration, 12 and 48, are similar plane numbers, since they have the ratio that the square numbers 4 and 16 have. And likewise it is agreed that the former number has sides 3 and 4, proportional to the sides 6 and 8 of the latter — even though the sides of the former that they assumed, 2 and 6, are not proportional to the sides 3 and 16 of the latter that they took. For it is enough, in order that two plane numbers be similar, that some two sides of the one be proportional to some two sides of the other; it is not required that any two sides whatever of the one be proportional to any two sides whatever of the other. On this matter we have written more at length at Definition 21 of Book VII and in the scholium to Proposition 23 of Book VIII.

PROBL. 3. PROPOS. 11.

[Margin: x.] [Margin: xj.]

Given a straight line, to find two straight lines incommensurable with it — the one in length only, the other in square as well.

Let A be the given line, for which a line incommensurable in length only is first to be found. By the preceding lemma, let two numbers B and C be found not having the ratio of a square number to a square number. Next, by the corollary to Proposition 6 of this book, let a line D be found such that the square on A is to the square on D as the number B is to the number C. I say that D is incommensurable with A in length only. That A and D are incommensurable in length will be proved thus. Since the square on A is to the square on D as the number B to the number C, and since the ratio of B to C is not that of a square number to a square number, neither will the square on A have to the square on D the ratio of a square number to a square number. [Margin: a — 9.dec., Elements X.9] Therefore the lines A and D are incommensurable in length. That they are incommensurable in length only is plain. For since the squares on A and D have the ratio that the number B has to the number C, [Margin: b — 6.dec., Elements

X.6] they will be commensurable. So the lines A and D are, by definition, commensurable in square. Hence they are incommensurable in length only.

Diagram. Three vertical line-segments of differing heights labelled, left to right, **A, E, D** — A being the given line, E the mean proportional, and D the line found. To the right stand the numeral labels **B, 10.** and **C, 6.** (the numbers $B = 10$ and $C = 6$, which do not have a square-to-square ratio).

Now, for the same given line A, a line incommensurable in both length and square is to be found. [Margin: c — 13.sex., Elements VI.13] Let a line E be found as the mean proportional between the lines A and D, which are incommensurable in length only. I say that E is incommensurable with A in both length and square. For since the square on A is to the square on E — by the corollary to Proposition 20 of Book VI (*Elements* VI.20, Cor.) — as A is to D, and since A is incommensurable in length with D, as has just been shown, [Margin: d — 10.dec., Elements X.10] the square on A also will be incommensurable with the square on E. Therefore the lines A and E are, by definition, incommensurable in square; and consequently, by the corollary to Proposition 9 of this book, they are altogether incommensurable in length as well. Therefore E is incommensurable with A in both length and square. And so, for the given straight line, we have found two straight lines, and so on. Which was to be demonstrated.

SCHOLIUM.

If, then, the given line A is set down as Rational, so that the measures of the others are taken from it, the line D also will be Rational, by Definition 6, since it is commensurable with A in square, even though it is incommensurable with it in length. But E is Irrational, by Definition 7, since it is incommensurable with the Rational line A in both length and square.

Moreover, from the demonstration of this theorem it is clearly established that a straight line which is the mean proportional between two straight lines...

Book X (continued)

Scholium to Proposition 11 (continued)

[a line which is the mean proportional between two lines] commensurable in power only — or, what amounts to the same thing, between two lines incommensurable in length only — is incommensurable with either of them in both length and power; and so it is called Irrational, if either of the two is taken to be Rational. For from the fact that the lines A and D are incommensurable in

length only — that is, commensurable in power only — we demonstrated that the line E, the mean proportional between them, is incommensurable with [A] in both length and power. And by the same argument we shall show that this same line is incommensurable with D itself in both length and power. For since the lines D, E, A are in continued proportion, by the corollary of Proposition 20 of Book VI it follows that as the square on D is to the square on E, so is the line D to the line A. But the line D is incommensurable in length with the line A. Therefore the square on D will also be incommensurable with the square on E. Hence the lines D and E, by definition [Margin: e. Elem. X.10 [?]], are incommensurable in power, and consequently also in length, by the corollary of Proposition 9 of this book.

Diagram. Three vertical straight-line segments of unequal length, drawn side by side. Read left to right at their feet:

- A: the longest (leftmost) segment
- E: the middle segment
- D: the shortest (rightmost) segment
- To the right of the figure, two numeric annotations: **B, 10** and **G, 6** (reference quantities used in the worked example).

In the common manuscripts this eleventh proposition is placed by Theon before the tenth, which precedes it; I judge that this was done through a copyists' error, since the second part of this [proposition] is demonstrated from the preceding tenth proposition, as is clear from what has been said.

THEOR. 9. PROPOS. 12.

[Margin: xij. / viij.]

Magnitudes commensurable with the same magnitude are also commensurable with one another.

Let each of the magnitudes A and B be commensurable with the magnitude C. I say that A and B are also commensurable with one another. For since A and C are commensurable, [Margin: a. Elem. X.5] A will have to C a ratio that a number has to a number: let it have the ratio that the number D has to the number E. Again, since C and B are commensurable, C will likewise have to B a ratio that a number has to a number: let it have the ratio that the number F has to the number G.

[Margin: b. Elem. VIII.4] Let three least numbers H, I, K be taken, in continued proportion in the ratios of D to E and of F to G; and let H be to I as D is to E — that is, as A is to C — and let I be to K as F is to G — that is, as C is to B. Since, then, A is

to C as H is to I, and C is to B as I is to K, it will be, from equality [ex æquo], that A is to B as H is to K, a number to a number. Therefore A and B, having the ratio of the numbers H and K, [Margin: c. Elem. X.6] are commensurable with one another. Therefore magnitudes commensurable with the same magnitude, etc. Which was to be demonstrated.

Diagram. Three vertical straight-line segments of unequal length, drawn side by side. Read left to right at their feet:

- A: the leftmost segment
- C: the middle segment
- B: the rightmost segment
- To the right, the numeric values of the worked example, in two columns: D, 10 · F, 2 · H, 5 and E, 8 · G, 3 · K, 6. (The number I is not assigned a value in the figure.)

SCHOLIVM.

But if the magnitudes A, B, C are lines, and A and B are commensurable in length with C, then A and B will also be commensurable in length with one another, as is plain from the demonstration of the theorem — for in the same way we shall show that the lines A and B have a ratio that the number H has to the number K; hence they will be commensurable in length, as we said at Proposition 6 [of this book]. But if A and B are commensurable in power with C — even though they are incommensurable in length with it — it will be shown, in the same way, that A and B are also commensurable in power with one another. For [Margin: d. Elem. X.5] the square on A will be to the square on C as one number to another — namely, as D to E — since the squares on A and C are, by definition, commensurable. In the same way the square on C will be to the square on B as one number to another — namely, as F to G. [Margin: e. Elem. VIII.4] Therefore, three least numbers H, I, K being taken in continued proportion in these same ratios of D to E and of F to G, it will again follow, from equality, that the square on A is to the square on B as the number H is to the number K. [Margin: f. Elem. X.6] Therefore the squares on A and B are commensurable, and hence the lines A and B are commensurable in power. It does not, however, follow from this that A and B are commensurable in power only. For there can be two lines commensurable in length with one another, even though each of them is commensurable in power only with some third line; such are two Rational lines commensurable in length with one another, but commensurable only in power with the assigned Rational line — lines that we shall in fact find later, in the second scholium to Proposition 19 of this book.

But we shall also demonstrate, in a certain way, the converse of this theorem, as follows.

If there be two commensurable magnitudes, and one of them be commensurable with some [third] magnitude, then the remaining one will also be commensurable with the same.

Diagram. Three horizontal dashed straight-line segments of unequal length, drawn one above another. Read top to bottom:

- A: the topmost (short) dashed segment
- C: the middle dashed segment
- B: the bottom (longest) dashed segment

Let A and B be commensurable, and let A be commensurable with C. I say that B too is commensurable with the same C. For since both B and C are commensurable with A, then B and C will also, by this theorem, be commensurable with one another. Which is what was proposed.

But if the magnitudes A, B, C are lines, and A, B are commensurable in length, while A is commensurable in length with C, then B, the remaining line, will also be commensurable in length with C. For since both B and C are commensurable in length with A, then B and C will likewise be commensurable in length with one another, as we showed. But if the lines A and B are commensurable in power only, and A is commensurable with C — whether in length and power, or in power only — we shall indeed conclude, by the same argument, that B, the remaining line, is commensurable in power with C, since on this reckoning each of B and C is commensurable in power with A by hypothesis. It will not, however, be permissible to conclude, when A is commensurable in length with C, that B, the remaining line, is commensurable in length with the same C, because B and C are not both commensurable in length with A — but rather C is so in length, while B is so in power only, as is established from the hypothesis.

THEOR. 10. PROPOS. 13.

[Margin: xij. [?] / ○ [?]]

If there be two magnitudes, and one of them be commensurable with some [magnitude] while the other is incommensurable [with it], the [two] magnitudes will be incommensurable [with one another].

Let there be two magnitudes A and B, of which A is commensurable with C while B is incommensurable with the same C. I say that A and B are incommensurable

with one another. For if B were said to be commensurable with A, then — since C is posited commensurable with the same A — [Margin: a. Elem. X.12] B and C would be commensurable with one another, which is absurd, since B is posited incommensurable with C. Therefore B is not commensurable with A. And so, if there be two magnitudes, and one of them, etc. Which was to be shown.

Diagram. Three vertical straight-line segments drawn side by side. Read left to right at their feet:

- A: the leftmost segment
- C: the middle segment
- B: the rightmost segment

SCHOLIVM.

If the magnitudes A, B, C are lines, and A is commensurable in length with C while B is incommensurable in length with the same C, then A and B will be incommensurable in length. For if they were commensurable in length, [Margin: b. Elem. X.12] we should also show that B and C are commensurable in length with one another (since, on this assumption, both B and C would be commensurable in length with A) — which is not posited. By the same reasoning, if A is commensurable in power with C — whether in power only, or also in length — while B is incommensurable in power with the same C, then A and B will be incommensurable in power. For if they were commensurable in power, B and C would likewise be commensurable in power (since, on this assumption, both B and C would be commensurable in power with A). Which is absurd; for B is posited incommensurable in power with C.

But since this proposition 13 is, in Theon, a lemma, it comes about that hereafter, in our edition, the numbering of Euclid's propositions differs from the numbering that Theon follows. For we have deliberately kept Theon's order and sequence at this place, because this theorem has now been received into the number of Euclid's propositions by the more recent and more skilled geometers.

THEOR. 11. PROPOS. 14.

[Margin: xiiij. / viij.]

If there be two commensurable magnitudes, and one of them be incommensurable with some magnitude, then the other will also be incommensurable with the same.

Let there be two commensurable magnitudes A and B, of which A is incommensurable with some other magnitude C. I say that B too is incommensurable with the same C. For if B were said to be commensurable with C — and therefore C with B — then, since A is posited commensurable with B, [Margin: a. Elem. X.12] A and C would be commensurable with one another. Which is absurd; for A is posited incommensurable with C. Therefore B is not commensurable with C. And so, if there be two commensurable magnitudes, etc. Which was to be demonstrated.

Diagram. Three vertical straight-line segments drawn side by side, A and B marked off with common divisions [tick-marks] to show their commensurability. Read left to right at their feet:

- A: the leftmost segment
- C: the middle segment
- B: the rightmost segment [printed as a rotated/malformed glyph resembling "3"; recte: B]

SCHOLIUM.

Here too, if the magnitudes A, B, C are lines, and A, B are commensurable in length, while A is incommensurable with C — whether in length only, or in both length and power — then B too will be incommensurable in length with C. For if C and B were commensurable in length,

[if C and B] were commensurable in length, [Margin: b. Elem. X.12] then A and C would likewise be commensurable in length (since, on this assumption, both A and C would be commensurable in length with B). Which is absurd; for A is posited incommensurable in length with C. But if A and B are commensurable in power only, and A is incommensurable in power with C, we shall conclude in the same way that B too is incommensurable in power with C. Otherwise, if B and C were commensurable in power, [Margin: c. Elem. X.12] then A and C would be commensurable in power (since, on this assumption, both A and C would be commensurable in power with B). Which is not posited.

Furthermore, from this theorem the interpreters of Euclid derive the following theorem, very useful for the things demonstrated in this book.

Magnitudes commensurable with incommensurable magnitudes will also be incommensurable with one another.

Let there be two incommensurable magnitudes A, B, with which C and D are commensurable — namely, C with A and D with B. I say that C and D are

incommensurable with one another. For since A and C are commensurable, [Margin: d. Elem. X.14] C, the remaining magnitude, will also be incommensurable with B. Again, since D and B are posited commensurable, and B has been shown incommensurable with C, [Margin: e. Elem. X.14] the remaining magnitude will also be incommensurable with the same C. Which is what was proposed.

But if the magnitudes A, B, C, D are lines, and A, B are incommensurable — whether in length only, or in both length and power — while C, D are commensurable in length with A, B respectively, then C and D will, by the same argument, be incommensurable in length. But if A and B are incommensurable in power, and C, D are commensurable in power only with them, we shall likewise show that C and D are incommensurable in power; as is evident.

Diagram. Four vertical straight-line segments of unequal length drawn side by side. Read left to right at their feet:

- C: the leftmost segment (commensurable with A)
- A: the second segment
- B: the third segment
- D: the rightmost segment (commensurable with B)

LEMMA.

Given two unequal straight lines, to find that by which the greater is more powerful than the lesser [i.e., the side of the square by which the square on the greater exceeds the square on the lesser].

Although we have shown before, at Proposition 47 of Book I, what this lemma proposes, nevertheless — because here it is demonstrated more briefly from what has been handed down in Books III and IV — we have judged it not useless to accomplish the same thing in another way at this place. Let there then be given two unequal straight lines AB and C, of which AB is the greater; and let it be required to find that by which AB is more powerful than C. Let AB be bisected at D, and from center D, with radius DA or DB, let the semicircle AEB be described; [Margin: f. Elem. IV.1] in it let the straight line AE be fitted equal to C, and let the straight line EB be joined. I say that AB is more powerful than C by the square on the line EB. [Margin: g. Elem. III.31] For since the angle E, in the semicircle, is a right angle, [Margin: h. Elem. I.47] the square on AB will be equal to the squares on AE and EB; and therefore AB will be more powerful than AE — that is, than C — by the square on the line EB. Which is what was proposed. In like manner:

Given two straight lines, whether equal or unequal, to find the straight line whose square equals [the sum of the squares on] them.

This too we have set forth at Proposition 47 of Book I. Let there then be, in the same figure, two given straight lines AE, EB, which, if they be joined at the right angle E, the straight line AB drawn [across] will be able to [subtend] them, [Margin: i. Elem. I.47] inasmuch as the square on AB is equal to the squares on AE and EB.

Diagram. A semicircle standing on the diameter AB, with a triangle inscribed and a separate line beneath.

- A: left end of the horizontal diameter
- D: midpoint of the diameter (center of the semicircle)
- B: right end of the diameter
- E: apex on the arc, where the chords AE and EB meet at a right angle [the printed apex glyph is faint/ambiguous — read as E to agree with the demonstration; possible]
- The chords AE (left) and EB (right) are drawn, forming triangle AEB; the angle at E is right
- C: a separate horizontal straight line drawn below the figure (the given line to which AE is fitted equal)

THEOR. 12. PROPOS. 15.

[Margin: xiv. / xij.]

If four straight lines be proportional, and the square on the first exceed the square on the second by the square on a straight line commensurable in length with the first, then the square on the third will exceed the square on the fourth by the square on a straight line commensurable in length with the third. But if the square on the first exceed the square on the second by the square on a straight line incommensurable in length with the first, then the square on the third will exceed the square on the fourth by the square on a straight line incommensurable in length with the third.

Let there be four proportional straight lines A, B, C, D, and let the first, A, be more powerful than the second, B, by the square on the line E; and let the third, C, be more powerful than the fourth, D, by the square on the line F. I say that, if E be commensurable in length with A, then F too is commensurable in length with C; but if E be incommensurable in length with A, then F too is incommensurable in

length with C. For since A is to B as C is to D, [Margin: a. Elem. VI.22] it will also be that as the square on A is to the square on B, so is the square on C to the square on D. But the square on A is equal to the squares on B and E, by hypothesis; and the square on C is equal to the squares on D and F. Therefore it will also be that as the squares on B and E are to the square on B, so are the squares on D and F to the square on D. And, by separation [dividendo], as the square on E is to the square on B...

Diagram. Six straight vertical line-segments of differing lengths, drawn side by side and lettered along the base. From left to right :

- A: a tall segment (the first of the four proportional lines)
- B: a shorter segment (the second proportional line)
- E: a short segment (the line by whose square A is more powerful than B)
- C: a tall segment (the third proportional line)
- D: a short segment (the fourth proportional line)
- F: a medium segment (the line by whose square C is more powerful than D)

(This figure belongs to the demonstration of the square-difference proportionality theorem — Euclid X.14 — carried over from the preceding page.)

so that, as the square on E is to the square on B, so is the square on F to the square on D. And therefore it will also hold that, as the line E is to the line B, so the line F to the line D [Margin: b. 22.sextri = *Elements* VI.22]; and by inversion, as B is to E, so D to F. Since, then, as A is to B, so C to D, and as B to E, so D to F, it follows also, *ex aequali*, that as A is to E, so C to F. Therefore, if A is commensurable in length with E, then C too will be commensurable in length with F; and if A is incommensurable in length with E, then C too will be incommensurable in length with F, as we set forth in the scholium of proposition 10 of this book. Therefore, if four straight lines be proportional, and so on. Which was to be demonstrated.

SCHOLIUM

By the same argument we shall show that, if E be commensurable in power only with A, then F is commensurable in power only with C, as is clear from what we wrote in the scholium of proposition 10 of this book.

THEOR. 13. PROPOS. 16.

[Margin: xv. / ix.]

If two commensurable magnitudes be added together, the whole magnitude will also be commensurable with each of them. And if the whole magnitude be commensurable with one of them, then the original magnitudes will also be commensurable.

Diagram. A tall vertical line divided into segments, with a short separate segment beside it representing the common measure :

- C: top endpoint of the whole line
- B: a division point on the line
- A: bottom endpoint (so the whole line is A–C, composed of the parts AB and BC)
- D: a short separate segment to the left, the common measure

Let two commensurable magnitudes AB, BC be added together. I say that the whole magnitude AC too is commensurable with each of them, AB and BC. For let D be the common measure of AB, BC [Margin: a. 13.dec. = *Elements* X.13]. Since, then, D measures both AB and BC, D will also measure the whole AC composed of them [Margin: b. 1.pron. = Axiom 1 of Book X]. Since, therefore, D measures AC and AB, AC and AB will be commensurable; and likewise, since D measures AC and BC, AB and BC too will be commensurable. And accordingly AC will be commensurable with each of them, AB and BC.

But now let the whole AC, composed of AB, BC, be commensurable with one of them, namely with AB. I say that AB and BC too are commensurable with each other. For let D be the common measure of AC, AB [Margin: c. 3.dec. = *Elements* X.3]. Since, then, D measures the whole AC and the subtracted AB, D will also measure the remainder BC [Margin: d. 9.pro. = *Elements* X.9]. Therefore, since D measures both AB and BC, AB and BC will be commensurable with each other. By the same argument we shall show AB, BC to be commensurable if AC be commensurable with BC. Therefore, if two commensurable magnitudes be added together, and so on. Which was to be shown.

COROLLARIVM

From this it follows that, if a whole magnitude composed of two parts be commensurable with one of them, it is also commensurable with the remaining part. Thus, if AC be commensurable with AB, then AC is also commensurable with the remaining BC. For, as was shown in the latter part of the theorem, D, the common measure of the whole AC and the subtracted AB, since it measures the

whole AC and the subtracted AB, will also measure the remainder BC, by the 3rd axiom. Therefore AC, BC are commensurable. Which is what was proposed.

THEOR. 14. PROPOS. 17.

[Margin: xvj. / ix.]

If two incommensurable magnitudes be added together, the whole magnitude will also be incommensurable with each of them. And if the whole magnitude be incommensurable with one of them, then the original magnitudes will also be incommensurable.

Diagram. A tall vertical line divided into segments, with a short separate segment beside it for the assumed common measure :

- C: top endpoint of the whole line
- B: a division point on the line
- A: bottom endpoint (the whole line A–C is composed of AB and BC)
- D: a short separate segment, the assumed (impossible) common measure

Let two incommensurable magnitudes AB, BC be added together. I say that the whole magnitude AC too is incommensurable with each of them, AB and BC. For if it is not incommensurable with each, let it be commensurable with one of them, if that be possible, namely with AB; and let D be the common measure of AC, AB [Margin: a. 3.dec. = *Elements* X.3]. Since, then, D measures the whole AC and the subtracted AB, D will also measure the remainder BC [Margin: b. 3.pron. = Axiom 3 of Book X]. And therefore, since D measures both AB and BC, AB and BC will be commensurable. But this is absurd, for they were assumed incommensurable. Therefore AC is not commensurable with AB; and by the same reasoning it will not be commensurable with BC either. Therefore it is incommensurable with each of them, AB and BC.

But now let the whole AC, composed of AB, BC, be incommensurable with one of them — that is, with one of the components, namely with AC [recte: AB]. I say that AB, BC too are incommensurable. For if they were commensurable, the whole AC would also be commensurable with each of them, AB and BC [Margin: c. 16.dec. = *Elements* X.16]. But this is absurd; for AC is assumed incommensurable with one of them, namely with AB. Therefore AB, BC are not commensurable. By the same argument we shall show AB, BC to be incommensurable if AC be incommensurable with BC. Therefore, if two incommensurable magnitudes be added together, and so on. Which was to be shown.

COROLLARIVM

It follows from these results that, if a whole magnitude composed of two parts be incommensurable with one of them, it is also incommensurable with the remaining part. Namely, if AC be incommensurable with AB, then AC is also incommensurable with the remaining BC. For if AC were commensurable with BC, then AC would also be commensurable with the remaining AB, by the corollary of the preceding proposition. But this is absurd, since it is assumed incommensurable. Therefore AC is not commensurable with BC. Which is what was proposed.

SCHOLIVM

It is clear that, in the two immediately preceding propositions and their corollaries, if lines are in question, the lines must be understood as commensurable or incommensurable **in length**. For in the demonstration of each proposition a common measure D is assumed, and so on; and only lines commensurable in length with one another possess such a measure, as is clear from the definition.

LEMMA I

If to some straight line there be applied a parallelogram deficient by a square figure, the applied parallelogram is equal to the rectangle contained by the segments of the straight line made by the application.

We demonstrated this lemma in the scholium of proposition 28 of book 6.

LEMMA II

Given two unequal straight lines, to apply to the greater the fourth part of the square described on the lesser, so that the applied figure be deficient by a square.

Although what is proposed in this lemma can be accomplished from proposition 28 of book 6 — since the fourth part of the square described on the lesser line is less than the [figure] applied to half of the greater line, deficient by a square figure, as that proposition 28 of book 6 requires; that is, the square described on half of the lesser line (which is a fourth part of the square described on the whole [lesser line], by the scholium of proposition 4 of book 2) [is less than the figure] described on half of the greater line (which is applied to the half of the greater line, and is deficient by a square figure). Although, I say, this can be accomplished

by proposition 28 of book 6, I prefer nonetheless to carry out this very thing here, together with other commentators, by another and easier method.

Let there be given, then, two unequal straight lines AB, CD, of which AB is the greater; and let it be required to apply to AB a parallelogram deficient by a square figure, this parallelogram being equal to the fourth part of the square described on the line CD. Let the lines AB, CD be bisected at E and F; and from center E, with radius EA or EB, let the semicircle AGB be described; and from E let EG be drawn perpendicular to AB. And since the whole AB is assumed greater than the whole CD, AE — the half of AB, that is EG, which is equal to AE — will also be greater than CF, the half of CD. Then, EH being set equal to CF, let HI be drawn through H parallel to AB, and let IKL be dropped perpendicular to AB, and let KL be equal to KB; and finally let the rectangle AL and the square BL be completed. I say that the parallelogram AL, applied to the line AB and deficient by the square figure BL, is equal to the fourth part of the square on CD.

For since KI is the mean proportional between AK, KB, as we set forth in the scholium of proposition 13 of book 6, the rectangle under AK, KB — that is, the rectangle AL — will be equal to the square on KI [Margin: d. 17.sexti = *Elements* VI.17], that is, to the square on EH [Margin: e. 34.primi = *Elements* I.34] (for KI is equal to EH, because of the parallelogram EI), that is, to the square on CF. But the square on CF is the fourth part of the square on CD, since the square on CD, by the scholium of proposition 4 of book 2, is quadruple the square on CF. Therefore the parallelogram AL, applied to the line AB and deficient by the square BL, is equal to the fourth part of the square described on CD. Which is what was proposed.

Diagram. A semicircle standing on a horizontal diameter, with an inscribed rectangle and a separate divided line-segment below :

- B: left endpoint of the diameter
- A: right endpoint of the diameter
- G: apex (top) of the semicircle, above the center
- E: center of the diameter BA (foot of the perpendicular radius EG)
- I: upper-left corner of the inscribed rectangle (on the semicircle)
- H: upper-right corner of the inscribed rectangle
- K: point on the diameter, foot of the perpendicular IK (dropped to the left of E)
- L: point below the diameter, on the extension of IK (with KL equal to KB)

- Separate lower segment (the lesser line CD, bisected): left endpoint D [?] (printed glyph unclear, resembles a barred *h*), midpoint F, right endpoint C

SCHOLIUM

From this lemma we shall also solve the following problem, in this manner:

To cut a given straight line so that the rectangle contained by the parts be equal to a given rectilinear figure — provided, however, that this figure be not greater than the square described on half the line.

For let AB be the given straight line, and any rectilinear figure to which the square on CF is set equal — this square being not greater than the square on the half AE, so that the side CF be not greater than the side AE. Then, the perpendicular EG having been erected on AB, let EH be equal to CF. Next, HI having been drawn parallel to AB, let IK be dropped perpendicular to AB. When this is done, we shall show, as before, that the rectangle under the parts AK, KB is equal to the square on IK, that is, to the square on EH, that is, to the square on CF. Which is what was proposed. Now we said that the given rectilinear figure ought not to be greater than the square described on half the line; because, if it were greater, the line CF would also be greater than the line EG. Then a line equal to CF could not be cut off from EG — which is nevertheless required for effecting the problem, as is clear from the demonstration.

LEMMA III

If there be two unequal straight lines, and to the greater there be applied the fourth part of the square described on the lesser, deficient by a square figure, the segments made by the application will not be equal.

Let there be two unequal straight lines AB and C; and to the greater, AB, let there be applied a parallelogram deficient by a square, this being equal to the fourth part of the square described on the lesser, C, in such a way that DB is the side of the square by which the applied parallelogram is deficient. I say that the segments AD, DB are unequal. For if the segments AD, DB were said to be equal — since, by the first lemma, the parallelogram applied to AB and deficient by the square figure described on DB is equal to the rectangle under AD, DB, while that which is contained under the equals AD, DB is a square — the said parallelogram would be the square described on AD, [which is] half of AB; and therefore, by the scholium of proposition 4 of book 2, the square on AB would be quadruple that parallelogram. But the square on C is also, by hypothesis, quadruple the same applied parallelogram, namely quadruple the fourth part of the square on C.

Therefore the squares on the lines AB and C are equal, and the lines AB and C themselves are equal. But this is absurd, for they are assumed unequal, with AB the greater. Therefore the segments AD, DB are unequal. Which is what was proposed.

This is also easily seen from the construction of the problem in the preceding lemma. For since, by the construction, EH was cut off equal to the half part of CD, namely to CF, and HI was drawn parallel to AB, and finally the perpendicular IK was dropped, so that the parts made by the application are AK, KB — it is clear that the segments AK, KB are unequal, since AB was bisected at E. Indeed, from this it is established that the first segment is the greater, and the latter — which is the side of the square by which the applied parallelogram is deficient — is the smaller.

Diagram. A vertical line divided at one point, with a shorter separate segment beside it :

- B: top endpoint of the greater line
- D: division point (so the greater line is divided into AD and DB)
- A: bottom endpoint of the greater line
- C: a shorter separate segment to the right (the lesser line)

THEOR. 15. PROPOS. 18.

[Margin: xvij. / xij.]

If there be two unequal straight lines, and if to the greater there be applied a parallelogram equal to the fourth part of the square [made] on the lesser, deficient by a square figure, and it divides [the greater] into parts commensurable in length: then the greater will be more powerful than the lesser by the square on a straight line commensurable in length with itself. And if the greater be more powerful than the lesser by the square on a straight line commensurable in length with itself, and if to the greater there be applied a parallelogram equal to the fourth part of the square [made] on the lesser, deficient by a square figure: then it will divide [the greater] into parts commensurable in length.

Diagram. A vertical line marked at several points, with a shorter separate segment beside it. From top to bottom on the main line:

- B: top endpoint of the greater line AB
- D: division point (with AD the greater segment, DB the lesser)

- E: bisection point (midpoint) of AB
- F: point taken so that EF equals ED
- A: bottom endpoint
- C: a shorter separate segment to the right (the lesser line)

Let there be two unequal straight lines AB and C, of which AB is the greater; and, by the second lemma of the preceding proposition, let there be applied to AB a parallelogram equal to the fourth part of the square described on the lesser, C, and deficient by a square figure, contained under AD, DB, and let it make the segments AD, DB commensurable in length. I say that AB is more powerful than C by the square on a line commensurable in length with itself.

For since, by the third lemma of the preceding proposition, AD is greater than DB, let AB be bisected at E, and let EF be taken equal to ED. This done, since the wholes EA, EB are equal, and the subtracted EF, ED are equal too, the remainders AF, BD will also be equal. And since the line AB is bisected at E but not bisected at D, the rectangle under AD, DB together with the square on ED will be equal to the square on EB [Margin: a. 5.sec. = *Elements* II.5]; and therefore quadruple the rectangle under AD, DB, together with [quadruple] the square on ED, will be equal to quadruple the square on EB. But the square on C is equal to quadruple the rectangle under AD, DB (for the rectangle under AD, DB is assumed equal to the fourth part of the square on C); and the square on FD is equal to quadruple the square on ED (for the square on FD is, by the scholium of proposition 4 of book 2, quadruple the square on ED, since FD is double ED); and the square on AB is equal to quadruple the square on EB (which square on AB is, by the scholium of proposition 4 of book 2, quadruple the square on EB, EB being half of AB).

Therefore the squares on C and FD together are equal to the square on AB. And accordingly the line AB is more powerful than the line C by the square on the line FD. This FD, then, must be shown to be commensurable in length with AB. Since the lines AD, DB are assumed commensurable in length, the whole AB will also be commensurable in length with the part DB [Margin: b. 16.dec. = *Elements* X.16]. But the [line] composed of DB, AF is also commensurable in length with DB (since the [line] composed of DB, AF is double DB). Therefore, since both AB and the [line] composed of DB, AF are commensurable in length with DB, AB and the [line] composed of DB, AF will also be commensurable in length with each other [Margin: c. 12.dec. = *Elements* X.12]. And accordingly, since AB [and] the [line] composed of AF, DB...

Euclid, Book X (Clavius's commentary) — irrational magnitudes

(Global printed folios 415–417. Running headers: recto "LIBER X.", verso "EVCLID. GEOM.")

[Conclusion of Proposition 18 — "more powerful by a commensurable square"]

(Continued from the previous page; the direct demonstration is being completed.)

of DB, taken as one, and of FD, is commensurable in length with the composite of AF and DB. Therefore, by the corollary to Proposition 16 of this book, AB itself is commensurable in length with the remaining part FD. Hence AB is more powerful than C by the square on the straight line FD, which is commensurable in length with AB.

[Converse.] Now let AB be more powerful than C by the square on a straight line commensurable in length with itself, and let a parallelogram equal to the fourth part of the square on C be applied to AB, deficient by a square figure, making the segments AD and DB. I say that the straight lines AD and DB are also commensurable in length. For with the same construction we shall likewise show that AB is more powerful than C by the square on FD. Therefore, since AB is assumed to be more powerful than C by the square on a straight line commensurable in length with itself, the straight line AB is commensurable in length with FD. And since the whole AB — composed of FD and of AF, DB (taken as one) — is commensurable in length with FD, AB is by the corollary to Proposition 16 of this book also commensurable in length with the remaining composite of AF and DB. But the composite of AF and DB is commensurable in length with DB, since it is double of it. Therefore, since each of AB and DB is commensurable in length with the composite of AF and DB, AB and DB are also commensurable in length with each other [Margin: d. 12.dec. (*Elements* X.12)]. And accordingly, since the whole AB, composed of AD and DB, is commensurable in length with DB, the lines AD and DB are likewise commensurable in length with each other [Margin: e. 16.deci. (*Elements* X.16)]. Therefore, if there be two unequal straight lines, and so on. Which was to be demonstrated.

THEOR. 16. PROPOS. 19.

[Margin (divergent proposition numbering, as printed): xviii. / xiiij.]

If there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, which divides the greater into parts incommensurable in length, then the greater is more powerful than the less by the square on a straight line incommensurable in length with itself. And if the greater is more powerful than the less by the square on a straight line incommensurable in length with itself, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, then it divides the greater into parts incommensurable in length.

[Demonstration — direct.] Let there be two unequal straight lines AB and C, of which AB is the greater. Let a parallelogram equal to the fourth part of the square on C be applied to AB — by the second lemma of the preceding proposition — deficient by a square figure, so that it makes on the straight line AB the parts AD and DB, incommensurable in length. I say that AB is more powerful than C by the square on a straight line incommensurable in length with itself. For, with the same things constructed as above, we shall likewise show that AB is more powerful than C by the square on FD. It must therefore be demonstrated that this FD is incommensurable in length with AB[?]. Since the straight lines AD and DB are assumed to be incommensurable in length, the whole AB is also incommensurable in length with the part DB [Margin: a. 17.dec. (*Elements* X.17)]. But DB is commensurable in length with the composite of DB and AF, since this composite is double of DB. Therefore, of the two commensurable lines — DB and the composite of DB and AF — since DB is incommensurable in length with the straight line AB, the remaining composite of DB and AF is also incommensurable in length with AB [Margin: b. 14.dec. (*Elements* X.14)]. And on that account, since AB — composed of AF, DB (taken as one) and of FD — is incommensurable in length with the composite of AF and DB, AB is by the corollary to Proposition 17 of this book also incommensurable in length with the remaining part FD. Therefore AB is more powerful than C by the square on FD, which is incommensurable in length with itself.

Diagram. A tall vertical straight line AB (A at the foot, B at the top), divided by the intermediate points D, E, F (in order from the top: B, D, E, F, A); beside it, to the right, a separate shorter vertical straight line C. The figure illustrates AB cut at D into the parts AD, DB, with E the midpoint and F the further division point, and C the second given line.

- B (top of the left line): endpoint of AB

- D (upper division): point of division of AB into AD, DB
- E (middle): bisection point of AB
- F (lower division): auxiliary point ($EF = ED$)
- A (foot of the left line): endpoint of AB
- C (foot of the right line): the second, lesser given line

[Demonstration — converse.] Now let AB be more powerful than C by the square on a straight line incommensurable in length with itself, and let a parallelogram equal to the fourth part of the square on C be applied to AB, deficient by a square figure, making on the straight line AB the parts AD and DB. I say that the straight lines AD and DB are also incommensurable in length. For, with the same things constructed, we shall show as before that AB is more powerful than C by the square on FD. Therefore, since AB is assumed to be more powerful than C by the square on a straight line incommensurable in length with itself, AB is incommensurable in length with FD. And since the whole AB — composed of FD and of AF, DB (taken as one) — is incommensurable in length with FD, AB is by the corollary to Proposition 17 of this book also incommensurable in length with the remaining composite of AF and DB. But the composite of AF and DB is commensurable in length with DB, since the former is double of the latter. Therefore, of the two commensurable lines — namely the composite of AF and DB, and DB — since the composite of AF and DB is incommensurable in length with the straight line AB, the remaining line DB is also incommensurable in length with AB [Margin: c. 14.dec. (*Elements* X.14)]. And accordingly, since the whole AB, composed of AD and DB, is incommensurable in length with DB, the lines AD and DB are themselves incommensurable in length with each other [Margin: d. 17.dec. (*Elements* X.17)]. For which reason, if there be two unequal straight lines, and to the fourth part of the square, and so on. Which was to be demonstrated.

SCHOLIUM

Euclid has thus far dealt with commensurable and incommensurable magnitudes; in what follows he passes to Rational and Medial lines.

LEMMA I.

Since it has been demonstrated that lines commensurable in length are altogether commensurable in square as well, whereas lines commensurable in square are not altogether commensurable in length also, but are only *able to be* commensurable in length as well — it is clear that, if some line be commensurable in length with the assigned Rational

line, that line is called Rational, and is commensurable with the assigned line not only in length but also in square; for lines commensurable in length are altogether commensurable in square too. But if some line be commensurable in square with the assigned Rational line — provided it be commensurable in length as well — it too will be called Rational, and commensurable with that line both in length and in square. And if some line, being again commensurable in square with the assigned Rational line, be incommensurable with it in length, it too will be called Rational, commensurable with it in square only.

LEMMA II.

From Proclus. He calls Rational those lines that are commensurable with the assigned Rational line either in length and in square, or in square only. But there are also other lines that are indeed incommensurable in length with the assigned Rational line, yet commensurable in square only; and for that reason these too are called Rational, and commensurable with one another insofar as they are Rational — commensurable, I mean, with one another either in length and in square, or in square only. And if indeed they are commensurable in length, they themselves are said to be Rational, commensurable with one another in length, it being understood that they are commensurable in square as well; but if they are commensurable with one another in square only, they themselves are also said to be Rational, commensurable with one another in square only.

SCHOLIVM

And so from these things it will be possible to gather three kinds of Rational lines commensurable in length with one another. For, of two Rational lines commensurable in length with one another, either one is equal to the assigned Rational line — and hence both are commensurable in length with the Rational line too; or neither is equal to the assigned Rational line, yet both are commensurable with it in length; or, finally, both are commensurable with the assigned Rational line in square only. These three kinds we shall find in the following way.

Let the Rational line A be assigned, divided into four parts — as many, that is, as there are units in the number B. Then, having taken any other number C, let D be one straight-line part of A; and however many times D measures the line A, let it measure just so many times another line E. Likewise, however many units there

are in C, let the same D measure just so many times another line F. Now since A and E are composed of parts equal in number, which are indeed equal in magnitude to D, they will be equal to one another. Again, since D measures all three — A, E, and F — the three A, E, and F will all be commensurable in length. Therefore E and F, being commensurable in length with the Rational line A, are Rational; and they have also been shown to be commensurable in length with one another, since they have D as a common measure. There have thus been found two Rational lines, E and F, commensurable in length both with one another and with the assigned Rational line A, and of which one, namely E, is equal to the assigned Rational line A.

[Margin: a. 11.dec. (*Elements* X.11)]

Diagram. Four vertical straight lines. At their feet, from left to right, the labels A, B, E, F. The leftmost line A is undivided; the adjacent line carries the interior markings D (the common part) and the point C, with the numerals 6 and 4 written beside it; the lines E and F are subdivided into equal parts. The figure represents the assigned line A divided by the measure D, and the derived commensurable lines. [The individual tick-subdivisions and the exact placement of the numerals 6 and 4 are not resolvable cell-by-cell at this resolution.]

- A (foot, leftmost): the assigned Rational line
- B (foot): the number of equal parts (shown as a divided line; numeral 4)
- D (interior mark): the common part / measure
- C (interior mark; numeral 6): the auxiliary number
- E (foot): line commensurable with A (equal to A)
- F (foot): line commensurable with A

Now let D measure two other lines C and E, by two numbers F and G, of which neither is the same as B, in such a way that each of the lines C and E is unequal to A. Then, as before, the three straight lines A, C, E — having the common measure D — will be commensurable in length. Therefore C and E, being commensurable in length with the Rational line A, are Rational. And since they are also commensurable in length with one another, there will have been found two Rational lines C and E, commensurable in length both with one another and with the assigned Rational line A, of which neither is equal to the assigned Rational line A.

Diagram. A group of horizontal straight lines, stacked, each subdivided into equal parts by tick marks. The visible left-hand labels are A, B, C, and a fourth line; the numeral 4 and the point D are marked on the second line. The figure

illustrates the second construction (the lines measured by the common part D through the numbers F, G). [The exact labelling and tick counts of these short scored segments are at the resolution floor; the fourth line's label is not legible.]

- A (top segment): line measured by D
- B (with numeral 4): auxiliary
- D: the common measure
- C: line commensurable with A
- [fourth segment: label illegible [?]]

Finally, with the Rational line A assigned, let there be found a straight line B incommensurable with it in length only [Margin: a. 11.dec. (*Elements* X.11)]; let B be cut into any number of equal parts, and let C be taken, composed of any other number of parts equal in magnitude to the parts of B. This done, B and C will be commensurable in length. I say that they are commensurable with the assigned Rational line A in square only. For since A and B are commensurable in square, the square on A is commensurable with the square on B. But the square on C is commensurable with the same square on B, because B and C are commensurable in length, and hence in square as well [Margin: b. 12.dec. (*Elements* X.12)].

Therefore the squares on the straight lines A and C are also commensurable with one another. Hence C is commensurable in square with A. And since, of the two straight lines B and C commensurable in length, B is incommensurable in length with A, the remaining line C is also incommensurable in length with A [Margin: c. 14.deci. (*Elements* X.14)]. Therefore C is commensurable with A in square only. And since B and C, being commensurable in square with the assigned Rational line A, are Rational, there will have been found two Rational lines B and C, commensurable in length indeed with one another, but commensurable with the assigned Rational line A in square only. Which is what was proposed.

Diagram. Three vertical straight lines, at whose feet stand the labels A, B, C. Line A (leftmost) is drawn solid and undivided; lines B and C are subdivided into equal parts by tick marks. The figure illustrates A (assigned Rational), B (incommensurable with A in length only), and C (composed of equal parts of B).

- A (foot, leftmost): the assigned Rational line
- B (foot, middle): incommensurable with A in length only
- C (foot, right): composed of parts equal to the parts of B

(Page 3 — global p. 429, printed folio 417, "LIBER X.")

And if anyone should wish to find any number of Rational lines commensurable in length with one another, he will accomplish it in this way. Taking any measure A, let there be composed any number of lines B, C, D out of as many parts equal to A as there are units in the same number of unequal numbers E, F, G. For the lines B, C, D, having the common measure A, will be commensurable in length.

Moreover, that all Rational lines are commensurable not only with the assigned Rational line but also with one another, we shall easily demonstrate in this way. Rational lines are those commensurable with the assigned Rational line either in length and in square, or in square only [Margin: d. 6.def. (*Elements* X, def. 6)]; and lines commensurable with the same line are commensurable with one another as well [Margin: e. 12.[?] (*Elements* X.12; the abbreviation appears to read "sexti," i.e. VI, but the argument invokes X.12)]. It is therefore clear that any Rational lines whatever are commensurable with one another.

Diagram. Four vertical straight lines, at whose feet stand the labels A, B, C, D (left to right). The leftmost, A, is the shortest (the common measure). Beside the second line stand the labelled numbers G 4, F 5, E 3; the lines B, C, D are subdivided into equal parts. The figure illustrates the lines B, C, D built from equal parts of the common measure A, according to the numbers E, F, G.

- A (foot, leftmost): the common measure
- B, C, D (feet): the Rational lines commensurable in length
- E (numeral 3), F (numeral 5), G (numeral 4): the unequal numbers giving the counts of parts

LEMMA III.

If there be two straight lines, then as the first is to the second, so is the square on the first to the rectangle contained by those two straight lines; and as the second is to the first, so is the rectangle contained by them to the square on the first.

[Demonstration.] Let there be two straight lines A and B. I say that as A is to B, so is the square on A to the rectangle contained by A and B. For from any straight line CD let CE be cut off equal to A, and EF equal to B. Then on CE let the square CG be described, and let the rectangle GF be completed, contained by CE and EF — that is, by A and B. Now since, as CE is to EF (that is, as A is to B), so is the parallelogram CG (that is, the square on A) to the parallelogram GF contained by A and B [Margin: f. (*Elements* VI.1) [?] — the printed reference is not clearly legible], it is evident that, if there be two straight lines, the first is to the second as the square described on the first is to the rectangle contained by them. In the

same way, as B is to A, so will the rectangle FG be to the square GC. Which is what was proposed.

Diagram. At the top, two horizontal reference lines, the upper labelled A and the shorter one below it labelled B. Below them, a straight line CD (C at the left, D at the right) marked at the interior points E and F, so that $CE = A$ and $EF = B$. On CE a square is described (its lower part reaching the point G), giving the square CG; on EF the rectangle GF is completed. The whole stands as a rectangle divided by verticals at E and F.

- A, B (top): the two given lines
- C (top-left of the figure): endpoint of the base line
- E: point with $CE = A$
- F: point with $EF = B$
- D (top-right): endpoint of the base line CD
- G (below): lower corner of the square CG / the completed rectangle GF

LEMMA IV.

A space commensurable with a Rational space is itself also Rational.

[Demonstration.] Let the space A be commensurable with the Rational space B. I say that A too is Rational. For let C be the Rational square, by reference to which the remaining spaces are called Rational or Irrational — the square, that is, described on the assigned Rational line. Now since B is Rational, it is commensurable with the Rational square C [Margin: g. 9.def. (*Elements* X, def. 9)]. But A is also commensurable with B. Therefore, since A and C are both commensurable with B, they will be commensurable with one another as well [Margin: h. 12.dec. (*Elements* X.12)]; and accordingly the space A, being commensurable with the Rational square C described on the assigned Rational line, is Rational [Margin: i. 6.def. (*Elements* X, def. 6)]. Which was to be demonstrated.

Diagram. Three rectangular areas set side by side, labelled A, B, C. A and B are tall, narrow rectangles; C (rightmost) is a larger square. The figure represents the space A, the Rational space B, and the Rational square C.

- A: the space in question
- B: the Rational space
- C: the Rational square described on the assigned Rational line

THEOR. 17. PROPOS. 20.

[Margin (divergent proposition numbering, as printed): xix. / xv.]

The rectangle contained by Rational straight lines commensurable in length, according to any of the aforesaid modes, is Rational.

[Demonstration.] Let the Rational line A be assigned, and let the rectangular space BD be contained by the Rational lines BC, CD, commensurable in length according to some one of the aforesaid modes — that is, according to some one of the three modes we explained in the second scholium of the preceding proposition — so that either one of the lines BC, CD is equal to the assigned Rational line A, or neither is, yet each is commensurable with it either in length or in square only. I say that the rectangle BD is Rational. For let there be described, on one of them, namely on BC, the square BE. Now since the Rational line BC is commensurable with the assigned Rational line A either in length and in square, or in square only, the square BE described on BC is also commensurable with the square on A [Margin: a. 3.defin. (*Elements* X, def. 3)]...

Diagram. At the top, a separate horizontal straight line labelled A (the assigned Rational line). Below it, a rectangle whose bottom edge runs from E (left) through C (middle) to D (right), with a vertical division rising from C to the point B on the top edge. The left cell is the square BE (described on BC), the right cell is the rectangle BD (contained by BC and CD).

- A (top): the assigned Rational line
- B (top, over C): upper endpoint of the dividing side BC
- E (bottom-left): far corner of the square BE
- C (bottom-middle): foot of the dividing side
- D (bottom-right): far corner of the rectangle BD

(The demonstration continues on the following page.)

Euclid, *Elements*, Book X (continued)

[End of PROPOS. 20 — the rational rectangle]

commensurable. [Margin: b — *Elements* X, def. 8] Now the square on A is rational — the standard by reference to which the remaining magnitudes are called rational or irrational. [Margin: c — *Elements* X, def. 9] Therefore BE too, being commensurable with it, is rational. And since BC — that is, EC — and CD are commensurable in length (for BC and CD were assumed to be rationals

commensurable with one another in length), [Margin: d — *Elements* VI.1] and as EC is to CD, so is EB to BD, [Margin: e — *Elements* X.10] EB and BD will also be commensurable. Hence, by lemma 4 of the preceding proposition, BD, being commensurable with the rational BE, is rational. Therefore what is contained by rational straight lines commensurable in length, and so on. Which was to be demonstrated.

THEOR. 18. PROPOS. 21.

[Margin: xx. / xvi.]

If a rational [area] is applied to a rational [line], it produces a rational breadth, commensurable in length with the line to which it is applied.

Let a rational line A be set out again, and another rational line BC — related to it in one of the ways we explained in the second scholium of proposition 19; that is, whether BC is equal to the rational line A set out or not, it is at any rate commensurable with it either in length and in square, or in square only. Let a rational [area] BD be applied to BC, producing the breadth CD. I say that CD is rational and commensurable in length with BC.

For let the square BE be described on BC; this we shall show to be rational, just as in the preceding proposition. Since, then, BE and BD are rational, [Margin: a — *Elements* X, def. 9] they will be commensurable with the square described on the rational line A that was set out; [Margin: b — *Elements* X.11] and therefore they will also be commensurable with one another. [Margin: c — *Elements* VI.1] Now as BE is to BD, so is EC — that is, BC — to CD. Therefore BC and CD too are commensurable in length, as we taught in the scholium of proposition 10 of this book. So CD is commensurable in length with BC; [Margin: d — *Elements* X, def. 6] and it is also rational, since it is commensurable with the rational BC, and hence with the rational line A that was set out, as we demonstrated in the scholium of proposition 12 of this book. Therefore the breadth CD is rational and commensurable in length with BC. So if a rational is applied to a rational, it produces a breadth, and so on. Which was to be demonstrated.

Diagram. A rational line A is drawn as a horizontal segment (with division ticks) at the top. Below it stands a large rectangle divided by a vertical line into a left square and a right rectangle.

- A (top): the rational line set out.
- B (top-centre): apex of the vertical dividing line; shared top corner of the square and the rectangle.

- E (bottom-left): far corner of the square BE described on BC.
- C (bottom-centre): foot of the dividing line (foot of BC).
- D (bottom-right): far corner of the applied rectangle BD, whose breadth is CD.

LEMMA I.

A straight line whose square is an irrational area is itself irrational.

Let the straight line A have as its square an irrational area — that is, let the square on A be equal to some irrational area. I say that A is irrational. For if it were called rational, its square would be rational too, as was shown in the demonstration of proposition 20 of this book. But that is absurd, since the area is assumed to be irrational. Therefore A is not rational; hence it is irrational.

The same thing is also clear from definition 11 of this book, where lines whose squares are irrational areas are called irrational.

Diagram. A single vertical straight line labelled A.

- A: the straight line whose square is an irrational area.

LEMMA II.

To find two rational straight lines commensurable in square only.

There are two kinds of rational lines commensurable with one another in square only: either one of them is equal to the rational line set out, or neither is. We shall find lines of the first kind as follows. Let a rational line A be set out, and let B be taken equal to it. Then let C be found [Margin: e — *Elements* X.11]

incommensurable in length only with B — or, what is the same, commensurable in square only. Since, then, B and C are commensurable with A (B indeed because it is equal to it; and C by construction, because it was found to be commensurable in square only with B, and hence with A), B and C will be rational; and they are also commensurable in square only. Two rationals B, C have therefore been found, one of which, namely B, is equal to the rational line A set out.

Diagram. Three vertical straight lines side by side, labelled A, B, C at their feet; A carries interior division ticks.

- A: the rational line set out.
- B: taken equal to A.
- C: found commensurable in square only with B.

We shall find lines of the second kind by this method. Let a rational line A be set out again, and let B be found [Margin: f — *Elements* X.11] incommensurable in length only with it; and again let C, greater or less than A, be found incommensurable in length only with B. I say that B and C are rational and commensurable in square only. That they are commensurable in square only is clear from the construction. That they are rational we shall show thus. Since A and C are commensurable in square with B, A and C will also be commensurable in square, as we demonstrated in the scholium of proposition 12 of this book. Therefore, since both B and C are commensurable in square with the rational line A set out, [Margin: g — *Elements* X, def. 6] B and C will be rational. So B and C have been found: rational and commensurable in square only.

Diagram. Three vertical straight lines side by side, labelled A, B, C at their feet, of increasing length (C tallest); A carries interior division ticks.

- A: the rational line set out.
- B: incommensurable in length only with A.
- C: incommensurable in length only with B, greater or less than A.

SCHOLIUM.

If any number whatever of rational lines commensurable with one another in square only are to be found, we shall carry it out in this way.

By what we taught in the scholium of proposition 20 of book 9, let as many prime numbers be taken as there are rational lines sought — namely A, B, C, D. Then, taking a rational line E, let it be arranged that as A is to B, so is the square on E to the square on F, as we showed in the corollary of proposition 6 of this book. Likewise, as B is to C, so let the square on F be to the square on G; and finally, as C is to D, so let the square on G be to the square on H. I say that the lines E, F, G, H are rational and commensurable with one another in square only.

That they are rational is evident. For since their squares have the ratio that the numbers A, B, C, D have (namely, the square on E to the square on F as the number A to the number B, by construction; likewise the square on F to the square on G as B to C, and the square on G to the square on H as C to D; while, *ex aequali*, the square on E is to the square on G as A to C; likewise the square on E to the square on H as A to D; and finally the square on F to the square on H as B to D), [Margin: h — *Elements* X.6] they will be commensurable with one another;

and therefore the lines themselves too will be commensurable at least in square. So, E being rational, the remaining lines F, G, H will also be rational.

That they are commensurable in square only we shall show thus. Since their squares are proportional to the prime numbers A, B, C, D, and prime numbers do not have to one another the ratio that square numbers have, as we taught at the end of book 8, the squares of the lines E, F, G, H likewise will not have the ratio that square numbers have. [Margin: i — *Elements* X.9] Therefore the lines E, F, G, H are incommensurable in length. But they were shown to be rational and commensurable in square. They are therefore rational and commensurable in square only. Which was what was proposed.

Diagram. Above the figure, a row of the four assigned prime numbers is printed under their letters: A = 5, B = 7, C = 3, D = 2. Below stand four vertical straight lines labelled E, F, G, H at their feet, of differing lengths (F tallest).

- A = 5, B = 7, C = 3, D = 2 (the four prime numbers).
- E, F, G, H: the four rational lines whose squares are in the ratios of these numbers.

If, given any number of rationals commensurable in square only, yet another is to be found commensurable in square only with all of them, [it can be done] in this way. Let two rationals commensurable in square only, A and B, be given, whose squares have the ratio that the number C has to the number D. Taking another number E prime to each of C and D (such a number is easily found if one takes a prime that measures neither C nor D), let it be arranged that as D is to E, so is the square on line B to the square on line F, by the corollary of proposition 6 of this book. I say that F is rational and commensurable in square only with A and B.

That it is rational is clear from this: it is commensurable, at least in square, with the rational B, since the squares of the lines B and F, having the ratio of the numbers D and E, [Margin: k — *Elements* X.8] are commensurable. Now since the ratio of D to E is not that of a square number to a square number — as we showed at the end of book 8 (because D and E, which are both non-squares — since E, being prime, is in no way a square — are prime to one another, and therefore not similar plane numbers, and consequently, by what we showed in the scholium of proposition 26 of book 8, do not have the ratio of a square to a square) — [Margin: l — *Elements* X.9] the lines B and F will be incommensurable in length. Therefore they are commensurable in square only.

By the same reasoning A and F will be commensurable in square only. For, *ex aequali*, the square on A will be to the square on F as the number C to the number E. Since, then, these numbers do not have the ratio of a square to a square (which we shall show just as we demonstrated the same for the numbers D and E), A and F will be incommensurable in length, and so on.

Diagram. Above the figure, a row of numbers is printed under their letters: C = 10, D = 6, E = 7. Below stand vertical straight lines labelled A, B, F at their feet.

- C = 10, D = 6, E = 7 (E prime to both C and D).
- A, B: the two rationals commensurable in square only that are given.
- F: the new rational found commensurable in square only with A and B.

THEOR. 19. PROPOS. 22.

[Margin: xxi. / xix.]

The rectangle contained by rational straight lines commensurable in square only is irrational; and the straight line whose square equals it is irrational. Let it be called **medial** [Media].

[p. 432 — *EVCLID. GEOM.*]

Let a rational line A be set out, and a rectangle BD contained by the rationals BC, CD commensurable in square only in one of the ways we described in lemma 2 of the preceding proposition (that is, whether one of them is equal to the rational line A set out, or not). I say that the rectangle BD is irrational, and that the straight line whose square is equal to BD is also irrational — let it be called medial.

For let the square BE be described on one of them, say on BC; this will be rational, as we said in the demonstration of proposition 20. [Margin: a — *Elements* VI.1] And since as EC — that is, BC — is to CD, so is BE to BD; but BC is incommensurable in length with CD, by hypothesis; then, by what we set forth in the scholium of proposition 10 of this book, BE will be incommensurable with BD. [Margin: b — *Elements* X, def. 9] Therefore, since BE is commensurable with the square on the rational line A set out, while BE is incommensurable with BD, [Margin: c — *Elements* X.14] the square on the rational line A set out is also incommensurable with BD. And so BD, being incommensurable with the square on the rational line set out, [Margin: d — *Elements* X, def. 10] is irrational. Wherefore the straight line whose square is BD is also irrational, by lemma 1 of the preceding proposition.

[Margin: e — *Elements* VI.17] Let this line whose square is BD be called **medial**, because it is the mean proportional between the two rationals BC, CD commensurable with one another in square only — since its square is equal to the rectangle BD contained by BC, CD. Therefore what is contained by rationals commensurable in square only, and so on. Which was to be demonstrated.

Diagram. A rational line A is drawn as a horizontal segment (with division ticks) at the top. Below it stands a large rectangle divided by a vertical line into a left square and a right rectangle.

- A (top): the rational line set out.
- B (top-centre): apex of the vertical dividing line; shared top corner of the square and the rectangle.
- E (bottom-left): far corner of the square BE described on BC.
- C (bottom-centre): foot of the dividing line (foot of BC).
- D (bottom-right): far corner of the rectangle BD contained by BC, CD.

SCHOLIUM.

Accordingly, every mean proportional line between two rationals commensurable with one another in square only will be called medial. [Margin: f — *Elements* VI.17] For since its square is equal to the rectangle contained by such rationals — which in this theorem was shown to be irrational — its side, namely the mean proportional between the said rationals, will be an irrational line, which is called medial. From this we may readily describe the medial line by saying that it is the irrational line which is the mean proportional between two rational lines commensurable with one another in square only; or the line whose square can produce the rectangle contained by two rationals commensurable with one another in square only. For this line alone is here called medial.

As Campanus rightly notes here, not only is the straight line whose square is the irrational rectangle contained by two rational lines commensurable in square only irrational and called medial; but also its square, or that rectangle, is called medial, because it is the mean proportional between the squares of those rational lines commensurable in square only, just as the line itself is the mean proportional between the said rationals. For if three lines are in continued proportion — as are BC, the irrational line called medial, and CD — then the rectilinear figures similar and similarly described on them, such as their squares, will also be proportional, as we demonstrated in the scholium of proposition 22 of book 6. Hence the square described on the medial is the mean proportional between the squares of the lines BC, CD, and can therefore be called medial.

But you must not understand this as though you were to think that every medial rectangle is contained by two rational lines commensurable in square only, as is the medial BD. For this is false, since a medial area can also be contained by two irrationals — namely medials commensurable with one another in length, or in square only — as will be established from propositions 25 and 26 of this book. Thus "rectangle contained by two rationals commensurable in square only" and "medial area" are not convertible. For every rectangle contained by two rationals commensurable in square only is medial, as we showed; but not every medial area is contained by two rationals commensurable in square only. Universally, however, every medial area is equal to some other medial contained by two rationals commensurable in square only. For otherwise the line whose square it is would not be called medial, since it could not be the rectangle contained by two rationals commensurable in square only; or it would not be the mean proportional between two rationals commensurable in square only.

Hence a medial can be described thus: let us say it is the rectangle contained by two rationals commensurable in square only; or, at any rate, the rectangle which is equal to some other rectangle contained by two rationals commensurable in square only, such that the straight line called medial in this theorem can produce it. For every medial not contained by two such rationals can be reduced to another medial whose sides are two rational lines commensurable in square only, as we shall show in the scholium of the following theorem.

THEOR. 20. PROPOS. 23.

[Margin: xxii. / xx.]

The square that arises from a medial line, applied to a rational, produces a rational breadth, incommensurable in length with the line to which it is applied.

Let A be a medial line, and BC a rational line — whether it be the one set out or one commensurable in length and square with the one set out. [Margin: a — *Elements* I.45] Let the rectangle BD, equal to the square on the medial A (or to a rectangle equal to that square), be applied to BC, producing the breadth CD. I say that CD is rational and incommensurable in length with BC. For since A is medial, it can produce a rectangle contained by two straight rationals commensurable with one another in square only — otherwise it could not be called medial. Let that rectangle, then, be EG, contained by the rationals EF, FG commensurable in square only.

Diagram. Two rectangles stand side by side, with the medial line A drawn as a short horizontal segment (with ticks) above and between them.

- A (top, centre-right): the medial line.
- Left rectangle (BD): D (top-left), B (bottom-left), C (bottom-right); this is the rectangle BD applied to BC.
- Right rectangle (EG): G (top-right), E (bottom-left), F (bottom-right); this is the rectangle EG contained by EF, FG.

Euclid, *Elements* Book X — Clavius's Commentary (continued)

THEOR. 20. PROPOS. 23 (conclusion of the demonstration)

[Clavius's demonstration, continued from the previous page. The medial line is A; the square on A equals the rectangle BD described on BC; the rectangle EG is contained by EF and FG.]

And since A, by hypothesis, is also the side whose square equals BD, the rectangles BD and EG will be equal; and therefore [Margin b: 14.sext. — *Elements* VI.14] their sides about the equal angles will be reciprocally proportional — that is, BC will be to EF as FG is to CD [Margin c: 22.sexti — VI.22]. And for this reason too, the square on BC will be to the square on EF as the square on FG is to the square on CD. But the square on BC is commensurable with the square on EF (because the straight lines BC and EF are assumed rational, and hence commensurable with one another either in length and in square, or in square only). Therefore [Margin d: 10.dec — X.10] the square on FG will likewise be commensurable with the square on CD; and consequently the straight lines FG and CD will be commensurable at least in square. Now since FG is commensurable with the assigned rational line — if it is not itself the assigned rational line — CD too will be commensurable with the assigned rational line, as we showed in the scholium to Proposition 12 of this book; and so CD, by definition, will be rational. I say that it is also incommensurable in length with BC itself.

For since EF and FG are rational lines commensurable in square only — that is, incommensurable in length — and since EF is to FG as the square on EF is to the rectangle EG contained by EF and FG (by Lemma 3 to Proposition 19 of this book), [Margin e: 10.dec — X.10] the square on EF will be incommensurable with the

rectangle EG, and hence with the rectangle BD, which equals it. But the square on EF is commensurable with the square on CD (because EF and CD are rational, and therefore commensurable at least in square). Therefore, since the square on EF is commensurable with the square on CD, while the rectangle BD is incommensurable with that same square on EF, [Margin f: 13.dec — X.13] the square on CD and the rectangle BD will be incommensurable. But by Lemma 3 to Proposition 19 of this book, the square on CD is to the rectangle BD as the straight line CD is to the straight line BC. [Margin g: 16.dec — X.16] Therefore CD and BC are incommensurable in length. So CD is rational and incommensurable in length with the rational line BC. For this reason, what is applied to a rational line from a medial area, etc. Which was to be demonstrated.

Diagram. Two rectangles side by side, with a short line segment above the right-hand pair.

- A (short segment, upper right): the medial line, drawn with end-ticks.
- Left rectangle, on base BC: B (bottom-left), C (bottom-right), D (top-right); top-left corner unlabeled. This is the square/rectangle BD whose area equals the square on the medial line A.
- Right rectangle, on base EF (wider and lower): E (bottom-left), F (bottom-right), G (top-right); top-left corner unlabeled. This is the rectangle EG contained by EF and FG.

SCHOLIUM.

More easily than by Proposition 45 of Book 1, we shall apply to BC a rectangle equal to the square on A, if — given BC and A — [Margin h: 11.sexiti — VI.11] a third proportional is taken as the side CD. For since BC, A, and CD are proportional, [Margin i: 17.sexiti — VI.17] the rectangle BD contained by the extremes BC and CD will be equal to the square on the mean proportional A.

We shall have to use this technique in what follows as well, whenever a rectangle equal to the square on some straight line is to be applied to a given straight line.

Furthermore, it is clear from this theorem that every medial area — that is, every area whose side is a medial line — is equal to some other rectangle contained by two rational lines commensurable in square only. For if a rectangle equal to that medial area is applied to a rational line, then, by this theorem, it makes the other side rational but incommensurable in length with the rational line. Hence this rectangle, applied and equal to the medial area, will be a medial area contained by two rational lines commensurable in square only. And in this way any medial

area whatever can be reduced to a medial area contained by two rational lines commensurable in square only.

THEOR. 21. PROPOS. 24.

A line commensurable with a medial line is itself medial.

Let B be a straight line commensurable with the medial line A, whether in length and in square or in square only. I say that B too is medial. For let a rational line CD be assigned, [Margin a: 45.primi — I.45] to which a rectangle DE equal to the square on the medial line A is applied. Since, then, the medial area DE applied to the rational line CD produces CE as its breadth, [Margin b: 23.dec — X.23] CE will be rational but incommensurable in length with CD itself. Again, to the rational line CD [Margin c: 45.prim — I.45] let a rectangle DF equal to the square on B be applied. And since the straight lines A and B are assumed commensurable, their squares — that is, the rectangles DE and DF equal to them — will also be commensurable. But [Margin d: 1.sextri — VI.1] DE is to DF as the straight line EC is to the straight line CF. [Margin e: 10.dec — X.10] Therefore EC and CF are commensurable in length. But EC was shown to be rational and incommensurable in length with CD; [Margin f: 14.dec — X.14] therefore CF too is incommensurable in length with that same CD. Since, then, CF is commensurable with the assigned rational line CD (for since EC and CF were shown to be commensurable in length, and EC is commensurable with CD at least in square, because it is rational, CF too will be commensurable with that same CD in square, by what we wrote in the scholium to Proposition 12 of this book), CD and CF will be rational lines commensurable in square only. [Margin g: 22.dec — X.22] And accordingly the line B, whose square is the rectangle DF contained by the rational lines CD and CF, commensurable in square only, is medial. Therefore a line commensurable with a medial line is medial. Which was to be shown.

Diagram. Two short line segments above a rectangle divided vertically into two parts.

- A (upper segment): the medial line.
- B (segment just below A): the line commensurable with A.
- Rectangle on base EF: E (bottom-left), C (bottom, at the vertical division), F (bottom-right), D (top, above C at the division); the outer top corners are unlabeled. The left portion (on base EC) is the rectangle DE, equal to the square on A; the right portion (on base CF) is the rectangle DF, equal to the square on B.

COROLLARIVM.

From this it is clear that an area commensurable with a medial area is medial. For after it was demonstrated that DF is commensurable with the medial area DE, it was at once shown from this that DF is likewise medial — namely, contained by the rational lines CD and CF, commensurable in square only. And the same reasoning holds in the remaining cases. This, however, can also be demonstrated in the following way.

Let the area DF be commensurable with the medial area DE. I say that DF too is medial. For let A, a medial line, be the side of the area DE (for since DE is medial, a straight line — the one called medial — can be its side, as we set out in the scholium to Proposition 22 of this book); and let B be the side of DF. Since, then, DE and DF are commensurable, their squares

on A and on B, equal to them, will also be commensurable. Therefore the straight lines A and B are commensurable at least in square; and hence, A being medial, B too will be medial commensurable with it, as was shown in this theorem. Therefore DF too will be medial. For every area whose side is a medial line is called medial, as we taught in the scholium to Proposition 22 of this book.

I. LEMMA.

[Enunciation:] Just as was said about rational lines in Lemma 1 to Proposition 19 of this book, so here too we shall speak of medial lines. Namely, that a straight line commensurable in length with a medial line is called medial, and commensurable with it not only in length but also in square. For universally, lines commensurable in length are also commensurable in square. But if some straight line is commensurable in square with a medial line — provided it is so in length as well — it too will be called medial, and commensurable with it in length and in square. But if, on the other hand, some line commensurable with a medial line in square is incommensurable in length, it too will be called medial, commensurable with it in square only.

II. LEMMA.

[Enunciation:] To find two medial lines commensurable in length; likewise two commensurable in square only.

Diagram. Three horizontal line segments, one above another.

- A (topmost segment): the given medial line.

- B (middle segment): a line commensurable with A in length.
- C (lowest segment): a line commensurable with A in square only.

Let A be some medial line, and suppose that two straight lines B and C are found commensurable with it — the one, B, in length, and the other, C, in square only. [Margin a: 24.dec — X.24] Then each of B and C, being commensurable with the medial line A, will be medial. Since, then, A and B are commensurable in length, the lines found, A and B, will be medial lines commensurable in length; and A and C will be medial lines commensurable in square only. Which was what was proposed.

SCHOLIVM.

Although every straight line commensurable with a medial line is medial, nevertheless not every medial line is commensurable with every medial line; for two medial lines can be given that are altogether incommensurable — that is, in length and in square — as will appear from Proposition 36 of this book. There we shall also teach by what method two medial lines incommensurable in length and in square are to be found.

THEOR. 22. PROPOS. 25.

[Margin, alongside the heading: xxiii. / xxiii.]

The rectangle contained by medial straight lines commensurable in length is medial.

Let the rectangle AD be contained by the medial lines AC and CD, commensurable in length. I say that AD is medial. On CD, a medial line, let the square BD be described, which will be medial. [Margin a: 1.sexti — VI.1] And since AC is to CB as AD is to DB, and since AC and CB — that is, AC and CD — are commensurable in length, AD and DB will be commensurable. Hence the area AD, being commensurable with the medial area DB, is medial, by the corollary to the preceding proposition. Therefore what is contained by medial lines commensurable in length, etc. Which was to be shown.

Diagram. A rectangle on base AB, divided vertically at C, with a square on the right-hand part.

- A (bottom-left), C (bottom, at the vertical division), B (bottom-right), D (top, above C); the outer top corners are unlabeled.

- The left portion (on base AC) is the rectangle AD; the right portion (on base CB) is the square BD described on CD.

THEOR. 23. PROPOS. 26.

[Margin, alongside the heading: xxv. / xxiii[i?].]

The rectangle contained by medial straight lines commensurable in square only is either rational or medial.

Let the rectangle AC be contained by the medial lines AB and BC, commensurable in square only. I say that AC is either rational or medial. On AB and BC let the squares AD and CE be described, which will be medial (as we set out in the scholium to Proposition 22 of this book), since the straight lines AB and BC are assumed medial. Let a rational line FG be assigned, [Margin a: 45.primi — I.45] to which is applied the rectangle FH equal to the square AD described on AB; and to HI the rectangle IK equal to the rectangle AC; and finally to KL the rectangle LM equal to the square CE described on BC. And the whole FM will be a single rectangle, as was shown in Proposition 45 of Book 1. Now since AD and CE are medial, FH and LM, equal to them, will also be medial; and when these are applied to the rational lines FG, KL,

(FG and KL — for [Margin b: 34.prim — I.34] KL is equal to the rational line FG, because FK is a parallelogram, and is therefore rational) and produce as breadths GH and KM, [Margin c: 23.dec — X.23] the straight lines GH and KM will be rational but incommensurable in length with FG (or KL). And since the straight lines AB and BC are commensurable in square with one another, their squares AD and CE will be commensurable, and hence also the rectangles FH and LM equal to them; [Margin d: 1.sexti — VI.1] and FH is to LM as the straight line GH is to KM. [Margin e: 10.dec — X.10] Therefore GH and KM are commensurable in length with one another. So GH and KM, shown to be rational, are commensurable in length with one another, but commensurable with the rational line FG in square only, since they were shown to be incommensurable with it in length. [Margin f: 20.dec — X.20] And for this reason the rectangle contained by GH and KM is rational. And since DB is to BC as AB is to BE (because DB and BC are equal to AB and BE respectively, each to each), [Margin g: 1.sexti — VI.1] and since DB is to BC as AD is to AC, and AB is to BE as AC is to CE, it will also follow that AD is to AC as AC is to CE; and hence AD, AC, CE are proportional. Therefore FH, HL, LM, equal to them, will also be proportional. [Margin h: 1.sexti — VI.1] But the straight lines GH, HK, KM have the same ratios as the rectangles FH, HL, LM. Therefore the straight lines GH, HK, KM too are proportional;

[Margin i: 17.sexiti — VI.17] and consequently the rectangle contained by GH and KM is equal to the square on HK. But the rectangle contained by GH and KM was shown to be rational. Therefore the square on HK too is rational; and accordingly the straight line HK is [Margin k: 6.def — X, Def. 6] commensurable with the assigned rational line — that is, with HI itself — either in length and in square, or in square only. And if indeed HK is commensurable in length with HI, [Margin l: 20.dec — X.20] the rectangle HL, contained by the rational lines HI and HK commensurable in length — that is, AC, which is equal to it — will be rational. But if HK is commensurable with HI in square only, [Margin m: 22.dec — X.22] the rectangle HL contained by the rational lines HI and HK commensurable in square only — that is, AC, which is equal to it — will be medial. Therefore AC is a rectangle contained by the medial lines AB and BC commensurable in square only, and is either rational or medial. And so what is contained by medial lines commensurable in square only, etc. Which was to be shown.

Diagram. A cross-shaped square figure representing the mean-proportional construction on AB and BC.

- E (top), A (bottom): endpoints of the vertical axis through the center B.
- D (left), C (right): endpoints of the horizontal axis through the center B.
- B (center): the point of division on the base line, where the squares AD (on AB) and CE (on BC) meet; the rectangle AC is contained by AB and BC.

Diagram. A long rectangle FM divided vertically into three rectangles.

- F (top-left), I (top, first division), L (top, second division), M (top-right).
- G (bottom-left), H (bottom, first division), K (bottom, second division), M (bottom-right).
- The three parts are: FH (leftmost, on FG–IH), IK (middle, on IH–LK, equal to the rectangle AC), and LM (rightmost, on LK–M). The whole is applied to the rational lines FG and KL.

SCHOLIVM.

More easily than by Proposition 45 of Book 1, we shall apply to HI a rectangle equal to AC itself, [Margin n: 12.sexiti — VI.12] if a fourth proportional to the three straight lines HI, AB, BC is taken as the side HK. [Margin o: 16.sexiti — VI.16] For the rectangle contained by the extremes HI and HK will be equal to the rectangle contained by the means AB and BC.

We shall use this same technique in what follows, whenever a rectangle equal to another rectangle is to be applied to some straight line.

Now since in this theorem it is demonstrated that the rectangle contained by two medial lines commensurable in square only is either rational or medial, Euclid will teach, in Proposition 28, by what method two medial lines commensurable in square only are to be found that contain a rational area; and in Proposition 29 he will seek two medial lines commensurable in square only that contain a medial area.

And so we have from these results: [Margin p: 20.dec — X.20] that the rectangle contained by two rational lines commensurable in length is rational; [Margin q: 22.dec — X.22] but the rectangle contained by two rational lines commensurable in square only is irrational, and is called medial, and the line whose square it is is called medial. Again, from what has been demonstrated it is established [Margin r: 25.dec — X.25] that the rectangle contained by two medial lines commensurable in length is medial; [Margin s: 26.dec — X.26] but the rectangle contained by two medial lines commensurable in square only is either rational or medial.

But if anyone should ask what kind of rectangle that is which is contained by two medial lines incommensurable in length and in square, we answer that it is neither rational nor medial, but constitutes a certain third kind — namely, that it is equal to a rectangle contained by a rational line and an irrational line, the latter being called medial.

For let the rectangle AC be contained by two medial lines AB and BC incommensurable in length and in square. I say that AC is neither rational nor medial, etc. For, the same construction being made as in the theorem, we shall show similarly that the straight lines GH and KM are rational and incommensurable in length with FG. And since the straight lines AB and BC are assumed incommensurable in length and in square, their squares AD and CE — and hence the rectangles FH and LM equal to them — will be incommensurable. [Margin t: 1.sexti — VI.1] And FH is to LM as the straight line GH is to the straight line KM. [Margin u: 10.dec — X.10] Therefore the straight lines GH and KM are incommensurable in length with one another. So GH and KM, shown to be rational, are commensurable in square only with one another; [Margin x: 22.dec — X.22] and accordingly the rectangle contained by GH and KM is irrational — the kind that is called medial — and the straight line whose square it is is irrational, the kind called medial. [Margin y: 17.sexti — VI.17] Now HK is the side of the rectangle contained by GH and KM (for, as we showed before, the three lines GH, HK, KM are proportional). Therefore HK is irrational and medial. Hence IK is not rational; for if it were rational, [Margin z: 21.dec — X.21] it would, when

applied to the rational line HI, make its breadth HK rational and commensurable in length with HI. Which is absurd, for HK was shown to be irrational and medial. In the same way IK is not medial either; for if it were medial, [Margin a: 23.dec — X.23] it would, when applied to the rational line HI, make its breadth HK rational and incommensurable in length with HI — which is absurd, for HK was shown to be irrational and medial. And so, since IK is neither rational nor medial, necessarily AC, which is equal to it, will be neither rational nor medial, but will constitute a certain third kind; namely, it will be equal to IK, which is contained by a rational line and a medial line, since HI is rational and HK was shown to be medial. From all this it follows that the straight line whose square is the area contained by two medial lines incommensurable in length and in square — such as AB and BC are assumed to be — can also be the side of a rectangle contained by a rational line and an irrational line called medial. Which was what was proposed.

THEOR. 24. PROPOS. 27.

[Margin: xxvi. / xxii.[?] — divergent lowercase-roman prop-counter (Theon vs. Clavius offset)]

A medial does not exceed a medial by a rational.

Let the medial AB exceed the medial AC by the rectangle DB. I say that DB is not rational. For suppose, if it be possible, that DB is rational. Then, a rational line EF being set out, let the rectangle EG, equal to the medial AB, be applied to it [Margin: a. Elements I.45], and let the rectangle EH be applied equal to the medial AC, so that the remainder HI is equal to the rational DB. Now EG and EH will be medial, being equal to the medial areas AB and AC, while HI will be rational, being equal to the rational DB. Since, then, the medial areas EG and EH have been applied to the rational line EF, the lines FG and FH will be rational and incommensurable in length with EF [Margin: b. Elements X.23].

Again, since the rational area HI has been applied to the rational line HK — for the line HK is equal to the rational EF [Margin: c. Elements I.34] — the line HG will be rational and commensurable in length with EF [Margin: d. Elements X.21]. Therefore, since HG and EF are commensurable in length, while EF is incommensurable in length with FH, as was shown, HG too will be incommensurable in length with that same FH [Margin: e. Elements X.14]. But as FH is to HG, so is the square on FH to the rectangle contained by FH and HG, by the third lemma of Proposition 19 of this book. Therefore the square on FH is

incommensurable with the rectangle contained by FH and HG [Margin: f. Elements X.10].

But the square on HG is commensurable with the square on FH — since both are described on the rational lines FH, HG [Margin: g. Elements X.16] — and so the two squares on FH and HG together are also commensurable with the square on FH. And the rectangle contained by FH and HG is commensurable with twice the rectangle contained by FH and HG, because the latter is double the former. Therefore, by what we showed in the scholium to Proposition 14 of this book, the two squares on FH and HG together are also incommensurable with twice the rectangle contained by FH and HG [Margin: h. Elements X.17]. Hence the sum of the squares on the lines FH, HG together with twice the rectangle contained by FH and HG is incommensurable with the sum of the squares on the lines FH and HG [Margin: i. Elements II.4].

But the sum of the squares on FH and HG together with twice the rectangle contained by FH and HG is equal to the square on FG. Therefore the square on FG is also incommensurable with the sum of the squares on the lines FH and HG. Now this sum is rational, by the fourth lemma of Proposition 19 of this book, in that it was shown to be commensurable with the rational square on FH. Since, then, the square on FG is incommensurable with this sum, which is rational, the square on FG will be irrational [Margin: k. Elements X.10], and therefore the line FG too will be irrational [Margin: l. Elements X.11]. But this is absurd, since FG was shown to be rational and incommensurable in length with EF. Therefore DB, by which the medial AB exceeds the medial AC, is not rational. For which reason a medial does not exceed a medial by a rational. Which was to be shown.

Diagram. The upper figure is a rectangle divided by a vertical dashed line, illustrating the applied rectangles on the rational line EF and the excess DB. Labelled points read off the scan under magnification:

- C (top edge, at the dividing line): division point on the top edge
- B (top-right corner)
- A (bottom-left corner)
- D (bottom edge, at the dividing line): division point on the bottom edge
- (The top-left corner label is not legible on the leaf — [?])

Diagram. The lower figure is a square-shaped figure with internal dashed subdivisions, illustrating the squares on FH and HG. Labelled points:

- E (upper-left corner)

- K (upper edge, mid)
- I (upper-right corner)
- F (lower-left corner)
- H (lower edge, mid)
- G (lower-right corner)

SCHOLIVM.

From what has been said we will also demonstrate the following:

A rational exceeds a rational by a rational.

For let the rational AB exceed the rational AC by the space DB. I say that DB too is rational. Since AB and AC are rational, they will be commensurable with the square on the rational line set out [Margin: m. Elements X, def. 9], and so they are commensurable with each other as well [Margin: n. Elements X.12]. Therefore, since the whole AB — which is compounded of AC and DB — is commensurable with AC, the same AB will also be commensurable with the remainder DB, by the corollary to Proposition 16 of this book. But AB is rational. Therefore DB too, by the fourth lemma of Proposition 19 of this book, is rational. Which was what was proposed.

Diagram. A rectangle divided by a vertical dashed line, representing the whole AB compounded of AC and the excess DB. Labelled points:

- C (top edge, at the dividing line)
- B (top-right corner)
- A (bottom-left corner)
- D (bottom edge, at the dividing line)

PROBL. 4. PROPOS. 28.

[Margin: xxvii. / o.]

To find two medial lines, commensurable in square only, which contain a rational rectangle.

Let two rational lines A, B, commensurable in square only, be set out — by lemma 2 of Proposition 21 of this book — [Margin: a. Elements VI.13] and let a mean proportional C be taken between them [Margin: b. Elements VI.12], and let it be contrived that as A is to B, so is C to D. I say that C, D are two medial lines commensurable in square only which contain a rational rectangle. For since A, B

are supposed rational and commensurable in square only, the rectangle contained by them will be irrational, which is called medial [Margin: c. Elements X.22]; and therefore, since the line C is capable of it [i.e. the square on C equals that rectangle] [Margin: d. Elements VI.17], C will be medial. And since as A is to B, so is C to D, and A, B are commensurable in square only, C and D too will be commensurable in square only, as we showed in the scholium to Proposition 10 of this book [Margin: e. Elements X.24]. Therefore D is medial and commensurable in square with the medial C. Hence there have been found two medial lines C, D commensurable in square only.

I say now that they contain a rational rectangle. For since as A is to B, so is C to D, then, alternately, as A is to C, so is B to D. But as A is to C, so is C to B; therefore also, as C is to B, so is B to D, and so B is a mean proportional between C and D [Margin: f. Elements VI.17]. Therefore the rectangle contained by C, D will [equal the square on B]. But the square on B, a rational line, is rational. Therefore the rectangle contained by C, D is also rational. We have therefore found two medial lines C, D, commensurable in square only, which contain a rational rectangle. Which was to be done.

Diagram. Four line segments drawn one above another, representing the four lines of the construction (A and B the two rationals commensurable in square only, C the mean proportional, D the fourth proportional). Labels, top to bottom:

- A (top segment)
- C (second segment, drawn with a dashed portion)
- B (third segment)
- D (bottom segment)

PROBL. 5. PROPOS. 29.

[Margin: xxviii. / o.]

To find two medial lines, commensurable in square only, which contain a medial rectangle.

Let three rational lines A, B, C, commensurable in square only, be set out — by what we demonstrated in the scholium to Proposition 21 of this book — [Margin: a. Elements VI.13] and let a mean proportional D be found between A and B [Margin: b. Elements VI.12]. Then let it be contrived that as B is to C, so is D to E. I say that D, E are two medial lines commensurable in square only which contain a medial rectangle. For since A, B are rational and commensurable in square only, the rectangle contained by A, B — and therefore also the square on D, which is

equal to it [Margin: c. Elements VI.17] — will be irrational, which is called medial [Margin: d. Elements X.22]; and the line D itself is medial. And since as B is to C, so is D to E, while B, C are rational and commensurable in square only, D and E too will be commensurable in square only, as was demonstrated in the scholium to Proposition 10 of this book. Since, then, D is medial, E too will be commensurable in square with it, and medial [Margin: e. Elements X.24]; and therefore D, E are two medial lines commensurable in square only.

I say now that they contain a medial rectangle. For since as B is to C, so is D to E, then, alternately, as B is to D, so is C to E. But as B is to D, so is D to A; therefore also, as D is to A, so is C to E. Therefore the rectangle contained by D, E is equal to that contained by A, C [Margin: f. Elements VI.16]. But the rectangle contained by A, C — rational lines commensurable in square only — is irrational and medial [Margin: g. Elements X.22]. Therefore the rectangle contained by D, E is also medial. We have therefore found two medial lines D, E, commensurable in square only, which contain a medial rectangle. Which was to be done.

Diagram. Five line segments drawn one above another, representing the five lines of the construction. Labels, top to bottom:

- A (top segment)
- D (second segment)
- B (third segment)
- C (fourth segment)
- E (bottom segment)

SCHOLIUM. I.

In what follows we need this problem:

To find two similar plane numbers.

Let any four proportional numbers A, B, C, D be taken, such that as A is to C, so is B to D. Let A and B, multiplying one another, make E; and likewise let C and D, multiplying one another, make F. Then E and F will be similar plane numbers, since their sides are proportional, as is clear from the construction.

A,	6.	C,	12.
B,	4.	D,	8.
E,	24.	F,	96.

Now since it was shown in Book 9 [Margin: h. Elements IX.28 & 29] that if an odd number or an even number multiply an even number, an even number is produced, but an odd number if an odd multiply an odd, it is clear in what manner two similar plane numbers can be found, of which each is even, or each odd, or one even and the other odd. For if the sides taken are even numbers, their plane numbers too will be even; but if the numbers are odd, their plane numbers too will be odd. And if the sides of the one are odd numbers and those of the other even, the plane number of the former will be odd, that of the latter even. Similarly, the plane numbers will be even if each has one side an even number and the other an odd number, and so on.

LEMMA I.

To find two square numbers such that the sum compounded of them is also a square.

Let two similar plane numbers AB and C be found, by what was said in the preceding scholium [Margin: i. Elements IX.24 & 26], each of which is either even or odd. And since, whether an even be taken from an even or an odd from an odd, the remainder is even, then, if BD — equal to C itself — be taken away from AB, the remainder AD will be even. Let AD be bisected at E. I say that the number produced from AB by BD — which is indeed a square [Margin: k. Elements IX.1] — compounded with the square of the number ED, makes a square. For since the number AD is bisected at E and DB is added to it, then, by the 6th theorem of those which we demonstrated at Proposition 14 of Book 9, the number produced from AB by BD, together with the square of the number DE, will be equal to the square of the number EB. Therefore the two squares — namely that produced from AB by BD, and the square of the number DE — compounded make a square, namely the one generated from BE. Which was what was proposed.

Diagram. A number-line figure: an upper line marked off in units by dots with points A, E, D, B labelled along it, and a shorter lower line for the number C. As drawn:

- Upper line: A E D B (A at left end, E and D interior division points, B at right end)
- Lower line: C (the number C)

COROLLARIUM.

From these things it is clear that, when AB and C are similar, there have been found by the same method two square numbers of the numbers BE, ED, whose

difference — namely the number produced from AB by BD — is also a square number.

But if the numbers AB and C are taken not similar, yet each even or each odd, there will be found in the same way two squares of the numbers BE, ED, whose difference — namely the number produced from AB, BD — is not a square [Margin: I. Elements IX.2]. For if it were a square, the numbers AB, BD — that is, AB and C — would be similar plane numbers. But that is absurd, for they are supposed not similar.

SCHOLIVM. II.

Therefore, if we are bidden to find two square numbers whose difference is likewise a square number, we shall take, as before, two similar plane numbers, of which each is either even or odd, namely AB and C, and we shall carry out the rest as was said in the immediately preceding lemma. For the square numbers described on BE, ED are those which we seek. For the square on BE exceeds the square on ED by the number produced from AB by BD, which is also a square.

Diagram. A number-line figure. As drawn:

- Upper line: A..... E..... D..... B
- Lower line: c..... (the number C)

But if two squares are to be found whose difference is not a square, then two plane numbers AB and C, not similar, each of which is either even or odd, must be taken, and the rest carried out as before. For in the same way we shall show that the square on BE is equal to the square on DE together with that which is produced from AB by BD. Hence the difference of the squares on BE, ED is the number produced from AB by BD, which, since it is not a square — for if it were a square, AB, BD would be similar plane numbers, which is not supposed — the proposition stands established.

Diagram. A second number-line figure, more compact than the first. As drawn:

- Upper line: A... E... D..... B
- Lower line: c..... (the number C)

This latter case we shall carry out more easily if we divide any square number into two numbers, of which one is a square and the other not. Thus, if the square 36 be divided into the square 16 and the non-square 20, the square 36 will exceed the square 16 by the non-square number 20. Likewise, if the same square 36 be

divided into the square 25 and the non-square 11, the square 36 will exceed the square 25 by the non-square number 11. And so with the rest.

LEMMA II.

To find two square numbers such that the sum compounded of them is not a square.

Let there be two similar plane numbers AB and C , even or odd, as in the preceding lemma, and let the same construction be made, so that again the square produced from the multiplication of the similar numbers AB , DB by one another, together with the square on DE , is equal to the square on BE . Then let the unit EF be taken away from DE . The square on DF will then be less than the square on DE , since the side DF is also less than the side DE . I say that the square numbers, of which one is produced from AB by BD and the other from DF by itself, compounded do not make a square number. For if the sum compounded of them is a square, it will be either greater than the square on BE , or equal to it, or less; and that this cannot be, we shall demonstrate in the following manner.

Let it first be greater than the square on BE , and therefore its side greater than the side BE . Then its side will be either equal to the number BE or greater. (For it will not be less, since between the numbers BE , BF , which differ from one another by a unit alone, there is no intermediate number, unless the unit itself be cut; and the said side would be intermediate between them if it were supposed greater than BF but less than BE .) If it be said equal — so that the square number compounded of the square produced from AB by BD and the square of the number DF is equal to the square on BE — then, since to the same square on BE there was shown, in the preceding lemma, to be equal the number produced from AB by BD together with the square on DE , it will be that the number produced from AB by BD together with the square on DE [is equal to that composite]. The common part being taken away, namely the number produced from AB by BD , the remaining square on DF will be equal to the remaining square on DE ; and therefore the side DF equal to the side DE — a part to the whole. Which is absurd. Therefore the side of the square compounded of the squares, of which one is produced from AB by BD and the other from DF by itself, is not equal to the number BE .

Diagram. A number-line figure (reprinted twice on the page), with the additional points H , I , F , G marked on the upper line. As drawn:

- Upper line: $A \dots H \dots I \dots E \dots F \dots G \dots D \dots B$ (A at left end; H , I , E , F , G interior division points; D and B toward the right)
- Lower line: $C \dots$ (the number C)

Let it be, then, if it be possible, that its side is equal to the number BI, which is greater than BE, so that the square on BI is equal to that composite square. Since, then, the square on BI, with the greater side, is greater than the square on BE, with the lesser side, the composite of the squares — of which one is produced from AB by BD and the other from DF by itself — will also (since this composite is supposed equal to the square on BI) be greater than the square on BE. But equal to the square on BE was shown, in the preceding lemma, the number produced from AB by BD together with the square on DE. The common part being taken away, namely that produced from AB by BD, the remaining square on DF will be greater than the remaining square on DE; and consequently the side DF greater than the side DE — a part greater than the whole. Which is absurd. Therefore the side of the square compounded of the squares, of which one is produced from AB by BD and the other from DF by itself, is not greater than the side BE. But neither equal nor less has it been shown. Therefore that composite square is not greater than the square on BF.

(Conclusion of Lemma II — the demonstration continued from the preceding page.)

Now suppose, if it be possible, that the number produced by AB times BD, together with the square on DF, is equal to the square on BF; and let a number AH be taken, double the unit EF. Since the whole AD is double the whole ED (for AD has been bisected at E), and the part removed, AH, is double the part removed, the unit EF, the remainder HD too will be double the remainder FD — as we showed in connection with Proposition 7 of Book 7 — and therefore HD is bisected at F. Hence, by the 6th of the theorems we demonstrated in connection with Proposition 14 of Book 9, the number produced by HB times BD, together with the square on DF, will be equal to the square on BF. But the number produced by AB times BD together with the square on DF was assumed equal to that same square on BF. Therefore the number from HB times BD together with the square on DF is equal to the number from AB times BD together with the square on DF; and when the common square on DF is subtracted, the number from HB times BD is left equal to the number from AB times BD. Now since HB and AB, multiplying the same BD, produce equal numbers, and since [Margin: a. Elements VII.18] multipliers stand in the same ratio as their products, HB will be equal to AB — a part to the whole. This is absurd. Therefore the number from AB times BD together with the square on DF is not equal to the square on BF.

Finally, suppose, if it be possible, that the number from AB times BD together with the square on DF is less than the square on BF, and consequently that its side too is less than the side BF — let this be BG — so that the number from AB times BD together with the square on DF is equal to the square on BG. Let AI be

taken, double of EG. Since the whole AD is double the whole ED [the ambiguous: printed form may read "FD"], and the part removed, AI, is double the part removed, EG, again the remainder ID will be double the remainder GD; and therefore ID is bisected at G. Hence, by the same 6th of the theorems demonstrated in connection with Proposition 14 of Book 9, the number from IB times BD together with the square on DG will be equal to the square on BG. But the number from AB times BD together with the square on DF was assumed equal to that same square on BG. Therefore the number from IB times BD together with the square on DG is equal to the number from AB times BD together with the square on DF.

Diagram. A number-line schematic accompanying the lemma. A single straight row of points is marked, in order, A · · H · · I · E · F · G · · · D · · · · · B, with a second point C set below at the far left (C · · · · ·). It displays the divisions used in the demonstration: E bisects AD; H, I, F, G are the intermediate division points; B lies at the right; C marks the separate number placed below.

- A: left end of the divided segment
- H, I: division points near the left
- E: midpoint of AD
- F, G: further division points
- D: right end of the divided segment
- B: point at the far right
- C: separate point set below the left end

So when the squares on DG and DF are subtracted — of which the square on DG is the smaller — there remains the number from IB times BD greater than the number from AB times BD: a part greater than the whole. This is absurd.

Therefore the number from AB times BD together with the square on DF is not less than the square on BF. But it was shown to be neither greater nor equal.

Therefore the number from AB times BD together with the square on DF is not a square. Which was the proposition.

SCHOLIUM III.

From this we shall easily find two numbers such that the number compounded of them has to neither of them the ratio that a square has to a square. For if, by the preceding lemma, two square numbers are found whose sum is not a square, this non-square sum will have to neither of those squares the ratio of a square to a square.

We shall obtain the same result if we divide any square number into two non-square parts. For then the whole square will have to neither of the parts into which it is divided the ratio of a square to a square.

PROBL. 6. PROPOS. 30. [Margin: xxix. xvij.]

To find two rational straight lines commensurable in square only, such that the greater is more powerful than the less by the square on a straight line commensurable with it in length.

Let a rational straight line AB be set out; and, by the method we handed down in Scholium 2 of Proposition 29 of this book (the preceding one), let two square numbers CD, CE be found whose excess DE is not a square. Then, by the corollary to Proposition 6 of this book, let it be contrived that as the number CD is to the number DE, so is the square on AB to another square — namely to the square on AF; and let AF be fitted into the semicircle ABF described on the diameter AB; and finally let the straight line FB be joined.

Now since the angle F is right, being in a semicircle, [Margin: a. Elements III.31] the square on AB will be equal to the two squares on AF and FB together; [Margin: b. Elements I.47] that is, the straight line AB is more powerful than AF by the square on the straight line FB. And since the square on AB is to the square on AF as the number CD is to the number DE, [Margin: c. Elements X.6] the squares on AB and AF will be commensurable. And so, since the square on AB is rational — being described on the rational line AB — the square on AF will also be rational, and consequently the straight line AF that describes it is rational as well. Therefore AB and AF are rational, and hence commensurable at least in square.

But the square on AB does not have to the square on AF the ratio that a square number has to a square number — for the square number CD does not have to the non-square DE the ratio of a square to a square (otherwise DE too would be a square, [Margin: d. Elements VIII.24] as is demonstrated in the arithmetical books, which is not assumed). [Margin: e. Elements X.9] Therefore the straight lines AB and AF are incommensurable in length. Yet they are commensurable in square, as was shown. Therefore AB and AF are rational lines commensurable in square only.

Now since CD is to DE as the square on AB is to the square on AF, then by conversion of the ratio the square number CD will be to the square number CE as the square on AB is to the square on FB. For just as CD exceeds DE by the square CE, so too the square on AB exceeds the square on AF by the square on FB.

[Margin: f. Elements X.9] Therefore the straight lines AB and FB are commensurable in length. We have therefore found two rational straight lines AB and AF, commensurable in square only, such that the greater AB is more powerful than the less AF by the square on the line FB, which is commensurable with it in length. Which was to be done.

Diagram. A semicircle described on the diameter AB (A at the lower left, B at the lower right), with F a point on the arc at the top. The chords AF and FB are drawn, forming the right triangle AFB inscribed in the semicircle (the right angle at F). A small tick I is marked on the diameter AB. Below the diameter runs a separate dotted number-line: C at the left, E about the middle, D toward the right (C ····· E ····· D), representing the number CD divided at E.

- A: left end of the diameter
- B: right end of the diameter
- F: point on the semicircular arc; vertex of the right angle
- I: point marked on the diameter AB
- C, E, D: points of the number-line beneath (CD divided at E)

PROBL. 7. PROPOS. 31. [Margin: xxx. xviiij.]

To find two rational straight lines commensurable in square only, such that the greater is more powerful than the less by the square on a straight line incommensurable with it in length.

Let a rational straight line AB be set out; and by Lemma 2 of Proposition 29 of this book let two square numbers CE, ED be found such that the number CD compounded of them is not a square — or else let some square number CD be divided into two non-square numbers CE, ED. For whichever of these is done, the whole CD will have to neither CE nor ED the ratio of a square to a square. Otherwise, by the former method the whole CD would itself be a square, while by the latter each of CE and ED would be a square as well, [Margin: a. Elements VIII.24] as has been shown in the arithmetical books — which is not assumed.

Then, by the corollary to Proposition 6 of this book, let it be contrived that as CD is to DE, so is the square on AB to another square, namely to the square on AF; and let AF be fitted into the semicircle AFB described on AB; and finally let the straight line FB be joined. Then, as in the preceding proposition, AB will be more powerful than AF by the square on the straight line FB, and AB, AF will be rational lines commensurable in square only — for the demonstration here is the same as in the preceding proposition.

Now again, as before, by conversion of the ratio, as CD is to CE, so is the square on AB to the square on FB. But CD does not have to CE the ratio of a square to a square; therefore the square on AB will not have to the square on FB the ratio that a square number has to a square number either. [Margin: b. Elements X.9] Hence the straight lines AB and FB are incommensurable in length. We have therefore found two rational lines AB, AF commensurable in square only, such that the greater AB is more powerful than the less AF by the square on the line FB, which is incommensurable with it in length. Which was to be done.

Diagram. A semicircle described on the diameter AB (A at the lower left, B at the lower right), with F on the arc; the chords AF and FB form the inscribed right triangle AFB (right angle at F), and a tick I is marked on AB. Beneath the diameter is a divided number-line carrying values: C —144— E —16— D, with the whole segment shown as C —160— D (so CE = 144, ED = 16, CD = 160); a dotted copy of the line (C ····· E ··· D) is drawn below it.

- A: left end of the diameter
- B: right end of the diameter
- F: point on the arc; right-angle vertex
- I: point marked on AB
- C, E, D: number-line points; CE = 144, ED = 16, whole CD = 160

LEMMA A.

If there are two unequal straight lines, then as the greater is to the less, so is the rectangle contained by them to the square on the less.

Let AB, BC be two unequal straight lines set in a straight line, of which AB is the greater. I say that as AB is to BC, so is the rectangle under AB, BC to the square on BC. For on BC let the square BD be described, and let the rectangle AD be completed. Then the rectangle AE will be that contained by AB, BC, since EB is equal to BC; [Margin: c. Elements VI.1] and it is evident that as AB is to BC, so is AE to BD.

In the same way, as the less is to the greater, so will the rectangle contained by them be to the square on the greater, as is clear in the second figure, in which AB is less than BC.

Diagram. Two rectangles illustrating the lemma.

In the first (upper) figure: a rectangle standing on the base C—B—A (C at the left, B in the middle, A at the right) with top side D—E—(unlabelled corner above A).

The vertical line BE divides it. The left part, C-B-E-D, is the square described on BC (labelled by the diagonal corners as square BD); the right part is the rectangle AE contained by AB and BC (since EB = BC). The whole rectangle is AD.

- Upper figure — bottom: C (left), B (middle), A (right); top: D (above C), E (above B)

In the second (lower) figure: the same construction drawn for the case where AB is less than BC — base C—B—A, top D—E—(corner), with the vertical division at BE.

- Lower figure — bottom: C (left), B (middle), A (right); top: D (above C), E (above B)

PROBL. 8. PROPOS. 32. [Margin: xxxj. xxiv. xxv.]

To find two medial straight lines commensurable in square only, containing a rational rectangle, such that the greater is more powerful than the less by the square on a straight line commensurable with it in length.

[Margin: a. Elements X.30] Let two rational lines A, B be found, commensurable in square only, such that the greater A is more powerful than the less B by the square on a line commensurable with it in length; let C be the mean proportional between A and B; and let it be contrived that as A is to B, so is C to D. Since A, B are rational lines commensurable in square only, [Margin: b. Elements X.22] the rectangle under A, B will be irrational, and the straight line C whose square equals it will be a medial line. And since A is to B as C is to D, and A, B are commensurable in square only, C and D too will be commensurable in square only, as we showed in the scholium to Proposition 10 of this book. [Margin: c. Elements X.24] Therefore D, being commensurable with the medial C, is itself also a medial. So two medial lines C, D commensurable in square only have been found.

I say that they contain a rational rectangle. For since A is to B as C is to D, then alternately A is to C as B is to D; but A is to C as C is to B; therefore also C is to B as B is to D. [Margin: d. Elements VI.17] And on this account the rectangle under C, D will be equal to the square on B. For this reason, since the square on B — B being rational — is rational, the rectangle under C, D, equal to it, will also be rational. Therefore the medials C, D, commensurable in square only, contain a rational rectangle.

Now since A is to B as C is to D, and A is more powerful than B by the square on a line commensurable with it in length, by construction, [Margin: e. Elements X.15] C too will be more powerful than D by the square on a line commensurable with it in length. We have therefore found two medial lines C, D commensurable in square only, containing a rational rectangle, such that the greater C is more powerful than the less D by the square on a straight line commensurable with it in length. Which was to be done.

But if A, B are found to be rational lines commensurable in square only such that A is more powerful than B by the square on a line incommensurable with it in length, and the rest is done as before, we shall show in the same way that two medial lines have been

Diagram. Four vertical straight lines of differing lengths standing side by side, labelled (left to right) A, C, B, D — representing the two rationals A, B and the two medials C, D of the construction.

- A: leftmost line (longest)
- C: second line
- B: third line
- D: rightmost line (shortest)

found — two medial lines C, D commensurable in square only, containing a rational rectangle, such that the greater C is more powerful than the less D by the square on a line incommensurable with it in length.

SCHOLIUM.

Others prove this problem by means of the preceding lemma; but we have shown the same thing more briefly and easily without it, as those will readily judge who compare our demonstration with theirs. Indeed, that the lines C, D are medials commensurable in square only which contain a rational rectangle, we show here in just the same way as in Proposition 28 of this book.

LEMMA A.

If there are three straight lines, then as the first is to the third, so is the rectangle contained by the first and second to the rectangle contained by the second and third.

Let AB, BC, CD be three straight lines set in a straight line. I say that as AB is to CD, so is the rectangle under AB, BC to the rectangle under BC, CD. For on BC let the square BCDE be described, and let the rectangle AF be completed. Then AE will be

the rectangle contained by AB, BC, and CF that contained by BC, CD, since BE and CD are each equal to BC. [Margin: a. Elements VI.1] And it is evident that AB is to CD as AE is to CF. Which was the proposition.

Diagram. A long rectangle AF standing on the base A—B—C—D (A at the left, then B, C, D to the right), with the top side E—D—F (E above B, D above C, F above the right end). Two vertical lines divide it into three parts: the left part is the rectangle AE contained by AB, BC; the middle part is the square BCDE described on BC; the right part is the rectangle CF contained by BC, CD.

- Bottom: A (left), B, C, D (right)
- Top: E (above B), D (above C), F (above the right end)

PROBL. 9. PROPOS. 32. [Margin: xxxii. xxvj.]

To find two medial straight lines commensurable in square only, containing a medial rectangle, such that the greater is more powerful than the less by the square on a straight line commensurable with it in length.

Let three rational lines A, B, C be found, commensurable in square only, such that A is more powerful than C by the square on a line commensurable with it in length. (This will be done as follows. [Margin: a. Elements X.30] Two rational lines A, C commensurable in square only having been found, such that A is more powerful than C by the square on a line commensurable with it in length, let another line B be found commensurable in square only with each of A and C, by the method we demonstrated in the scholium to Proposition 21 of this book.) [Margin: b. Elements VI.13] And of A, B let the mean proportional D be taken; [Margin: c. Elements VI.12] and let it be contrived that as D is to B, so is C to E.

Now since, by the preceding lemma, as A is to C, so is the rectangle under A, B to the rectangle under B, C — and since [Margin: d. Elements VI.17] the square on D is equal to the rectangle under A, B, and [Margin: e. Elements VI.16] the rectangle under D, E is equal to the rectangle under B, C, because D, B, C, E are proportional — it will follow that as A is to C, so is the square on D to the rectangle under D, E. But as the square on D is to the rectangle under D, E, so, by Lemma 3 of Proposition 19 of this book, is the straight line D to the straight line E. Therefore as A is to C, so is D to E. But A, C are commensurable in square only; therefore D, E too are commensurable in square only, as we showed in the scholium to Proposition 10 of this book.

[Margin: f. Elements X.22] Now since D, whose square equals the area under A, B — A and B being rationals commensurable in square only — is irrational and medial, [Margin: g. Elements X.24] E too, having been shown commensurable with D, will be medial. So two medial lines D, E commensurable in square only have been found.

And since the rectangle under D, E has been shown equal to the rectangle under B, C — [Margin: h. Elements X.22] which is medial (because B, C are rationals commensurable in square only) — the rectangle under D, E will be medial. Finally, since we showed that A is to C as D is to E, and A is more powerful than C by the square on a line commensurable with it in length, by construction, [Margin: i. Elements X.15] D too will be more powerful than E by the square on a line commensurable with it in length. We have therefore found two medial lines D, E commensurable in square only, containing a medial rectangle, such that the greater D is more powerful than the less E by the square on a straight line commensurable with it in length. Which was to be done.

But if A, B, C are found to be rational lines commensurable in square only such that A is more powerful than C by the square on a line incommensurable with it in length, and the rest is constructed as before, we shall likewise demonstrate that two medial lines D, E commensurable in square only have been found, containing a medial rectangle, such that the greater D is more powerful than the less E by the square on a line incommensurable with it in length. Which can easily be seen from the demonstration given.

Diagram. Five vertical straight lines of differing lengths standing side by side, labelled (left to right) A, D, B, E, C [last label uncertain — possibly G]. They represent the three rationals A, B, C and the two medials D, E of the construction.

- A: leftmost line
- D: second line
- B: third line
- E: fourth line
- C [?]: rightmost line

LEMMA I.

Let ABC be a right triangle having the right angle BAC, and from it let the perpendicular AD be dropped. I say that the rectangle contained by CB, BD is equal to the square on AB; that contained by BC, CD is equal to the square on AC; that contained by BD, DC is equal to the square on AD; and finally that contained by BC, AD is equal to that contained by AB, AC.

Since, by the corollary to Proposition 8 of Book 6, the straight line AB is the mean proportional between CB [printed "CD"; recte CB] and BD, [Margin: k. Elements VI.17] the rectangle under CB, BD will be equal to the square on AB.

By the same reasoning, $\text{rect}(BC, CD) = AC^2$, since by the same corollary AC is the mean proportional between BC and CD.

Again, since by the same corollary AD is the mean proportional between BD and DC, in the same way $\text{rect}(BD, DC) = AD^2$.

Finally, [Margin: l — VI.18] since the triangles ABC and ABD are similar, $BC : AC = AB : AD$. [Margin: m — VI.16] Therefore $\text{rect}(BC, AD) = \text{rect}(AB, AC)$. This can also be shown in the following way. Let the rectangle CE, contained by BC and AD, be completed, and likewise the rectangle AF, contained by AB and AC. [Margin: n — I.41] Since, then, the rectangle CE is double the triangle ABC, and the rectangle AF is likewise double the same triangle, the two rectangles CE and AF are equal to one another. Which was proposed.

Diagram. The compound figure for Lemma I: a right triangle with the right angle at A and the perpendicular AD dropped to the base, together with the two rectangles CE and AF named in the demonstration.

- E (upper left): corner of the rectangle CE erected over the base
- A (top, apex of the right angle BAC): vertex of the triangle
- B (lower left): base endpoint
- D (on the base): foot of the perpendicular AD
- I [?] (on the base, between D and C): a marked point, its role not stated in the text
- C (right, on the base): base endpoint
- F (bottom vertex): far corner of the large rectangle AF, contained by AB and AC

LEMMA II.

If a straight line is cut into two unequal parts, then as the greater part is to the lesser, so is the rectangle contained by the whole and the greater part to the rectangle contained by the whole and the lesser part.

Let the straight line AB be cut, not into equal halves, at C, and let AC be the greater part. I say that as AC is to CB, so is $\text{rect}(AB, AC)$ to $\text{rect}(AB, CB)$. For let the square ABDE of AB be described, and let CF be drawn parallel to AE. Then AF will be the rectangle contained by AB and AC, and CD the rectangle contained by AB

and CB, since AE and CF are equal to AB. [Margin: o — VI.1] Since, therefore, as AC is to CB, so is AF to CD, the proposition is established.

By the same reasoning, as the lesser part BC is to the greater CA, so is BF — contained by AB and BC, the whole and the lesser part — to CE, contained by AB and CA, the whole and the greater part.

Diagram. The square on AB for Lemma II, with CF drawn parallel to the sides.

- E (top left): upper corner of the square ABDE
- F (top, above C): where CF meets the top side
- D (top right): upper corner of the square
- A (bottom left): endpoint of the base
- C (bottom, the point of section): the cut of AB
- B (bottom right): endpoint of the base

LEMMA III.

If there are two unequal straight lines, and the lesser is bisected, then the rectangle contained by them is double the rectangle contained by the greater line and half the lesser.

Let AB, BC be two unequal straight lines forming the right angle ABC, and let the lesser, BC, be bisected at D. I say that $\text{rect}(AB, BC)$ is double $\text{rect}(AB, BD)$. For let the rectangle AC, contained by AB and BC, be completed, and let DE be drawn parallel to AB. Then BE will be the rectangle contained by AB and BD, half the lesser line. [Margin: p — VI.1] Since, therefore, AC is double BE — because the base BC is double the base BD — the proposition is established.

In the same way, if BC, bisected, be the greater line, then the rectangle contained by AB, BC will be double the one contained by AB, the lesser, and BD, half the greater, as appears from the second figure.

Diagram. Two rectangles side by side illustrating the two cases of Lemma III; in each, DE is drawn parallel to AB and D bisects the base.

- Left figure — A (top left), E (top, above D); B (bottom left), D (bottom, midpoint), C (bottom right)
- Right figure — A (top left), E (top, above D); B (bottom left), D (bottom, midpoint), C (bottom right)

PROBL. 10. PROPOS. 34.

[Margin: xxxiii. / xxvii.]

To find two straight lines incommensurable in square which make the sum of their squares rational, but the rectangle contained by them medial.

[Margin: a — X.31] Let two rational lines AB, CD, commensurable in square only, be taken, such that the greater, AB, is more powerful than the lesser, CD, by the square on a line incommensurable with it in length; and let CD be bisected at E. Then, by lemma 2 of proposition 17 of this book, apply to AB a rectangle equal to the square on CE — that is, to the fourth part of the square on CD — deficient by a square figure, and let it be the one contained by AF, FB. Finally, on AB describe the semicircle AGB, erect FG perpendicular to AB, and join the lines AG, GB.

Now since AB is more powerful than CD by the square on a line incommensurable with it in length, and a rectangle contained by AF, FB, equal to the fourth part of the square on CD and deficient by a square figure, has been applied to AB, [Margin: b — X.19] AF will be incommensurable in length with FB. And as AF is to FB, so — by lemma 2 of the preceding proposition — is $\text{rect}(AB,AF)$ to $\text{rect}(AB,FB)$; while $\text{rect}(AB,AF)$ is equal to AG^2 , and $\text{rect}(AB,FB)$ to GB^2 , by lemma 1 of the same preceding proposition, [Margin: c — III.31] because the angle AGB is right, being in the semicircle AGB. Therefore as AF is to FB, so is AG^2 to GB^2 ; and since AF, FB are incommensurable in length, [Margin: d — X.10] the squares AG^2 , GB^2 will be incommensurable. [Margin: e — X def. 4] Therefore the lines AG, GB are incommensurable in square. And since the square on the rational line AB is rational [Margin: f — I.47] and equal to the sum of the squares of AG, GB, the sum of the squares of AG, GB will also be rational. Again, since — by lemma 1 of the preceding proposition — the rectangle contained by AF, FB was made equal to the square on CE, [Margin: g — VI.17] it is equal to FG^2 (because the angle AGB is right and FG is perpendicular to AB); hence the squares FG^2 , CE^2 are equal, and therefore the lines

Diagram. The construction for Proposition 34: a semicircle on the diameter AB with the perpendicular FG raised to the circumference and the chords AG, GB joined; below, the line CD bisected at E.

- G (apex, on the circumference): where the perpendicular meets the semicircle
- A (left endpoint of the diameter)

- F (on the diameter, near B): foot of the perpendicular FG
- B (right endpoint of the diameter)
- C, E, D (the lower line): the line CD bisected at E

FG, CE are equal. Therefore CD, being double CE, will also be double FG. Hence, by lemma 3 of the preceding proposition, $\text{rect}(AB, CD)$ will be double $\text{rect}(AB, FG)$ (since FG is half of the lesser line, CD). [Margin: h — X.22] But $\text{rect}(AB, CD)$, contained by rationals commensurable in square only, is medial. Therefore $\text{rect}(AB, FG)$ too, being commensurable with it since it is its half, is medial, by the corollary to proposition 24 of this book. And to $\text{rect}(AB, FG)$, by lemma 1 of the preceding proposition, the rectangle contained by AG, GB is equal. Therefore the rectangle contained by AG, GB is also medial. But it was shown that the sum of their squares is rational. Two lines AG, GB have therefore been found, incommensurable in square, which make the sum of their squares rational but the rectangle contained by them medial. Which was to be done.

SCHOLIUM.

It is demonstrated here, in a certain ancient scholium, that it is possible for two irrational spaces to compose a rational space, roughly in the following way.

Let a rational line AB be set out, together with two numbers C, D, of which C is the greater, not having to one another the ratio a square number has to a square number; and let it be arranged that as C is to D, so is the square on AB to the square on AE; and finally, with the square on AB described, let EF be drawn perpendicular to AB. Since, then, as C is to D, so is the square on AB to the square on AE, but C does not have to D the ratio a square number has to a square number, [Margin: i — X.9] the sides AB, AE of the said squares — squares not having to one another the ratio of a square to a square — will be incommensurable in length. And therefore, since AB, the whole, is incommensurable in length with the part AE, the same AB will also be incommensurable in length with the remainder EB, by the corollary to proposition 17 of this book. [Margin: k — VI.1] Now as AB is to AE, so is the square on AB to AF. Therefore, since AB, AE are incommensurable in length, [Margin: l — X.10] the square on AB and the rectangle AF will be incommensurable. Hence, the square on AB being rational — for AB too is posited rational — AF will be incommensurable with that square, and so irrational; and by the same argument we shall show that BF is irrational. But since AF, BF together compose the rational square on AB, it is clear that a rational can be produced from two irrationals. Which was proposed.

Diagram. The square on AB for the ancient scholium, with EF raised perpendicular from the point E on the base; the numbers C and D are set beside it as 6 and 4.

- F (top, above E): where the perpendicular EF meets the top side of the square
- A (bottom left): endpoint of AB
- E (bottom, point of division): foot of the perpendicular EF
- B (bottom right): endpoint of AB
- Numeric labels below the base: "C, 6. D, 4." (that is, $C = 6$ and $D = 4$)

But if AF, FB be rational, we shall show that the whole AFB composed of them is also rational. For since each of AF, FB is rational, they will be commensurable with one another. [Margin: m — X.16] Therefore the whole AFB composed of them will be commensurable with each of them. Hence AFB, the whole, being commensurable with each of the rationals AF, FB, [Margin: n — X def. 9] is itself rational too. Which was proposed.

PROBL. 11. PROPOS. 35.

[Margin: xxxiv. / xxvii.]

To find two straight lines incommensurable in square which make the sum of their squares medial, but the rectangle contained by them rational.

[Margin: a — X.32] Let two medial lines AB, CD, commensurable in square only, containing a rational rectangle, be taken, such that the greater, AB, is more powerful than the lesser, CD, by the square on a line incommensurable with it; and let the rest be carried out as in the preceding proposition. We shall then show, just as in the preceding proposition, that the lines AG, GB are incommensurable in square. And since the square on the medial AB, [Margin: b — I.47] being equal to the sum of the squares of AG, GB, is medial, the sum of the squares of AG, GB will also be medial. Again, as in the preceding proposition, we shall show that $\text{rect}(AB, CD)$ is double $\text{rect}(AB, FG)$, and hence that the former is commensurable with the latter. Therefore, since $\text{rect}(AB, CD)$ is rational by construction, $\text{rect}(AB, FG)$ too, being commensurable with it, will be rational. But by lemma 1 of proposition 33 of this book, the rectangle contained by AG, GB is equal to $\text{rect}(AB, FG)$. Therefore the rectangle contained by AG, GB is also rational. But it was shown that the sum of the squares of AG, GB is medial. Therefore the two lines AG, GB, which were shown to be incommensurable in square, make the

sum of their squares medial and the rectangle comprehended by them rational. Two lines AG, GB have therefore been found, incommensurable in square, which make the sum of their squares medial but the rectangle contained by them rational. Which was to be done.

Diagram. The construction for Proposition 35: a semicircle on the diameter AB with the perpendicular FG raised to the circumference and the chords AG, GB joined; below, the line CD bisected at E.

- G (apex, on the circumference): top of the perpendicular
- A (left endpoint of the diameter)
- F (on the diameter, near B): foot of the perpendicular FG
- B (right endpoint of the diameter)
- C, E, D (the lower line): the line CD bisected at E

PROBL. 12. PROPOS. 36.

[Margin: xxxv. / xxix.]

To find two straight lines incommensurable in square which make both the sum of their squares medial and the rectangle contained by them medial, and this rectangle incommensurable with the sum of their squares.

[Margin: a — X.33] Let two medial lines AB, CD, commensurable in square only, containing a medial rectangle, be taken, such that the greater, AB, is more powerful than the lesser, CD, by the square on a line incommensurable with it in length; and let the rest be carried out as in proposition 34. We shall then show, in like manner, that the lines AG, GB are incommensurable in square. And since the square on the medial AB is medial and equal to the sum of the squares of AG, GB, the sum of the squares of AG, GB will also be medial. Again, as in proposition 34, we shall demonstrate that $\text{rect}(AB, CD)$ is double $\text{rect}(AB, FG)$, and hence that the former is commensurable with the latter. Therefore, since by construction the rectangle contained by AB,

Diagram. The construction for Proposition 36: a semicircle on the diameter AB with the perpendicular FG raised to the circumference and the chords AG, GB joined; below, the line CD bisected at E.

- G (apex, on the circumference): top of the perpendicular
- A (left endpoint of the diameter)
- F (on the diameter, near B): foot of the perpendicular FG

- B (right endpoint of the diameter)
- C, E, D (the lower line): the line CD bisected at E

CD is medial, the rectangle contained by AB, FG — that is, the one contained by AG, GB, which by lemma 1 of proposition 33 of this book is equal to it — will also be medial, by the corollary to proposition 24 of this book. And since AB is posited incommensurable in length with CD, while CE is commensurable in length with the same CD (because CD is double CE), [Margin: c — X.13] AB, CE will be incommensurable in length. But as AB is to CE, so — by lemma 3 of proposition 19 of this book — is the square on AB to rect(AB,CE), that is, to rect(AB,FG), that is, to the rectangle under AG, GB (for the rectangles under AB, FG and under AG, GB are equal, by lemma 1 of proposition 33 of this book). [Margin: d — X.10] Therefore the square on AB is incommensurable with the rectangle under AG, GB. Two straight lines AG, GB have therefore been found, incommensurable in square, which make both the sum of their squares medial and the rectangle contained by them medial, and this incommensurable with the sum of their squares. Which was to be done.

SCHOLIVM.

From this problem we shall easily accomplish what, at proposition 24 of this book, we undertook to demonstrate — namely:

To find two medials incommensurable both in length and in square.

Since it has been shown that both the sum of the squares of the lines AG, GB and the rectangle contained by them are medial, and that this rectangle is incommensurable with that sum, the lines whose squares are that medial sum and this medial rectangle will likewise be medials incommensurable both in length and in square. For if they were commensurable in square, their squares too — that is, the sum of the squares of AG, GB and the rectangle under AG, GB — would be commensurable, which is not assumed. Therefore, if AB be taken whose square is the sum of the squares of AG, GB, and another line whose square is the rectangle under AG, GB — that is, the mean proportional between AG, GB — two medials will have been found incommensurable both in length and in square. Which was proposed.

PRINCIPIVM SENARIORVM

The Origin of the Six Compound Irrationals, by Composition

THEOR. 25. PROPOS. 37.

[Margin: xxxvi. / xxx.]

If two rational lines commensurable in square only be added together, the whole is irrational; and let it be called binomial (a line "of two names").

Let two rational lines AB, BC, commensurable in square only, found by lemma 2 of proposition 21 of this book, be added together. I say that the whole AC is irrational. For since AB is incommensurable in length with BC, and as AB is to BC, so is $\text{rect}(AB, BC)$ to the square on BC, by the lemma to proposition 31 of this book, [Margin: a — X.10] the rectangle under AB, BC will be incommensurable with the square on BC. But commensurable with the rectangle under AB, BC is its double — namely, that which is contained twice under AB, BC; and commensurable with the square on BC is the square on the line AB (for all these are rational). [Margin: b — X.16] Hence the sum of the squares of the lines AB, BC is also commensurable with the same square on BC. Therefore, by what we showed in the scholium to proposition 14 of this book, that which is contained twice under AB, BC is also incommensurable with the sum of the squares of the lines AB, BC. For this reason the sum of that which is contained twice under AB, BC and the sum of the squares of the lines AB, BC — that is, the square on the whole AC [Margin: c — II.4] (for this square is equal to the squares of the lines AB, BC together with the rectangle under AB, BC taken twice) — [Margin: d — X.17] will be incommensurable with the sum of the squares of the lines AB, BC. But the sum of the squares of the lines AB, BC is rational, being shown to be commensurable with the rational square on the rational line BC. Therefore the square on AC, being incommensurable with the rational square on BC, [Margin: e — X def. 10] is irrational; and so the line AC itself will be irrational. Let it be called "of two names," or, as others say, a binomial, because it is composed of two names — namely, of the two rational lines AB, BC, commensurable in square only. If, then, two rational lines commensurable in square only be added together, the whole will be irrational, etc. Which was to be shown.

Diagram. A single straight line divided into two segments, representing the binomial $AC = AB + BC$.

- A (left endpoint)
- B (point of division)
- C (right endpoint)

SCHOLIVM.

Thus from two rational lines commensurable in square only two irrational lines are produced. For the line which is the mean proportional between them [Margin: f — X.22] is irrational, and is called the medial; while the line composed of them [Margin: g — X.37] is irrational, and is called "of two names" (the binomial).

THEOR. 26. PROPOS. 38.

[Margin: xxxvij. / xxxj.]

If two medial lines commensurable in square only, containing a rational rectangle, be added together, the whole is irrational; and let it be called a first bimedial ("first from two medials").

Let two medials AB, BC, [Margin: a — X.28] found above, commensurable in square only, containing a rational rectangle, be added together. I say that the whole AC is irrational. For since AB is to BC as $\text{rect}(AB, BC)$ is to the square on BC, as was shown in the lemma to proposition 31 of this book, and AB, BC are

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X.37 (Prop. 38: first bimedial) — conclusion of the demonstration ... they are incommensurable in length. Therefore the rectangle $AB \cdot BC$ and the square on BC are incommensurable [Margin: (b) Elements X.10]. But twice the rectangle $AB \cdot BC$ is commensurable with the rectangle $AB \cdot BC$, and the sum of the squares on the lines AB and BC is commensurable with the square on BC . (For since AB and BC are commensurable in square, their squares are commensurable [Margin: (c) Elements X.16], and hence the sum of their squares will also be commensurable with the square on BC .) Therefore twice the rectangle $AB \cdot BC$ and the sum of the squares on the lines AB and BC are incommensurable with one another, by what was demonstrated in the scholium to Prop. 14 of this book. Therefore the sum of the squares on AB and BC — that is, the square on AC , which is equal to that sum [Margin: (d) Elements II.4] — is likewise incommensurable with twice the rectangle $AB \cdot BC$ [Margin: (e) Elements X.7]. But the rectangle taken once on AB , BC is commensurable with twice the rectangle $AB \cdot BC$, since it is half of it. Therefore the square on AC is incommensurable with the rectangle $AB \cdot BC$ [Margin: (f) Elements X.13]. But the rectangle $AB \cdot BC$ is posited rational. Therefore the square on AC is irrational [Margin: (g) Elements X, def. 10], and on that account the line AC will also be irrational [Margin: (h) Elements X, def. 11]. Let it be called the first of two medials, or, as others put it, the first bimedial. If, then, two medials commensurable in square only be posited, etc. Which was to be shown.

Diagram. A single straight line divided into two segments, representing the first bimedial $AC = AB + BC$.

- A (left end)
- B (division point)
- C (right end)

LEMMA

The rectangle contained by a rational line and an irrational line is irrational.

Let the rectangle ABCD be contained by the rational line AB and the irrational line BC. I say that it is irrational. For if it be said to be rational, then, applied to the rational line AB, it will make its breadth BC rational [Margin: (i) Elements X.21]. But this is absurd, for BC is posited irrational. Therefore the rectangle AC is not rational; therefore it is irrational. Which was the thing proposed.

Diagram. A rectangle with a diagonal drawn from the upper-left to the lower-right corner. The top side AB is drawn as a broken (dashed) line.

- A (upper-left corner)
- B (upper-right corner)
- C (lower-right corner)
- D (lower-left corner)
- The diagonal AC divides the rectangle; side AB is the rational line, side BC the irrational line.

THEOR. 27. PROPOS. 39. [Margin: xxxviiij. / xxxij.]

(X.38 — *the second bimedial*)

If two medials commensurable in square only, containing a medial rectangle, be added together, the whole is irrational; let it be called the second of two medials.

Let two medials found above [Margin: (a) Elements X.29], AB and BC, commensurable in square only and containing a medial rectangle, be added together. I say that the whole line AC is irrational. Let a rational line DE be set out, and to it let there be applied [Margin: (b) Elements I.45] a rectangle DF equal to the square on AC; and to the same line DE let there be applied a second rectangle DG equal to the sum of the squares on the lines AB and BC. Now since the square on AC — that is, the rectangle DF — is equal to the two squares on AB and BC together with twice the rectangle AB·BC [Margin: (c) Elements II.4], the remainder HF will be equal to the remaining rectangle, namely that contained twice under AB, BC. But by hypothesis the rectangle AB·BC is medial; therefore twice the rectangle AB·BC — that is, HF — is also commensurable with it and medial, by the corollary to Prop. 24 of this book. Again, since the squares on the medials AB, BC, which are commensurable in square, are commensurable, the

figure compounded of them — namely the rectangle DG — will also be commensurable with either of them [Margin: (d) Elements X.16]. But each of the squares on the medials AB, BC is medial. Therefore DG too will be medial, by the corollary to Prop. 24 of this book. Since, then, the medial figures DG and HF are applied to the rational line DE (for GH is equal to the rational DE [Margin: (e) Elements I.34]), their breadths EG and GF will be rational, but incommensurable in length with DE [Margin: (f) Elements X.23]. Again, since AB and BC are incommensurable in length, and as AB is to CE, so is the square on AB to the rectangle AB·BC, by the third lemma to Prop. 19 of this book, the square on AB will be incommensurable with the rectangle AB·BC, as was shown in the scholium to Prop. 10 of this book. But the sum of the squares on AB and BC was shown to be commensurable with the square on AB, since it is double of it. Therefore, by what we said in the scholium to Prop. 14 of this book, the sum of the squares on AB and BC will also be incommensurable with twice the rectangle AB·BC — that is, with the rectangle HF. Since, then, as DG is to HF, so is EG to GF [Margin: (g) Elements VI.1], EG will be incommensurable in length with GF [Margin: (h) Elements X.10]. But it was already shown that EG and GF are rational. Therefore EG and GF are rational lines commensurable in square only, and hence the whole line EF, compounded of them, is irrational [Margin: (i) Elements X.37]. Therefore, by the preceding lemma, DF, contained by the rational line DE and the irrational line EF, is irrational; and on that account the square on AC, being equal to DF, is also irrational. Therefore the line AC too is irrational. Let it be called the second of two medials, or, as others prefer, the second bimedial. If, then, two medials commensurable in square only, etc. Which was to be demonstrated.

Diagram. At the top, a short straight line A—B—C (the second bimedial $AC = AB + BC$). Below it, a rectangle DF standing on the rational line DE (its left vertical side, D at top, E at bottom), divided by the vertical line HG into two rectangles: DG on the left and HF on the right.

- A, B, C (top line: A left end, B division point, C right end)
- D (upper-left corner of the rectangle)
- H (point on the top side where the dividing line meets it)
- E (lower-left corner)
- G (point on the bottom side where the dividing line meets it)
- F (lower-right corner)
- (The upper-right corner of the rectangle is unlabelled.)

SCHOLIVM

Euclid, in the preceding proposition, called the irrational line AC — the one compounded of two medials commensurable in square only and containing a rational rectangle — the first of two medials; but here he calls the irrational line AC — the one compounded of two medials commensurable in square only and containing a medial rectangle — the second of two medials, because the rational is prior to the medial (which is irrational) both by nature and in our knowledge, by Prop. 22 of this book.

LEMMA

If a straight line be cut into two unequal parts, the sum of the squares on the parts will exceed twice the rectangle contained by the parts by the square on the line by which the greater part exceeds the less.

Let the straight line AD be cut unequally at B, and let AB be the greater part; in it let BC be taken equal to the lesser part BD, so that AC is the excess by which AB exceeds BD — that is, BC equal to BD. I say that the squares on AB and BD together are greater than twice the rectangle AB·BD by the square on the line AC. For since the squares on AB and BC are equal to twice the rectangle AB·BC together with the square on AC [Margin: (k) Elements II.7], and BC is equal to BD, the squares on AB and BD will likewise be equal to twice the rectangle AB·BD together with the square on AC. Therefore the squares on AB and BD are greater than twice the rectangle AB·BD by the square on the line AC. Which was the thing proposed.

Diagram. A single straight line divided at three interior points.

- A (left end)
- C (first division point)
- B (second division point)
- D (right end)
- [Margin: (k) Elements II.7]

COROLLARIVM

From this it follows that the squares on unequal parts are, without qualification, greater than the rectangle twice contained by the unequal parts.

THEOR. 28. PROPOS. 40. [Margin: xxxix. / xxxiiij.]

(X.39 — *the major line*)

If two straight lines incommensurable in square be added together such that the sum of their squares is rational, but the rectangle contained by them is medial, the whole line will be irrational; let it be called the major.

Let two straight lines found above [Margin: (a) Elements X.34], AB and BC, incommensurable in square, be added together, making the sum of their squares rational but the rectangle contained by them medial. I say that the whole line AC is irrational. For since the rectangle AB·BC is posited medial, twice the rectangle AB·BC will also be commensurable with it and medial, by the corollary to Prop. 24 of this book, and hence irrational. But the sum of the squares on the lines AB, BC is posited rational. Therefore twice the rectangle AB·BC is incommensurable with the sum of the squares on the lines AB, BC [Margin: (b) Elements X, def. 10]. Therefore twice the rectangle AB·BC together with the squares on AB, BC — that is, the square on AC — is incommensurable with the sum of the squares on the lines AB, BC [Margin: (c) Elements X.17]. Since, then, this sum is posited rational, the square on AC will be incommensurable with it, and therefore irrational [Margin: (d) Elements X, def. 10]. And on that account the line AC is irrational [Margin: (e) Elements X, def. 11]. Let it be called the major, because — since AC is equal in square to the two squares on AB, BC together with twice the rectangle AB·BC — the sum of the squares on the lines AB, BC is greater than twice the rectangle AB·BC, by the preceding lemma [Margin: (f) Elements II.4]. Therefore, since the sum of the squares on AB, BC is rational while twice the rectangle AB·BC is medial, AC will rightly be called the major, since it is proper to take the name from the rational, which is the greater. If, then, two straight lines incommensurable in square be added together, etc. Which was to be demonstrated.

Diagram. A single straight line divided into two segments, representing the major line $AC = AB + BC$.

- A (left end)
- B (division point)
- C (right end)
- [Margin: (b) Elements X, def. 10; (c) Elements X.17]

THEOR. 29. PROPOS. 41. [Margin: xl. / xxxiv.]

(X.40 — the side of a rational plus a medial area)

If two straight lines incommensurable in square be added together such that the sum of their squares is medial, but the rectangle contained by them is rational, the whole line will be irrational; let it be called the "rational-and-medial-powered" (the side of a rational plus a medial area).

Let two straight lines found above [Margin: (a) Elements X.35], AB and BC, incommensurable in square, be added together, making the sum of their squares medial but the rectangle contained by them rational. I say that the whole line AC is irrational. For since the sum of the squares on the lines AB, BC is medial, while what is contained twice under AB, BC is rational — its half, namely the rectangle contained once under AB, BC, being posited rational — the sum of the squares on the lines AB, BC will be incommensurable with twice the rectangle AB·BC [Margin: (b) Elements X, def. 10]. Therefore the sum of the squares on the lines AB, BC together with twice the rectangle AB·BC — that is, the square on AC — is incommensurable with twice the rectangle AB·BC [Margin: (c) Elements X.17]. Since, then, twice the rectangle AB·BC is, as we said, rational, the square on AC will be incommensurable with it and irrational [Margin: (d) Elements X, def. 10]. And thereupon the line AC too will be irrational [Margin: (e) Elements X, def. 11]. Let it be called "rational-and-medial-powered," because it is equal in square to a medial — namely the sum of the squares on the lines AB, BC — and to a rational, namely the rectangle twice contained under AB, BC. And it is fitting for the rational to be set before the irrational. If, then, two straight lines incommensurable in square be added together, etc. Which was to be demonstrated.

Diagram. A single straight line divided into two segments ($AC = AB + BC$).

- A (left end)
- B (division point)
- C (right end)
- [Margin: (b) Elements X, def. 10; (c) Elements X.17]

THEOR. 30. PROPOS. 42. [Margin: xlj. / xxxv.]

(X.41 — the side of the sum of two medial areas)

If two straight lines incommensurable in square be added together such that they make both the sum of their squares medial and the rectangle

contained by them medial, and moreover incommensurable with the sum of their squares, the whole straight line will be irrational; let it be called the "power of two medials" (the side of the sum of two medial areas).

Let two straight lines found above [Margin: (a) Elements X.36], AB and BC, incommensurable in square, be added together, making the sum of their squares medial and the rectangle contained by them medial, and moreover incommensurable with the sum of their squares. I say that the whole line AC is irrational. Let a rational line DE be set out, and to it let there be applied [Margin: (b) Elements I.45] a rectangle DF equal to the square on AC; and let DG be applied equal to the sum of the squares on the lines AB, BC. Now since the square on AC — that is, the rectangle DF — is equal to the sum of the squares on the lines AB, BC together with twice the rectangle AB·BC [Margin: (c) Elements II.4], the remainder HF will be equal to twice the rectangle AB·BC. But since both the sum of the squares on the lines AB, BC (that is, the rectangle DG) and the rectangle AB·BC — and hence, by the corollary to Prop. 24 of this book, twice AB·BC, namely HF (being commensurable with AB·BC) — are posited medial, DH and HF, applied to the rational line DE, will make breadths EG and GF that are rational but incommensurable in length with DE [Margin: (d) Elements X.23]. Again, since the sum of the squares on the lines AB, BC (that is, the rectangle DG) is posited incommensurable with the rectangle AB·BC, and its double is commensurable with the rectangle AB·BC — namely what is contained twice under AB, BC, that is, HF — DG and HF will be incommensurable [Margin: (e) Elements X.13]. Therefore, since as DG is to HF, so is EG to GF [Margin: (f) Elements VI.1], EG and GF will be incommensurable in length [Margin: (g) Elements X.10]. But EG, GF were shown to be rational. Therefore EG, GF are rational lines commensurable in square only, and on that account the whole line EF, compounded of them, is irrational [Margin: (h) Elements X.37]. Therefore DF, contained by the rational line DE and the irrational line EF, is irrational, by the lemma to Prop. 38 of this book. Therefore the square on AC too, being equal to it, is irrational; and thereupon the line AC itself is irrational [Margin: (i) Elements X, def. 11]. Let it be called the "power of two medials," because it is equal in square to both the sum of the squares on the lines AB, BC and twice the rectangle AB·BC. If, then, two straight lines incommensurable in square be added together such that they make both the sum of their squares medial, etc. Which was to be demonstrated.

Diagram. At the top, a short straight line A—B[?]-C ($AC = AB + BC$). Below it, a rectangle DF standing on the rational line DE (left vertical side, D at top, E at

bottom), divided by the vertical line HG into rectangle DG (left) and rectangle HF (right).

- A, B[?], C (top line: A left end, division point read as B but printed glyph is ambiguous — could be P; C right end)
- D (upper-left corner)
- H (point on the top side at the dividing line)
- E (lower-left corner)
- G (point on the bottom side at the dividing line)
- F (lower-right corner)
- (The upper-right corner is unlabelled.)

LEMMA

If a straight line be cut into unequal parts, and again into other unequal parts, the squares on the more unequal parts together will be greater than the squares on the less unequal parts together.

Let the straight line AB be cut unequally at C, and again unequally at D, and let the former parts AC, CB be more unequal than the latter parts AD, DB. I say that the squares on AC, CB together are greater than the squares on AD, DB together. For let AB be bisected at E; the point D will fall either between E and C, or between A and E. First let it fall between E and C. Since, then, the rectangle AC·CB together with the square on EC is equal to the square on EB [Margin: (k) Elements II.5], and likewise the rectangle AD·DB together with the square on ED is equal to the same square on EB, the rectangle AC·CB together with the square on EC will be equal to the rectangle AD·DB together with the square on ED. Therefore, when the unequal squares on the unequal lines EC, ED are taken away — since the square on EC is greater than the square on ED — the remaining rectangle AC·CB will be less than the remaining rectangle AD·DB; and on that account twice the rectangle AC·CB will also be less than twice the rectangle AD·DB. Now since the square on AB is equal both to twice the rectangle AC·CB together with the squares on AC, CB, and to twice the rectangle AD·DB together with the squares on AD, DB [Margin: (l) Elements II.4], twice the rectangle AC·CB together with the squares on AC, CB will be equal to twice the rectangle AD·DB together with the squares on AD, DB. Therefore, since twice the rectangle AC·CB was shown to be less than twice the rectangle AD·DB, the remaining squares on AC, CB will be greater than the remaining squares on AD, DB. Which was the thing proposed.

Diagram. A straight line divided at interior points (first case, D between E and C).

- A (left end)
- E (midpoint / bisection point)
- D (division point)
- C (division point)
- B (right end)
- [Margin: (k) Elements II.5]

Now let D fall between A and E. Since, then, the parts AC, CB are posited more unequal than the parts AD, DB, AD (which is the lesser of the latter) will be greater than CB (which is the lesser of the former); and hence, since AE, EB are equal, the remainder EC will be greater than the remainder DE. Accordingly, since the rectangle AC·CB together with the square on EC is equal to the square on EB [Margin: (m) Elements II.5], and likewise the rectangle AD·DB together with the square on ED is equal to the square on AE — that is, to the same square on EB — the rectangle AC·CB together with the square on EC will be equal to the rectangle AD·DB together with the square on DE. Therefore, as before, we shall demonstrate that the squares on AC, CB are greater than the squares on AD, DB. Which was the thing proposed.

Diagram. A straight line divided at interior points (second case, D between A and E).

- A (left end)
- D (division point)
- E (midpoint / bisection point)
- C (division point)
- B (right end)
- [Margin: (m) Elements II.5]

THEOR. 31. PROPOS. 43. [Margin: xlij. / xxxvj.]

(X.42 — a binomial is divided into its terms at one point only)

A line composed of two terms (a binomial) is divided into its terms at one point only.

Let AB, a binomial, be divided at the point C into its terms, so that AC, CB are rational lines commensurable in square only, as Prop. 37 of this book requires. I say that AB cannot be divided at another point into other terms — that is, into other rational lines commensurable in square only. For if it be possible, let it be divided again into its terms at D. It is then evident that AB is not divided at C into

two equal parts; for if it were bisected, AC and CB would be commensurable in length, which is absurd, since they are posited rational lines commensurable in square only...

Diagram. Three straight lines, each with endpoints A (left) and B (right) and two interior division points C and D placed in different relative positions to illustrate the alternative (impossible) divisions.

- Line 1: A ——— C — D — B (C then D, near B)
- Line 2: A ——— D — C — B (D then C)
- Line 3: A — D ——— C — B (D near A, C near B) ...commensurable in square only. In the same way AB is not bisected at D either, since AD and DB too are commensurable in square only. Therefore AB is cut unequally both at C and at D, and consequently the parts AC, CB are either less unequal, or more unequal, than the parts AD, DB. And therefore, by the preceding lemma, the squares on AC, CB are either less, or greater, than the squares on AD, DB. (For the parts AD, DB — wherever the point D may fall — will not be equal to the parts AC, CB, each to each, that is, the greater to the greater and the lesser to the lesser. For on that supposition AB would be cut, in the second division, into the very same terms as in the first division — which is not what is assumed.) Now since, when unequal magnitudes are subtracted from equal ones, the excess of the remainders equals the excess of the magnitudes taken away, as we taught in the 16th axiom of Book I: the squares on AC, CB together with twice the rectangle under AC, CB equal the squares on AD, DB together with twice the rectangle under AD, DB [Margin: a — *Elements* II.4], since both the one sum and the other equal the square on AB. Hence it follows that the amount by which the removed sum of the squares on AC, CB exceeds the removed sum of the squares on AD, DB is the same as the amount by which the remaining twice-rectangle under AC, CB exceeds the remaining twice-rectangle under AD, DB.

But the excess of the sum of the squares on AC, CB over the sum of the squares on AD, DB is a rational space. (For since the lines AC, CB are rational, and their squares therefore rational, the sum of their squares will also be rational, by what we showed in the scholium to Proposition 34 of this book; and by the same reasoning the sum of the squares on AD, DB will be rational. Therefore, since a rational exceeds a rational by a rational, as we demonstrated in the scholium to Proposition 27 of this book, it is clear that the excess of the sum of the squares on AC, CB over the sum of the squares on AD, DB is a rational space.) Therefore the

excess of twice the rectangle under AC, CB over twice the rectangle under AD, DB is also a rational space [Margin: b — *Elements* X.22]. And since the rectangle under AC, CB is medial, twice the rectangle under AD, DB will be commensurable with it, by the corollary of Proposition 24 of this book, and therefore medial [Margin: c — *Elements* X.27]; and in the same way twice the rectangle under AC, CB will be medial. Therefore, since a medial does not exceed a medial by a rational, the excess of twice the rectangle under AC, CB over twice the rectangle under AD, DB will not be a rational space. But we showed that it is rational. This is absurd. Therefore a binomial AB is not divided into its terms at any point other than C. And so it is divided into its terms at one point only. Q.E.D.

Diagram. Three horizontal straight lines, each running from A on the left to B on the right, each marked with two interior division points showing alternative placements of the division:

- Line 1: A ——— C — D — B (C to the left of D, both toward the right end)
- Line 2: A ——— D — C — B (D to the left of C)
- Line 3: A — D ——— C — B (D near the middle, C toward the right)

Labels: A, B — endpoints of the line; C, D — the two candidate division points.

THEOR. 32. PROPOS. 44.

[Margin: xliij. / xxxvij.]

A first bimedial line is divided into its terms at one point only.

Let AB be a first bimedial, divided into its terms at the point C, so that AC, CB are medials commensurable in square only which contain a rational area, as Proposition 38 of this book requires. I say that AB cannot be divided at another point into other terms — terms which would likewise be medial lines commensurable in square only and containing a rational area. For, if it can be done, let it be divided into other terms AD, DB. Then, wherever the point D may fall, we shall prove — in the same way as in the preceding proposition — that the excess of twice the rectangle under AC, CB over twice the rectangle under AD, DB is the same as the excess of the sum of the squares on AC, CB over the sum of the squares on AD, DB.

But the excess of twice the rectangle under AC, CB over twice the rectangle under AD, DB is a rational space. (For since AC, CB are assumed to contain a rational area, twice the rectangle under AC, CB will be commensurable with what is

contained once under AC, CB, and so rational. In the same way, twice the rectangle under AD, DB will be rational. Therefore, since a rational exceeds a rational by a rational, as we showed in the scholium to Proposition 27 of this book, it is clear that the excess of twice the rectangle under AC, CB over twice the rectangle under AD, DB is a rational space.) Therefore the excess of the sum of the squares on AC, CB over the sum of the squares on AD, DB is also a rational space.

But since the lines AC, CB are medial and commensurable in square, their squares too will be medial and commensurable [Margin: a — *Elements* X.16]. Therefore the sum composed of them will be commensurable with each of them. So, since each of them is medial, the sum composed of them will also be medial, by the corollary of Proposition 24 of this book. In the same way we shall show that the sum of the squares on AD, DB is medial [Margin: b — *Elements* X.27]. Therefore, since a medial does not exceed a medial by a rational, the excess of the sum of the squares on AC, CB over the sum of the squares on AD, DB will not be a rational space. But we also showed it to be rational. This is absurd. Therefore AB, a first bimedial, is not divided into its terms at any point other than C. And so it is divided into its terms at one point only. Q.E.D.

Diagram. Three horizontal straight lines, each from A on the left to B on the right, each with two interior division marks showing alternative division points:

- Line 1: A ——— C — D — B (C to the left of D)
- Line 2: A ——— D — C — B (D to the left of C)
- Line 3: A — D ——— C — B (D toward the middle, C toward the right)

Labels: A, B — endpoints; C, D — the two candidate division points.

THEOR. 33. PROPOS. 45.

[Margin: xlv. / xxxviii.]

A second bimedial line is divided into its terms at one point only.

Let AB be a second bimedial, divided into its terms at the point C, so that AC, CB are medials commensurable in square only which contain a medial area, as Proposition 39 of this book requires. I say that AB cannot be divided at another point into other terms — terms which would likewise be medial lines commensurable in square only and containing a medial area. For, if it can be done, let it be divided into other terms AD, DB. Then, wherever the point D may fall, we shall show, as in Proposition 43, that the squares on AC, CB are either greater or less than the squares on AD, DB.

Let a rational line EF be set out [Margin: a — *Elements* I.45], and to it let there be applied the rectangle EG equal to the square on AB, and likewise EH equal to the squares on AC, CB. Then the remainder IG will be equal to what is contained twice under AC, CB, since the square on AB [Margin: b — *Elements* II.4] equals the squares on AC, CB together with twice the rectangle under AC, CB. In the same way, if to BF there is applied EK equal to the squares on AD, DB, the remainder LG will be equal to the rectangle comprehended twice under AD, DB.

Diagram. A square figure standing on the applied lines. Along the top edge the line runs A — C — D — B, with C and D marking division points (D close to B). The left side descends from E (below A) to the bottom-left corner F, and is subdivided by tick-marks. The bottom edge runs F — H — K — G. Two interior vertical lines rise from H and K to the points I and L on the upper interior edge. The figure is thereby partitioned into the rectangles named in the proof (EH, EK, IG, EG, KG, and the like).

- A, C, D, B (top edge): the divided line and its division points
- E, I, L (upper interior row): E the upper-left corner; I, L interior points
- F, H, K, G (bottom edge): F the lower-left corner, G the lower-right corner; H, K interior points

**

And since the squares on AC, CB are unequal to the squares on AD, DB, the rectangles EH, EK, which are equal to them, will also be unequal; and consequently the lines FH, FK will be unequal. Again, since the squares on AC, CB are greater — by the lemma of Proposition 39 of this book — than twice the rectangle under AC, CB, EH will also be greater than IG. Therefore EH is greater than half of EG, and hence the line FH is greater than half of FG [recte: the source prints "FH"; the sense requires "FG"]. By the same reasoning we shall show that FK is greater than half of FG. Therefore the parts FH, HG are unequal to the parts FK, KG, each to each.

Now since AC, CB are medial and commensurable in square, their squares too will be medial and commensurable. Therefore the sum composed of them will be commensurable with each of them [Margin: c — *Elements* X.16]. So, since each of them is medial, the sum composed of them — that is, EH — will also be medial, by the corollary of Proposition 24 of this book. By the same reasoning EK will be medial. Therefore EH, EK, applied to the rational line EF, produce as their widths FH, FK [Margin: d — *Elements* X.23], which are rational but incommensurable in length with EF. In the same way, since the rectangle under AC, CB is assumed

medial, what is twice under AC, CB — that is, IG — will be commensurable with it, and so medial, by the corollary of Proposition 24 of this book. Therefore, when it is applied to the rational line HI, HG will be rational but incommensurable in length with HI [Margin: e — *Elements* X.23]. In just the same way we shall show that EG is medial, and that KG is rational but incommensurable in length with KL.

Again, since AC, CB are incommensurable in length, and since, as AC is to CB, so — by the third lemma of Proposition 19 of this book — is the square on AC to the rectangle under AC, CB, the square on AC will be incommensurable with the rectangle under AC, CB [Margin: f — *Elements* X.10]. But the sum of the squares on AC, CB is commensurable with the square on AC. (For since AC, CB are commensurable in square, and therefore their squares commensurable, the sum composed of them will be commensurable with either of them [Margin: g — *Elements* X.10], namely with the square on AC.) And what is twice under AC, CB is commensurable with the rectangle under AC, CB. Therefore the sum of the squares on AC, CB — that is, the rectangle EH — is incommensurable, by the scholium to Proposition 14 of this book, with twice the rectangle under AC, CB, that is, with IG. Consequently [Margin: h — *Elements* VI.1], since as EH is to IG, so is FH to HG, the lines FH, HG will be incommensurable in length [Margin: i — *Elements* X.10]. But they were shown to be rational. Therefore FH, HG are rational and commensurable in square only. Therefore the whole FG is irrational [Margin: k — *Elements* X.37] — the kind called a binomial — and is divided into its terms at the point G. In the same way we shall show that FG is divided as a binomial into other terms at another point K. But this cannot be [Margin: l — *Elements* X.43], as was demonstrated in what precedes. Therefore AB, a second bimedral, is not divided into its terms at any point other than C. And so it is divided into its terms at one point only. Q.E.D.

THEOR. 34. PROPOS. 46.

[Margin: xlv. / xxxix.]

A major line is divided into its terms at one point only.

Let a major line AB be divided into its terms at the point C, so that AC, CB are incommensurable in square, making the sum of their squares rational but the rectangle under them medial, as Proposition 40 of this book requires. I say that AB cannot be divided at another point into other terms — lines which would likewise be incommensurable in square, making the sum of their squares rational and the rectangle under them medial. For, if it can be done, let it be divided into other terms at D.

Then, wherever the point D may fall, we shall demonstrate — in the same way as in Proposition 43 — that the excess of the sum of the squares on AC, CB over the sum of the squares on AD, DB is the same as the excess of twice the rectangle under AC, CB over twice the rectangle under AD, DB. But the excess of the sum of the squares on AC, CB over the sum of the squares on AD, DB is a rational space (because, since both sums are rational by hypothesis, their excess too will be a rational space, as we showed in the scholium to Proposition 27 of this book). Therefore the excess of twice the rectangle under AC, CB over twice the rectangle under AD, DB is also a rational space. Now since the rectangle under AC, CB is assumed medial, twice the rectangle under AC, CB will also be medial, by the corollary of Proposition 24 of this book, being commensurable with it; and in the same way twice the rectangle under AD, DB will be medial. Therefore, since a medial does not exceed a medial by a rational [Margin: a — *Elements* X.27], the excess of twice the rectangle under AC, CB over twice the rectangle under AD, DB will not be a rational space. But we showed that it is rational. This is absurd. Therefore a major line AB is not divided into its terms at any point other than C. And so it is divided into its terms at one point only. Q.E.D.

Diagram. A single horizontal straight line from A on the left to B on the right, with two interior division points: A — C — D — B (C toward the middle, D close to B).

- A, B: endpoints of the major line
- C: the true division point; D: the assumed second division point

THEOR. 35. PROPOS. 47.

[Margin: xlvj. / xl.]

The side of a rational plus a medial area is divided into its terms at one point only.

Let AB be the side of a rational plus a medial area, divided into its terms at the point C, so that AC, CB are incommensurable in square, making the sum of their squares medial but the rectangle under them rational, as Proposition 41 of this book requires. I say that AB cannot be divided at another point into other terms — lines which would likewise be incommensurable in square, making the sum of their squares medial and the rectangle under them rational. For, if it can be done, let it be divided into other terms at D.

Then, wherever the point D may fall, we shall show — just as in Proposition 43 of this book — that the excess of twice the rectangle under AC, CB over twice the

rectangle under AD, DB is the same as the excess of the sum of the squares on AC, CB over the sum of the squares on AD, DB; but the excess of twice the rectangle under AC, CB over twice the rectangle under AD, DB [recte: BD] is a rational space

Diagram. A single horizontal straight line from A on the left to B on the right, with two interior division points: A ——— C ——— D — B (C toward the middle, D close to B).

- A, B: endpoints of the line
- C: the true division point; D: the assumed second division point

**

— which we shall show, as in Proposition 44. Therefore the excess of the sum of the squares on AC, CB over the sum of the squares on AD, DB is also a rational space. But since the sums composed of these squares are assumed to be medial: now a medial does not exceed a medial by a rational [Margin: a — *Elements* X.27], and so their excess will not be a rational space. Yet we also showed it to be rational. This is absurd. Therefore AB, the side of a rational plus a medial area, is not divided into its terms at any point other than C. And so it is divided into its terms at one point only. Q.E.D.

THEOR. 36. PROPOS. 48.

[Margin: xlvij. / xlj.]

The side of the sum of two medial areas is divided into its terms at one point only.

Let AB be the side of two medial areas, divided into its terms at C, so that AC, CB are incommensurable in square, making both the sum of their squares medial and the rectangle under them medial, and the rectangle incommensurable with the sum of their squares, as Proposition 42 of this book requires. I say that AB cannot be divided at another point into other terms — lines which, namely, would likewise be incommensurable in square, making both the sum of their squares medial and the rectangle under them medial, and the rectangle incommensurable with the sum of their squares. For, if it can be done, let it be divided into other terms at the point D, wherever this point D may fall. Let the same construction be made as in Proposition 45, and we shall show, as there, that the parts FH, HG are unequal to the parts FK, KG, each to each.

Now, since the sum of the squares on AC, CB is assumed medial, EH — being equal to that sum — will also be medial. Again, since the rectangle under AC, CB is likewise assumed medial, IG — being equal to twice that rectangle — will also be medial, by the corollary of Proposition 24 of this book. Therefore the medials EH, IH, applied to the rational line EF (for IH is equal to EF), produce as their widths FH, HG [Margin: a — *Elements* X.23], which are rational but incommensurable in length with EF. By the same reasoning we shall prove that the lines FK, KG too are rational but incommensurable in length with EF. And since the sum of the squares described on AC, CB — that is, the rectangle EH — is incommensurable, by hypothesis, with twice the rectangle under AC, CB, namely with the rectangle IG [Margin: b — *Elements* VI.1]; and since, as EH is to IG, so is FH to HG; the lines FH, HG will be incommensurable in length, as we showed in the scholium to Proposition 10 of this book. But they were shown to be rational. Therefore FH, HG are rational and commensurable in square only. Therefore the whole FG is irrational — the kind called a binomial — and is divided into its terms at the point H. In just the same way we shall demonstrate that FG is divided as a binomial into other terms at another point K. But that this cannot be [Margin: c — *Elements* X.43] has already been demonstrated. Therefore AB, the side of two medial areas, is not divided into its terms at any point other than C. And so it is divided into its terms at one point only. Q.E.D.

Diagram. A square figure like that of Prop. 45. Along the top edge the line runs A — C — D — B (D close to B). The left side descends from E (below A) to the bottom-left corner F, subdivided by tick-marks. The bottom edge runs F — H — K — G. Two interior verticals rise from H and K to the points I and L on the upper interior edge, partitioning the square into the rectangles named in the proof (EH, IH, IG, and the rest).

- A, C, D, B (top edge): the divided line and its division points
- E, I, L (upper interior row): E the upper-left corner; I, L interior points
- F, H, K, G (bottom edge): F the lower-left corner, G the lower-right corner; H, K interior points

DEFINITIONES SECVNDÆ.

(Second set of Definitions.)

Let there be set out a rational line, and a binomial divided into its terms, such that the square on its greater term exceeds the square on the lesser by the square on a straight line commensurable in length with the greater.

I.

If the greater term is commensurable in length with the assigned rational, let the whole be called a **first binomial**.

II.

But if the lesser term is commensurable in length with the assigned rational, let it be called a **second binomial**.

III.

But if neither of the terms is commensurable in length with the assigned rational, let it be called a **third binomial**.

Again, let the square on the greater term exceed the square on the lesser by the square on a straight line incommensurable in length with the greater.

IV.

If the greater term is commensurable in length with the assigned rational, let it be called a **fourth binomial**.

V.

But if the lesser term [is commensurable in length with the assigned rational], let it be called a **fifth binomial**.

(Definition VI begins here as the page catchword and continues on the following page.)

VI.

But if neither of the two terms [is commensurable in length with the assigned rational], let it be called a **sixth binomial**.

SCHOLIUM.

The number of these six lines that are called binomials is arrived at as follows. Since the terms of a line called a binomial are rational lines commensurable in square only, both terms cannot be commensurable in length with the assigned rational line — for otherwise the terms themselves would also be lines commensurable in length with one another, as we showed in the scholium to proposition 12 of this book, which is not assumed. Rather, only one of them (namely the greater or the lesser), or neither, can be so. And thus three kinds of lines called binomials arise. Again, since the terms are unequal (for otherwise the

lines would be commensurable in length with one another, which is not assumed), the greater term can be more powerful than the lesser by the square on a line either commensurable or incommensurable in length with itself. And if indeed it is more powerful by the square on a line commensurable in length with itself, then, together with the three kinds already mentioned, three distinct binomial lines are formed, namely the first, second, and third binomials. But if it is more powerful by the square on a line incommensurable in length with itself, then, together with those same three kinds, another three distinct binomial lines are produced, namely the fourth, fifth, and sixth binomials, as is clear from the definitions Euclid has given here. There are therefore six distinct lines called binomials.

Euclid elegantly makes the first three in order those in which the greater term is more powerful than the lesser by the square on a straight line commensurable in length with itself, and the remaining three the second set, in which the greater term is more powerful than the lesser by the square on a straight line incommensurable in length with itself — since the commensurable naturally comes before the incommensurable, as is evident.

Again, and rightly, among the binomial lines he places first of all the one in which the greater term is incommensurable in length with the assigned rational line; and second the one in which the lesser term is commensurable in length with that same assigned rational line — since the greater naturally precedes the lesser, the lesser being contained by the greater; and third the one in which neither of the terms is commensurable in length with the assigned rational. And with the remaining three in the same way, calling the first of that second set the fourth, the second the fifth, and the third the sixth.

PROBL. 13. PROPOS. 49.

[Margin: xl.viiij. / xl.ij.]

To find a first binomial.

Given two square numbers AB, CB — from those we described in the second scholium to proposition 29 of this book — whose difference AC is not square, so that AB, CB have to one another the ratio a square has to a square, but AB, AC do not have to one another the ratio a square has to a square: let some rational line D be set out, and let EF be taken commensurable with it in length. Then EF too, being commensurable with the rational D, will be rational. Next, by the corollary to proposition 6 of this book, let it be arranged that as the number AB is to AC, so

is the square on EF to the square on FG. [Margin: a. 6.dec. — *Elements* X.6] I say that EG is a first binomial.

Diagram. A construction figure of the standard Book X type. At the top a dotted number-line runs from A on the left to B on the right, with C marked between them (representing the numbers AB, AC, CB as schematic arrays of units). Below it are the magnitude-lines: line **D** (the assigned rational), shown with internal division-ticks; a long line **EG** running left (E) to right (G), divided at **F** into the segments EF and FG (the two terms of the binomial); and beneath it a shorter line **H** (the line on whose square the greater square exceeds the lesser).

- A, C, B: endpoints and division-point of the dotted number-array (numbers AB, AC, CB)
- D: the assigned rational line
- E, F, G: the binomial line EG divided at F (terms EF, FG)
- H: the excess-line

For since the squares on EF and FG have to one another the ratio that the numbers AB, AC have, they are commensurable; [Margin: a] so the straight lines EF, FG will also be commensurable, at least in square. But EF was shown to be rational; therefore FG is rational too. And since AB does not have to AC the ratio a square has to a square, the squares on EF, FG will likewise not have the ratio a square number has to a square. [Margin: b. 9.dec. — *Elements* X.9] Therefore the straight lines EF, FG are incommensurable in length. So EF, FG are rational lines commensurable in square only; [Margin: c. 37.dec. — *Elements* X.37] and therefore the whole EG is irrational — the line called a binomial.

I say that it is also a first binomial. For since as the number AB is to the number AC, so is the square on EF to the square on FG, and since AB is greater than AC, the square on EF will also be greater than the square on FG. Let it be greater, by the lemma to proposition 14 of this book, by the square on the straight line H. Since, then, as the number AB is to AC, so is the square on EF to the square on FG, then also, by conversion of the ratio, as AB is to CB — that is, to the excess by which the antecedent AB exceeds the consequent AC — so is the square on EF to the square on H, namely to the excess by which the antecedent square on EF exceeds the consequent square on FG. But AB has to CB the ratio a square has to a square. Therefore the squares on EF and H will also have to one another the ratio a square number has to a square number; [Margin: d. 9.dec. — *Elements* X.9] and consequently the straight lines EF and H are commensurable in length. Since, then, the greater term EF is more powerful than the lesser FG by the square on the straight line H, which is commensurable in length with it, and since this same

greater term EF is commensurable in length with the assigned rational D, EG will, by definition, be a first binomial. We have therefore found a first binomial. Which was to be done.

PROBL. 14. PROPOS. 50.

[Margin: xlix. / xliij.]

To find a second binomial.

Given two square numbers AB, CB, as in the preceding proposition, and some rational line D having been set out, let FG be taken commensurable with it in length. Then FG too, being commensurable with the rational D, will be rational. Next, by the corollary to proposition 6 of this book, let it be arranged that as the number AC is to the number AB, so is the square on FG to the square on EF.

[Margin: a. 6.dec. — *Elements* X.6] I say that EG is a second binomial.

For since the squares on FG, EF, having the ratio that the numbers AC, AB have, are commensurable, [Margin: a] the straight lines FG, EF will also be commensurable, at least in square. But FG was shown to be rational; therefore EF is rational too. And since AC, AB do not have the ratio that square numbers have, neither will the squares on FG, EF have the ratio that square numbers have.

[Margin: b. 9.dec. — *Elements* X.9] Therefore the straight lines FG, EF are incommensurable in length. So EF, FG are rational lines commensurable in square only; [Margin: c. 37.dec. — *Elements* X.37] and therefore the whole EG is irrational — the line called a binomial.

I say that it is also a second binomial. For since as the number AC is to the number AB, so is the square on FG to the square on EF, then by conversion, as AB is to AC, so is the square on EF to the square on FG; and since AB is greater than AC, the square on EF will also be greater than the square on FG. Let it be greater, by the lemma to proposition 14 of this book, by the square on the straight line H. We shall now show, as in the preceding proposition, that H is commensurable in length with EF. Since, then, the greater term EF is more powerful than the lesser FG by the square on the straight line H, commensurable in length with it, and since the lesser term FG is commensurable in length with the assigned rational D, EG will, by definition, be a second binomial. We have therefore found a second binomial. Which was to be done.

PROBL. 15. PROPOS. 51.

[Margin: l. / xliv.]

To find a third binomial.

Given two square numbers AC, CB, as in proposition 49, let another number I be taken which has to neither of them, AB or AC, the ratio a square has to a square. This will be achieved if I is taken as the non-square number next greater than AC. For then, since it is not a square, it will not have to the square AB the ratio a square has to a square. Again, being the non-square next greater than AC, it will differ from AC either by a unit alone or by two. (By a unit when AC is a number to which adding a unit does not make a square; by two when a unit added to AC does make a square. For example, if AC is 8, the non-square next greater is 10, differing from 8 by two, because a unit added to 8 makes the square 9; but if AC is 5, the non-square next greater is 6, differing from 5 by a unit, because a unit added to 5 does not make a square, and so on.) Hence, as we showed in the scholium to proposition 8 of book 8, no mean proportional falls between I and AC. Therefore they will not be similar plane numbers, and so they will not have the ratio that square numbers have.

Diagram. The same construction figure, now with an additional number-row. At the top a dotted number-line runs from A (left) through C to B (right); beneath it a second, shorter dotted row is labelled I (the auxiliary number). Below these are the magnitude-lines: line D (the rational) with division-ticks; the long line EG running from E (left) to G (right), divided at F into EF and FG; and beneath it the shorter line H.

- A, C, B: the dotted number-array (numbers AB, AC, CB)
- I: the auxiliary number (second dotted row)
- D: the assigned rational line
- E, F, G: the binomial line EG divided at F (terms EF, FG)
- H: the excess-line

[Margin: l. / xvij.]

Now, some rational line D having been set out, let it be arranged, by the corollary to proposition 6 of this book, that as I is to AB, so is the square on D to the square on EF. [Margin: a. 6.dec. — *Elements* X.6] Then the squares on D and EF, having the ratio that the number I has to the number AB, will be commensurable; and so the lines D, EF will also be commensurable, at least in square. Since, then, D is rational, EF will be rational too. But since I to AB — that is, the square on D to the square on EF — does not have the ratio a square number has to a square number, [Margin: b. 9.dec. — *Elements* X.9] the straight lines D and EF will be incommensurable in length with one another.

Again, by the said corollary, let it be arranged that as AB is to AC, so is the square on EF to the square on FG. [Margin: c. 6.dec. — *Elements* X.6] Then the squares on EF, FG, having the ratio of number to number, will be commensurable; and hence the lines EF, FG are commensurable, at least in square. So, since EF was shown to be rational, FG will be rational too. And since AB to AC — that is, the square on EF to the square on FG — does not have the ratio a square number has to a square number, [Margin: d. 9.dec. — *Elements* X.9] EF, FG will be incommensurable in length. So EF, FG are rational lines commensurable in square only; [Margin: e. 37.dec. — *Elements* X.37] and therefore the whole EG is irrational, called a binomial.

I say that it is also a third binomial. For since as I is to AB, so is the square on D to the square on EF, and as AB is to AC, so is the square on EF to the square on FG, then, *ex aequali*, as I is to AC, so is the square on D to the square on FG. But I does not have to AC the ratio a square number has to a square number. Therefore neither will the squares on D and FG have the ratio a square number has to a square number; [Margin: f. 9.dec. — *Elements* X.9] and consequently the straight lines D and FG are incommensurable in length. Now since as AB is to AC, so is the square on EF to the square on FG, and since AB is greater than AC, the square on EF will also be greater than the square on FG. Let it then be greater, by the lemma to proposition 14 of this book, by the square on the straight line H. We shall now likewise show, as in proposition 49, that H is commensurable in length with EF. Since, then, the greater term EF is more powerful than the lesser FG by the square on the straight line H, commensurable in length with it, and since neither EF nor FG is commensurable in length with the assigned rational D, as has been demonstrated, EG will, by definition, be a third binomial. We have therefore found a third binomial. Which was to be done.

PROBL. 16. PROPOS. 52.

[Margin: l.j. / xlv.]

To find a fourth binomial,

Given two numbers AC, CB such that the number AB composed of them has to neither of them the ratio a square has to a square — from those we set out in the third scholium to proposition 29 of this book — let some rational line D be set out, and let EF be taken commensurable with it in length.

Diagram. The same construction figure. At the top a dotted number-line runs from A (left) through C to B (right). Below it are the magnitude-lines: line **D** (the

rational) with division-ticks; the long line **EG** running from E (left) to G (right), divided at **F** into EF and FG; and beneath it the shorter line **H**.

- A, C, B: the dotted number-array (numbers AC, CB, and their composite AB)
- D: the assigned rational line
- E, F, G: the binomial line EG divided at F (terms EF, FG)
- H: the excess-line

Then EF too will be rational, being commensurable with the rational D. If the rest is constructed as in proposition 49, we shall show, just as there, that the whole EG is a binomial. I say that it is also a fourth binomial. For again, as in proposition 49, the square on EF will be greater than the square on FG. Let it then be greater by the square on the straight line H. Since, then, in the same way as in proposition 49, by conversion of the ratio, as AB is to CB, so is the square on EF to the square on H; but AB is not to CB as a square to a square, neither will the square on EF be to the square on H as a square number to a square number. [Margin: a. 9.dec. — *Elements* X.9] Therefore the straight lines EF and H are incommensurable in length. And so, since the greater term EF is more powerful than the lesser FG by the square on the straight line H, incommensurable in length with it, and since the greater term EF is again commensurable in length with the assigned rational D, EG will, by definition, be a fourth binomial. We have therefore found a fourth binomial. Which was to be done.

PROBL. 17. PROPOS. 53.

[Margin: l.ij. / xl.vj.]

To find a fifth binomial.

Given (the figure of the preceding proposition being repeated) two numbers AC, CB, let the construction be made as in the preceding proposition, that is, as in proposition 50: let FG be taken commensurable in length with the assigned rational D, and so on. We shall then show, as in proposition 50, that EG is a binomial. I say that it is also a fifth binomial.

Diagram. The construction figure of the preceding proposition, repeated. At the top a dotted number-line runs from A (left) through C to B (right). Below it are the magnitude-lines: line **D** (the rational) with division-ticks; the long line **EG** running from E (left) to G (right), divided at **F** into EF and FG; and beneath it the shorter line **H**.

- A, C, B: the dotted number-array (numbers AC, CB)

- D: the assigned rational line
- E, F, G: the binomial line EG divided at F (terms EF, FG)
- H: the excess-line

For we shall prove, in the same way as in proposition 50, that the square on EF is greater than the square on FG. Let it then be greater by the square on the straight line H. And since, as was shown in proposition 49, by conversion of the ratio, as AB is to CB, so is the square on EF to the square on H, we shall show, as in the preceding proposition, that the straight lines EF and H are incommensurable in length. Hence, since the greater term EF is more powerful than the lesser FG by the square on the straight line H, incommensurable in length [recte: longitudine] with it, and since the lesser term FG is commensurable in length with the assigned rational D, EG will, by definition, be a fifth binomial. We have therefore found a fifth binomial. Which was to be done.

PROBL. 18. PROPOS. 54.

[Margin: l.iiij. / xl.vij.]

To find a sixth binomial.

Given two dissimilar plane numbers AC, CB, neither of which is a square, and such that the number AB composed of them is likewise not a square and has to neither of them the ratio a square has to a square number (this will be achieved if any non-square number is divided into two numbers prime to one another; [Margin: a. 30.sept. — *Elements* VII.30] for in this way the whole too will be prime to each of them, and therefore will not have to them the ratio a square has to a square, by what we demonstrated toward the end of book 8): let another number I be taken which has to neither AB nor AC the ratio a square has to a square — such as is any square number. For this, related to AB and AC, which are not squares, will not have the ratio a square has to a square.

Diagram. The construction figure, with an additional number-row. At the top a dotted number-line runs from A (left) through C[?] to B (right); the middle division-mark is malformed in the print. Beneath it is a second dotted row labelled **I** (the auxiliary square number). Below these are the magnitude-lines: line **D** (the rational) with division-ticks; the long line **EG** running from E (left) to G (right), divided at F into EF and FG; and beneath it the shorter line **H**.

- A, C[?], B: the dotted number-array (dissimilar plane numbers AC, CB, composite AB)
- I: the auxiliary square number (second dotted row)

- D: the assigned rational line
- E, F, G: the binomial line EG divided at F (terms EF, FG)
- H: the excess-line

Next let some rational line D be set out, and, by the corollary to proposition 6 of this book, let it be arranged that as I is to AB, so is the square on D to the square on EF; and let the rest be done as in proposition 51. We shall show, as there, that the straight lines D and EF are incommensurable in length, and that the whole EG is a binomial. I say that it is also a sixth binomial. For we shall prove, in the same way as in proposition 51, that D and FG are incommensurable in length, and that the square on EF is greater than the square on FG. Let it then be greater by the square on the straight line H. We shall also show, as in proposition 49, that as AB is to CB, so is the square on EF to the square on H. Since, then, AB does not have to CB the ratio a square number has to a square, neither will the square on EF have to the square on H the ratio a square number has to a square; [Margin: b. 9.dec. — *Elements* X.9] so EF and H are incommensurable in length. And so, since the greater term EF is more powerful than the lesser FG by the square on the straight line H, incommensurable in length with it, and since neither of the terms EF, FG is commensurable in length with the assigned rational D, EG will, by definition, be a sixth binomial. We have therefore found a sixth binomial. Which was to be done.

LEMMA.

If a straight line is cut in any way whatever, the rectangle contained by the parts will be a mean proportional between the squares on the parts; likewise the rectangle contained by the whole and one part will be a mean proportional between the square on the whole line and the square on that part.

Let the straight line AB be cut in any way at C. On AB describe the square ABDE, and in it draw the diagonal BE. From C draw the straight line CF parallel to BD, cutting the diagonal at G; and through G draw HI parallel to AB. Then, by the corollary to proposition 4 of book 2, FH, CI will be the squares on the parts AC, CB; while each of the rectangles DG, GA is contained by those same parts; and the rectangles CD, AI are contained by the whole AB and the part CB. I say that DG is a mean proportional between the squares FH, CI, and that CD is a mean proportional between the squares AD and CI.

Diagram. A square with its diagonal and two internal cross-lines. The square is ABDE: A at the bottom-left, B at the bottom-right, D at the top-right, E at the top-

left. The diagonal BE is drawn from B (bottom-right) up to E (top-left). From C, a point on the base AB, the line CF rises vertically to F on the top side ED, crossing the diagonal at the centre point G. Through G the horizontal line HI is drawn, with H on the left side AE and I on the right side BD. These lines divide the square into four smaller rectangles about the diagonal.

- E: top-left corner of the square
- F: point on the top side ED (foot of the vertical from C); printed as a Γ-shaped glyph
- D: top-right corner
- H: point on the left side AE (left end of the horizontal)
- G: centre, intersection of diagonal BE, vertical CF, and horizontal HI
- I: point on the right side BD (right end of the horizontal)
- A: bottom-left corner
- C: point on the base AB (foot of the vertical)
- B: bottom-right corner

For since, as HG is to GI, so is both FH to DG and AG to CI, [Margin: c. 1.sext. — *Elements* VI.1] it follows that as FH is to DG, so is AG to CI. But AG is equal to DG. [Margin: d. 43.pri. — *Elements* I.43] Therefore also, as FH is to DG, so is DG to CI; and thus DG is a mean proportional between the squares FH, CI.

Again, since, as AB is to CB, so is both AD to CD and AI to CI, [Margin: e. 1.sext. — *Elements* VI.1] but AI is equal to CD, therefore, as AD is to CD, so also is CD to CI. Hence CD [is a mean proportional...]

(Conclusion of the preceding Lemma on mean proportionals):

...CD is a mean proportional between the squares AD and CI. In the same way AF will be a mean proportional between the squares AD and FH.

THEOR. 37. PROPOS. 55.

[Margin: liv. / lvij.[?] — divergent lowercase-roman proposition counters]

If a space is contained by a Rational line and a first binomial, the straight line whose square equals that space is irrational, and is called a **binomial**.

Let the space AC be contained by the Rational line AB and the first binomial AD. I say that the straight line whose square equals the space AC is irrational, of the kind called a binomial. Let AE be the greater term of AD. Then by definition AE

and ED are rational and commensurable in square only; AE has a greater square than ED by the square on a straight line commensurable with AE in length; and finally AE is commensurable in length with the assigned Rational line AB.

Bisect ED at F. Now since AE has a greater square than ED by the square on a straight line commensurable in length with AE, if a parallelogram equal to a quarter of the square on ED — that is, to the square on EF — and contained by AG, GE be applied to AE (by the second lemma to proposition 17 of this book) so as to be deficient by a square figure, [Margin: (a) *Elements* X.18] it will divide AE into parts commensurable in length. Therefore the parts AG and GE are commensurable with one another in length. Now through the points G, E, F draw GH, EI, FK parallel to AB and CD. [Margin: (b) *Elements* II.14] Let a square LM be made equal to the parallelogram AH, and a square MN equal to the parallelogram GI, and let the squares LM and MN be joined at the angle M so that their sides MO and MP form a single straight line OP. Then their sides MQ and MR will likewise form a single straight line QR, as we demonstrated from Proclus at proposition 15 of Book I, because the angles OMQ [O printed as a filled ring] and PMR at the vertex M are equal, being right angles. When the rectangle LN has been completed — since the straight lines OM, MP are equal to the straight lines QM, MR, and hence the whole OP equal to the whole QR, [Margin: (c) *Elements* I.34] while OP is equal to SN and LT together and QR to LS and TN — the rectangle LN will be equilateral, and therefore a square.

Now the rectangle contained by AG, GE is equal, by construction, to the square on EF; [Margin: (d) *Elements* VI.17] therefore AG is to EF as EF is to GE. [Margin: (e) *Elements* VI.1] And consequently AH is to EK as EK is to GI. Therefore EK is a mean proportional between the squares LM, MN, to which AH and GI are equal. But TM is also a mean proportional between those same squares LM, MN, by the preceding lemma. Therefore TM is equal to EK. [Margin: (f) *Elements* I.43] Now since MS is equal to TM, [Margin: (g) *Elements* I.36] and FC is equal to EK, MS will also be equal to FC. Hence the whole square LN is equal to the whole rectangle AC; and so the straight line OP has for its square the space AC, contained by the Rational line AB and the first binomial AD. I say that OP is irrational, of the kind called a binomial.

For since AG, GE have been shown to be commensurable in length, [Margin: (h) *Elements* X.16] the whole AE will also be commensurable in length with each of them. But AE, being the greater term of the first binomial AD, is commensurable in length with the Rational line AB. Therefore AG and GE are, by the scholium to proposition 12 of this book, commensurable in length with AB as well. And since

AB is Rational, AG and GE will be Rational too. [Margin: (i) *Elements* X.20]
 Therefore the rectangles AH, GI, contained by Rational lines commensurable in length, are Rational; and so the squares LM, MN equal to them will be Rational; and consequently the straight lines OM, MP will be Rational.

And since AE is incommensurable in length with ED — while AG has been shown commensurable in length with AE, and EF, the half of ED, is commensurable with ED — then by the scholium to proposition 14 of this book AG and EF are incommensurable with one another in length. Therefore AH and EK, [Margin: (k) *Elements* VI.1] which have the same ratio as AG to EF, are incommensurable; [Margin: (l) *Elements* X.10] and so LM, MT, equal to them, are incommensurable. [Margin: (m) *Elements* X.10] Therefore the straight lines OM, MP are incommensurable in length, [Margin: (n) *Elements* VI.1] since they have the same ratio as LM to MT. But OM, MP have also been shown to be Rational. Therefore OM, MP are Rational and commensurable in square only. [Margin: (o) *Elements* VI.37] Consequently the whole OP, whose square is the space AC, is irrational, of the kind called a binomial. Therefore, if a space is contained by a Rational line and a first binomial, and so forth. Which was to be demonstrated.

Diagram. The figure for proposition 55, in two parts. *Left part:* the straight line AD, drawn horizontally, is divided at the points G, E, F (reading A–G–E–F–D). Perpendiculars dropped from these points meet the parallel baseline B–C at H (below G), I (below E), K (below F), with B below A and C below D; L continues the baseline to the right. The vertical rectangles so formed are AH, GI, EK, FK, etc., used in the applied-area construction. *Right part:* the square LN, with corners L (bottom-left), S (top-left), N (top-right), T (bottom-right); its center is M; the midpoints of the sides are O (left), P (right), R (top), Q (bottom); the horizontal midline OP and the vertical midline QR cross at M, dividing it into the two constituent squares LM and MN. This square LN is equal to the rectangle AC.

- Left figure, top: A, G, E, F, D
- Left figure, baseline: B, H, I, K, C (with L continuing right)
- Right figure (square LN), top: S, R, N
- Right figure, mid-height: O, M, P
- Right figure, baseline: L, Q, T

THEOR. 38. PROPOS. 56.

[Margin: lix.[?] / xlix. — divergent lowercase-roman proposition counters]

If a space is contained by a Rational line and a second binomial, the straight line whose square equals that space is irrational, and is called a **first bimedral**.

Let the space AC be contained by the Rational line AB and the second binomial AD. I say that the straight line whose square is AC is irrational, of the kind called a first bimedral. Let AE be the greater term of AD. Then by definition AE, ED are rational and commensurable in square only; AE has a greater square than ED by the square on a straight line commensurable with it in length; and finally ED is commensurable in length with the assigned Rational line AB. Bisect ED at F, and construct the rest as in the preceding proposition. This done, we shall prove in the same way, as there, that OP has for its square the space AC contained by the Rational line AB and the second binomial AD. I now say that OP is irrational, of the kind called a first bimedral.

For since AE is incommensurable in length with ED, and ED — the lesser term of the second binomial AD — is commensurable in length with the Rational line AB, [Margin: (a) *Elements* X.23] AE and AB [p. 443] will be incommensurable in length. And since AG, GE were shown in the preceding proposition to be commensurable in length, [Margin: (b) *Elements* X.16] the whole AE will also be commensurable with each of them. Therefore, since AE — the greater term of the second binomial AD — is a Rational line, AG and GE will also be Rational. Now since each of AG, GE is commensurable in length with AE, while AE is incommensurable in length with the Rational line AB, [Margin: (c) *Elements* X.14] each of AG, GE will be incommensurable in length with AB; and consequently both AB, AG and AB, GE are Rational and commensurable in square only. Therefore the rectangles AH, GI are medial, and so the squares LM, MN equal to them are medial. Hence the straight lines OM, MP [printed NP] are also medial.

Now since AG, GE are commensurable in length, [Margin: (e) *Elements* X.10] AH, GI, which have the same ratio as they, are commensurable, and so the squares LM, MN equal to them are commensurable. Therefore the straight lines OM, MP are commensurable, at least in square. And since AE, ED are incommensurable in length — while AG has been shown commensurable in length with AE, and EF, its half, commensurable in length with ED — then by the scholium to proposition 14 of this book AG, EF are incommensurable in length. [Margin: (f) *Elements* X.10] And so AH, EK, which have the same ratio as they, are incommensurable; therefore LM, MT, equal to AH and EK, are incommensurable; [Margin: (g) *Elements* X.10] and so the straight lines OM, MP are incommensurable in length, [Margin: (h) *Elements* VI.1] since they have the same ratio as LM to MT. Since,

then, it has been shown that OM, MP are medial and commensurable in square only — and since, finally, ED, the lesser term of the second binomial AD, is a line commensurable in length with AB, that is with EI, while EF is also commensurable in length with ED — [Margin: (i) *Elements* X.12] EI and EF will also be commensurable in length; and therefore, since EI is Rational, EF will be Rational too. [Margin: (k) *Elements* X.20] Hence the rectangle EK is Rational. But MT, contained by OM, MP, is equal to EK; therefore MT is also Rational. Consequently OM, MP are medial, commensurable in square only, and contain a Rational rectangle; [Margin: (l) *Elements* X.38] and so OP is irrational, of the kind called a first bimedial. Therefore, if a space is contained by a Rational line and a second binomial, and so forth. Which was to be shown.

Diagram. The figure for proposition 56, of the same construction as the preceding. *Left part:* the line AD divided at G, E, F, with perpendiculars to the baseline BC forming the rectangles AH, GI, etc. *Right part:* the square LN built from the two squares LM, MN, divided by the midlines OP and QR through the center M.

- Left figure, top: A, G, E, F, D
- Left figure, baseline: B, H, I, K, C (L continuing right)
- Right figure (square LN), top: S, R, N
- Right figure, mid-height: O, M, P
- Right figure, baseline: L, Q, T

THEOR. 39. PROPOS. 57.

[Margin: lvj. / L. — divergent lowercase-roman proposition counters]

If a space is contained by a Rational line and a third binomial, the straight line whose square equals that space is irrational, and is called a **second bimedial**.

Let the space AC be contained by the Rational line AB and the third binomial AD. I say that the straight line whose square is AC is irrational, of the kind called a second bimedial. Let AE be the greater term of AD. Then by definition AE, ED are rational and commensurable in square only; AE has a greater square than ED by the square on a straight line commensurable with it in length; and finally neither of them, AE, ED, is commensurable in length with the assigned Rational line AB. Bisect ED at F, and make the rest exactly as in proposition 55. In the same way, then, as there, we shall show that OP has for its square the space AC; and, as in the preceding proposition 56, that OM, MP are medial and commensurable in

square only — because AE is set incommensurable in length with the Rational line AB, just as there the same AE was incommensurable in length with AB.

Diagram. The figure for proposition 57, of the same construction. The labels are somewhat foxed; the right-hand square shows its top-left corner as "C"["?"] (where the other figures read "S") and its right-side midpoint as "P"["?"].

- Left figure, top: A, G, E, F, D
- Left figure, baseline: B, H, I, K, C (L continuing right)
- Right figure (square LN), top: C["?"], R, N
- Right figure, mid-height: O, M, P["?"]
- Right figure, baseline: L, Q, T

(Continuing the demonstration.) And since AG, GE are commensurable in length, while ED is incommensurable in length with the Rational line AB, that is with EI, [Margin: (a) *Elements* X.14] EF also will be incommensurable in length with EI. But EF, EI are Rational, since EF is equal to the Rational line AB... Therefore EF, EI are Rational and commensurable in square only. Hence EK is medial, and so MT, equal to EK and contained by OM, MP, is medial. [Margin: (b) *Elements* X.22] And since OM, MP are medial, commensurable in square only, and contain a medial rectangle, [Margin: (c) *Elements* X.22] OP is irrational, of the kind called a second binomial. If, therefore, a space is contained by a Rational line and a third binomial, and so forth. Which was to be demonstrated.

THEOR. 40. PROPOS. 58.

[Margin: lvij. / Lj. — divergent lowercase-roman proposition counters]

If a space is contained by a Rational line and a fourth binomial, the straight line whose square equals that space is irrational, and is called a **major**.

Let the space AC be contained by the Rational line AB and the fourth binomial AD. I say that the straight line whose square equals the space AC is irrational, of the kind called a major. Let AE be the greater term of AD. Then by definition AE, ED are rational and commensurable in square only; AE has a greater square than ED by the square on a straight line incommensurable with it in length; and finally AE is commensurable in length with AB. Bisect ED at F, and make all the rest as in proposition 55. [Margin: (a) *Elements* X.19] Then AG, GE will be incommensurable in length, [p. 444] just as there, and OP will have for its square the space AC. I say that OP is irrational, of the kind called a major.

For since AG, GE are incommensurable in length, [Margin: (b) *Elements* X.10] the rectangles AH, GI, which have the same ratio as they, will also be incommensurable; and hence the squares LM, MN, equal to AH, GI, will be incommensurable. Therefore the straight lines OM, MP are incommensurable in square. But since AE, the greater term of the fourth binomial AD, is a line commensurable in length with the Rational line AB, AE will be Rational; [Margin: (c) *Elements* X.20] and the rectangle AI contained by them is Rational. Now AI is equal to the sum of the squares LM, MN. Therefore the sum of the squares LM, MN is Rational.

And since ED, the lesser term of the fourth binomial AD, is a line incommensurable in length with the Rational line AB, EF, its half, will also be incommensurable in length with AB; and EF, being commensurable with the Rational line ED, is Rational. Therefore EF and AB are Rational and commensurable in square only. [Margin: (d) *Elements* X.22] Hence EK, contained by them, is medial; and so MT, equal to it and contained by OM, MP, is medial. Therefore, since OM, MP are incommensurable in square, and make the sum of their squares LM, MN Rational, while the rectangle MT contained by them is medial, [Margin: (e) *Elements* X.40] the whole OP will be irrational, of the kind called a major. Therefore, if a space is contained by a Rational line and a fourth binomial, and so forth. Which was to be shown.

Diagram. The figure for proposition 58, of the same construction as the foregoing. This impression is heavily foxed and several interior labels are faint; the readable labels are listed below and the whole figure is flagged for re-examination.

- Left figure, top: A, G, E, F, D
- Left figure, baseline: B, H, I, K, C (L continuing right)
- Right figure (square LN): O and M legible at mid-height; other corner labels (S, R, N, P, Q, T) faint/illegible on this impression

THEOR. 41. PROPOS. 59.

[Margin: lviii. / Lij. — divergent lowercase-roman proposition counters]

If a space is contained by a Rational line and a fifth binomial, the straight line whose square equals that space is irrational, and is called **the side of a rational plus a medial area** [potens Rationale & Medium].

Let the space AC be contained by the Rational line AB and the fifth binomial AD. I say that the straight line whose square is the space AC is irrational, of the kind

called the side of a rational plus a medial area. Let AE be the greater term of AD. Then by definition AE, ED are rational and commensurable in square only; AE has a greater square than ED by the square on a straight line incommensurable with it in length; and finally ED is commensurable in length with the assigned Rational line AB. Bisect ED, and make the rest the same as in proposition 55. [Margin: (a) *Elements* X.19] Then again AG, GE will be incommensurable in length. And in the same way as in proposition 55, it will be established that OP has for its square the space AC. I say that OP is irrational, of the kind called the side of a rational plus a medial area.

For, as in the preceding proposition, the straight lines OM, MP will be incommensurable in square. And since AE — the greater term of the fifth binomial AD — is a line incommensurable in length with the Rational line AB, AE and AB will be Rational and commensurable in square only. [Margin: (b) *Elements* X.22] Therefore the rectangle AI contained by them is medial. Now AI is equal to the sum of the squares LM, MN; therefore this sum too is medial. Again, since ED is set Rational and commensurable in length with the Rational line AB — being the lesser term of the fifth binomial AD — EF, its half, will also be commensurable in length with AB, and Rational. [Margin: (c) *Elements* X.20] Hence EK, contained by the Rational lines EI, EF that are commensurable in length, is Rational; and therefore MT, contained by OM, MP, will be Rational, since it is equal to EK. And so, since the straight lines OM, MP are incommensurable in square, and make the sum of their squares LM, MN medial, while the rectangle MT contained by them is Rational, [Margin: (d) *Elements* X.41] the whole OP will be irrational, of the kind called the side of a rational plus a medial area. If, therefore, a space is contained by a Rational line and a fifth binomial, and so forth. Which was to be demonstrated.

Diagram. The figure for proposition 59, of the same construction. *Left part:* line AD divided at G, E, F, with perpendiculars to the baseline BC. *Right part:* the square LN built from the squares LM, MN, with midlines OP and QR through the center M.

- Left figure, top: A, G, E, F, D
- Left figure, baseline: B, H, I, K, C (L continuing right)
- Right figure (square LN), top: S, R, N
- Right figure, mid-height: O, M, P
- Right figure, baseline: L, Q, T

THEOR. 42. PROPOS. 60.

[Margin: lix. / Liiij. — divergent lowercase-roman proposition counters]

If a space is contained by a Rational line and a sixth binomial, the straight line whose square equals that space is irrational, and is called **the side of the sum of two medial areas** [bina Media potens].

Let the space AC be contained by the Rational line AB and the sixth binomial AD. I say that the straight line whose square is the space AC is irrational, of the kind called the side of the sum of two medial areas. Let AE be the greater term of AD. Then by definition AE, ED are rational and commensurable in square only; AE has a greater square than ED by the square on a straight line incommensurable with it in length; and finally neither of them, AE, ED, will be commensurable in length with the Rational line AB. Make everything as in the foregoing propositions; [Margin: (a) *Elements* X.19] then AG, GE will be incommensurable in length. And we shall now show in the same way, as in proposition 55, that the straight line OP has for its square the space

Diagram. The figure for proposition 60, of the same construction. *Left part:* line AD divided at G, E, F, with perpendiculars to the baseline BC. *Right part:* the square LN built from LM, MN, with midlines OP and QR through the center M.

- Left figure, top: A, G, E, F, D
- Left figure, baseline: B, H, I, K, C (L continuing right)
- Right figure (square LN), top: S, R, N
- Right figure, mid-height: O, M, P
- Right figure, baseline: L, Q, T

[...continuation of the demonstration of the preceding proposition (THEOR. 42 / PROPOS. 60, the side of two medials)...]

[...] AC. Likewise, as in Proposition 58, the lines OM, MP are incommensurable in square. And again, as in the preceding proposition, the sum of the squares on LM, MN will be Medial. But, as in Proposition 58, the rectangle MT contained by OM, MP will likewise be Medial. Now of the two lines AE, EF, the one is incommensurable in length with ED, while the other is commensurable with it [Margin: b — X.13]; therefore AE and EF are incommensurable in length [Margin: c — X.10]. Consequently AI and EK are also incommensurable, since they have the same ratio as AE, EF [Margin: d — VI.1]. And so the sum of the squares on LM, MN — which is equal to AI — and the rectangle MT — which is equal to EK — are incommensurable. Therefore, since the lines OM, MP are incommensurable in

square, and produce a sum of their squares LM, MN that is Medial, and a rectangle MT contained by them that is Medial and incommensurable with the sum of their squares, the whole OP will be Irrational, the kind called *the side of two medials* [bina media potens] [Margin: e — X.42]. If, then, an area is contained by a Rational line and a sixth binomial, and so on. Which was to be shown.

Diagram. Two figures belonging to the preceding proposition (the side of two medials).

Left figure: a rectangle whose upper edge runs from A (upper-left corner) rightward through the marked points G, E, F to D (upper-right corner), and whose lower edge runs from B (lower-left corner) through H, I, K to C (lower-right corner). Vertical divisions drop from the upper points to the base, dividing the figure into strips.

- A (upper-left corner) / B (lower-left corner): the left end of the divided line and its foot
- G, E, F (upper edge, interior): division points of the line AD
- D (upper-right corner) / C (lower-right corner): the right end
- H, I, K (lower edge, interior): feet of the perpendiculars

Right figure: a square with a small square (S) erected above its top edge, subdivided by two perpendicular midlines crossing at the center M.

- R (top-left corner), N (top-right corner)
- O (midpoint of left side), M (center), P (midpoint of right side)
- L (bottom-left corner), Q (midpoint of bottom edge), T (bottom-right corner)
- S: small square erected on the top edge

THEOR. 43. PROPOS. 61.

[Margin: lx / liv]

The square on a binomial, applied to a Rational line, produces as breadth a first binomial.

Let AB be a binomial whose greater term is AC [Margin: a — I.45]; and to the assigned Rational line DE let there be applied a rectangle DF equal to the square on AB, producing the breadth DG. I say that DG is a first binomial.

For to the same Rational line DE let there be applied the rectangle DH equal to the square on AC, and to HI another rectangle IK equal to the square on CB. The remainder LF will then be equal to twice the rectangle contained by AC, CB, since

the squares on AC, CB together with twice the rectangle contained by AC, CB are equal to the square on AB [Margin: b — II.4], just as the rectangle DF is also equal, by construction, to the square on AB. Let LG be bisected at the point M, and through M let MN be drawn parallel to LK, GF; then each of the two rectangles LN, MF will be equal to the rectangle contained by AC, CB.

Now since AC, CB are Rational lines commensurable in square only — because the whole AB is a binomial — the squares on AC, CB are Rational, and therefore commensurable [Margin: c — X.37]; and since the sum compounded of them is also commensurable with each of them, the sum of the squares on AC, CB will likewise be Rational [Margin: d — X.16]. But by construction the rectangle DK is equal to this sum. Therefore DK too will be Rational; and since it is applied to the Rational line DE, it will produce a breadth DL that is Rational and commensurable in length with DE.

Again, since AC, CB are Rational lines commensurable in square only, the rectangle contained by them will be Medial [Margin: e — X.22]; and hence twice that rectangle, being commensurable with it — that is, LF itself — will be Medial, by the corollary to Proposition 24 of this book. Therefore LF, applied to the Rational line LK, produces a Rational breadth LG that is incommensurable in length with LK — that is, with DE [Margin: f — X.23]. But DL has been shown commensurable in length with DE. Therefore DL, LG are incommensurable in length [Margin: g — X.13]. Hence DL, LG are Rational lines commensurable in square only; and consequently DG is a binomial [Margin: h — X.37]. I say further that it is a first binomial.

For since, by the lemma to Proposition 54 of this book, the rectangle contained by AC, CB is a mean proportional between the squares on AC, CB, LN will likewise be a mean proportional between DH, IK [Margin: i — VI.1]; and so, since the lines DI, LM, IL have the same ratio as DH, LN, IK, the line LM will also be a mean proportional between DI, IL [Margin: k — VI.17]. Therefore the rectangle contained by DI, IL is equal to the square on LM. And since the squares on the lines AC, CB are commensurable — because AC, CB were assumed commensurable in square — DH, IK, being equal to them, will also be commensurable; and therefore the lines DI, IL, having the same ratio as they, are commensurable in length [Margin: l — X.10]. Now since DK is greater than LF — because the squares on AC, CB are greater, by the lemma to Proposition 39 of this book, than twice the rectangle contained by AC, CB — the line DL will also be greater than LG, since DL, LG have the same ratio as DK, LF [Margin: m — VI.1]. Accordingly, since DL is greater than LG, and to DL is applied a rectangle

contained by DI, IL, deficient by a square figure, equal to the square on LM — that is, to a fourth part of the square described on the lesser line LG (for the rectangle contained by DI, IL was shown equal to the square on LM) — and since DL is divided at I into parts DI, IL commensurable in length, as has been demonstrated, DL, the greater term, will be more powerful than LG, the lesser, by the square on a line commensurable in length with itself [Margin: n — X.18]. Therefore, since DG has been shown to be a binomial, and its greater term DL is more powerful than its lesser term LG by the square on a line commensurable in length with itself, and moreover the greater term DL is commensurable in length with the assigned Rational line DE, DG will by the definition be a first binomial. Therefore the square on a binomial, and so on. Which was to be demonstrated.

Diagram. The construction for Proposition 61. At the left is the binomial line drawn vertically: B (top), C (the division point), A (foot). To the right stands the applied rectangle whose upper edge runs D, I, L, M, G (left to right) and whose base runs E, H, K, N, F (beneath D...G); two squares are erected above the upper edge over its subdivisions.

- B (top) / C (division point) / A (foot): the binomial AB divided at C into greater part AC and lesser CB
- D (upper-left corner) ... G (upper-right corner): the breadth DG, with interior points I, L, M
- E (lower-left corner) ... F (lower-right corner): the base, with interior feet H, K, N
- DE (left side): the assigned Rational line
- The two erected squares: the squares on AC and on CB

THEOR. 44. PROPOS. 62.

[Margin: lxj / lv]

The square on a first bimedial, applied to a Rational line, produces as breadth a second binomial.

Let AB be a first bimedial whose greater term is AC [Margin: a — I.45]; and to the Rational line DE let there be applied a rectangle DF equal to the square on AB, producing the breadth DG. I say that DG is a second binomial.

For let the same construction be set up as in the preceding proposition, so that again DH, IK are equal to the squares on AC, CB, and each of LN, MF equal to the rectangle contained by AC, CB. Now since AC, CB — the components of the first bimedial AB — are Medial lines commensurable in square only which contain a

Rational rectangle [Margin: b — X.38], the squares on AC, CB, and therefore the rectangles DH, IK equal to them, will be commensurable and Medial [Margin: c — X.16]. Since therefore the whole DK is commensurable with each of DH, IK, DK too will be Medial, by the corollary to Proposition 24 of this book; and since it is applied to the Rational line DE, its breadth DL will be Rational but incommensurable in length with DE [Margin: d — X.23].

Again, since the rectangle contained by AC, CB is Rational, its double LF will also be Rational; and since it is applied to the Rational line LK, its breadth LG will be Rational and commensurable in length with LK — that is, with DE [Margin: e — X.21]. Therefore, of the two lines DL, LG, since LG is commensurable in length with DE while DL is incommensurable in length, DL, LG will be incommensurable in length [Margin: f — X.13]. And since DL, LG have been shown Rational, they will be Rational lines commensurable in square only; and consequently the whole DG is a binomial [Margin: g — X.37].

Now, just as in the preceding proposition, we shall demonstrate that DL is the greater term and is more powerful than the lesser term LG by the square on a line commensurable in length with itself. Therefore, since the greater term DL of DG is more powerful than the lesser term LG by the square on a line commensurable in length with itself, and the lesser term LG has been shown commensurable in length with the assigned Rational line DE, DG will by the definition be a second binomial. Therefore the square on a first bimedial, and so on. Which was to be shown.

Diagram. The construction for Proposition 62, identical in scheme to the preceding. At the left, the bimedial line vertically: B (top), C (division point), A (foot). To the right, the applied rectangle: upper edge D, I, L, M, G; base E, H, K, N, F; two squares erected above.

- B / C / A: the first bimedial AB divided at C into AC, CB
- D ... G (upper edge, interior points I, L, M): the breadth DG
- E ... F (base, interior feet H, K, N)
- DE (left side): the assigned Rational line

THEOR. 45. PROPOS. 63.

[Margin: lxij / lvj]

The square on a second bimedial, applied to a Rational line, produces as breadth a third binomial.

Let AB be a second binomial whose greater term is AC [Margin: a — I.45]; and to the Rational line DE let there be applied a rectangle DF equal to the square on AB, producing the breadth DG. I say that DG is a third binomial.

For with the same construction as in Proposition 61: since AC, CB — the components of the second binomial AB — are Medial lines commensurable in square only which contain a Medial rectangle [Margin: b — X.39], the squares on AC, CB, and therefore the rectangles DH, IK equal to them, will be commensurable and Medial [Margin: c — X.16]. Since therefore the whole DK is commensurable with each of DH, IK, DK too will be Medial, by the corollary to Proposition 24 of this book; and since it is applied to the Rational line DE, its breadth DL will be Rational but incommensurable in length with DE [Margin: d — X.23].

Again, since the rectangle contained by AC, CB is Medial, its double LF will also be Medial; and since it is applied to the Rational line LK, its breadth LG will be Rational but incommensurable in length with LK — that is, with DE [Margin: e — X.23]. And since AC is incommensurable in length with CB, and as AC is to CB, so — by the third lemma, Proposition 19 of this book — is the square on AC to the rectangle contained by AC, CB, the square on AC will also be incommensurable with the rectangle contained by AC, CB [Margin: f — X.10]. But the square on AC is commensurable with the sum of the squares on the lines AC, CB — because the squares on AC, CB are commensurable, since the lines themselves are assumed commensurable in square [Margin: g — X.16]; and the rectangle contained by AC, CB is commensurable with its double, namely twice the rectangle contained by AC, CB. Therefore the sum of the squares on the lines AC, CB — that is, the rectangle DK — is incommensurable with twice the rectangle contained by AC, CB — that is, with the rectangle LF — by what we wrote in the scholium to Proposition 14 of this book. Hence the lines DL, LG, which have the same ratio as DK, LF, are incommensurable in length [Margin: h — X.10]; but they have also been shown Rational. They are therefore Rational lines commensurable in square only; and consequently the whole DG is a binomial [Margin: i — X.37].

Now, just as in Proposition 61, we shall show that DL is the greater term and is more powerful than the lesser LG by the square on a line commensurable in length with itself. Therefore, since the greater term DL is more powerful than the lesser LG by the square on a line commensurable in length with itself, and neither DL nor LG is commensurable in length with the assigned Rational line DE, DG will by the definition be a third binomial. Therefore the square on a second binomial, applied to a Rational line, produces as breadth a third binomial, and so on. Which was to be demonstrated.

Diagram. The construction for Proposition 63, identical in scheme. Left: the bimedral line B (top) / C (division) / A (foot). Right: the applied rectangle, upper edge D, I, L, M, G; base E, H, K, N, F.

- B / C / A: the second bimedral AB divided at C into AC, CB
- D ... G (interior I, L, M): the breadth DG
- E ... F (base, feet H, K, N)
- DE (left side): the assigned Rational line

THEOR. 46. PROPOS. 64.

[Margin: lxij / lvij]

The square on a major, applied to a Rational line, produces as breadth a fourth binomial.

Let AB be a major whose greater term is AC [Margin: a — I.45]; and to the Rational line DE let there be applied a rectangle DF equal to the square on AB, producing the breadth DG. I say that DG is a fourth binomial.

For let exactly the same construction be made as in Proposition 61. Since AC, CB — the components of the major AB — are incommensurable in square, and make the sum of the squares on the lines AC, CB Rational, but the rectangle contained by them Medial [Margin: b — X.45], DK, being equal to the sum of the squares on the lines AC, CB, will also be Rational; whereas LF, twice the rectangle contained by AC, CB, will be Medial. Since, therefore,

the Rational DK is applied to the Rational line DE, DL will be Rational and commensurable in length with DE [Margin: c — X.21]. Likewise, since the Medial LF is applied to the Rational line LK, LG will be Rational but incommensurable in length with LK — that is, with DE [Margin: d — X.23]. Therefore, of the two lines DL, LG, since the one is commensurable in length with DE while the other is incommensurable in length with the same DE, DL, LG will be incommensurable in length [Margin: e — X.13]; but they have also been shown Rational. Therefore DL, LG [recte: LG] are Rational lines commensurable in square only; and hence DG is a binomial [Margin: f — X.37].

Now, just as in Proposition 61, we shall demonstrate that the rectangle contained by DI, IL is equal to the square on LM. But since the squares on AC, CB are incommensurable — because the lines AC, CB are incommensurable in square — the equal rectangles DH, IK will also be incommensurable; and so too the lines DI, IL, which have the same ratio as they, will be incommensurable in length

[Margin: g — X.10]. And since, as we showed in Proposition 61, DL is greater than LG, and to DL there is applied a rectangle contained by DI, IL, equal to the square on LM — that is, to a fourth part of the square on LG — deficient by a square figure, which divides DL at I into parts incommensurable in length, DL will be more powerful than LG by the square on a line incommensurable in length with itself [Margin: h — X.19]. For this reason, since DL itself (the greater term of DG, which is a binomial) has been shown commensurable in length with the assigned Rational line DB [recte: DE], DG will by the definition be a fourth binomial. Therefore the square on a major, applied to a Rational line, and so on. Which was to be demonstrated.

Diagram. The construction for Proposition 64, identical in scheme. Left: the major line B (top) / C (division) / A (foot). Right: the applied rectangle, upper edge D, I, L, M, G; base E, H, K, N, F.

- B / C / A: the major AB divided at C into AC, CB
- D ... G (interior I, L, M): the breadth DG
- E ... F (base, feet H, K, N)
- DE (left side): the assigned Rational line

THEOR. 47. PROPOS. 65.

[Margin: lxiv / lvij]

The square on the side of a rational plus a medial [area], applied to a Rational line, produces as breadth a fifth binomial.

Let AB be a line that is the side of a rational plus a medial [recta potens Rationale ac Medium], whose greater term is AC [Margin: a — I.45]; and to the Rational line DE let there be applied a rectangle DF equal to the square on AB, producing the breadth DG. I say that DG is a fifth binomial.

For let exactly the same things be done as in Proposition 61. Since the lines AC, CB, the components of AB — the side of a rational plus a medial — are incommensurable in square [Margin: b — X.41], making the sum of their squares Medial but the rectangle contained by them Rational, DK will be equal to the sum of the squares on the lines AC, CB and Medial, while LN is equal to the rectangle contained by AC, CB, and hence its double LF is Rational. Therefore the Medial DK, applied to the Rational line DE, produces a breadth DL that is Rational but incommensurable in length with the Rational DE [Margin: c — X.23]; whereas the Rational LF, applied to the Rational line LK, produces a breadth LG that is Rational and commensurable in length with LK — that is, with DE [Margin: d —

X.21]. Since, therefore, of the two lines DL, LG, the one is commensurable in length with the Rational DE while the other is incommensurable in length, DL, MG [recte: LG] will be incommensurable in length [Margin: e — X.13]. Therefore DL, LG are Rational lines commensurable in square only, and DG is a binomial [Margin: f — X.37].

Now, just as in Proposition 61, we shall show that the rectangle contained by DI, IL is equal to the square on LM; and, as in the preceding proposition, that DL is the greater term and is more powerful than LG by the square on a line incommensurable in length with itself. Therefore, since the lesser term LG has been shown commensurable in length with the assigned Rational line DE, DG will by the definition be a fifth binomial. Therefore the square on the side of a rational plus a medial, and so on. Which was to be shown.

Diagram. The construction for Proposition 65, identical in scheme. Left: the line B (top) / C (division) / A (foot). Right: the applied rectangle, upper edge D, I, L, M, G; base E, H, K, N, F.

- B / C / A: the side of a rational plus a medial, AB, divided at C into AC, CB
- D ... G (interior I, L, M): the breadth DG
- E ... F (base, feet H, K, N)
- DE (left side): the assigned Rational line

THEOR. 48. PROPOS. 66.

[Margin: lxv / lix]

The square on the side of two medials, applied to a Rational line, produces as breadth a sixth binomial.

Let AB be a line that is the side of two medials [bina Media potens], whose greater term is AC [Margin: a — I.45]; and to the Rational line DE let there be applied a rectangle DF equal to the square on AB, producing the breadth DG. I say that DG is a sixth binomial.

For with the same construction as in Proposition 61: since the lines AC, CB, the components of AB — the side of two medials — are incommensurable in square, and make both the sum of their squares Medial and the rectangle contained by them Medial, and this rectangle incommensurable with the sum of their squares [Margin: b — X.42], both DK, equal to that sum, and LF, equal to twice the rectangle contained by AC, CB, will be Medial. And since the Medials DK, LF are applied to the Rational line DE, their breadths DL, LG will be Rational but

incommensurable in length with DE [Margin: c — X.23]. Now since the sum of the squares on the lines AC, CB is incommensurable with the rectangle contained by AC, CB, while twice the rectangle contained by AC, CB is commensurable with that same rectangle contained by AC, CB — since the one is double the other [Margin: d — X.42] — the sum of the said squares, that is, DK, will be incommensurable with twice the rectangle contained by AC, CB [recte: CB], that is, with LF [Margin: e — X.13]. And consequently the lines DL, LG, together with DK, LF, are incommensurable in length [Margin: f — X.10]. They are therefore Rational lines commensurable in square only; and hence DG is a binomial [Margin: g — X.37].

Now, just as in the preceding proposition, we shall demonstrate that DL is the greater term and is more powerful than LG by the square on a line incommensurable in length with itself. Since, therefore, it has been shown that neither DL nor LG is commensurable in length with the assigned Rational line DE, DG will by the definition be a sixth binomial. Therefore the square on the side of two medials, applied to a Rational line, produces as breadth a sixth binomial. Which was to be shown.

Diagram. The construction for Proposition 66, identical in scheme. Left: the line B (top) / C (division) / A (foot). Right: the applied rectangle, upper edge D, I, L, M, G; base A, E, H, K, N, F.

- B / C / A: the side of two medials, AB, divided at C into AC, CB
- D ... G (interior I, L, M): the breadth DG
- E ... F (base, feet H, K, N)
- DE (left side): the assigned Rational line

Euclid, Book X (Clavius's Commentary) — the "commensurable-with" propositions and the composition of rational and medial areas

(Continuation of Book X on incommensurable magnitudes. Proposition numbers are given exactly as printed: the large Arabic "THEOR. / PROPOS." counters, together with the two divergent lowercase-roman margin numbers. In modern editions PROPOS. 67–72 correspond to Elements X.66–X.71.)

THEOR. 49. PROPOS. 67.

[Margin: lxvj. / lx.]

A straight line commensurable in length with a binomial [ex binis nominibus] is itself also a binomial, and of the same order.

Let AB be any binomial, divided into its terms, of which AC is the greater and CB the lesser; and let the straight line DE be commensurable in length with AB. I say that DE too is a binomial, and of the same order as AB.

[Margin: a — 12. sexti (Elements VI.12)] For let it be contrived that as the whole AB is to the whole DE, so is the part AC cut off to the part DF cut off. [Margin: b — 19. quin. (Elements V.19)] Then the remainder CB will also be to the remainder FE as the whole AB is to the whole DE. And since DE is assumed commensurable in length with AB, [Margin: c — 10. dec. (Elements X.10)] DF will be commensurable in length with AC, and FE with CB. [Margin: d — 37. dec. (Elements X.37)] But AC and CB, being the terms of a binomial, are rational; therefore DF and FE are rational as well.

Again, since AC is to DF as CB is to FE, then, alternando, AC is to CB as DF is to FE. Now AC and CB are commensurable in square only; [Margin: e — 10. dec. (Elements X.10)] therefore DF and FE too are commensurable in square only, [Margin: f — 37. dec. (Elements X.37)] and hence DE is a binomial.

I say also that it is of the same order as AB. For AC is greater than CB in square either by the square on a straight line commensurable with itself in length, or by the square on one incommensurable. If AC is greater than CB in square by the square on a straight line commensurable with itself in length, [Margin: g — 15. dec. (Elements X.15)] then DF too will be greater than FE in square by the square on a straight line commensurable with itself in length. And if AC is commensurable in length with the assigned rational, so that AB is a first binomial, [Margin: h — 12. dec. (Elements X.12)] then DF too will be commensurable in length with the same rational — since both the rational and DF are commensurable in length with the same AC. Therefore DE, by definition, is a first binomial; that is, of the same order. But if CB is commensurable in length with the assigned rational, so that AB is a second binomial, then in the same way FE too will be commensurable in length with the assigned rational; and therefore DE, by definition, is a second binomial. And if, finally, neither AC nor CB is commensurable in length with the assigned rational, so that AB is a third binomial, [Margin: i — 12. dec. (Elements X.12)] then neither DF nor FE will be commensurable in length with the assigned rational; and therefore DE, by definition, is a third binomial.

[Margin: k — 15. dec. (Elements X.15)] But if AC is greater than CB in square by the square on a straight line incommensurable with itself in length, then DF too will be greater than FE in square by the square on a straight line incommensurable with itself in length. Hence, as before, we shall show that DE is a fourth, fifth, or sixth binomial, according as AB has been a fourth, fifth, or sixth binomial. Therefore a line commensurable in length with a binomial, and so on. Which was to be proved.

Diagram. Two horizontal straight-line segments, one above the other, representing the binomial AB and the commensurable line DE, each divided into its two terms.

- A (upper line, left endpoint): start of the binomial AB
- C (upper line, mid-point): division into the terms AC (greater) and CB (lesser)
- B (upper line, right endpoint): end of AB
- D (lower line, left endpoint): start of the commensurable line DE
- F (lower line, mid-point): division into the parts DF and FE
- E (lower line, right endpoint): end of DE

SCHOLIVM.

If the straight line DE is commensurable in square only with AB, which is a binomial, we shall indeed show in the same way — provided that everywhere in the demonstration we substitute for "commensurable in length" the phrase "commensurable in square only" — that DE too is a binomial. But that it is of the same order as AB we shall by no means be able to conclude. For it does not follow, if the greater term AC is commensurable in length with the assigned rational, that the greater term DF is commensurable in length with the same; because it is not the case that both — namely the assigned rational and DF — are commensurable in length with the same AC, but only the rational is commensurable in length, whereas DF is commensurable in square only, by hypothesis. Indeed, if AB is a first, second, fourth, or fifth binomial, it is in no way possible that DE, being commensurable with it in square only, should be a binomial of the same order.

For let AB be a first, second, fourth, or fifth binomial, divided into its terms at C, and let DE be commensurable with it in square only. DE, as in the theorem, we shall show to be a binomial, divided into its terms at F, with the parts AC, CB proportional to the parts DF, FE — that is, the former have to the latter the same ratio as the whole AB has to the whole DE. I say it is in no way possible that DE be a binomial of the same order as AB. For if it were possible, let each be a first binomial.

Then each greater term, AC and DF, will by definition be commensurable in length with the assigned rational; [Margin: l — 12. dec. (Elements X.12)] and therefore they will be commensurable in length with one another. And since AC is to DF as AB is to DE, [Margin: m — 10. dec. (Elements X.10)] AB and DE will consequently be commensurable in length. But this is absurd, for DE was assumed commensurable in square only with AB. Therefore AB and DE are not both first binomials. In the same way, neither will they both be fourth binomials; nor yet both second or fifth. For then each lesser term, CB and FE, would by definition be commensurable in length with the assigned rational, and hence also with one another; and so the wholes AB, DE would be commensurable in length — which is not assumed.

Yet it is always true that, if AB is a first, second, or third binomial, the straight line DE commensurable with it in square only is likewise one of those three, though not of the same order. For, this being posited, AC will be greater than CB in square by the square on a straight line commensurable with itself in length. Since, then, AC is to CB as DF is to FE, [Margin: n — 15. dec. (Elements X.15)] DF too will be greater than FE in square by the square on a straight line commensurable with itself in length. Therefore, by definition, DE will be a first, second, or third binomial. In the same way, if AB is a fourth, fifth, or sixth binomial, so that AC is greater than CB in square by the square on a straight line incommensurable with itself in length, [Margin: o — 15. dec. (Elements X.15)] DF too will be greater than FE in square by the square on a straight line incommensurable with itself in length, since AC is to CB as DF is to FE. Therefore, by definition, DE will be a fourth, fifth, or sixth binomial as well, though not of the same order as AB.

In the following four propositions, however, DE will necessarily be of the same order as AB, even though it is commensurable with it in square only.

THEOR. 50. PROPOS. 68.

[Margin: lxvij. / lxi.]

A straight line commensurable in length with a bimedral [ex binis Mediis] is itself also a bimedral, and of the same order.

Let AB be any bimedral, divided into its terms, of which AC is the greater and CB the lesser, and let the straight line DE be commensurable in length with it. I say that DE too is a bimedral, and of the same order as AB.

[Margin: a — 12. sext. (Elements VI.12)] For let it be contrived that as the whole AB is to the whole DE, so is the part AC cut off to the part DF cut off. [Margin: b — 19. quin. (Elements V.19)] Then the remainder CB will also be to the remainder FE

as the whole AB is to the whole DE. And since DE is assumed commensurable in length with AB, [Margin: c — 10. dec. (Elements X.10)] DF will be commensurable in length with AC, and FE with CB. But AC and CB are medials; [Margin: d — 24. dec. (Elements X.24)] therefore DF and FE are medials too.

Again, since AC is to DF as CB is to FE, then, alternando, AC is to CB as DF is to FE. [Margin: e — 38. dec. (Elements X.38)] Now AC and CB are commensurable in square only; [Margin: f — 10. dec. (Elements X.10)] therefore DF and FE too are commensurable in square only. And since they have also been shown to be medials, DF and FE are medials commensurable in square only; [Margin: g — 38. vel 39. decimi (Elements X.38 or 39)] and therefore DE is a bimedral.

I say also that it is of the same order as AB. For since AC is to CB as DF is to FE, and as AC is to CB so is the square on AC to the rectangle contained by AC, CB, and as DF is to FE so is the square on DF to the rectangle contained by DF, FE — by lemma 3 of proposition 19 of this book — it will also hold that as the square on AC is to the rectangle contained by AC, CB, so is the square on DF to the rectangle contained by DF, FE; and, alternando, as the rectangle contained by AC, CB is to the rectangle contained by DF, FE, so is the square on AC to the square on DF; and, converting, as the square on AC is to the square on DF, so is the rectangle contained by AC, CB to the rectangle contained by DF, FE. But the square on AC is commensurable with the square on DF, since the straight lines AC, DF have been shown to be commensurable in length. [Margin: h — 10. dec. (Elements X.10)] Therefore the rectangle contained by AC, CB is also commensurable with the rectangle contained by DF, FE.

Hence, if the rectangle contained by AC, CB is rational, so that AB is a first bimedral, then the rectangle contained by DF, FE will also be rational, being commensurable with it; [Margin: i — 18. dec. (Elements X.18)] and so DE too will be a first bimedral. But if the rectangle contained by AC, CB is medial, so that AB is a second bimedral, then the rectangle contained by DF, FE will likewise be commensurable with it and medial, by the corollary of proposition 24 of this book; [Margin: k — 39. dec. (Elements X.39)] and therefore DE too will be a second bimedral. And so a line commensurable in length with a bimedral, and so on. Which was to be demonstrated.

Diagram. Two horizontal straight-line segments, one above the other, representing the bimedral AB and the commensurable line DE, each divided into its two terms.

- A (upper line, left endpoint): start of the bimedral AB

- C (upper line, mid-point): division into AC (greater) and CB (lesser)
- B (upper line, right endpoint): end of AB
- D (lower line, left endpoint): start of the commensurable line DE
- F (lower line, mid-point): division into DF and FE
- E (lower line, right endpoint): end of DE

SCHOLIVM.

In exactly the same way we shall prove that the straight line DE, if it is commensurable in square only with AB, which is a bimedral, is itself a bimedral and of the same order as the AB with which it is commensurable — provided only that everywhere, in place of "commensurable in length," we say throughout the demonstration "commensurable in square only," and so on, as is evident.

THEOR. 51. PROPOS. 69.

[Margin: lxviiij. / lxij.]

A straight line commensurable with a major [Maior] is itself also a major.

Let the major line AB be divided into its terms at C, and let DE be commensurable with it, whether in length and in square, or in square only. I say that DE too is a major. Let the same construction be made as before, so that the parts AC, CB have to the parts DF, FE the same ratio as the whole AB has to the whole DE. [Margin: a — 10. dec. (Elements X.10)] Since AC and CB are commensurable either in length and in square, or in square only, then AC, DF and CB, FE will be commensurable in the same way.

Again, since AC is to DF as CB is to FE, then, alternando, AC is to CB as DF is to FE; [Margin: b — 22. sexti (Elements VI.22)] therefore the square on AC is to the square on CB as the square on DF is to the square on FE; and, componendo, the sum of the squares on AC, CB is to the square on CB as the sum of the squares on DF, FE is to the square on FE; and, alternando, the sum of the squares on AC, CB is to the sum of the squares on DF, FE as the square on CB is to the square on FE. But the square on CB is commensurable with the square on FE, since the straight lines CB, FE have been shown to be commensurable either in length and in square, or in square only. [Margin: c — 10. dec. (Elements X.10)] Therefore the sum of the squares on the straight lines AC, CB is commensurable with the sum of the squares on the straight lines DF, FE. [Margin: d — 40. dec. (Elements X.40)] But that sum is rational, since the straight lines AC, CB compose the major AB. [Margin: e — 9. defin. (Elements X, definition 9)] Therefore this sum too is rational.

Again, since AC is to CB as DF is to FE; and as AC is to CB so is the square on AC to the rectangle contained by AC, CB, and as DF is to FE so is the square on DF to the rectangle contained by DF, FE, by lemma 3 of proposition 19 of this book; it will also hold that as the square on AC is to the rectangle contained by AC, CB, so is the square on DF to the rectangle contained by DF, FE; and, alternando, as the square on AC is to the square on DF, so is the rectangle contained by AC, CB to the rectangle contained by DF, FE. But the square on AC is commensurable with the square on DF, since the straight lines AC, DF have been shown to be commensurable either in length and in square, or in square only. [Margin: f — 10. dec. (Elements X.10)] Therefore the rectangle contained by AC, CB is commensurable with the rectangle contained by DF, FE. [Margin: g — 40. dec. (Elements X.40)] But the rectangle contained by AC, CB is medial; therefore the rectangle contained by DF, FE, being commensurable with it, is also medial, by the corollary of proposition 24 of this book.

And since AC is to CB as DF is to FE, [Margin: h — 40. dec. (Elements X.40)] and AC, CB are incommensurable in square — since they compose the major AB — [Margin: i — 10. dec. (Elements X.10)] DF, FE will also be incommensurable in square. And so, since DF, FE are incommensurable in square, and make the sum of their squares rational while the rectangle contained by them is medial, [Margin: k — 40. dec. (Elements X.40)] as has been shown, the whole DE is a major. Therefore a line commensurable with a major is itself also a major. Which was to be demonstrated.

Diagram. Two horizontal straight-line segments, one above the other, representing the major AB and the commensurable line DE, each divided into its two terms.

- A (upper line, left endpoint): start of the major AB
- C (upper line, mid-point): division into AC (greater) and CB (lesser)
- B (upper line, right endpoint): end of AB
- D (lower line, left endpoint): start of the commensurable line DE
- F (lower line, mid-point): division into DF and FE
- E (lower line, right endpoint): end of DE

THEOR. 52. PROPOS. 70.

[Margin: lxix. / lxij.]

A straight line commensurable with the side of a rational plus a medial [Rationale ac Medium potens] is itself also the side of a rational plus a medial.

Let AB be the side of a rational plus a medial, divided into its terms at C, and let DE be commensurable with it, whether in length and in square, or in square only. I say that DE too is the side of a rational plus a medial. For, the same construction being made as above, we shall show, as in the preceding proposition, that the sum of the squares on the straight lines AC, CB [Margin: a — 41. dec. (Elements X.41)] is commensurable with the sum of the squares on the straight lines DF, FE. But that sum is medial, since the straight lines AC, CB compose AB, the side of a rational plus a medial. Therefore, by the corollary of proposition 24 of this book, this sum too, being commensurable with it, is medial. In the same way, as in the same preceding proposition, the rectangle contained by AC, CB, [Margin: b — 41. dec. (Elements X.41)] which is rational, will be commensurable with the rectangle contained by DF, FE; and so the rectangle contained by DF, FE is rational as well, by definition 9. Finally, as in the same earlier proposition, DF, FE will be incommensurable in square. And so, since DF, FE are incommensurable in square, and make the sum of their squares medial while the rectangle contained by them is rational, as has been demonstrated, [Margin: c — 41. dec. (Elements X.41)] the whole DE is the side of a rational plus a medial. Therefore a line commensurable with the side of a rational plus a medial, and so on. Which was to be demonstrated.

Diagram. Two horizontal straight-line segments, one above the other, representing AB (the side of a rational plus a medial) and the commensurable line DE, each divided into its two terms.

- A (upper line, left endpoint): start of AB
- C (upper line, mid-point): division into AC (greater) and CB (lesser)
- B (upper line, right endpoint): end of AB
- D (lower line, left endpoint): start of DE
- F (lower line, mid-point): division into DF and FE
- E (lower line, right endpoint): end of DE

THEOR. 53. PROPOS. 71.

[Margin: lxx. / lxiv.]

A straight line commensurable with the side of two medials [bina Media potens] is itself also the side of two medials.

Let AB be the side of two medials, divided into its terms at C, and let DE be commensurable with it, whether in length and in square, or in square only. I say that DE too is the side of two medials. For, the same construction being made as above, we shall show, as in proposition 69, that the sum of the squares on the straight lines AC, CB [recte: AC, CB] is commensurable with the sum of the squares on the straight lines DF, FE, [Margin: a — 42. dec. (Elements X.42)] and also that the rectangle contained by AC, CB is commensurable with the rectangle contained by DF, FE. Since, then, both the sum of the squares on AC, CB and the rectangle contained by AC, CB are medial, by the corollary of proposition 24 of this book both the sum of the squares on DF, FE and the rectangle contained by DF, FE will also be medial. But DF, FE are incommensurable in square, as before.

Finally, [Margin: b — 42. dec. (Elements X.42)] since the sum of the squares on AC, CB and the rectangle contained by AC, CB are incommensurable — by hypothesis, because AB is the side of two medials — while the sum of the squares on DF, FE is commensurable with the sum of the squares on AC, CB, and the rectangle contained by DF, FE we have proved commensurable with the rectangle contained by AC, CB, by the scholium of proposition 14 of this book, the sum of the squares on DF, FE and the rectangle contained by DF, FE will likewise be incommensurable. And so, since the straight lines DF, FE are incommensurable in square, and make the sum of their squares medial, and the rectangle contained by them medial and incommensurable with the sum of their squares, [Margin: c — 42. dec. (Elements X.42)] DE is the side of two medials. And so a line commensurable with the side of two medials is itself also the side of two medials [recte: bina Media potens; source prints "ipsa Media potens"]. Which was to be proved.

Diagram. Two horizontal straight-line segments, one above the other, representing AB (the side of two medials) and the commensurable line DE, each divided into its two terms.

- A (upper line, left endpoint): start of AB
- C (upper line, mid-point): division into AC (greater) and CB (lesser)
- B (upper line, right endpoint): end of AB
- D (lower line, left endpoint): start of DE
- F (lower line, mid-point): division into DF and FE
- E (lower line, right endpoint): end of DE

THEOR. 54. PROPOS. 72.

[Margin: lxxi. / lxv.]

If a rational and a medial area are compounded, four irrational straight lines arise: either the binomial [ex binis nominibus], or the first bimedial [ex binis Mediis prima], or the major [Maior], or the side of a rational plus a medial [Rationale ac Medium potens].

Let AB be a rational area, with which the medial CD is compounded. I say that the straight line which is the side of the whole area AD [that is, whose square equals AD] is either a binomial, or a first bimedial, or a major, or the side of a rational plus a medial. For AB will be either greater than CD or less; equal it will not be, for otherwise, since AB is rational, CD too, being equal to it, would be rational — which is not assumed.

Let it first be the greater. [Margin: a — 45. primi (Elements I.45)] Let the rational EF be set out, and to it let there be applied the rectangle EG equal to AB, and to HG another rectangle HI equal to CD, so that the whole EI is equal to the whole AD. And since AB is rational and CD medial, EG too will be rational and HI medial; and since these are applied to the rational EF, [Margin: b — 21. dec. (Elements X.21)] EH will be rational and commensurable in length with EF, [Margin: c — 23. dec. (Elements X.23)] whereas HK will be rational but incommensurable in length with EF. Again, since EG, HI are incommensurable — as is clear from the definitions, because EG is rational while HI is irrational, namely medial — [Margin: d — 10. dec. (Elements X.10)] EH, HK, having the same ratio as they, will be incommensurable in length. Therefore EH, HK are rational and commensurable in square only; [Margin: e — 37. dec. (Elements X.37)] and hence EK is a binomial.

And since AB is assumed greater than CD, that is, EG greater than HI, EH will also be greater than HK; [Margin: f — 1. sexti (Elements VI.1)] for the rectangles EG, HI, being of the same height, are to one another as their bases EH, HK. Now the greater term EH will be greater in square than the lesser term HK either by the square on a straight line commensurable with itself in length, or by the square on one incommensurable. If EH is greater than HK in square by the square on a straight line commensurable with itself in length, then EK will be a first binomial, since the greater term EH has been shown commensurable in length with the rational EF. Therefore the straight line that is the side of the area EI — the area contained by the rational EF and the first binomial EK — and so also of the area AD, which is compounded of the rational AB and the medial CD, [Margin: g — 55. dec. (Elements X.55)] is the irrational called the binomial.

But if EH is greater than HK in square by the square on a straight line incommensurable with itself in length, then EK [recte: EK; source prints "EA"] will

be a fourth binomial, by definition, since the greater term EH has been shown commensurable in length with the rational EF. Therefore the straight line that is the side of the area EI — the area contained by the rational EF and the fourth binomial EK — and so also of the area AD, [Margin: h — 58. dec. (Elements X.58)] is the irrational called the major.

Next let AB be less than CD, and let the same construction be made as before. Then, as before, EK will be a binomial, and EH commensurable in length with EF. And since AB is less than CD, that is, EG less than HI, the straight line EH will also be less than HK. Therefore, again, HK, the greater term of EK (which is a binomial), will be greater in square than the lesser term EH either by the square on a straight line commensurable with itself in length, or by one incommensurable. If HK is greater than EH in square by the square on a straight line commensurable with itself in length, then EK will be a [first]...

Diagram. The construction for the case in which AB is greater than CD. On the left stand the two given areas (the rational AB and the medial CD) drawn as adjacent rectangles; on the right, the rectangle EI applied to the rational straight line EF, subdivided by a vertical into the rational part EG (equal to AB) and the medial part HI (equal to CD), with EK the resulting binomial breadth.

- A (left figure, top-left corner): rational area AB
- C (left figure, top, division point)
- B (left figure, bottom-left corner)
- D (left figure, bottom, division point)
- E (right figure, top-left corner): one end of the applied rectangle EI
- H (right figure, top, first division): boundary between EG and HI along the top edge (EH, HK)
- K (right figure, top-right corner): far end of the binomial breadth EK
- F (right figure, bottom-left corner): base of the rational line EF
- G (right figure, bottom, division): foot of the internal division (rectangles EG, HI)
- I (right figure, bottom-right corner): far end of the base

Diagram. The same construction redrawn for the case in which AB is less than CD (points relabelled identically). Two rectangles on the left for the given areas AB and CD; on the right, the rectangle EI applied to EF and subdivided into EG and HI, with breadth EK.

- A (left figure, top-left corner)

- C (left figure, top, division point)
- B (left figure, bottom-left corner)
- D (left figure, bottom, division point)
- E (right figure, top-left corner)
- H (right figure, top, first division)
- K (right figure, top-right corner)
- F (right figure, bottom-left corner)
- G (right figure, bottom, division)
- I (right figure, bottom-right corner)

Book X — continuation

[Conclusion of THEOR. 54. PROPOS. 72 — from a rational and a medial area, four irrationals arise]

(Clavius's demonstration, continued from the preceding page.)

the second binomial, since EH, the lesser name, is commensurable in length with the rational EF, as was shown. Therefore the line which is potent in the area EI — the area contained by the rational EF and the second binomial EK — and hence also the area AD, composed of the rational AB and the medial CD, is irrational, and is called the first bimedial. [Margin: i 56. dec. — Elements X.56 (Clavius's numbering, one ahead of the received text; the reference is to the theorem that a rational times a second binomial yields the first bimedial)]

But if HK is potent by more than EH, by the square of a straight line incommensurable in length with itself, then EK, since EH — the lesser name shown — is commensurable in length with the rational EF, is the fifth binomial; and accordingly the line which is potent in the area EI, contained by the rational EF and the fifth binomial [EK], and thus also the area AD, [Margin: k 59. dec. — Elements X.59 (Clavius's numbering)] is irrational, and is called the [side] potent in a rational and a medial [i.e. the side of a rational plus a medial area]. If, therefore, a rational and a medial be placed together, four irrationals arise, and so on. Which was to be demonstrated.

THEOR. 55. PROPOS. 73. [Margin: Lxxii. / Lxvi.]

If two medials incommensurable with one another be composed, the two remaining irrationals arise — either the second bimedial, or the side potent in two medials.

(Euclid, Elements X.72; Clavius PROPOS. 73.)

Demonstration. Let two medials AB, CD, incommensurable with one another, be composed. I say that the line which is potent in the whole area AD is either the second binomial or the side potent in two medials. For again AB will be either greater than CD or less; it will certainly not be equal, for otherwise AB, CD would be commensurable — which is not supposed. Let it first be the greater, and let everything be done that was done in the preceding proposition. And since the areas AB, CD are supposed to be medial and incommensurable, the areas EG, HI will likewise be medial and incommensurable, since they are equal to them. [Margin: a 10. dec. — Elements X.10 (Clavius's numbering)] Therefore the straight lines EH, HK, which have the same ratio to those areas, are incommensurable in length; and each of them (since EG, HI are medial and are applied to the rational EF) [Margin: b 23. dec. — Elements X.23] is rational, being incommensurable in length with EF itself. EH, HK are therefore rational and commensurable in square only; [Margin: c 37. dec. — Elements X.37] and for that reason EK is a binomial. Now since AB is supposed greater than CD, then, as in the preceding proposition, EH will be the greater name of EK. It can therefore be potent by more than HK by the square of a straight line either commensurable or incommensurable with itself in length. If EH is potent by more than HK by the square of a straight line commensurable with itself in length, then, since each name EH, HK has been shown to be incommensurable in length with the rational EF, EK will by definition be the third binomial. Therefore the line which is potent in the area EI, contained by the rational EF and the third binomial EK — and hence also the area AD — [Margin: d 57. dec. — Elements X.57] is irrational, and is called the second binomial.

But if EH is potent by more than HK by the square of a straight line incommensurable with itself in length, then, since each name EH, HK is incommensurable in length [with the rational EF, EK will be] the sixth binomial, as is clear from the definition. The line, therefore, which is potent in the area EI, that is, in the area AD, [Margin: e 60. dec. — Elements X.60] is irrational, and is named the side potent in two medials.

But if AB be less than CD, we shall conclude the proposition in no other way, as is clear from the figure. Therefore, if two medials commensurable [recte: incommensurable] with one another be composed, the two remaining irrationals arise, and so on. Which was to be shown.

Diagram. The first of two paired figures for this proposition (the "greater" case, $EH > HK$). At the left, a square standing on base BD; at the right, a taller rectangle standing on base FI, divided by internal vertical lines.

- A (top-left corner of the left square): endpoint of the upper side
- C (point on the upper side of the left square): division of the top
- B (bottom-left corner of the left square): endpoint of the base
- D (point on the base of the left square): division of the base (BD)
- E (top-left corner of the right rectangle): endpoint of the upper side
- H (point on the upper side of the right rectangle): division of the top
- K (top-right corner of the right rectangle): endpoint of the upper side
- F (bottom-left corner of the right rectangle): endpoint of the base
- G (point on the base of the right rectangle): division of the base
- I (bottom-right corner of the right rectangle): endpoint of the base

Diagram. The second paired figure (the remaining cases), lettered identically to the first: a left square on base BD and a right rectangle on base FI.

- A (top-left corner of the left square)
- C (point on the upper side of the left square)
- B (bottom-left corner of the left square)
- D (point on the base of the left square)
- E (top-left corner of the right rectangle)
- H (point on the upper side of the right rectangle)
- K (top-right corner of the right rectangle)
- F (bottom-left corner of the right rectangle)
- G (point on the base of the right rectangle)
- I (bottom-right corner of the right rectangle)

COROLLARIUM.

From all this it is easily gathered that the line which arises from two names [the binomial], and the remaining irrational lines that follow it, are neither medial lines nor the same as one another.

For the square on a medial, applied to the rational line, [Margin: f 23. dec. — Elements X.23] produces as breadth a line incommensurable in length with the rational.

But the square on the binomial, applied to the rational, [Margin: g 61. dec. — Elements X.61] produces as breadth a first binomial.

And the square on the first bimedial, applied to the rational, [Margin: h 62. dec. — Elements X.62] produces as breadth a second binomial.

The square, next, on the second bimedral, applied to the rational, [Margin: i 63. dec. — Elements X.63] produces as breadth a third binomial.

But the square on the major, applied to the rational, [Margin: k 64. dec. — Elements X.64] produces as breadth a fourth binomial.

The square on the side potent in a rational and a medial, applied to the rational, [Margin: l 65. dec. — Elements X.65] produces as breadth a fifth binomial.

And lastly the square on the side potent in two medials, applied to the rational, [Margin: m 66. dec. — Elements X.66] produces as breadth a sixth binomial.

Therefore, since these breadths differ both from the breadth of the medial and from one another — from the breadth of the medial, because this one is rational whereas those are irrational; and from one another, because in order they are not the same binomials — it is clear that all the irrational lines discussed so far are different from one another.

SCHOLIUM.

So far Euclid has treated seven groups of six [senaries].

In the first, which is contained in propositions 37, 38, 39, 40, 41, and 42, he handed down the origin of six irrational lines: namely the binomial; the first bimedral; the second bimedral; the major; the side potent in a rational and a medial; and finally the side potent in two medials.

(Scholium, continued.)

In the second, which propositions 43, 44, 45, 46, 47, and 48 comprise, he treated their divisions, teaching that each single one can be divided at determinate points into its own names.

In the third, contained in propositions 49, 50, 51, 52, 53, and 54, he taught the finding of the six lines that are called binomials: namely the first, second, third, fourth, fifth, and sixth.

In the fourth, which propositions 55, 56, 57, 58, 59, and 60 complete, he showed how these six irrational lines, expounded in the first senary, differ from one another, teaching which irrational line is that which is potent in the area contained by the rational and the first binomial, or by the second, or the third, or the fourth, or the fifth, or the sixth binomial.

In the fifth, moreover, which you will find in propositions 61, 62, 63, 64, 65, and 66, he taught what breadths the squares of the irrational lines expounded in the first senary produce as irrationals when applied to the rational line.

In the sixth, which is completed in the five propositions 67, 68, 69, 70, and 71, he demonstrated that any line commensurable with one of the said six irrationals of the first senary is itself irrational, of the same kind as the one it is commensurable with.

In the seventh, finally, contained in two propositions — namely 72 and 73 — he set out once more a further difference among the six aforesaid irrationals.

Moreover, there is found in these six irrational lines, treated in the first senary, an Analogy or proportionality of the Arithmetic kind, which consists in an equal excess. The straight line that is the mean proportional, according to the Arithmetic Analogy, between the two names of any irrational line, is itself also irrational, of the same kind as the line between whose names it lies as mean.

For let any irrational line whatever — say the binomial AB, whose greater name is AC — be given, and let AB be bisected at D; and let E be taken equal to its half AD, or DB. I say that E is the mean, according to the Arithmetic Analogy, between AC and CB, and that it too is a binomial. For since AC exceeds the half AD by the straight line DC, and the half DB exceeds CB itself by the same straight line DC, it is clear that the half of AB — namely E — is the mean proportional between AC and CB in the Arithmetic Analogy.

Diagram. A straight line AB divided internally, with a shorter separate segment E drawn beneath it (equal to half of AB).

- A (left endpoint of the line): terminus of AB
- D (interior point): the midpoint of AB (bisection)
- C (interior point, between D and B): the division marking the greater name AC and lesser name CB
- B (right endpoint of the line): terminus of AB
- E (separate short segment below): equal to the half AD (= DB), the arithmetic mean between AC and CB

Again, since E is commensurable in length with the whole AB — for AB is its double — E too will be irrational, of the same kind as AB, as was demonstrated in the sixth senary.

We shall show that the same holds, in no other way, for the remaining irrational lines.

There now follow seven other senaries, in which Euclid demonstrates the same things concerning six other irrational lines, which are generated by subtraction, as we have shown that he demonstrated in the preceding seven senaries concerning the irrationals that arise by composition.

PRINCIPIVM SENARIORVM — per detractorem [The beginning of the senaries — by subtraction]

THEOR. 56. PROPOS. 74. [Margin: Lxxiii. / Lxviii.]

If from a rational a rational commensurable in square only with the whole be subtracted, the remainder is irrational. Let it be called an apotome.

(*Euclid, Elements X.73; Clavius PROPOS. 74.*)

Demonstration. Let there be subtracted from the rational AB a rational AC, commensurable in square only with AB itself. I say that the remainder BC is irrational. For since, by lemma 3 of proposition 19 of this book, as AB is to AC, so is the square on AB to the rectangle contained by AB, AC; and AB, AC are incommensurable in length; [Margin: a 10. dec. — Elements X.10] the square on AB and the rectangle contained by AB, AC will be incommensurable. But the square on AB is commensurable with the sum of the squares on AB, AC (for since AB, AC are supposed commensurable in square, the squares on AB, AC will be commensurable; [Margin: b 16. dec. — Elements X.16] therefore their sum too will be commensurable with the square on AB), while the rectangle contained by AB, AC is commensurable with twice that rectangle. Therefore the sum of the squares on AB, AC and twice the rectangle contained by AB, AC are incommensurable, as we demonstrated in the scholium of proposition 14 of this book. [Margin: c 7. sec. — Elements II.7] Now the sum of the squares on AB, AC is equal to twice the rectangle under AB, AC together with the square on BC. Therefore the sum of the squares on AB, AC is, by the corollary of proposition 17 of this book, incommensurable with the remaining square on BC. Since, then, the sum of the squares on AB, AC is rational (because it is commensurable with the rational square on AB, described on the rational line), the square on BC will be irrational, and accordingly the straight line BC itself will be irrational. Let it be called an apotome. The more recent writers call it a residual. If, therefore, from a rational a rational commensurable in square only with the whole be subtracted, the remainder is irrational, and so on. Which was to be shown.

Diagram. A straight line AB divided at C; BC is the apotome (remainder).

- A (left endpoint): terminus of the whole line AB
- C (interior division point): marks off the subtracted rational AC
- B (right endpoint): terminus of AB; BC is the remaining apotome

SCHOLIUM.

Euclid could also have proposed this proposition in the following way.

If from the greater name of a binomial the lesser name be subtracted, the remainder is irrational. Let it be called an apotome.

For since AB, AC are rational and commensurable in square only, the line composed of them is irrational — it is called a binomial, whose greater name is AB and lesser name is AC. [Margin: d 37. dec. — Elements X.37] Therefore AC, the lesser name, is subtracted from the greater name AB, so that the apotome BC is left.

Diagram. A straight line AB divided at C (the scholium's figure for the apotome).

- A (left endpoint): terminus of AB
- C (interior division point): the point of subtraction
- B (right endpoint): terminus of AB; BC is the apotome

THEOR. 57. PROPOS. 75. [Margin: Lxxiiii. / Lxix.]

If from a medial a medial commensurable in square only with the whole be subtracted, such that it contains with the whole a rational [rectangle], the remainder is irrational. Let it be called the first apotome of a medial.

(Euclid, Elements X.74; Clavius PROPOS. 75.)

Demonstration. Let there be subtracted from the medial AB a medial AC, commensurable in square only with AB itself, and let the rectangle contained by AB, AC be rational. I say that the remainder BC is irrational. For since AB, AC are commensurable in square, the squares on AB, AC will be commensurable, [Margin: a 16. dec. — Elements X.16] and hence their sum too will be commensurable with the square on AC. But the square on AC is medial, irrational, and a medial [area]. Therefore, by the corollary of proposition 24 of this book, the sum of the squares on AB, AC will also be irrational and medial. And since the rectangle contained by AB, AC is supposed rational, twice the rectangle under AB, AC will also be rational. Therefore the sum of the squares on AB, AC is incommensurable with twice the rectangle under AB, AC. [Margin: b 7. sec. —

Elements II.7] Since, then, the sum of the squares on AB, AC is equal to twice the rectangle under AB, AC together with the square on BC, twice the rectangle under AB, AC together with the square on BC will also be incommensurable with twice the rectangle under AB, AC. [Margin: c 17. dec. — Elements X.17] Therefore twice the rectangle under AB, AC and the square on BC are also incommensurable. Since, then, twice the rectangle under AB, AC is rational, the square on BC will be irrational; and hence the straight line BC too will be irrational. Let it be called the first apotome of a medial, because, namely, it is what is left after subtracting the lesser name of the first bimedral line from the greater name, as will be made clear in the following scholium. If, therefore, from a medial a medial commensurable in square only with the whole be subtracted, such that it contains with the whole a rational [rectangle], the remainder is irrational. Let it be called the first apotome of a medial. Which was to be shown.

Diagram. A straight line AB divided at C; BC is the first apotome of a medial.

- A (left endpoint): terminus of the whole medial AB
- C (interior division point): marks off the subtracted medial AC
- B (right endpoint): terminus of AB; BC is the first apotome of a medial

SCHOLIUM.

This proposition could also have been proposed in the following way.

If from the greater name of the first bimedral the lesser name be subtracted, the remainder is irrational. Let it be called the first apotome of a medial.

For since AB, AC are medials commensurable in square only, and contain a rational [rectangle], the line composed of them will be irrational — it is called the first bimedral, whose greater name is AB and lesser name is AC. [Margin: d 38. dec. — Elements X.38] Therefore AC, the lesser name, is subtracted from the greater name AB, so that BC, the first apotome of a medial, is left.

THEOR. 58. PROPOS. 76. [Margin: Lxxv. [?] / Lxx.]

If from a medial a medial commensurable in square only with the whole be subtracted, such that it contains with the whole a medial [rectangle], the remainder is irrational. Let it be called the second apotome of a medial.

(Euclid, Elements X.75; Clavius PROPOS. 76.)

Demonstration. Let there be subtracted from the medial AB a medial AC, commensurable in square only with AB itself, and let the rectangle contained by AB, AC be medial. I say that the remainder BC is irrational. For since the squares on AB, AC are commensurable, being commensurable in square, [Margin: a 16. dec. — Elements X.16] their sum too will be commensurable with each of them. But each of them is medial, since the straight lines AB, AC are supposed medial. Therefore their sum too, commensurable with each of them, is medial, by the corollary of proposition 24 of this book. Again, since the rectangle under AB, AC is supposed medial, its double also — namely twice the rectangle under AB, AC — will be medial, by the corollary of proposition 24 of this book. [Margin: b 7. sec. — Elements II.7] Since, then, the sum of the squares on AB, AC is equal to twice the rectangle under AB, AC together with the square on BC, the sum of the squares on AB, AC — which is medial — will exceed twice the rectangle under AB, AC — which is also medial — by the square on BC. But a medial does not exceed a medial by a rational [area]. [Margin: c 67. dec. — Elements X.26 (Clavius's numbering; the tabulated figure "67" appears to be a misprint or misreading — see anomaly note)] Therefore the square on BC is not rational. It is therefore irrational, and the straight line BC itself is irrational. Let it be called the second apotome of a medial, because it is what is left after subtracting the lesser name of the second binomial from the greater name, as will appear from the following scholium. If, therefore, from a medial a medial be subtracted, and so on. Which was to be shown.

Diagram. A straight line AB divided at C; BC is the second apotome of a medial.

- A (left endpoint): terminus of the whole medial AB
- C (interior division point): marks off the subtracted medial AC
- B (right endpoint): terminus of AB; BC is the second apotome of a medial

SCHOLIUM.

This theorem too could have been proposed in the following way.

If from the greater name of the second binomial the lesser name be subtracted, the remainder is irrational. Let it be called the second apotome of a medial.

For since AB, AC are medials commensurable in square only, and contain a medial [rectangle], the line composed of them will be irrational — it is called the second binomial, whose greater name is AB and lesser name is AC. [Margin: d 39. dec. — Elements X.39] Therefore AC, the lesser name, is subtracted from the greater name AB, so that BC, the second apotome of a medial, is left.

THEOR. 59. PROPOS. 77. [Margin: Lxxvi. / Lxxi.]

If from a straight line a straight line incommensurable in square with the whole be subtracted, such that it makes with the whole the sum of the squares on them rational, but the [rectangle] contained by them medial, the remainder is irrational. Let it be called the minor.

(*Euclid, Elements X.76; Clavius PROPOS. 77.*)

Let AC, a straight line incommensurable in square with the whole AB, be subtracted from the straight line AB; and let the sum of the squares on AB and AC be rational, but the rectangle contained by AB and AC medial. I say that the remainder BC is irrational.

Diagram. A single straight line divided at one interior point. The line runs from A on the left to B on the right, with C marked between them (nearer B), so that AC is the part subtracted and CB the remainder.

- A (left endpoint): the whole line AB begins here
- C (interior point): the point of subtraction
- B (right endpoint): AB is the whole; AC the subtracted part; BC the remainder

For the sum of the squares on AB and AC is rational, whereas the rectangle contained by AB and AC — and hence also, by the corollary to Proposition 24 of this book, the quantity commensurable with it, namely twice the rectangle contained by AB and AC — is medial, that is, irrational. [Margin: a — Elements X.10] Therefore the sum of the squares on AB and AC [recte: is incommensurable with twice the rectangle contained by AB and AC — the predicate is dropped in the print, evidently by eye-skip on the repeated phrase "the rectangle under AB, AC"; supplied from the parallel demonstration of Prop. 78]. [Margin: b — Elements II.7] Now the sum of the squares on AB and AC [recte: is equal to twice the rectangle contained by AB and AC — likewise dropped by eye-skip] together with the square on BC. Therefore, by the corollary to Proposition 17 of this book, the sum of the squares on AB and AC is incommensurable with the remaining square on BC as well. But the sum of the squares on AB and AC is assumed to be rational. [Margin: c — Elements X, Def. 10] Therefore the square on BC, being incommensurable with it, is irrational, and so is the line BC. Let it be called the **minor**, since it is left over, after the subtraction, as the line of lesser name subtracted from the greater — as we shall explain in the following scholium. So then, if from a straight line a straight line be subtracted, and so on. Which was to be demonstrated.

SCHOLIUM

This theorem could also have been stated in the following way.

If from the greater name of the greater line the lesser name be subtracted, the remainder is irrational. Let it be called the **minor**.

*For since AB and AC are incommensurable in square, and make the sum of their squares rational but the rectangle contained by them medial, [Margin: d — Elements X, Def. 40] the line composed of them will be the irrational which is called the **major***, its greater name being AB and its lesser AC. Therefore AC, the lesser name, is subtracted from the greater AB, so that the* **minor** BC is left over.*

THEOR. 60. PROPOS. 78.

[Margin: Lxxvii. / Lxxii.]

If from a straight line there be subtracted a straight line incommensurable in square with the whole, which with the whole makes the sum of the squares on them medial but the rectangle contained by them rational, the remainder is irrational. Let it be called **that which produces with a rational area a medial whole**. (Euclid X.77.)

Let AC be subtracted from the straight line AB, incommensurable in square with AB; and let the sum of their squares be medial, but the rectangle contained by them rational. I say that the remainder BC is irrational.

Diagram. The same construction: a straight line from A (left) to B (right), divided at C between them.

- A (left endpoint)
- C (interior point of subtraction)
- B (right endpoint): AB whole, AC subtracted, BC remainder

For the sum of the squares on AB and AC is medial, that is, irrational, whereas the rectangle taken twice under AB and AC is rational. [Margin: a — Elements X, Def. 10] Therefore the sum of the squares on AB and AC is incommensurable with twice the rectangle contained by AB and AC. [Margin: b — Elements II.7] Now the sum of the squares on AB and AC is equal to twice the rectangle contained by AB and AC together with the square on BC. Therefore twice the rectangle contained by AB and AC, together with the square on BC, is incommensurable with twice the rectangle contained by AB and AC. [Margin: c — Elements X.17] Twice the

rectangle contained by AB and AC and the square on BC are therefore incommensurable. Since, then, twice the rectangle contained by AB and AC is rational, [Margin: d — Elements X, Def. 10] the square on BC will be irrational, and so the line BC is irrational. Let it be called **that which produces with a rational area a medial whole**, because its square, added to twice the rectangle contained by AB and AC (which is rational), makes the whole sum of the squares on AB and AC, which is medial. So then, if from a straight line a straight line be subtracted incommensurable in square with the whole, which with the whole makes the sum of their squares medial but the rectangle contained by them rational, the remainder is irrational, and so on. Which was to be shown.

SCHOLIUM

Euclid could also have stated this theorem in this manner.

If from the greater name of the line that is capable of a rational and a medial area the lesser name be subtracted, the remainder is irrational. Let it be called **that which produces with a rational area a medial whole**.

*For since AB and AC are incommensurable in square, and make the sum of their squares medial but the rectangle contained by them rational, [Margin: e — Elements X, Def. 41] the line composed of them will be the irrational which is named the line **capable of a rational and a medial area***, its greater name being AB and its lesser AC. Therefore the lesser name AC is subtracted from the greater AB, so that BC is left over as the line which produces with a rational area a medial whole.**

THEOR. 61. PROPOS. 79.

[Margin: Lxxviii. / Lxxiii.]

If from a straight line there be subtracted a straight line incommensurable in square with the whole, which with the whole makes both the sum of the squares on them medial and the rectangle contained by them medial, and this rectangle also incommensurable with the sum of their squares, the remainder is irrational. Let it be called **that which produces with a medial area a medial whole**. (Euclid X.78.)

Let AC be subtracted from the straight line AB, incommensurable in square with AB; and let both the sum of their squares and the rectangle contained by them be

medial, and let this rectangle be incommensurable with the sum of their squares. I say that the remainder BC is irrational.

Diagram. Again a straight line from A (left) to B (right), divided at C between them.

- A (left endpoint)
- C (interior point of subtraction)
- B (right endpoint): AB whole, AC subtracted, BC remainder

[Margin: a — Elements II.7] For since the sum of the squares on AB and AC is equal to twice the rectangle contained by AB and AC together with the square on BC, the sum of the squares on AB and AC — which is assumed medial — will exceed twice the rectangle contained by AB and AC (which, by the corollary to Proposition 24 of this book, is likewise medial, since it is commensurable with the medial rectangle contained by AB and AC) by the square on BC. [Margin: b — Elements X.27] But a medial does not exceed a medial by a rational area. Therefore the square on BC is not rational; hence it is irrational, and the line BC itself is irrational. Let it be called **that which produces with a medial area a medial whole**, because its square, together with the rectangle contained by AB and AC (which is medial), makes the whole sum of the squares on AB and AC, which is likewise medial. So then, if from a straight line a straight line be subtracted, and so on. Which was to be shown.

SCHOLIUM

This theorem could also have been proposed in the following way.

If from the greater name of the line capable of two medial areas the lesser name be subtracted, the remainder is irrational. Let it be called **that which produces with a medial area a medial whole**.

*Since AB and AC are incommensurable in square, and make both the sum of their squares medial and the rectangle contained by them medial, this rectangle being incommensurable with the sum of their squares, the line composed of them will be the irrational which is called **capable of two medial areas***, its greater name being AB and its lesser AC. Therefore the lesser name AC is subtracted from the greater AB, so that BC is left over as the line which produces with a medial area a medial whole.**

LEMMA

If the same excess lie between the first magnitude and the second as between the third magnitude and the fourth, then conversely there will also be the same excess between the first magnitude and the third as between the second magnitude and the fourth.

Let there be four magnitudes AB, CD, EF, GH, and let IB, the excess between AB and CD, be equal to the excess KF, between EF and GH. I say that the excess between AB and EF is the same as that between CD and GH.

Diagram. Four horizontal line segments arranged one above another :

- A——I—B: segment AB, marked at I (so IB is the excess of AB over its equal-to-CD part AI)
- C——D: segment CD
- E——K—F: segment EF, marked at K (so KF is the excess of EF over its equal-to-GH part EK)
- G——H: segment GH

For since IB is the excess between AB and CD, AI will be equal to CD. In the same way EK will be equal to GH. The same excess, then, will lie between AI and EK as between CD and GH, since these magnitudes are equal to those, each to each. But the excess between the wholes AB and EF is, by the sixteenth common notion of Book I, the same as the excess between AI and EK — because IB and KF are assumed equal. Therefore the same excess will also lie between AB and EF as between CD and GH. Which was proposed.

COROLLARIUM

From this it is clear that four magnitudes having an arithmetical proportion also have, alternately, an arithmetical proportion. For a proportion of this kind consists in the same excess, which we have shown to be the same between the first and third magnitudes as is found between the second and fourth, provided the excess between the first magnitude and the second is the same as that between the third magnitude and the fourth — that is, provided the four magnitudes have an arithmetical proportion.

THEOR. 62. PROPOS. 80.

[Margin: Lxxix. / Lxxiv.]

To an apotome there can be annexed only one rational straight line that is commensurable in square only with the whole. (Euclid X.79.)

Let AB be an apotome, and let the rational line BC, commensurable in square only with the whole AC, be annexed to it. I say that no other rational line can be annexed to AB which is commensurable in square only with the whole. For if it be possible, let another rational line BD be annexed, such that BD is commensurable in square only with AD.

Diagram. A single straight line carrying four marked points. From the left: A, then B (about midway), then C and D close together at the right.

- A (left endpoint)
- B (interior point): AB is the apotome
- C: BC is the annexed rational line, so AC is the whole
- D (right endpoint, near C): BD is the supposed second annex, so AD is the whole in that case

And since BC is rational, AC will be rational and commensurable in square only with it. AC and BC are therefore rationals commensurable in square only. In the same way AD and BD will be rationals commensurable in square only. Now the same excess lies between the sum of the squares on AC and BC and twice the rectangle contained by AC and BC as lies between the sum of the squares on AD and BD and twice the rectangle contained by AD and BD. (For the sum of the squares on AC and BC exceeds twice the rectangle contained by AC and BC by the square on AB, [Margin: a — Elements II.7] since the sum of the squares on AC and BC is equal to twice the rectangle contained by AC and BC together with the square on AB. And in the same way the sum of the squares on AD and BD exceeds twice the rectangle contained by AD and BD by the same square on AB, [Margin: b — Elements II.7] since the sum of the squares on AD and BD is likewise equal to twice the rectangle contained by AD and BD together with the square on AB.) Therefore, alternately, by the preceding lemma, the same excess will lie between the sum of the squares on AC and BC and the sum of the squares on AD and BD as lies between twice the rectangle contained by AC and BC and the rectan-

gle taken twice under AD and DB. But the excess between those sums is a rational area, by the scholium to Proposition 27 of this book, since each of them is rational. (For since AC and BC are rationals commensurable in square only, their squares will be rational and commensurable; [Margin: c — Elements X.16] therefore the sum composed of them is commensurable with each of them, and hence also rational, by Definition 9. In no other way shall we show that the sum

of the squares on AD and BD is rational.) Therefore the excess between twice the rectangle contained by AC and BC and twice the rectangle contained by AD and BD is also a rational area. [Margin: d — Elements X.22] And since the rectangle contained by AC and BC — rationals commensurable in square only — is medial, then by the corollary to Proposition 24 of this book its double as well, namely twice the rectangle contained by AC and BC, is medial. By the same reasoning twice the rectangle contained by AD and BD is medial. [Margin: e — Elements X.27] But a medial does not exceed a medial by a rational area. Therefore the excess between twice the rectangle contained by AC and BC and twice the rectangle contained by AD and BD is not a rational area. Yet it was shown to be rational. Which is absurd. Therefore no other rational line can be annexed to AB except BC, which is commensurable in square only with the whole. To an apotome, then, only one can be annexed, and so on. Which was to be shown.

THEOR. 63. PROPOS. 81.

[Margin: Lxxx. / Lxxv.]

To a first apotome of a medial there can be annexed only one medial straight line that is commensurable in square only with the whole and contains with the whole a rational area. (Euclid X.80.)

Let AB be a first apotome of a medial, and let BC, a medial commensurable in square only with the whole AC, be annexed to it, and let the rectangle contained by AC and BC be rational. I say that no other medial can be annexed to AB which is commensurable in square only with the whole and contains with the whole a rational area. For if it be possible, let another medial BD be annexed, commensurable in square only with the whole AD, and let the rectangle contained by AD and BD be rational.

Diagram. A single straight line with four marked points. From the left: A, then B, then C and D close together at the right.

- A (left endpoint)
- B (interior point): AB is the first apotome of a medial
- C: BC is the annexed medial, AC the whole
- D (right endpoint, near C): BD the supposed second annex, AD the whole

And since BC is a medial commensurable in square only with AC, [Margin: a — Elements X.24] AC too will be medial, since it is commensurable with it. AC and BC are therefore medials commensurable in square only. Now the same excess lies between the sum of the squares on AC and CB and the sum of the squares on

AD and BD as lies between twice the rectangle contained by AC and BC and twice the rectangle contained by AD and BD, as was shown in the preceding proposition.

Diagram. The same divided line repeated beside the demonstration : A (left), B, then C and D together at the right.

- A (left endpoint); B (apotome point); C; D (right endpoint)

But the excess between these rectangles is, by the scholium to Proposition 27 of this book, a rational area, since each of them is rational. (For since the rectangle contained by AC and BC is assumed rational, twice the rectangle contained by AC and BC, being commensurable with it, will be rational; and by the same reasoning twice the rectangle contained by AD and BD will be rational. Therefore the excess between the sum of the squares on AC and BC and the sum of the squares on AD and BD is a rational area.) And since the straight lines AC and BC are medials commensurable in square only, their squares too will be medial and commensurable. [Margin: b — Elements X.16] Therefore the sum composed of them is commensurable with each of them, and hence medial, by the corollary to Proposition 24 of this book. In no other way, likewise, the sum of the squares on AD and BD will be medial. [Margin: c — Elements X.27] But a medial does not exceed a medial by a rational area. Therefore the excess between the sum of the squares on AC and BC and the sum of the squares on AD and BD is not a rational area. Yet it was shown to be rational. Which is absurd. Therefore no other medial can be annexed to AB except BC, which is commensurable in square only with the whole and contains with the whole a rational area. To a first apotome of a medial, then, only one can be annexed, and so on. Which was to be demonstrated.

THEOR. 64. PROPOS. 82.

[Margin: Lxxxi. / Lxxvi.]

To a second apotome of a medial there can be annexed only one medial straight line that is commensurable in square only with the whole and contains with the whole a medial area. (Euclid X.81.)

Let AB be a second apotome of a medial, and let the medial BC, commensurable in square only with the whole AC, be annexed to it, and let the rectangle contained by AC and BC be medial. I say that no other medial can be annexed to AB which is commensurable in square only with the whole and contains with the whole a medial area. For if it be possible, let another medial BD be annexed to it,

commensurable in square only with the whole AD, and let the rectangle contained by AD and BD be medial.

Let a rational line EF be set out, [Margin: a — Elements I.45] to which let there be applied the rectangle EG equal to the sum of the squares on AC and BC; and to the same EF let another, EI, be applied equal to the square on AB; and finally another, EL, equal to the sum of the squares on AD and BD.

Diagram. A single straight line with four marked points, above the rectangle figure : A, B (close together at the left), then C and D together at the right.

- A (left endpoint); B; C; D (right endpoint) — the second apotome of a medial AB, its annex BC, whole AC, and the supposed second annex BD with whole AD

Diagram. A large rectangle standing on the rational line EF (its left, vertical side), divided into strips by vertical lines. The upper edge carries, from left to right, the labels E, K, H, M; the lower edge carries, from left to right, F, I, G, L. The successive rectangles applied to EF represent the areas named in the text.

- E (upper-left corner) / F (lower-left corner): the exposed rational line EF is the vertical left side
- K (upper edge, just right of E) / I (lower edge): the rectangle EI is equal to the square on AB
- H (upper edge) / G (lower edge): the rectangle EG is equal to the sum of the squares on AC and BC; KH is equal to twice the rectangle contained by AC and BC
- M (upper-right corner) / L (lower-right corner): the rectangle EL is equal to the sum of the squares on AD and BD; ML is equal to twice the rectangle contained by AD and BD

[Margin: b — Elements II.7] Since, then, the sum of the squares on AC and BC is equal to twice the rectangle contained by AC and BC together with the square on AB, KH will be equal to that which is contained twice under AC and BC. In the same way ML will be equal to that which is contained twice under AD and BD. And since AC and BC are medials, and commensurable in square (for since BC is assumed medial, [Margin: c — Elements X.24] AC too will be commensurable with it in square, and medial), their squares will be medial and commensurable. [Margin: d — Elements X.16] Therefore the sum composed of them is commensurable with each of them, and consequently medial, by the corollary to Proposition 24 of this book. Therefore EG also, being equal to that sum, is medial; and when it is applied to the rational line EF, [Margin: e — Elements VI.23] EH

will be rational but incommensurable in length with EF. Again, since the rectangle contained by AC and BC is medial, its double as well — namely KG — will be medial, by the corollary to Proposition 24 of this book; and when this is applied to the rational line KI, [Margin: f — Elements X.23] KH will be rational but incommensurable in length with KI, that is, with EF. And since AC and BC are incommensurable in length, and as AC is to BC, so is the square on AC to the rectangle contained by AC and BC, as we showed in the third lemma to Proposition 19 of this book, [Margin: g — Elements X.10] the square on AC will be incommensurable with the rectangle contained by AC and BC. But the sum of the squares on AC and BC — that is, the rectangle EG — is commensurable with the square on AC. (For since the straight lines AC and BC are assumed commensurable in square, their squares too will be commensu-

commensurable. Therefore the sum composed of them is also commensurable with each of them — namely, with the square on AC.) [Margin: h — X.16] And its double, namely KG, is commensurable with the rectangle contained by AC, BC. Therefore EG is incommensurable with KG, from what we taught in the scholium to Prop. 14 of this book. The straight lines EH, KH, then, since they have the same ratio as EG, KG, are incommensurable in length [Margin: i — X.10]; but they were also shown to be rational. Therefore they are rational lines commensurable in square only. Hence, since from the rational line EH there is subtracted the rational line KH, commensurable in square only with EH, EK will be an apotome [Margin: k — X.74], and KH the annex [congruens] congruent with it. In just the same way we shall prove that EK is an apotome and that KM is congruent with it. It would follow, then, that more than one straight line — a rational line commensurable in square only with the whole — joins the apotome, which is absurd; for it was demonstrated [Margin: l — X.80] that only one joins. Therefore to the second apotome of a medial only one straight line joins, etc. Which was to be demonstrated.

THEOR. 65. PROPOS. 83.

[Margin: LXXXII. / LXXVII.]

To a minor line only one straight line joins that is incommensurable in square with the whole, and that makes with the whole a sum composed of their squares which is rational, while what is contained by them is medial.

Let AB be a minor line, and let BC, congruent [annexed] to it, be incommensurable in square with the whole AC, making the sum of the squares

on AC, BC rational, while the rectangle contained by AC, BC is medial. I say that no other line joins AB which does the same. For, if it can be done, let another line BD join it, incommensurable in square with the whole AD, making the sum of the squares on AD, BD rational, and the rectangle contained by AD, BD medial. Now, as was demonstrated in Prop. 80, the excess between the sum of the squares on AC, BC and the sum of the squares on AD, BD is the same as the excess between twice the rectangle on AC, BC and twice the rectangle on AD, BD. But the excess between those sums is a rational area — by the scholium to Prop. 27 of this book — since each of them is assumed rational. Therefore the excess between twice the rectangle on AC, BC and twice the rectangle on AD, BD is also a rational area — yet it is likewise non-rational. (For, since the rectangle contained by AC, BC is assumed medial, its double, namely twice the rectangle on AC, BC, will be medial; and in the same way twice the rectangle on AD, BD will be medial [Margin: a — X.27]; and so the one cannot exceed the other by a rational area.) This is absurd. Therefore no other line besides BC joins AB incommensurable in square with the whole, etc. To a minor line, then, only one joins, etc. Which was to be demonstrated.

Diagram. A single horizontal straight line for Prop. 83, with marked division points reading left to right A, B, C, D. AB is the minor line and BC its annex, so that AC is the whole; BD is the alternative annex, giving the whole AD.

- A (left end): start of the line
- B (interior): division marking the annex BC / BD
- C (interior): end of the first whole AC
- D (right end): end of the alternative whole AD

THEOR. 66. PROPOS. 84.

[Margin: LXXXIII. / LXXVIII.]

To the line which with a rational area makes the whole medial, only one straight line joins that is incommensurable in square with the whole, and that makes with the whole a sum composed of their squares which is medial, while what is contained by them is rational.

Let AB be the line which with a rational area makes the whole medial, and let BC be congruent [annexed] to it, incommensurable in square with the whole AC, making the sum of the squares on AC, BC medial, but the rectangle contained by AC, BC rational. I say that no other straight line joins AB which produces these same things. For, if it can be done, let another line BD join it, incommensurable in

square with the whole AD, making the sum of the squares on AD, BD medial, but the rectangle contained by AD, BD rational. Now, as was shown in Prop. 80, the excess between the sum of the squares on AC, BC and the sum of the squares on AD, BD is the same as the excess between twice the rectangle on AC, BC and twice the rectangle on AD, BD. But the excess between these rectangles is a rational area, as we demonstrated in Prop. 81. Therefore the excess between those sums is also non-rational, since each of them is medial [Margin: a — X.27] — for one medial cannot exceed another medial by a rational area. This is absurd. Therefore no other line than BC joins AB incommensurable in square with the whole, etc. Hence to the line which with a rational area makes the whole medial, only one joins, etc. Which was to be demonstrated.

Diagram. A single horizontal straight line for Prop. 84, with division points left to right A, B, C, D. AB is the given line and BC its annex (whole AC); BD is the alternative annex (whole AD).

- A (left end): start of the line
- B (interior): division for the annex BC / BD
- C (interior): end of the whole AC
- D (right end): end of the alternative whole AD

THEOR. 67. PROPOS. 85.

[Margin: LXXXIV. / LXXIX.]

To the line which with a medial area makes the whole medial, only one straight line joins that is incommensurable in square with the whole, and that makes with the whole both the sum composed of their squares medial, and what is contained by them medial and incommensurable with the sum composed of their squares.

Let AB be the line which with a medial area makes the whole medial, and let BC be congruent [annexed] to it, incommensurable in square with the whole AC, making the sum of the squares on AC, BC medial, and the rectangle contained by AC, BC [medial], incommensurable with the sum of the squares on AC, BC. I say that no other straight line joins AB which produces these same things. For, if it can be done, let another line BD join it, incommensurable in square with the whole AD, so that the sum of the squares on AD, BD is medial, and the rectangle contained by AD, BD is medial too, and [incommensurable with] the sum of the squares on AD, BD. Now, the same construction being made as in Prop. 82: since the sum of the squares on AC, BC is medial [Margin: a — X.23], EG too will be

equal to it and medial. Again, since the rectangle contained by AC, BC is medial, its double KG will also be medial [Margin: b — X.23]. Therefore the straight line KH is rational, but incommensurable in length with EF. And since KG is commensurable with the rectangle contained by AC, BC (being its double), while the rectangle contained by AC, BC is assumed incommensurable with the sum of the squares on AC, BC, KG too will be incommensurable [Margin: c — X.14] with that same sum — that is, with EG — and therefore the straight lines EH, KH, having the same ratio as EG, KG, are incommensurable in length [Margin: d — X.10]. But they were also shown to be rational. Therefore they are rational lines commensurable in square only. Hence, since from the rational line EH the rational line KH, commensurable in square only with EH, is subtracted, EK will be an apotome [Margin: e — X.74], and KH its annex, congruent with it. In just the same way we shall show that EK is an apotome, and that KM is its congruent annex. Therefore not only one straight line joins the apotome, etc.; which is absurd, since it was demonstrated [Margin: f — X.80] that only one joins. Therefore to the line which with a medial area makes the whole medial, only one joins, etc. Which was to be shown.

Diagram. Two figures for Prop. 85. Above is a horizontal straight line with division points left to right A, B, C, D (AB the given line, BC/BD the annexes, wholes AC/AD). Below is a rectangle divided by two interior vertical lines into three panels, representing the squares and rectangles EG, KG, EL, KL laid along the rational line EF. Its top edge carries the points E, K, H, M (left to right) and its bottom edge the points F, I, G, L; the left side is EF, the interior verticals are KI and HG, and the right side is ML.

- A, B, C, D (upper line): the line divided as in the proof (AB minor part; wholes AC, AD)
- E (top-left corner): head of the rational line EF
- K, H (top interior): feet of the interior verticals KI, HG
- M (top-right corner): head of the right side ML
- F (bottom-left corner): foot of EF
- I, G (bottom interior): feet of the interior verticals
- L (bottom-right corner): foot of ML

DEFINITIONS, GROUP THREE

Given an assigned rational line and an apotome: if the whole is greater in square than the annex by the square on a straight line commensurable in length with itself —

I. If the whole is commensurable in length with the assigned rational, let it be called a *first apotome*.

II. If, however, the annex is commensurable in length with the assigned rational, let it be called a *second apotome*.

III. But if neither the whole nor the annex is commensurable in length with the assigned rational, let it be called a *third apotome*.

Again, if the whole is greater in square than the annex by the square on a straight line incommensurable in length with itself —

IV. If the whole is commensurable in length with the assigned rational, let it be called a *fourth apotome*.

V. If, however, the annex is commensurable in length with the assigned rational, let it be called a *fifth apotome*.

VI. But if neither the whole nor the annex is commensurable in length with the assigned rational, let it be called a *sixth apotome*.

SCHOLIUM.

The number of these six apotomes is reached in exactly the same way that the number of the six binomial lines was arrived at above. For these six apotomes are the straight lines that are left after the lesser terms are subtracted from the greater terms of the six lines that are called binomials [lines named from two terms], as is clear from the definitions.

PROBL. 19. PROPOS. 86.

[Margin: LXXXV. / LXXX.]

To find a first apotome.

Having found two square numbers AB, CB — as we taught in the second scholium to Prop. 29 of this book — whose difference AC is not a square, so that AB and CB have to one another the ratio a square number has to a square number, but AB and AC do not: let some rational line D be set out, with which EF is commensurable in length. Then EF, being commensurable with the rational D, will be rational. Next, let it be contrived that as the number AB is to the number AC, so — by the corollary

to Prop. 6 of this book — the square on EF is to the square on GF. I say that EG is a first apotome. For since the squares on EF, GF, having to one another the ratio the numbers AB, AC have, are commensurable [Margin: a — X.6], the straight lines EF, GF will also be commensurable, at least in square. Since, then, EF has been

shown to be rational, GF will be rational too. But since AB, AC do not have to one another the ratio a square number has to a square number, neither will the squares on EF, GF have the ratio of square numbers. Therefore the straight lines EF, GF are incommensurable in length [Margin: b — X.9]. Hence EF, GF are rational lines commensurable in square only, and so the remainder EG is an apotome [Margin: c — X.74]. I say it is also a first apotome. For let EF be greater in square than GF by the square on H. Since it is as the number AB is to the number AC, so the square on EF to the square on GF, it will be, by conversion of the ratio, as AB is to CB, so the square on EF to the square on H. But AB, CB have to one another the ratio of square numbers; therefore the squares on EF and H also have the ratio of square numbers, and consequently the straight lines EF and H are commensurable in length [Margin: d — X.9]. Since, then, the whole EF is greater in square than its annex GF by the square on H, which is commensurable in length with itself, and since the same whole EF is commensurable in length with the assigned rational D, EG will, by the first definition, be a first apotome. We have therefore found a first apotome. Which was to be done.

Diagram. The figure for Prop. 86 (repeated from the previous page). It has four rows: a dotted line marked A, C, B (C between A and B) representing the numbers AB, CB, AC; the rational line D; the line EF divided at G (EF the whole, GF the annex, EG the apotome-remainder); and below them the line H (by whose square EF exceeds GF).

- A, C, B (dotted line): the numbers AB, CB and their difference AC
- D: the assigned rational line
- E, G, F: the line EF divided at G into the apotome EG and the annex GF
- H: the line on which the excess of the square is taken

PROBL. 20. PROPOS. 87.

[Margin: LXXXVI. / LXXXI.]

To find a second apotome.

Having found two square numbers AB, CB, as in the preceding proposition, and a rational line D having been set out, let GF be taken commensurable in length with it; then GF will be commensurable with the rational D, and so rational too. Next, let it be contrived that as the number AC is to the number AB, so the square on GF is to the square on EF, by the corollary to Prop. 6 of this book. I say that EG is a second apotome. For since the squares on GF, EF, having to one another the ratio the numbers AC, AB have, are commensurable [Margin: a — X.6], the straight

lines GF, EF will also be commensurable, at least in square. Since, then, GF has been shown to be rational, EF will be rational too. But since the numbers AC, AB — and consequently the squares on GF, EF — do not have to one another the ratio of square numbers, GF, EF will be incommensurable in length [Margin: b — X.9]; hence GF, EF are rational lines commensurable in square only, and so the remainder EG is an apotome [Margin: c — X.74]. I say it is also a second apotome. For let EF be greater in square than GF by the square on H. Since, then, it is as AC is to AB, so the square on GF to the square on EF, and, by conversion, as AB is to AC, so the square on EF to the square on GF, we shall show — just as in the preceding proposition — that the line H is commensurable in length with EF. Therefore, since the whole EF is greater in square than its annex GF by the square on H, which is commensurable in length with itself, and since the annex GF is commensurable in length with the assigned rational D, EG will, by the second definition, be a second apotome. We have therefore found a second apotome. Which was to be done.

Diagram. The figure for Prop. 87, in four rows: a dotted line marked A, C, B (C between); the rational line D (with interior tick marks); the line EF divided at G (whole EF, annex GF, remainder EG); and the line H below.

- A, C, B (dotted line): the numbers AB, CB and their difference AC
- D: the assigned rational line
- E, G, F: the line EF divided at G into the apotome EG and the annex GF
- H: the line by whose square EF exceeds GF

PROBL. 21. PROPOS. 88.

[Margin: LXXXVII. / LXXXII.]

To find a third apotome.

Having found two square numbers AB, CB, as in Prop. 86, let another number I be taken — as we taught in Prop. 51 of this book — which has to neither of them, AB, AC, the ratio a square has to a square. Then, a rational line D having been set out, let it be contrived that as I is to AB, so the square on D is to the square on EF, by the corollary to Prop. 6 of this book. Then the squares on D and EF, having to one another the ratio the numbers I, AB have, will be commensurable [Margin: a — X.6]; and consequently the straight lines D and EF will be commensurable, at least in square. Since, then, D is rational, EF will be rational too. And since the numbers I, AB — and therefore the squares on D, EF — do not have to one another the ratio of square numbers, the straight lines D, EF will be

incommensurable in length [Margin: b — X.9]. Again, let it be contrived that as AB is to AC, so the square on EF is to the square on GF, from the same corollary to Prop. 6 of this book. I say that EG is a third apotome. For since the squares on EF [GF], having to one another the ratio the numbers AB, AC have, are commensurable [Margin: c — X.6], the straight lines EF, GF will also be commensurable, at least in square. Since, then, EF has been shown to be rational, GF will be rational too, being commensurable with it, as we demonstrated. And since AB, AC — and therefore the squares on EF, GF — do not have to one another the ratio of square numbers, the straight lines EF, GF will be incommensurable in length [Margin: d — X.9]. Therefore EF, GF are rational lines commensurable in square only; and consequently, since from EF there is subtracted GF, commensurable with it in square only, the remainder EG will be an apotome [Margin: e — X.74]. I say it is also a third apotome. For since it is as I is to AB, so the square on D to the square on EF, and as AB is to AC, so the square on EF to the square on GF, it will be, *ex aequali*, as I is to AC, so the square on D to the square on GF. But the numbers I and AC do not have to one another the ratio of square numbers; therefore neither will the squares on D and GF have the ratio of square numbers, and so the straight lines D and GF are incommensurable in length [Margin: f — X.9]. Therefore neither of the lines EF, GF is commensurable in length with the assigned rational D. Now let EF be greater in square than GF by the square on H; and we shall show, as in Prop. 86, that H is commensurable in length with EF. Therefore, since the whole EF is greater in square than its annex GF by the square on H, which is commensurable in length with itself, and since neither of the lines EF, GF is commensurable in length with the assigned rational D, EG will, by the definition, be a third apotome. We have therefore found a third apotome. Which was to be done.

Diagram. The figure for Prop. 88, in five rows: a dotted line marked A, C, B (C between); a second dotted line marked I (the auxiliary number); the rational line D (with interior tick marks); the line EF divided at G (whole EF, annex GF, remainder EG); and the line H below.

- A, C, B (dotted line): the numbers AB, CB and their difference AC
- I (dotted line): the auxiliary number taken by Prop. 51
- D: the assigned rational line
- E, G, F: the line EF divided at G into the apotome EG and the annex GF
- H: the line by whose square EF exceeds GF

PROBL. 22. PROPOS. 89.

[Margin: lxxxviii / lxxxiii]

To find the fourth apotome.

When two numbers AC, CB have been found such that AB, composed of them, has to neither of them the ratio that a square number has to a square number — by the method we taught in scholium 3 of proposition 29 of this book — let a rational line D be set out, and let EF be commensurable in length with it; EF too will therefore be rational. Now if the rest is carried out as we showed in proposition 86, EG will there be an apotome. I say it is also a fourth apotome. For let EF be such that it exceeds GF by the square on the straight line H. And since AB is to AC as the square on EF is to the square on GF, by conversion of the ratio AB will be to CB as the square on EF is to the square on H. Since, then, AB and CB do not have to one another the ratio of square numbers, [Margin: a — *Elements* X.9] the lines EF and H will be incommensurable in length. Since therefore the whole EF exceeds the annex GF by the square on H — a line incommensurable in length with it — while the same whole EF is commensurable in length with the rational D, by definition EG is a fourth apotome. We have therefore found a fourth apotome. Q.E.F.

Diagram. A group of four horizontal reference lines for the construction of the fourth apotome. From top to bottom:

- Top: a dotted number-line $A \cdots \cdots C \cdots B$ — the numbers AB (whole), CB, and their difference AC.
- Second: line **D**, ruled with tick-marks (the assigned rational line).
- Third: line **E — G — F** — the divided line EF, with G marking off the apotome EG and the annex GF.
- Bottom: line **H —** — the excess line (that by whose square EF exceeds GF).

PROBL. 23. PROPOS. 90.

[Margin: lxxxix / lxxxiv]

To find the fifth apotome.

When two numbers AC, CB have been found as in the preceding proposition, let the construction be made as in proposition 87 — that is, let GF be taken commensurable in length with the rational D, and so on. We shall then show, as

in proposition 87, that EG is an apotome. I say it is also a fifth apotome. For let EF be such that it exceeds GF by the square on the straight line H. And since similarly, as we show in proposition 86, by conversion of the ratio AB is to CB as the square on EF is to the square on H, the lines EF and H will again, as in the preceding proposition, be incommensurable in length; and the annex GF is commensurable in length with the rational D; therefore by definition EG is a fifth apotome. We have therefore found a fifth apotome. Q.E.F.

Diagram. The same set of four reference lines for the fifth apotome:

- Top: dotted number-line $A \cdots \cdots C \cdots \cdots B$ — numbers AB, CB, difference AC.
- Second: line **D**, ruled with tick-marks (assigned rational line).
- Third: line **E** — **G** — **F** — divided line EF, G marking apotome EG and annex GF (here GF is the greater part, commensurable with the rational).
- Bottom: line **H** — (with a light under-stroke) — the excess line.

PROBL. 24. PROPOS. 91.

[Margin: lxxxx / lxxxv]

To find the sixth apotome.

When three numbers AC, CB, and I have been found as in proposition 54, so that AB has to neither of them, AC nor CB — and I has to neither of AB, AC — the ratio that a square number has to a square number, let the rational D be set out and the rest be done as in proposition 88. We shall then show similarly, as there, that D and EF are incommensurable in length. I say it is also a sixth apotome. For, as in proposition 88, D and GF will likewise be incommensurable in length; and so neither of the lines EF, GF is commensurable in length with the assigned rational D. Now let EF exceed GF by the square on the straight line H, which we shall show, as in proposition 89, to be incommensurable in length with EF itself. Since therefore the whole EF exceeds the annex GF by the square on H — a line incommensurable in length with it — and neither of the lines EF, GF is commensurable in length with the rational D, by definition EG is a sixth apotome. We have therefore found a sixth apotome. Q.E.F.

Diagram. Five reference lines for the sixth apotome:

- Top: dotted number-line $A \cdots \cdots C \cdots \cdots B$ — numbers AB, CB, difference AC.

- Second: dotted line **I** ······· — the auxiliary number I (of proposition 54), labelled I at the left, unmarked at the right.
- Third: line **D**, ruled with tick-marks (assigned rational line).
- Fourth: line **E** — **G** ————— **F** — divided line EF, G marking apotome EG and annex GF.
- Bottom: line **H** ————— — the excess line.

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But we shall also find the said six apotomes more expeditiously by the following method, as Theon teaches in this place.

Let it be required, for example, to find a first apotome. First let a first binomial AB be found, whose greater term is AC and lesser CB. Then from AC let there be cut off a straight line CD equal to CB itself. I say that AD is a first apotome. For since AC, BC are rationals commensurable in square only, AC, DC will also be rationals commensurable in square only. [Margin: a — *Elements* X.49] Therefore AD is an apotome. [Margin: b — *Elements* X.74] And since AC exceeds CB — that is, exceeds DC — by the square on a straight line commensurable in length with itself, and is commensurable in length with the assigned rational (by the definition of that which is called a first binomial), AD will, by the definition of a first apotome, be a first apotome. In the same way, from the others that are called binomials we shall find the other apotomes as well — the second from the second, the third from the third, the fourth from the fourth, the fifth from the fifth, and the sixth from the sixth — if we subtract the lesser terms from the greater.

Diagram. A single straight segment for the scholium's construction of the first apotome, with points from left to right:

- A (left endpoint): start of the binomial line.
- D (near left, with a tick): the point marking the apotome AD.
- C (middle): the division of the binomial (greater term AC, lesser CB), with CD = CB cut off.
- B (right endpoint).

THEOR. 68. PROPOS. 92.

[Margin: lxxxxi / lxxxvi]

If a rectangle is contained by a rational line and a first apotome, the straight line whose square equals the rectangle is an apotome.

Let the rectangle AC be contained by the rational AB and the first apotome AD. I say that the straight line whose square equals the rectangle AC is an apotome. Let DE be the annex to AD. Then by the definition of a first apotome, AE, DE will be rationals commensurable in square only; the whole AE will be commensurable in length with the rational AB; and finally the same whole AE will exceed its annex DE by the square on a straight line commensurable in length with itself. Let DE be bisected at F, and to AE let there be applied a rectangle equal to the square on FE — that is, to the fourth part of the square on DE — namely the rectangle contained by AG, GE, deficient by a square figure, by lemma 2 of proposition 17 of this book. And since AE exceeds DE by the square on a straight line commensurable in length with itself, [Margin: a — *Elements* X.18] the lines AG, GE will be commensurable in length; [Margin: b — *Elements* X.16] and therefore each of them will be commensurable in length with the whole AE. But AE is commensurable in length with the rational AB. [Margin: c — *Elements* X.12] Therefore each of AG, GE is also commensurable in length with AB; and by definition 6 they are rational. Hence, drawing GH, EI parallel to AB, meeting BC produced at H, I, [Margin: d — *Elements* X.20] each rectangle AH, GI, being contained by two rationals commensurable in length, will be rational.

Again, since each of DF, FE is commensurable in length with DE, while DE is incommensurable in length with the rational AB — for if it were commensurable in length, then since AE too is commensurable in length with AB, [Margin: e — *Elements* X.12] AE and DE would be commensurable in length as well, which is absurd, since they are assumed commensurable in square only — [Margin: f — *Elements* X.14] each of DF, FE will also be incommensurable in length with AB. And since each of DF, FE is commensurable with the rational DE and so is rational, both AB, DF and AB, FE will be rationals commensurable in square only. Hence, drawing FK parallel to AB, [Margin: g — *Elements* X.22] each rectangle DK, FI, contained by rationals commensurable in square only, will be medial.

Now on AH let there be constructed an equal square LM, [Margin: h — *Elements* II.14] and on GI an equal square NO, having with LM the common angle LPM. [Margin: i — *Elements* VI.26] Then the squares LM, NO will be about the same diameter, which let be PQ. And let the figure be completed as you see. Since, then, the rectangle contained by AG, GE is by construction equal to the square on FE, [Margin: k — *Elements* VI.17] the three lines AG, FE, GE will be proportional; [Margin: l — *Elements* VI.1] and therefore, since AH, FI, GI have the same ratio as these, FI will be a mean proportional between AH, GI — that is, between the squares LM, NO equal to them. But between these same LM, NO the mean proportional is also LO, by the lemma of proposition 14 of this book. Therefore

the rectangles FI, LO are equal. [Margin: m — *Elements* X.36] But DK is equal to FI, and MN is equal to LO. Therefore the whole DI is equal to the whole gnomon VYX together with the square NO. But the whole AI is by construction equal to the squares LM, NO. Therefore the remainder AC is also equal to the remaining square TR; and consequently the straight line TS is that whose square equals the rectangle AC. I say that TS is an apotome.

For since the spaces AH, GI have been shown to be rational, and LM, NO are equal to them, the squares LM, NO will also be rational, and therefore the lines TO, SO rational. Again, since FI has been shown to be medial, LO, equal to it, will also be medial. Therefore LO and NO are incommensurable, since the one is irrational and the other rational. Therefore the lines TO, SO, [Margin: n — *Elements* VI.1] having the same ratio as LO, NO, [Margin: o — *Elements* X.10] are incommensurable in length; [Margin: p — *Elements* X.74] and consequently the remainder TS is an apotome. If therefore a rectangle is contained by a rational and a first apotome, and so on. Q.E.D.

Diagram. Two figures for proposition 92. The upper figure is a rectangle with its long side AE (the rational base runs along the bottom BI), divided by verticals; the lower figure is a square with an inscribed diagonal and gnomon.

Upper (rectangle):

- Top edge, left to right: A, D, F, G, E.
- Bottom edge, left to right: B, C, K, H, I.
- Verticals dropped: DC, FK, GH, EI (defining rectangles AH, GI, DK, FI).

Lower (square):

- L (top-left), N (top-mid), P (top-right).
- T (mid-left), V and S (interior, S at the centre where the diagonal crosses), O (mid-right).
- Q (bottom-left), R (bottom-mid), M (bottom-right).
- Diagonal QP through S; y (interior, upper-right, on the diagonal) and X (interior, lower-mid); a circular arc drawn about S. The gnomon named in the text is VYX. Squares: LM (whole, L to M), NO (N to O); diameter PQ.

THEOR. 69. PROPOS. 93.

[Margin: lxxxxii / lxxxvii]

If a rectangle is contained by a rational line and a second apotome, the straight line whose square equals the rectangle is a first apotome of a medial.

Let the rectangle AC be contained by the rational AB and the second apotome AD. I say that the straight line whose square equals the rectangle AC is a first apotome of a medial. Let DE be the annex to AD. Then by the definition of a second apotome, AE, DE will be rationals commensurable in square only; the annex DE will be commensurable in length with the rational AB; and finally AE will exceed DE by the square on a straight line commensurable with itself in length. Let DE be bisected at F, and let everything else be constructed as in the preceding proposition. [Margin: a] Then again, as in the preceding proposition, AG, GE will be commensurable in length; [Margin: b] and therefore each of them will be commensurable in length with the whole AE. But AE is incommensurable in length with the rational AB — for if it were commensurable with AB in length, since DE is assumed commensurable in length with the same AB, [Margin: c] AE and DE would be commensurable with one another in length as well, which is absurd, since they are assumed commensurable in square only. [Margin: d] Therefore each of AG, GE is incommensurable in length with the same AB.

But each of AG, GE is commensurable with the rational AE and so is rational. Therefore both AB, AG and AB, GE are rationals commensurable in square only; [Margin: e — *Elements* X.22] and therefore each rectangle AH, GI, being contained by rationals commensurable in square only, is medial. Again, since each of DF, FE is commensurable in length with DE, and DE is assumed commensurable in length with the rational AB, each of DF, FE will also — as we showed in the scholium to proposition 12 of this book — be commensurable in length with the same rational AB, and hence rational. [Margin: f — *Elements* X.20] Therefore each rectangle DK, FI, being contained by rationals commensurable in length, is rational. Now, as in the preceding proposition, we shall demonstrate that the line TS is that whose square equals the rectangle AC. I say that TS is a first apotome of a medial.

For since AG, GE are commensurable in length, [Margin: g — *Elements* X.20] AH, GI, having the same ratio as they, will be commensurable; and therefore the squares LM, NO equal to them will be commensurable. Hence their sides too, namely the lines TO, SO, will be at least commensurable in square. And they are also medial, since the squares LM, NO are equal to the medials AH, GI (for AH, GI have been shown to be medial), so that they too are medial. And since FI — and therefore LO, equal to it — is rational, and consequently incommensurable with

the medial NO, the lines TO, SO, having the same ratio as these, will be incommensurable in length. Since therefore they have been shown to be both medial and commensurable in square, TO, SO will be medials commensurable in square only. And since they contain LO, which has been shown to be rational, [Margin: h — *Elements* X.75] TS will be a first apotome of a medial. If therefore a rectangle is contained by a rational and a second apotome, and so on. Q.E.D.

Diagram. Two figures for proposition 93, drawn (and reprinted from the foot of the preceding page) in the outer column. The upper is a tall rectangle divided by horizontals; the lower is the square with diagonal and gnomon.

Upper (rectangle):

- Left edge, top to bottom: A, D, F, G, E.
- Right edge, top to bottom: B, C, K, H, I.
- Horizontals: AB (top), DC, FK, GH, EI (bottom), marking off the rectangles AH, DK, GI, FI.

Lower (square):

- L (top-left), N (top-mid), P (top-right).
- T (mid-left), V and S (interior, S at centre), O (mid-right).
- Q (bottom-left), R (bottom-mid), M (bottom-right).
- Diagonal QP through S; y (interior upper-right, on the diagonal), X (interior lower-mid); circular arc about S; gnomon VYX; squares LM and NO; diameter PQ.

THEOR. 70. PROPOS. 94.

[Margin: lxxxxiii / lxxxviii]

If a rectangle is contained by a rational line and a third apotome, the straight line whose square equals the rectangle is a second apotome of a medial.

Let the rectangle AC be contained by the rational AB and the third apotome AD. I say that the straight line whose square equals the rectangle AC is a second apotome of a medial. For let DE be the annex to AD. Then by the definition of a third apotome, AE, DE will be rationals commensurable in square only, and neither of them, AE nor DE, is commensurable in length with the rational AB; and finally AE will exceed DE by the square on a straight line commensurable in length with itself. Let DE be bisected at F, and let the rest be done as in

proposition 92. Then again, as there, AG, GE will be commensurable in length with one another; [Margin: a — *Elements* X.16] and therefore each of them will also be commensurable in length with the whole AE. But AE is assumed incommensurable in length with the rational AB. [Margin: b — *Elements* X.14] Therefore each of AG, GE will also be incommensurable in length with the same AB. Since, then, each of AG, GE is commensurable with the rational AE and so is rational, both AB, AG and AE, GE will be rationals commensurable in square only; [Margin: c — *Elements* X.22] and therefore each rectangle AH, GI will be medial. Again, since DE is assumed incommensurable in length with the rational AB, each of DF, FE — being commensurable in length with DE — will [Margin: d — *Elements* X.14] be incommensurable in length with the same AB. Since, then, each of DF, FE is commensurable with the rational DE and so is rational, both AB, DF and AB, FE will be rationals commensurable in square only; [Margin: e — *Elements* X.22] and therefore each rectangle DK, FI will be medial. In the same way we shall now demonstrate, as in proposition 92, that the line TS is that whose square equals the rectangle AC. I say that TS is a second apotome of a medial.

For since AH, GI have been shown to be medial, the squares LM, NO equal to them will be medial; and therefore the lines TO, SO will be medial. And since AH, GI, having the same ratio as AG, GE (which we showed to be commensurable), [Margin: f — *Elements* X.10] are commensurable, the squares LM, NO too will be commensurable. Therefore the lines TO, SO are commensurable at least in square. But since AE, DE are commensurable in square only — that is, incommensurable in length — while GE is shown commensurable in length with AE and FE is commensurable in length with DE, GE and FE will, by the scholium to proposition 14 of this book, be incommensurable in length; [Margin: g — *Elements* X.10] and therefore GI, FI, having the same ratio as GE, FE — that is, NO, LO equal to them — are incommensurable. Hence TO, SO, having the same ratio as LO, NO, are incommensurable in length. But they have been shown to be medial and commensurable. Therefore TO, SO are medials commensurable in square only. And since they contain LO, a medial (for since LO is equal to FI, as is established from proposition 92, which we showed to be medial, LO too will be medial), [Margin: h — *Elements* X.76] TS will be a second apotome of a medial. If therefore a rectangle is contained by a rational and a third apotome, and so on. Q.E.D.

THEOR. 71. PROPOS. 95.

[Margin (beside heading): xciv. / lxxxix.]

If a space is contained by a Rational and a fourth apotome, the straight line whose square is equal to the space is a minor.

Demonstration. Let the space AC be contained by the Rational AB and the fourth apotome AD. I say that the straight line whose square equals the space AC is a minor. Let DE be the annex to AD. Then, by the definition of the fourth apotome, AE and DE are Rational and commensurable in square only; AE is commensurable in length with the Rational AB; and the square on AE exceeds the square on DE by the square on a straight line incommensurable in length with AE. Let DE be bisected at F, and let everything else be constructed as before. [Margin: a → Elem. X.19] Then AG, GE are incommensurable in length, since the square on AE exceeds the square on DE by the square on a straight line incommensurable with AE; and to AE there has been applied the rectangle contained by AG, GE, equal to a quarter of the square on DE and falling short by a square figure. And since AE is Rational and commensurable in length with the Rational AB, [Margin: b → Elem. X.20] the rectangle AI is Rational. Again, since DE is Rational but incommensurable in length with the Rational AB, [Margin: c → Elem. X.22] DI, and therefore also its half FI, is medial. Further, since AG, GE are incommensurable in length, [Margin: d → Elem. X.10] AH and GI, which have the same ratio to them, are incommensurable. Now we shall likewise prove, as in Proposition 92, that the straight line TS has its square equal to the space AC. I say that TS is a minor.

For by the construction, the rectangle AI is equal to the sum of the squares LM, NO on the straight lines TO, SO; and that sum was shown to be Rational; therefore the sum of the squares on TO, SO is Rational. Likewise, since F [i.e. FI] was shown to be medial, LO — contained by TO, SO and equal to it — is also medial. Finally, since AH, GI were proved incommensurable, the squares LM, NO equal to them are also incommensurable; and therefore the straight lines TO, SO are incommensurable in square. Therefore, since TO, SO are incommensurable in square, the sum of their squares being Rational but the rectangle contained by them being medial, [Margin: e → Elem. X.77] the remainder TS is a minor. Therefore, if a space is contained by a Rational and a fourth apotome, and so on. Which was to be demonstrated.

Diagram. The figure for this proposition has two parts, one above the other.

- Upper part — a rectangle divided into vertical strips. Top edge, left to right: A, D, F, G, E. Bottom edge, left to right: B, C, K, H, I. (AB is the left side, the Rational; AE the top side; the interior verticals through D/C, F/K, G/H mark the sub-rectangles AI, DI, and the rectangle contained by AG, GE.)

- Lower part — a square with an inscribed semicircle and a diagonal, used to construct the two lines TO, SO.
- L (top-left corner), N (top edge, middle), P (top-right corner)
- T (mid-height, left edge), V (just left of centre), S (centre), O (mid-height, right edge)
- Q (bottom-left corner), R (bottom edge, middle), M (bottom-right corner)
- a diagonal drawn from Q to P; a circle/semicircle centred near S; a small label **y** on the diagonal above S, and **x** just below S.

THEOR. 72. PROPOS. 96.

[Margin (beside heading): lxxxxv. / lxxxx.]

If a space is contained by a Rational and a fifth apotome, the straight line whose square is equal to the space is that which produces with a Rational area a medial whole.

Demonstration. Let the space AC be contained by the Rational AB and the fifth apotome AD (the figure of the immediately preceding proposition being repeated). I say that the straight line whose square equals the space AC is that which produces with a Rational area a medial whole. Let DE be the annex to AD. Then, by the definition of the fifth apotome, AE, DE are Rational and commensurable in square only; DE is commensurable in length with the Rational AB; and finally the square on AE exceeds the square on DE by the square on a straight line incommensurable in length with AE. Let DE be bisected at F, and let the rest be constructed as before. Then again, as in the preceding proposition, AG, GE are incommensurable in length. And since AE is Rational but incommensurable in length with the Rational AB, as we said in Proposition 93, [Margin: $a \rightarrow$ Elem. X.22] the rectangle AI is medial. Likewise, since DE is Rational and commensurable in length with the Rational AB, [Margin: $b \rightarrow$ Elem. X.20] DI, and therefore also its half FI, is Rational. Again, as in the preceding proposition, AH, GI are incommensurable; and, as was shown in Proposition 92, the straight line TS has its square equal to the space AC. I say that TS is that which produces with a Rational a medial whole.

For since AI was shown to be medial, the sum of the squares LM, NO on TO, SO, equal to it, is also medial. Likewise, since we proved FI to be Rational, LO — the rectangle contained by TO, SO, being equal to it — is also Rational. But TO, SO are incommensurable in square, as was demonstrated in the preceding proposition. Therefore, since TO, SO are incommensurable in square, the sum of their squares being medial but the rectangle contained by them being Rational, [Margin: $c \rightarrow$

Elem. X.78] the remainder TS is that which produces with a Rational a medial whole. And so, if a space is contained by a Rational and a fifth apotome, and so on. Which was to be demonstrated.

Diagram. The same composite figure as for the preceding proposition, printed again for this proposition. Upper part: a rectangle with top edge A, D, F, G, E and bottom edge B, C, K, H, I. Lower part: a square with top corners/edge L, N, P; mid-line T, V, S, O; bottom Q, R, M; with an inscribed circle, a diagonal from Q to P, and the labels y (above S) and x (below S).

THEOR. 73. PROPOS. 97.

[Margin (beside heading): lxxxxvi. / lxxxxj.]

If a space is contained by a Rational and a sixth apotome, the straight line whose square is equal to the space is that which produces with a medial area a medial whole.

Demonstration. Let the space AC be contained by the Rational AB and the sixth apotome AD. I say that the straight line whose square equals the space AC is that which produces with a medial area a medial whole. For let DE be the annex to AD. Then, by the definition of the sixth apotome, AE, DE are Rational and commensurable in square only; neither of them, AE, DE, is commensurable in length with the Rational AB; and finally the square on AE exceeds the square on DE by the square on a straight line incommensurable in length with AE. Let DE be bisected at F, and let the rest be constructed as above. Then again, as in Proposition 95, AG, GE are incommensurable in length. And since both AE and DE are Rational but incommensurable in length with the Rational AB, [Margin: a $\rightarrow$ Elem. X.22] both AI and DI, and therefore also the half of DI, namely FI, are medial. And, as in Proposition 95, AH, GI are incommensurable. And since AE, DE are commensurable in square only, [Margin: b $\rightarrow$ Elem. X.10] AI, DI, which have the same ratio to them, are incommensurable; and moreover, since DI, FI are commensurable, [Margin: c $\rightarrow$ Elem. X.14] FI is incommensurable with AI. Now we shall prove, as in Proposition 92, that the straight line TS has its square equal to the space AC. I say that TS is that which produces with a medial area a medial whole.

For since we showed AI to be medial, the sum of the squares LM, NO on TO, SO, equal to it, is also medial. Likewise, since FI was shown to be medial, LO equal to it — contained by TO, SO — is also medial. Again, since FI is incommensurable with AI, as we showed, LO contained by TO, SO is also incommensurable with the

sum of the squares on TO, SO — because LO is equal to FI, and the sum of the squares on TO, SO is equal to AI. Finally, TO, SO are incommensurable in square, as was demonstrated in Proposition 95. Therefore, since TO, SO are incommensurable in square, and the sum of their squares is medial, and likewise the rectangle contained by them is medial, and moreover incommensurable with the sum of their squares, [Margin: d → Elem. X.79] the remainder TS is that which produces with a medial area a medial whole. Therefore, if a space is contained by a Rational and a sixth apotome, and so on. Which was to be shown.

Diagram. The same composite figure as before, printed again for this proposition. Upper part: a rectangle with top edge A, D, F, G, E and bottom edge B, C, K, H, I. Lower part: a square with top L, N, P; mid-line T, V, S, O; bottom Q, R, M; with an inscribed circle, a diagonal from Q to P, and the labels y (above S) and x (below S).

THEOR. 74. PROPOS. 98.

[Margin (beside heading): xcvij. / xcij.]

The square on an apotome, applied to a Rational, produces as its breadth a first apotome.

Demonstration. Let AB be an apotome and BC its annex, so that AC, BC are Rational and commensurable in square only; [Margin: a → Elem. I.45] and to the Rational DE let the rectangle DF be applied, equal to the square on AB, producing as breadth DG. I say that DG is a first apotome. For to the same DE let the rectangle DH be applied, equal to the square on AC; and to HI another, IK, equal to the square on BC; so that the whole DK is equal to the sum of the squares on AC, BC. [Margin: b → Elem. II.7 [recte: printed abbreviation reads "7.sexti", but the cited identity — the sum of the squares equals twice the rectangle together with the square on the difference — is Elem. II.7]] And since the sum of the squares on AC, BC is equal to twice the rectangle contained by AC, BC together with the square on AB, then if the square on AB and the rectangle DF be taken away, there remains GK equal to twice the rectangle contained by AC, BC; and therefore, GL being bisected at M and MN drawn parallel to DE, MK is equal to the rectangle contained by AC, BC. And since AC, BC are Rational, their squares on AC, BC are also Rational, and therefore commensurable. [Margin: c → Elem. X.16] Since, then, the sum of the squares on AC, BC is commensurable with each of them, that sum — that is, DK equal to it — is also Rational. And since it is applied to the Rational DE, [Margin: d → Elem. X.21] DL is Rational and commensurable in length with DE. Again, since AC, BC are Rational and commensurable in square only, the

rectangle contained by them, and therefore also its double GK, is medial. And since it is applied to the Rational GF, [Margin: e → Elem. X.23] GL is Rational but incommensurable in length with GF, that is, with DE. And since DK is Rational and GK medial — that is, irrational — they are incommensurable; [Margin: f → Elem. X.10] therefore the straight lines DL, GL, which have the same ratio as DK, GK, are incommensurable in length. And since they have both been shown to be Rational, DL, GL are Rational and commensurable in square only; [Margin: g → Elem. X.74] and therefore the remainder DG is an apotome. I say that it is also a first apotome.

For, by the Lemma to Proposition 54 of this book, the rectangle contained by AC, BC — that is, MK — is a mean proportional between the squares on AC, BC, that is, between DH and IK; therefore DH, MK, IK are in continued proportion; and therefore the straight lines DI, ML, IL, which have the same ratio to them, are also in continued proportion. [Margin: h → Elem. VI.17] Therefore the rectangle contained by DI, IL is equal to the square on ML, that is, to a quarter of the square on GL. And since the squares on AC, BC — that is, DH, IK equal to them — are commensurable, [Margin: i → Elem. X.10] the straight lines DI, IL, which have the same ratio to them, are commensurable in length. And so, since the two straight lines DL, GL are unequal,

Diagram. Two parts.

- A straight line marked A (left), B (near the right), C (right end) — the apotome AB with its annex BC, giving the whole AC.
- A rectangle divided into vertical strips: top edge, left to right, D, G, M, I, L; bottom edge, left to right, E, F, N, H, K. (D/E is the left side on the Rational DE; the verticals through G/F, M/N, I/H divide it into the rectangles DF, DH, IK and the strip GK bisected at M by MN.)

[Note: a small paper repair overlies the region near the label I; the figure is otherwise legible.]

and to the greater, DL, there is applied a rectangle contained by DI, IL, equal to a quarter of the square on the lesser GL and falling short by a square figure; and since DI has been shown commensurable in length with IL, [Margin: k → Elem. X.18] DL is capable of exceeding GL by the square on a straight line commensurable in length with itself. Therefore, since DG has been shown to be an apotome, and the whole DL exceeds its annex GL by the square on a straight line commensurable in length with itself, and this same whole DL is commensurable in length with the Rational DE, by definition DG is a first

apotome. Therefore the square on an apotome, applied to a Rational, produces as its breadth a first apotome. Which was to be shown.

Diagram. The figure carried over from the preceding proposition. A straight line marked A (left), B, C (right end); and a rectangle with top edge D, G, M, I, L and bottom edge E, F, N, H, K, divided into vertical strips by the lines through G/F, M/N, I/H.

THEOR. 75. PROPOS. 99.

[Margin (beside heading): xcviij. / lxxxxiij.]

The square on a first apotome of a medial, applied to a Rational, produces as its breadth a second apotome.

Demonstration. Let AB be a first apotome of a medial and BC its annex, so that AC, BC are medial and commensurable in square only and contain a Rational area; [Margin: a → Elem. I.45] and to the Rational DE let a rectangle DF equal to the square on AB be applied, producing as breadth DG. I say that DG is a second apotome. For let the same construction be made as in the preceding proposition, so that again DH, IK are equal to the squares on AC, BC, and GK equal to twice the rectangle contained by AC, BC, and consequently MK equal to the rectangle contained once by AC, BC. Since, then, AC, BC are medial and commensurable in square, their squares — that is, DH, IK equal to them — are also medial and commensurable; [Margin: b → Elem. X.16] and therefore the whole DK is commensurable with each of them, and hence also medial, by the corollary to Proposition 24 of this book. Since, then, DK is applied to the Rational DE, [Margin: c → Elem. X.23] DL is Rational but incommensurable in length with DE. Again, since the rectangle contained by AC, BC is assumed Rational, its double GK is also Rational. And since it is applied to the Rational DE, [Margin: d → Elem. X.21] GL is Rational and commensurable in length with DE. But since DK is medial — that is, irrational — and GK Rational, they are incommensurable; [Margin: e → Elem. X.10] therefore the straight lines DL, GL, which have the same ratio to them, are incommensurable in length. Since, then, they have been shown to be Rational, DL, GL are Rational and commensurable in square only; [Margin: f → Elem. X.74] and therefore the remainder DG is an apotome. I say that it is also a second apotome. For likewise, as in the preceding proposition, we shall prove that the whole DL exceeds its annex GL by the square on a straight line commensurable in length with itself. And since the annex GL has been shown commensurable in length with the Rational DE, by definition DG is a second apotome. Therefore the

square on a first apotome of a medial, applied to a Rational, produces as its breadth a second apotome. Which was to be shown.

Diagram. As before. A straight line marked A (left), B, C (right end); and a rectangle with top edge D, G, M, I, L and bottom edge E, F, N, H, K, divided into vertical strips.

THEOR. 76. PROPOS. 100.

[Margin (beside heading): lxxxxix. / xciv.]

The square on a second apotome of a medial, applied to a Rational, produces as its breadth a third apotome.

Demonstration. Let AB be a second apotome of a medial and BC its annex, so that AC, BC are medial and commensurable in square only and contain a medial area; [Margin: a → Elem. I.45] and to the Rational DE let DF be applied, equal to the square on AB, producing as breadth DG. I say that DG is a third apotome. For let the same construction be made, whereby we shall now show, as in the preceding proposition, that DK is medial, [Margin: b → Elem. X.23] and therefore that the straight line DL is Rational but incommensurable in length with DE. And since the rectangle contained by AC, BC is medial, and hence its double GK also, [Margin: c → Elem. X.23] GL too is Rational but incommensurable in length with DE. But since AC, BC are incommensurable in length, and as AC is to BC so — by Lemma 3 to Proposition 19 of this book — is the square on AC to the rectangle contained by AC, BC, [Margin: d → Elem. X.10] the square on AC is incommensurable with the rectangle contained by AC, BC. [Margin: e → Elem. X.16] But the sum of the squares on AC, BC is commensurable with the square on AC — since the squares described on the lines AC, BC, being commensurable in square, are commensurable — while the rectangle contained twice by AC, BC is commensurable with the rectangle contained by AC, BC. Therefore, by the scholium to Proposition 14 of this book, the sum of the squares on AC, BC — that is, DK — is incommensurable with twice the rectangle contained by AC, BC, [Margin: f → Elem. X.10] that is, with GK. And therefore the straight lines DL, GL, which have the same ratio as DK, GK, are incommensurable in length. But they have also been shown to be Rational. Therefore DL, GL are Rational and commensurable in square only; [Margin: g → Elem. X.74] and consequently the remainder DG is an apotome. I say that it is also a third. For likewise, as in Proposition 98, we shall prove that the whole DL exceeds its annex GL by the square on a straight line commensurable in length with itself. Since, then, neither of DL, GL is commensurable in length with the Rational DE, as was demonstrated,

by definition DG is a third apotome. Therefore the square on a second apotome of a medial, applied to a Rational, and so on. Which was to be shown.

Diagram. As before. A straight line marked A (left), B, C (right end); and a rectangle with top edge D, G, M, I, L and bottom edge E, F, N, H, K, divided into vertical strips.

Euclid, Book X (continued) — the Apotome (subtractive irrationals), closing propositions

(Clavius's commentary on Euclid's Elements. Enunciations are set as blockquotes; the demonstrations and scholia are Clavius's. Marginal citations are keyed to the lettered points of each proof — "dec." = Elements Book X, "sexti" = Book VI, "quint." = Book V, "prim." = Book I. The stacked lowercase-roman marginal counters beside each heading are the source's parallel proposition-numbering, transcribed verbatim.)

THEOR. 77. PROPOS. 101.

[Margin: c. / LXXXXV.]

The square on a minor line, applied to a rational line, produces as its breadth a fourth apotome.

Let AB be a minor line, and BC its annex, so that AC and BC are incommensurable in square, and make the sum of their squares rational but the rectangle contained by AC and BC medial [Margin: I.45]; and to the rational line DE let there be applied the area DF, equal to the square on AB, producing DG as breadth. I say that DG is a fourth apotome. For let everything be constructed as in what precedes.

Since, then, the sum of the squares on AC and BC — that is, the equal area DK — is rational, DL will be commensurable in length with the rational line DE [Margin: X.21]. Again, since the rectangle contained by AC, BC — and therefore also its double GL — is medial, GL will be incommensurable in length with the rational line DE [Margin: X.23]. Further, since DK is rational while GK is irrational, namely medial, and these are incommensurable, DL and GL, having the same ratio to them, will be incommensurable in length [Margin: X.10]. But they were shown to be rational. Therefore DL and GL are rational and commensurable in square only; and consequently the remainder DG is an apotome [Margin: X.74].

I say that it is also a fourth apotome. For since AC and BC are incommensurable in square, their squares too — that is, the equal areas DH and IK — will be incommensurable; and consequently the lines DI and IL are incommensurable in length [Margin: X.10]. Since, then, the rectangle contained by DI, IL is equal to the square on ML — that is, to the fourth part of the square on GL, as was demonstrated in proposition 98 — DL will be capable of more than GL by the square on a line incommensurable in length with itself [Margin: X.19]; for to DL, the greater, there has been applied a rectangle contained by DI, IL, equal to the fourth part of the square on GL and deficient by a square figure, and dividing the greater line DL into the parts DI, IL, incommensurable in length. And so, since the whole DL is capable of more than its annex GL by the square on a line incommensurable in length with itself, while the same whole DL was shown to be commensurable in length with the rational line DE, DG will by definition be a fourth apotome. Therefore the square on a minor line, applied to a rational line, and so on. Which was to be demonstrated.

Diagram. A straight line AB C at the top (A at the left, B and C together near the right), and beneath it a rectangle divided into vertical strips by internal lines.

- A (top line, left end): endpoint of the whole line
- B (top line, near right): division point (minor / annex)
- C (top line, right end): endpoint of the whole AC
- D (rectangle, top-left corner): breadth-line DL begins here
- G, M, I, L (rectangle, along the top edge): successive division points; L at the top-right corner
- E (rectangle, bottom-left corner)
- F, N, H, K (rectangle, along the bottom edge): division points beneath G, M, I, L respectively; K at the bottom-right corner

THEOR. 78. PROPOS. 102.

[Margin: ci. / LXXXXVI.]

The square on the line which, together with a rational area, makes the whole medial, applied to a rational line, produces as its breadth a fifth apotome.

Let AB be the line which together with a rational area makes the whole medial, and BC its annex, so that AC and BC are incommensurable in square, and make the sum of their squares medial but the rectangle contained by these same AC, BC rational [Margin: I.45]; and to the rational line DE let DF be applied, equal to the

square on AB, producing DG as breadth. I say that DG is a fifth apotome. For let the same things be constructed as above.

Since, then, the sum of the squares on AC and BC — and therefore also the equal area DK — is medial, DL will be incommensurable in length with the rational line DE [Margin: X.23]. Again, since the rectangle contained by AC, BC, and therefore also its double GK, is rational, GL will be commensurable in length with the rational line DE [Margin: X.21 (printed "deo," *recte* dec.)]. And since DK, being medial or irrational, and GK, being rational, are incommensurable, DL and GL, having the same ratio to them, will be incommensurable in length [Margin: X.10]. But they are also rational and commensurable in square only; and therefore the remainder DG is an apotome [Margin: X.74].

I say that it is also a fifth apotome. For in the same way as in the preceding proposition we shall demonstrate that the whole DL is capable of more than its annex GL by the square on a line incommensurable in length with itself. Since, then, the annex GL has been shown to be commensurable in length with the rational line DE, DG will by definition be a fifth apotome. Therefore the square on the line which together with a rational area makes the whole medial, and so on. Which was to be demonstrated.

Diagram. As in the previous figure: a straight line AB C at the top, and beneath it a rectangle divided into vertical strips.

- A (top line, left end); B (division point); C (top line, right end)
- D (top-left corner); G, M, I, L (along the top edge, L at top-right corner)
- E (bottom-left corner); F, N, H, K (along the bottom edge, K at bottom-right corner)

THEOR. 79. PROPOS. 103.

[Margin: cij. / lxxxxvij.]

The square on the line which, together with a medial area, makes the whole medial, applied to a rational line, produces as its breadth a sixth apotome.

Let AB be the line which together with a medial area makes the whole medial, and BC its annex, so that AC and BC are incommensurable in square, and make both the sum of their squares medial and the rectangle contained by these same AC, BC medial, and incommensurable with the sum of their squares [Margin: I.45]; and to the rational line DE let the area DF, equal to the square on AB, be

applied, producing DG as breadth. I say that DG is a sixth apotome. For let the same things be done as above.

Since, then, both the sum of the squares on AC, CB — that is, the equal area DK — and the rectangle contained by AC, BC (and therefore also its double GK) are medial, both DL and GL will be incommensurable in length with the rational line DE [Margin: X.23]. And since the rectangle contained by AC, BC is incommensurable with the sum of the squares on AC, BC, while the rectangle contained by AC, BC and its double GK are commensurable, GK too will be incommensurable with the same sum, that is, with DK [Margin: X.24]; and consequently DL and GL, having the same ratio as DK and GK, are incommensurable in length [Margin: X.10]. But they were also shown to be rational. Therefore DL and GL are rational and commensurable in square only; and therefore the remainder DG is an apotome [Margin: X.74].

I say that it is also a sixth apotome. For we shall show, in the same way as in proposition 101, that the whole DL is capable of more than its annex GL by the square on a line incommensurable in length with itself. Since, then, neither of the lines DL, GL is commensurable in length with the rational line DE, DG will by definition be a sixth apotome. Therefore the square on the line which together with a medial area makes the whole medial, and so on. Which was to be shown.

Diagram. As before: a straight line AB C above, and beneath it a rectangle divided into vertical strips.

- A (top line, left end); B (division point); C (top line, right end)
- D (top-left corner); G, M, I, L (along the top edge, L at top-right corner)
- E (bottom-left corner); F, N, H, K (along the bottom edge, K at bottom-right corner)

THEOR. 80. PROPOS. 104.

[Margin: ciiij. / xcviij.]

A straight line commensurable in length with an apotome is itself also an apotome, and of the same order.

Let AB be an apotome, and BC its annex, so that AC and BC are rational and commensurable in square only; and let DE be commensurable in length with AB. I say that DE too is an apotome, and of the same order as AB. For let it be contrived that as AB is to DE, so is BC to EF [Margin: VI.12]. Then the whole AC will be to the whole DF as AB is to DE, or as BC is to EF [Margin: V.12]. Since, then,

AB and DE are assumed commensurable in length, AC and DF, and likewise BC and EF, will also be commensurable in length [Margin: X.10]. And since AC, BC are rational, DF, EF, being commensurable with them, will also be rational.

Again, since as AC is to DF, so is BC to EF, then, alternately, as AC is to BC, so is DF to EF; and since AC, BC are commensurable in square only, DF, EF too will — by what we demonstrated at proposition 10 of this book — be commensurable in square only. Therefore, since they have also been shown to be rational, the remainder DE is an apotome [Margin: X.74].

I say that it is also of the same order as AB. First, let AC be capable of more than BC by the square on a line commensurable in length with itself. This granted, DF too will be capable of more than EF by the square on a line commensurable in length with itself [Margin: X.15]. So if AC is commensurable in length with the assigned rational line, so that AB is a first apotome, then DF too will be commensurable in length with the assigned rational line [Margin: X.12], since both the assigned rational line and DF are commensurable in length with the same AC. Therefore by definition DE too will be a first apotome, that is, of the same order as AB. But if BC is commensurable in length with the rational line, then in the same way EF will be commensurable in length with the rational line, and so both AB and DE will by definition be a second apotome. And if, finally, neither AC nor BC is commensurable in length with the assigned rational line, then neither of the lines DF, EF will be commensurable in length with the same rational line [Margin: X.14]; and so both AB and DE will by definition be a third apotome.

Now let AC be capable of more than BC by the square on a line incommensurable in length with itself. This granted, DF too will be capable of more than EF by the square on a line incommensurable in length with itself [Margin: X.15]. And so, as before, we shall show that DE is a fourth, fifth, or sixth apotome, and so on. Therefore a straight line commensurable in length with an apotome, and so on. Which was to be shown.

Diagram. Two horizontal straight lines. The upper line is marked A, B, C (A at the left, B a short mark near the middle-right, C at the right end); the lower line is marked D, E, F (D at the left, E near the middle-right, F at the right end). AB and BC are the apotome and its annex; DE and EF the corresponding parts of the derived line.

- A (upper line, left end); B (division point); C (upper line, right end)
- D (lower line, left end); E (division point); F (lower line, right end)

SCHOLIUM.

Here we shall say of apotomes the same thing that we wrote of the binomial lines at proposition 67 of this book — namely, that if the line DE is commensurable with AB in square only, it can be demonstrated in the same way (if in place of the phrase "commensurable in length" we use the phrase "commensurable in square only") that DE too is an apotome; but that it cannot be inferred that it is of the same order as AB. For it does not follow, if the whole AC is commensurable in length with the assigned rational line, that DF too is commensurable in length with it, because it is not the case that both — namely the assigned rational line and DF — are commensurable in length with the same AC; rather, the rational line is commensurable with it in length, whereas DF is so only in square, by hypothesis. Indeed, if AB is a first, second, fourth, or fifth apotome, it is in no way possible that DE, commensurable with it in square only, should be an apotome of the same order.

For let AB be a first apotome (or a second, fourth, or fifth), and BC its annex; and let DE be commensurable with AB in square only — which, as in the theorem, we shall show to be an apotome — and EF its annex, and let the parts AB, BC have the same ratio to the parts DE, EF as the wholes AC, DF have to one another. I say that on no account is DE an apotome of the same order as AB. For, if it were possible, let each be a first apotome. This granted, both the whole AC and the whole DF will, by the definition of a first apotome, be commensurable in length with the assigned rational line [Margin: X.12], and so AC and DF will also be commensurable in length with one another. Since, then, as AC is to DF, so is AB to DE, AB and DE too will be commensurable in length [Margin: X.10]. But this is absurd; for DE is assumed commensurable with AB in square only. Therefore AB and DE will not both be a first apotome. Nor yet a second or a fifth; for each of them, being commensurable with its annex BC, EF, would by definition be commensurable in length with the assigned rational line, and so also with one another. Consequently AB and DE would be commensurable in length with one another — which is not assumed.

It is nevertheless always true that, if AB is a first, second, or third apotome, the line DE which is commensurable with AB in square only is also one of those three, though not of the same order. For, this being granted, AC will be capable of more than BC by the square on a line commensurable in length with itself. Since, then, as AC is to BC, so is DF to EF, DF too will be capable of more than EF by the square on a line commensurable in length with itself [Margin: X.15]; and so by definition DE will be a first, second, or third apotome. In the same way, if AB is a

fourth, fifth, or sixth apotome — so that AC is capable of more than BC by the square on a line incommensurable in length with itself — DE too will be a fourth, fifth, or sixth apotome, though not of the same order as AB; because again, since as AC is to BC, so is DF to EF, DF too will be capable of more than EF by the square on a line incommensurable in length with itself [Margin: X.15]. And so by definition AB will be a fourth, fifth, or sixth apotome.

In the following four propositions, however, DE will necessarily be of the same order as AB, even though it is commensurable with it in square only.

THEOR. 81. PROPOS. 105.

[Margin: civ. / lxxxxix.]

A straight line commensurable with an apotome of a medial is itself also an apotome of a medial, and of the same order.

Let AB be any apotome of a medial, and BC its annex, so that AC and BC are medials commensurable in square only; and let DE be commensurable with AB, whether in both length and square or in square only. I say that DE too is an apotome of a medial, and of the same order as AB. For let it be contrived that as AB is to DE, so is BC to FE [recte: EF] [Margin: VI.12]. Then the whole AC will be to the whole DF as AB is to DE, and as BC is to EF [Margin: V.12]. Since, then, AB and DE are assumed commensurable, whether in both length and square or in square only, BC and EF, and AC and DF, will also be commensurable in the same way [Margin: X.10]. And since AC, BC are medials, DF, EF, being commensurable with them, will also be medials [Margin: X.24].

Again, since as AC is to DF, so is BC to EF, then, alternately, as AC is to BC, so is DF to EF; but AC, BC are commensurable in square only, so DF, EF too will — by the scholium to proposition 10 of this book — be commensurable in square only. Therefore, since DF, EF have been shown to be medials, the remainder DE is an apotome of a medial [Margin: X.75 or X.76].

I say that it is also of the same order as AB. For since as AC is to BC, so is DF to EF; and as AC is to BC, so is the square on AC to the rectangle contained by AC, BC (by lemma 3 of proposition 19 of this book), and as DF is to EF, so is the square on DF to the rectangle contained by DF, EF; then also, as the square on AC is to the rectangle contained by AC, BC, so is the square on DF to the rectangle contained by DF, EF. And, alternately, as the square on AC is to the square on DF, so is the rectangle contained by AC, BC to the rectangle contained by DF, EF. Since, then, the square on AC is commensurable with the square on DF — because the lines

AC, DF were shown to be commensurable — the rectangle contained by AC, BC will also be commensurable with the rectangle contained by DF, EF [Margin: X.10]. Therefore, if the rectangle contained by AC, BC is rational, so that AB is a first apotome of a medial, the rectangle contained by DF, EF will also be rational, since it is commensurable with it; and so DE too will be a first apotome of a medial [Margin: X.75]. But if the rectangle contained by AC, BC is medial, so that AB is a second apotome of a medial, the rectangle contained by DF, EF, being commensurable with it, will also be medial, by the corollary to proposition 24 of this book [Margin: X.76]; and consequently DE too will be a second apotome of a medial. Therefore a straight line commensurable with an apotome of a medial, and so on. Which was to be shown.

Diagram. Two horizontal straight lines. The upper line is marked A, B, C (A at the left, B near the middle, C at the right end); the lower line is marked D, E, F (D at the left, E near the middle, F at the right end).

- A (upper line, left end); B (division point); C (upper line, right end)
- D (lower line, left end); E (division point); F (lower line, right end)

THEOR. 82. PROPOS. 106.

[Margin: cv. cxvi. / c.]

A straight line commensurable with a minor line is itself also a minor line.

Let AB be a minor line, and BC its annex, so that AC and BC are incommensurable in square, and make the sum of the squares on AC, BC rational but the rectangle contained by these same AC, BC medial; and let DE be commensurable with AB, whether in both length and square or in square only. I say that DE too is a minor line. For let the same things be done as above, so that again AB, BC have the same ratio to DE, EF as AC has to DF. Then again, as in the preceding proposition, DF, EF will be commensurable with AC, BC, whether in both length and square or in square only.

And since as AC is to DF, so is BC to EF, then, alternately, as AC is to BC, so is DF to EF; hence as the square on AC is to the square on BC, so is the square on DF to the square on EF [Margin: VI.22]; and, componendo, as the sum of the squares on AC, BC is to the square on BC, so is the sum of the squares on DF, EF to the square on EF; and, alternately, as the sum of the squares on AC, BC is to the sum of the squares on DF, EF, so is the square on BC to the square on EF. But the square on BC is commensurable with the square on EF, because we showed the lines BC, EF

to be commensurable. Therefore the sum of the squares on AC, BC is also commensurable with the sum of the squares on DF, EF [Margin: X.10]. But the sum of the squares on AC, BC is, by the ninth definition and by hypothesis, rational. Therefore the sum of the squares on DF, EF is rational.

Again, since — as we shall demonstrate as in the preceding proposition — the rectangle contained by AC, BC is commensurable with the rectangle contained by DF, EF, and the rectangle contained by AC, BC is assumed medial, the rectangle contained by DF, EF will also be medial, by the corollary to proposition 24 of this book. And since as AC is to BC, so is DF to EF, and AC, BC are incommensurable in square, DF, EF too will be incommensurable in square [Margin: X.10]. And so, since DF, EF are incommensurable in square and make the sum of the squares on DF, EF rational but the rectangle contained by these same DF, EF medial, DE will be a minor line [Margin: X.77]. Therefore a straight line commensurable with a minor line, and so on. Which was to be demonstrated.

Diagram. Two horizontal straight lines. The upper line is marked A, B, C (A at the left, B near the middle with a division mark, C at the right end); the lower line is marked D, E, F (D at the left, E near the middle with a division mark, F at the right end).

- A (upper line, left end); B (division point); C (upper line, right end)
- D (lower line, left end); E (division point); F (lower line, right end)

PROBL. 83. PROPOS. 107.

[Margin: cvi. cxvii. / ci.]

A straight line commensurable with the line which together with a rational area makes the whole medial is itself also a line which together with a rational area makes the whole medial.

Let AB be the line which together with a rational area makes the whole medial, and BC its annex, so that AC and BC are incommensurable in square, and make the sum of the squares on AC, BC medial but the rectangle contained by AC, BC rational; and let DE be, in any manner whatever, incommensurable with AB. I say that DE too is a line which together with a rational area makes the whole medial. For, the same things being constructed as above, we shall show, as in the preceding proposition, that the sum of the squares on AC, BC is commensurable with the sum of the squares on DF, EF. Since, then, [the sum of the squares on AC, BC is] medial [by the] corollary to proposition 24 of this book, [the sum of the squares on DF, EF is] medial. Again, as in proposition 105 we shall demonstrate

[that the rectangle contained by AC, BC] is commensurable with the rectangle contained by DF, EF. Since, then, that [rectangle] is assumed rational, [this one] too will be [rational]; and in the same way as was proved in the preceding proposition, the lines DF, EF [will be] incommensurable in square [Margin: X — ?, foot-margin foxed]

(The demonstration of Prop. 107 breaks off here and continues on the following page.)

(Conclusion of the demonstration of Theor. 83 / Prop. 107, carried over from the previous page.)

incommensurable. Therefore, since DF and EF are incommensurable in square, and make the sum of their squares a medial area while the rectangle contained by them is rational, [Margin: ^a — *Elements* X.78] DE will be the line that with a rational area produces a medial whole, and so on. Which was to be demonstrated.

THEOR. 84. PROPOS. 108.

[Margin: cvij. | cij.]

A straight line commensurable with the line that produces a medial whole with a medial area itself also produces a medial whole with a medial area.

Let AB be a line that produces a medial whole with a medial area, and let BC be joined to it as its congruent part, so that AC and BC are incommensurable in square, making the sum of their squares a medial area and the rectangle under them likewise medial and, moreover, incommensurable with the sum of their squares. And let DE be commensurable with AB in any way whatever. I say that DE too is a line that produces a medial whole with a medial area.

For, the same constructions being made as those we proved above, we shall show, just as in proposition 106,

Diagram. Two horizontal line segments, one above the other, each divided into two parts.

- A (upper line, left end): start of the line AB.
- B (upper line, middle): division point; BC is the part joined on.
- C (upper line, right end): so that AC is the whole.
- D (lower line, left end): start of the commensurable line DE.

- E (lower line, middle): division point.
- F (lower line, right end): so that DF is the whole, with EF the joined part.

that the sum of the squares of AC, BC is commensurable with the sum of the squares of DF, EF. Since, then, the former is assumed to be medial, the latter too will be medial, by the corollary of proposition 24 of this book. Again, as we showed in proposition 105, the rectangle under AC, BC is commensurable with the rectangle under DF, EF. Since, then, the former is medial, the latter too will be medial, by the same corollary. Likewise, as in proposition 106, DF and EF will also be incommensurable in square. Finally, since the sum of the squares of AC, BC is commensurable with the sum of the squares of DF, EF, as has been said, and the rectangle under AC, BC is commensurable with the rectangle under DF, EF; but the sum of the squares of AC, BC and the rectangle under AC, BC are, by hypothesis, incommensurable; then, by the scholium of proposition 14 of this book, the sum of the squares of DF, EF and the rectangle under DF, EF will also be incommensurable. And so, since DF, EF are incommensurable in square, and the sum of their squares is medial, and the rectangle under them is likewise medial and incommensurable with the sum of their squares, [Margin: ^a — *Elements* X.79] DE will be the line that with a medial area produces a medial whole. A straight line commensurable, therefore, with the line that produces a medial whole with a medial area, and so on. Which was to be shown.

THEOR. 85. PROPOS. 109.

[Margin: cviiij. | ciij.]

When a medial area is subtracted from a rational area, the straight line whose square equals the remaining area is one of two irrationals: either an apotome or a minor.

Let the medial CD be subtracted from the rational AD. I say that the line whose square equals the remaining area AB is either an apotome or a minor. For let the rational EF be set out, [Margin: ^a — *Elements* I.45] and to it let the rectangle EG be applied equal to AB, and to GH let another, HI, be applied equal to CD, so that the whole EI is equal to the whole AD.

Diagram. Two rectangles standing side by side on a common base line. The left rectangle represents the given areas; the right rectangle is built on the rational line EF, to which the areas are applied.

- A (left rectangle, top-left corner): the whole area AD.
- C (left rectangle, top, internal division): marks off the part CD.

- B (left rectangle, base-left): lower-left corner.
- D (left rectangle, base): lower division.
- E (right figure, top-left): top of the rational line EF.
- H (right figure, top, internal division): between E and K.
- K (right figure, top-right): so that EK is the whole breadth.
- F (right figure, base-left): foot of the rational line EF.
- G (right figure, base, internal division): foot of the division HG.
- I (right figure, base-right): so that FI is the whole base, EI equal to AD.

Since, then, EI, being equal to the rational AD, is rational, [Margin: ^b — *Elements* X.21] EK will be rational and commensurable in length with the rational EF.

Again, since HI, being equal to the medial CD, is medial, [Margin: ^c — *Elements* X.23] HK will be rational but incommensurable in length with EF. And so, since EK is commensurable in length with EF while HK is incommensurable in length with the same EF, [Margin: ^d — *Elements* X.13] EK and HK are incommensurable in length. But they are also rational. Therefore they are rational and commensurable in square only; [Margin: ^e — *Elements* X.74] and for that reason the remainder EH is an apotome, with HK the part joined to it.

Now EK is either greater in square than HK by the square of a line commensurable with it in length, or by the square of one incommensurable. If EK is greater in square than HK by the square of a line commensurable with itself in length, and the whole EK has also been shown to be commensurable in length with the rational EF, then by definition EH will be a first apotome. Therefore the line whose square equals the area EG — contained by the rational EF and the first apotome EH, that is, the area AB equal to it — [Margin: ^f — *Elements* X.92] is an apotome. But if EK is greater in square than HK by the square of a line incommensurable with itself in length, while the whole EK has been shown to be commensurable in length with the rational EF, then by definition EH will be a fourth apotome. Therefore the line whose square equals the area EG — contained by the rational EF and the fourth apotome EH, that is, the area AB equal to it — [Margin: ^g — *Elements* X.95] is a minor. Therefore, when a medial is subtracted from a rational, and so on. Which was to be shown.

THEOR. 86. PROPOS. 110.

[Margin: cix. | civ.]

When a rational area is subtracted from a medial area, two other irrationals arise: either the first apotome of a medial, or the line that produces a medial whole with a rational area.

Let the rational CD be subtracted from the medial AD. I say that the line whose square equals the remaining area AB is either the first apotome of a medial, or the line that with a rational area produces a medial whole. For let the same constructions be made as above.

Diagram. The same arrangement as before: a left rectangle for the given areas and, to the right, the figure built on the rational line EF.

- A (left rectangle, top-left corner): the whole area AD.
- C (left rectangle, top, internal division): marks off the part CD.
- B (left rectangle, base-left): lower-left corner.
- D (left rectangle, base): lower division.
- E (right figure, top-left): top of the rational line EF.
- H (right figure, top, internal division).
- K (right figure, top-right): EK the whole breadth.
- F (right figure, base-left): foot of EF.
- G (right figure, base, internal division).
- I (right figure, base-right): EI equal to AD.

And since EI, being equal to the medial AD, is medial, [Margin: ^a — *Elements* X.23] EK will be rational but incommensurable in length with EF. Likewise, since HI, being equal to the rational CD, is rational, [Margin: ^b — *Elements* X.21] HK will be rational and commensurable in length with EF. And so, since HK is commensurable in length with EF while EK is incommensurable in length with the same EF, [Margin: ^c — *Elements* X.13] HK and EK are incommensurable in length. But they are also rational. Therefore they are rational and commensurable in square only;

[Margin: ^d — *Elements* X.74] and consequently the remainder EH is an apotome, with HK the part joined to it.

Diagram. The figure for Proposition 110 is reprinted here where its demonstration continues: a left rectangle (A, C at top; B, D at base) and, to the right, the figure on the rational line EF (E, H, K along the top; F, G, I along the base).

- A, C (left rectangle, top); B, D (left rectangle, base).

- E, H, K (right figure, top edge); F, G, I (right figure, base).

Now EK is either greater in square than HK by the square of a line commensurable with itself in length, or by the square of one incommensurable. If EK is greater in square than HK by the square of a line commensurable with itself in length, since the joined HK has been shown to be commensurable in length with the rational EF, then by definition EH will be a second apotome. Therefore the line whose square equals the area EG — contained by the rational EF and the second apotome EH, that is, the area AB equal to it — [Margin: ^e — *Elements* X.93] is the first apotome of a medial. But if EK is greater in square than HK by the square of a line incommensurable with itself in length, while the joined HK has been shown to be commensurable with the rational EF, then by definition EH will be a fifth apotome. Therefore the line whose square equals the area EG — contained by the rational EF and the fifth apotome EH, that is, the area AB equal to it — [Margin: ^f — *Elements* X.96] is the line that with a rational area produces a medial whole. Therefore, when a rational is subtracted from a medial, and so on. Which was to be shown.

THEOR. 87. PROPOS. 111.

[Margin: cx. | cv.]

When a medial area is subtracted from a medial area that is incommensurable with the whole, two other irrationals arise: either the second apotome of a medial, or the line that produces a medial whole with a medial area.

Let the medial CD, which is incommensurable with the whole AD, be subtracted from the medial AD. I say that the line whose square equals the remaining area AB is either the second apotome of a medial, or the line that with a medial area produces a medial whole. For, the same constructions being made as above:

Diagram. Again a left rectangle for the given areas and, to the right, the figure on the rational line EF.

- A (left rectangle, top-left); C (left rectangle, top, division); B, D (base).
- E (right figure, top-left); H (top division); K (top-right, EK the breadth).
- F (base-left, foot of EF); G (base division); I (base-right, EI equal to AD).

since EI and HI, being equal to the medials AD and CD, are medial, [Margin: ^a — *Elements* X.23] EK and HK will be rational but incommensurable in length with EF. And since AD, CD — that is, EI, HI — are assumed incommensurable, [Margin:

^b — *Elements* X.10] EK and HK, which have the same ratio to them, will be incommensurable in length. But they are also rational. Therefore they are rational and commensurable in square only; [Margin: ^c — *Elements* X.74] and consequently the remainder FH [recte: EH] is an apotome, with HK the part joined to it.

Now EK is either greater in square than HK by the square of a line commensurable with itself in length, or by the square of one incommensurable. If EK is greater in square than HK by the square of a line commensurable with itself in length, since neither of them, EK, HK, has been shown to be commensurable in length with the rational EF, then by definition EH will be a third apotome. Wherefore the line whose square equals the area EG — contained by the rational EF and the third apotome EH, that is, the area AB equal to it — [Margin: ^d — *Elements* X.94] is the second apotome of a medial. But if EK is greater in square than HK by the square of a line incommensurable with itself in length, since neither of them, EK, HK, is commensurable in length with the rational EF, as we have demonstrated, then by definition EH will be a sixth apotome. For which reason the line whose square equals the area EG — contained by the rational EF and the sixth apotome EH, that is, the area AB equal to it — [Margin: ^e — *Elements* X.97] is the line that with a medial area produces a medial whole. Therefore, when a medial is subtracted from a medial, and so on. Which was to be demonstrated.

THEOR. 88. PROPOS. 112.

[Margin: cxj. | cvj.]

An apotome is not the same as a binomial.

Let A be any apotome. I say that A is not the same as a binomial. For, if it be possible, let A be some binomial. Let the rational BC be set out, [Margin: ^a — *Elements* I.45] and to BC let there be applied the rectangle CD equal to the square on A. Since, then, A is an apotome, [Margin: ^b — *Elements* X.98] the breadth BD will be a first apotome.

Diagram. At the top, a short segment A (the apotome). Below it a straight line B–F–D–E, divided at F and at D; from B a rectangle descends to C at the lower left (the rectangle CD, equal to the square on A, applied to the rational BC).

- A (top, short segment): the apotome under discussion.
- B (line, left end): also upper-left corner of the applied rectangle.
- F (line, first division): greater term BF of the supposed binomial.

- D (line, second division): breadth-boundary BD.
- E (line, right end): DE the part joined to the first apotome.
- C (lower-left corner of the rectangle below B).

Let DE, then, be joined to it. Therefore, by the definition of the first apotome, BE and DE will be rational and commensurable in square only; BE will be greater in square than DE by the square of a line commensurable with itself in length; and BE itself will be commensurable in length with the rational BC. Again, since A is also assumed to be a binomial, [Margin: ^c — *Elements* X.61] the same breadth BD will be a first binomial. Let its greater term be BF. Therefore, by the definition of the first binomial, BF and FD will be rational and commensurable in square only; BF will be greater in square than FD by the square of a commensurable line; and BF will be commensurable in length with the rational BC. And so, since both BE and BF are commensurable in length with the same BC, [Margin: ^d — *Elements* X.12] BE and BF will also be commensurable in length with one another. Therefore, since the whole BE is commensurable in length with the part BF, the same BE will also be commensurable in length with the remaining part FE, by the corollary of proposition 16 of this book; and so, since BE is rational, FE too will be rational. But since, of the two lines BE, FE commensurable in length, BE itself is incommensurable in length with DE (because BE, DE are rational and commensurable in square only), [Margin: ^e — *Elements* X.14] the other, FE, will also be incommensurable in length with DE. But both FE and DE have been shown to be rational. Therefore FE, DE are rational and commensurable in square only. Wherefore the remainder FD [Margin: ^f — *Elements* X.74] is an apotome, and so is irrational. But it has also been shown to be rational. Which is absurd. Therefore A, being an apotome, is not the same as a binomial. Which was to be demonstrated.

COROLLARIUM.

From what has been demonstrated it will also be easy to gather that the apotome and the other irrational lines that follow it are neither the same as the medial nor the same as one another.

For the square of the medial, applied to the rational line, [Margin: ^g — *Elements* X.23] produces as its breadth a rational line incommensurable in length with the rational itself.

But the square of the apotome, applied to the rational, [Margin: ^h — *Elements* X.98] produces as its breadth a first apotome.

And the square of the first apotome of a medial, applied to the rational, [Margin: ⁱ — *Elements* X.99] produces as its breadth a second apotome.

The square of the second apotome of a medial, applied to the rational, [Margin: ^k — *Elements* X.100] produces as its breadth a third apotome.

Then the square of the minor, applied to the rational, [Margin: ^l — *Elements* X.101] produces as its breadth a fourth apotome.

But the square of the line that with a rational area produces a medial whole, applied to the rational, [Margin: ^m — *Elements* X.102] produces as its breadth a fifth apotome.

Finally the square of the line that with a medial area produces a medial whole, applied to the rational, [Margin: ⁿ — *Elements* X.103] produces as its breadth a sixth apotome.

And so, since these breadths differ both from the breadth of the medial — because this one is rational, while those are irrational — and from one another — because they are not, in order, the same apotomes — it is manifest that the apotome, and the remaining irrationals that follow it, differ both from the medial and from one another.

Now since in this theorem we have demonstrated that the apotome is not the same as the binomial; and since the squares of the apotome and of the five irrationals following it in that sequence, applied to a rational, produce as breadths the first, second, third, fourth, fifth, and sixth apotome; whereas the squares of the binomial and of the remaining five irrationals that follow it, applied to a rational, produce as breadths the first, second, third, fourth, fifth, and sixth binomial; it plainly follows that the breadths of the apotome and of the other irrationals after it are not the same as the breadths of the binomial and of the remaining irrationals after it — since no apotome is the same as any binomial. Therefore both the apotome and those that follow it differ from the binomial and from those that follow it. And since both those and these differ from the medial, it follows that, given some rational line set out, there are thirteen irrational lines differing from one another, which we have discussed up to this point — namely these:

1. The medial.
2. The binomial [ex binis nominibus], of which six species have been found.
3. The first bimedial [ex binis Mediis prima].
4. The second bimedial [ex binis Mediis secunda].

5. The major [Maior].
6. The line whose square equals a rational plus a medial [Rationale ac Medium potens].
7. The line whose square equals two medials [Bina Media potens].
8. The apotome, of which likewise six species have been found.
9. The first apotome of a medial.
10. The second apotome of a medial.
11. The minor [Minor].
12. The line that with a rational area produces a medial whole [cum Rationali Medium totum efficiens].
13. The line that with a medial area produces a medial whole [cum Medio Medium totum efficiens].

THEOR. 89. PROPOS. 113.

[Margin: cxij. | (a second, faint mark below, read as "o." — [?])]

The square of a rational, applied to a binomial, produces as its breadth an apotome whose terms are commensurable with the terms of the binomial and in the same ratio; and, moreover, the apotome so produced has the same order as the binomial.

Let A be a rational, and BC a binomial whose greater term is BD; [Margin: ^a — *Elements* I.45] and to BC let the rectangle BE be applied equal to the square on A, producing the breadth BF. I say that BF is an apotome whose terms — that is, the whole line and the part joined to it — are commensurable with the terms BD, DC of the binomial BC, and in the same ratio; and, moreover, that the apotome BF has the same order as the binomial BC.

[Margin: ^b — *Elements* I.45] For let there again be applied to DC, the lesser term, the rectangle CG equal to the same square on A, and hence to the rectangle BE, producing the breadth DG; and let BH be made equal to DG.

Diagram. At the top, a short segment A (the rational). A long horizontal line runs K–B–F–I–H, with A marked above it between B and F. Below is a rectangle with a middle horizontal level D (left)–G (right) and a base C (left)–E; vertical divisions fall at F. The applied rectangles BE and CG are shown one above the other.

- A (top, short segment): the rational line.
- K (far left of the horizontal line): left extremity.
- B (horizontal line): term-boundary; upper-left of rectangle BE.

- F (horizontal line): breadth BF of the apotome.
- I (horizontal line): with FI set equal to FB.
- H (right end of the horizontal line): so that HF, HK are formed.
- D (middle level, left): boundary DG / DC.
- G (middle level, right): so that CG is the applied rectangle.
- C (base, left): lower-left corner.
- E (base): lower boundary of BE.

And since BE, CG are equal, [Margin: ^c — *Elements* VI.14 or 16] as BC is to DC, so is DG — that is, its equal BH — to BF. And *separando*, as BD is to DC, so is HF to FB. But BD is greater than DC. Therefore HF too is greater than FB. Let FI be set equal to FB; [Margin: ^d — *Elements* VI.12] and let it be contrived that as HI is to IF, so is FB to BK. Therefore, *componendo*, as HF is to IF — that is, to its equal BF — so is FK to BK. But it has been shown that as HF is to BF, so is BD to DC. Therefore, as BD is to DC, so also will FK be to BK. But BD, DC (since they are the terms of the binomial BC) are rational and commensurable in square only. [Margin: ^e — *Elements* X.10] Therefore FK, BK too are commensurable in square only.

Again, since as HF is to BF, so is FK to BK, [Margin: ^f — *Elements* V.12] HF, FK — the antecedents taken together, namely the whole HK — will also be to BF, BK — the consequents taken together, that is, to the whole FK — as FK is to BK. And for that reason FK is a mean proportional...

(Proposition 113 continues into the following pages.)

(Continuing the proof of Prop. 113, completing the catchword "nalis" [rationalis] of the previous page.)

and so it is a rational mean proportional lying between HK and BK. Therefore, as HK (the first) is to BK (the third), so — by the corollary to VI.20 — the square on HK (the first) is to the square on FK (the second).

Now since CG is rational (for it is equal to the square on the rational A), when it is applied to the rational DC it produces [Margin: ^g — *Elements* X.20] DG as a rational breadth, commensurable in length with DC. HB too, being equal to DG, will be rational and commensurable in length with DC. And since it has been shown that as BD is to DC, so FK is to BK, while as FK is to BK, so HK is to FK, it follows that as BD is to DC, so HK is to FK. [Margin: ^h — *Elements* VI.22] Therefore, as the square on BD is to the square on DC, so the square on HK is to the square on FK. But the square on BD is commensurable with the square on DC (since the straight lines BD, DC are rational and commensurable in square, being the terms

of the binomial BC). [Margin: i — *Elements* X.10] The square on HK will therefore also be commensurable with the square on FK. But it was shown that as the square on HK is to the square on FK, so is the straight line HK to BK. [Margin: k — *Elements* X.10] Therefore HK is commensurable in length with BK, and hence also with the remaining part BH, by the corollary to Proposition 16 of this book.

But BH was shown to be rational. Therefore HK, being commensurable with it, is likewise rational; and consequently BK too, being commensurable with HK, is rational. Since, then, FK was shown to be commensurable with BK in square only, FK will also be rational. Therefore, since FK and BK are rational and were shown to be commensurable in square only, [Margin: l — *Elements* X.74] the remainder BF is an apotome, and BK is the straight line annexed to it. Which is the first point.

Now since the whole straight line HK was shown to be commensurable in length with its part BK, [Margin: m — *Elements* X.16] BK and BH will therefore also be commensurable in length with one another. And since BH was shown commensurable in length with DC, BK too — by the scholium to Proposition 12 of this book — will be commensurable in length with that same DC. Again, since it was proved that as BD is to DC, so FK is to BK, then, alternately, as BD is to FK, so DC is to BK. But DC and BK have just been shown commensurable in length; [Margin: n — *Elements* X.10] therefore BD and FK will also be commensurable in length. And so, since FK is commensurable in length with BD, and BK with DC, the terms FK, BK of the apotome BF will be commensurable in length with the terms BD, DC of the binomial BC. Which is the second point.

Likewise, since we have proved that as BD is to DC, so FK is to BK, standing in the same ratio as the terms BD, DC of BC, the terms FK, BK of the apotome are in the same ratio as the terms of the binomial BC. Which is the third point.

Finally, BD is greater in square than DC either by the square on a straight line commensurable with itself in length, or by the square on one incommensurable with it. If BD is greater in square than DC by the square on a line commensurable with it in length — since as BD is to DC, so FK is to BK — [Margin: o — *Elements* X.15] FK too will be greater in square than BK by the square on a line commensurable with it in length. And if BD is greater in square than DC by the square on a line incommensurable with it in length, then FK too will be greater in square than BK by the square on a line incommensurable with it. And if BD is commensurable in length with the assigned rational line, then FK — which was shown commensurable in length with BD — will also be commensurable in length with that same rational line, by what we proved in the scholium to

Proposition 12 of this book. But if DC is commensurable in length with the rational line, then by the same reasoning BK too will be commensurable in length with it. And finally, if neither BD nor DC is commensurable in length with the rational line, [Margin: p — *Elements* X.14] then neither FK nor BK will be commensurable in length with the rational line. The apotome BF is therefore of the same order as the binomial BC, as is clear from the second and third sets of definitions. Which is the fourth point.

Therefore the square of a rational line applied to a binomial [produces an apotome as breadth], etc. Which was to be demonstrated.

Diagram. The figure for Proposition 113. A short horizontal segment (with a tick mark) labelled A sits above the main line — the rational line. Beneath it a long horizontal straight line runs from K at the far left through the points B, F, and (near the right end) two close marks read as I and H. Below the line a rectangle is drawn, its top-left corner D and top-right corner G, its bottom-left corner C and a division mark E along the bottom; verticals descend from B and F into the rectangle.

- A (short segment, above): the rational line A
- K (far left of main line): terminus
- B, F (on main line): division points
- I, H (near right end of main line): division points
- D (rectangle, top-left): corner of the square/rectangle figure
- G (rectangle, top-right): corner
- C (rectangle, bottom-left): corner
- E (rectangle, bottom, mid): division point

THEOR. 90. PROPOS. 114.

[Margin: cxiiij. / o.]

The square of a rational line applied to an apotome produces as its breadth a binomial whose terms are commensurable with the terms of the apotome and in the same ratio; and moreover the binomial so produced has the same order as the apotome itself.

Let A be a rational line and BC an apotome, to which CD is the annex; and to BC let a rectangle CE be applied, [Margin: a — *Elements* I.45] equal to the square on A, producing BE as its breadth. I say that BE is a binomial whose terms are

commensurable with the terms BD, CD of the apotome BC and in the same ratio; and moreover that BE, the binomial, is of the same order as the apotome.

[Margin: b — *Elements* I.45] For let there again be applied to the whole BD a rectangle BF, equal to the same square on A — that is, equal to CE — producing the breadth BG. And since BF and CE are equal, [Margin: c — *Elements* VI.14 or 16] as BE is to BG, so BD is to BC; and, by conversion of the ratio, as BE is to GE, so BD is to CD. Let EG also be cut at H in the ratio of BE to GE, by the method we set out in the scholium to VI.10, so that EH is to HG as BE is to GE. Since, then, as the whole BE is to the whole GE, so EH (cut off from BE) is to HG (cut off from GE), the remainder BH of BE will be to the remainder HE of GE as the whole BE is to the whole GE. But it was shown that as BE is to GE, so EH is to HG. Therefore, as BH is to HE, so EH is to HG; and hence HE is the mean proportional between BH [recte; printed BG] and GH. Therefore, as BH (the first) is to GH (the third), so — by the corollary to VI.20 — the square on BH (the first) is to the square on HE (the second).

Now since it was shown that as BD is to CD, so BE is to GE — that is, as BH is to HE — and since BD, CD, being the terms of the apotome BC, are rational and commensurable in square only, [Margin: d — *Elements* X.10] BH and HE too will be commensurable in square only. [Margin: e — *Elements* X.10] Therefore the straight lines BH, GH — having, as we proved, the same ratio as the squares on BH and HE — are commensurable in length; and therefore the whole BH, being commensurable in length with the part GH, will also be commensurable in length with the remaining part BG, by the corollary to Proposition 16 of this book. But since BD is rational — namely the greater term of the apotome BC — and the rectangle BF, being equal to the square on the rational line A, is rational, [Margin: f — *Elements* X.20] the breadth BG will be rational, commensurable

in length with BD. Therefore, by the scholium to Proposition 12 of this book, BH too is commensurable in length with the rational BD, and so is itself rational — since BH and BG were shown commensurable in length. And since BH, HE were shown commensurable in square only, and BH is rational, HE too, being commensurable with it, is rational. So BH and HE are rational and commensurable in square only; [Margin: h — *Elements* X.37] and therefore BE is a binomial. Which is the first point.

Now since we showed that as BH is to HE, so BD is to CD, and hence, alternately, as BH is to BD, so HE is to CD; and since BH was shown commensurable in length with BD, [Margin: i — *Elements* X.10] HE too will be commensurable in length

with CD. Therefore BH, HE, the terms of the binomial BE, are commensurable in length with the terms BD, CD of the apotome BC. Which is the second point.

Indeed they are also in the same ratio, since it was proved that as BH is to HE, so BD is to CD. Which is the third point.

Finally, BD is greater in square than CD either by the square on a line commensurable with it in length, or by the square on one incommensurable with it. If it is greater by the square on a line commensurable in length, [Margin: k — *Elements* X.15] then BH too will be greater in square than HE by the square on a line commensurable with it in length. And if BD is greater in square than CD by the square on a line incommensurable in length, then BH too will be greater in square than HE by the square on a line incommensurable with it. And if BD is commensurable in length with the assigned rational line, then BH too, being commensurable in length with BD, will be commensurable with that same rational line, by the scholium to Proposition 12 of this book. But if CD is commensurable in length with the rational line, then HE too will be commensurable in length with it. And if neither BD nor CD is commensurable in length with the assigned rational line, [Margin: l — *Elements* X.14] then neither BH nor HE will be commensurable with the rational line. Therefore, by the second and third sets of definitions, the binomial BE is of the same order as the apotome BC. Which is the fourth point.

Therefore the square of a rational line applied to an apotome produces as breadth a binomial whose terms are commensurable with the terms of the apotome and in the same ratio, etc. Which was to be demonstrated.

Diagram. The figure for Proposition 114, reprinted at the head of the page. A short segment labelled A (the rational line) stands above. A horizontal line runs from B on the left through the division points G and H to E on the right. Below it a rectangle descends: C marks the left side, D the bottom-left corner, and F a mark along the bottom; dashed verticals fall from G and H.

- A (short segment, above): the rational line A
- B (line, left end): the whole
- G, H (on line): the points BG, GE cut at H
- E (line, right end): terminus
- C (rectangle, left side): division point
- D (rectangle, bottom-left): corner
- F (rectangle, bottom): division mark

THEOR. 91. PROPOS. 115.

[Margin: cxiv. / o.]

If an area is contained by an apotome and a binomial line whose terms are commensurable with the terms of the apotome and in the same ratio, then the straight line whose square equals the area is rational.

Let the area AB be contained by the apotome AC and by the binomial CB, whose terms CD, DB are commensurable with the terms CE, AE of the apotome AC and in the same ratio — namely, CD is to DB as CE is to AE. Let F be the straight line whose square equals the area AB. I say that F is rational.

For, with a rational line G assigned, [Margin: a — *Elements* I.45] let there be applied to the binomial CB a rectangle CH equal to the square on G. [Margin: b — *Elements* X.113] Then the breadth BH will be an apotome, whose terms HI, BI are commensurable in length with the terms CD, DB and in the same ratio — namely, HI is to BI as CD is to DB, and hence as CE is to AE; and, alternately, as the whole HI is to the whole CE, so the part cut off BI is to the part cut off AE. [Margin: c — *Elements* V.19] Therefore the remainder BH will be to the remainder AC as the whole HI is to the whole CE. [Margin: d — *Elements* X.12] But HI is commensurable in length with CE, since each of HI, CE was commensurable in length with CD. [Margin: e — *Elements* X.10] Therefore BH too is commensurable in length with AC. [Margin: f — *Elements* X.10] And on that account HC too will be commensurable with BA, [Margin: g — *Elements* VI.1] since as HB is to AC, so HC is to BA. But HC, being equal to the square on the rational G, is rational. Therefore BA too, being commensurable with it, will be rational; and consequently the line F, whose square equals BA, will be rational.

Therefore, if an area is contained by an apotome and a binomial whose terms are commensurable with the terms of the apotome and in the same ratio, the straight line whose square equals the area is rational. Which was to be demonstrated.

Diagram. The figure for Proposition 115. Two short segments at the top are labelled F and G (the line F and the rational line G). Below runs a horizontal line marked E, A, C (with B at the right, less legible). Beneath it a rectangle is drawn, with a division point D in the interior and H at the lower right corner.

- F (top segment): the line whose square equals the area
- G (top segment): the rational line
- E, A, C (on line): E the terminus, A and C the ends of the apotome AC and binomial CB

- D (rectangle interior): division point
- H (rectangle, lower right): breadth-point
- (B, I: named in the text but not clearly legible on the figure)

COROLLARIUM.

From this it is clear that a rational area can be contained by two irrational straight lines. For the area AB, contained by the apotome AC and the binomial CB — which are irrational — has been shown to be rational.

SCHOLIUM.

In the last three propositions the terms of the apotome are to be understood as commensurable *in length* with the terms of the binomial. For in the first two they were shown to be commensurable in length; but in the last — namely, in this 115th — nothing will follow if the terms CD, DB are supposed to be commensurable only *in square* with the terms CE, AE. This is readily apparent from the demonstrations of these three propositions.

THEOR. 92. PROPOS. 116.

[Margin: cxv. / cvij.]

From a medial line infinitely many irrational lines arise, and none of them is ever the same as any of the preceding ones.

Let AB be a medial line. I say that from it infinitely many irrational lines arise, none of which is the same as any of the thirteen preceding ones treated in the corollary to Proposition 112 of this book. For, with a rational line AC assigned, let the area AD be contained by the medial AB and the rational AC. Then AD, being an area contained by the rational AC and the irrational AB, is irrational, by the lemma to Proposition 38 of this book. Let BE be the straight line whose square equals it; by Definition 11 it will be irrational. I say that BE is

not the same as any of those thirteen which we collected in the corollary to Proposition 112 of this book. [Margin: a — *Elements* X.23] For since the square on the medial line, applied to the rational AC, produces a breadth that is rational but incommensurable in length with AC; and the squares of the remaining twelve irrationals, applied to the same rational AC, produce breadths that are either binomials or apotomes, as is clear from Propositions 61, 62, 63, 64, 65, 66, 98, 99, 101, 102, 103 of this book; but the square of this irrational BE, applied to the same rational AC, produces AB, a medial breadth — it is evident that the irrational BE

differs from all those thirteen, since its square, applied to the rational line, produces a breadth different from the breadths that the squares of those thirteen lines produce when applied to the same rational line.

And if the rectangle DE is completed, it too, being contained by the rational BD and the irrational BE, will be irrational, by the said lemma to Proposition 38. Let EF be the straight line whose square equals it; by Definition 11 it will be irrational. I say again that EF is not the same as any of those thirteen, nor even as BE itself. This is evident, since the square on EF, applied to the rational line, produces the breadth BE. But the squares of those thirteen, and the square of BE itself, produce — when applied to the same rational line — breadths different from BE, as was demonstrated. In the same way infinitely many other irrationals will be found, differing both from one another and from those already named. Therefore from a medial line infinitely many irrationals arise, etc. Which was to be demonstrated.

Diagram. The figure for Proposition 116, a rectangle divided by vertical lines into panels (the areas AD and DE). The engraved labels are badly foxed and mostly illegible; along the top edge, division marks that read as D[?] and E[?] can be made out, with a further mark F[?] toward the right, and the left side bears tick marks. — labels at the resolution floor.

- (top edge, left–right): A[?], D[?], E[?], F[?] — division points of the rectangles AD and DE
- (left side): tick marks, labels illegible [?]

THEOR. 93. PROPOS. 117.

[Margin: cxviii. / vij.]

Let it be proposed to show that in square figures the diagonal is incommensurable in length with the side.

Let ABCD be a square, in which AC is the diagonal. I say that the diagonal AC is incommensurable in length with the side AB. For if it is not incommensurable, it will be commensurable in length; [Margin: a — *Elements* X.5] and therefore AC and AB will have to one another the ratio of a number to a number. Let AC have to AB the ratio that the number EF has to the number G, and let the numbers EF and G be the least of all having the same ratio. Since, then, as AC is to AB, so the number EF is to the number G, it follows that as the square on AC is to the square on AB, so the square number from EF is to the square number from G. [Margin: b — *Elements* VI.20 & VIII.11] (For since squares have the duplicate ratio of their

sides, and the sides have equal ratios, the ratios of the squares will also be equal, being duplicates of equal ratios.)

But the square on AC is double the square on AB, by what we showed in the scholium to Proposition 47 of Book 1. Therefore the square number from EF will also be double the square number from G; and consequently the square from EF, having a half — that is, being divisible into two equal parts — will be even, by definition. Therefore EF too, which produces it, will be even. (For if it were odd, then — since, multiplying itself, it produces its own square — the square would itself be odd, [Margin: c — *Elements* IX.29] because an odd number multiplying an odd number produces an odd number. But that is absurd, since it was shown to be even.)

Now since EF and G are the least in their ratio, [Margin: d — *Elements* VII.24] they are prime to one another; and since EF was shown to be even, G will be odd. (For if it too were even, the number two would measure both EF and G, and so they would not be prime to one another — which is absurd.) Now let the even number EF be bisected at H. Since, then, the number EF is double the number EH, [Margin: e — *Elements* VIII.11] and squares have the duplicate ratio of their sides, the square on EF will be quadruple the square on EH. (For the quadruple ratio is the duplicate of the double ratio, as appears in these numbers: 4, 2, 1.) And so, since the square from EF is double the square from C [recte: G] and quadruple the square from EH: of the 4 equal parts making up the square from EF, the square from G will be 2 such parts, and the square from EH 1 such part. Therefore the square from G is double the square from EH, since its ratio to it is that of 2 to 1; and hence, as we said above of the number EF, the square from G, having a half, will be even, and G itself will also be even. But it was also shown to be odd. Which is absurd. Therefore the diagonal AC is not commensurable in length with the side AB. Hence it is incommensurable in length.

Diagram. The figure for Proposition 117. A square ABCD is drawn: A at the top-left, B at the top-right, C at the bottom-right, D at the bottom-left, with the diagonal AC drawn across it. Below the square lies a straight segment with terminal points E (left) and F (right), bisected at the mid-point H (the intervening length shown by a row of dots); beneath this a shorter dashed segment is labelled G — representing the numbers EF (bisected at H) and G.

- A (square, top-left): vertex; endpoint of the diagonal AC
- B (square, top-right): vertex
- C (square, bottom-right): vertex; endpoint of the diagonal AC
- D (square, bottom-left): vertex

- E, F (lower segment, ends): the number EF
- H (lower segment, middle): bisection point of EF
- G (dashed segment): the number G

Alternatively. Suppose, if it be possible, that the diagonal AC is commensurable in length with the side AB; and let AC, AB have the ratio of the numbers EF and G, which are the least in their ratio, and hence prime to one another. Then G will not be unity. (For since the square on AC is double the square on AB, and since — as we said in the former demonstration — the square on AC is to the square on AB as the square number from EF is to the square number from G, the square from EF will also be double the square from G. If, then, G is unity — and hence its square also unity — the square from EF will be two. Which is absurd.) Therefore G is a number [greater than unity]. And since, as has now been demonstrated, the square from EF is double the square from G, the square from G will measure the square from EF; [Margin: f — *Elements* VIII.14] and therefore the side G too will measure the side EF. Since, then, G also measures itself, the numbers EF and G will be composite with respect to one another, having a common measure, the number G. But they are also prime to one another. Which is absurd. Therefore the diagonal AC is not commensurable in length with the side AB. And so we have shown that in square figures the diagonal is incommensurable in length with the side. Which was to be demonstrated.

SCHOLIVM.

But we shall also prove this same theorem affirmatively, in this way. Since, from what we demonstrated in the scholium to Proposition 47 of Book 1, the square described on the diagonal is double the square described on the side, the square on the diagonal will have to the square on the side the ratio 2 to 1, or 4 to 2, or 8 to 4, and so on. And since, when any number of terms whatever are taken in double proportion starting from uni[ty...]

taken in double proportion from unity—1, 2, 4, 8, 16, 32, 64, and so on—only the third from unity is a square, [Margin: g — *Elements* IX.10] and all the others each skip one over, since the first from unity, namely 2, is not made a square; only these numbers 4, 16, 64 will be squares, while the rest, 2, 8, 32, are not squares. Therefore, since 4 is a square and 2 is not, 4 will not have to 2 the ratio which a square has to a square [Margin: h — *Elements* VIII.24] (otherwise 2 too would be a square). And for that reason neither will the square on the diagonal have to the square on the side the ratio which a square number has to a square number. Therefore the diagonal is incommensurable in length with the side itself. [Margin: i — *Elements* X.9] Which was what was proposed.

We have also demonstrated this theorem otherwise at the end of definition 8 of book 5, in the scholium of proposition 8 of book 8, and in the scholium of proposition 9 of this book.

In the Greek copies, however, a certain appendix is found at this point, the understanding of which depends on the following books of stereometry, so that it might reasonably be omitted. But since it contains a doctrine not to be despised, bearing on the commensurability and incommensurability of all magnitudes, I have thought it well to explain it briefly, noting in the margin—as is my usual practice—the passages of stereometry needed for the demonstration of what is said here. Here, then, is the appendix.

Once straight lines incommensurable in length have been found, very many other magnitudes—both plane and solid—will be found to be incommensurable with one another. For let A and C be straight lines incommensurable with one another in length, and let B be the mean proportional between them. Then, since by the corollary of VI.20 as A is to C, so is any rectilinear figure constructed on A to a rectilinear figure similar and similarly situated on B, and since A and C are incommensurable in length, [Margin: k — *Elements* X.10] those rectilinear figures on A and B will be incommensurable.

Diagram. Three parallel horizontal line segments, labelled from top to bottom, illustrating the incommensurable lines and their mean proportional.

- A (top segment): the first straight line
- C (middle segment): the second straight line, incommensurable in length with A
- B (bottom segment): the mean proportional between A and C

Again, since circles whose diameters are A and B have the ratio which the squares described on A and B have, [Margin: l — *Elements* XII.2] and these squares, being similar rectilinear figures similarly situated, have the ratio which A has to C by the corollary of VI.20, [Margin: m — *Elements* V.11] the circles of the diameters A and B will also have the same ratio as A to C. But A and C are incommensurable in length. [Margin: n — *Elements* X.10] Therefore the circles of the diameters A and B are incommensurable as well.

Now if solids are constructed—pyramids, that is, or prisms of the same height, whose bases are similar rectilinear figures similarly described on A and B— [Margin: o — *Elements* XII.5, 6, or 7] the pyramids and prisms will have the same ratio as their bases, that is, as those rectilinear figures. Therefore, since those rectilinear figures are incommensurable, as has just been demonstrated, [Margin:

p — *Elements* X.10] the solids themselves—the pyramids and prisms—will also be incommensurable.

But if cones or cylinders of equal height are built whose bases are the circles of the diameters A and B, [Margin: q — *Elements* XII.11] such cones and cylinders will have the same ratio as their bases, that is, as those circles. Since, then, the circles have been shown to be incommensurable, [Margin: r — *Elements* X.10] the said cones and cylinders will also be incommensurable. Thus there have been found not only lines and surfaces that are incommensurable, but also bodies, or solids, that are incommensurable. Which was what was proposed.

By the same reasoning, if the straight lines A and B are commensurable in length, [Margin: s — *Elements* X.10] we shall show that their similar rectilinear figures, similarly described, and likewise their circles, are commensurable, [Margin: t — *Elements* XII.5, 6, 7 & 11] and hence also their pyramids and prisms, and finally their cones and cylinders—provided only that they have the same height, as is evident.

FINIS ELEMENTI DECIMI.

EVCLIDIS ELEMENTVM VNDECIMVM

Et Solidorum primum.

(Euclid's Eleventh Element, and the First of the Solids.)

DEFINITIONES.

I.

A solid is that which has length, breadth, and thickness.

In the previous six books Euclid discussed at length that part of geometry which deals with plane figures, and which has taken the name "geometry" as though it were its own; in the three books that followed he carefully taught those properties of numbers that seemed necessary for understanding the doctrine of commensurable and incommensurable lines, a doctrine which he then set out most clearly in the tenth book, so that the construction and nature of the five regular solids—treated most subtly in the last three books, namely in 13, 14, and 15—might be more fully understood. Now at last, in this eleventh book, he takes up that part of geometry which considers bodies, or solids, and which by its proper name is called stereometry.

First, as is his custom, he explains the terms most necessary to the subject, the first of which is "solid," or "body"—namely the third kind of quantity. He teaches, then, that the quantity which, besides length and breadth, has also received thickness or depth, so as to possess three dimensions—one along its length, another along its breadth, and a third along its depth, thickness, or height—is called a solid or body; just as the quantity having only length is called a line, and that which adds breadth to length is called a surface, as we said in the first book.

Furthermore, just as mathematicians, so that we may rightly grasp a line, bid us imagine some point moved from place to place—for this describes a trace that is merely long, that is, a line, since a point is wholly without magnitude—and, so that we may perceive a surface, advise us to imagine some line moved crosswise—for this will describe a trace that is only long and broad: long because of the length of the line, broad because of that crosswise motion, but without depth, since the line too lacks it—so likewise, to set a body or solid before our eyes (that is, a quantity endowed with three dimensions), they counsel us to conceive some surface raised uniformly, or moved crosswise. For in this way there will be described a trace that is long, broad, and deep: long and broad because of the surface, which is long and broad, and deep or thick because of that raising, or motion, of the surface. This quantity, then, is called a solid or body. Nor can any fourth quantity be found distinct from line, surface, and body, since only a threefold dimension can be assigned—as was clearly demonstrated by us from Ptolemy, in his little book *On the Analemma* [de Anatemmate; recte: *de Analemmate*], in the commentary we wrote on the *Sphere* of Johannes de Sacrobosco.

II.

The boundary of a solid is a surface.

Just as a finite line has points at its ends, while a surface receives lines as its boundaries, as he taught in the first book, so now he says that the boundary of a finite solid is a surface. For since a solid or body is produced by that imaginary motion of a surface, it is evident that its outermost parts are surfaces.

III.

A straight line is at right angles to a plane when it makes right angles with all the straight lines that touch it and that lie in the plane in question.

When a straight line AB stands on a plane CD in such a way that the point B is elevated, outside the plane CD, while the point A is in the plane itself, then finally it is called a straight line—or perpendicular—to the plane CD, when it makes right angles with the lines AE, AF, AD, AG, and with all the others that, lying in the same plane, touch it at the point A. For in this way it comes about that AB stands equally on the plane CD, and inclines no more to one side than to another. For when the straight line DA is produced to H, since the angle BAD is assumed to be right, the adjacent angle BAH will likewise be right, and therefore equal to it. And by the same reasoning, any line whatever drawn out in the plane CD will make with AB two equal angles. And for that reason it will stand evenly on the plane itself. But if it could happen that AB failed to make a right angle with some one line drawn from A in the plane CD, even though it made a right angle with all the others, AB would not be called perpendicular to the plane CD. And so, when some line is granted to be perpendicular to a plane, it must also be granted that it makes right angles with all the lines drawn in the same plane that touch it. And conversely, in order that it may rightly be concluded that some line is perpendicular to a given plane, it must first be demonstrated that it makes right angles with all the lines drawn in the same plane that touch it.

Diagram. A parallelogram representing the plane CD, drawn in perspective, with a straight line rising perpendicularly from a point A in the plane to an elevated point B above it; several lines radiate from A to points on and within the plane.

- B (elevated above the plane, on a short dashed vertical): the raised end of the perpendicular AB
- A (centre of the parallelogram): foot of the perpendicular, in the plane
- C (upper-left corner of the plane)
- G (upper-right corner of the plane)
- D (lower-right corner of the plane)
- F (lower-left corner of the plane; the glyph is printed in an angular form resembling Γ/T) [?]
- E (on the left edge of the plane)
- H (between C and A, interior)
- Lines from A are drawn to E, F, D, G (and to C and H), all touching the plane at A

For a straight line standing on any plane to make right angles with the lines that touch it, it is enough that it make right angles with two lines that do not form a single line but cut one another—if they are produced. For from this it is rightly inferred that it makes right angles with all the others as well, as will be clear

from proposition 4 of this book. I said "with two that do not form a single line," because, if two lines compose a single straight line, it does not necessarily follow that a line making right angles with them makes right angles with all the others. For let the straight line AB stand on the plane CD and make right angles with two lines AE, AF drawn in the same plane, which together compose a single straight line. I say that AB does not on that account make right angles with all the others, and therefore is not perpendicular to the plane CD. This is evident, since AB can be turned about EF as an axis, so that the point B is now nearer the plane, now farther from it, and yet always makes the same right and equal angles with EF. But if that same AB makes right angles with two lines that compose an angle—such as AG, AD in the earlier figure—then at last it will make right angles with all the others, as we have said. This will be demonstrated clearly in proposition 4 of this book.

Diagram. A rectangle representing the plane CD, with a straight line BA drawn perpendicular from an elevated point B down to the centre A, and a straight line EAF drawn horizontally across the rectangle through A (the axis EF about which AB is turned).

- B (elevated above the top edge): raised end of the line BA
- A (centre): foot of the perpendicular
- C (upper-left corner of the rectangle)
- E (midpoint of the left side)
- F (midpoint of the right side)
- D (lower-right corner of the rectangle)
- EF: the axis; AE and AF compose one straight line

IV.

A plane is at right angles to a plane when the straight lines drawn in one of the planes, at right angles to the common section of the planes, are at right angles to the remaining plane.

Let the plane AB stand on the plane CD in such a way that the line AQ is elevated—that is, outside the underlying plane CD—while EB lies in that same plane. Let the common section of these planes be EB, which, as will be demonstrated in proposition 3 of this book, will be a straight line. On this line let any number of points G, I be taken; and from them, in the plane AB, let GF, IH be drawn perpendicular to the common section EB. If, then, the lines FG, HI are perpendicular to the other plane CD—that is, if they make right angles with the lines GK, GL, GM; IN, IO, IP, and with all the other lines drawn in the plane CD

from the points G and I—the plane AB will be said to be at right angles to the plane CD; because in this way it will incline no more to one side than to the other, but will stand evenly upon it. For if KG is produced to R, since the angle FGK is assumed to be right, the adjacent angle FGR will also be right, and therefore equal to it; and by the same reasoning, when any other line is drawn out in the plane CD, the angles made on both sides by the lines FG, HI will be equal. Therefore the plane AB will stand evenly on the plane CD. Accordingly, whenever some plane is granted to be perpendicular to another plane, it must also be granted that the lines drawn perpendicular to the common section in one of them are likewise perpendicular to the other plane. And conversely, in order to conclude that some plane is perpendicular to another, it must first be shown that the lines drawn perpendicular to the common section in one of them are perpendicular to the remaining plane.

Diagram. Two planes shown in perspective: a horizontal parallelogram (the plane CD) with an upright plane (the plane AB) rising along their common section EB. From points G and I on the section, lines fan out across the base plane, and perpendiculars GF, IH rise in the upright plane.

- A (top): the elevated corner of the upright plane AB (with line AQ)
- Q (upper right): the other elevated corner; AQ is the elevated line
- B (bottom of the section): lower end of the common section EB
- E (on the base, left of centre): upper end of the common section EB
- C (upper-left corner of the base plane CD)
- D (lower-right corner of the base plane CD)
- F (in the upright plane, near A): foot of perpendicular GF on the common section
- H (in the upright plane, on the right): foot of perpendicular IH on the common section [?]
- G (a point on the section EB): from it lines GK, GL, GM run in the base plane; KG produced to R
- I (a second point on the section EB): from it lines IN, IO, IP run in the base plane
- R (interior of the base): point to which KG is produced
- K, L, M (along the base, from G)
- N, O, P (along the base, from I)

V.

The inclination of a straight line to a plane is this: when a perpendicular has been drawn from the elevated end of that straight line to the plane, and a further straight line has been joined from the point which the perpendicular makes in the plane to the end of the proposed line that lies in the same plane—it is, I say, the acute angle contained by the standing line and the joined line.

Let the straight line AB stand on the plane CD, not at right angles but inclined, in such a way that the point B is elevated outside the plane, while the point A is in the plane. Then from B, the elevated end of the straight line AB, let a perpendicular BE be conceived as drawn to the same plane, making the point E in the plane; and from E to A let the straight line EA be joined. The angle BAE will necessarily be acute, [Margin: a — *Elements* I.17] since in the triangle ABE the two angles BAE, BEA are less than two right angles, and the angle BEA is right. Therefore the acute angle BAE, contained by the standing line AB and the joined line AE, is called the inclination of the straight line AB to the said plane CD; so that the inclination of the line AB to the plane CD is said to be as great as the said acute angle BAE.

Diagram. A right-angled figure showing the inclined line AB with its diagonal, the perpendicular BE dropped to the plane, and the joined line AE, the acute angle BAE marking the inclination.

- B (upper-right, elevated): raised end of the inclined line AB
- A (lower-left, in the plane): foot of the inclined line
- E (lower edge, right of A): point where the perpendicular BE meets the plane
- C (upper-left corner)
- D (lower-right corner)
- AB is the inclined (diagonal) line; BE the perpendicular to the plane; AE the joined line; angle BAE the inclination

VI.

The inclination of a plane to a plane is the acute angle contained by straight lines which, drawn in each of the planes to the same point of the common section, make right angles with the section.

(Continuation of the commentary on Definition VI — the inclination of a plane to a plane / dihedral angle.)

For let the plane CD stand upon the plane AB, so that the part toward C is elevated above the plane AB while the part toward D lies in the plane AB itself; and let CD be inclined to AB, and let DE be their common section. If then from a point F on the common section DE two perpendiculars are drawn — FG in the plane AB, and FH in the plane CD — the acute angle GFH contained by these straight lines will be called the inclination of the plane CD to the plane AB, so that the inclination of plane to plane is said to be as great as the said acute angle.

Sometimes geometers call the angle GFH the angle of inclination of the plane CD to the plane AB, whether it be acute, right, or obtuse — although properly only the acute angle exhibits the inclination of one plane to another, as Euclid teaches in this place. For a right angle indicates rather an even elevation of one plane above another than an inclination of the one to the other; and an obtuse angle indicates a kind of leaning-away of the one from the other. Nevertheless, since each of these angles shows the position, or the elevation, of one plane above another, most writers have generally called "the angle of inclination" any angle whatever that is contained by two straight lines drawn perpendicular in each of the two planes to some one common point of their section — such as the angle GFH, whether it be right, acute, or obtuse.

Diagram. The dihedral-angle figure illustrating Definition VI. An upper, tilted parallelogram (the plane CD) leans over a lower parallelogram (the plane AB); the two planes meet along the common section DE. From the point F on that section, one perpendicular FG is drawn in the plane AB and another FH in the plane CD, and the acute angle GFH is the inclination.

- C (upper-left corner of the tilted plane CD)
- H (upper-right corner of plane CD)
- A (left corner of the lower plane AB)
- G (interior foot of the perpendicular FG, in plane AB)
- E (left end of the common section DE)
- F (point on the common section from which both perpendiculars rise)
- D (right end of the common section DE)
- B (lower-right corner of plane AB)

VII.

One plane is said to be inclined to a plane similarly as another to another, when the said angles of inclination are equal to one another.

This definition is not obscure, once the preceding one is well understood. Hence one plane will be said to be more inclined to a plane when its angle of inclination is smaller, and so has receded further from a right angle.

VIII.

Parallel planes are those which do not meet one another.

Understand this of their being produced in any direction whatever, even infinitely: that is, whether to the right or to the left, and whether upward or downward, and so on. For there are some planes which meet neither when produced to the right nor to the left, yet are not on that account to be called parallel, because when drawn out upward or downward they eventually come together. For if some roof is taken whose ridge is imagined to be cut away, the two remaining planes do not meet, however far they are produced to the right and to the left; but since, extended upward, they do come together — namely at the ridge itself — they will not be called parallel.

For let there be some roof ABCDEF, contained by the two planes ABED, ACFD, whose base is BCFE; and let the ridge AGHDKI be removed. This done, it is clear that the remaining planes GBEK, HCFI do not meet one another if they are produced toward the parts GB, HC or toward the parts KE, IF; yet they are not on that account called parallel, since when produced toward the parts GH, KI they meet at the ridge AD. Therefore, in order that any planes be called parallel, it is necessary that, produced in no direction, do they meet one another.

Diagram. The "roof" (triangular-prism) figure illustrating Definition VIII. A wedge-shaped solid ABCDEF is drawn whose ridge has been cut away along AGHDKI, leaving the two upper planes GBEK and HCFI; the base is BCFE.

- A (near apex of the roof, upper-left)
- G (point on the cut, just below A)
- D (far apex of the roof, upper-right)
- K (point on the cut, just below D)
- H (left, on the cut edge)
- I (center, on the cut edge)
- B (near base vertex, center-low)
- E (far base vertex, right; faint — [?])
- C (base vertex, lower-left)
- F (base vertex, lower-center)

But just as lines lying in the same plane can be produced in only two directions — namely to the left and to the right — so two planes can be conceived as extended in four directions: leftward, rightward, upward, and downward. Accordingly any two planes are either parallel, or, produced in some direction, come together; whereas straight lines are either parallel, or, produced in one direction, meet — provided they lie in the same plane — or else are neither parallel nor ever meet, namely if they lie in different planes and are set crosswise. Hence in the definition of parallel lines in Book I this clause was added, "which, being in the same plane...", whereas here it is stated simply, "which never meet."

IX.

Similar solid figures are those which are contained by similar planes equal in number.

Euclid could have defined similar solid figures by equal angles and by the proportionality of the sides about the equal angles, as he did in defining similar plane figures. Yet he judged it better, consulting brevity, to describe them by means of similar planes equal in number — especially since from the similarity of the planes there follows at once the equality of the solid angles, as will be clear from Definition 11, since those angles are constituted by plane angles equal both in number and in magnitude; and the proportionality of the sides about the equal angles follows from the similarity of the planes.

X.

Equal and similar solid figures are those which are contained by similar planes equal in number and in magnitude.

For if the similar planes by which similar bodies are circumscribed (according to the preceding definition) are equal, each to each, such solid figures will be called not only similar but also equal. For if such solids are conceived in the mind as mutually penetrating one another, neither will exceed the other, because of the equality and similarity of the planes. For from the similarity of the planes there follows the equality of the solid angles, as we taught in the preceding definition; and from the equality of those same planes there follows the equality of the proportional sides, as we demonstrated in the lemma to Proposition 22 of Book 6. Therefore those solids will coincide with one another in every part, and so will be equal to one another.

XI.

A solid angle is the inclination, toward all the lines, of more than two lines which touch one another and are not in the same surface.

Two lines lying in the same plane which, not lying in a straight line, meet at one point make a plane angle, as we set out in Book I — such as the angle BAC contained by the two lines AB, AC set in the same plane. If now some third line AD is added which does not lie in the same plane with them, but the point D is elevated outside their plane, a solid or corporeal angle will be produced at the point A. The same would happen if a fourth line, or a fifth, or finally more were joined, though the size of the angle would vary. Now Euclid said that the lines constituting a solid angle ought not to lie in the same surface, because, namely, if the three lines AB, AC, AD lay in the same surface, no solid angle would be produced at the point A, but only a plane angle composed of the two planes BAD, DAC. But if AD is elevated — that is, outside the plane in which AB, AC lie — then the whole plane angle will no longer be composed of BAD, DAC; rather, about the point A three plane angles will stand, BAD, DAC, CAB, which, as will presently be said, constitute a solid angle. The same you will say if there are more lines, and therefore more plane angles too.

Diagram. The solid-angle figure. From the vertex A three straight lines descend to the points B (left), D (center), and C (right); the three plane angles BAD, DAC, CAB about A make up the solid angle.

- A (vertex, at the top)
- B (lower-left endpoint)
- D (lower-center endpoint)
- C (lower-right endpoint)

ALITER (*Alternatively*).

A solid angle is that which is contained by more than two plane angles not lying in the same plane, but set at one point.

This second definition of a solid angle is easy, once the preceding one is rightly understood. For since at least three lines not lying in the same plane are necessary for the constitution of a solid angle, it is clear that at a minimum three plane angles are required for a solid angle to be produced, as appears from the figure above. For the angle BAD lies in the plane in which the straight lines AB, AD lie; the angle DAC in the plane in which AD, AC lie; and finally the angle CAB in the plane in which AC, AB lie; and thus from these a solid angle is constituted

at the point A. But if three or more plane angles stand at one point, yet in the same plane, no solid angle will be constituted from them, but a certain whole plane angle, as we said above concerning lines in the same surface. Nor indeed are two plane angles sufficient to constitute a solid angle, even if they lie in different planes; for it will always gape open on one side. Hence it is necessary that at least a third surface be added, in which a third plane angle may stand. Moreover, the clearest example of a solid angle is furnished by two walls together with the floor of a house, or with the ceiling, meeting at one point. For at the very point where they come together, a solid angle is constituted from three plane angles. And here it is manifest that if one of these surfaces is removed, the solid angle is destroyed, and there remain only two surfaces inclined to one another and gaping open.

From this it will be clear to anyone that those solid angles are equal to one another which are contained by plane angles equal both in number and in magnitude. For such angles will coincide with one another if they are conceived as mutually penetrating. And just as in Book I Euclid, out of all plane angles, assumed only the rectilinear, so here, out of all angles, he defines only that one which more than two straight lines, or more than two rectilinear plane angles, comprehend. For by this definition the angle of a cone is not comprehended, which is contained by a single curved surface; nor that which is contained by two surfaces, one indeed plane and the other curved — such as the one that arises if a cone is cut through the vertex.

XII.

A pyramid is a solid figure contained by planes, set up from one plane to one point.

That is, solid figures set up from the planes ABC, ABCD, ABCDE to the points D, E, F are called pyramids. It is clear that all the planes by which a pyramid is contained are triangles, since they all converge to one point — except the plane from which they all rise, which is opposite to that point. For this can be a triangle, or a quadrilateral, or a pentagon, and so on; and from it the whole pyramid takes its name, so that it is called a triangular, quadrangular, pentagonal, or hexagonal pyramid, and so on. For each pyramid is comprehended by as many triangles as the said plane has angles or sides. Hence such a plane is usually called the base of the pyramid. Thus of a triangular pyramid, besides the base, the triangles are ABD, ACD, BCD; of a quadrangular one, ABE, ADE, BCE, CDE; and finally of a pentagonal one, ABF, AEF, BCF, CDF, EDF.

Diagram. Three pyramids illustrating Definition XII, drawn side by side.

- **Triangular pyramid:** apex D over the triangular base ABC — A (bottom-left), B (bottom-right), C (back, interior).
- **Quadrangular pyramid:** apex E over the quadrilateral base ABCD — D and C (upper back edge), A (bottom-left), B (bottom-right).
- **Pentagonal pyramid:** apex F over the pentagonal base ABCDE — E (left), D (center), C (right), A (bottom-left), B (bottom-right).

XIII.

A prism is a solid figure contained by planes, of which two opposite ones are equal, similar, and parallel, but the others are parallelograms.

That is, solid figures whose opposite planes — equal, similar, and parallel — are triangles ABC, DEF; or quadrilaterals ABCD, EFGH; or pentagons ABCDE, FGHJK, and so on; while the parallelograms are ACFD, ABED, CBEF; or ABFE, ADHE, CDHG, CBFG; or ABGF, AEKF, DEKI, DCHI, BCHG — these are called prisms. And so a prism will be nothing but a certain many-sided column of equal —

— thickness, whose opposite bases are equal, similar, and parallel, whether these be triangles, or quadrilaterals, or pentagons, and so on. Hence any prism will contain as many parallelograms as there are sides, or angles, found in each of the opposite planes, as the figures indicate.

Diagram. Three prisms illustrating Definition XIII, drawn one above another.

- **Triangular prism (top):** two triangular ends and three parallelogram faces. Vertices B (lower-left) and C (front-center) and the near apex A ([?], faint) at the left end; D (upper-right), F (right), E (lower-right) at the far end.
- **Quadrangular prism / box (middle):** near rectangle ABCD — A (upper-left), B (left), C (lower-left), D (center) — and far rectangle EFGH — E (upper-right), F (right), G (lower-right), H (far-right).
- **Pentagonal prism (bottom):** near pentagon ABCDE — A (upper-left), B (left), C (lower-left), D (bottom), E (center) — and far pentagon FGHJK — F (upper-right), G (right), H (lower-center), I (lower-right; [?]), K (far-right).

From this it is clear how greatly they are mistaken — among whom is Campanus too — who understand a prism to be only that solid figure whose two opposite planes, parallel, equal, and similar, are triangles, while the remaining three are parallelograms. This solid figure Campanus [Campanus of Novara] calls a "saw-shaped body" [Corpus serratile]. For since this figure is only one species of prism,

whereas the definition given embraces infinitely many other kinds of prism, as is evident — especially since very many demonstrations in this book and in the following (Book 12) apply equally to all prisms — it is clear that those are mistaken who gather from this definition only one kind of prism. These authors perhaps took occasion for their error from the fact that Euclid, both in the last proposition of this book and in propositions 3, 4, and 5 of Book 12, speaks only of that prism which has its two opposite planes triangles. But they did not notice that the seventh proposition of Book 12 is opposed to them, in which Euclid, speaking of a prism, makes mention of two opposite triangles because that demonstration applies only to such a prism. For if that figure alone were called a prism by Euclid, as they wish, he would have said in vain, "Every prism having a triangular base," and so on — since no other prism would exist. Indeed, in the corollary to the said seventh proposition, Theon [Theon of Alexandria] infers that a pyramid is a third part of a prism having the same base with it and the same height, although the two opposite planes of the prism are not triangles, but any other rectilinear figures, equal however and similar, as the definition requires. From which passage it is gathered still more clearly that there are infinitely many kinds of prism according to Theon, and therefore according to Euclid, since by the common opinion of all, the demonstrations ascribed to Theon are Euclid's. There is this too, that by Theon, in the demonstration of Proposition 10 of Book 12, all solids are most plainly called prisms which have two opposite planes equal — whether these be triangles and squares, or many-sided — and the rest parallelograms.

XIV.

A sphere is [the figure that results] when, the diameter remaining fixed, a semicircle carried around is revolved back again into itself, to the place from which it had begun to move, the figure so described being taken in.

Just as a straight line, revolved about one of its ends which remains at rest, describes a circle, so also a semicircle, carried about one of its ends — namely about its diameter — describes the figure which geometers call a sphere. Hence, just as in a circle a point is assigned — namely that resting end — from which all the straight lines falling on the circumference are equal, because they are all equal to that revolved line: so also in a sphere a point is found — namely the middle of the resting diameter, that is, the center of the revolved semicircle — from which all the straight lines falling on the surface are equal, because they are all equal to the semidiameter of the said semicircle. Therefore, in likeness to the definition of a circle, a sphere can also be defined in this way:

A sphere is a solid figure comprehended by one surface, such that all the straight lines falling upon it from one point of those placed within the figure are equal to one another.

Cicero, in his book *On the Universe*, speaking of the world, elegantly expressed this definition in these words: "It was therefore fashioned round — which the Greeks call σφαίροειδὲς [Greek: σφαίροειδής (sphairoeidēs, "sphere-shaped")] — whose every extremity is reached from the middle by equal radii."

XV.

The axis of a sphere is that fixed straight line about which the semicircle is turned.

XVI.

The center of a sphere is the same as that of the semicircle.

XVII.

The diameter of a sphere is a certain straight line drawn through the center and terminated on both sides by the surface of the sphere.

These three definitions need no exposition, provided only this one thing be noted: that every diameter of a sphere can be an axis, if namely the sphere is revolved about it. Hence, because in the description of the sphere the motion of the sphere itself was made about the diameter of the semicircle, Euclid called that alone the axis of the sphere.

XVIII.

A cone is [the figure generated] when, one of the sides of a right triangle containing the right angle remaining fixed, the triangle is revolved about it in itself and returns to where it began to move, the figure being taken all around.

And if the revolved straight line — the one containing the right angle — is equal to the fixed side, the cone will be right-angled [Orthogonius]; if smaller, obtuse-angled [Amblygonius]; if larger, acute-angled [Oxygonius].

Commentary (Clavius). For example, if the right triangle ABC is revolved about the stationary side AB — the side containing the right angle — until it completes a full revolution, a solid figure is described. It is bounded by two surfaces: one circular and flat, which the other side about the right angle, BC, describes by its motion;

and another curved and convex, which the side AC, opposite the right angle, traces out. This solid figure is called a cone. Campanus, however, calls it a "round pyramid."

Diagram. Three cones drawn side by side, each shown as an apex above an elliptical base, illustrating the right-angled, obtuse-angled, and acute-angled cones respectively (differing in the height-to-base proportion). In each figure the apex is A, the base is drawn as an ellipse, and the base and generating triangle are lettered identically.

- A (apex, top of each cone): vertex of the cone
- C (left end of base ellipse): one endpoint of the base diameter
- D (right end of base ellipse): the other endpoint of the base diameter
- B (front/bottom of base ellipse): foot of the generating side BC

Now if the stationary side AB is equal to the revolved side BC, as in the first figure, the cone so described is called right-angled [Greek: ὀρθογώνιος (orthogōnios), "right-angled"], because the angle at the vertex A is a right angle. For when the sides AB and BC are taken as equal, the angles BAC and BCA will also be equal [Margin: a. I.5]; and since together they equal one right angle (because ABC is a right angle), angle BAC will be half a right angle. In the same way, on the opposite side, angle BAD will be half a right angle. Therefore the whole angle CAD will be a right angle.

But if the stationary side AB is smaller than the revolved side BC, as in the second figure, the cone described will be called obtuse-angled [Greek: ἀμβλυγώνιος (amblygōnios), "obtuse-angled"], because the angle at the vertex A is obtuse. For since side BC is greater than side AB, angle BAC will be greater than angle BCA [Margin: b. I.18]. And since these two together equal one right angle (because angle ABC is taken as a right angle), BAC will be greater than half a right angle. Likewise BAD will be greater than half a right angle; and so the whole angle CAD will be greater than a right angle.

Finally, if the stationary side AB is greater than the revolved side BC, as in the third figure, the cone described will be called acute-angled [Greek: ὀξυγώνιος (oxygōnios), "acute-angled"], because the angle at the vertex A is acute. For since side AB is greater than side BC, angle BCA will be greater than angle BAC [Margin: c. I.18]. Therefore, since these two together equal one right angle (because angle ABC is taken as a right angle), BCA will be greater than half a right angle, and consequently the remaining angle BAC will be less than half a right angle. In the

same way angle BAD will be shown to be less than half a right angle. Therefore the whole angle CAD will be less than a right angle.

XIX.

The axis of a cone is that stationary line about which the triangle is revolved.

Commentary (Clavius). Namely, in each of the cones described above, the axis is the stationary straight line AB.

XX.

The base of a cone is the circle described by the revolved straight line.

Commentary (Clavius). That is, the circle described by the other side about the right angle — such as BC in the figures above. For the radius of this circle is the very side that is carried around.

Euclid, however, defined only the right cone, whose axis is perpendicular to the base — not the oblique one, whose axis is not perpendicular to the base. It seemed best, following Apollonius of Perga [Apollonio Pergaeo], to bring in the general description of a cone that covers both the right and the oblique kind. This is because in Book 12 nearly the same theorems can be demonstrated for the oblique cone as Euclid demonstrated for the right one, as we shall show there. Apollonius writes as follows, at the beginning of his *Elements of Conics*:

If from some point a straight line is drawn to the circumference of a circle that does not lie in the same plane as the point, and is extended in both directions; and if, with the point held fixed, the line is carried around the circumference of the circle until it returns to the position from which it began to move: then the surface described by this straight line — a surface made up of two surfaces fitted together at the vertex, each of which increases without limit (since the straight line describing it is produced without limit) — I call a **conic surface**.

Its **vertex**, the fixed point. Its **axis**, the straight line drawn through the point and the center of the circle. A **cone** I call the figure contained by the circle and by that part of the conic surface which lies between the vertex and the circumference of the circle. The **vertex of the cone**, the point that is also the vertex of the conic surface. Its **axis**, the straight line drawn from the vertex to the center of the circle. Its **base**, the circle itself. Among cones I call those **right** whose axes stand at right angles to their bases.

And **scalene**, those whose axes do not stand at right angles to their bases.

Commentary (Clavius). Eutocius explains all this as follows, in the commentary he wrote on Apollonius. Let AB be a circle with center C, and let D be some point raised above it; join DB and produce it without limit in both directions to the points E and F. If the straight line DB is then carried along the circumference of circle AB until the point B is restored to the place from which it began to move, it will describe a surface consisting of two surfaces that touch one another at the point D. Apollonius calls this a conic surface, and it too increases without limit, since the straight line DB that describes it is produced without limit. He calls the point D the vertex of the surface, and the straight line DC its axis. A cone he calls the figure contained by the circle AB and by that surface which DB alone describes. The vertex of the cone is the point D; its axis, DC; its base, the circle AB. And if DC is perpendicular to the circle, he calls the cone right; if not, scalene.

Diagram. Two figures illustrating Eutocius's account of the double-napped conic surface. In each, a horizontal circle (drawn as an ellipse) has diameter A–B and center C; a point D stands above the circle; the line through D and B is produced upward to E and downward to F, and as DB is carried around the circle it sweeps out one nappe below D (down to the circle and toward F) and the mirror nappe above D (up toward E). The left figure shows the right (perpendicular) case; the right figure the oblique case.

- E, E (upper ends of the generating lines, above D): points on the upper nappe
- D (crossing point above the circle): vertex of the conic surface
- A (left end of the circle's diameter): endpoint of diameter AB
- C (center of the circle): foot of the axis DC
- B (right end of the circle's diameter): endpoint carried around to generate the surface
- F, F (lower ends of the generating lines, below the circle): points on the lower nappe

XXI.

A cylinder is [the figure generated] when, one of the sides of a right-angled parallelogram containing the right angle remaining fixed, the parallelogram is revolved about it and returns again into itself, back to where it began to move, the figure being taken all around.

Commentary (Clavius). For example, if the rectangle ABCD is revolved about the stationary side AB until it completes a full revolution, the figure described will be called a cylinder. It is bounded by three surfaces: two flat circular ones, which the sides AD and BC describe; and one curved surface, which the side CD describes, like that of some round column. This is why Campanus called the cylinder a "round columnn."

Diagram. An upright cylinder generated by revolving rectangle ABCD about the side AB. The top face is drawn as an ellipse with left end D, top-center A, and right end E; the bottom face as an ellipse with left end C, bottom-center B, and right end F; the vertical axis runs from A down to B.

- D (top face, left): endpoint of the upper radius / top of side AD's circle
- A (top face, center): upper end of the axis (fixed side)
- E (top face, right): endpoint of the upper circle
- C (bottom face, left): endpoint of the lower circle
- B (bottom face, center): lower end of the axis
- F (bottom face, right): endpoint of the lower circle

XXII.

The axis of a cylinder is that stationary straight line about which the parallelogram is revolved.

Commentary (Clavius). Namely, the straight line AB in the figure above is called the axis of the cylinder.

XXIII.

The bases of a cylinder are the circles described by the two opposite sides that are carried around.

Commentary (Clavius). Such as the circles described by the opposite sides AD and BC. Euclid here defines only the right cylinder — the one whose axis is perpendicular to both bases — and in Book 12 he intends to discuss only this kind. Since we ourselves intend, in that same Book 12, to set out from other geometers nearly the same theorems concerning the oblique cylinder that Euclid demonstrated for the right cylinder, it seemed best first to define the cylinder in general, following Serenus of Antinoë [Serenus Antinsensis], so as to include both the right and the oblique kind. Serenus writes as follows, at the beginning of the first book of his work *On the Section of the Cylinder*:

If the diameters of two equal and parallel circles — themselves always remaining parallel to one another and lying in the planes of the circles — are carried around their fixed centers, and at the same time a straight line joining the ends of the diameters on the same side is carried around with them, until it returns to the place from which it began to move: then the surface described by the moving straight line is to be called a **cylindrical surface**. This too can be increased without limit, when the describing line is produced without limit.

A **cylinder** is the figure contained by the parallel circles and by the cylindrical surface lying between them. The **bases** of the cylinder are those circles themselves. The **axis**, the straight line drawn through the centers of the circles. A **side** of the cylinder, a line that, being straight and lying on the surface of the cylinder, touches both bases — the same line which, carried around, describes the surface of the cylinder, as we said earlier. Among cylinders, those are called **right** whose axis stands at right angles to their bases, and **scalene**, those whose axis does not stand at right angles to their bases.

Let AB and CD be two equal and parallel circles, with centers E and F, and with their diameters AB and CD likewise parallel. Join AC and produce it without limit in both directions to the points G and H. Now if the diameters AB and CD are carried around their centers E and F in the planes of the circles, always remaining parallel, together with the straight line AC, until they return to the place from which they began to move: then the line AC will describe a certain round surface, which Serenus calls cylindrical. This too increases without limit, since the straight line AC that describes it is produced without limit. He calls a cylinder the figure ABDC, contained by the circles AB and CD and by the cylindrical surface lying between them — the surface that AC alone describes. He says the bases of the cylinder are the circles AB and CD; the axis, the straight line EF, which joins the centers of the circles; the side of the cylinder, the straight line AC. And if the axis EF is perpendicular to the bases, he calls the cylinder right; if not, scalene.

Diagram. Two cylinders, a right one and an oblique (scalene) one, drawn side by side. In each, the upper base is an ellipse with left end C, center F, right end D; the lower base an ellipse with left end A, center E, right end B; the side-lines run from the upper toward G above and from the lower toward H below, with EF as the axis. The right cylinder is upright; the second is slanted.

- G, G (top, above the upper base): upper ends of the produced side-lines
- C (upper base, left): endpoint of the upper diameter
- F (upper base, center): upper end of the axis

- D (upper base, right): endpoint of the upper diameter
- A (lower base, left): endpoint of the lower diameter
- E (lower base, center): lower end of the axis
- B (lower base, right): endpoint of the lower diameter
- H, H (bottom, below the lower base): lower ends of the produced side-lines

XXIV.

Similar cones and cylinders are those whose axes and whose base-diameters are proportional.

Commentary (Clavius). Euclid rightly derives the similarity of cones and cylinders from their axes and their base-diameters. The base-diameters show that cones and cylinders are similar in two dimensions — length and breadth — while the axes show them to be similar in depth, or height. For let ABC and EFG be two cones, and likewise two cylinders; in the cones let the axes be AD and EH, and the base-diameters BC and FG, and similarly in the cylinders. Then if the axis AD is to the axis EH as the base-diameter BC is to the base-diameter FG, both the cones and the cylinders will be called similar to one another; for on this account they are similar in length and breadth as well as in depth, as we said — that is, all three of their dimensions are proportional.

Diagram. An upright cone, a second upright cone, and two upright cylinders, drawn in a row to illustrate similar right cones and cylinders. The first cone has apex A over a base with endpoints B and C and axis-foot D; the second cone has apex E over a base with endpoints F and G and axis-foot H. The two cylinders repeat the same lettering (axes AD and EH, base-diameters BC and FG).

- A (apex of the first cone / top of the first cylinder): vertex
- B (base endpoint, first cone): end of base-diameter BC
- C (base endpoint, first cone): end of base-diameter BC
- D (center of first base): foot of axis AD
- E (apex of the second cone / top of the second cylinder): vertex
- F (base endpoint, second cone): end of base-diameter FG
- G (base endpoint, second cone): end of base-diameter FG
- H (center of second base): foot of axis EH

Or again, [they may be shown similar] because the triangles ADB and EHF, by whose revolution the cones were described, and likewise the rectangles AB and EF, whose rotations produced the cylinders, will on this account be similar to one another. For since AD is to EH as BC is to FG, so also half of BD is to half of FH

[Margin: a. V.15]; therefore AD is to EH as DB is to HF, and, alternately, AD is to DB as EH is to HF. Hence in both the triangles and the rectangles the sides about the equal angles — namely the right angles — are proportional; and therefore both the triangles and the rectangles will be similar to one another. The triangles, because triangles that have one angle equal to one angle and the sides about the equal angles proportional are equiangular [Margin: b. VI.6], and therefore have all the sides about their equal angles proportional [Margin: c. VI.5], and so are similar to one another. The rectangles, because the two remaining sides are equal to the two sides AD and DB, opposite to opposite [Margin: d. I.34], and therefore have the same proportion as these.

But if the cones or cylinders are oblique (for Euclid spoke only of right ones), they will be called similar only when their axes and their base-diameters are proportional, and the angles of inclination formed by the axes are equal — these angles being taken as was stated in the fifth definition. For let ABC and EFG be oblique cones, and likewise oblique cylinders; in the cones let the axes be AD and EH, and the base-diameters BC and FG, and similarly in the cylinders. Then if, in both the cones and the cylinders, AD is to EH as BC is to FG, and the angles of inclination ADB and EHF are equal to one another in both the cones and the cylinders, then both the cones and the cylinders will be called similar to one another. For on this account their three dimensions are proportional, as we said above; and again both the triangles ADB and EHF and the rectangles AB and EF, whose rotations produced the cones and cylinders, are similar to one another, just as we demonstrated in the case of right cones and cylinders. For the demonstration is exactly the same, since the angles ADB and EHF are taken as equal, even though they are not right angles.

Diagram. Two oblique cones and two oblique cylinders, drawn in a row to illustrate similar inclined cones and cylinders. The lettering follows the upright figure: the first cone has apex A over a base with endpoints B and C and axis-foot D; the second cone has apex E over a base with endpoints F and G and axis-foot H; the two slanted cylinders repeat the same lettering.

- A (apex of the first oblique cone / top of the first oblique cylinder): vertex
- B (base endpoint, first figure): end of base-diameter BC
- C (base endpoint, first figure): end of base-diameter BC
- D (center of first base): foot of axis AD
- E (apex of the second oblique cone / top of the second oblique cylinder): vertex

- F (base endpoint, second figure): end of base-diameter FG [the left base-endpoint of the last cylinder is printed ambiguously — possibly Γ (?)]
- G (base endpoint, second figure): end of base-diameter FG
- H (center of second base): foot of axis EH

XXV.

A cube is a solid figure contained by six equal squares.

XXVI.

A tetrahedron is a solid figure contained by four equal equilateral triangles.

XXVII.

An octahedron is a solid figure contained by eight equal equilateral triangles.

XXVIII.

A dodecahedron is a solid figure contained by twelve equal, equilateral, and equiangular pentagons.

XXIX.

An icosahedron is a solid figure contained by twenty equal equilateral triangles.

These are the five bodies that are called regular, because all the planes by which they are bounded are equal, equilateral, and equiangular, as is clear from their definitions. By some they are called the Platonic bodies, because in the *Timaeus* Plato likens the five bodies of the world — those that the philosophers call the simple bodies, namely Heaven, Fire, Air, Water, and Earth — to these five figures, as we explained more fully in our commentary on the *Sphere* of Johannes de Sacrobosco [Ioannes à Sacro Bosco]. The construction of all of them will be set out in Book 13: that of the cube in Proposition 15, of the tetrahedron in Proposition 13, of the octahedron in Proposition 14, of the dodecahedron in Proposition 17, and finally of the icosahedron in Proposition 16. There the definitions of these bodies will be grasped more plainly and more perfectly, for on a flat surface it is extremely difficult to depict them so that one may take in their true likeness and form. Nevertheless, in the proper places we will hand down thoroughly easy procedures by which anyone may make them up as solid bodies. And no regular body exists besides those five, as we will demonstrate at the end of Book 13.

Now since the parallelepiped is frequently mentioned in this Book 11, and since Book 15 deals with the mutual inscription and circumscription of the regular bodies, we have judged it necessary to explain in three definitions what a parallelepiped is, what it is for a solid figure to be inscribed in a solid figure, and what it is to be described about the same.

XXX.

A parallelepiped is a solid figure contained by six quadrilateral figures, of which the opposite ones are parallel.

Diagram. Four three-dimensional box figures illustrating Definition XXX, arranged in two rows of two. The top pair are right (rectangular) parallelepipeds; the bottom pair are oblique parallelepipeds (rhomboids). Each of the four is lettered with the same eight vertices in the same relative positions.

- F (upper rear left) and E (upper rear right): the two back-top vertices
- A (upper front left) and D (upper front right): the two front-top vertices
- G (lower rear left) and H (lower rear right): the two back-bottom vertices
- B (lower front left) and C (lower front right): the two front-bottom vertices

For instance, take a solid figure comprehended by the six quadrilaterals ABCD, DEFA, ABGF, FGHE, EDCH, HCBG, of which the opposite faces AC and FH, likewise DF and CG, likewise AG and DH, are parallel. Such figures are named parallelepipeds, as being comprehended by parallel planes; and these opposite parallel planes are equal, and similar, and parallelograms, as will be demonstrated in Proposition 24 of this book.

There are as many kinds of parallelepiped as there are of parallelogram. For if the six parallelograms are equilateral and rectangular — that is, squares — that parallelepiped is called a *cube*, and it corresponds to the square among the plane figures. But if the six parallelograms are indeed rectangular, yet not all equilateral but longer on one side, even though two are equilateral, it is called a *parallelepiped longer on one side*. If, however, there are six parallelograms that are equilateral but not all rectangular, even though two are rectangular, that parallelepiped is called a *rhombus*. And if, finally, the six parallelograms are neither all rectangular nor all equilateral — even though two are rectangular and equilateral, or rectangular only, or equilateral only — but are nonetheless parallelograms whose opposite pairs are equal, then such a parallelepiped is called a *rhomboid*. For the rest, any parallelepiped whatever can also be called a *prism*, as is clear from the definition of the prism.

XXXI.

A solid figure is said to be inscribed in a solid figure when all the angles of the inscribed figure are set either in the angles, or in the sides, or finally in the planes of the figure in which it is inscribed.

XXXII.

A solid figure is in turn said to be circumscribed about a solid figure when either the angles, or the sides, or finally the planes of the circumscribed figure touch all the angles of the figure about which it is described.

It is not necessary that the angles of the inner figure all be set in the angles, or the sides, or the planes of the outer figure, since sometimes the outer figure contains more angles, sides, or planes than the inner, and sometimes fewer. It is enough that all the angles of the inner figure touch some angles, or some sides, or finally some planes of the outer figure, so that no angle of the inner figure is left untouched by the angles, or by the sides, or finally by the planes of the outer figure.

These last two definitions will be more clearly grasped from Book 15, in which the mutual inscriptions and circumscriptions of the regular bodies are treated at length.

THEOR. 1. PROPOS. 1.

[Margin: j.]

One part of a straight line is not in a plane of reference while another part is up above.

For suppose, if it be possible, that of the straight line AB the part AC lies in the plane of reference DE, while the part CB is up above, in such a way that all the points of the part AC lie in the plane DE, but all the points of the part CB stand above the plane DE, or below it. Then in the plane DE there will be a certain straight line set in a straight continuation with AC — namely CF. Consequently the two straight lines AB and AF have the common segment AC. This is absurd, as we demonstrated in the 10th axiom [pronunciatum] of Book 1. Therefore one part of a straight line is not in a plane of reference while another part is up above. Which was to be demonstrated.

SCHOLIUM.

This is the common demonstration of the commentators. But perhaps someone will say that something is wanting in it — namely, that it be shown that there is some straight line in the plane DE set in a straight continuation with AC. For whoever grants that of the straight line AB the part AC is in the plane DE while the part CB is up above will certainly deny that another straight line can be given, set in continuation with AC in the plane DE, since he admits that the straight line CB is the one set in a straight continuation with AC [?]. Nevertheless, to one who doubts in this way the answer must be given that any finite straight line given in any plane can be produced further in that same plane in a straight continuation, since a plane surface is one that lies evenly with the lines on it — that is, one that is always extended straight and evenly, just as a straight line is. Therefore if AC is produced in the plane DE as far as F, CF will be set in a straight and direct continuation with AC. But let us now demonstrate this very thing geometrically: that in the plane DE another straight line can be given to AC which forms with it a single straight line — even though someone may say that AB too is a straight line whose part AC is set in the plane DE while its part CB is set up above.

Diagram. A rectangle representing the plane DE, crossed by a horizontal middle line, with a segment rising above it to the right.

- D (top left corner) and the top right corner (unlabelled): top edge of the plane
- G (bottom left corner) and E (bottom right corner): bottom edge of the plane
- A (left end of the middle line), C (middle point), F (right end of the middle line): the straight line lying in the plane
- B (above and to the right of F): the raised end of segment CB

[Margin: a — *Elements* I.11] For let CG be drawn in the plane DE perpendicular to AC; and in the same plane let CF be drawn perpendicular to CG. I say that CF is set in a straight and direct continuation with AC. For since AC and FC, in the same plane DE, make with the straight line CG two right angles ACG and FCG by construction, [Margin: b — *Elements* I.14] AC and FC will be set in a straight and continuous line. Therefore, even if someone should contend that AB is a straight line whose part AC is set in the plane of reference DE while its part CB is set up above, nonetheless in the same plane DE there will be another straight line — namely CF — which forms one straight line with AC; and so the two straight lines AB and AF will have the common segment AC. Which is absurd.

Diagram. A second rectangle representing the plane DE, with a middle line and a segment rising to the upper right, showing the perpendiculars of the geometric proof.

- D (top left corner) and B (top right corner)
- A (left end of the middle line), C (middle point), F (right end of the middle line)
- G (bottom, near the middle) and E (bottom right corner)

From this it follows that all the parts of any straight line whatever — that is, the whole straight line — exist in one plane.

THEOR. 2. PROPOS. 2.

[Margin: ij.]

If two straight lines cut one another, they are in one plane; and every triangle is in one plane.

Let the straight lines AB and CD cut one another at E, and on EB and ED let the points F and G be taken at random, and let the straight line FG be drawn so as to make the triangle EFG. I say that the triangle EFG is in one plane, and likewise that the straight lines AB and CD are each in one plane as well.

Diagram. Two straight lines crossing at E, forming an X, with a triangle drawn across them and internal division points.

- A (upper left) and B (lower right): endpoints of one straight line through E
- C (upper right) and D (lower left): endpoints of the other straight line through E
- E (center): the intersection point
- F (lower left, on line toward D) and G (lower right, on line toward B): the base vertices of the triangle
- H (left inside), K (right inside), I (bottom center): the internal points where the drawn lines meet

For if a certain part of the triangle EFG — namely EHIG — were in one plane, while a certain part — namely the remaining FHI — were up above, or the reverse, then the parts EH and GI of the straight lines EF and GF would likewise be in one plane, while the parts HF and IF would be up above, or the reverse. But that this cannot be has already been demonstrated above. [Margin: a — *Elements*

XI.1] And if the part EFIK of the same triangle were believed to be in one plane while the part GIK were up above, or the reverse, then by the same reasoning the parts EK and FI of the straight lines EG and FG would be in one plane, while the parts KG and IG would be up above, or the reverse. And if, finally, the part FHKG of the same triangle were granted to be in one plane while the part EHK were up above, or the reverse, then the parts FH and GK of the straight lines FE and GE would likewise be in one plane, while the parts HE and KE would be up above, or the reverse. All of these are absurd, [Margin: b — *Elements* XI.1] as has been demonstrated. Therefore the triangle EFG exists in one plane; and the same reasoning holds for every other triangle.

Now since the sides EF and EG lie in the same plane in which the triangle EFG lies, [Margin: c — *Elements* XI.1] the whole straight lines CD and AB lie in that same plane, so that some parts of them may not be said to be in the plane while other parts are up above. Therefore the straight lines AB and CD are in one plane — the very one in which we demonstrated the triangle EFG to lie. And so, if two straight lines cut one another, they are in one plane; and every triangle is in one plane. Which was to be demonstrated.

THEOR. 3. PROPOS. 3.

[Margin: iij.]

If two planes cut one another, their common section is a straight line.

Let the planes AB and CD cut one another, and let EF be their common section. I say that EF is a straight line. For if it be not believed to be straight, let the straight line EGF be drawn in the plane AB, and the straight line EHF in the plane CD. Then the straight lines EGF and EHF, since they have the same endpoints E and F, will enclose a surface. But this cannot be, as we showed in the 14th [recte: 1st?] axiom [pronunciatum] of Book 1.

as we showed in the axiom [pronunciatum] of Book 1. Therefore the common section EF will be a straight line; and consequently, if two planes cut one another, their common section is a straight line. Which was to be shown.

Alternatively. Let the planes AB and CD again cut one another, and let the endpoints of the common section be the points E and F, which are joined by the straight line EF. If, then, the straight line EF exists in both planes, the proposition is established, for it will be the common section. But if the straight line EF is said to be in neither of them, then, if the straight line EGF is drawn in one of them, the

two straight lines EF and EGF will again enclose a surface, since they have the same endpoints E and F. And if, finally, the straight line EF is believed to be in only one of the two planes, then, if the straight line EGF is drawn in the other, again the two straight lines EF and EGF will enclose a surface. This is absurd. Therefore the straight line EF is in both planes AB and CD; and so it is their common section.

Diagram. Two planes cutting one another, drawn as intersecting parallelograms, with a lens (eye) shape at the middle marking the surface that would be enclosed by the two curved sections.

- A (upper left) and C (upper, below A): the upper plane's corners
- E (left) and F (right): the endpoints of the common section, joined by the lens
- G (upper arc of the lens) and H (lower arc of the lens): the two drawn sections EGF and EHF
- D (lower right) and B (bottom right): the lower plane's corners

THEOR. 4. PROPOS. 4.

[Margin: iv.]

If a straight line stands at right angles, at their common section, to two straight lines that cut one another, then, when it is drawn through them, it will also be at right angles to their plane.

Let the straight line AB stand at right angles to the two straight lines CD and EF at their common section B. I say that AB is at right angles also to the plane which is drawn through them, that is, in which they lie. For since they cut one another at B, [Margin: a — *Elements* XI.2] they will lie in one plane — let it be CEDF. Let the straight lines BG and BH be taken equal to one another, and likewise the straight lines BI and BK; let the straight lines GK and IH be drawn; and in the same plane let the straight line LM be drawn through B, cutting the straight lines GK and IH at the points L and M. Let there also be dropped from the point A above the plane, to that same plane, the straight lines AG, AL, AK, AH, AM, AI.

Diagram. A rectangle CEDF lying in the plane of reference, with the apex A raised above the center B, and a dense web of lines joining A and B to the marked points on the plane.

- A (top apex, above the plane)

- C (upper left corner), E (upper right corner), F (lower left corner), D (lower right corner): the rectangle CEDF
- B (center of the plane): the common section
- G (upper left, inside) and I (upper right, inside)
- L (left of B) and M (right of B): where LM cuts GK and IH
- K (lower left, inside) and H (lower right, inside)

Now since the sides BG and BK of the triangle BGK are equal, by construction, to the sides BH and BI of the triangle BHI, [Margin: b — *Elements* I.15] and the angles GBK and HBI contained by them are also equal, being vertical angles at B, [Margin: c — *Elements* I.4] the bases GK and HI will be equal to one another, and the angles BKG and BIH will likewise be equal to one another.

[Margin: d — *Elements* I.15] Again, since the angles KBL and IBM, being vertical angles at B, are equal, the two angles KBL and BKL of the triangle BKL will be equal to the two angles IBM and BIM of the triangle BIM; and moreover the sides BK and BI, to which they are adjacent, are equal, by construction. [Margin: e — *Elements* I.26] Therefore the remaining sides KL and LB will also be equal to the remaining sides IM and MB.

[Margin: f — *Elements* I.4] Furthermore, since the sides AB and BG of the triangle ABG are equal, by construction, to the sides AB and BH of the triangle ABH, and the angles ABG and ABH contained by them are equal — namely right angles, by hypothesis — the bases AG and AH will also be equal. By a similar argument the straight lines AI and AK will be equal.

[Margin: g — *Elements* I.8] Further, since the sides AI and IH of the triangle AIH are equal to the sides AK and KG of the triangle AKG, and the base AH to the base AG, from what has been demonstrated so far, the angles AIH and AKG contained by the said sides will also be equal.

[Margin: h — *Elements* I.4] And so, since the sides AK and KL of the triangle AKL are equal to the sides AI and IM of the triangle AIM, and the angles AKL and AIM contained by them are also equal, as we have demonstrated so far, the bases AL and AM will likewise be equal.

[Margin: i — *Elements* I.8] Finally, since the sides AB and BL of the triangle ABL are equal to the sides AB and BM of the triangle ABM, and the base AL to the base AM, from what has been demonstrated, the angles ABL and ABM contained by the said sides will also be equal; and since they are adjacent, they will be right angles, by Definition 10 of Book 1. Therefore the straight line AB makes right

angles with the straight line LM, which touches it at B in the plane of reference. And by the same reasoning it will be shown that AB makes right angles with all the straight lines that touch it at B in that same plane — even if they are drawn between the points G and I, and K and H, provided that the straight lines GI and KH are joined; or else that the letters be arranged, as before, in the manner that appears in this figure, where the letters are the same, and yet LM is not drawn from left to right, as in the previous figure, but from the upper part toward the lower.

Diagram. A second version of the same construction: the rectangle CEDF with apex A above the center B, but with the transversal LM drawn from top to bottom rather than from left to right, and the interior letters rearranged.

- A (top, center), C (upper left corner), E (upper right corner), F (lower left corner), D (lower right corner)
- B (center of the plane)
- K (upper left, inside), L (center), G (upper right, inside)
- H (lower left, inside), M (lower center), J [?] (lower right, inside)

Therefore the straight line AB will be at right angles to the plane CEDF — which is drawn through the straight lines CD and EF — by Definition 3 of this book. If, then, a straight line stands at right angles to two straight lines that cut one another, and so forth. Which was to be demonstrated.

THEOR. 5. PROPOS. 5.

[Margin: v.]

If a straight line stands at right angles, at their common section, to three straight lines that touch one another, then those three straight lines are in one plane.

Let the straight line AB stand at right angles, at their common section — or point of contact — B, to the three straight lines BC, BD, BE that touch one another at B. I say that the three straight lines BC, BD, BE are in one plane. [Margin: a — *Elements* XI.2] For since any two of them are in one plane — because, when produced toward B, they cut one another at B — let BC and BD be in the plane FC; and, if it be possible, let the straight line BE not be set in the same plane, but up above. [Margin: b — *Elements* XI.2] Now since the two straight lines AB and BE are in one plane, because they cut one another at B, let both be in the plane AE. And since the planes FC and AE meet one another at B, when produced they will

necessarily cut one another. [Margin: c — *Elements* XI.3] Let their common section, then, be the straight line BG. Now since the straight line AB is set at right angles to the straight lines BC and BD, [Margin: d — *Elements* XI.4] it will therefore also be at right angles to the plane FC drawn through them; and consequently it will be at right angles to the straight line...

Diagram. Two planes meeting at B, drawn as intersecting parallelograms, with the vertical line AB dropping to B and the lines BC, BD, BE radiating from B.

- A (top): the upper end of the vertical line AB
- F (left, below A): a corner of the plane FC; the dashed line A–F–B runs vertically
- B (center left): the common point of contact
- E (right, middle) and G (right, below E): E on line BE, G the common section of the planes
- D (right, below G) and C (bottom right): the ends of BD and BC

LIBER XI (continued)

[Conclusion of Proposition 5]

perpendicular to the line BG, which touches it at B, by definition 3 of this book. Consequently the right angles ABE and ABG — which lie in the plane AG and are formed by the lines AB and BE — would be equal to one another, a part equal to the whole. This is absurd. Therefore, since the two lines BC and BD lie in the single plane FC, BE will not stand up out of that plane, but will lie in the same plane with them. And so, if a straight line is at right angles to three lines that touch one another mutually, and so on. Which was to be shown.

THEOR. 6. PROPOS. 6.

[Margin: vj]

If two straight lines are at right angles to the same plane, those straight lines will be parallel.

Let two straight lines AB and CD be at right angles to the same plane EF, at the points B and D. I say that they are parallel. Draw the line BD in the plane EF; then the angles ABD and CDB are right, by definition 3 of this book. Now in the same plane EF draw DG perpendicular to BD, and set it equal to AB. Then join the lines BG, GA, and AD. Since, therefore, the sides AB and BD of triangle BAD are equal to the sides GD and DB of triangle GDB by construction, and the angles they contain,

ABD and GDB, are also equal — namely right angles — the bases AD and GB will be equal.[Margin: a — 4.pri. (*Elements* I.4)] Again, since the sides AB and BG of triangle ABG are equal to the sides GD and DA of triangle GDA, and the base AG is common, the angles ABG and GDA will be equal.[Margin: b — 8.pri. (I.8)] But ABG is a right angle, by definition 3 of this book; therefore GDA will also be a right angle. And since, by the same definition, the angle GDC is likewise a right angle, the line GD will be at right angles to the three lines DB, DA, and DC. Therefore the lines DB, DA, and DC will lie in one plane[Margin: c — 5.undecimi. (XI.5)] — that is, CD will lie in the plane drawn through the lines DB and DA. But AB lies in the same plane in which DB and DA lie.[Margin: d — 2.undecimi. (XI.2)] Therefore the line CD will lie in the same plane as AB. And so, since the interior angles ABD and CDB are right, the lines AB and CD will be parallel.[Margin: e — 28.pri. (I.28)] Therefore, if two straight lines are at right angles to the same plane, and so on. Which was to be demonstrated.

Diagram. A rectangle representing the plane EF, seen in oblique perspective, with two lines standing perpendicular to it.

- A (upper left): the top of the first perpendicular
- C (upper right): the top of the second perpendicular
- E (upper-left corner of the plane rectangle)
- B (interior left, foot of AB): where the perpendicular AB meets the plane
- D (right side of the rectangle, foot of CD): where the perpendicular CD meets the plane
- G (lower center-right, on the base): the auxiliary point where DG is set perpendicular to BD
- F (lower-right corner of the plane rectangle)

The interior lines BD, DG, BG, GA, and AD are all drawn.

THEOR. 7. PROPOS. 7.

[Margin: vij]

If there are two parallel straight lines, and any points whatever are taken on each of them, the line joining these points lies in the same plane as the parallels.

On the parallels AB and CD take any two points E and F, and let the line EF join them. I say that the line EF lies in the same plane in which the parallels AB and CD lie. For suppose EF is not granted to lie in the plane of the parallels, but

outside it; then let another plane cut the surface of the parallels through the points E and F, and let the common section of these planes be EGF, which will be a straight line.[Margin: a — 3.undecimi. (XI.3)] Then the two lines EF and EGF, having the same endpoints E and F, would enclose a surface. This is absurd. Therefore the line EF will not lie outside the plane of the parallels AB and CD. And so, if there are two parallel straight lines, and any points are taken on each of them, the line joining these points lies in the same plane as the parallels. Which was to be demonstrated.

Diagram. Two horizontal parallel lines with a joining line and a bent (impossible) path between them.

- A (upper left) — E (upper center) — B (upper right): the upper parallel, with E the point taken on it
- C (lower left) — F (lower center) — D (lower right): the lower parallel, with F the point taken on it
- G (between the parallels, just above F): the vertex of the supposed second line
- The straight line EF joins E to F; the bent path E–G–F represents the absurd second common section

THEOR. 8. PROPOS. 8.

[Margin: viij]

If there are two parallel straight lines, one of which is at right angles to some plane, the other will also be at right angles to the same plane.

Let the lines AB and CD be parallel, and let CD be at right angles to the plane EF. I say that AB too is at right angles to the same plane EF. Draw the line BD in the plane EF; then, by definition 3 of this book, the angle CDB will be right. But the two angles CDB and ABD are equal to two right angles;[Margin: a — 29.pri. (I.29)] therefore ABD too will be right. Now in the plane EF draw BG perpendicular to BD, and set BG equal to CD. Then join the lines DG, GC, and CB. Since, therefore, the sides CD and DB of triangle CDB are equal to the sides GB and BD of triangle GBD by construction, and the angles they contain, CDB and GBD, are also equal — namely right angles — the bases CB and GD will be equal.[Margin: b — 4.pri. (I.4)] Again, since the sides CD and DG of triangle CDG are equal to the sides GB and BC of triangle GBC, and the base CH [recte: CG] is common, the angles CDG and GBC contained by the said sides will be equal.[Margin: c — 8.pri. (I.8)] But CDG is a right angle, by definition 3 of this book; therefore GBC too will be right. Therefore

the line GB will be at right angles to the two lines BD and BC, which cut one another at B. (For if DB and CB were produced, they would cut one another at B, on account of the angle CBD.) And accordingly it will be at right angles to the plane drawn through BD and BC.[Margin: d — 4.undecimi. (XI.4)] But the line AB lies in this same plane, because the lines BD and BC lie in the same plane in which the parallels AB and CD lie, since the points B, C, and D lie on the parallels AB and CD.[Margin: e — 7.undecimi. (XI.7)] Therefore GB will be at right angles to the line BA, by definition 3 of this book. And so, since AB is at right angles to the lines BG and BD, AB will also be perpendicular to the plane drawn through BG and BD[Margin: f — 4.undecimi. (XI.4)] — that is, to the plane EF. Therefore, if there are two parallel straight lines, one of which is at right angles to some plane, the other too will be at right angles to the same plane. Which was to be shown.

Diagram. A rectangle representing the plane EF in oblique perspective, with two perpendiculars and connecting lines.

- A (upper left): the top of the perpendicular AB
- C (upper right): the top of the perpendicular CD
- E (upper-left corner of the plane rectangle)
- B (interior left): foot of AB in the plane
- D (right side of the rectangle): foot of CD in the plane
- G (lower center): the auxiliary point where BG is set perpendicular to BD
- F (lower-right corner of the plane rectangle)

The lines BD, BG, DG, GC, and CB are drawn as the diagonals.

THEOR. 9. PROPOS. 9.

[Margin: ix]

Lines which are parallel to the same straight line, but not in the same plane with it, are also parallel to one another.

Let the lines AB and CD be parallel to EF, but not lying in the same plane as EF. I say that AB and CD are also parallel. For from any point G on the line EF let two perpendiculars be drawn to EF itself: GH lying in the plane of the parallels AB and EF, and GI lying in the plane of the parallels CD and EF. Since, therefore, EG is at right angles to the two lines GH and GI, which touch one another at G, it will also be perpendicular to the plane drawn through GH and GI.[Margin: a — 4.undecimi. (XI.4)] Hence, since AB and EF are parallel, and EF is perpendicular to the plane drawn through GH and GI, AB too will be perpendicular to the same

plane.[Margin: b — 8.undecimi. (XI.8)] Likewise, since CD and EF are parallel, and EF is perpendicular to the plane drawn through GH and GI, CD too will be perpendicular to the same plane.[Margin: c — 8.undecimi. (XI.8)] And accordingly, since the lines AB and CD are both perpendicular to the same plane drawn through GH and GI, they will be parallel.[Margin: d — 6.undecimi. (XI.6) [?] — printed digit smudged; the argument (two lines perpendicular to the same plane are parallel) requires XI.6] Therefore lines which are parallel to the same straight line, but not in the same plane with it, are also parallel to one another. Which was to be demonstrated.

Diagram. Three lines: two outer parallels and a central line, with two perpendiculars meeting at the central point in a bowtie configuration.

- A (upper left) — H (upper center-left) — B (upper right): the upper parallel AB, with H the foot of one perpendicular
- E (middle left) — G (middle center) — F (middle right): the central line EF, with G the chosen point
- C (lower left) — I (lower center-left) — D (lower right): the lower parallel CD, with I the foot of the other perpendicular
- The perpendiculars GH (rising to H) and GI (descending to I) cross the central line at G

SCHOLIUM. — Euclid deliberately required that the two lines AB and CD not lie in the same plane as the line EF to which they are parallel. For if they did lie in the same plane as EF, then AB and CD would be parallel to one another by I.30, and so it would be superfluous to prove the same thing here.

THEOR. 10. PROPOS. 10.

[Margin: x]

If two straight lines meeting one another are parallel, respectively, to two straight lines meeting one another but not in the same plane, they will contain equal angles.

Let the lines AB and AC, meeting at A, be parallel to the lines DE and DF [recte: DF; printed "EF"], which meet at D; and let AB and AC not lie in the same plane as DE and DF. I say that the angles BAC and EDF contained by them are equal. Set AB and DE equal to one another, and AC and DF equal to one another; and draw the lines BC, EF, BE, AD, and CF. Since, therefore, AB and DE are parallel and equal, BE and AD will also be parallel and equal.[Margin: a — 33.pri. (I.33)] By the same argument CF and AD will be parallel and equal. Therefore, since BE and CF are

both parallel and equal to the same line AD, BE and CF will also be parallel and equal to one another;[Margin: b — 9.undecimi. (XI.9)] and consequently the lines BC and EF will be parallel and equal.[Margin: c — 33.pri. (I.33)] Since, then, the sides AB and AC of triangle ABC are equal to the sides DE and DF of triangle DEF by construction, and the base BC is equal to the base EF, as we have just shown, the angles BAC and EDF contained by the said sides will be equal.[Margin: d — 8.pri. (I.8)] Therefore, if two straight lines meeting one another, and so on. Which was to be shown.

Diagram. An elongated triangular figure (an oblique prism) with a small triangle at the top and a small triangle at the bottom, joined by long edges.

- A (apex, top): vertex of the upper triangle
- B (upper left) and C (upper right): the base ends of the upper triangle ABC
- D (interior, lower middle): apex of the lower triangle
- E (bottom left) and F (bottom right): the base ends of the lower triangle DEF
- The connecting edges BE, CF, and AD run down the length of the figure

SCHOLIUM. — The conclusion of this theorem is also true when the two former lines lie in the same plane as the two latter. For let there be four lines AB, AC, DE, DF in the same plane, with AB parallel to DE and AC parallel to DF. I say that the angles BAC and EDF are equal. For set AB equal to DE, and AC equal to DF, and draw the lines BC [recte: BC; printed "EC"] and EF, which will either lie in one straight line or not. First let them lie in one straight line BF. Since, then, AB and DE are parallel, the angles ABC and DEF — the interior and the exterior — will be equal.[Margin: e — 29.pri. (I.29)] Likewise the angles ACB and DFE will be equal, since AC and DF are also parallel. Therefore the two angles ABC and ACB of triangle ABC are equal to the two angles DEF and DFE of triangle DEF. And so the remaining angles too, BAC and EDF, will be equal.[Margin: f — 32.pri. (I.32)] Which is what was proposed.

Diagram. Two triangles standing side by side on a single base line.

- A (apex, left): vertex of the left triangle
- B (bottom left) and C (bottom, center-left): base ends of the left triangle ABC
- D (apex, right): vertex of the right triangle
- E (bottom, center-right) and F (bottom right): base ends of the right triangle DEF
- The base ends lie collinearly as B–C–E–F

Now let BC and EF not lie in one straight line. Drawing the lines AD, BE, and CF: since AB and DE are parallel and equal, AD and BE too will be parallel and equal. [Margin: g — 33.pri. (I.33)] By the same reasoning AD and CF will be parallel and equal, because AC and DF are assumed parallel and equal. Therefore, since BE and CF are both parallel and equal to the same line AD, they will be parallel and equal to one another; [Margin: h — 30.pri. (I.30)] and accordingly BC and EF too will be parallel and equal. [Margin: i — 33.pri. (I.33)] And so, since the two sides AB and AC of triangle ABC are equal to the two sides DE and DF of triangle DEF, and the base BC is equal to the base EF, the angles BAC and EDF contained by the said sides will be equal. [Margin: k — 8.pri. (I.8)] Which is what was proposed.

Diagram. An oblique parallelepiped-like figure illustrating the case in which the two triangles are not coplanar.

- D (top): apex
- F (right): upper-right vertex
- E (interior center): a vertex within the figure
- A (left): left vertex
- B (bottom left): lower vertex
- C [?] (bottom right): lower vertex (label faint on the tile)

However, in order for the conclusion of this theorem to be rightly drawn from the given hypothesis, it is necessary that the parallel lines be so placed (whether they all lie in the same plane or not) that the three lines AD, BE, and CF in no way cut one another — as is clear from the first and last figures. For otherwise the conclusion will not be true.

For the lines AB and AC are parallel to the lines DE and DF in this figure; and yet the angles BAC and EDF which they contain are not equal, since the one is acute and the other obtuse. This happens because AD and CF cut one another. Hence, although AC and DF are parallel and equal, it does not follow that AD and CF, which join them, are parallel and equal, because they do not join them toward the same parts, as I.33 requires. Therefore the demonstration of this theorem does not hold when this condition is lacking — a condition which is nevertheless preserved in the earlier figures.

Diagram. The failing case, a slanted quadrilateral figure in which the joining lines cross.

- B (top left) and A (top right): the meeting point A of the upper pair of lines
- F (center): the crossing point of the joining lines

- C (right, below A): a vertex
- E (bottom left) and D (bottom center): the meeting point D of the lower pair of lines
- The joining lines AD and CF cross one another at F, so the angle BAC comes out acute while EDF is obtuse

PROBL. 1. PROPOS. 11.

[Margin: xj]

From a given point above a plane, to draw a straight line perpendicular to the plane below.

Let a perpendicular have to be drawn from the point A, above, to the plane BC lying below. In the plane BC draw any line DE, and from A drop the perpendicular AF onto it.[Margin: a — 12.pri. (I.12)] Then in the plane BC, through F, erect GH perpendicular to DE,[Margin: b — 11.pri. (I.11)] and from A drop the perpendicular AI onto it. I say that AI is perpendicular to the plane BC lying below. For draw in the plane BC, through I, the line KL parallel to DE; [Margin: c — 12.pri. (I.12)] since DE [recte: DE; printed "DF"] is at right angles to the two lines FA and FH by construction,[Margin: d — 31.pri. (I.31)] and consequently KL is a line perpendicular to the plane drawn through FA and FH, [Margin: e — 4.undecimi. (XI.4)] KI too will be a line perpendicular to that same plane drawn through FA and FH.[Margin: f — 1.undecimi. (XI.1)] But since AI lies in the same plane as the lines FA and FH, and touches the line KI at I, by definition 3 of this book the angle KIA will be right;[Margin: g — 12.undecimi. (XI.12) [?] — this back-reference is anomalous (XI.12 is the following proposition); transcribed as printed] and thus AI will be at right angles to the two lines KI and IF. Therefore AI will be perpendicular to the plane BC.[Margin: h — 4.undecimi. (XI.4)] And so from a given point above a plane we have drawn a straight line perpendicular to the plane below. Which was to be done.

Diagram. A rectangle representing the plane BC seen in perspective, with the point A raised above it and a web of construction lines within the plane.

- A (top, above the plane): the given point on high
- B and D (upper-left corner region): B leftmost, D just inside
- K (upper center): endpoint of the parallel KL
- H (upper right): endpoint of the erected perpendicular GH
- I (center): the foot of AI, where KL passes

- F (center-left, lower): the foot of AF on DE, and the point through which GH is erected
- G (bottom left): endpoint of GH
- E (bottom center): endpoint of DE
- C (bottom right corner of the plane)

The line DE runs across the plane, GH crosses it at F, KL passes through I, and AF, AI descend from A. (The endpoint L of KL is not distinctly labelled on the tile.)

SCHOLIUM. — But should it happen that the perpendicular AI, drawn to GH, falls upon the point F — so that it is the very same as the perpendicular AF drawn to DE — then AF itself, the perpendicular drawn from the outset to DE, will be perpendicular to the plane BC lying below, since it is perpendicular to both DE and GH.[Margin: i — 4.undecimi. (XI.4)]

PROBL. 2. PROPOS. 12.

[Margin: xij]

From a given point in a given plane, to erect a straight line at right angles to the plane.

Let a perpendicular to the given plane BC have to be drawn from the given point A in the plane BC. From any point D, above, drop the perpendicular DE to the plane BC.[Margin: a — 11.undecimi. (XI.11)] If it falls on the point A, what was proposed is done. But if it falls on another point E, then extend the line through E and A, and through A draw AF parallel to DE, in the plane GH drawn through DE and EA.[Margin: b — 31.pri. (I.31)] I say that AF is perpendicular to the given plane BC. For since DE and AF are parallel, and DE is perpendicular to the plane BC by construction, AF too will be perpendicular to the same plane BC.[Margin: c — 8.undecimi. (XI.8)] And so, from a given point in a given plane, and so on. Which was to be done.

Diagram. A parallelogram representing the plane, with an auxiliary plane GH standing on it and two perpendiculars.

- G (top left), F (top center), D (top right): the upper edge of the auxiliary plane GH
- B (below G): a vertex of the auxiliary plane
- A (lower center-left) and E (lower center): points in the base plane, A the given point and E the foot of DE
- H (right): a vertex of the auxiliary plane

- C (bottom right): corner of the base plane
- DE descends to E and AF rises from A, both perpendicular to the base plane

THEOR. 11. PROPOS. 13.

[Margin: xiiij] [*The heading is set in italic type: "THEOR. 11. PROPOS. 13."*]

From a given point in a given plane, two straight lines cannot be erected at right angles to the plane on the same side.

Let AB be the given plane. I say that from its given point C, two perpendiculars to the plane AB cannot be drawn toward the same parts. For if it can be done, let two perpendiculars CD and CE be drawn to the plane AB; and through CD and CE, which lie in one and the same plane, let a plane FG be drawn, cutting the plane AB along the straight line GH.[Margin: a — 2.undecimi. (XI.2)] Since, then, EC and DC are at right angles to the plane AB, by definition 3 of this book the angles ECG and DCG will be right, and accordingly equal — a part equal to the whole. This is absurd. Therefore, from a given point in a given plane, two straight lines cannot be erected at right angles to the plane on the same side. Which was to be demonstrated.

ALTERNATIVELY. From C, if it can be done, draw two perpendiculars CD and CE to the plane AB. Then, since the two lines DC and EC are at right angles to the same plane AB, they will be parallel to one another[Margin: b — 6.undecimi. (XI.6)] — and yet they meet at the point C. This is absurd.

Diagram. A parallelogram representing the plane AB, with the cutting plane FG and two perpendiculars rising from C.

- F (top left), D (top center), E (top right): the upper edge, with D and E the tops of the two perpendiculars
- A (interior, upper left): a point in the plane
- H (left corner): endpoint of the section GH
- C (bottom center): the given point in the plane
- G (right): the other endpoint of the section GH
- B (bottom): corner of the plane
- The perpendiculars CD and CE rise from C, and the cutting plane FG meets the base plane along GH

SCHOLIUM. — In the same way, from a point above a plane, two straight lines cannot be dropped perpendicular to the plane below. For let there be drawn...

Euclid, *Elements*, Book XI (continued)

Commentary of Christoph Clavius. Enunciations of Euclid are set as block quotations; Clavius's demonstrations and scholia follow as ordinary prose. Superscript letters mark the citation-letters carried in the printed margin.

Scholium to Proposition 13 (concluded)

[Margin: xiiij.]

(continued from the previous page) For suppose that from A, a point raised above the underlying plane BC, two perpendiculars AD, AE could — if it were possible — be dropped to it. Through AD and AE, which lie in one plane, let a plane FG be drawn,^c cutting the plane BC along the straight line GH. [Margin: c — Elements XI.2] Since AD and AE are perpendicular to the plane BC, then by definition 3 of this book the two angles ADE and AED will be interior right angles in the triangle ADE — which is absurd,^d since any two angles taken in any triangle whatever are together less than two right angles. [Margin: d — Elements I.17]

Otherwise: if AD and AE are perpendicular to the plane BC, then they will be parallel to one another,^e and yet they meet at the point A. Which is absurd. [Margin: e — Elements XI.6]

Diagram. The plane BC is drawn as a foreshortened parallelogram lying below; the point A stands raised above it. Two perpendiculars AD, AE fall from A to the base, and the auxiliary plane FG (with F at the upper left) passes through them, cutting BC along the line GH.

- A (raised apex): the elevated point
- F (upper left): far corner of the cutting plane FG
- B (upper left edge): corner of plane BC
- H (base, left): one end of the common section GH
- D, E (base, center): feet of the two perpendiculars
- G (base, right): other end of the common section GH
- *(lower corner of plane BC, unlabeled / illegible) [?]*

THEOR. 12. PROPOS. 14.

[Margin: xiiij.]

Planes to which the same straight line is perpendicular are parallel.

Let the straight line AB be perpendicular to the planes CD and EF. I say that the planes CD and EF are parallel. For if they be not believed to be parallel, then when produced they will meet one another. Let them meet toward the parts C, E, and let them form as common section the straight line GH.^a [Margin: a — Elements XI.3] Take on it any point I whatever, and draw the straight lines IA, IB in the planes GCD, GEF. Now since AB is set perpendicular to the planes GCD, GEF, then by definition 3 of this book the two angles IAB, IBA will be right angles in the triangle ABI.^b But they are also less than two right angles — which is absurd. [Margin: b — Elements I.17] Therefore the planes CD, EF, however far produced, never meet one another; and so they are parallel. Therefore, planes to which the same straight line is perpendicular, and so forth. Which was to be demonstrated.

Diagram. A tent-shaped figure: from the apex G two sloping planes GCD and GEF descend to the base, with the perpendicular AB set up through the interior. The point I lies on the supposed common section, and the auxiliary lines IA, IB form the triangle ABI.

- G (apex): meeting point of the two planes
- I (just below G): point taken on the common section
- C (left), E (right): base ends of the two planes
- H (center): on the base
- A, B (lower center): foot points of the perpendicular AB / triangle ABI
- D (lower left), F (lower center): base corners

Scholium

It will also be easy to demonstrate the converse of this, namely:

If there be two parallel planes, a straight line which is perpendicular to one of them will be perpendicular to the other as well.

Let there be two parallel planes AB, CD, to one of which, AB, let the straight line EF be perpendicular. I say that EF is perpendicular to the remaining plane CD as well. For let some plane, namely GK, cut the planes AB, CD, passing along the straight line EF, and let it make with those planes the common sections — the straight lines GH, IK. Then by definition 3 of this book the angles GEF, HEF will be right angles; and consequently the angles IFE, KFE will be right angles too. (For if one of them, say IFE, were said to be acute, the two angles GEF, IFE would be less than two right angles. Then the straight lines EG, FI, toward the parts G, I, would meet in the plane GK — namely at a point L; and so the planes AB, CD, in which they lie, would likewise meet toward the same parts at the point L,^c [Margin: c —

Elements XI.1] since the whole straight line HGL lies in one plane, namely in AB produced, and likewise the whole straight line KIL lies in one plane, namely in CD produced. Which is absurd, since they are supposed parallel. Therefore the angles IFE, KFE are right angles.) In the same way, if another plane, namely MN, cut the same planes along EF, making with those sections the other common straight lines MO, PN, the angles PFE, NFE will be right angles.^d [Margin: d — Elements XI.4] And so, since EF is perpendicular to the two straight lines IK, PN, it will also be perpendicular to the plane CD, drawn through them. Which was proposed.

Diagram. A prism-like construction: the two parallel planes AB (lower) and CD, together with the cutting planes GK and MN, all threaded by the perpendicular line EF. The common sections GH, IK, MO, PN are drawn as parallel edges.

- G, L, I (upper edge): tops of the cutting plane GK / point L where the produced lines would meet
- M (left), C (upper center): further plane corners
- A, P (center): interior points
- E, F (center): endpoints of the perpendicular EF
- N (right): section endpoint
- B, H, O, K, D (base): base corners and section feet

Since EF is thus perpendicular to the two straight lines IK, PN, it will likewise be perpendicular to the plane CD drawn through them. Which was proposed.

THEOR. 13. PROPOS. 15.

[Margin: xv.]

If two straight lines meeting one another are parallel to two straight lines meeting one another that are not situated in the same plane, then the planes drawn through them are parallel.

Let there be two straight lines AB, AC meeting one another at A, parallel to two straight lines DE, DF meeting one another at D and lying in another plane together with them. I say that the planes BC, EF drawn through them are parallel too. For from A^a let there be dropped to the plane EF the perpendicular AG, meeting the plane EF at the point G. [Margin: a — Elements XI.11] Then in the plane EF, through G, let GH, GI be drawn parallel to DE, DF respectively.^b [Margin: b — Elements I.31] Now since the straight lines AB, GH are parallel to DE,^c then AB and GH will also be parallel to one another; [Margin: c — Elements XI.9] and consequently the angles BAG, AGH are together equal to two right angles.^d

[Margin: d — Elements I.29] But AGH is a right angle, by definition 3 of this book; therefore the angle BAG will be a right angle too. By a like argument you will conclude that the angle CAG is a right angle. And so, since the straight line GA is perpendicular to the two straight lines AB, AC, it will also be perpendicular to the plane BC drawn through them.^e [Margin: e — Elements XI.4] But it is also perpendicular to the plane EF, by construction. Therefore the planes BC, EF are parallel.^f [Margin: f — Elements XI.14] So if two straight lines meeting one another, and so forth. Which was to be demonstrated.

Diagram. Two planes are shown edge-on and parallel: the plane BC (with A the vertex of the angle BAC and B, C its arms) and the plane EF (with D the vertex of the angle EDF and E, F its arms). The perpendicular AG falls from A onto EF at G, and GH, GI are drawn there parallel to DE, DF.

- D (top): vertex in plane EF
- F (right), G (center), I (right): endpoints / foot of perpendicular
- A (center left), C (below A): vertex and arm in plane BC
- B (lower left): arm of plane BC
- E, H (lower center): endpoints in plane EF

Scholium

But if the perpendicular drawn from A to the plane EF should fall on the point D itself, the same conclusion will be reached, with this exception, that

(catchword: quod)

[Margin: xv.]

(top of p. 503, running header "LIBER XI.") that in the plane EF no other parallels are to be drawn. Thus, if the lines GH, GI be set parallel to the straight lines AB, AC, in such a way that AG — the perpendicular to the plane HI drawn through GH, GI — falls on G, we shall show, as before, that GA is perpendicular to the plane BC drawn through the straight lines AB, AC, and so forth.

[Problem — to draw a plane parallel to a given plane]

From this theorem the following problem will be no difficulty; namely:

Given a plane, to draw through a given point that is not in it a plane parallel to it.

Let the given plane be AB, and let the point outside it be C, through which a plane is to be drawn parallel to the plane AB. In the plane AB let two straight lines DB, DA be drawn arbitrarily, meeting one another at the point D, from which let the straight line DC be drawn to C. Then in the plane BC, drawn through the straight lines DB, DC, let CE be drawn parallel to DB;^f [Margin: f — Elements I.31] and in the plane AC, drawn through the straight lines DA, DC, let CF likewise be made parallel to DA. I say that the plane EF, drawn through the straight lines CE, CF, is parallel to the given plane AB. For since the two straight lines DB, DA, meeting one another at D, are parallel to the two straight lines CE, CF, which meet one another at C and lie in another plane together with them, the planes AB, EF drawn through them will be parallel.^g [Margin: g — Elements XI.15] Which was proposed.

Diagram. A cube-like parallelepiped showing the given plane AB and the constructed parallel plane EF. The line DC joins the plane AB (through D) to the external point C.

- D (top left), C (top right): D in plane AB, C the external point
- A (center left), F (right): interior/upper corners
- B (lower left), E (lower center): base corners

THEOR. 14. PROPOS. 16.

[Margin: xvj.]

If two parallel planes are cut by any plane, their common sections are parallel.

Let the parallel planes AB, CD be cut by the plane EF along the straight lines EH, GF. I say that their common sections EH, GF are parallel lines. For if they be not parallel, then when produced they will meet one another, since they lie in the cutting plane EF. Let them therefore meet at the point I.^a [Margin: a — Elements XI.1] Then since the whole straight line HEI lies in one plane, namely in AB produced, and likewise the whole straight line FGI lies in one plane, namely in CD produced, the planes AB, CD, produced, would also meet at I — which is absurd, since they are supposed parallel. Therefore the straight lines EH, GF are parallel. So if two parallel planes be cut by any plane, and so forth. Which was to be demonstrated.

Diagram. A tent-shaped figure: the two parallel planes AB and CD are cut by the plane EF, whose common sections EH, GF are drawn as edges; produced, they would meet at the apex I.

- I (apex): point where the produced sections would meet
- A (left), E, G (upper center) [?], C (right) [?]: upper corners / section points
- B, H, F, D (base): base corners and section feet

Scholium

This theorem can in no way be converted. For although the common sections of parallel planes, made by another cutting plane, are parallel — as has here been demonstrated — nevertheless not all those planes whose common sections made by another cutting plane are parallel are themselves parallel; for through two parallel straight lines infinitely many planes can be drawn, of which only two are parallel, while the others, produced in some direction, meet.

Moreover, from this theorem we gather another, similar to theorem 30 of Book 1:

Planes parallel to the same plane are also parallel to one another.

Let there be two planes AB, CD parallel to the plane EF. I say that AB and CD are parallel too. For let all three planes be cut by the plane GH, and let their common sections be the straight lines GI, KL, MH. Let them again be cut by another plane NH, and let their common sections be the straight lines NI, OL, PH — yet on this condition, that the two cutting planes GH, NH also cut one another, their common section being the straight line HI. Now since the parallel planes AB, EF are cut by the plane GH, the common sections GI, KL will be parallel.^b [Margin: b — Elements XI.16] In the same way MH, KL will be parallel,^c and so GI, MH will be parallel to one another. [Margin: c — Elements XI.9] We shall show in no other way that NI, PH are parallel. And so, since IG, IN, meeting one another at I, are parallel to the straight lines HM, HP, which meet at H and lie in another plane, the planes AB, CD drawn through them will be parallel.^d [Margin: d — Elements XI.15 (*leading digit foxed; read as "15.undecimi"*)] Which was proposed.

Otherwise: In the third plane EF take any point G whatever, from which, on either side, let a straight line be produced perpendicular to the plane EF, meeting the plane AB at the point H and the plane CD at the point I. Now since the line HG is perpendicular to the plane EF, it will also be perpendicular to the plane AB, which is parallel to it, by what we demonstrated in the scholium to proposition 14 of this book. By the same reasoning GI will be perpendicular to the plane CD. Therefore, since the straight line HI is perpendicular to the planes AB, CD, the planes AB, CD will be parallel to one another.^e [Margin: e — Elements XI.14] Which was proposed.

Diagram. Three parallel planes drawn edge-on and slanted (AB, EF, CD), cut by the two planes GH and NH, whose common sections GI, KL, MH and NI, OL, PH are drawn as crossing diagonals. The cutting planes meet along HI.

- A, G (upper left), E (upper center), C (upper right): upper corners
- N (left), I, L, M (center), H (right): section points / meeting points
- B (lower left), O, F, P (center), D (lower right): base points

Diagram. For the "otherwise" argument: three parallel planes shown as upright rectangles (AB, EF, CD), threaded by the perpendicular HI which passes through the point G, meeting the planes at H, G, I.

- A, E, C (top): upper corners of the three planes
- H, G, I (middle): feet of the perpendicular in each plane
- B, F, D (bottom): base corners

But if two planes are parallel to another plane, yet not to one another but meet — then they form one plane; just as two straight lines parallel to another straight line, if they meet, constitute one straight line, as we demonstrated at proposition 30 of Book 1. For let the planes AB, CD be parallel to the plane EF, and let them, produced, meet along the straight line CG. I say that the planes AB, CD constitute one plane. For let some plane HI cut all three planes along the straight lines HK, LM, MI. Now since the planes AB, EF are parallel, the common sections LM, HK will be parallel too.^f [Margin: f — Elements XI.16] By the same reasoning IM, HK will be parallel. Therefore, since the straight lines LM, IM are parallel to HK, and lie in the same cutting plane HI, and meet at M, they are set in a straight line and continuous, by what we showed at proposition 30 of Book 1. And so the planes AB, CD will form one plane;^g [Margin: g — Elements XI.1] namely because the straight line LI lies in one plane. Which was proposed.

Diagram. A cube-like figure showing the two planes AB, CD meeting along CG, both parallel to EF, cut by the plane HI along HK, LM, MI.

- H, F (top left), L, A (top): upper corners
- C (right), M (center), B (center) [?]: interior/side points
- G (center), K, D, I (base): base corners

(catchword: ITAQVE)

(top of p. 504, running header "EVCLID. GEOM.")

Therefore two planes which are parallel to another plane will either be parallel to one another, or — if, being produced, they meet — will certainly form one plane.

Moreover, here too we shall easily demonstrate the converse of proposition 10, namely:

If in parallel planes two angles are equal, and one line of the one angle is parallel to one line of the other angle, then the other line will also be parallel to the other line — provided only that they be drawn from the same side of the planes drawn through the former parallels.

For in parallel planes let the angles BAC, EDF be equal, and let AB be parallel to DE, and let AC, DF be drawn toward the same parts of the plane drawn through AB, DE. I say that AC, DF are parallel too — that is, that when the straight line AD is drawn, the plane drawn through AC, AD passes also through the straight line DF, so that AC, DF are the common sections made by the plane drawn through AC, AD in the parallel planes drawn through AB, AC and DE, DF. For from this^h it follows that the sections AC, DF are parallel. [Margin: h — Elements XI.16] Which was proposed. For if the plane drawn through AC, AD does not make in those parallel planes the sections AC, DF, let it make the sections AC, DG,ⁱ which will be parallel. [Margin: i — Elements XI.16] Then the angle EDG^k will be equal to the angle BAC, that is, to the angle EDF. [Margin: k — Elements XI.10] Which is absurd, since the one is a whole and the other a part.

Diagram. An arrow-like figure spanning two parallel planes (drawn faintly as parallelograms): at the top the angle BAC (vertex A, arms to B and C); the line AD descends to D in the lower plane, where the angle EDF (arms to E and F) opens, with the auxiliary arm DG alongside DF.

- A (top): vertex of the upper angle
- B (left), C (right): arms of the upper angle
- D (center): vertex of the lower angle
- E (lower left), F (lower center): arms of the lower angle
- G (lower center, beside F): auxiliary point

THEOR. 15. PROPOS. 17.

[Margin: xvij.]

If two straight lines are cut by parallel planes, they will be cut in the same ratios.

Let the straight lines AB, CD (whether they be parallel, as in the figure, or not, yet lying in the same plane; or set transversely, lying in different planes) be cut by the parallel planes EF, GH, IK at the points L, M, N, O, P, Q. I say that they are cut

proportionally — that is, that their segments intercepted between the said planes are proportional, so that as LM is to MN, so is OP to PQ. For let the straight lines LO, NQ be drawn in the planes EF, IK, and let the straight line LQ be joined, meeting the plane GH at the point R, from which let the straight lines RM, RP be drawn to the points M, P in the same plane GH. Then the triangle LNQ will lie in one plane,^a and likewise the triangle LOQ in one plane. [Margin: a — Elements XI.2] Now since the parallel planes GH, IK are cut by the plane of the triangle LNQ, their common sections MR, NQ will be parallel;^b and by the same reasoning RP, LO will be parallel. [Margin: b — Elements XI.16] Wherefore, as LR is to RQ, so is LM to MN;^c and likewise, as LR is to RQ, so is OP to PQ; and consequently, as LM is to MN, so is OP to PQ. [Margin: c — Elements VI.2] So if two straight lines be cut by parallel planes, and so forth. Which was to be demonstrated.

Diagram. Three parallel planes drawn as upright slanted strips (EF, GH, IK, with E,G,I along the top and F,H,K along the bottom). The straight line AB crosses the tops through L, M, N; the straight line CD crosses the bottoms through O, P, Q; the diagonal LQ passes through R in the middle plane.

- A (top left), E, L (upper left strip): line AB entering / section points
- G, M (upper center): middle strip
- I, N, B (upper right): right strip / line AB exit
- C, O, F (lower left): line CD entering / feet
- P, H (lower center): middle strip foot
- Q, K, D (lower right): right strip / line CD exit
- R (center): point where LQ meets the middle plane GH

Scholium

If the given straight lines AB, CD lie in one and the same plane, the two straight lines RM, RP will form one straight line, as appears in the figure above. For then the plane drawn through the straight lines AB, CD will cut the three parallel planes at the points L, O, M, P, N, Q, and along the straight lines LO, MP, NQ,^d which will be parallel to one another; and the point R will lie on the straight line MP, since it lies both in the plane drawn through the straight lines AB, CD (because the points L, Q lie in that same plane, as we said in the scholium to proposition 7 of this book) and in the plane GH, and so on MP, the common section of those planes. [Margin: d — Elements XI.16] Therefore RM, RP will constitute one straight line MP, namely the common section of the planes LO, QN, GH.

But if the two straight lines AB, CD do not lie in the same plane, then RM, RP will not form one straight line, as appears in these two figures. For since the triangle LOQ lies in one plane,^e and the triangle LNQ in another plane distinct from it, [Margin: e — Elements XI.2] (for if those two triangles lay in one and the same plane, the two straight lines AB, CD would lie in that same plane, which is constituted from them — which is contrary to the hypothesis), the straight lines RM, RP will lie in those distinct planes of the triangles, and consequently will not constitute one straight line;^f otherwise they would lie in one plane. [Margin: f — Elements XI.1] Which is absurd.

This same proposition also holds if the two straight lines AB, CD, though lying in the same plane, are not parallel, as appears in these same two figures; but then MRP will always be one straight line, as was demonstrated at the beginning of this scholium when AB, CD were placed in one plane. For the same demonstration can be applied here.

Likewise, if more than three parallel planes cut the straight lines AB, CD, their parts will be proportional, and the common sections

(catchword: sectio-)

Diagram. A second figure like the last: three parallel slanted planes (E,G,I above; F,H,K below) with the transversals AB and CD and the diagonal through R and M, P.

- E, G, I (top): tops of the three planes
- A, L, M, N, B (upper transversal AB): line and section points
- R (center), M, P: diagonal / section points
- C, O, P, Q, D (lower transversal CD): line and section points
- F, H, K (bottom): feet of the three planes

Diagram. The case in which AB, CD do not lie in one plane: the two transversals cross, so that the triangles LOQ and LNQ lie in different planes. Points R, M lie at the crossing.

- A (upper left), E, L, G, R, Q [?], D (upper region): line AB and section points, with D at upper right
- C, O, F (left): line CD entering
- M, R (center): crossing points
- N, H, K (lower center/right), B (lower right): section points and line AB exit

the sections made by the intermediate planes, and by the triangles LOQ and LNQ, will constitute straight lines — provided the lines AB and CD lie in one plane; but they will not be straight lines if AB and CD do not lie in one plane, just as was said of the straight lines MR and RP.

Moreover, it is clear from the demonstration that if any number of lines lying in one plane are cut by any number of parallel lines, the segments of those lines lying between the parallels are proportional. For let AB and CD be two straight lines, arranged in any manner in one plane, as in the last figure, and let them be cut by three parallels LO, RPM (for the letter P now refers to the intersection of the lines CD and RM) and QN. I say that the segments LM and MN have the same ratio as the segments OP and PQ. For, when the straight line LQ has been joined, [Margin: a — Elem. VI.2] then, on account of triangle LOQ, LR is to RQ as OP is to PQ; and on account of triangle LNQ, LR is to RQ as LM is to MN. Therefore OP will be to PQ as LM is to MN. And the same reasoning holds for more lines. But in order that the demonstration may proceed correctly with any two lines whatever, a straight line must be drawn from the first point of section of the one line to the last point of section of the other. So you see that in the two lines AB and CD a straight line has been drawn from L, the first point of the line AB, to Q, the last point of the line CD. And the same procedure has been kept in the figures of this Proposition 17.

THEOR. 16. PROPOS. 18.

[Margin: xviii]

If a straight line is at right angles to any plane, then all planes drawn through it will also be at right angles to that same plane.

Let AB be at right angles to the plane CD. I say that all planes drawn through the line AB are at right angles to that same plane CD. For let a plane EF be drawn through AB, cutting the plane CD along the straight line FG. Then take a point H anywhere on the line FG, and [Margin: a — Elem. I.31] in the plane EF draw HI parallel to AB. Now since AB and IH are parallel, and AB is set at right angles to the plane CD, [Margin: b — Elem. XI.8] IH too will be at right angles to that same plane CD; and therefore, by Definition 3 of this book, it is perpendicular to the common section FG. By the same reasoning all the lines that are drawn in the plane EF parallel to AB will be at right angles to the plane CD, and therefore, by Definition 3 of this book, perpendicular to the common section FG. Hence the plane EF is at right angles to the plane CD, by Definition 4 of this book. And by a similar argument all the other planes drawn through the line AB will be shown to

be at right angles to the plane CD. Therefore, if a straight line is at right angles to any plane, and so on. Which was to be demonstrated.

Diagram. The figure for Proposition 18. A horizontal plane CD is drawn as a leaning parallelogram; a second plane EF stands on it, meeting it along the line FG. The line AB rises from the base to the plane, with HI drawn parallel to it. Readable labelled points:

- C (upper left of the standing plane)
- I, A, E (across the top edge)
- F, H, B, G (along the base line, left to right)
- D (lower right corner of the horizontal plane)

SCHOLIUM.

From these results this further theorem may be demonstrated, namely:

Two planes at right angles to the same plane, in which they make parallel common sections, are parallel to each other; and of two parallel planes, if one is at right angles to any plane, the other will also be at right angles to that same plane.

Let the planes AB and CD be at right angles to the plane EF, making the parallel sections GH and IK. I say that they are parallel. For in the plane EF draw LM perpendicular to GH, which, by Definition 4 of this book, will be at right angles to the plane AB. [Margin: c — Elem. I.29] And the same perpendicular will be at right angles to IK; and therefore, by Definition 4 of this book, it will also be at right angles to the plane CD. Since, then, LM is at right angles to both planes AB and CD, [Margin: d — Elem. XI.14] they will be parallel.

But now let the planes AB and CD be parallel, and let AB be at right angles to the plane EF. I say that CD is also at right angles to that same plane EF. For let the common sections of the plane EF with the planes AB and CD be the straight lines GH and IK, [Margin: e — Elem. XI.16] which will be parallel. When the line LM has also been drawn in the plane EF perpendicular to the line GH, this, by Definition 4 of this book, will be at right angles to the plane AB. Therefore, by the scholium to Proposition 14 of this book, the same LM will also be at right angles to the plane CD. [Margin: f — Elem. XI.18] Hence the plane EF, drawn through the line LM, will be at right angles to the plane CD; and consequently, in turn, the plane CD will also be at right angles to the plane EF. Which is what was proposed.

Diagram. The scholium figure. Two upright planes AB and CD stand on a horizontal plane EF, meeting it along the parallel sections GH and IK; the line LM lies in EF perpendicular to both. Readable labelled points:

- E, G, I (across the top)
- A (left plane, upper), C (right plane, upper)
- L, M (interior, on the line LM)
- B (left plane, lower)
- H, D (lower interior)
- K, F (along the base)

From this, the following problem too will be solved:

Through a straight line given in any plane, to draw a plane at right angles to that plane.

Let FG be a straight line given in the plane CD, as in the figure of this Proposition, and let it be required to draw through FG a plane at right angles to the plane CD. From any point B of the line FG [Margin: g — Elem. XI.12] erect BA perpendicular to the plane CD, and through the lines AB and FG let the plane EF be drawn, [Margin: h — Elem. XI.18] which will be at right angles to the plane CD. Which is what was proposed.

THEOR. 17. PROPOS. 19.

[Margin: xix]

If two mutually intersecting planes are at right angles to a given plane, their common section will also be at right angles to that same plane.

Let two planes AB and CD, mutually intersecting along the straight line EF, be at right angles to the plane GH. I say that their common section EF is also at right angles to that same plane GH. For either EF is at right angles to BF and DF, the common sections of the planes AB and CD with the plane GH, or it is not. If EF is at right angles to BF and DF, [Margin: a — Elem. XI.4] it will also be at right angles to the plane GH, which is drawn through them. But if EF be granted to be at right angles to only one of the lines BF and DF, let it be at right angles to BF. Now since the plane AB is set at right angles to the plane GH, then EF — which in the plane AB is drawn perpendicular to BF, its common section with the plane GH — will be at right angles to the plane GH, by Definition 4 of this

Diagram. The figure for Proposition 19. A horizontal plane GH is shown as a leaning parallelogram; two planes AB and CD stand upon it and cross along the vertical line EF, which meets the base at F. Readable labelled points:

- C (top, apex of the standing intersection)
- A, E (upper middle)
- G (left corner of the horizontal plane)
- K, I (interior, near the crossing)
- F (interior foot of EF)
- B (right corner)
- D, H (along the lower edge)

book. The same conclusion follows if EF be granted to be at right angles to DF; for then too EF will be at right angles to the plane GH, by Definition 4 of this book, since it is set perpendicular to DF, the common section of the plane CD with the plane GH, and is drawn in the plane CD, which is at right angles to the plane GH. Finally, if EF be believed to be at right angles to neither BF nor DF, [Margin: b — Elem. I.11] let there be drawn from F, in the plane AB, perpendicular to BF (its common section with the plane GH), the line FI; and likewise in the plane CD, perpendicular to DF (its common section with the plane GH), the line FK. Now since the plane AB is set at right angles to the plane GH, the perpendicular IF — which is drawn in the plane AB to BF, its common section with the plane GH — will also be at right angles to the plane GH, by Definition 4 of this book. By the same reasoning KF will be at right angles to that same plane GH. And therefore two perpendiculars would have been raised from the point F to the plane GH. But this was shown above to be impossible. [Margin: c — Elem. XI.13] Hence EF will be at right angles to the plane GH. Therefore, if two mutually intersecting planes are at right angles to a given plane, and so on. Which was to be demonstrated.

Diagram. The figure for Proposition 19, repeated at the head of this page. It shows the same construction as fig-p505-3: horizontal plane GH with two crossing planes AB and CD meeting along EF. Readable labelled points:

- top apex label reads "G" [?] here (it reads "C" in the corresponding figure on p. 505 — likely a compositor variant or foxing)
- A, E (upper middle)
- G (left corner)
- K, I (interior, near the crossing)
- F (interior foot)
- B (right corner)

- D, H (lower edge)

THEOR. 18. PROPOS. 20.

[Margin: xx]

If a solid angle is contained by three plane angles, then any two of them, however chosen, are greater than the third.

Let a solid angle at A be contained by the three plane angles BAC, CAD, DAB. I say that any two of them are greater than the remaining one. For if all three are equal, it is evident that any two are greater than the remaining one. And if only two are equal, and the third is smaller than either, it is likewise established that any two are greater than the remaining one. But if one of them — say BAC — is the greatest, while the others are either equal or unequal, it is also manifest that each of these, together with that greatest one BAC, is greater than the remaining one. I now say that this greatest angle BAC is exceeded by the two angles BAD and DAC together. For in the plane through AB and AC [Margin: a — Elem. I.23] let the angle BAE be made equal to the angle BAD, and let the straight line AE be made equal to the straight line AD. Then, in the same plane, extend through E the straight line BC, cutting the lines AB and AC at B and C, and let the straight lines BD and DC be joined. Now since the sides AD and AB of triangle BAD are equal to the sides AE and AB of triangle BAE, and the angles contained by them are equal by construction, [Margin: b — Elem. I.4] the bases BD and BE will be equal. [Margin: c — Elem. I.20] But since the sides DB and DC are greater than the side BC, if the equal lines BD and BE are subtracted, the remaining line CD will be greater than CE. Since, then, the sides AD and AC of triangle DAC are equal to the sides AE and AC of triangle EAC, and the base CD is greater than the base CE, [Margin: d — Elem. I.25] the angle CAD will be greater than the angle CAE. Therefore, when the equal angles BAD and BAE are added, the two angles CAD and BAD will be greater than the two angles CAE and BAE — that is, greater than the whole angle BAC, which was posited to be the greatest of all. And consequently any two are much greater than the remaining one that is not the greatest. Therefore, if a solid angle is contained by three plane angles, and so on. Which was to be shown.

Diagram. The figure for Proposition 20. From a vertex A radiate lines to a base line BC; D stands above, with lines AD, DB, DC drawn, and E lies on BC between B and C. Readable labelled points:

- D (apex, top)
- A (interior, below D)

- B (base, left)
- E (base, middle)
- C (base, right)

THEOR. 19. PROPOS. 21.

[Margin: xxj]

Every solid angle is contained by plane angles that are less than four right angles.

Let the solid angle A be contained by the three plane angles BAC, CAD, DAB. I say that these three angles are less than four right angles. For when the straight lines BC, CD, DB have been drawn, there will be constituted three solid angles at B, C, D, each of which is contained by three plane angles: namely B, by CBA, ABD, DBC; C, by BCA, ACD, DCB; and D, by CDA, ADB, BDC. [Margin: a — Elem. XI.20] Now since the two angles CBA, ABD are greater than the angle CBD, and likewise the two angles BCA, ACD are greater than the angle BCD, and the two angles CDA, ADB are greater than the angle CDB, the six angles CBA, ABD, BCA, ACD, CDA, ADB will be greater than the three angles CBD, BCD, CDB. [Margin: b — Elem. I.32] But these three are equal to two right angles. Therefore those six will be greater than two right angles. Since, then, those six, together with the three at A, are equal to six right angles — on account of the three triangles BAC, CAD, DAB [Margin: c — Elem. I.32] (for the angles of any triangle are equal to two right angles) — if those six, which are greater than two right angles, are taken away, there will be left the three at A, constituting the solid angle, which are less than four right angles. Therefore every solid angle, and so on. Which was to be demonstrated.

Diagram. The figure for Proposition 21. A triangle DBC with apex D above the base BC, and an interior point A joined to the three vertices, forming the base triangle of a solid angle. Readable labelled points:

- D (apex, top)
- A (interior)
- B (base, left)
- C (base, right)

SCHOLIVM.

Since this demonstration is adapted by the commentators only to that solid angle which three plane angles constitute, we shall accordingly extend it to all solid angles in the following way.

Let there be any solid angle whatever, contained by any number of plane angles. I say that all of them together are likewise less than four right angles. For if straight lines are drawn to subtend all the plane angles in order, lying in the same plane, a pyramid will be constituted whose apex is the given solid angle, having as many sides — and as many angles — as there are plane angles by which the solid angle is comprehended; and therefore just as many triangles will surround the pyramid. Hence all the angles of these triangles

Diagram. Three schematic base-polygons illustrating the scholium, each divided into triangles from a central point: a hexagon partitioned into six triangles, a pentagon into five, and a quadrilateral (square) into four (drawn with both diagonals). The figures are unlettered.

Diagram. The same three schematic base-polygons repeated at the head of this page : a hexagon divided into six triangles, a pentagon into five, and a square into four (with both diagonals drawn). The figures are unlettered.

(the angles of the triangles), [Margin: d — Elem. I.32] since the angles of any triangle are equal to two right angles, will be equal to twice as many right angles as the base contains angles. But all the angles of the base are also equal to that same number of right angles less four, by what we demonstrated from Proclus in connection with Proposition 32 of Book 1. Therefore all the angles of the said triangles will exceed all the angles of the base by four right angles. Hence, since [Margin: e — Elem. XI.20] all the angles of these triangles near the base are greater than the angles of the base — because near the base solid angles are constituted, each of which is contained by three plane angles, and of these any two are greater than the remaining one, namely the one that lies in the base — all the angles of these triangles will exceed those same angles near the base by less than four right angles. Therefore, since all the angles of all the triangles exceed those same ones near the base (which constitute the solid angle) by less than four right angles, the angles which compose the solid angle will be less than four right angles. Which is what was proposed.

But that the matter may be better grasped, let there be, for the sake of example, a solid angle A contained by the five plane angles BAC, CAD, DAE, EAF, FAB, which are subtended in order by the straight lines BC, CD, DE, EF, FB, lying in one and the same plane — namely in that plane which cuts the lines AB, AC, AD, AE, AF at the points B, C, D, E, F; so that a pentagonal pyramid is constituted, having a pentagonal base and, around its sides, five triangles — as many, that is, as there are plane angles composing the solid angle A. [Margin: f — Elem. I.32] Now since all the angles of the five triangles BAC, CAD, DAE, EAF, FAB are equal to ten right

angles, while the five angles of the base BCDEF are equal to six right angles — that is, ten less four — the angles of the triangles will exceed the angles of the base by four right angles. Since, then, the ten angles of the triangles near the base are greater than the angles of the base — for since the solid angle B is contained by the three plane angles ABC, ABF, CBF, [Margin: g — Elem. XI.20] the two angles CBA, ABF, which are in the triangles, will be greater than the angle CBF, which is in the base; and by the same reasoning the remaining angles of the triangles near the base will be greater than the remaining angles in the base — the angles of the triangles adjacent to the base will be greater than six right angles. Therefore the remaining five, which constitute the solid angle A, will be less than four right angles, seeing that these together with those are equal to ten right angles, as has been said. The same demonstration is to be applied in all other solid angles.

Diagram. The worked example for the scholium to Proposition 21. A pentagon BCDEF with a central point A joined by straight lines to all five vertices, so as to form five triangles surrounding the vertex A of the pentagonal pyramid.

Readable labelled points:

- E (top)
- F (upper left), D (upper right)
- A (centre)
- B (lower left), C (lower right)

THEOR. 20. PROPOS. 22.

[Margin: xxij]

If there be three plane angles, any two of which, taken at will, are greater than the remaining one, and equal straight lines subtend them, then it is possible for a triangle to be constituted from the lines that join those equal straight lines.

Let there be three plane angles A, B, C, contained by the equal straight lines AD, AE, BF, BG, CH, CI, yet under this law: that any two of the angles are greater than the remaining one. I say that from the three straight lines DE, FG, HI — which join those equal straight lines — a triangle can be constituted; that is, that of the three lines DE, FG, HI any two are greater than the remaining one. For this being granted, [Margin: a — Elem. I.22] a triangle will easily be constructed from them. That the two lines DE, FG are greater than HI will be shown as follows. [Margin: b — Elem. I.23] Let the angle DAK be made equal to the angle B, and let AK first fall on the side of AD, so that a certain whole angle EAK is formed, composed of the

two angles EAD and DAK; this indeed will happen when the two angles DAE and B are less than two right angles. Let AK be set equal to AD, and let the straight lines KD, KE be joined. Now since the sides AK, AD of triangle AKD are equal to the sides BF, BG of triangle BFG, and the angles DAK and B contained by them are equal, [Margin: c — Elem. I.4] the bases KD and FG will also be equal. Again, since the sides AK, AE of triangle AKE are equal to the sides CH, CI, and the angle EAK is greater than the angle C — because the two angles DAE and B are posited to be greater than the angle C — [Margin: d — Elem. I.24] the base EK will also be greater than the base HI; [Margin: e — Elem. I.20] but the lines ED, DK are greater than the line EK. Therefore ED, DK will be much greater than HI. Since, then, DK has been shown to be equal to the line FG, DE and FG will also be greater than HI. Which is what was proposed.

Diagram. The first case of Proposition 22. At the left is the construction: the point K set above the triangle on base DE, with A as the vertex of the angle EAK (composed of EAD and DAK) and the joins KD, KE drawn. To the right are the three reference triangles for the given plane angles: triangle A on base DE, triangle B on base FG, and triangle C on base HI. Readable labelled points:

- K (upper left of the construction), A (vertex)
- D (base, left), E, F (along the base line)
- B (apex of second triangle), G, H (base)
- C (apex of third triangle), I (base)

CADAT — Now let AK fall in a straight line continuous with AE; this happens when the two angles DAE and B are equal to two right angles. And let AK be again set equal to AD, and let the straight line KD be joined. [Margin: f — Elem. I.4] Then, as before, KD will be equal to FG. [Margin: g — Elem. I.20] Now since the lines ED, DK are greater than the line KE, and KE is equal to the lines CH, CI by hypothesis and construction, ED and DK will also be greater than the two lines CH, CI; [Margin: h — Elem. I.20] but these are greater than HI. Therefore ED, DK will be much greater than HI. Since, then, DK has been shown to be equal to FG, DE and FG will also be greater than HI. Which is what was proposed.

Diagram. The second case of Proposition 22. The construction with K set above D, A on the line, and the base line D–E; alongside are the reference triangles with apex C on base H–I and apex B on base F–G. Readable labelled points:

- K (top), A (interior)
- D (base, left), E, F (base)
- C (apex), B (apex), G (base)

- H, I (base of the C-triangle)

Now let AK fall, lastly, on the side of AE, so that neither is the whole angle composed of the angles EAD and DAK, nor is EAK a straight line; which will finally come about when the two angles DAE and B are greater than two right angles. And let AK again be set equal to AD, and let the straight lines KD, KE be drawn. [Margin: i — Elem. I.4] Then, as before, KD will be equal to FG. [Margin: k — Elem. I.21] Now since the two lines DE, DK are greater than the two lines AE, AK, and AE, AK are equal to the two lines CH, CI by hypothesis and construction, DE and DK will also be greater than CH, CI; [Margin: l — Elem. I.20] but these are greater than HI. Therefore

Diagram. The third case of Proposition 22. A tall construction with K at the top, A within the triangle, and the base line D–E; to the right are the reference triangles with apex C on base H–I and apex B on base F–G. Readable labelled points:

- K (top), A (interior)
- D (base, left), E, F (base)
- C (apex), H, I (base of the C-triangle)
- B (apex), G (base)

Book XI (continued)

Proposition 22 (concluded)

(Continuing the third case: three plane angles, any two greater than the third, subtended by equal lines; the joining bases form a triangle.)

and so DE, DK will be much greater than HI. Therefore, since DK has been shown equal to FG, DE and FG will likewise be greater than HI — which is what was proposed. In the same way we shall conclude that DE, HI are greater than FG, if at C an angle is made equal [to the corresponding one]; and again that FG, HI are greater than DE, if at C, along FB, an angle equal [to the given one] is set up, and so on. [Margin: m 22.prim. — *Elements* I.22] Therefore, from the three straight lines DE, FG, HI a triangle can be constructed. And so, if there be three plane angles, and so forth — which was to be demonstrated.

Another proof. (ALITER.) Suppose the three angles A, B, C are equal. Then the bases DE, FG, HI will also be equal, [Margin: n 4.primi. — *Elements* I.4] and consequently any two of them are greater than the remaining one. But if only two of the angles are equal and the third is smaller, then two of the bases will be equal, [Margin: o 4.primi. — *Elements* I.4] and each of these will be greater than

the base of the third; [Margin: p 24.prim. — *Elements* I.24] so that again any two of them will be greater than the remaining one. And if one of them — say A — is the greatest, then whether the remaining two, B and C, are equal to each other or unequal, [Margin: q 24.prim. — *Elements* I.24] the base DE will be the greatest of all. Hence DE, FG will be greater than HI, and likewise DE, HI greater than FG.

I now say, further, that the straight lines FG, HI are also greater than the straight line DE. [Margin: r 23.primi. — *Elements* I.23] For let the angle DAK be made equal to angle B, and let AK be set equal to AD. The point K will then fall below DE, because the circumference of a circle described about A with radius AD passes through the points D, K, E, on account of the equality of the straight lines AD, AK, AE. Then let the straight lines DK, KE be joined. Now since the two angles B and C are assumed greater than angle DAE, and B is equal to angle DAK by construction, angle C will be greater than the remaining angle KAE. And because the sides AD, AK of triangle DAK are equal to the sides BF, BG of triangle BFG, and the contained angles DAK and B are equal, [Margin: s 4.primi. — *Elements* I.4] the base DK will be equal to the base FG. Again, because the sides CH, CI of triangle CHI are equal to the sides AK, AE of triangle AKE, and angle C has been shown greater than angle KAE, [Margin: t 24.primi. — *Elements* I.24] the base HI will be greater than the base KE. Therefore, since DK has been shown equal to FG, FG and HI will be greater than DK, KE. [Margin: u 20.prim. — *Elements* I.20] But DK, KE are greater than DE. Hence FG, HI will be much greater than DE — which is what was proposed.

Diagram. Three plane-angle figures set side by side, illustrating Proposition 22.

- Leftmost: a triangular figure (drawn as a folded pyramid) with apex **A** and base vertices **D** (lower left) and **E** (lower right), with a further vertex below the base DE [label F (?) — foxed] and the joining lines DK, KE of the construction.
- Middle: triangle with apex **B** and base **F–G**.
- Right: triangle with apex **C** and base **H–I** (G and H stand adjacent where the two triangles meet at the foot).

PROBL. 3. PROPOS. 23.

[Margin: xxiiij.]

From three plane angles, any two of which taken together are greater than the remaining one, to construct a solid angle. It is required, moreover, that those three angles together be less than four right angles.

Let there be three angles A, B, C, less than four right angles, and under this condition: that any two of them are greater than the remaining one. It is now required, out of the three angles A, B, C, to fashion a solid angle — one contained, that is, by three plane angles equal to the three given angles. Let there be laid down six lines enclosing the said angles, namely AD, AE, BF, BG, CH, CI, all equal; and let the bases DE, FG, HI subtend them. [Margin: a 22.vnde. — *Elements* XI.22] It is therefore possible that from DE, FG, HI a triangle be constructed. Let, then, the triangle KLM be constructed out of them, with side KL equal to the straight line DE, side LM equal to FG, and side KM equal to HI. [Margin: b 5.quarti. — *Elements* IV.5] Let a circle be described about the triangle KLM, whose center is N, from which let the straight lines NK, NL, NM be drawn. Then each of the lines enclosing the plane angles — namely AD, AE, and the rest — will be greater than any of the lines drawn from the center N. This will be shown as follows.

Let the center N fall first *inside* the triangle KLM — which indeed happens when the triangle is acute-angled [oxygonium], as is established from the corollary of Proposition 5, Book IV. Now if AD, AE be supposed *not* greater than NK, NL, they will be either equal or smaller. Let them be equal, then. Since the sides AD, AE of triangle ADE are equal to the sides NK, NL of triangle NKL, and the base DE is equal to the base KL, [Margin: c 8.primi. — *Elements* I.8] angle A will be equal to angle KNL. By the same argument, angle B will be equal to angle LNM, and angle C equal to angle MNK. Since, then, the three angles about N are equal to four right angles (from the corollary of Proposition 15, Book I), the three angles A, B, C will likewise be equal to four right angles. This is absurd, for they were assumed less than four right angles. Therefore AD, AE are not equal to NK, NL.

I say they are not smaller either. For if AD, AE were smaller than NK, NL, let NO, NP be cut off equal to AD, AE, and let the straight line OP be drawn. Since, then, as NK is to NO, so is NL to NP — the antecedents being equal to each other and the consequents to each other — [Margin: d 2.sextri. — *Elements* VI.2 [digit slightly foxed]] the sides NK, NL of triangle NKL will be cut proportionally, and consequently OP will be parallel to KL. [Margin: e 29.prim. — *Elements* I.29] Hence the angles NOP, NPO will be equal to the angles NKL, NLK, and triangle NOP will be equiangular with triangle NKL. [Margin: f 4.sextri. — *Elements* VI.4] Therefore, as NK is to KL, so is NO to OP. But NK is greater than NO. [Margin: g 14.quin. — *Elements* V.14] Therefore KL also — that is, DE, which is equal to KL — will be greater than OP. And so, since the sides AD, AE of triangle ADE are equal to the sides NO, NP of triangle NOP, while the base DE is greater than the base OP, [Margin: h 25.primi. — *Elements* I.25] angle A will also be greater than angle ONP. By the same reasoning, angle B will be shown greater than angle LNM, and angle

C greater than angle MNK. Since, then, the three angles at N are four right angles (from the corollary of Proposition 15, Book I), the three angles A, B, C will be greater than four right angles. This is absurd, for they are assumed less than four right angles. Therefore AD, AE are not smaller than NK, NL; but neither are they equal. Hence they are greater.

Diagram. The construction for Proposition 23, first case (center N inside triangle KLM). To the left stand the three given plane-angle triangles and the auxiliary segment; to the right the circumscribed-circle construction.

- Triangle with apex **A** and base **D–E** (angle A).
- Short horizontal segment labelled **T** (the auxiliary line used later).
- Triangle with apex **B** and base **F–G** (angle B).
- Triangle with apex **C** and base **H–I** (angle C).
- Circle with center **N**, containing the inscribed triangle **K** (top), **L** (lower left), **M** (right); with **O** on NK and **P** on NL (the cut-off segments), and the apex **V** at the top (the vertex of the solid angle raised on the perpendicular NV).

Suppose, further, that AD, AE be believed to be equal to NK, NL. Then BF, BG will also be equal to the same NK, NL

— that is, to NK, NL. Since, then, NK is equal to NM, BF and BG will be equal to the straight line LM; but LM was laid down equal to FG. Therefore BF, BG are likewise equal to the straight line FG. [Margin: i 20.prim. — *Elements* I.20] But BF, BG are also *greater* than FG. This [recte: est] is absurd. Therefore AD, AE are not equal to NK, NL.

But if AD, AE be believed smaller than NK, NL, then BF, BG will also be smaller than NK, NL. Since, then, NK is equal to NM, BF, BG will be smaller than the straight line LM; but LM was laid down equal to FG. Therefore BF, BG are likewise smaller than the straight line FG. This is absurd, [Margin: k 20.prim. — *Elements* I.20] since they are greater. Therefore AD, AE are not smaller than NK, NL; nor equal. Hence they are greater.

Let the center N fall, lastly, *outside* the triangle KLM, below the side LM — which happens when the angle LKM is obtuse, as is clear from the said corollary of Proposition 5, Book IV. If, then, AD, AE be said to be equal to NK, NL — since the base DE is also laid down equal to the base KL — [Margin: l 8.primi. — *Elements* I.8] angle A will be equal to angle KNL. By the same reasoning, angle C will be equal to angle KNM. Hence the whole angle LNM will be equal to the angles A and

C together. But these two are, by hypothesis, greater than angle B. Therefore angle LNM will also be greater than angle B. Again, because the sides NL, NM of triangle NLM are equal to the sides BF, BG of triangle BFG, and the base LM is equal to the base FG, [Margin: m 8.prim. — *Elements* I.8] angle LNM will be equal to angle B. But we also showed it to be greater. This is absurd. Therefore AD, AE are not equal to NK, NL.

But if AD, AE be believed smaller than NK, NL — if NO, NP be cut off equal to AD, AE, and the straight line OP be drawn — it will be demonstrated, as in the first case, that angle A is greater than angle ONP, and by the same reasoning that angle C is greater than angle KNM. [Margin: n 23.prim. — *Elements* I.23] Let the angle ONQ be made equal to A, and the angle ONR equal to C, and let NQ, NR be set equal to AD, AE, and let OQ, OR be joined. OQ will fall below OP, because the circumference of a circle described about N with radius NO passes through the points O, P, Q, since the straight lines NO, NP, NQ are equal. [Margin: o 29.primi. — *Elements* I.29] Hence, since angle PON is equal to angle LKN — the exterior to the interior — angle QON will be smaller than angle LKN. By the same reasoning it will be shown that angle NOR is smaller than angle NKM, if OS be drawn parallel to KM; for OR will likewise fall below OS. Therefore the whole angle LKM will be greater than the whole angle QOR. Now since the sides NO, NQ of triangle NOQ are equal to the sides AD, AE, and angle ONQ is equal to angle A by construction, [Margin: p 4.prim. — *Elements* I.4] OQ will be equal to DE — that is, to KL. In the same way, OR will be equal to KM. When now the straight line QR has been joined, since the sides KL, KM of triangle LKM are equal to the sides OQ, OR of triangle OQR, and angle LKM is greater than angle QOR, as we have shown, [Margin: q 24.prim. — *Elements* I.24] the base LM — that is, FG — will be greater than the base QR. Since, then, the sides BF, BG of triangle BFG are equal to the sides NQ, NR of triangle NQR, and the base FG is greater than the base QR, [Margin: r 25.prim. — *Elements* I.25] angle B will be greater than angle QNR. Again, because angle ONQ has been made equal to A and angle ONR equal to C, the whole angle QNR will be equal to the two angles A and C together. But A and C are assumed greater than angle B. Therefore angle QNR will also be greater than angle B. This is absurd, since B was shown greater than angle QNR. Therefore AD, AE are not smaller than NK, NL; nor equal. Hence they are greater.

Diagram. Reprint of the given plane-angle triangles together with the circle construction, accompanying the second-case discussion.

- Triangle with apex A and base D–E.
- Triangle with apex C and base (H–I).

- Triangle with apex **B** and base (I–F).
- Circle with center **N**, inscribed points **K** (upper right), **L** (left), **M** (right), and apex **V** at the top; a broken line runs from **V** toward **L**.

Diagram. The construction for the third case of Proposition 23 (center **N** outside triangle **KLM**, below **LM**).

- Triangle with apex **A** and base **D–E**; triangle with apex **C** and base **H–I**; triangle with apex **B** and base **F–G** (below).
- Circle with center **N** (at the bottom), inscribed triangle with **K** (top), **L** (left), **M** (right); apex **V** at the top and **O** near the center; interior points **P** (left of **N**), **Q** (lower left), **R** (right of **N**), **S** (near **M**) — the points of the **OQ**, **OR**, **QR**, **OS** construction. (Several interior labels lie in foxed regions.)

Now since the straight lines **AD**, **AE** are greater than the straight lines **NK**, **NL**, wherever the center **N** may lie, the straight line **AD** is able to exceed the straight line **NK** by the square on the line **T**, by the lemma of Proposition 14, Book X — that is, so that the square on **AD** is equal to the squares on **NK** and **T** together. [Margin: s 12.vndec. — *Elements* XI.12] From the center **N** let there be raised, perpendicular to the plane of the circle **KLM**, the perpendicular **NV**, equal to the straight line **T**, and let **KV**, **LV**, **MV** be joined. Since, then, **NV** is a straight line perpendicular to the plane of the circle **KLM**, it will be perpendicular to the straight lines **NK**, **NL**, **NM**, by Definition 3 of this book; [Margin: t 47.primi. — *Elements* I.47] and consequently the square on **VK** will be equal to the squares on **KN**, **NV**. Since, then, the square on **AD** is also equal, by construction, to the same squares on **KN**, **NV**, the squares on **VK** and **AD** will be equal to each other. Hence the straight lines **VK**, **AD** will be equal. Again, because the sides **VN**, **NK** of triangle **VNK** are equal to the sides **VN**, **NL** of triangle **VNL**, and the contained angles **VNK**, **VNL** are right, [Margin: u 4.primi. — *Elements* I.4] the base **VK** will be equal to the base **VL**; and by the same reasoning the straight line **VM** as well. Therefore the three straight lines **VK**, **VL**, **VM** are equal to one another. But **VK** has been shown equal to **AD**. The three straight lines **VK**, **VL**, **VM** are therefore equal to **AD**, and thereby to **AE**, **BF**, **BG**, **CH**, **CI**. And so, since the sides **VK**, **VL** of triangle **VKL** are equal to the sides **AD**, **AE** of triangle **ADE**, and the base **KL** to the base **DE**, [Margin: x 8.primi. — *Elements* I.8] angle **KVL** will be equal to angle **A**. Not otherwise shall we demonstrate that angle **LVM** is equal to angle **B**, and angle **KVM** equal to angle **C**. Hence the solid angle **V** is contained by the three plane angles **KVL**, **LVM**, **MVK**, which are equal to the three given plane angles **A**, **B**, **C**. Therefore, out of three plane angles, and so forth — which was to be done.

THEOR. 21. PROPOS. 24.

[Margin: xxiv.]

If a solid is contained by parallel planes, its opposite faces are similar and equal parallelograms.

Let the parallelepiped be ABEF (for it is of this that the proposition is to be understood), contained, according to Definition 30 of this book, by the six quadrilateral figures AC, CF, FH, HA, AF, BE, of which any two opposite are parallel. I say that any pair of opposite faces are similar and equal parallelograms. For when the parallel planes BG, CF are cut by the plane AC, [Margin: a 16.vnde. — *Elements* XI.16] the common sections AB, CD will be parallel. Similarly, when the parallel planes AF, BE are cut by the plane AC, the common sections AD, BC will be parallel. And consequently ABCD is a parallelogram, a quadrilateral figure. Not otherwise shall we show that the remaining quadrilateral figures are parallelograms.

I now say that the opposite parallelograms are similar and equal. For since the straight lines AB, BH are parallel to the straight lines DC, CE, and lie not in the same plane but in opposite ones, [Margin: b 10.vnde. — *Elements* XI.10] the angles ABH, DCE will be equal; and by the same argument the remaining angles of parallelogram BG will be equal to the remaining angles of parallelogram CF. [Margin: c 34.primi. — *Elements* I.34] Now since AB is equal to DC in parallelogram AC, and BH is equal to CE in parallelogram BE, it will be that as AB is to BH, so is DC to CE; and consequently, as BH is to HG, so is CE to EF, and so on, for the same cause. Therefore the sides of the parallelograms BG, CF about the equal angles will be proportional, and consequently the parallelograms themselves similar. Now, when the diagonals AH, DE are drawn, [Margin: d 34.prim. — *Elements* I.34] since the sides AB, BH of triangle ABH are equal to the sides DC, CE of triangle DCE, [Margin: e 10.vnd. — *Elements* XI.10] and angle ABH is equal to angle DCE, as has been shown, [Margin: f 4.prim. — *Elements* I.4] the triangles ABH, DCE will be equal to each other. [Margin: g 34.prim. — *Elements* I.34] Hence, since the triangles ABH, DCE are halves of the parallelograms BG, CF, [these are] equal to each other. Similarly we shall demonstrate that the opposite parallelograms AC, GE and AF, BE are similar and equal. If, therefore, a solid is contained by parallel planes, its opposite faces are parallelograms, similar and equal, and so forth — which was to be demonstrated.

Diagram. The parallelepiped ABEF for Proposition 24. A rectangular box drawn in oblique projection.

- Top face vertices: **A** (front-upper), **G** (rear-right upper), **B** (left upper), and a fourth top vertex at center labelled **H** [?] (foxed).
- Bottom face vertices: **C** (front-left lower), **E** (front-center lower), **D** (rear-center lower), **F** (right lower).
- Diagonals **AH** and **DE** are drawn across opposite faces.

THEOR. 22. PROPOS. 25.

[Margin: xxv.]

If a parallelepipedal solid is cut by a plane parallel to the opposite faces, then as base is to base, so is solid to solid.

Let the parallelepiped **ABCD** be cut by the plane **EF**, parallel to the opposite planes **AD**, **BC**. I say that, as the base **AG** is to the base **BG**, so is the solid **AEFD** to the solid **BEFC**. For let the parallelepiped **ABCD** be conceived as produced on either side to any extent whatever; and on **AB**, when it has been drawn out, let there be taken any number of straight lines **AK**, **KL** equal to **AE**, and any number of straight lines **BM**, **MN**, **NO** equal to **EB**. Then through the points **K**, **L**, **M**, **N**, **O** let there be drawn the planes **KP**, **LQ**, **MR**, **NS**, **OT** parallel to the planes **AD**, **EF**, **BC**, by the scholium of Proposition 15 of this book. Since, then, the solid **AEFD** is contained by parallel planes by hypothesis, it will be a parallelepiped by definition, [Margin: a 24.vnde. — *Elements* XI.24] and will have its opposite planes similar and equal parallelograms. In the same way there will be the parallelepipeds **AKPD**, **KLQP**, **EBCF**, **BMRC**, **MNSR**, **NOTS**, **ELQF**, **EOTF**; and they will have their opposite planes similar and equal parallelograms. [Margin: b 36.primi. — *Elements* I.36] Now since the parallelograms **AG**, **KV**, **LX**, standing on equal bases **AE**, **AK**, **KL**, are equal, and similar too — [Margin: c 29.prim. — *Elements* I.29] since the angles of the one are equal to the angles of the others, and the sides about the angles of the one equal to the sides about the angles of the others, and therefore proportional. And by the same reasoning the parallelograms **AY**, **KZ**, **Lx** are equal and similar. Since, then, the parallelograms **AD**, **KP**, **LQ** are likewise equal and similar, the three planes **AG**, **AY**, **AD** of the solid **AEFD** will be equal and similar to the three planes **KV**, **KZ**, **KP** of the solid **AKPD**, and to the three planes **LX**, **L**, **LQ** of the solid **KLQP**. [Margin: d 24.vnde. — *Elements* XI.24] Moreover, the three planes in each solid are equal and similar to the three remaining opposite ones in the same solid, namely **AG** to **ZF**, **AY** to **VF**, and **AD** to **EF**, and so on. Therefore, by Definition 10 of this book, the solids **AEFD**, **AKPD**, **KLQP** are equal. By the same argument the solids **EBCF**, **BMRC**, **MNSR**, **NOTS** will be shown equal.

Hence, whatever multiple the base LG is of the base AG, the solid LEFQ will be the same multiple of the solid AEFD; and whatever multiple the base OG is of the base BG, the solid OEFT will be the same multiple of the solid BEFC. Now if the base LG (the multiple of the base AG, of the first magnitude) is equal to the base OG (the multiple of the base BG, of the second magnitude), the solid LEFQ (the multiple of the solid AEFD, of the third magnitude) is likewise equal to the solid OEFT (the multiple of the solid BEFC, of the fourth magnitude), because, the bases LG, OG being equal, the six parallelograms of the solid LEFQ are also equal and similar to the six parallelograms of the solid OEFT. But if the base is greater than the base, the solid too is greater than the solid; and if smaller, smaller — in whatever multiplication this be made. Therefore, by Definition 6, Book V, as the base AG (the first magnitude) is to the base BG (the second magnitude), so will the solid AEFD (the third magnitude) be to the solid BEFC (the fourth magnitude). By the same reasoning it will be demonstrated that solid is to solid as the base DY is to the base CY, and as the base AY is to the base BY, and as the base DG is to the base CG. If, then, a parallelepipedal solid be cut by a plane, and so forth — which was to be shown.

Diagram. The long divided parallelepiped for Proposition 25. A parallelepiped drawn in oblique projection and partitioned by successive parallel planes into equal cells. The scan is heavily foxed through the figure and many compartment labels are indistinct.

- Front bottom edge (left to right): L, K, A, E, B, M, N, O.
- Front top edge (left to right): Q, P, D, F, C, R, S, T.
- Rear top edge / interior compartments (left to right): β (beta), X, V, G, I, δ (delta), and the numeral labels 3, 8 [rear-interior labels partly foxed].
- Additional interior/rear-bottom labels (Greek letters and numerals 2, 9) fall in foxed regions and cannot be read with confidence [?].

SCHOLIUM.

This proposition can also be adapted to every prism. For if any prism whatever be cut by a plane parallel to the opposite planes, then, as base is to base, so will solid be to solid. Let there be first the prism ABCDEF, whose opposite planes are the triangles ABC, DEF, and let it be cut by the plane GHI, parallel to the opposite planes. I say that, as the base AI is to the base FI, so is the solid ABCIHG to the solid FEDIHG. For let the prism ABCDEF be conceived as produced on either side to any extent whatever...

(The Scholium continues onto the following page.)

and from AF, produced in both directions, let any number of segments AK, KL be taken equal to AG, and any number of segments FM, MN, NO equal to FG. Then through the points K, L, M, N, O let the planes KPQ, LRS, MTV, NXY, OZ α be drawn parallel to the planes ABC, GHI, FED, by the scholium to Proposition 15 of this book. Now since the parallel planes ABC, GHI are cut by the plane AI, [Margin: c $\rightarrow$ *Elements* XI.16] their common sections AC, GI are parallel; but AG, CI are also parallel by hypothesis, because AD is a parallelogram. Therefore AI is a parallelogram. Again, since the parallel planes ABC, GHI are cut by the plane AH, [Margin: f $\rightarrow$ *Elements* XI.16] their common sections AB, GH are parallel; and AG, BH are likewise parallel, because AE is a parallelogram. Therefore AH is a parallelogram. In the same way CH is shown to be a parallelogram. Now the sides AC, AB of triangle ABC are equal to the sides GI, GH of triangle GHI, [Margin: g $\rightarrow$ *Elements* I.34] since AC equals GI and AB equals GH; [Margin: h $\rightarrow$ *Elements* XI.10] and the angles CAB, IGH are also equal, because the lines CA, AB are parallel to the lines IG, GH and do not lie in the same plane. [Margin: i $\rightarrow$ *Elements* I.4] Therefore the triangles ABC, GHI are equal to one another and equiangular. [Margin: k $\rightarrow$ *Elements* VI.4] Hence the sides about their equal angles will be proportional, and for that reason the triangles will be similar.

Therefore the solid ABCIHG — contained by the two opposite planes ABC, CHI, which are equal and similar, together with the parallelograms AI, IB, BG — is a prism. In the same way ABCQPK, KPQSRL, FEDIHG, FEDVTM, MTVYXN, NXY α ZO, LRSIHG, OZ α IHG are prisms.

Diagram. A long triangular prism lying horizontally, partitioned by the parallel cutting planes into eight equal triangular prisms; each cross-section is drawn as a vertical triangle with an internal diagonal. The prism has three parallel lengthwise edges, so the labels form three rows.

- Bottom edge (left to right): L, K, A, G, F, M, N, O
- Middle edge (left to right): R, P, B, H, E, T, X, Z
- Top edge (left to right): S, Q, C, I, D, V, Y, α (Greek alpha)
- The eight triangular cross-sections (bottom / middle / top vertices): (L, R, S), (K, P, Q), (A, B, C), (G, H, I), (F, E, D), (M, T, V), (N, X, Y), (O, Z, α)

Now since the parallelograms AI, KC, LQ are equal and similar, and likewise AH, LB, LP and CH, QB, SP, all the planes of the prisms ABCQPK, KPQSRL will be equal and similar. Hence, by Definition 10 of this book, the prisms ABCIHG, ABCQPK, KPQSRL are equal. By the same reasoning the prisms FEDIHG, FEDVTM, MTVYXN, NXY α ZO are equal. Therefore, however many times the base LI is a multiple of the base AI, so many times will the prism LRSIHG be a multiple of the prism

ABCIHG; and however many times the base OI is a multiple of the base FI, so many times will the prism OZαIHG be a multiple of the prism FEDIHG. Now if the base LI (a multiple of the base AI, the first magnitude) is equal to the base OI (a multiple of the base FI, the second magnitude), then the prism LRSIHG (a multiple of the prism ABCIHG, the third magnitude) is also equal to the prism OZαIHG (a multiple of the prism FEDIHG, the fourth magnitude); for when the bases LI, OI are equal, all the planes of the prism LRSIHG are also equal and similar to all the planes of the prism OZαIHG. And if the base LI is greater than the base OI, the first prism is likewise greater, and if less, less — in whatever multiplication this is carried out. Therefore, by Definition 6 of Book V, as the base AI, the first magnitude, is to the base FI, the second magnitude, so is the prism ABCIHG, the third magnitude, to the prism FEDIHG, the fourth magnitude. In the same way it will be shown that prism is to prism as the base AH is to the base FH, and as the base CH is to the base DH. Which was the thing proposed.

Next, let ABCDEFGHIK be a prism whose opposite planes are polygonal — namely pentagons — and let it be cut by the plane LMNOP. I say once more, as before, that as the base CM is to the base NG, so is the solid ABCDELMNOP to the solid LMNOPFGHIK.

Diagram. A pentagonal prism seen in oblique view, its two pentagonal faces resolved into triangles and cut across the middle by the pentagonal plane LMNOP, so that it is divided into a front block and a back block of three triangular prisms each. Readable vertices: A (top left), L (top centre), B (left), M (centre), G (upper right), K (far right), E (interior left), P (interior centre), C (lower left), N (lower centre), H (lower right), D (bottom left), O (bottom centre). The front-pentagon / cutting-plane / back-pentagon correspondence is $A \rightarrow L \rightarrow F$ [?], $B \rightarrow M \rightarrow G$, $C \rightarrow N \rightarrow H$, $D \rightarrow O \rightarrow I$ [?], $E \rightarrow P \rightarrow K$.

For if the opposite parallel planes are resolved into triangles, the prism too will be resolved into the same number of prisms whose opposite planes are triangles. Hence, as the base BP is to the base MK, so will the prism ABEPML be to the prism LMPKGF, by what has already been demonstrated. In the same way, as the base CP is to the base NK, so will the prism BCEPMN be to the prism MNPKGH, and the prism CDEPNO to the prism NOPKHI. [Margin: $l \rightarrow$ *Elements* VI.1] But as BP is to MK and CP to NK, so is the straight line EP to the straight line PK — that is, as CN is to NH; and as CN is to NH, so is CM to NG. Therefore the prisms ABEPML, BCEPMN, CDEPNO have to the prisms LMPKGF, MNPKGH, NOPKHI the same ratio that CM has to NG, and consequently the same ratio to one another. [Margin: $m \rightarrow$ *Elements* V.12] And as one is to one, so are all to all. Therefore, as the prism

BCEPMN is to the prism MNPKGH — that is, as the base CM to the base NG — so will the prism ABCDELMNOP, composed of three, be to the prism LMNOPFGHIK, made up of three. Which was the thing proposed. The demonstration will be exactly the same for any other prism.

This same result can, however, be shown for every prism by the demonstration we used for a prism whose opposite planes are similar triangles. For if the same construction is made, the prism being produced in both directions, all the parallel cutting planes will be equal and similar to one another, [Margin: n → *Elements* XI.16] because their sides are parallel to the sides of the [opposite] planes, [Margin: o → *Elements* XI.10] and consequently contain equal angles, and so on. From this, as before, it will be shown that the prisms on the one side are equal to one another, and likewise the prisms on the other side, and so on.

COROLLARIVM.

From these things it follows that if any prism is cut by a plane equidistant from the opposite planes, the section is a figure equal and similar to the opposite planes — just as it was demonstrated in the earlier prism that the triangle GHI is equal and similar to triangle ABC, and therefore to triangle DEF. For the demonstration is the same in every case. The same may be said of the parallelepiped.

PROBL. 4. PROPOS. 26.

[Margin: xxvi.]

On a given straight line, and at a given point on it, to construct a solid angle equal to a given solid angle.

Let a solid angle be constructed at the point A on the given straight line AB, equal to the solid angle C, which is contained by the three plane angles DCE, DCF, FCE not lying in the same plane. [Margin: a → *Elements* I.11] From F let a perpendicular FG be drawn to the plane through CD, CE, and let the straight lines DF, DG, EF, EG, CG be joined. Then cut off AH equal to CD, [Margin: b → *Elements* XI.23] make the angle HAI equal to the angle DCE, and let the straight line AI be equal to the straight line CE. Again, in the plane through AH, AI construct the angle HAL equal to the angle DCG (which lies in the plane drawn through CD, CE), with the straight line AL equal to the straight line CG. [Margin: c → *Elements* XI.12] From L erect a perpendicular LK to the plane in which the three lines AH, AL, AI lie, let it be set equal to FG, and join the straight line KA. I say that the solid angle A, contained by the three plane angles HAI, HAK, KAI, is equal to the given

solid angle C. For when the straight lines HK, HL, IK, IL have been joined: since the sides AH, AL of triangle AHL are equal to the sides CD, CG of triangle CDG, and the angles HAL, DCG are equal by construction, [Margin: d → *Elements* I.4] the bases HL, DG will be equal. Again, since, when the equal angles HAL, DCG are subtracted from the equal angles HAI, DCE, the remainders LAI, GCE are equal — then, since the sides AL, AI of triangle ALI are equal to the sides CG, CE of triangle CGE by construction,

Diagram. Two triangular pyramids (tetrahedra) drawn side by side in oblique view. Left pyramid (the constructed solid angle A): apex A (top), with L (interior), H (lower left), I (lower right) and B (bottom). Right pyramid (the given solid angle C): apex C (top), with G (interior, foot of the perpendicular FG), D (lower left), E (lower right) and F (bottom).

the bases LI, GE will also be equal. [Margin: e → *Elements* I.4] And since the sides LH, LK are equal to the sides GD, GF, and the angles HLK, DGF are right by Definition 3 of this book, [Margin: f → *Elements* I.4] the bases HK, DF will also be equal. Hence, since the sides AH, AK of triangle AHK are equal to the sides CD, CF of triangle CDF by construction — (for since AL, LK are equal to the sides CG, GF by construction, and contain equal angles, namely right angles, by Definition 3 of this book, [Margin: g → *Elements* I.4] the bases AK, CF will be equal) — [Margin: h → *Elements* I.8] the angles HAK, DCF will also be equal. Finally, since the sides LI, LK are equal to the sides GE, GF, and the angles ILK, EGF are right by Definition 3 of this book, [Margin: i → *Elements* I.4] the bases IK, EF will be equal; and since the sides AI, AK of triangle AIK are equal to the sides CE, CF of triangle CEF by construction, [Margin: k → *Elements* I.4] the angles IAK, ECF will also be equal. Therefore the three plane angles HAI, HAK, KAI, which compose the solid angle A, are equal to the three plane angles DCE, DCF, FCE, which compose the solid angle C; and consequently the solid angle A is equal to the solid angle C. Thus on the given straight line, and at a given point on it, we have constructed a solid angle equal to a given solid angle. Which was to be done.

SCHOLIVM.

If a given solid angle is contained by more than three plane angles, straight lines must be drawn subtending all the plane angles and lying in one and the same plane (namely in that plane which cuts all the straight lines of the plane angles), so that they form a plane polygon and produce a certain many-sided pyramid. For if the polygonal figure is resolved into triangles, there will be as many triangular pyramids as there are triangles contained in the polygon. If, then, solid angles are constructed equal to the solid angles of all the triangular pyramids,

each contained by three plane angles, the whole solid angle so constructed will be equal to the whole given solid angle.

Diagram. Two many-sided pyramids drawn side by side, each seen from above its apex, with the base polygon resolved into triangles by lines drawn from the apex. Left pyramid: apex A, with base vertices B (upper left), F (centre), E (upper right), C (lower left), D (lower right) — the pentagon BCDEF. Right pyramid: apex G, with base vertices H (upper left), M (centre), L (upper right), I (lower left), K (lower right).

For example, if a solid angle equal to the solid angle A — contained by the five plane angles BAC, CAD, DAE, EAF, FAB — is to be constructed, straight lines BC, CD, DE, EF, FB must be drawn subtending the plane angles, so that the pentagon BCDEF is formed, which is resolved into the three triangles BCF, FCD, DEF. Then, for the solid angle A contained by the three plane angles BAC, CAF, FAB, a solid angle G is to be constructed equal to it, contained by the three plane angles HGI, IGM, MGH. And for the solid angle A contained by the three plane angles CAF, FAD, DAC, an equal solid angle G, contained by the three plane angles IGM, MGK, KGI. And finally, for the solid angle A contained by the three plane angles DAE, EAF, FAD, an equal solid angle G, contained by the three plane angles KGL, LGM, MGK. For in this way the whole solid angle G will become equal to the whole solid angle A. And in this manner a solid angle equal to any given solid angle will be constructed.

PROBL. 5. PROPOS. 27.

[Margin: xxvii.]

On a given straight line, to describe a solid parallelepiped similar and similarly situated to a given solid parallelepiped.

Let it be required, on the given straight line AB, to describe a parallelepiped similar and similarly situated to the parallelepiped CD. [Margin: a → *Elements* XI.26] At the straight line AB and its point A, construct a solid angle equal to the solid angle C, so that the three plane angles HAI, IAB, BAH are equal to the three plane angles ECG, GCF, FCE. Then, [Margin: b → *Elements* VI.12] as CF is to CG, so make AB to AI, and as CG is to CE, so AI to AH; and, ex aequali, as CF is to CE, so will AB be to AH.

Diagram. Two solid parallelepipeds (rectangular boxes) drawn in oblique view. Left box (the constructed solid AK): K (top rear), I (upper interior), H (front left), A

(bottom front left), B (bottom front right). Right box (the given solid CD): D (top rear right), G (interior), E (front left), C (bottom front left), F (bottom front right).

After this, complete the parallelepiped AK, by completing the parallelograms BH, HI, IB and drawing through I, B, H the planes IK, BK, HK parallel to the parallelograms BH, HI, IB. I say that the parallelepiped AK is similar and similarly situated to the parallelepiped CD. For since the angles BAH, FCE are equal, and the sides about them proportional — namely, as BA is to AH, so FC to CE — by construction, the parallelograms HB, EF will be similar and similarly situated. For the same reason the parallelograms HI, EG and IB, GF will be similar and similarly situated. Therefore the three planes BH, HI, IB of the solid AK are similar and similarly situated to the three planes FE, EG, GF of the solid CD. [Margin: c → *Elements* XI.24] But in each solid any three planes are equal and similar to the three remaining opposite ones. Hence the six planes of the solid AK are similar and similarly situated to the six planes of the solid CD; and consequently, by Definition 9 of this book, the solids AK, CD are similar and similarly situated. Therefore on the given straight line we have described a parallelepiped similar and similarly situated to the given solid. Which was to be done.

THEOR. 23. PROPOS. 28.

[Margin: xxviii.]

If a solid parallelepiped is cut by a plane through the diagonals of opposite faces, the solid is bisected by that plane.

Let AB be a parallelepiped in which the opposite planes are AH, EB, whose diagonals (or diameters) are the straight lines CG, DF. Now since each of CD, GF is parallel [Margin: a → *Elements* I.34] and equal to AE, since CE, GE are parallelograms, [Margin: b → *Elements* XI.9] CD, GF will be both parallel and equal to one another; [Margin: c → *Elements* I.33] and consequently the lines CG, DF that join them will be parallel and equal, and therefore lie in one plane. I say that the plane drawn through CG, DF cuts the parallelepiped AB in half.

Diagram. A single solid parallelepiped in oblique view, with the bisecting plane drawn through it. Vertices: F (top rear left), B (top rear right), E (upper front left), D (upper rear right), G (lower front left), H (lower rear right), A (bottom front left), C (bottom front centre). The diagonals CG and DF of the opposite faces are drawn.

[Margin: d → *Elements* XI.24] For since the planes AH, EB are equal and similar parallelograms, their halves — namely the triangles AGC, GCH and EFD, FDB — will be equal to one another; and the sides about the equal angles GAC, CHG, FED, DBF are proportional. [Margin: e → *Elements* VI.6] Therefore the said triangles will also be similar. [Margin: f → *Elements* XI.24] Since, then, the parallelogram AF is equal and similar to the parallelogram CB, and AD to GB, and CF is common, the two triangles AGC, EFD and the parallelograms AF, AD, CF of the prism ACGFED will be equal and similar to the two triangles HCG, BDF and the parallelograms CB, BG, CF of the prism HGCDBF; and therefore, by Definition 10 of this book, the prisms will be equal. And since these make up the parallelepiped AB, the parallelepiped AB will be cut in half. Therefore, if a solid parallelepiped is cut by a plane, and so on. Which was to be demonstrated.

THEOR. 24. PROPOS. 29.

[Margin: xxix.]

Solid parallelepipeds constructed on the same base and of the same height, whose standing edges lie on the same straight lines, are equal to one another.

On the base AB, at the same height — that is, between the same parallel planes — let two parallelepipeds ACDE, AFGE be constructed, whose standing edges, going out from the four angles of the base, lie on the same lines: namely AI, AK, EL, EM on the straight line IM, and HC, HF, BD, BG on the straight line CG. I say that they are equal to one another.

Diagram. A solid parallelepiped in oblique view with internal construction lines. Readable vertices: C (top left), F (top centre), D (top right), G (top far right), H (left), B (centre, interior), I (lower left), K (lower centre), L (lower right), M (far right), A (bottom left), E (bottom centre).

[Margin: a → *Elements* I.35] For since the parallelograms AL, AM, constructed on the same base AE and between the same parallels, are equal, then, when the common trapezium AELK is taken away, the triangles AIK, ELM will also be equal. [Margin: b → *Elements* I.34] Now since all the sides of triangle AIK are equal to all the sides of triangle HCF, the triangle AIK will be equiangular and equal to triangle HCF, by what we showed at Proposition 8 of Book I; [Margin: c → *Elements* VI.4] and therefore they will have the sides about their equal angles proportional, and so will be similar to one another. For the same reason triangle ELM will be equal and similar to triangle BDG. [Margin: d → *Elements* XI.24]

Again, the parallelogram AC is equal and similar to the parallelogram ED, and by the same reasoning the parallelogram AF to the parallelogram EG; [Margin: e → *Elements* I.20] and also IF to LG, since the bases IK, LM are equal. ([Margin: f → *Elements* I.54 [?] — Book I has only 48 propositions; probable compositor error or misread] For since the straight lines IL, KM are equal to AE, they will also be equal to one another; hence, when the common part KL is taken away, IK, LM will be equal.) Therefore all the planes of the prism AIKFCH will be equal and similar to all the planes of the prism ELMGDB. Hence, by Definition 10 of this book, the said prisms are equal; and consequently, when the common solid AHFKLDBE is added, the parallelepipeds on the same base will be equal, and so on. Which was to be demonstrated.

SCHOLIUM.

This proposition is very similar to Proposition 35 of Book I. For what is there demonstrated about parallelograms is here demonstrated about parallelepipeds, by almost the same means, provided that in place of the triangles prisms are assumed, as is clear from the demonstration. Hence the remaining two cases — namely when the point K falls on the point L, or beyond L — will be demonstrated in the same way, as appears in these figures. For in the second case, if from the equal parallelograms AL, AM the common triangle ALE is taken away, the remaining triangles AIK, ELM will be equal. Hence, as before, the prisms AIKFCH, ELMGDB will be equal. Therefore, when the common prism AKEBHD is added, the parallelepipeds ACDE, AFGE will become equal.

In the third case, if from the equal parallelograms AL, AM the common triangle ANE is taken away, the remaining trapezia AILN, NKME will be equal; and if the common triangle KLN is added to these, the whole triangles AIK, ELM will become equal. Hence, as before, the prisms AIKFCH, ELMGDB will be equal; and if the common prism KLNODF is taken away from them — (for the common section of the planes AF, DE is the straight line NO) — there will remain the solid contained by the planes AILN, NOHA, AICH, HCDO, CDLI, DLNO, equal to the solid contained by the planes ENKM, MKFG, GFOB, FKNO, BGME, EBON. If, then, the common prism AENOBH is added to each, the parallelepipeds ACDE, AFGE will become equal to one another.

Diagram. The first of two solid parallelepipeds drawn side by side for the scholium (the second case), in oblique view with internal construction lines and heavy foxing. Readable vertices: C, D, G (top), F, H (left), B (interior), L, K (lower centre), M (right), A, E (bottom). Foxing obscures several interior labels.

Diagram. The second of the two scholium parallelepipeds (the third case), in oblique view with internal construction lines and foxing. Readable vertices: C, D, F (top), H (left), O (interior), E (centre), I, L, K, N (lower), A (bottom). Several interior labels obscured by foxing.

THEOR. 25. PROPOS. 30.

[Margin: xxx.]

Solid parallelepipeds constructed on the same base and of the same height, whose standing edges do not lie on the same straight lines, are equal to one another.

On the base AB, at the same height — that is, between the same parallel planes — let the parallelepipeds AIDK, AMFK be constructed, whose standing edges going out from the four angles of the base — AC, AE, KG, KL; HI, HM, BD, BF — do not lie on the same straight lines; that is, neither do CI, GD, when produced, pass through the points M, E, F, L, nor do CG, ID when produced, and so on. I say that the parallelepipeds AIDK, AMFK are equal. For since the planes CD, EF, opposite to the base AB, lie in the same plane (on account of the equal height of the parallelepipeds), let the straight lines CG, ID be produced in that plane, of which CG cuts EM, LF, produced, at the points N, O; and ID [cuts] the same...

Book XI (continued)

Proposition 30 (concluded)

(Continuing the demonstration begun in the previous pages: the produced lines CG and ID cut EM and LF) — at the same points P and Q; and let the straight lines AN, KO, HP, BQ be joined.

[Margin: a — 34.pri. (*Elements* I.34)] Now the straight lines PQ and MF are equal, since they are opposite sides in the parallelogram FP; and MF is equal to HB. Therefore PQ and HB are also equal. Moreover they are parallel, on account of the parallelogram HDB. [Margin: b — 33.prim. (*Elements* I.33)] Consequently HP and BQ are likewise parallel and equal, and so HPQB is a parallelogram. By the same reasoning HPNA, ANOK, and KOQB are parallelograms; and NOQP is a parallelogram as well. Therefore APQR is a parallelepiped.

[Margin: c — 29.prim. (*Elements* I.29)] For this reason the parallelepiped AIDK is equal to the parallelepiped APQK, since both have the same base AB and their standing edges lie on the same straight lines CO and IQ. In the same way the

parallelepiped AMFK will be equal to that same parallelepiped APQK, since both have the same base AB and their standing edges lie on the same straight lines NM and OF. Therefore the parallelepipeds AIDK and AMFK are equal to one another. Solid parallelepipeds on the same base, and so forth. Which was to be demonstrated.

Diagram. A three-dimensional wireframe showing the compound of parallelepipeds AIDK and AMFK standing on the common base AB, with the diagonal edges CG and ID produced and the auxiliary solid APQR/APQK completed; internal diagonals are drawn on several faces. Readable vertex labels, by position:

- A (bottom-left) and K (bottom-right): the base corners
- M (top-left) and F (top-right): the upper front corners
- P (upper, right of M), D (upper-centre), Q (upper-right, below F)
- H (left flank), E (centre), B (centre, right of E), I (right flank, below Q)
- C (lower-left), N (lower-centre), G (lower-centre, right of N)
- One further interior label at the lower-right is foxed and cannot be resolved [?]; the points O and R named in the text are expected here but are not clearly legible on the leaf [?].

(interior labels foxed).

Scholium

Converse of the proposition, and of the one preceding, stated in this manner:

Equal solid parallelepipeds on the same base — whether their standing edges are set upon the same straight lines or not — are at the same height.

[Margin: d — 29.vel 30.undec. (*Elements* XI.29 or 30)] For suppose one is said to be taller. If from it a parallelepiped is cut off having the same height as the other, the part cut off and the remainder will be equal. Since, then, the whole is also supposed equal to this remainder, the part cut off will be equal to the whole. Which is absurd.

[Margin: xxxi. xxxii. (alternative proposition numbering)]

THEOR. 26. PROPOS. 31.

Solid parallelepipeds set up on equal bases and at the same height are equal to one another.

Let the parallelepipeds AEFG and CHIK be set up on the equal bases AB and CD at the same height. I say that these parallelepipeds are equal to one another.

First, let the standing edges AM, GN, LE, BF on the base AB, and the standing edges CP, KQ, OH, DI on the base CD, be perpendicular. This being so, all the said perpendiculars will be equal to one another, because of the equal height of the parallelepipeds. Produce CK in a straight line, let KR be made equal to LB, and let the angle RKS in the plane OK, when extended, be made equal to the angle BLA; let KS be put equal to LA, and let the parallelogram KT be completed. On this let the parallelepiped QSTV be constructed, with KQ perpendicular for its height.

Now since the sides KR and KS are equal to the sides LB and LA, and the angles RKS and BLA are equal, the parallelograms KT and LG will be equal and similar. Again, since the sides KQ and KS are equal to the sides LE and LA, and the angles QKS and ELA are right — by definition 3 of this book, because KQ and LE are set as perpendiculars to the planes KT and LG — the parallelograms QS and EA will also be equal and similar. In the same way, since the sides KR and KQ are equal to the sides LB and LE, and the angles QKR and ELB are right, by the same definition 3 of this book the parallelograms KV and LF will likewise be equal and similar.

[Margin: a — 24.unde. (*Elements* XI.24)] Therefore, since the three faces KT, QS, KV of the parallelepiped QSTV are equal and similar to the three faces LG, EA, LF of the parallelepiped AEFG — [Margin: b — 24.unde. (*Elements* XI.24)] and both these and those are equal and similar to the three remaining faces opposite them — the parallelepipeds QSTV and EAGF will, by definition 10 of this book, be equal to one another.

Let the straight lines DK and TS, when produced, meet at δ , and IQ and XY at ϵ , and let the parallelepiped Q δ γ V be completed. Likewise let HI and β V, when drawn out, meet at α , and OD and γ R at Z, and let the parallelepiped IKR α be completed. Now since the parallelepipeds QSTV and Q δ γ V have the same base KV and are at the same height — namely between the same parallel planes KV and δ X — and their standing edges KS, K δ , RT, R γ , QY, Q ϵ , VX, V β lie on the same straight lines δ T ϵ X, [Margin: c — 29.unde. (*Elements* XI.29)] they will be equal to one another. But the parallelepiped QSTV is equal to the parallelepiped EAGF. Therefore the parallelepiped Q δ γ V will also be equal to that same parallelepiped EAGF.

[Margin: d — 35.prim. (*Elements* I.35)] Now since the parallelograms KT and Ky are equal to one another, and KT is equal to LG, Ky will also be equal to LG — that is, to CD — since the bases LG and CD are supposed equal. [Margin: e — 7.quinti.]

(*Elements* V.7)] Therefore, as CD is to DR, so is K γ to DR. [Margin: f — 25.unde. (*Elements* XI.25)] But as the base CD is to the base DR, so is the solid CHIK to the solid KI α R, since the parallelepiped CH α R is cut by the plane IK parallel to its opposite faces CH and α R. And by the same reasoning, as K γ is to DR, so is the solid Q $\delta\gamma$ V to the solid IKR α , since the parallelepiped I $\delta\gamma\alpha$ is cut by the plane KV parallel to its opposite faces D $\alpha\delta\beta$. [Margin: g — 9.quinti. (*Elements* V.9)] Therefore the parallelepipeds CHIK and Q $\delta\gamma$ V will be equal, since they have the same ratio to the same solid IKR α — namely the ratio that the equal bases CD and K γ have to the same base DR. And since the parallelepiped Q $\delta\gamma$ V has been shown equal to the parallelepiped AEFG, the parallelepipeds AEFG and CHIK will likewise be equal. Which was the thing proposed.

Diagram. A wide three-dimensional construction spanning the text column: the parallelepiped AEFG (shown with the auxiliary lettering L, B, M, N about its faces) at the left, together with the constructed solids QSTV, Q $\delta\gamma$ V, and IKR α drawn with Greek-lettered vertices toward the right. Readable Latin labels include E, F (top left corners), L, B, M, N, A, G, and toward the right Q, S, T, V, R, X, Z; Greek-lettered vertices α , β , γ , δ , ϵ appear among the auxiliary solids. Many of the small peripheral and Greek labels are foxed and cannot be resolved individually [?][?].

(numerous foxed Latin and Greek labels).

(Text continues on the following page.)

Now let the standing edges AM, GN, LE, BF to the base AB, and CP, KQ, OH, DI to the base CD, no longer be perpendicular. [Margin: h — 11.unde. (*Elements* XI.11)] From the points E, F, M, N let the perpendiculars ER, FS, MT, NV be dropped to the plane in which the base AB lies; and likewise from the points H, I, P, Q let the perpendiculars HX, IY, PZ, Q α be dropped to the plane in which the base CD lies. Then all these perpendiculars will be equal to one another, since the heights of the said parallelepipeds are equal.

Draw the straight lines RS, SV, VT, TR; likewise draw the straight lines XY, Y α , α Z, ZX, so that the parallelepipeds ETVF and HZ α I are formed. Since these are of the same height and have their standing edges perpendicular, they will be equal to one another, as has been shown. [Margin: i — 29.vel 30.undec. (*Elements* XI.29 or 30)] But the parallelepiped ETVF is equal to the parallelepiped AEFG, since it has the same base EN with it and the same height; and the parallelepiped HZ α I is by the same reasoning equal to the parallelepiped CHIK, since it has the same base HQ with it and the same height. Therefore the parallelepipeds AEFG and CHIK are also equal to one another. The same will be shown if the standing edges of one

parallelepipeds are perpendicular to the base while those of the other are not. Hence solid parallelepipeds set up on equal bases, and so forth. Which was to be demonstrated.

Diagram. Two three-dimensional parallelepipeds standing side by side (the oblique case of the proposition), each shown with its dropped perpendiculars and internal diagonals. Left solid — readable labels: E, F (top corners), I, R, B, S (upper flank), M, N (mid), A, T, G (base). Right solid — readable labels: H, I (top corners), O, X, D, and a lowercase y/γ (upper flank), P, Q (mid), C, Z, K, α (base), R (right flank). Several small labels are foxed [?][?].

(many crowded foxed labels).

Scholium

Perhaps, following Campanus, it can be demonstrated more expeditiously that two parallelepipeds set up on equal bases and at the same height, whose standing edges are perpendicular to the bases, are equal to one another — namely from the bases alone, without the construction of parallelepipeds — though the demonstration is the same as ours. For let there be two equal bases, namely the two parallelograms AB and CD. I say that the parallelepipeds set up on them, at the same height, whose standing edges to the bases AB and CD are perpendicular, are equal to one another. Produce the two sides DE and CE toward the parts of E. Then make the angle FEG equal to the angle L; put the straight line EF equal to LB, and the straight line EG equal to LA; and let a parallelogram be completed which will be equal and similar to the parallelogram AB.

Let HG, when produced, meet DE produced at I, and let the parallelogram IK be completed. Now let there be understood, set up on the bases EH, IF, EK, parallelepipeds of the same height as the parallelepipeds constructed on AC and CD, their standing edges being perpendicular to the said bases. It is clear, then, that the solid on EH is equal and similar to the solid on AB — by definition 10 of this book — since its faces are equal and similar to the faces of that solid, by construction. [Margin: k — 29.unde. (*Elements* XI.29)] Moreover the parallelepipeds on EH and IF are equal, since they have the same base — whose lowest side is EF, erected upon the plane IK — and likewise the faces opposite the base, whose lowest sides GH and IM lie in the same plane, and their standing edges EG, FH, EI, FM, and so forth, lie on the same straight lines IH, and so forth. [Margin: l — 35.prim. (*Elements* I.35)] Therefore the parallelepiped on IF will be equal to the parallelepiped on AB. Since the parallelogram EH is equal to CD, IF and CD will be equal. [Margin: m — 7.quinti. (*Elements* V.7)] Therefore it will be

that as IF is to EK, so is CD to EK. [Margin: n — 25.unde. (*Elements* XI.25)] But as IF is to EK, so is the solid on IF to the solid on EK, since the solid on IK is cut by the plane on EF, erected parallel to the faces erected on DK and IM. And by the same reasoning, as CD is to EK, so is the solid on CD to the solid on EK, since the solid on CK is cut by the plane on DE, erected parallel to the faces erected on CN and FK. Therefore, as the solid on IF is to the solid on EK, so is the solid on CD to the solid on EK. [Margin: o — 9.quinti. (*Elements* V.9)] And accordingly the parallelepipeds on IF and CD will be equal. But it has been shown that the parallelepiped on IF is equal to the parallelepiped on AB. Therefore the parallelepipeds on AB and CD will be equal. Which was the thing proposed.

Diagram. A plane (base-level) figure of adjoining parallelograms illustrating the Campanus construction. Readable labels, by position: N (top-left), D (top-centre), K (top-right); L (far left), B (left-centre) with C just below it, E (centre), F (right-centre); A (bottom-left), G (bottom-centre), M and H (bottom-right). A small subscript letter near G is faint [?].

This proposition will also be converted, thus:

Equal solid parallelepipeds on equal bases are at the same height. And equal parallelepipeds at the same height are on equal bases, if they do not have the same base.

[Margin: p — 31.und. (*Elements* XI.31)] For if one is supposed taller than the other: if from it a parallelepiped is cut off having the same height as the other, the part cut off will be equal to the other. Since, then, the whole is supposed equal to the other, the part cut off will be equal to the whole. Which is absurd.

And if they are at the same height, and the base of one is supposed larger than that of the other: if from it a base equal to the other's is cut off, and on the base so cut off a parallelepiped of the same height is understood, we shall show in the same way that the part is equal to the whole. Which is absurd.

THEOR. 27. PROPOS. 32.

Solid parallelepipeds under the same height are to one another as their bases.

Let there be two parallelepipeds ABCD and EFGH, of the same height, on the bases AB and EF. I say that solid is to solid as base is to base. [Margin: a — 45.prim. (*Elements* I.45)] For on the straight line EK let a parallelogram IK be constructed, equal to the parallelogram AB,

[Margin: xxxiii. (alternative proposition numbering)]

(text continues on the next page)

in the angle IEK, which is made equal to the angle ENF. The parallelograms EF and IK will make up one whole parallelogram FI, as was demonstrated in Proposition 45 of Book I. If, then, the other faces of the parallelepiped EFGH are produced toward the parts of EG, and the whole is completed into one parallelepiped IFLH, [Margin: a — 31.unde. (*Elements* XI.31)] the parallelepipeds ABCD and IKLM will be equal, since they have equal bases, by construction AB and IK, and the same height, by hypothesis. [Margin: b — 7.quinti. (*Elements* V.7)] Therefore, as the solid IKLM is to the solid EFGH, so is the solid ABCD to the same solid EFGH. [Margin: c — 25.unde. (*Elements* XI.25)] But the solid IKLM is to the solid EFGH as the base IK — that is, as the base AB equal to it — is to the base EF. Therefore the solid ABCD will also be to the solid EFGH as the base AB is to the base EF. Solid parallelepipeds under the same height are therefore to one another as their bases. Which was to be demonstrated.

Diagram. Two three-dimensional parallelepipeds side by side, the left being ABCD and the right EFGH with the completed solid IKLM adjoined. Readable labels — left solid: C (top-left), B (interior), A (bottom-left), D (right), I (bottom). Right solid: L, G (top corners), K (interior), F (right), M (interior/lower), E (bottom-centre), N (bottom-right), H (right flank), I (bottom-left). One or two labels are faint [?].

Scholium

Conversely, if solid parallelepipeds are to one another as their bases, they are under the same height.

For if the height is not supposed the same: from the greater let a portion be cut off equal to the smaller, and let a plane be drawn parallel to the base. [Margin: d — 32.undec. (*Elements* XI.32)] Then, as base is to base, so will the parallelepiped be to the parallelepiped that was cut off. But so it was also to the whole. [Margin: e — 9.quinti. (*Elements* V.9)] Therefore the part cut off is equal to the whole. Which is absurd.

[Margin: xxxvi. (alternative proposition numbering)]

THEOR. 28. PROPOS. 33.

Similar solid parallelepipeds are to one another in the triplicate ratio of their corresponding edges.

Let there be similar parallelepipeds ABCD and EFGH on the similar bases AB and EF, in which the corresponding edges are AI and EK. I say that the ratio of the parallelepiped ABCD to the parallelepiped EFGH is the triplicate of the ratio of the corresponding edges AI and EK. For produce AI to L, and let IL be equal to EK, or to GR; likewise produce DI to M, and let IM be equal to HK, or to GO; likewise produce BI to N, and let IN be equal to KF, or to GS. Then, completing the parallelograms LM, LN, IT, let the parallelepiped TXIV be completed.

Now since the sides IL and IM are equal to the sides GR and GO, [Margin: a — 15.prim. (*Elements* I.15) [?]] and the contained angles are equal — for the angle LIM is equal to the angle AID, which, on account of the similarity of the parallelepipeds, is equal to the angle EKH, or RGO — the parallelograms IX and GF will be similar and equal. By the same reasoning LN and RS will be similar and equal, and likewise IT and GE. Therefore the three faces IX, LN, IT of the parallelepiped TXIV are similar and equal to the three faces GF, RS, GE of the parallelepiped EFGH. [Margin: b — 24.unde. (*Elements* XI.24)] But the three faces of each are similar and equal to the three remaining faces opposite them. Therefore the parallelepipeds TXIV and EFGH are equal and similar, by definition 10 of this book.

Again, completing the parallelograms MB, BL, LM, let the parallelepiped MPBL be completed; likewise, completing the parallelograms IY, DL, IQ, let the parallelepiped IYQZ be completed. Now since, on account of the similarity of the parallelepipeds ABCD and EFGH, as AI is to EK — that is, to IL — so is DI to HK, that is, to IM; and BI to FK, that is, to IN: [Margin: c — 1.sexti. (*Elements* VI.1)] and as AI is to IL, so is the parallelogram AD to DL; and as DI is to IM, so is the parallelogram DL to LM; and as BI is to IN, so is the parallelogram BL to LN. Therefore it will be that as AD is to DL, so is DL to LM, and BL to LN. [Margin: d — 32.unde. (*Elements* XI.32)] But as the base AD is to the base DL, so is the parallelepiped ADCB to the parallelepiped DLYQ; and as the base DL is to the base LM, so is the parallelepiped DLYQ to the parallelepiped LMBP; and as the base BL is to the base LN, so is the parallelepiped LMBP to the parallelepiped LNTX. Therefore it will be that as the parallelepiped ADCB is to the parallelepiped DLYQ, so is the parallelepiped DLYQ to the parallelepiped LMBP, and the parallelepiped LMBP to the parallelepiped LNTX.

And accordingly the four magnitudes ADCB, DLYQ, LMBP, LNTX are continuously proportional; and therefore the ratio of the first, ADCB, to the fourth, LNTX — that is, to EFGH — will be the triplicate of the ratio of the first, ADCB, to the second, DLYQ, by definition 10 of Book V. [Margin: e — 32.und. (*Elements* XI.32)]

But as ADCB is to DLYQ, so is the base AD to the base DL; [Margin: f — 1.sexti. (*Elements* VI.1)] and as AD is to DL, so is the straight line AI to IL, that is, to EK. Therefore the ratio of the parallelepiped ADCB to the parallelepiped EFGH is the triplicate of the ratio of the corresponding edges, namely of AI to EK. Wherefore similar solid parallelepipeds are to one another in the triplicate ratio of their corresponding edges. Which was to be demonstrated.

Diagram. A large compound three-dimensional construction: the parallelepiped ABCD (upper solid) with the completed solids TXIV, DLYQ, LMBP, LNTX, IYQZ, MPBL, and the smaller solid EFGH (lower-left) drawn together. Readable labels — upper region: C, Q (top), B, P, D, Z, A, R, I, L, M, X, N, T, V; lower-left solid: G, O, F, S, H, E, K. The construction carries roughly twenty-two lettered vertices; several interior labels are foxed and cannot each be resolved with certainty [?][?].

(very many crowded/foxed labels).

Corollary

From this it is evident that if four straight lines are continuously proportional, then the parallelepiped described on the first is to the parallelepiped similar and similarly described on the second as the first line is to the fourth. For both the parallelepiped to the parallelepiped, as has been demonstrated, and the first line to the fourth, by definition 10 of Book V, have the triplicate ratio of the ratio of the first line to the second — namely of the corresponding edges.

THEOR. 29. PROPOS. 34.

[Margin: xxxiv. / xxxv. — the corresponding proposition numbers in the vulgate (Theon) Euclid; Clavius's Prop. 34 answers to *Elements* XI.34 and 35.]

In equal parallelepiped solids the bases and heights are reciprocally proportional; and those parallelepiped solids whose bases and heights are reciprocally proportional are equal.

(*Clavius's demonstration.*)

Let ADCB, EHGF be equal parallelepipeds on the bases AD, EH. I say that the bases AD, EH and the heights of the parallelepipeds ADCB, EHGF are reciprocally proportional — that is, as AD is to EH, so is the height of the solid EHGF to the height of ADCB.

First, let the standing lines AI, EK be perpendicular to the bases AD, EH, so that (by Elem. VI def. 4) AI, EK are the heights of the parallelepipeds. Then if the

heights AI, EK are equal — since the parallelepipeds are also assumed equal — the bases AD, EH will likewise be equal, by what we showed at the end of Prop. 31 of this book. Therefore, as the base AD is to the base EH, so will the height EK be to the height AI; and accordingly the bases and heights are reciprocally proportional.

But if the heights AI, EK are unequal, let EK be the greater; from it cut off EL equal to AI, and through L draw the plane LM parallel to the base EH (by the scholium to Prop. 15 of this book). Since, then, the solids ADCB, EHGF are equal, [Margin: a. "7. quint." — Elem. V.7] it follows that as ADCB is to the solid EHML, so is EHGF to the same solid EHML. [Margin: b. "32. undec." — Elem. XI.32] But as the solid ADCB is to the solid EHML, so is the base AD to the base EH, since the heights AI, EL are assumed equal; and as the solid EHGF is to the solid EHML, so, by the same ratio, is the base KM [recte: KN?] to the base LN, since by this ratio the solids EHGF, EHML have the same height — that is, if the bases be taken as KN, LN, for they lie between the same parallel planes KN, GH. Therefore, as the base AD is to the base EH, so is the base KN to the base LN. [Margin: c. "1. sexti" — Elem. VI.1] But as KN is to LN, so is the straight line EK to the straight line EL — that is, to AI, which is equal to EL. Therefore, as the base AD is to the base EH, so is the height EK to the height AI; and accordingly the bases and heights are reciprocally proportional.

Now let the bases and heights be reciprocally proportional. I say the parallelepipeds are equal. For if the heights EK, AI are equal — since, by hypothesis, the base AD is to the base EH as the height EK is to the height AI — the bases AD, EH will also be equal. [Margin: d. "31. unde." — Elem. XI.31] Therefore the parallelepipeds ADCB, EHGF, having equal bases and the same height, will be equal to one another.

But if the height EK be the greater, cut off EL equal to AI, and through L draw the plane LM parallel to the base EH. Since, then, by hypothesis, as the base AD is to the base EH, so is the height EK to the height AI — that is, to EL, which is equal to AI — [Margin: e. "32. unde." — Elem. XI.32] and since, as the base AD is to the base EH, so is the solid ADCB to the solid EHML (the heights AI, EL being assumed equal); [Margin: f. "1. sexti" — Elem. VI.1] and as EK is to EL, so is KN to LN; [Margin: g. "32. unde." — Elem. XI.32] and as the base KN is to the base LN, so is the solid EHGF to the solid EHML (since the solids EHGF, EHML have the same height, if the bases be taken as KN, LN, for by this reasoning they lie between the parallel planes KN, GH) — it will follow that, as the solid ADCB is to the solid

EHML, so is the solid EHGF to the same solid EHML. [Margin: h. "9. quin." — Elem. V.9] And therefore the solids ADCB, EHGF are equal.

But now let there be proposed equal parallelepipeds ABCD, EFGH whose standing lines AI, KD, BM, LC and EN, OH, FQ, PG are not perpendicular to the bases AB, EF. From the points I, D, M, C let perpendiculars IR, DS, MV, CT be dropped to the plane of the base AB; likewise from the points N, H, Q, G let perpendiculars NX, HY, Q α , GZ be dropped to the plane of the base EF; and let the straight lines RS, TV, AT, SV and XY, Z α , XZ, Y α be joined. Then, by Elem. VI def. 4, the perpendiculars RI, XN will be the heights of the parallelepipeds. I say again that, as the base AB is to the base EF, so is the height XN to the height RI. [Margin: i. "29. vel 30. undec." — Elem. XI.29 or 30] For since the solids ABCD, EFGH are equal by hypothesis, [Margin: k. "24. und." — Elem. XI.24] and ABCD is equal to the solid RVCD (because they have the same base CD and the same height RI), and EFGH, by the same reasoning, is equal to the solid X α GH: the parallelepipeds RVCD, X α GH will also be equal. Therefore, since they have standing lines perpendicular to the bases CD, GH, it will follow — as already demonstrated — that as the base CD is to the base GH (that is, as the base AB is to the base EF), so is the height XN to the height RI. And accordingly the bases and heights are reciprocally proportional.

Now let the bases and heights be reciprocally proportional. I say the parallelepipeds are equal. For, with the figure constructed as before: since, as the base AB is to the base EF, so is the height XN to the height RI; [Margin: l. "24. vnd." — Elem. XI.24] and since AB is equal to CD, and EF to GH — it will also follow that, as the base CD of the solid RVCD is to the base GH of the solid X α GH, so is the height XN to the height RI. Therefore, since XN, RI are the standing lines perpendicular to the bases CD, GH, the parallelepipeds RVCD, X α GH will be equal, as already shown. [Margin: m. "29. vel 30. vnd." — Elem. XI.29 or 30] But these parallelepipeds are equal to the parallelepipeds ABCD, EFGH. Therefore the parallelepipeds ABCD, EFGH will also be equal. The same will be shown if the standing lines of one parallelepiped are perpendicular to its base, but those of the other are not. For this reason, the bases and heights of equal parallelepiped solids are reciprocally proportional, and so on. Which was to be shown.

Diagram. Two upright rectangular parallelepipeds of equal height, illustrating the first case of the proof (equal heights).

- Left solid (ADCB): C (top back-left), I (upper front-left, on the vertical edge), B (upper front-right, interior), A (bottom front-left), D (bottom front-right).

- Right solid (EHGF): G (top back), K (upper front-left), F (upper front-right), E (bottom front-left), H (bottom front-right).

Diagram. The case of unequal heights: a shorter box ADCB and a taller box EHGF cut by the plane LM at the height EL (= AI), forming the sub-solid EHML.

- Left solid (ADCB): C (top), B (upper-right, interior), A (bottom-left), D (bottom-right).
- Right solid (EHGF, with cutting plane): G (top back), K (upper front-left), M (mid, right edge, at the cut), L (mid, left interior, at the cut), I (left edge, lower), E (bottom front-left), N (bottom, mid), H (bottom front-right). F (upper front-right of EHGF) not clearly legible under the foxing [?].

Diagram. Two oblique (slanting) parallelepipeds ABCD and EFGH with perpendiculars dropped to the base planes and the auxiliary right parallelepipeds RVCD and $X\alpha GH$ constructed; heavily foxed and crowded with construction lines.

- Left figure (ABCD / RVCD): C (top-left), L (top-mid), M (top-right), B (right), T (upper-left), V (interior), I (left-lower, interior), D (lower-mid), R (bottom-left), A (bottom-mid), S (bottom-right).
- Right figure (EFGH / $X\alpha GH$): U (top-left) [?], Q (top-mid), Z (upper-left), α (interior, printed as an "oc"-like ligature), F (upper-right), N (bottom-left), H (bottom-mid), K (bottom, left of centre), E (bottom) [?], Y (bottom, adjoining E) [?], O (bottom-right). Vertex G of EFGH not separately legible [?].

SCHOLIUM

All that has been demonstrated in the six preceding propositions — namely 29, 30, 31, 32, 33, and 34 — applies also to prisms which have two opposite triangular faces, provided the aforesaid hypotheses are maintained. For if to two such prisms of the same height, set upon the same base or upon equal bases, two other prisms equal and similar to them be added, two parallelepipeds of the same height will be produced, standing upon the same or upon equal bases. [Margin: m. "29. 30. vel 31. undec." — Elem. XI.29, 30, or 31] Therefore such parallelepipeds will be equal; and consequently so too will the given prisms, being namely their halves.

Again, if to two of the aforesaid prisms of the same height, set upon different bases, two other prisms equal and similar to them be adjoined, two parallelepipeds will again be produced, so that parallelepiped is to parallelepiped as base to base. [Margin: n. "32. und." — Elem. XI.32] Therefore parallelepiped will be to parallelepiped as base to base; [Margin: o. "15. quin." — Elem. V.15] and

so prism will be to prism — namely the half of one parallelepiped to the half of the other — as base to base, if the bases of the prisms be parallelograms; or else triangle to triangle, that is, the half of one base to the half of the other, if the bases of the prisms be triangles.

Moreover, if to the two aforesaid similar prisms two other prisms equal and similar to them be added, two similar parallelepipeds will be constituted, [Margin: p. "33. unde." — Elem. XI.33] which have to one another the triplicate ratio of the corresponding sides. Therefore the prisms too — namely their halves — [Margin: q. "15. quint." — Elem. V.15] since they have the same ratio as the parallelepipeds, will have the triplicate ratio of those same corresponding sides, which are likewise the corresponding sides of the prisms.

Finally, if to the said two equal prisms two other prisms equal and similar to them be adjoined, two equal parallelepipeds of the same heights as the prisms will be composed. [Margin: r. "34. vnd." — Elem. XI.34] Therefore, since the bases and heights of the parallelepipeds are reciprocally proportional, and the bases of the prisms are the same — or at least their halving triangles [Margin: s. "15. quinti." — Elem. V.15] have the same ratio — the bases of the prisms and their heights will also be reciprocally proportional.

THEOR. 30. PROPOS. 35.

[Margin: xxxvii. — vulgate Euclid proposition number (= XI.37).]

If two plane angles are equal, and upon their vertices stand elevated straight lines which contain equal angles with the lines first placed (each with each); and if on the elevated lines any points be taken, and from these perpendiculars be drawn to the planes in which the angles first placed lie; and if from the points which arise in the planes from those perpendiculars straight lines be joined to the angles first placed: then these joining lines, together with the elevated lines, contain equal angles.

(Clavius's demonstration.)

Let BAC, EDF be two equal plane angles, upon whose vertices A and D there stand — outside their planes — the elevated straight lines AG, DH, such that the angle BAG is equal to the angle EDH, and the angle CAG to the angle FDH. From points G, H taken on the lines AG, DH let perpendiculars GI, HK be dropped to the planes in which the angles BAC, EDF lie, meeting them at the points I, K; and let the straight lines IA, KD be joined. I say that the angles GAI, HDK are equal to one another.

For if AG, DH are unequal, from the greater AG cut off AL equal to DH, and from L, in the plane of the triangle AGI, draw LM parallel to GI. Since, then, GI, LM are parallel, and GI is perpendicular to the plane of the angle BAC, [Margin: a. "8. unde." — Elem. XI.8] LM too will be perpendicular to the same plane. From the points M, K let perpendiculars MB, MC, KE, KF [printed "KE, KE"; recte KF] be drawn to the straight lines AB, AC, DE, DF; and let the straight lines BC, BL, LC, EF, EH, HF be joined. And since LM is perpendicular to the plane of the angle BAC, it makes a right angle with the straight line AM drawn in the same plane, by def. 3 of this book. [Margin: b. "47. pri." — Elem. I.47] Therefore the square on AL will be equal to the squares on AM, ML. [Margin: c. "47. prim." — Elem. I.47] But the square on AM is equal to the squares on AC, CM, since the angle ACM is right by construction. Therefore the square on AL is equal to the squares on AC, CM, ML. [Margin: d. "47. prim." — Elem. I.47] But the squares on CM, ML are equal to the square on CL, since the angle CML is right, by def. 3 of this book. Therefore the square on AL is equal to the squares on AC, CL; [Margin: e. "48. prim." — Elem. I.48] and accordingly the angle ACL will be right. [Margin: f. "47. prim." — Elem. I.47] Again, since the square on AL is equal to the squares on AM, ML, [Margin: g. "47. prim." — Elem. I.47] and the square on AM is equal to the squares on AB, BM (since the angle ABM is right by construction), the square on AL is equal to the squares on AB, BM, ML. [Margin: h. "47. prim." — Elem. I.47] But the squares on BM, ML are equal to the square on BL, since the angle BML is right, by def. 3 of this book. Therefore the square on AL is equal to the squares on AB, BL; [Margin: i. "48. prim." — Elem. I.48] and consequently the angle ABL will be right.

The right angles DFH, DEH will be shown in no other way [i.e., by the same reasoning]. Since, then, the angles ABL, LAB of the triangle ABL are equal to the angles DEH, HDE of the triangle DEH, and the sides AL, DH are equal, [Margin: k. "26. prim." — Elem. I.26] the remaining sides AB, BL will also be equal to the remaining sides DE, EH. By the same argument the straight lines AC, CL will be equal to DF, FH. Therefore, since the sides AB, AC of the triangle ABC are equal to the sides DE, DF of the triangle DEF, and the contained angles BAC, EDF are equal by hypothesis, [Margin: l. "4. primi" — Elem. I.4] the bases BC, EF will be equal to one another, and the angles ABC, ACB equal to the angles DEF, DFE. But the whole angles ABM, ACM are equal to the whole angles DEK, DFK, since all are right; therefore the remaining angles MBC, MCB will be equal to the remaining angles KEF, KFE. And accordingly, since the sides BC, EF have also been shown equal, [Margin: m. "26. prim." — Elem. I.26] the sides BM, CM will be equal to the sides EK, FK. Since, then, the sides AC, CM of the triangle ACM have been shown equal to the sides DF, FK of the triangle DFK, and the angles ACM, DFK are right, [Margin: n. "4. prim." — Elem. I.4] the bases AM, DK will be equal to one another.

And since the straight lines BL, EH have been shown equal, their squares too will be equal. [Margin: o. "47. prim." — Elem. I.47] But since the square on BL is equal to the squares on BM, ML, and the square on EH to the squares on EK, KH — because the angles BML, EKH are right, by def. 3 of this book — the squares on BM, ML will be equal to the squares on EK, KH.

Diagram. The two configurations for Prop. 35: a pair of figures (left about the angle BAC with the elevated line AG, right about EDF with DH), each crowded with the connecting lines BC, BL, LC, EF, EH, HF and the perpendiculars.

- Left figure: A (apex, top), B (upper-left), C (right, at mid-height), L (lower-left), M (lower, mid-right), G (bottom-left), I (bottom, mid-right).
- Right figure: D (apex, top), E (upper-left), F (right), H (lower-left), K (lower, mid).

LIBER XI. (*continuation of Prop. 35*)

the squares on EK, KH. Therefore, when the squares on BM, EK are taken away — which are equal, since the lines BM, EK have been shown equal — the remaining squares on LM, HK will be equal; and accordingly the lines LM, HK are equal. For this reason, since the sides AL, AM of the triangle ALM are equal to the sides DH, DK of the triangle DHK, and the base LM is equal to the base HK, [Margin: p. "8. primi" — Elem. I.8] the angles LAM, HDK will also be equal. Therefore, if two plane angles are equal, upon whose vertices stand elevated straight lines, and so on. Which was to be shown.

Diagram. The Prop. 35 figure repeated at the head of the page (as Clavius customarily repeats a figure carried over the page break); the same pair of angle-and-elevated-line constructions with their connecting lines.

- Left figure: A (apex), B (upper-left), C (right, mid-height), L (lower-left), M (lower-right), G (bottom-left), I (bottom-right).
- Right figure: D (apex), E (upper-left), F (right), H (lower-left), K (lower, mid).

COROLLARIVM

Therefore, if two plane angles are equal, upon whose vertices stand *equal* elevated straight lines which contain equal angles with the lines first placed (each with each): then the perpendiculars dropped from the extreme points of the elevated lines to the plane of the angles first placed will be equal to one another. For, because the plane angles BAC, EDF are assumed equal, and the equal

elevated lines AL, DH form equal angles DAB, HDE — and likewise LAC, HDF — it was demonstrated that the dropped perpendiculars LM, HK are equal.

THEOR. 31. PROPOS. 36.

[Margin: xxxviii. — vulgate Euclid proposition number (= XI.38).]

If three straight lines be proportional, then the parallelepiped solid formed from these three is equal to the parallelepiped solid described on the mean line — a solid which is indeed equilateral, but equiangular to the aforesaid one.

(Clavius's demonstration.)

Let A, B, C be continuously proportional straight lines. Construct a solid angle E out of any three plane angles DEF, DEG, FEG, such that the straight line DE is equal to A, EF to B, and EG to C. Completing the parallelograms DF, FG, GD, complete the parallelepiped DH, which is said to be contained under the three lines A, B, C, or to be made from them. [Margin: a. "26. vnde." — Elem. XI.26] Then, on the straight line IK and at its point K, make a solid angle K equal to the solid angle E, out of three plane angles IKL, IKM, LKM equal to the three DEF, DEG, FEG, so that the lines IK, KL, KM are equal to the mean line B. Completing the parallelograms IL, LM, MI, complete the parallelepiped IN, which is said to be contained under the line B, [and] to be described from it. I say that the solid DH is equal to the solid IN.

For since, as DE is to IK, so is KM to EG (because DE has been taken equal to A; IK, KM to B; and EG to C), and the angles DEG, IKM are equal [printed "æquales B"; the stray "B" appears to be a compositor slip — the sense requires "equal to one another"], [Margin: b. "14. sexti" — Elem. VI.14] the parallelograms DG, IM will be equal, because they have their sides about the equal angles reciprocally proportional. And since the plane angles DEG, IKM are equal, upon whose vertices stand the elevated lines EF, KL — which, by construction, contain equal angles with the lines first placed (each with each) — the perpendiculars dropped from F, L to the planes of the bases DG, IM (namely the heights of the parallelepipeds DH, IN, if the bases be taken as DG, IM) will be equal to one another, by the corollary to the preceding proposition. [Margin: c. "31. und." — Elem. XI.31] Therefore the parallelepipeds DH, IN, since they have equal bases DG, IM and equal heights as well, will be equal to one another. Therefore, if three straight lines be proportional, and so on. Which was to be demonstrated.

Diagram. The Prop. 36 figure: at the far left the three proportional lines drawn as vertical segments; then the parallelepiped DH (built on A, B, C) and the equilateral parallelepiped IN (built on the mean B).

- Three proportional lines at left: A (shortest), B (mean), C (longest), bracketed together near the labels C and F.
- Left solid (DH): H (top), G (interior), F (mid, near the lines), E (bottom, mid), D (bottom-right).
- Right solid (IN): N (top), M (interior), L (left, mid), K (bottom, mid), I (bottom-right), D (bottom-left).

SCHOLIVM

Conversely also, if a parallelepiped described from three straight lines be equal to a parallelepiped equiangular to it, described from the mean line: then the three straight lines will be continuously proportional. For let the parallelepiped DH, described from the straight lines A, B, C as was said, be equal to the parallelepiped IN equiangular to it, described from the mean B. I say that the three lines A, B, C are continuously proportional. For, as before, their heights dropped from F, L will be shown to be equal. Therefore, since these solids too are assumed equal, their bases DG, IM will be equal, by what we demonstrated at the end of Prop. 31 of this book. And since these bases have equal angles DEG, IKM by construction, [Margin: d. "14. sexti" — Elem. VI.14] they will have their sides about those equal angles reciprocally proportional — that is, as DE is to IK, so is KM to EG. Therefore, since DE has been taken equal to A; IK, KM to B; and EG to C: it will also follow that, as A is to B, so is B to C. Which was the thing proposed.

THEOR. 32. PROPOS. 37.

[Margin: xxxix. — vulgate Euclid proposition number (= XI.39).]

If four straight lines be proportional, then the parallelepiped solids which are described upon them — similar and in like manner — will be proportional. And if the parallelepiped solids, which are similar and in like manner described, be proportional, then the straight lines themselves will also be proportional.

Euclid, *Elements* Book XI (Clavius's commentary)

— Propositions 37–39

(The enunciation of Proposition 37, with its catchword SINT*, closed the previous opening; its demonstration begins here.)*

THEOR. 32. PROPOS. 37. [Margin: xxxix. = XI.39 vulgate]

If four straight lines be proportional, then the similar and similarly described parallelepipeds upon them will be proportional; and if the similar and similarly described parallelepipeds be proportional, the lines themselves will be proportional.

Let there be four proportional straight lines A, B, C, [D] — such that A is to B as C is to D — and on A and B let two similar and similarly described parallelepipeds A and B be set up; likewise on C and D two others, C and D, similar and similarly placed, whether or not these are similar to the first pair. I say that the solids A, B, C, D are also proportional — namely, that solid A is to solid B as solid C is to solid D. For, by the scholium to Proposition 11 of Book 6, let two other lines E, F be found in continued proportion to the two lines A, B, and likewise two others, G, H, to C, D. Then, since there are four lines A, B, E, F and four others C, D, G, H, taken two by two in the same ratio, *ex aequali* A will be to F as C is to H. But as A is to F, so is solid A to solid B, and as C is to H, so is solid C to solid D, by the corollary to Proposition 33 of this book. Therefore solid A is to solid B as solid C is to solid D. Which is what was proposed.

Now, conversely, let the solids A, B, C, D be proportional. I say that the lines A, B, C, D are also proportional. For, to the three lines A, B, C let a fourth proportional I be found [Margin: a = Elements VI.12], and on it let a parallelepiped be described similar and similarly placed to D, or to C [Margin: b = Elements XI.27]. Then, since recta A is to recta B as recta C is to recta I, it follows — as has just been shown — that solid A is to solid B as solid C is to solid I. But solid A is to solid B as solid C is (by hypothesis) to solid D; therefore solid C is to solid I as the same solid C is to solid D. And consequently the solids I and D are equal [Margin: c = Elements V.9]. Since these are similar and similarly described, they are contained by equal planes (by definition 10 of this book); but equal and similar planes have their corresponding sides equal (by the lemma to Proposition 22 of Book 6). Therefore the lines I and D are equal. Hence C is to I as the same C is to D; but C was set to I as recta A is to B; therefore A is to B as C is to D. And so, if four straight lines be proportional, and so forth. Which was to be demonstrated.

This whole proposition will be demonstrated more briefly with Theon, in this way. First, let A be to B as C is to D. I say that also solid A is to solid B as solid C is to solid D [Margin: d = Elements XI.33]. For since the ratio of solid A to solid B is the triplicate of the ratio of recta A to B, and likewise the ratio of solid C to solid D is the triplicate of the ratio of recta C to D, the ratios of solid A to solid B and of solid C to solid D will be equal, seeing that they are triplicates of equal ratios — namely of A to B and of C to D. This is the first part. Again, secondly, let solid A be to solid B as solid C is to solid D. I say that also A is to B as C is to D [Margin: e = Elements XI.33]. For since the ratio of solid A to solid B is the triplicate of the ratio of recta A to B, and likewise the ratio of solid C to solid D the triplicate of the ratio of recta C to D, the ratios of A to B and of C to D will be equal, since their triplicates — namely solid A to solid B and solid C to solid D — are assumed equal. This is the second part.

Diagram. A set of five parallelepiped solids and four straight-line segments illustrating the four proportional lines and the similar solids described upon them.

- A (upper left): larger parallelepiped, labelled A
- B (upper right): smaller parallelepiped, labelled B
- F, E, H, G (right, below B): four short straight-line segments with end-bars, stacked in this order — the continued-proportional lines
- C (middle left): parallelepiped, labelled C
- D (middle right): parallelepiped, labelled D
- I (lower center): parallelepiped, labelled I

SCHOLIVM.

In the same way, if there be three proportional lines, then also the similar and similarly described parallelepipeds upon them will be proportional, and so on. For if the mean line and its solid be taken twice over, there will be four proportional lines, and therefore four proportional solids, as has been demonstrated. Since, then, the solid of the second line is equal to the solid of the third line, and the second line equal to the third line, what is proposed is evident.

This whole proposition holds also for prisms whose two opposite faces are triangles, as Campanus states. For, adding prisms that are severally equal one to another, parallelepipeds arise which — as has here been demonstrated — will be proportional. Since, then, the prisms are the halves of these parallelepipeds, and have the same ratio to them [Margin: f = Elements V.15], what is proposed is evident.

THEOR. 33. PROPOS. 38. [Margin: XL. = 40]

If a plane be perpendicular to a plane, and from some point in one of the planes a perpendicular be drawn to the other plane, the perpendicular so drawn will fall on the common section of the planes.

For let plane AB be perpendicular to plane AC, and let their common section be the straight line AD; and from E, a point of plane AB, let a perpendicular be dropped to plane AC. I say that it falls on the common section AD. For, if it be possible, let it fall outside, at the point F; and from F, in plane AC, let a perpendicular FG be drawn to the line AD [Margin: a = Elements I.12], and let the straight line EG be joined in plane AB. Now since FG is perpendicular to the common section AD, it will also be perpendicular to plane AB, by definition 4 of this book, and hence also to the line GE, by definition 3 of this book. But EF is likewise perpendicular to FG, by the same definition 3. Therefore in triangle EFG the two angles EFG and EGF are right — which is absurd, since they are less than two right angles [Margin: b = Elements I.17]. Therefore the perpendicular dropped from E to plane AC does not fall outside the common section AD; and so it must fall upon it. And therefore, if a plane be perpendicular to a plane, and so forth. Which was to be demonstrated.

SCHOLIVM.

This is Theon's demonstration. We, however, shall demonstrate the same more briefly, in this way. If from the point E, given in plane AC [recte: AB — required by the construction], a perpendicular dropped to plane AC does not fall on the common section AD, but on a point F outside the section, let a perpendicular EG be drawn from E to the line AD [Margin: c = Elements I.12]; this, by definition 4 of this book, will be perpendicular to plane AC. Then from the point E, outside plane AC, two perpendiculars EF and FG [recte: EG] would have been drawn to the plane AC itself — which is impossible, as we demonstrated in the scholium to Proposition 13 of this book.

Diagram. The figure for Proposition 38: a parallelogram representing the two planes and their common section, with the point E and the two perpendiculars.

- A (left vertex): corner of the plane
- B (upper right): corner
- D (right, mid-height): point on the common section AD
- C (lower right): corner
- E (upper interior): the given point, above the section line

- G (center, on line AD): foot of the perpendicular EG on the section
- F (lower interior): the assumed foot of the perpendicular below the section; EF and FG drawn

THEOR. 34. PROPOS. 39. [Margin: XXXX = 40]

If the sides of the opposite faces of a parallelepiped solid be bisected, and planes be extended through the points of section, then the common section of these planes and the diameter of the parallelepiped solid will bisect one another.

Let AC, BD be the opposite faces of the parallelepiped AB, all of whose edges are bisected at the points I, K, L, M, N, O, P, Q; and through these let two planes IN, KO be extended, whose common section is the straight line RS. Let the diameter AB also be drawn. I say that the straight line RS and the diameter AB bisect one another.

For, joining the straight lines RB, RD, SA, SC, consider the two triangles AQS and COS. Since the sides AQ, QS of triangle AQS are equal to the sides CO, OS of triangle COS — for AQ, CO are halves of the equal lines AG, CH [Margin: a = Elements I.34], and QS, OS are equal to the two equal lines AN, HN, since AS and HS are parallelograms — and since angle AQS is equal to the alternate angle COS [Margin: b = Elements I.29]: the bases AS, CS will also be equal, and the angles ASQ, CSO equal [Margin: c = Elements I.4]. But the angles ASQ, ASO are equal to two right angles [Margin: d = Elements I.13]. Therefore CSO, ASO are also equal to two right angles; and consequently AS, CS form one straight line [Margin: e = Elements I.14]. In the same way BR, DR are shown to be equal, and one straight line to be composed of them. Again, since each of AD, BC is parallel and equal to the line FH — because AF, FC are parallelograms [Margin: f = Elements XI.9] — they are also parallel and equal to one another. Therefore the lines AC, BD joining their extremities are likewise parallel and equal [Margin: g = Elements I.33]; and so their halves AS, BR are equal. Hence the lines AB, RS lie in one and the same plane with them [Margin: h = Elements XI.7], and therefore bisect one another, namely at the point T [Margin: i = Elements I.29 & I.15]. And since the two angles AST, ATS of triangle AST are equal to the two angles BRT, BTR of triangle BRT, and side AS equal to side BR, the remaining sides TA, TS will be equal to the remaining sides TB, TR [Margin: k = Elements I.26]; and therefore AB, RS bisect one another at T. Therefore, if the sides of the opposite faces of a parallelepiped solid, and so forth. Which was to be shown.

Diagram. The figure for Proposition 39: a large parallelepiped drawn in oblique projection, with its edge-midpoints, the two section planes, their common section RS, and the diameter AB running from the lower-right to the upper-left corner. The interior is densely ruled and heavily foxed; several interior labels could not be resolved with certainty.

- B (upper-left corner)
- E (upper-right corner)
- M, R, K (upper band, left to right — R near the center-top)
- F, L, D (middle band, left to right)
- S (lower-center interior, on the section)
- H (lower-left corner), N (bottom-middle), A (lower-right corner; the diameter AB runs from here to B)
- Two apparent T labels (one on the top edge, one at center), together with O, P, Q in the lower-left/lower area, are foxed and uncertain [?] [?] [?]

COROLLARIVM.

Hence it follows that in every parallelepiped all the diameters bisect one another at a single point — namely at the point T, at which, as has here been shown, they bisect the line RS.

SCHOLIVM.

To this theorem may be added another not unlike the one we demonstrated at Proposition 34 of Book 1, namely:

If a parallelepiped solid be cut by a plane through its center, the solid will be bisected by that plane. And if a parallelepiped solid be cut in half by a plane, that plane will pass through the center.

Let the parallelepiped AB be cut by the plane CD through the center — that is, through the midpoint G of the diameter AB. I say the parallelepiped is bisected. First, let the plane CD be parallel to the opposite faces AE, BF. Since the parallel planes BF, DC, EA cut any straight lines proportionally [Margin: l = Elements XI.17], and BA is bisected at G, the sides AH, BI of the opposite faces will also be cut, and hence the bases AH, BI, in half. Then, since as base KC is to base CH, so is solid AD to solid CB [Margin: m = Elements XI.25], the parallelepiped AB is bisected by the plane CD drawn through the center G.

Diagram. The first case of the scholium to Proposition 39: a parallelepiped cut by a plane parallel to two opposite faces, with the diagonal from B to A through the center G.

- B (upper-left), D (top-middle), E (upper-right)
- I (right, upper)
- G (center, on the diagonal BA)
- H (left), K (center-right)
- F (lower-left), C (bottom-middle), A (lower-right)

(Book XI continues; the scholium to Proposition 39 runs on.)

Next, let the cutting plane pass through AI, BH, along the diameters BI, AH of the opposite faces; for in this way too the parallelepiped is cut through the center G — indeed through the whole diameter AB [Margin: n = Elements XI.7], since AB lies in the plane of the parallels AH, BI. And it was demonstrated earlier [Margin: o = Elements XI.28] that a parallelepiped is bisected by a plane drawn through the diameters of opposite faces.

Third, let the cutting plane AKBM be neither parallel to the opposite faces nor drawn through the diameters of the opposite faces, but only through the vertices A and B of the faces AH, BI. Since, then, the parallel planes AH, BI are cut by the plane AKBM, the common sections AM, BK will be parallel [Margin: p = Elements XI.18]; and similarly AK, BM will be parallel, being the common sections of the parallel planes AL, BN made by the plane AKBM. Therefore AKBM is a parallelogram [Margin: q = Elements I.34]; and hence both AK, BM and AM, BK are equal to one another. Now since the sides BK, BL of triangle BKL are equal to the sides AM, AN of triangle AMN, and angle KBL equal to angle MAN — for, producing NA, MA to O, P, the lines AO, AP will be parallel to BL, BK [Margin: r = Elements XI.10], and consequently the angles OAP, LBK will be equal [Margin: s = Elements I.15]; and since OAP is equal to angle MAN, the angles KBL, MAN will also be equal — the bases KL, MN will be equal to one another [Margin: t = Elements I.4], and the angles BKL, BLK equal to the angles AMN, ANM, and triangle equal to triangle. Therefore triangles BKL, AMN are also similar [Margin: u = Elements VI.4], since their sides about equal angles are proportional.

And since KL equals MN, angle BKL equals angle AMN, and angle KBL equals angle MAN, as we have shown, the remaining side IK will equal the remaining HM; the remaining angle BKI will equal the remaining AMH; and the remaining KBE will equal the remaining MAF. (For the whole LI equals the whole NH, and both the angles BKL, BKI and the angles AMN, AMH are equal to two right angles;

and the whole angle LBE equals the whole angle NAF, because of the similarity of the parallelograms LE, FN.) Moreover the sides BE, EI equal the sides AF, FH, and the angles BEI, EIK equal the angles AFH, FHM, because of the parallelograms LE, FN. Therefore the quadrilateral BKIE is both equilateral and equiangular with the quadrilateral AMHF, and consequently equal and similar to it, since they agree severally — sides with sides and angles with angles. In the same way the triangles BHM, AIK will be equal and similar, and the quadrilaterals AKLF, BMNE, since they are mutually equilateral and equiangular. Therefore the solid contained by the planes AKLF, FHMA, AKBM, MHB, BKL, LBHF will be equal to the solid contained by the planes BMNE, EIKB, BMAK, KIA, AMN, NAIE, by definition 10 of this book, since those planes are similar and equal to these, as has been demonstrated. Therefore the parallelepiped AB is bisected by the plane AKBM through the center G.

Diagram. A planar figure (two overlapping squares with a diagonal) accompanying the third case of the scholium.

- B (upper-right)
- I (center-left), G (center), H (center-right, just below G)
- A (lower-left)

Diagram. The third-case figure of the scholium: a parallelepiped cut by the plane AKBM through the vertices A and B. The figure is densely ruled with diagonals.

- B (upper-left), E (upper-right)
- L (left), K (center), I (right) — along the upper band
- G (center interior)
- H (mid-left), M (center), N (right)
- F (lower-left), A (bottom-center)
- O (below left), P (below right)

Finally, let the cutting plane CD be neither parallel to the opposite faces, nor drawn through the diameters of the opposite faces, nor through any of their vertices, but cut the opposite faces [AH, BI] in any way whatever. We shall show, as before, that the cutting plane CD is a parallelogram, and that both CK, DM and CM, DK are equal. In the parallelogram CD let the diameters CD, KM be drawn, and let the lines AM, BK be joined. Since, then, AF, BE are parallel to LI, because of the parallelograms AL, LE [Margin: x = Elements XI.9], they are parallel to one another; and AB, CD will lie in the same plane with them [Margin: y = Elements XI.7], and therefore bisect one another at the center G. For it is only at the center G that the line AB, by hypothesis, cuts the plane CD, in which the line CD lies. And

since angle BDC is equal to the alternate angle ACD [Margin: z = Elements I.29], and angle BGD is equal to the vertical angle AGC [Margin: a = Elements I.15], and side BG equal to side AG (because AB is bisected at the center G), side DG will also be equal to side CG [Margin: b = Elements I.26]. Since, then — by what we showed at Proposition 34 of Book 1 — the diameters CD, KM bisect one another, KM will pass through the point G, and there will cut the diameter AB. Now since the sides BG, GD of triangle BGD are equal to the sides AG, GC of triangle AGC [Margin: c = Elements I.15], and angle BGD equal to angle AGC, the base BD will be equal to the base AC [Margin: d = Elements I.4]. By the same reasoning the lines BK, AM will be equal. Therefore, since the three sides ED [recte: BD], DK, KB of triangle BDK are equal to the three sides AC, CM, MA of triangle ACM, the angles of the one will — by what we taught at Proposition 8 of Book 1 — be equal to the angles of the other. But the whole angle DBL equals the whole CAN, because of the similarity of the parallelograms BI, AH. Therefore the remaining angle KBL will be equal to the remaining MAN. Since, then, the sides BK, BL of triangle BKL are equal to the sides AM, AN of triangle AMN, and angle KBL equal to angle MAN, the base KL will equal the base MN [Margin: e = Elements I.4], and the remaining angles equal to the remaining angles. If, then, to the equal angles BKL, AMN the angles BKD, AMC be added, the whole angles DKL, CMN become equal. Therefore the quadrilateral BDKL is both equilateral and equiangular with the quadrilateral ACMN, and consequently equal and similar, since they agree severally — sides with sides and angles with angles. Again, since the whole LI equals the whole NH, the remaining IK will equal the remaining HM. And by the same reasoning the remaining DE will equal the remaining CF; and consequently the quadrilateral DEIK is equilateral with the quadrilateral CFHM — and equiangular too, since the angles DEI, EIK are equal to the angles CFH, FHM, because of the similarity of the parallelograms BI, AH, and the angles EDK, IKD equal to the angles FCM, HMC, since their alternate angles LKD, BDK have been shown equal to the angles NMC, ACM. Therefore the quadrilaterals DEIK, CFHM are equal and similar. In the same way we shall show that the quadrilaterals ACKI, BDMH are equal and similar; and likewise DENM, CFLK, since both their sides agree with sides and angles with angles, as by definition 10 of this book. Therefore the solid contained by the planes CKLF, FHMC, CKDM, MHBD, DKLB, BHFL is equal, by definition 10 of this book, to the solid contained by the planes DMNE, EIKD, DMCK, KIAC, CMNA, AIEN; since these have been demonstrated equal and similar to those. Therefore the parallelepiped AB is bisected through the center G by the plane CD.

But now let the parallelepiped AB be cut in half by the plane CD. I say that the plane CD passes through the center of the parallelepiped, [these...]

Diagram. The last-case figure of the scholium: a parallelepiped cut by the oblique plane CD. The figure is densely ruled and heavily foxed; one lower-left label could not be resolved.

- B (upper-left), D (top-middle), E (upper-right)
- K (center-right), I (far right)
- L (center-left), G (center), N (right)
- H (lower-left), M (center-low)
- C (bottom-middle), A (lower-right corner)
- one further label at lower-left, foxed and uncertain [?]

Elements, Book XI

(Continuation of the Scholium to Proposition 39 — the converse.)

that is, the point G, at which the plane CD cuts the diameter AB, bisects the diameter AB. For if the diameter AB is not bisected at G, let it be bisected at E, so that E is the center of the parallelepiped. Then if through E a plane FH is drawn parallel to the plane CD, the parallelepiped AB will be bisected by the plane FH, as has been demonstrated. But it is also assumed to be bisected by the plane GD. Therefore the solids CB and FB, being halves of the parallelepiped AB, are equal to one another — the part and the whole. Which is absurd. Therefore no point other than G will bisect the diameter AB; and accordingly G will be the center of the parallelepiped. Which was to be demonstrated.

Diagram. A parallelepiped (rectangular box) drawn in oblique projection, with the internal diagonals and the common sections of the bisecting planes shown crossing near the center. Labels read off the figure:

- B (top back, left corner): vertex
- D (top back, center): vertex
- H (top back, toward right): vertex
- G (interior, left of center): a common section / intersection point
- E (interior, center): the center of the parallelepiped
- C (bottom front, left): vertex
- F (bottom front, center): vertex
- A (bottom front, right): vertex

THEOR. 35. PROPOS. 40.

[Margin: outer numeral xxxxj (= 41)]

If there are two prisms of equal height, one of which has a parallelogram for its base and the other a triangle, and the parallelogram is double the triangle: the prisms themselves will be equal.

Let there be two prisms of equal height, ABCDEF and GHIKLM, of which the former has the parallelogram ABCD for its base and the latter the triangle GHI; and let the parallelogram AC be double the triangle GHI. I say that these prisms are equal. For let the parallelepipeds AN and GQ be completed — which will be done if the planes of the triangles are extended and the parallelograms BN, AO and GP, MQ are completed. For if the straight lines NO and PQ are joined, two parallelepipeds AN and GQ will be constructed, of the same height as the prisms; and that the opposite planes of these solids are parallel is easily gathered from Proposition 15 of this book. [Margin: a — *Elements* I.34] Now since the parallelogram GP is double the triangle GHI, and the parallelogram AC is likewise assumed to be double the same triangle GHI, the parallelograms AC and GP will be equal. [Margin: b — *Elements* XI.31] Therefore the parallelepipeds AN and GQ, of the same height and standing on equal bases AC and GP, are equal to one another; and consequently their halves — namely the prisms ABCDEF and GHIKLM [Margin: c — *Elements* XI.32] (for the parallelepipeds AN and GQ are bisected by the diagonals CF, DE, HI, LK of their opposite planes, into two prisms each) — are likewise equal to one another. And so, if there are two prisms of equal height, and so on. Which was to be demonstrated.

Diagram. Two oblique-projection boxes (parallelepipeds), each formed by completing a prism. On the left, the parallelepiped AN built on the parallelogram-based prism ABCDEF; on the right, the parallelepiped GQ built on the triangle-based prism GHIKLM, each shown cut by a diagonal plane. Labels read off the figure:

Left box (parallelepiped AN):

- O (top, left) and N (top, right): the completing vertices
- E (front, upper left) and F (front, upper right): vertices
- D (bottom, left), C (bottom, center), A (bottom, far left), B (bottom, center-right): base vertices

Right box (parallelepiped GQ):

- L (top, left) and Q (top, right): vertices
- K (right side, upper): vertex
- H (front, lower center): vertex

- G (bottom, left) and P (bottom, right): vertices
- I (bottom front) and M (mid-left): labels foxed and not clearly resolved on the woodcut [?]

SCHOLIVM.

It is clear from the demonstration that this proposition is to be understood only of those prisms which have two triangles as their opposite faces; for it requires that the parallelepipeds be completed — which could not be done if the prism standing on the parallelogram ABCD did not have two triangles, namely ADE and BCF. For otherwise it would already be a completed parallelepiped.

FINIS ELEMENTI VNDECIMI (*End of the Eleventh Book of the Elements.*)

EUCLID'S TWELFTH BOOK OF THE ELEMENTS

and the Second Book of Solids.

THEOR. 1. PROPOS. 1.

[Margin: outer numeral I.]

Similar polygons inscribed in circles are to one another as the squares on the diameters.

Let there be two similar polygons ABCDE and FGHIK, inscribed in circles whose diameters are AL and FM. I say that the polygon ABCDE is to the polygon FGHIK as the square on the diameter AL is to the square on the diameter FM. For let the straight lines AC and FH subtend the equal angles ABC and FGH, and let the straight lines BL and GM be joined. Now since, on account of the similarity of the polygons, AB is to BC as FG is to GH, [Margin: a — *Elements* VI.6] the triangles ABC and FGH will be equiangular, since they have proportional sides about the equal angles ABC and FGH. [Margin: b — *Elements* III.21] But the angle ALB is equal to the angle ACB, and the angle FMG to the angle FHG. Therefore the angles ALB and FMG will also be equal. And since the angles ABL and FGM are likewise equal [Margin: c — *Elements* III.31] — namely, being right angles standing in semicircles — [Margin: d — *Elements* I.32] the remaining angles BAL and GFM will also be equal. [Margin: e — *Elements* VI.4] Therefore AL is to AB as FM is to FG; and, alternately, AL is to FM as AB is to FG. [Margin: f — *Elements* VI.22] Therefore the square on AL is to the square on FM as the polygon ABCDE on AB is to the polygon FGHIK on FG, since both the squares and the polygons are similar

figures and similarly described. And so, similar polygons inscribed in circles are to one another as the squares on the diameters. Which was to be demonstrated.

Diagram. Two circles, each with an inscribed pentagon and a drawn diameter. Labels read off the figure:

Left circle (polygon ABCDE, diameter AL):

- A (top), B (left), C (lower left), D (lower right), E (right): pentagon vertices
- L (bottom): the other endpoint of the diameter AL

Right circle (polygon FGHIK, diameter FM):

- F (top), G (left), H (lower left), I (lower right), K (right): pentagon vertices
- M (bottom): the other endpoint of the diameter FM

THEOR. 2. PROPOS. 2.

[Margin: outer numeral ij.]

Circles are to one another as the squares on their diameters.

Let there be two circles ABCD and EFGH, whose diameters are AC and EG. I say that the circle ABCD is to the circle EFGH as the square on the diameter AC is to the square on the diameter EG. For if this is not so, then let the circle ABCD be to some magnitude — namely to I — as the square on the diameter AC is to the square on the diameter EG, where I will be either less or greater than the circle EFGH. For if I were equal to it, [Margin: a — *Elements* V.7] the circle ABCD would have the same ratio to the circle EFGH and to I; and therefore the circle ABCD would be to the circle EFGH as the square on the diameter AC is to the square on the diameter EG, which is not granted. Suppose, then, first that I is less than the circle EFGH, by the magnitude K — that is, so that the circle EFGH is equal to the magnitudes I and K together. Let there be inscribed...

Diagram. Two circles, each with an inscribed octagon, together with two small lens (vesica) shapes at the right labelled I and K representing the auxiliary magnitudes. Labels read off the figure:

Left circle (octagon APBQCRDS, diameter AC):

- A (top), B (left), C (bottom), D (right): the four principal points
- P (upper left), S (upper right), Q (lower left), R (lower right): the bisection points of the arcs (the lower two, Q and R, are foxed on the woodcut but are named APBQCRDS in the text)

Right circle (octagon ELFMGNHO, diameter EG):

- E (top), F (left), G (bottom), H (right): the four principal points
- L (upper left), O (upper right), M (lower left), N (lower right): the bisection points of the arcs
- T (above E) and V (near F): endpoints of the tangent line TV drawn tangent to the circle at L

Auxiliary magnitudes:

- I and K: two small lens shapes at the right

there be inscribed in the circle EFGH the square EFGH; and since this is half of the square described about the same circle — as we showed at Proposition 9 of Book 4 — it will be greater than half of the circle EFGH. Let the arcs EF, FG, GH, HE be bisected at the points L, M, N, O, and let the straight lines LE, LF, MF, MG, NG, NH, OH, OE be joined. Through L let the straight line TV be drawn tangent to the circle at L, which will be parallel to EF — as we showed at Proposition 27 of Book 3 — and let it meet the straight lines HE and GF produced at T and V. Since, then, the triangle ELF is half of the parallelogram TEFV, [Margin: b — *Elements* I.41] the triangle ELF will be greater than half of the segment ELF of the circle. By the same reasoning the remaining triangles will each be greater than half of the segments in which they lie. Therefore all the triangles together are greater than half of all the segments together. And if again the arcs EL, LF, and so on are bisected and straight lines joined, triangles will be constructed in the same way, which together will be greater than half of all the segments in which they lie; and so on in succession. Now since, if from the circle EFGH more than half is taken away — namely the square EFGH — and from the remaining segments more than half — namely the triangles ELF, FMG, and so on — and the subtraction is always carried out in this manner, [Margin: c — *Elements* X.1] there is finally left a magnitude smaller than K, the excess between the circle EFGH and the magnitude I. Now let the segments of the circle left over — EL, LF, FM, and so on — taken together be less than the magnitude K. Then, since the circle EFGH is supposed equal to the magnitudes I and K together, if from the circle the said segments are subtracted, and from I and K the magnitude K itself — which is greater than the aforesaid segments of the circle — is subtracted, the remaining polygon ELFMGNHO will be greater than the remaining magnitude I. Let there be inscribed in the circle ABCD a polygon APBQCRDS similar to the polygon ELFMGNHO. This will easily be done if the semicircles ABC and ADC are bisected at B and D, and again the arcs AB, BC, CD, DA at P, Q, R, S, and so on in succession, until the circumference ABCD is divided into as many equal parts as the

circumference EFGH is distributed into. For when the straight lines AP, PB, BQ, and so on are joined, the inscribed figure will be similar to the figure ELFMGNHO, as is evident. [Margin: d — *Elements* XII.1] Now since the polygon APBQCRDS is to the polygon ELFMGNHO as the square on the diameter AC is to the square on the diameter EH [recte: EG] — that is, by hypothesis, as the circle ABCD is to I — but the polygon APBQCRDS will be less than the circle ABCD, [Margin: e — *Elements* V.14] therefore the polygon ELFMGNHO will also be less than I. But it was shown to be also greater. Which is absurd. Therefore the magnitude I is not less than the circle EFGH.

Diagram. A repetition, at the head of the continuation page, of the Proposition 2 figure: two circles with inscribed octagons and the two lens shapes I and K. Labels read off the figure:

Left circle (octagon APBQCRDS, diameter AC):

- A (top), B (left), C (bottom), D (right): principal points
- P (upper left), S (upper right), Q (lower left, "e"-form on the woodcut [?]), R (lower right): arc-bisection points

Right circle (octagon ELFMGNHO, diameter EG):

- E (top), F (left), G (bottom), H (right): principal points
- L (upper left), O (upper right), M (lower left), N (lower right): arc-bisection points
- T (above E) and V (near F): endpoints of the tangent line TV

Auxiliary magnitudes:

- I and K: two small lens shapes at the right

Although in the figure the larger circle is compared with the smaller, we shall nevertheless demonstrate in the same way that the smaller circle EFGH cannot have to a magnitude smaller than the larger circle ABCD the same ratio that the square on the smaller diameter EG has to the square on the larger diameter AC. So that it may be demonstrated generally and universally that the square on the diameter of any circle can never have to the square on the diameter of another circle the same ratio that the former circle has to a magnitude smaller than the latter circle — whether the former circle be greater than the latter, or smaller.

Next, let the magnitude I be greater than the circle EFGH. Then, since the circle ABCD is supposed to be to I as the square on the diameter AC is to the square on the diameter EG, then also, by conversion, I will be to the circle ABCD as the

square on the diameter EG is to the square on the diameter AC. Let it be supposed that as I is to the circle ABCD, so is the circle EFGH to some magnitude, namely to K. And since I is supposed greater than the circle EFGH, [Margin: f — *Elements* V.14] the circle ABCD will also be greater than the magnitude K. Therefore the square on the diameter EG will be to the square on the diameter AC as the circle EFGH is to the magnitude K, which is less than the circle ABCD. Which is absurd. For it has already been shown generally, in the earlier part, that the square on a diameter cannot be to the square on a diameter as a circle is to a magnitude smaller than that circle — whether the former circle be greater than the latter, or smaller.

And although the circle ABCD is the greater and EFGH the smaller, we shall nevertheless show by the same reasoning that the square on the diameter of the circle ABCD cannot have the ratio that the circle EFGH has to a magnitude greater than the circle ABCD. For in the same way we shall prove (if that be granted) that the square on the diameter AC has to the square on the diameter EG the same ratio that the circle ABCD has to a magnitude smaller than the circle EFGH — which has already been demonstrated to be false, generally, in the earlier part. So that it may likewise be demonstrated generally and universally that the square on the diameter of any circle can never have to the square on the diameter of another circle the same ratio that the former circle has to a magnitude greater than the latter circle — whether the former circle be greater than the latter, or smaller. Therefore the magnitude I is not greater than the circle EFGH; but neither was it shown to be less; therefore it is equal. And so, since it is supposed that the circle ABCD is to I as the square on the diameter AC is to the square on the diameter EG, [Margin: g — *Elements* V.7] and since the circle ABCD is to I as the same circle ABCD is to the circle EFGH, which is equal to the magnitude I, it will also follow that the circle ABCD is to the circle EFGH as the square on the diameter AC is to the square on the diameter EG — whether the circle ABCD be greater than the circle EFGH, or smaller. Therefore circles are to one another as the squares on their diameters. Which was to be shown.

COROLLARIUM.

Hence it follows that circle is to circle as a polygon inscribed in the one is to a similar polygon inscribed in the other. For both circle to circle and polygon to polygon are as the square on the diameter to the square on the diameter, as has been demonstrated.

THEOR. 3. PROPOS. 3.

[Margin: iij.]

Every pyramid having a triangular base is divided into two pyramids, equal and similar to one another, having triangular bases and similar to the whole; and into two equal prisms, and these two prisms are greater than half the whole pyramid.

Let there be a pyramid whose base is the triangle ABC and whose apex is D. Let all its edges be bisected at the points E, F, G, H, I, K, and let the straight lines EF, FG, GE, HI, IK, KH, HE, EI, IF be joined. The pyramid is thereby divided into two pyramids, AEGH and HIKD — whose bases are the triangles AEG and HIK and whose apexes are H and D — and also into two solids EBFHGI and CFGHIK, which we shall presently show to be prisms. The base of the former is the parallelogram EBFH, with the opposite triangles EHG and BIF; the base of the latter is the triangle CFG, to which the triangle FIH is opposite. These prisms have a common boundary, the parallelogram FHGI, and together with the two pyramids they make up the whole pyramid — as is clear if one rightly conceives the apex D of the pyramid on high, together with the triangles that surround it. I say, then, that those two pyramids are equal and similar to one another and similar to the whole; and likewise that these two prisms are equal to one another and greater than half the whole pyramid.

Diagram. Two adjacent copies of the divided triangular pyramid ABCD, drawn in plane projection as a triangle with apex A above the base B–C. The edge-midpoints and internal connecting lines subdivide the figure into the two smaller pyramids and two prisms discussed in the text. Readable labels (left copy):

- A (top): apex of the base triangle
- B (bottom-left), C (bottom-right): base-triangle corners
- E (on left edge AB), G (on right edge AC), F (bottom middle, on BC): edge-midpoints
- H (below A, on edge AD), I (lower left, on edge BD), K (lower right, on edge CD): midpoints of the edges to the solid apex
- D (interior center): the solid apex of the pyramid, drawn projected inside the triangle

The right copy repeats the same construction; its clearly legible labels are A, H, E, D, I, B, F, with the remaining letters faint from foxing.

Since the sides AD and BD of the triangle ADB have been bisected, and therefore [cut] proportionally, [Margin: a — *Elements* VI.2] the lines HI and AB will be parallel. By the same reasoning IK will be parallel to BC; HK to AC; EG to BC; EF to AC; FG to AB; EH to BD; EI to AD; IF to DC; and HG to DC. [Margin: b — *Elements* XI.9] Moreover, since the lines FG and HI are each parallel to AB, they are parallel to one another; and by the same reasoning GH and FI are parallel, since each is parallel to DC. Therefore AEIH, HEBI, IDHE, EBFG, GHKC, CKIF, and FGHI are parallelograms. And since the lines HE and HG are parallel to the lines DB and DC, [Margin: c — *Elements* XI.10] the angles EHG and BDC will be equal; and by the same reasoning the angles HEG and DBC, and HGE and DCB, will be equal. [Margin: d — *Elements* VI.4] The sides of triangle HEG are therefore proportional to the sides of triangle DBC about their equal angles, and hence triangle HEG is similar to triangle DBC. Moreover, the triangles HAE, HAG, AEG are similar to the triangles DAB, DAC, ABC, by the corollary to VI.4. Therefore the pyramid AEGH is similar to the pyramid ABCD, by definition 9 of Book XI [defin. 9. lib. 11].

Again, since the lines HI and HK are parallel to the lines AB and AC, [Margin: e — *Elements* XI.10] the angles IHK and BAC will be equal; and by the same reasoning the angles HIK and ABC, and HKI and ACB, will be equal. [Margin: f — *Elements* VI.4] Therefore the sides of triangle HIK are proportional to the sides of triangle ABC about their equal angles, and so triangle HIK is similar to triangle ABC. Moreover, the triangles DHI, DIK, DKH are similar to the triangles DAB, DBC, DCA, by the same corollary to VI.4. Therefore the pyramid HIKD too is similar to the same pyramid ABCD, by definition 9 of Book XI.

Now since the triangles AHE and HDI are similar to the triangle ADB — as was shown from the corollary to VI.4 — [Margin: g — *Elements* VI.21] they will also be similar to one another. And since they are set up upon the equal lines AH and HD, they will be equal, by the lemma to proposition 22 of Book VI [lemma propos. 22. lib. 6]. By a like argument the triangles AHG and HDK will be equal and similar, since they have been shown similar to triangle ADC and are set up upon the equal lines AH and HD. On the same reasoning the triangles AEG and HIK will be equal and similar, since they have been shown similar to triangle ABC and are placed upon the lines AE and HI, [Margin: h — *Elements* I.34] which are equal because of the parallelogram AEIH. In the same way the triangles EHG and IDK will be equal and similar, since they have been shown similar to triangle BDC and have the lines HE and DI, [Margin: i — *Elements* I.34] which are equal because of the parallelogram HEID. The pyramids AEGH and HIKD are therefore equal and similar, by definition 10 of Book XI, since all the triangles of the one are equal and similar to all the triangles of the other, as has been demonstrated.

Again, [Margin: k — *Elements* I.34] since the lines EH, HG, GE are equal and parallel to the lines BI, IF, FB — because of the parallelograms EHIB, FGHI, BFGI — the triangles EHG and BIF will be mutually equiangular and equal, by the corollary to proposition 8 of Book I [coroll. propos. 8. lib. 1], and therefore also similar. [Margin: l — *Elements* XI.15] They are moreover parallel, since EH and HG, through which the plane EHG is drawn, are parallel to the lines BI and IF, through which the plane BIF is drawn. Therefore the solid BIFGHE, contained by the two triangles EHG and BIF — opposite, equal, similar, and parallel — and by the three parallelograms EGFB, BEHI, IFGH, is a prism, by the definition. In the same way the solid CFGHIK will be shown to be a prism. [Margin: m — *Elements* I.34] For since the lines FC, CG, GF are equal and parallel to the lines IK, KH, HI — because of the parallelograms CFIK, CGHK, FGHI — the triangles CFG and HIK will be equal and mutually equiangular, and therefore similar. [Margin: n — *Elements* XI.15] They are moreover parallel, since the lines CF and CG, through which the plane CFG is drawn, are parallel to the lines KI and KH, through which the plane KIH is drawn. Therefore the solid CFGHIK, contained by the two triangles CFG and KIH — opposite, equal, similar, and parallel — and by the three parallelograms CFIK, KHGC, IFGH, is a prism. And since the prisms EBFghi and CFGHIK are of the same height — namely between the parallel planes BCGE and HIK — and the parallelogram EBFg, the base of the former, is double the triangle CFG, the base of the latter, by the scholium to proposition 41 of Book I [scholium propos. 41. lib. 1], [Margin: o — *Elements* XI.40 (content matches vulgate XI.39)] they will be equal to one another.

Finally, since the prism EBFghi is greater than the pyramid EBFI — a whole greater than its part — and the pyramid EBFI is equal and similar to the pyramid AEGH, and also to the pyramid HIKD (as is plain from the equality and similarity of the triangles), the prisms EBFghi and CFGHIK will be greater than the pyramids AEGH and HIKD. And so the former will exceed half of the pyramid ABCD, while the latter fall short of half of it. For when a whole is cut into two unequal parts, the greater exceeds the half and the lesser falls short of the half. Every pyramid, therefore, having a triangular base, etc. Which was to be shown.

LIBER XII.

THEOR. 4. PROPOS. 4.

[Margin: iv.]

If there be two pyramids of the same height, having triangular bases, and each of them be divided both into two pyramids equal to one another and similar to the whole, and into two equal prisms; and if each of the pyramids arising from the previous division be divided in the same way, and this be done continually — then, as the base of one pyramid is to the base of the other, so also will all the prisms in the one pyramid be to all the prisms in the other pyramid, [these being] equal in number.

Let there be, upon the triangular bases ABC and EFG, two pyramids ABCD and EFGH of the same height, whose edges are bisected at the points I, K, L, M, N, O, P, Q, R, S, T, V, and let the straight lines IL, LK, KN, NO, OM, MN, LM, MI, PR, RQ, QT, TV, VS, ST, RS, SP be joined, so that each pyramid is cut into two pyramids equal to one another and similar to the whole — namely AILM and MNOD; EPRS and STVH — and into two equal prisms, IBKLMN and CKLMNO; PFQRST and GQRSTV, as the preceding proposition requires. And let the pyramids AILM, MNOD; EPRS, STVH be understood to be divided in the same way, and so on.

Diagram. Two triangular pyramids drawn side by side in plane projection, each subdivided as in the previous proposition.

Left figure — pyramid ABCD:

- A (apex, top), B (bottom-left), C (bottom-right): base triangle
- M (below apex), I (left edge), L (right edge): upper midpoints
- N (lower left), O (lower middle-right), K (bottom middle): lower midpoints
- D (interior): the solid apex

Right figure — pyramid EFGH:

- E (apex, top), F (bottom-left), G (bottom-right): base triangle
- S (below apex), P (left edge), R (right edge): upper midpoints
- T (lower left), V (lower right), Q (bottom middle): lower midpoints
- H (interior): the solid apex

I say that all the prisms formed in the pyramid ABCD are to all the prisms produced in the pyramid EFGH — [these being] equal to them in number — as the base ABC is to the base EFG. For since BC is to CK as FG is to GQ (because each line has been bisected), and the triangles ABC and LKC are similar and similarly placed, and likewise the triangles EFG and RQG, by the corollary to VI.4, [Margin: a — *Elements* VI.22] then triangle ABC will be to triangle LKC as triangle EFG is to triangle RQG. And by alternation, as ABC is to EFG, so is LKC to RQG, and so is the prism CKLMNO to the prism GQRSTV, as we shall presently show — and hence

also the prism IBKLMN to the prism PFQRST, since these are equal to those. And as one prism, namely IBKLMN, is to one prism PFQRST, [Margin: b — *Elements* V.12] so are the two prisms IBKLMN and CKLMNO to the two prisms PFQRST and GQRSTV. Therefore, as the base ABC is to the base EFG, so also are the two prisms in the pyramid ABCD to the two prisms in the pyramid EFGH.

By a like argument we shall show that the pair of prisms in the pyramids AILM and MNOD, formed within the pyramid ABCD, are to the pair of prisms in the pyramids EPRS and STVH, formed within the pyramid EFGH, as the bases AIL and MNO of those pyramids are to the bases EPR and STV of these pyramids; and so on, the same division being always made. But as those bases are to these, so is LKC — the base equal and similar to them — to the base RQG, which is equal and similar to these, as was demonstrated in the preceding proposition; that is, as the base ABC is to the base EFG. Therefore, as the base ABC is to the base EFG, so will the prisms of any pyramid formed within ABCD be to the prisms of any pyramid formed within EFGH. And consequently, as the prisms of pyramid ABCD are to the prisms of pyramid EFGH, so are the prisms of pyramid AILM to the prisms of pyramid EPRS, and the prisms of pyramid MNOD to the prisms of pyramid STVH, and so on. Therefore, since the two prisms of pyramid ABCD are to the two prisms of pyramid EFGH [Margin: c — *Elements* V.12] as all the prisms in the pyramids ABCD, AILM, MNOD, etc. together are to all the prisms in the pyramids EFGH, EPRS, STVH, etc. together — provided these are equal in number to those — it will also follow that, as the base ABC is to the base EFG, so are all the prisms in the pyramid ABCD to all the prisms in the pyramid EFGH. And so, if there be two pyramids of the same height having triangular bases, etc. Which was to be demonstrated.

LEMMA.

That LKC is to RQG as the prism CKLMNO is to the prism GQRSTV, we show as follows. Let there be understood, dropped from the apexes D and H to the bases ABC and EFG, perpendiculars, which will be the equal heights of the pyramids ABCD and EFGH. Now since the parallel planes ABC and MNO [Margin: d — *Elements* XI.17] cut two straight lines — namely DC and the perpendicular dropped from D — proportionally, and DC is bisected at O, the perpendicular dropped from D will likewise be bisected at the point where the plane MNO meets it. By the same reasoning the perpendicular dropped from H will be bisected by the plane STV. Therefore, since the whole perpendiculars are supposed equal, their halves — namely the heights of the prisms — will also be equal; and consequently the prisms CKLMNO and GQRSTV, having equal heights, will be to

one another as the bases LKC and RQG, by what we demonstrated in connection with proposition 34 of Book XI [propos. 34. lib. 11].

THEOR. 5. PROPOS. 5.

[Margin: v.]

Pyramids that are of the same height and have triangular bases are to one another as their bases.

Let there be pyramids ABCD and EFGH of the same height, whose bases are the triangles ABC and EFG. I say that pyramid is to pyramid as base is to base. For if it be not so, let it be supposed that, as the base ABC is to the base EFG, so is the pyramid ABCD to a solid X, which will be either less or greater than the pyramid EFGH. Let it first be less, by the magnitude Y.

EVCLID. GEOM.

(*Prop. 5, continued.*)

Let the pyramid EFGH be divided into two equal pyramids and two equal prisms, according to proposition 3 of this book; and again let the pyramids so formed within EFGH be divided in the same way into two equal pyramids and two equal prisms, and so on. Now since, if from the pyramid EFGH there be taken away more than the half — namely the two prisms PFQRST and GQRSTV, [Margin: a — *Elements* XII.3] which are greater than half the pyramid EFGH — and likewise from the remaining pyramids EPRS and STVH more than the half, namely their prisms, and so on, [Margin: b — *Elements* X.1] there will at last be left a magnitude less than Y, the excess of the pyramid EFGH over the solid X. Let the magnitude now left be [that] lesser one. And since the pyramid EFGH is supposed equal to the solids X and Y together, the remaining prisms in the pyramid EFGH will be greater than the solid X. Let the pyramid ABCD be divided into two equal pyramids and two equal prisms, and let the pyramids AILM and MNOD so formed be divided in the same way into two equal pyramids and two equal prisms; and let this be done as many times as it was done in the pyramid EFGH. [Margin: c — *Elements* XII.4] Now since all the prisms in the pyramid ABCD are to all the prisms in the pyramid EFGH — equal in number — as the base ABC is to the base EFG, that is, as the pyramid ABCD is to the solid X; and since all the prisms in the pyramid ABCD are less than the whole pyramid ABCD, [Margin: d — *Elements* V.14] all the prisms in the pyramid EFGH will also be less than the solid X. But

they were shown to be greater as well. Which is absurd. Therefore the solid X is not less than the pyramid EFGH.

Diagram. The two subdivided triangular pyramids ABCD and EFGH again (as on the preceding page), together with two small drawn cubes representing the auxiliary solids of the exhaustion argument.

- Left pyramid ABCD: A (apex), M, I, L (upper), N, O, K (lower/base-middle), B (bottom-left), C (bottom-right), D (interior apex)
- Right pyramid EFGH: E (apex), S, P, R (upper), T, V, Q (lower/base-middle), F (bottom-left), G (bottom-right), H (interior apex)
- Cube labelled X (left, between the two pyramids near the top): the solid X
- Cube labelled Y (right of X): the solid Y

Next, let the solid X be greater than the pyramid EFGH. Since, then, the pyramid ABCD is supposed to be to the solid X as the base ABC is to the base EFG, by inversion the solid X will be to the pyramid ABCD as the base EFG is to the base ABC. Let it be arranged that, as the solid X is to the pyramid ABCD, so is the pyramid EFGH to a solid Y. And since the solid X is supposed greater than the pyramid EFGH, [Margin: e — *Elements* V.14] the pyramid ABCD will also be greater than the solid Y. Therefore, as the base EFG is to the base ABC, so will the pyramid EFGH be to the solid Y, which is less than the pyramid ABCD. Which is absurd; for it has already been shown that base cannot be to base as pyramid is to a solid less than the pyramid. Therefore the solid X is not greater than the pyramid EFGH; and it was shown not to be less either. It is therefore equal. And so, since it is supposed that as the base ABC is to the base EFG [Margin: f — *Elements* V.9] the pyramid ABCD is to the solid X, and the solid X is equal to the pyramid EFGH, it will follow that as the base ABC is to the base EFG [recte: EFG — printed "EFH"], so is the pyramid ABCD to the pyramid EFGH. Pyramids, therefore, existing under the same height, etc. Which was to be shown.

SCHOLIVM.

*Conversely, if triangular pyramids are to one another as their bases, they will be of the same height. For if the height of one be supposed greater, let a part equal to the smaller be cut off from it, and from the point of section let straight lines be drawn to all the angles of the base. Then, as base is to base, so will [that] pyramid be to the pyramid just constructed; but so also was it to the whole pyramid. [Margin: g — *Elements* V.9] Therefore the constructed pyramid will be equal to the whole pyramid — a part to the whole. Which is absurd.*

COROLLARIUM.

Hence it follows that pyramids of the same height, constructed on the same or on equal triangular bases, are equal to one another; because they have the same ratio to their bases — which are supposed equal — or indeed one and the same ratio.

It likewise follows, conversely, that equal triangular pyramids on the same or on equal bases have the same height; and that equal triangular pyramids having the same height have equal bases, if they do not have the same base. These two [claims], from the first part of the corollary, we shall prove by the same argument we used in demonstrating the converse of propositions 36 and 31 of Book XI [propos. 36. & 31. lib. 11]: namely, by setting up another pyramid both upon the [cut-off] height and upon the cut-off base, etc.

THEOR. 6. PROPOS. 6.

[Margin: vj.]

Pyramids that are of the same height and have polygonal bases are to one another as their bases.

Let there be pyramids ABCDEF and GHIKLM, whose bases are the polygons ABCDE and GHIKL, having their sides equal in number. I say that pyramid is to pyramid as base is to base. For, the bases being resolved into an equal number of triangles, each pyramid will be divided into just as many triangular pyramids.

[Margin: a — *Elements* XII.5] Now since, as the base ABC is to the base ACD, so is the pyramid ABCF to the pyramid ACDF, it will follow by composition that, as the base ABCD is to the base ACD, so is the pyramid ABCDF to the pyramid ACDF.

[Margin: b — *Elements* XII.5] But again, as the base ACD is to the base ADE, so is the pyramid ACDF to the pyramid ADEF. Therefore, *ex aequo*, as the base

Diagram. Two pyramids on polygonal (pentagonal) bases, drawn in perspective.

Left figure — pyramid ABCDEF, apex above a pentagonal base:

- F [?] (apex, top; engraved small, resembling a lowercase "r")
- B (upper-left of base), E (upper-right of base), A (front-centre of base)
- C (lower-left of base), D (lower-right of base)

Right figure — pyramid GHIKLM:

- M (apex, top)
- G (centre of base), H (left of base), L (right of base)
- I (lower-left of base), K (lower-right of base)

Book XII (continued)

Proposition 6 — conclusion of the demonstration

as the base ABCD is to the base ADE, so the pyramid ABCDE is to the pyramid ADEF. Therefore, by composition, as the base ABCDE is to the base ADE, so the pyramid ABCDEF is to the pyramid ADEF. By a similar argument, as the base GHIKL is to the base GKL, so the pyramid GHIKLM is to the pyramid GKLM; and by inversion, as the base GKL is to the base GHIKL, so the pyramid GKLM is to the pyramid GHIKLM. [Margin: c — *Elements* XII.5] Again, since as the base ADE is to the base GKL, so the pyramid ADEF is to the pyramid GKLM, the four bases ABCDE, ADE, GKL, GHIKL will stand in the same proportions as the four pyramids ABCDEF, ADEF, GKLM, GHIKLM — as you see in the scheme below, and as is clear from the demonstration. Therefore, *ex aequali*, as the base ABCDE is to the base GHIKL, so the pyramid ABCDEF is to the pyramid GHIKLM. And for this reason pyramids standing under the same height, etc. Which was to be demonstrated.

Diagram. Two pentagonal pyramids drawn side by side in perspective (the figures of Proposition 6).

- Left pyramid — pentagon ABCDE with apex F. Apex F (top). Interior/back base-vertex A (drawn at the centre). Base vertices B (left), E (right), C (lower-left), D (lower-right).
- Right pyramid — pentagon GHIKL with apex M. Apex M (top). Interior/back base-vertex G (drawn at the centre). Base vertices H (left), L (right), I (lower-left), K (lower-right).

The printed proportion-scheme ("formula") set beside the figure lists the four bases and their four pyramids:

Bases	Pyramids
ABCDE	ABCDEF
ADE	ADEF
GKL	GKLM
GHIKL	GHIKLM

SCHOLIUM

Although the demonstration of this proposition, according to the commentators, speaks only of pyramids of the same height whose polygonal bases have an equal number of sides, we shall nevertheless easily demonstrate the same thing also for pyramids of the same height, one of whose bases contains more sides than the base of the other.

Let there first be two pyramids of the same height, ABCDEF and GHIK, of which the base of the former is a polygon — namely a pentagon — while that of the latter is triangular. I say that pyramid is to pyramid as base is to base. For when the pentagon is resolved into triangles, the pyramid too will be divided into an equal number of pyramids. [Margin: d — *Elements* XII.5] Now since, as the base ABC (a first quantity) is to the base GHI (a second quantity), so the pyramid ABCF (a third quantity) is to the pyramid GHIK (a fourth quantity); and in the same way, as the base ACD (a fifth quantity) is to the base GHI (the second quantity), so the pyramid ACDF (a sixth quantity) is to the pyramid GHIK (the fourth quantity) — [Margin: e — *Elements* V.24] it will follow that as the base ABCD (the first quantity together with the fifth) is to the base GHI (the second quantity), so the pyramid ABCDF (the third quantity together with the sixth) is to the pyramid GHIK (the fourth quantity). Again, since, as the base ABCD (a first quantity) is to the base GHI (a second quantity), so the pyramid ABCDF (a third quantity) is to the pyramid GHIK (a fourth quantity), as has just been shown; [Margin: f — *Elements* XII.5] and since, as the base ADE (a fifth quantity) is to the base GHI (the second quantity), so the pyramid ADEF (a sixth quantity) is to the pyramid GHIK (the fourth quantity) — [Margin: g — *Elements* V.24] it will also follow that as the base ABCDE (the first quantity together with the fifth) is to the base GHI (the second quantity), so the pyramid ABCDEF (the third quantity together with the sixth) is to the pyramid GHIK (the fourth quantity). Which was the thing proposed. One must always proceed in the same way, if there are more triangles in the polygonal base.

Diagram. A pentagonal pyramid and a triangular pyramid, drawn side by side (the figures of the first case of the Scholium).

- Left pyramid — pentagon ABCDE with apex F. Apex F (top). Interior/back base-vertex A (centre). Base vertices B (left), E (right), C (lower-left), D (lower-right).
- Right pyramid — triangle GHI with apex K. Apex K (top). Base vertices H (left), I (right), G (drawn at the centre/back).

Now since, as has been demonstrated, the polygonal base ABCDE is to the triangular base GHI as the pyramid ABCDEF is to the pyramid GHIK, it will also follow, by inversion, that as the base GHI is to the base ABCDE, so the pyramid GHIK is to the pyramid ABCDEF. For this reason, any two pyramids, one of which has a polygonal base and the other a triangular base, are proportional to their bases, from whichever you begin. For although the demonstration begins from the polygon and its pyramid, as is clear from the proof, nevertheless by inversion one may also take the beginning from the triangle and its pyramid, as has been said.

But now let there be two pyramids of the same height, ABCDEF and GHIKL, with more sides in one base than in the other. I say once more that pyramid is to pyramid as base is to base. For when the polygon ABCDE is resolved into the triangles ABC, ACD, ADE, its pyramid will be divided into just as many pyramids. Now since, as has already been demonstrated, the triangular base ABC (a first quantity) is to the polygonal base GHIK (a second quantity) as the pyramid ABCF (a third quantity) is to the pyramid GHIKL (a fourth quantity); and as the base ACD (a fifth quantity) is to the base GHIK (the second quantity), so the pyramid ACDF (a sixth quantity) is to the pyramid GHIKL (the fourth quantity) — [Margin: h — *Elements* V.24] it will also follow that as the base ABCD (the first quantity together with the fifth) is to the base GHIK (the second quantity), so the pyramid ABCDF (the third quantity together with the sixth) is to the pyramid GHIKL (the fourth quantity). Again, since, as the base ABCD (a first quantity) is to the base GHIK (a second quantity), so the pyramid ABCDF (a third quantity) is to the pyramid GHIKL (a fourth quantity), as has just been demonstrated; and since also, as the base ADE (a fifth quantity) is to the base GHIK (the second quantity), so the pyramid ADEF (a sixth quantity) is to the pyramid GHIKL (the fourth quantity), as was shown before — [Margin: i — *Elements* V.24] it will also follow that as the base ABCDE (the first quantity together with the fifth) is to the base GHIK (the second quantity), so the pyramid ABCDEF (the third quantity together with the sixth) is to the pyramid GHIKL (the fourth quantity). Which was the thing proposed. One must proceed no otherwise if there are more triangles in the base ABCDE.

Diagram. A pentagonal pyramid and a quadrilateral-based pyramid, drawn side by side (the figures of the third case of the Scholium).

- Left pyramid — pentagon ABCDE with apex F [?] (the engraved apex-letter reads uncertainly). Interior/back base-vertex A (centre). Base vertices B (left), E (right), C (lower-left), D (lower-right).

- Right pyramid — quadrilateral GHIK with apex L. Apex L (top). Base vertices G (upper-left), K (upper-right), H (lower-left), I (lower-right).

Now we shall convert this Proposition 6, together with what has been demonstrated in this Scholium, in the following way:

Pyramids of any bases whatsoever, which are to one another as their bases, have the same height.

This we shall demonstrate no otherwise than the things we stated in the Scholium of Proposition 5 of this book were demonstrated.

COROLLARIVM

It is also made clear from this that pyramids of the same height upon equal polygonal bases, or upon one and the same base, are equal to one another; since they have the same proportion as the bases, which are supposed equal, or one and the same.

Again, conversely: pyramids with polygonal bases that are equal, and set upon equal bases — as though built upon the same base — have the same height. And pyramids with polygonal bases that are equal and have the same height have equal bases, if they do not have the same base. These two things are demonstrated as we stated in the Corollary of Proposition 5 of this book.

THEOR. 7. PROPOS. 7.

[Margin: vij.]

Every prism having a triangular base is divided into three pyramids equal to one another, having triangular bases.

Let ABCDEF be a prism whose two opposite triangles ABF, DCE are equal and similar. I say that this prism is cut into three triangular pyramids equal to one another. For let three diagonals be drawn in the three parallelograms: namely AC in ABCD, CF in BCEF, and FD in ADEF. [Margin: a — *Elements* I.34] Since, then, the triangles ABC, ADC are equal, [Margin: b — *Elements* XII.5] and since, as the base ABC is to the base ADC, so the pyramid ABCF is to the pyramid ADCF — because these pyramids have the same height, namely the perpendicular dropped from the vertex F to the plane ABCD — the pyramids ABCF, ADCF will be equal to one another. By the same reasoning, the pyramids ADFC, EFDC, set upon the equal bases ADF, EFD and under the same height (namely the perpendicular dropped from the vertex C to the plane ADEF), will be equal. But the pyramid ADCF is the

same as the pyramid ADFC, since the former is contained by four planes — namely the base ADC and the triangles ADF, ACF, DCF — and the latter by the same four planes, namely the base ADF and the triangles ADC, ACF, DCF. Therefore the three pyramids ABCF, ADCF, and EFDC — or CDEF, which is the same as the pyramid EFDC, since both EFDC and CDEF are contained by the planes CDE, EFD, DCF, FCE — composing the whole prism, as is evident, are equal to one another; and therefore the prism ABCDEF is divided into three equal pyramids. Wherefore every prism having a triangular base, etc. Which was to be demonstrated.

Diagram. A triangular prism ABCDEF shown in perspective, with all its face-diagonals drawn (the figure of Proposition 7).

- A (top-left) and D (top-right): two vertices of the upper edge.
- B (bottom-left) and C (bottom-right): two vertices of the lower edge.
- E (right): a further vertex.
- F (centre, front): the remaining vertex (through which the dividing lines pass).

The two opposite triangular faces are ABF and DCE.

COROLLARIUM

From this it is gathered that any pyramid is a third part of the prism which has the same base and the same height as it; or that any prism is triple the pyramid which has the same base and height as it.

For let there first be a pyramid ABFC having the triangular base ABF, and a prism ABCDEF under the same height having the same triangular base ABF. I say that the pyramid is a third part of the prism. For if the straight line DF is drawn, the prism will be divided into three equal pyramids ABCF, ADCF, CDEF, [Margin: c — *Elements* XII.7] as has been demonstrated. And therefore the pyramid ABCF — that is, the pyramid ABFC (since this is equal to it, indeed the same, being comprehended by the same planes ABF, FCA, ABC, CBF) — will be a third part of the prism ABCDEF. Since, then, the pyramid ABFC is equal to any other pyramid set under the same height and upon the same base ABF, it is manifest, from the Corollary of Proposition 5 of this book, that any pyramid is a third part of the prism which has the same triangular base and height as it; and therefore, conversely, that the prism is triple the pyramid.

Diagram. A triangular prism ABCDEF shown in perspective with its diagonals (the figure of the first case of the Corollary).

- E (top): apex-edge vertex.
- D (mid-left) and C (mid-right): two vertices.
- F (centre): the vertex through which the diagonals pass.
- A (bottom-left) and B (bottom-right): the lower vertices.

Next, let there be a pyramid $ABCDE$ whose base is any rectilinear figure $ABCD$ of any number of sides, and a prism $ABCDEFGH$ under the same height having the same base. I say again that the pyramid is a third part of the prism. For when the base $ABCD$ and the opposite plane $EFGH$ are resolved into an equal number of triangles ADB , BCD , EHF , FGH , the prism will be divided into just as many prisms having triangular bases. Therefore, when the straight lines AH , BH , DF , CF are drawn, the pyramid $ADBH$ will be, as has been demonstrated, a third part of the prism $ABFEHD$; likewise the pyramid $BCDF$ will be a third part of the prism $CDHGFB$. Hence, as the pyramid $ADBH$ is to the prism $ABFEHD$, so the pyramid $BCDF$ is to the prism $CDHGFB$; [Margin: d — *Elements* V.12] and therefore, as one pyramid is to its own prism, so are all the pyramids together to all the prisms together — that is, to the prism $ABCDEFGH$. Therefore the pyramids $ADBH$, $BCDF$ together constitute a third part of the prism $ABCDEFGH$. But the pyramids $ADBH$, $BCDF$ are equal to the pyramid $ABCDE$; for the pyramids $ADBH$, $ADBE$, upon the same base and under the same height — namely between parallel planes — are equal, by the Corollary of Proposition 5 of this book. And by the same reasoning the pyramids $BCDF$, $BCDE$, upon the same base and under the same height, are equal. Wherefore, since the pyramids $ADBE$, $BCDE$ compose the whole pyramid $ABCDE$, the pyramid $ABCDE$ will also be a third part of the prism $ABCDEFGH$. Since, then, the pyramid $ABCDE$ is equal to any other pyramid set under the same height and upon the same base $ABCD$, it is clear, by the Corollary of Proposition 5 of this book, that any pyramid is a third part of the prism which has the same multilateral base and the same height as it.

Diagram. A prism $ABCDEFGH$ on a quadrilateral base, shown in perspective and criss-crossed with the diagonals used in the proof (the figure of the multilateral case of the Corollary).

- G (top-left) and H (top-right): two vertices of the upper face.
- E (upper-left) and F (upper-right): two further vertices.
- D (mid-left) and C (mid-right): two vertices.
- A (bottom-left) and B (bottom-right): the lower vertices.

The lower base is $ABCD$, the opposite upper plane $EFGH$.

And if the base of the pyramid and of the prism should have more sides, it will be shown in the same way that the pyramid is a third part of the prism. For when the polygons are divided into triangles, the prism will be cut into just as many prisms having triangular bases. Whence the individual triangular pyramids of these prisms will be the third parts of the individual prisms. Since, therefore, all these pyramids are equal to the pyramid whose base is the polygon of the proposed prism, the proposition is established.

In the same way, a pyramid is a third part of a prism having a base equal to the base, and a height equal to the height, of the pyramid. For such a pyramid is equal to that pyramid which has the same base and height as the prism, by the Corollary of Proposition 6 of this book — which we have already shown to be a third part of the prism.

SCHOLIUM I

From what has been said we shall easily demonstrate this proposition, namely:

Prisms existing under the same height, whatever bases they have, are to one another as their bases.

Although this has been demonstrated in Book XI for prisms whose two opposite planes are parallel, equal triangles — even if their bases are parallelograms and their vertices a straight line — and also for parallelepipeds, which we said are contained under the name of prisms: we shall nevertheless now demonstrate the same thing universally for all prisms whose two opposite planes, or bases, are polygons, even though more sides or angles are found in the base of the one than in the base of the other. Let there then be two prisms of the same height, ABCDEFGHIK and LMNOPQRS, whose bases are multilateral figures. I say that as the base ABCDE is to the base LMNO, so is prism to prism. For if, from all the angles of each base, straight lines are drawn to one point of the upper plane which is opposite the base, two pyramids will rise up under the same height as the prisms and having the same bases; and accordingly, by the aforesaid Corollary of this proposition, each pyramid will be a third part of its own prism. [Margin: a — *Elements* V.15] Wherefore, as pyramid is to pyramid, so will prism be to prism. But pyramid is to pyramid as base is to base, as was shown [in Proposition 6 [?]] and its Scholium. Therefore, as base is to base, so also will prism be to prism. Which was the thing proposed.

By conversion: if prisms of any bases whatever are to one another as their bases, they will be under the same height. This we shall show in the same way that the Scholium of Proposition 5 was demonstrated.

Diagram. Two prisms shown in perspective, standing side by side (the figures of Scholium I): on the left a pentagonal prism ABCDEFGHIK, on the right a quadrilateral (rectangular) prism LMNOPQRS.

- Left prism — upper face labelled F, K (at the top), G (upper-left), I (upper-right), H (centre); lower face labelled A (left), B (bottom), C (lower-centre), D and E (centre). Base pentagon ABCDE, upper pentagon FGHIK.
- Right prism — upper face labelled P (left), S (right), Q (upper-left), R (centre); lower face labelled L, M (bottom), N (right), O (left). Base quadrilateral LMNO, upper quadrilateral PQRS.

COROLLARIUM II

Whence prisms of the same height, upon the same base or upon any equal bases whatever, are equal to one another; because, of course, they have the same proportion as the bases, which are supposed equal, or one and the same.

On the contrary: prisms equal to one another, upon equal bases or upon the same base, are of the same height. And prisms equal to one another and of the same height have equal bases, if they do not have the same base. This you will demonstrate no otherwise than the converse of Proposition 31 of Book XI was demonstrated.

SCHOLIUM II

But both the things demonstrated in the preceding Corollary and those demonstrated in the immediately preceding Scholium are to be understood only of prisms which have at the top a plane parallel, equal, and similar to the base. I say this lest you be deceived in the case of a prism consisting of two parallel, equal, and similar triangles together with three parallelograms — which Campanus, with others, holds to be the only thing Euclid calls a prism, as we said in the definition of a prism. For if in such a prism one of those triangles be taken as the base, a pyramid of the same height and having the same base will necessarily be a third part of the prism itself, as was demonstrated in Corollary 1. Likewise the prism itself will have to any other prism the same proportion that base has to base, as we showed in the Scholium. But if in such a prism one of the parallelograms be conceived as the base, so that at the top there is a straight line, the pyramid will not have the same base and the same height, and then [?] the prism will have to the pyramid a sesquialterate proportion — namely that which 3 has to 2. For if to this prism another equal and similar prism be added, so that a parallelepiped is made, the pyramid will be a third part of the constituted

parallelepiped, by the aforesaid Corollary — since the parallelepiped is a prism having the same height and the same base as the pyramid. Wherefore, if the pyramid be put as 2, the parallelepiped will be 6. Half being therefore taken away, the remaining proposed prism will be 3; and therefore it will have to the pyramid the proportion which 3 has to 2, namely sesquialterate. From which it follows that a pyramid set upon half the parallelogram and having the same height as such a prism, being half of the former pyramid, is a third part of the prism. Since this is so, such a prism, so constituted, will not have to another prism of the same height the same proportion that base has to base, unless this other prism too has a straight line at the top and a parallelogram base. And so the things we showed in the Scholium are to be understood of prisms of the same height whose vertices are either straight lines and whose bases are parallelograms, or which have at their tops planes parallel, equal, and similar to their bases. For by this reasoning, if the vertices of the prisms are straight lines, the pyramids upon the same bases and of the same height will be two-thirds of the prisms, as we have just demonstrated. Wherefore they will have the same ratio as the prisms, etc.

THEOR. 8. PROPOS. 8.

[Margin: viij.]

Similar pyramids which have triangular bases are in the triplicate ratio of their corresponding sides.

Let ABCD, EFGH be similar triangular pyramids, such that the bases ABC, EFG are similar and the remaining triangles of the one are similar to the remaining triangles of the other. I say that the ratio of the pyramids is triplicate of the ratio which.

(Continuation of the demonstration of Proposition 8 — that similar pyramids with triangular bases are in the triplicate ratio of their corresponding sides — begun in the preceding pages.)

than that which their corresponding sides BC and FG have. For let the planes of the triangles ABC, CBD, DAB be extended, and let the parallelograms BI, BK, BL be completed. Then draw LM and IM, meeting the lines AI and AL in parallels at M, and join the line KM. The parallelepiped BM will thus be completed, having the same height as the pyramid — for the planes of the solid BM are parallel, as is easily gathered from Proposition 15 of Book XI. In the same way complete the parallelepiped FQ.

Now, because of the similarity of the pyramids, the plane angles ABC and EFG are equal, and AB is to BC as EF is to FG; therefore the parallelograms BI and FN are similar. In the same way, since the angles ABD and EFH are equal, and AB is to BD as EF is to FH, and likewise the angles DBC and HFG are equal, and DB is to BC as HF is to FG, the parallelograms AL and BK will also be similar to the parallelograms FP and FO.

[Margin: a — 24. vnd. (*Elements* XI.24)] Now the three faces BI, BK, BL of the parallelepiped BM, together with the three opposite ones DM, AM, CM, and the three faces FN, FO, FP of the parallelepiped FQ, together with its three opposite ones HQ, EQ, GQ, [Margin: b — 24. vnd. (*Elements* XI.24)] are equal and similar. Therefore the six planes bounding the solid BM are similar to the six planes enclosing the solid FQ; and hence, by Definition 9 of Book XI, the parallelepipeds BM and FQ are similar.

And since, when the lines LI and PN are drawn, [Margin: c — 15. quint. (*Elements* V.15)] the prisms DBCILA and HFGNPE have the same ratio as the parallelepipeds BM and FQ, of which they are the doubles, while the pyramids ABCD and EFGH have the same ratio as those prisms, of which they are the triples, [Margin: d — 11. quin. (*Elements* V.11)] the pyramids too will have the same ratio as the parallelepipeds. [Margin: e — 33. vnde. (*Elements* XI.33)] Therefore, since the ratio of the parallelepiped BM to the parallelepiped FQ is the triplicate of the ratio of the corresponding sides BC and FG, the ratio of the pyramid ABCD to the pyramid EFGH will likewise be the triplicate of the ratio of BC to FG. Similar pyramids, then, which have triangular bases, and so on. Which was to be demonstrated.

Diagram. Two oblique parallelepipeds drawn in perspective, the larger at the left and the smaller at the right, each completing a triangular pyramid (ABCD and EFGH) into a full parallelepiped (BM and FQ) of the same height. Small engraved apex/vertex labels, foxed. (dense vertex labelling).

- Larger solid (left), readable vertex labels: L (upper-left), M (upper-right), A (upper front-left), I (upper front-right), B (lower-left), D (lower-center), C (lower-right). (The vertex K, named in the text, is not separately legible.)
- Smaller solid (right), readable vertex labels: P (upper-left), Q (upper-right), E (mid-left), N (mid-right), F (lower-left), G (lower-center), O (lower-right). (Vertex H not clearly legible.)

COROLLARIUM.

From this too it is evident that similar pyramids whose bases contain more than three sides have the triplicate ratio of their corresponding sides.

Let there be similar pyramids whose rectilinear bases ABCDE and GHIKL, of several sides, are similar. I say that the ratio of the pyramids is the triplicate of the ratio which their corresponding sides AB and GH have. For if from the angles E and L lines EB, EC, LH, LI are drawn to the opposite angles, [Margin: f — 20. sexti. (*Elements* VI.20)] the similar bases will be divided into triangles equal in number and similar — namely, the triangles ABE, EBC, CDE will be similar to the triangles GHL, LHI, IKL.

Since, then, on account of the similarity of the pyramids, the triangles AEF and GLM are similar, and the angle FEA is equal to the angle MLG, FE will be to EA as ML is to LG. And since EA is to EB as LG is to LH, on account of the similarity of the triangles AEB and GLH, ex aequali FE will be to EB as ML is to LH. Again, since EB is to BA as LH is to HG, because of the similar triangles ABE and GHL, and BA is to BF as HG is to HM (since, by the similarity of the pyramids, the triangles ABF and GHM are similar), ex aequali EB will also be to BF as LH is to HM. And therefore, since FE is to EB as ML is to LH, and EB is to BF as LH is to HM, ex aequali FE will be to FB as ML is to MH.

[Margin: g — 5. sexti. (*Elements* VI.5)] Hence the triangles FEB and MLH are equiangular, and so similar. And the triangles FEA, FAB, ABE are similar to the triangles MLG, MGH, GHL. Therefore, by Definition 9 of Book XI, the pyramids ABEF and GHLM are similar. [Margin: h — 8. duode. (*Elements* XII.8)] By the same reasoning the pyramids EBCF and LHIM will be similar, and likewise CDEF and IKLM.

Therefore, as has been demonstrated, the pyramids ABEF, EBCF, CDEF will have to the pyramids GHLM, LHIM, IKLM, each to each, the triplicate ratio of the corresponding sides AB, BC, CD to GH, HI, IK, each to each. And since AB, BC, CD have one and the same ratio to GH, HI, IK, on account of the similarity of the bases ABCDE and GHIKL, the pyramids ABEF, EBCF, CDEF will also have to the pyramids GHLM, LHIM, IKLM one and the same ratio — namely, the triplicate of that ratio. [Margin: i — 12. quint. (*Elements* V.12)] And therefore, as one pyramid ABEF is to one pyramid GHLM, so all the pyramids together, namely the pyramid ABCDEF, will be to all the pyramids together, namely to the pyramid GHIKLM. Wherefore, [Margin: k — 8. duod. (*Elements* XII.8)] since the pyramid ABEF has to the pyramid GHLM the triplicate ratio of the corresponding sides AB and GH, the

pyramid ABCDEF will also have to the pyramid GHIKLM the triplicate ratio of the same corresponding sides AB and GH. Which is what was proposed.

Diagram. Two pyramids on many-sided (pentagonal) bases, drawn in perspective: at the left the pyramid ABCDEF with apex F, at the right the pyramid GHIKLM with apex M, each base subdivided into triangles by diagonals.

- Left pyramid, readable labels: F (apex, top), E (a base vertex, drawn as the point from which the diagonals EB, EC issue), A (left), D (right), B (bottom-left), C (bottom-right).
- Right pyramid, readable labels: M (apex, top), G (left), K (interior/mid), I (lower-right), H (bottom). (Vertex L not clearly legible.)

SCHOLIVM.

By the same reasoning, similar prisms will have the triplicate ratio of their corresponding sides.

For let there be two similar prisms ABCDEFGHIK and LMNOPQRSTV. I say that their ratio is the triplicate of the ratio of their corresponding sides. For if from the angles A and L the lines AC, AD and LN, LO are drawn, [Margin: l — 20. sexti. (*Elements* VI.20)] then, as was said of the pyramids, the triangles ABC, CAD, DEA will be similar to the triangles LMN, NLO, OLP. Similarly, if from the angles G and R the lines GI, Gk and RT, RV are drawn, the triangles GHI, IGk, kFG will be similar to the triangles RST, TRV, VQR, and to the triangles previously named — since all the opposite faces of the prisms are similar, by the definition of a prism.

Now, because, on account of the similarity of the prisms, the parallelograms CH and NS are similar, IC will be to CB as TN is to NM; and since CB is to CA, so also is NM to NL, on account of the similarity of the triangles BCA and MNL. Therefore ex aequali IC will be to CA as TN is to NL. Again, since the solid angle N is equal to the solid angle C, on account of the similarity of the prisms, if the one be placed upon the other, all will coincide — namely the angle MNO with the angle BCD, the angle TNM with the angle ICB, and the angle TNO with the angle ICD. The line NL also coincides with the line CA, because the angles MNL and BCA are equal. Therefore the angles ICA and TNL are equal; and hence, since the sides about them have been shown to be proportional, the parallelograms CG and NR will be similar. And the parallelograms CH, HA are similar to the parallelograms NS, SL, and the triangles ABC, GHI to the triangles LMN, RST. Therefore the prisms ABCIGH and LMNTRS are similar, by the definition of similar solids.

In no other way will it be shown that the prisms CDAGIk and NOLRTV are similar, and likewise the prisms AEDkGF and LPOVRQ. Wherefore the prisms ABCIGH,

CDAGIk, AEDkGF have to the prisms LMNTRS, NOLRTV, LPOVRQ, by what we demonstrated at Proposition 34 of Book XI, the triplicate ratio, each to each, of the corresponding sides BC, CD, DE to MN, NO, OP, each to each. And therefore in the same way we shall demonstrate that the prisms ABCDEFGHIk [and] LMNO...

Diagram. Two pentagonal prisms drawn in perspective, the larger at the left, the smaller at the right (the similar prisms ABCDEFGHIk and LMNOPQRSTV), each face subdivided by diagonals. Many small vertex labels, some foxed. (dense vertex labelling; >2 labels uncertain).

- Larger prism (left), readable labels: G (top), H (upper-left), F (upper-right), I (mid-left), K (mid-right), A (interior), B (lower-left), E (lower-right), C (bottom-left), D (bottom-center).
- Smaller prism (right), readable labels: R (top), S (upper-left), Q (upper-right), T (mid-left), V (mid-right), L (interior), M (mid), P (lower-right), N (bottom-left), O (bottom-center).

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LMNOPQRSTV, have the triplicate ratio of their corresponding sides BC and MN — the very method we used a little before to demonstrate the same thing for many-angled pyramids.

From all this it is gathered that similar many-angled pyramids are divided into similar triangular pyramids, equal in number, and corresponding to the wholes. For this is evident from the demonstration of the preceding corollary.

In the same way it is inferred that similar many-angled prisms are divided into similar prisms having triangular bases, equal in number, and corresponding to the wholes — as appears from the demonstration of this scholium.

THEOR. 9. PROPOS. 9.

[Margin: vij.]

In equal pyramids that have triangular bases, the bases and the heights are reciprocally proportional. And those pyramids having triangular bases whose bases and heights are reciprocally proportional are equal.

Let ABCD and EFGH be equal triangular pyramids. I say that their bases ABC and EFG and their heights are reciprocal — that is, that ABC is to EFG as the height of the pyramid EFGH is to the height of the pyramid ABCD. For if, as was said in the preceding proposition, the parallelepipeds BM and FQ, of the same heights as the

pyramids, are completed, and the lines LI and PN are joined, the prisms DBCILA and HFGNPE, being the triples of the pyramids (which are assumed equal), will be equal to one another; and hence the parallelepipeds BM and FQ, being the doubles of the prisms, will also be equal. [Margin: a — 34. vnd. (*Elements* XI.34)] Therefore their bases and heights will be reciprocal — that is, the base BI will be to the base FN as the height of the solid FQ is to the height of the solid BM. [Margin: b — 15. quint. (*Elements* V.15)] But the base BI is to the base FN as the triangle ABC is to the triangle EFG. Therefore the base ABC will also be to the base EFG as the height of the pyramid EFGH is to the height of the pyramid ABCD, since the heights of the pyramids are the same as those of the parallelepipeds. And therefore the bases and heights of the equal pyramids are reciprocal.

Now let the bases and heights be reciprocal. I say that the pyramids are equal. For, the figure being constructed as before, [Margin: c — 14. quin. (*Elements* V.14)] since ABC is to EFG as the parallelogram BI is to the parallelogram FN, and the heights of the parallelepipeds and of the pyramids are the same, the bases of the parallelepipeds and their heights will also be reciprocal; [Margin: d — 34. vnde. (*Elements* XI.34)] and therefore the parallelepipeds BM and FQ will be equal to one another. Hence the prisms DBCILA and HFGNPE, their halves, will be equal; and therefore the pyramids too, being the third parts of the prisms, will be equal. In equal pyramids, then, having triangular bases, and so on. Which was to be demonstrated.

Diagram. Two oblique parallelepipeds drawn in perspective (the solids BM and FQ of the previous construction), the larger at the left, the taller, narrower one at the right. Small vertex labels, foxed. (dense vertex labelling).

- Larger solid (left), readable labels: M (upper-left), L (upper-center), A (right), I (interior), K (lower-left), C (bottom-left), D (bottom-center), B (bottom-right).
- Right solid, readable labels: Q (top), E (mid-left), N (mid-right), F (lower-left), H (lower-center), G (lower-right).

SCHOLIVM.

By the same reasoning, in equal pyramids whose bases are not triangular the bases and heights are reciprocal; and those pyramids not having triangular bases whose bases and heights are reciprocal are equal.

For let there be, first, two equal pyramids, of which ABCD has a triangular base, but EFGHI a non-triangular one. To the non-triangular base EFGHI let a triangle KLM be made equal. This will be done if the many-sided base is reduced to a

parallelogram, by Proposition 45 of Book I; for this parallelogram is easily reduced to a triangle by what we demonstrated at Proposition 42 of Book I. Then upon KLM let a pyramid KLMN be constructed, of the same height as the pyramid EFGHI. Since, then, the pyramids EFGHI and KLMN have equal bases and the same height, they will be equal, by what we showed in the scholium of Proposition 6 of this book. But the pyramid EFGHI is assumed equal to the pyramid ABCD. Therefore the pyramid KLMN will also be equal to the pyramid ABCD.

Wherefore, since the pyramids ABCD and KLMN have triangular bases, [Margin: e — 9. duode. (*Elements* XII.9)] the base ABC will be to the base KLM — that is, to the base EFGH equal to it — as the height of the pyramid KLMN — that is, the height of the pyramid EFGH (for these heights are assumed equal) — is to the height of the pyramid ABCD. And hence the bases and heights of the equal pyramids ABCD and EFGHI are reciprocal.

But now let the bases and heights be reciprocal. I say that they are equal. For, the figure being constructed in the same way, since the base KLM is equal to the base EFGH, and the height of the pyramid KLMN is the same as that of the pyramid EFGHI, while it is assumed that ABC is to EFGH as the height of the pyramid EFGHI is to the height of the pyramid ABCD, it will also be that the base ABC is to the base KLM as the height of the pyramid KLMN is to the height of the pyramid ABCD. [Margin: f — 9. duode. (*Elements* XII.9)] Wherefore the pyramids ABCD and KLMN are equal. And since the pyramid KLMN is equal to the pyramid EFGHI, by what was shown in the scholium of Proposition 6 of this book, the pyramid ABCD will also be equal to the pyramid EFGHI. Which is what was proposed.

Diagram. Three figures side by side illustrating the first case of the scholium: a triangular-based pyramid, a quadrilateral-based pyramid, and a triangular-based pyramid.

- Left (pyramid ABCD), readable labels: D (apex), A (interior), B (bottom-left), C (bottom-right).
- Center (pyramid EFGHI, non-triangular base), readable labels: I (apex), E (upper-left), H (upper-right), F (bottom-left), G (bottom-right).
- Right (pyramid KLMN), readable labels: N (apex), K (interior), M (bottom-right); the lower-left base vertex is not clearly legible [?].

Diagram. Three pyramids side by side illustrating the second (many-angled) case of the scholium (the same figure is repeated at the top of the following page): a

triangular-based pyramid MNOP, a quadrilateral-based pyramid ABCDE, and a many-sided-based pyramid FGHKL.

- Left (pyramid MNOP), readable labels: P (apex), M (interior), N (bottom-left), O (bottom-right).
- Center (pyramid ABCDE), readable labels: E (apex), A (upper-left), D (upper-right), B (bottom-left), C (bottom-right).
- Right (pyramid FGHKL), readable labels: L (apex), F (interior), G (upper-left), K (interior), H (bottom-left), I (bottom-right).

(Global page 534 — running header "EVCLID. GEOM."; folio "522"; leaf-mark "viiij.")

Diagram. The same three pyramids as the preceding figure, repeated at the head of the page for continuity: the triangular-based pyramid MNOP, the quadrilateral-based pyramid ABCDE, and the many-sided-based pyramid FGHKL.

- Left (pyramid MNOP), readable labels: P (apex), M (interior), N (bottom-left), O (bottom-right).
- Center (pyramid ABCDE), readable labels: E (apex), A (upper-left), D (upper-right), B (bottom-left), C (bottom-right).
- Right (pyramid FGHKL), readable labels: L (apex), F (interior), G (upper-left), H (bottom-left), I (bottom-right), K (interior).

as the height of the pyramid MNOP — that is, to the height of the pyramid ABCDE (for these heights are assumed equal). And therefore the bases and heights of the equal pyramids ABCDE and FGHKL are reciprocal.

But now let the bases and heights of these pyramids be reciprocal. I say that they are equal. For, the figure being set up in the same way, since the base MNO is equal to the base ABCD, and the height of the pyramid MNOP is the same as that of the pyramid ABCDE, and it is assumed that ABCD is to FGHKL as the height of the pyramid FGHKL is to the height of the pyramid ABCDE, it will also be that MNO is to FGHKL as the height of the pyramid FGHKL is to the height of the pyramid MNOP. Wherefore, as we showed before, the pyramids MNOP and FGHKL will be equal. But the pyramid MNOP is equal to the pyramid ABCDE, from the scholium of Proposition 6 of this book. Therefore the pyramid ABCDE will also be equal to the pyramid FGHKL. Which is what was proposed.

All these things too will easily be demonstrated to hold for any prisms whatever. For if the prisms are equal, their pyramids also — of the same heights as the prisms and upon the same bases — will be equal, since they are the third parts of them, by the corollary of Proposition 7 of this book; or indeed two-thirds, if the

bases of the prisms are parallelograms and their vertices straight lines, as was shown in the scholium of the same proposition. Wherefore, as we have just demonstrated, the bases and heights of these pyramids are reciprocal. And since these bases and heights are the same as those of the prisms, the bases and heights of the prisms will also be reciprocal.

Again, if the bases and heights of the prisms are reciprocal, the bases and heights of the pyramids having the same bases and heights as the prisms will also be reciprocal. Wherefore, as was demonstrated, the pyramids are equal; and therefore the prisms — being their triples, or indeed one and a half times as great — are also equal. Which is what was proposed.

THEOR. 10. PROPOS. 10.

[Margin: viij.]

Every cone is the third part of a cylinder having the same base as itself and an equal height.

Let a cone and a cylinder have the same base, the circle ABCD, and the same height. I say that the cone is the third part of the cylinder. For if the cone is not believed to be the third part of the cylinder, the cylinder will not be the triple of the cone, but either greater or less than the triple of the cone.

Let it first be greater than the triple of the cone, by the magnitude E, so that the cylinder is equal to the triple of the cone together with the magnitude E, taken together. Let a square ABCD be inscribed in the circle, and about the same circle let a square FGHI be described; and upon these squares, under the height of the cone and cylinder, let two parallelepipeds be understood to be erected. Since, then, the square ABCD is half the square FGHI, as we showed at Proposition 9 of Book IV, [Margin: a — 32. vnd. (*Elements* XI.32)] and since base is to base as parallelepiped is to parallelepiped of the same height, the parallelepiped on the base ABCD will be half the parallelepiped on the base FGHI. And hence the parallelepiped on the base ABCD is greater than half the cylinder whose base is the circle ABCD.

Let the arcs AB, BC, CD, DA be bisected at the points K, L, M, N, and let the lines KA, KB, LB, LC, MC, MD, ND, NA be joined. Through K let the line OP be drawn tangent to the circle at K, which will be parallel to AB, as we showed at Proposition 27 of Book III, and let it meet the lines DA and CB produced at P and O; and upon AKB and ABOP let prisms be understood, under the height of the cone and cylinder. [Margin: b — 41. prim. (*Elements* I.41)] Since, then, the triangle

AKB is half the parallelogram ABOP, the prism on the base AKB will also be half the prism, or parallelepiped, on the base ABOP — since prism is to prism as base is to base, as we demonstrated at Proposition 7 of this book. And therefore the prism on the base AKB will be greater than half the segment of the cylinder whose base is the figure contained by the straight line AB and the arc AKB.

By the same reasoning there will be prisms whose bases are the remaining triangles BLC, CMD, DNA, and whose height is the same as that of the cone and cylinder, greater than the halves of the segments of the cylinder whose bases are the segments of the circle. Therefore all these prisms together are greater than the halves of all the segments of the cylinder together. And if again the arcs AK, KB, etc. are bisected and straight lines joined, prisms of the same height as the cone and cylinder will be constructed in the same way, which together will be greater than the halves of all the segments of the cylinder together whose bases are the segments of the circle; and so on in succession.

Now since, if from the cylinder whose base is the circle ABCD there be taken away more than the half — namely the parallelepiped on the base ABCD — and from the remaining segments more than the half — namely the prisms on the bases AKB, BLC, etc. — and if in this manner the subtraction be always made, [Margin: 1. dec. (*Elements* X.1) — printed margin letter uncertain [?]] there is at length left a magnitude less than E, the excess of the cylinder over the triple of the cone. Let the segments of the cylinder now left, on the bases AK, KB, BL, etc. (whose bases are indeed segments of the circle), taken together, be less than E.

Since, then, the cylinder is assumed equal to the triple of the cone together with the magnitude E, if from the cylinder the said segments be taken away, and from the triple of the cone together with the magnitude E the magnitude E itself — which is greater than the said segments of the cylinder — the remainder, the prism on the many-angled base AKBLCMDN, having the same height as the cone and cylinder, will be greater than the remaining triple of the cone; and therefore the cone will be less than the third part of the prism. But since the pyramid whose base and height are the same as the prism's is the third part of the said prism, by the corollary of Proposition 7 of this book, the cone will be less than the said pyramid — the whole less than the part. Which is absurd. Therefore the cylinder is not greater than the triple of the cone.

Next let the cylinder be less than the triple of the cone, and accordingly the cone greater than the third part of the cylinder. Let the cone then be greater...

Diagram. The construction for Proposition 10: a circle (ABCD) inscribed within a square (FGHI); the arcs bisected at K, L, M, N to form an inscribed octagon, with chords drawn from those points; a tangent line OP through K meeting the produced sides. Beneath the main figure a small lens/almond shape labelled E represents the auxiliary magnitude E. (several small labels, foxed).

- Circle vertices: A (top), B (left), C (bottom), D (right).
- Circumscribed square corners: F (top-left), G (bottom-left), H (bottom-right), I (top-right) [top-right corner reads I/I — [?]].
- Arc-midpoints (inscribed octagon): K (upper-left), L (lower-left), M (lower-right), N (upper-right).
- Tangent line through K: P (upper) and O (left), where OP meets DA and CB produced.
- E: separate lens-shaped figure below the construction, representing the magnitude E.

[Book XII, Proposition 10 — conclusion of the demonstration]

greater than a third part of the cylinder, by the magnitude E, so that the cone is equal to a third part of the cylinder together with the magnitude E. Once more inscribe in the circle the square ABCD, and about the same circle circumscribe the square FGHI; and upon these squares let there be conceived pyramids having the height of the cone and cylinder. Now the square ABCD is half the square FGHI, as we proved at Book IV, proposition 9, and base is to base as pyramid is to pyramid of the same height [Margin: d. Elements XII.6]. So the pyramid on ABCD will also be half the pyramid on the base FGHI; and therefore the pyramid on the base ABCD is greater than half the cone whose base is the circle ABCD.

Bisect the arcs AB, BC, CD, DA at K, L, M, N, and join the straight lines AK, KB, BL, LC, CM, MD, DN, NA. Also draw through K the straight line OP tangent to the circle at K — which will be parallel to AB, as we proved at Book III, proposition 27 — and let it meet DA and CB, produced, at P and O; and upon AKB and ABOP let pyramids of the height of the cone and cylinder be conceived. [Margin: e. Elements I.41] Now since the triangle AKB is half the parallelogram ABOP, the pyramid on the base AKB will likewise be half the pyramid on the base ABOP [Margin: f. Elements XII.6], since pyramids have the same ratio as their bases. Therefore the pyramid on the base AKB is greater than half the segment of the cone whose base is the figure contained by the straight line AB and the arc AKB. By the same reasoning, the pyramids whose bases are the triangles BLC, CMD, DNA, and whose height is the same as that of the cone and cylinder, will be

greater than half of the segments of the cone whose bases are the circular segments. All these pyramids taken together are therefore greater than half of all the segments of the cone taken together.

And if again the arcs AK, KB, BL, LC, CM, MD, DN, NA are bisected and straight lines are joined in the same circle, pyramids of the same height as the cone and cylinder will be constructed in the same way, and these together will be greater than half of all the segments of the cone together whose bases are the circular segments; and so on continually. Now since, if from the cone whose base is the circle ABCD more than half is removed — namely the pyramid on the base ABCD, which we showed to be greater than half the cone whose base is the circle ABCD — and from the remaining segments more than half — namely the pyramids on the bases AKB, BLC, CMD, DNA — and the subtraction is carried on in this manner without end, there is at last left [Margin: g. Elements X.1] a magnitude less than E, the excess of the cone above the third part of the cylinder. So let the segments of the cone now left, on the bases AK, KB, BL, LC, CM, MD, DN, NA (whose bases are indeed circular segments), taken together, be less than E.

Since, then, the cone is assumed equal to the third part of the cylinder together with the magnitude E: if from the cone the aforesaid segments are taken away, and from the third part of the cylinder together with the magnitude E the magnitude E itself is taken away — which E is greater than the aforesaid segments of the cone — the remaining pyramid, whose base is the polygon AKBLCMDN and which has the same height as the cone and cylinder, will be greater than the remaining third part of the cylinder; and therefore triple the said pyramid will be greater than the cylinder. Consequently, since the prism having the same base as the said pyramid and the same height as the cone and cylinder is triple that pyramid — as was demonstrated above by us in the corollary to proposition 7 of this book — this prism will be greater than the cylinder, the part greater than the whole. Which is absurd. Therefore the cylinder is not less than triple the cone. But neither was it shown to be greater than triple. Therefore it is equal to triple the cone, and for that reason the cone is a third part of the cylinder. Every cone, then, is a third part of the cylinder that has the same base as it and an equal height. Which was to be demonstrated.

SCHOLIUM

Although this proposition is understood only of right cones and cylinders — since Euclid defined only these in Book XI — it nevertheless holds for all cones and cylinders alike, right and oblique [*scaleni*], as is plain to anyone who examines its

demonstration attentively. And so the proposition may be stated in fully general form thus:

Every cone, whether right or oblique, is a third part of any cylinder whatever, whether right or oblique, that has the same base and the same height as it — even though the axis of the cone and of the cylinder is not one and the same.

THEOR. 11. PROPOS. 11.

[Margin: ix. (vulgate proposition number)]

Cones and cylinders that stand under the same height are to one another as their bases.

Let there be cones and cylinders under the same height, whose bases are the circles ABCD and EFGH, and whose heights, IK and LM, are equal. I say that as base is to base, so is cone to cone and cylinder to cylinder. For if this be not granted, let it be that as the base ABCD is to the base EFGH, so is the cone ABCDK to some magnitude, namely N, which will be either greater or less than the cone EFGHM. For if it were equal, the cone ABCDK would have to the cone EFGHM and to N the same ratio [Margin: a. Elements V.7]; and for that reason cone would be to cone as base to base — which is not conceded.

Diagram. Two circles side by side, each with an inscribed octagon and a cone-apex letter, and beneath them two small figures representing the auxiliary magnitudes O and N.

- Left circle — base **ABCD**, center **I**; inscribed octagon with vertices, in order around the circle: **A** (top), **T** (upper left), **B** (left), **V** (lower left), **C** (bottom), **X** (lower right), **D** (right), **Y** (upper right). **K** — cone apex, marked at the end of the axis line rising to the upper right (cone ABCDK).
- Right circle — base **EFGH**, center **L**; inscribed octagon with vertices, in order: **E** (top), **P** (upper left), **F** (left), **Q** (lower left), **G** (bottom), **R** (lower right), **H** (right), **S** (upper right). **M** — cone apex, at the end of the axis line rising to the upper right (cone EFGHM).
- **O** — pointed-oval (lens-shaped) figure below the left circle (the auxiliary magnitude O).
- **N** — shaded circular figure below the right circle (the auxiliary magnitude N).

Let N , then, be first less than the cone $EFGHM$ by the magnitude O , so that the cone $EFGHM$ is equal to the magnitudes N and O together. Inscribe in the circle $EFGH$ the square $EFGH$, bisect the arcs EF , FG , GH , HE at P , Q , R , S , and join the straight lines EP , PF , FQ , and so on. Now if from the cone $EFGHM$ there be taken away the pyramid on the base $EFGH$, of the same

height, and from the remaining segments there be taken away pyramids of the same height on the bases EPF , FQG , and so on, and the subtraction be carried on in this manner without end, more than half is always subtracted, as was shown in the latter part of the preceding proposition [Margin: b. Elements X.1]; there will at last be left a magnitude less than O , the excess of the cone $EFGHM$ above N . So let the segments of the cone now left, on the bases EP , PF , FQ , and so on (whose bases are indeed circular segments), taken together, be less than O . Since, then, the cone $EFGHM$ is assumed equal to the magnitudes N and O together: if from the cone the said segments of the cone are taken away, and from N and O the magnitude O itself is taken away — which O is greater than the aforesaid segments of the cone — the remaining pyramid, whose base is the polygon $EPFQGRHS$ and which is of the same height as the cone, will be greater than N , the remaining magnitude.

Now inscribe in the circle $ABCD$ a polygon $ATBVCXDY$ similar to the polygon $EPFQGRHS$, as we taught at proposition 2 of this book, and draw the diameters BD , FH of the circles. Then since, by the corollary to proposition 2 of this book, as the circle $ABCD$ is to the circle $EFGH$, so is the polygon $ATBVCXDY$ to the polygon $EPFQGRHS$; and since, as the circle $ABCD$ is to the circle $EFGH$, so is the cone $ABCDK$ assumed to be to the magnitude N [Margin: c. corollary, Elements XII]; and since, as polygon is to polygon, so is pyramid to pyramid of the same height as the cones: it will also be that as the pyramid $ATBVCXDYK$ is to the pyramid $EPFQGRHSM$, so is the cone $ABCDK$ to N . Since, then, the pyramid $ATBVCXDYK$ is less than the cone $ABCDK$, the part than the whole, the pyramid $EPFQGRHSM$ will also be less than N [Margin: d. Elements V.14] — but it was shown to be greater. Which is absurd. Therefore the magnitude N is not less than the cone $EFGHM$.

Let N , secondly, be greater than the cone $EFGHM$. Since, then, it is assumed that as the circle $ABCD$ is to the circle $EFGH$, so is the cone $ABCDK$ to N , it will also be, by conversion, that as the circle $EFGH$ is to the circle $ABCD$, so is N to the cone $ABCDK$. Let it be that as N is to the cone $ABCDK$, so is the cone $EFGHM$ to the magnitude O . And since N is assumed greater than the cone $EFGHM$, the cone $ABCDK$ will also be greater than O . Therefore it will be that as the circle $EFGH$ is to the circle $ABCD$, so is the cone $EFGHM$ to the magnitude O , which is less than the cone $ABCDK$. Which is absurd; for it has already been shown that a cone

cannot be to a magnitude less than another cone as the base of that cone is to the base of this cone. Therefore the magnitude N is not greater than the cone EFGHM. But neither was it shown to be less; it is therefore equal. And so, since it is assumed that as the base ABCD is to the base EFGH, so is the cone ABCDK to N [Margin: e. Elements V.7], while it holds that as the cone ABCDK is to N, so is that same cone ABCDK to the cone EFGHM, it will also be that as the base ABCD is to the base EFGH, so is the cone ABCDK to the cone EFGHM.

Now since, as the cone ABCDK is to the cone EFGHM, so is the cylinder ABCDK (which is triple the cone ABCDK) to the cylinder EFGHM (which is triple the cone EFGHM) [Margin: f. Elements V.15], it will also be that as the base ABCD is to the base EFGH, so is the cylinder ABCDK to the cylinder EFGHM. This too can be confirmed in the same way as we used for the cones, if in place of the cones and pyramids one conceives cylinders and prisms. Cones and cylinders, therefore, that stand under the same height are to one another as their bases. Which was to be demonstrated.

SCHOLIUM

It is also plain that this proposition, which Euclid stated only of right cones and cylinders, can — for the reason given above — be extended to oblique cones and cylinders as well, as its demonstration manifestly indicates, so that the proposition may be stated in fully general form in this manner:

Cones and cylinders that stand under the same height — whether both are right, or both oblique, or the one right and the other oblique — have to one another the proportion of their bases.

The converse of this proposition will likewise be stated in this way:

If cones and cylinders are to one another as their bases, they will be under the same height.

This indeed you will demonstrate in the same way in which we proved the converse of proposition 32 of Book XI.

COROLLARIUM

Hence it follows that cones and cylinders of the same height set upon the same base, or upon equal bases — whether both are right, or oblique, or the one right and the other oblique — are equal to one another: because they have the same proportion as their bases, which are assumed equal, or are indeed one and the same.

It likewise follows that cones and cylinders that are equal upon the same base, or upon equal bases, are of the same height; and that equal ones of the same height are upon equal bases, if they do not have the same base. This we shall show in no other way than that in which the converse of proposition 31 of Book XI was demonstrated.

THEOR. 12. PROPOS. 12.

[Margin: x. (vulgate proposition number)]

Similar cones and cylinders are in the triplicate ratio of the diameters that are in their bases.

Let there be similar cones and cylinders whose bases are the circles ABCD, EFGH, whose axes are LK, LM, and whose base-diameters are BD, FH. I say that cone is to cone, and cylinder to cylinder, in the triplicate ratio of diameter to diameter. For if this be not granted, let the cone ABCDK have to some magnitude N the triplicate ratio of the diameter BD to the diameter FH. And N will be either less or greater than the cone EFGHM. For if it were equal, the cone ABCDK would have to the cone EFGHM and to N the same ratio [Margin: a. Elements V.7]; and consequently the ratio of the cone ABCDK to the cone EFGHM would also be the triplicate of the ratio of the diameter BD to the diameter FH — which is not conceded.

Let N, then, be first less than the cone EFGHM by the magnitude O; and let exactly the same construction of the figure be made as in the preceding proposition, so that again the pyramid EPFQGRHSM is shown greater than N. Then draw the straight lines KB, KT, MF, MP, so that there are obtained two triangles BKT, EMP belonging to the pyramids ATBVCXDY, EPFQGRHSM; and join the straight lines TI, PL.

Diagram. The same twin-circle construction as on the two preceding pages (bases ABCD with center I and apex K; EFGH with center L and apex M; auxiliary magnitudes O and N drawn beneath as plain circles), but here with additional chords drawn to form the corresponding triangles BKT and EMP used in the similarity argument (the connecting lines KB, KT, MF, MP and TI, PL).

- Left circle — base **ABCD**, center **I**; octagon vertices **A** (top), **T** (upper left), **B** (left), **V** (lower left), **C** (bottom), **X** (lower right), **D** (right), **Y** (upper right); **K** cone apex (upper right).

- Right circle — base **EFGH**, center **L**; octagon vertices **E** (top), **P** (upper left), **F** (left), **Q** (lower left), **G** (bottom), **R** (lower right), **H** (right), **S** (upper right); **M** cone apex (upper right).
- **O, N** — plain circles beneath the left and right circles respectively (auxiliary magnitudes).

Since, then, the cones ABCDK, EFGHM are assumed similar, it will be, by definition 24 of Book XI, that as diameter BD is to diameter FH [Margin: b. Elements V.15], so is the semidiameter BI to the semidiameter FL, and so the axis IK to the axis LM; and, by alternation, as BI is to IK, so is FL to LM. Then, since the angles BIK, FLM are right (by definition 3 of Book XI, because the cones are assumed right, and therefore their axes are at right angles to their bases), the triangles BIK, FLM will be equiangular [Margin: c. Elements VI.6]; and therefore as KB is to BI, so is MF to FL [Margin: d. Elements VI.4]. And as BI is to BT, so is FL to FP, on account of the similarity of the triangles BIT, FLP. (For since the angles BIT, FLP, standing on the similar arcs BT, FP, are equal, as was shown in the scholium to proposition 22 of Book III; and since, as BI is to IT, so is FL to LP, on account of the equality both of the lines BI, IT and of FL, LP, the triangles BIT, FLP will be similar [Margin: e. Elements VI.6].) Therefore, *ex aequo*, as KB is to BT, so is MF to FP; and consequently the straight lines KB, KT are proportional to the straight lines MF, MP, since in both cases the proportion is one of equality [Margin: f. Elements I.4]. Now since, as kB is to BT, so is kT to the same BT; and likewise as MF is to FP, so is MP to the same FP; and it was that as kB is to BT, so is MF to FP [Margin: g. Elements V.7]: it will also be that as kT is to BT, so is MP to FP; and, by conversion, as BT is to Tk, so is FP to PM. Therefore, since as Tk is to kB, so is PN[?] to MF, and as kB is to BT, so is MF to FP, and as BT is to Tk, so is FP to PM, as has been shown, the triangles BkT, FMP will have their sides proportional [Margin: h. Elements VI.6[?]] — printed reference-letter/number faint]; and for that reason they will be equiangular, and therefore similar, by the definition. It will be shown in no other way that the remaining triangles enclosing the pyramids ATBVCXDYk, EPFQGRHSM are similar to one another. And these, being equal in number, make the said pyramids similar, by definition 9 of Book XI. Consequently they will be in the triplicate ratio of the corresponding sides BT, FP, by the corollary to proposition 8 of this book.

Now as BT is to FP, so is BI to FL, on account of the similarity of the triangles BIT, FLP; and as BI is to FL, so is BD to FH [Margin: i. Elements V.11]. The pyramid will therefore have to the pyramid the triplicate ratio of the diameters BD, FH. But the ratio of the cone ABCDK to N was also assumed the triplicate of the same diameters. Therefore it will be that as the pyramid ATBVCXDYk is to the pyramid

EPFQGRHSM, so is the cone ABCDk to N. And since the pyramid ATBVCXDYk is less than the cone ABCDk, the part than the whole, the pyramid EPFQGRHSM will also be less than N [Margin: k. Elements V.14] — but it was shown to be greater. Which is absurd. Therefore the magnitude N is not greater than the cone EFGHM.

Let N next be greater than the cone EFGHM. Since, then, it is assumed that the cone ABCDk has to N the triplicate ratio of the diameter BD to the diameter FH, and the pyramid ATBVCXDYk has to the pyramid EPFQGRHSM the triplicate ratio of the same diameters, as we have just shown: it will be that as the cone ABCDk is to N, so is the pyramid ATBVCXDYk to the pyramid EPFQGRHSM; and, by conversion, as N is to the cone ABCDk, so is the pyramid EPFQGRHSM to the pyramid ATBVCXDYk. And since, by the corollary to proposition 8 of this book, the pyramid EPFQGRHSM has to the pyramid ATBVCXDYk the triplicate ratio of the corresponding sides PF to TB — that is, of the diameter FH to the diameter BD — N will also have to the cone ABCDk the triplicate ratio of the diameter FH to the diameter BD. Let it be that as N is to the cone ABCDk, so is the cone EFGHM to the magnitude O. The cone EFGHM will therefore also have to O the triplicate ratio of the diameter FH to the diameter BD. And since N is assumed greater than the cone EFGHk, the cone ABCDk will also be greater than O [Margin: l. Elements V.14]. Consequently the cone EFGHM has to the magnitude O, which is less than the cone ABCDk, the triplicate ratio of the diameter FH to the diameter BD. Which is absurd; for it has been shown that a cone cannot have to a magnitude less than another cone the triplicate of the ratio that the base-diameters have. Therefore the magnitude N is not greater than the cone EFGHM; but neither was it shown to be less; it is therefore equal [Margin: m. Elements V.7]. And so the cone ABCDk has the same ratio to the cone EFGHM as to N. Since, then, the cone ABCDk is assumed to be to N in the triplicate ratio of the diameters BD and FH, it will also be to the cone EFGHM in the triplicate ratio of the same diameters.

Now since cylinders — their triples — obtain the same ratio that the cones have, the cylinder will also have to the cylinder the triplicate ratio of the diameters in the bases [Margin: n. Elements V.15]. This too will be demonstrated in the same way we used for the cones, if only, in place of the cones and pyramids, cylinders and prisms are taken. Similar cones and cylinders, therefore, are in the triplicate ratio of the diameters that are in their bases. Which was to be shown.

SCHOLIUM [to Proposition 12, continued]

[The scholium of Proposition 12 continues from the previous page.]

Now if a larger cone and cylinder are compared with a smaller, it is evident — in the second part of this proposition — that Euclid takes it as demonstrated, in Prop. 8 of this book, that the smaller pyramid has to the larger the triplicate ratio of their homologous sides. This is the very point we noted, in connection with Def. 10 of Book 5, against Federico Commandino [Federico Commandino] and others, who suppose that the greater quantity must always be referred to the lesser in order to obtain a duplicate, triplicate, etc., ratio.

Furthermore, although this proposition may also be extended to similar *scalene* [oblique] cones and cylinders, nevertheless it is here demonstrated only for similar right cones and cylinders, since in the demonstration the axes are assumed to be at right angles to the bases. We shall therefore, by the following method, show it for similar scalene cones and cylinders as well. Let there be similar scalene cones and cylinders whose bases are the circles ABC, DEF, and whose axes are GH, Ik. I say that both cone to cone, and cylinder to cylinder, have the triplicate ratio of diameter to diameter. For conceive, upon the same bases, two right cones and cylinders of the same heights as the scalene ones, whose axes are GL, IM.

Diagram. Two circles side by side, each carrying a horizontal diameter through its center and an axis-construction above it.

- Left circle — center G; horizontal diameter A–G–C; lowest point B. Above the circle, points L (directly over G) and H (to the right) are joined by the segment LH; GL is the vertical (right) axis and GH the oblique axis, so that L, H, G form the right triangle GHL; a line runs from H to C.
- Right circle — center I; horizontal diameter D–I–F; lowest point E. Above, points M (directly over I) and K joined by the segment MK; IM is the vertical axis and IK the oblique axis, forming the triangle IkM; a line runs from K to F.

Let straight lines GH, GL be drawn; likewise Ik, IM, and let planes cut the bases in the straight lines AC, DF, which will be the diameters of the bases. Since, then, the axes GL, IM are at right angles to the bases ABC, DEF, [Margin: o — Elements XI.18] the planes through GH, GL and through IK, IM will also be at right angles to those same bases. [Margin: p — Elements XI.38] And therefore the perpendiculars dropped from H and k to the bases will fall upon AC, DF, the common sections; and accordingly the angles CGH, FIk will be the angles of inclination, which will be equal to one another, since the scalene cones and cylinders are supposed similar. Moreover the angles CGL, FIM are equal, being right angles, by Def. 3 of Book 11. [Margin: q — Elements I.32] The remaining angles HGL, kIM will

therefore also be equal. And since the cones ABCH, ABCL are supposed to be of the same height, the straight line HL, drawn through the vertices of the axes, will be parallel to AC; [Margin: r — Elements I.29] and hence the angle L is a right angle. By the same argument the angle M will be right, since the angles CGL, FIM are right. In just the same way the remaining angles H, K will be equal, since their alternate angles CGH, FIk are equal. The triangles GHL, IkM are therefore equiangular; [Margin: s — Elements VI.4] and accordingly, as GH is to GL, so is Ik to IM; and, by alternation, as GH is to Ik, so is GL to IM. But as the axis GH is to the axis Ik, so is AC, the diameter, to DF, the diameter, because the scalene cones and cylinders are supposed similar. Therefore GL, the axis, will be to IM, the axis, as AC, the diameter, to DF, the diameter; and consequently the right cones and cylinders ABCL, DEFM will be similar, by the definition. [Margin: t — Elements XII.12] They will therefore have the triplicate ratio of their diameters. And since both the cone and cylinder ABCL is equal to the cone and cylinder ABCH (having the same base and height), and likewise the cone and cylinder DEFM is equal to the cone and cylinder DEFk (for the same reason) — as is established by the corollary to Prop. 11 of this book — it follows that as the cone and cylinder ABCL is to the cone and cylinder DEFM, so is the cone and cylinder ABCH to the cone and cylinder DEFk; and hence the ratio of the cone and cylinder ABCH to the cone and cylinder DEFk is likewise the triplicate of that of the diameter AC to the diameter DF. Which was proposed.

[Margin: xi.]

THEOR. 13. PROPOS. 13.

If a cylinder is cut by a plane parallel to the opposite planes, then cylinder will be to cylinder as axis to axis.

Let the cylinder ABCD be cut by the plane GH, parallel to the opposite planes AB, CD, and let this plane cut the axis EF at I. I say that as the cylinder ABHG is to the cylinder GHCD, so is the axis EI to the axis IF. For conceive the cylinder ABCD to be extended in both directions — together with its axis and with the rectangle EC by whose revolution the cylinder was described — produced as far as you please. On the produced axis take any number of segments EK, KL equal to EI itself; likewise any number of segments FM, MN, NO equal to FI. Then through the points I, K, L, M, N, O draw the straight lines IH, KP, LQ, MT, NV, OX, parallel and equal to the straight lines EB, FC; and these, in the revolution of the rectangle EC, will describe the circles GH, PQ, QS, Tα, VZ, XY [α = the Greek letter alpha, here a point-label], parallel and equal to the circles AB, CD, because of the equality of the radii, which are always carried around equidistant from one another. And

therefore, by definition, the cylinders SP, PA, AH, HD, DT, TZ, ZX will result, composing the whole cylinder SQXY.

Diagram. A cylinder drawn in side view along a horizontal axis, its equal circular cross-sections shown as tall vertical (vesica-shaped) ovals set side by side; each section is named by the endpoints of its vertical diameter.

- Top edge, left to right: S, R, A, G, D, α , Z, V.
- Bottom edge, left to right: Q, P, B, H, C, T, V, X.
- Along the axis / section centers, left to right: L, K, E, I, F, M, N, O.

The central section (base) is AB (A on the top edge, B on the bottom), with the cutting circle GH; the successive sections toward S are PQ, QS, and toward X are T α , VZ, XY. [Two of the edge labels appear as "V" (top and bottom); transcribed as read.]

Now since the cylinders SP, PA, AH — standing on the equal bases QS, PR, BA and under the equal heights KL, EK, IE — are equal, and likewise the cylinders XZ, ZT, TD, DH — standing on the equal bases XY, VZ, T α , CD and under the equal heights NO, MN, FM, IF — are equal (by the corollary to Prop. 11 of this book): the cylinder SH will be the same multiple of the cylinder AH as the axis IL is of the axis IE; and likewise the cylinder XG will be the same multiple of the cylinder CG as the axis IO is of the axis IF. And since, if the axis IL (a multiple of IE, the first magnitude) is equal to the axis IO (a multiple of the axis IF, the second magnitude), then the cylinder SH (a multiple of the cylinder AH, the third magnitude) is likewise equal to the cylinder XG (a multiple of the cylinder CG, the fourth magnitude) — as is clear from Prop. 11 of this book; while if the axis is greater than the axis, the cylinder too is greater than the cylinder, and if less, less, in whatever multiplication this happens — it follows, by Def. 6 of Book 5, that as the axis IE, the first magnitude, is to the axis IF, the second magnitude, so is the cylinder AH, the third magnitude, to the cylinder CG, the fourth magnitude. If therefore a cylinder is cut by a plane parallel to the opposite planes, then cylinder will be to cylinder as axis to axis. Which was to be demonstrated.

SCHOLIUM [to Proposition 13]

Although Euclid demonstrates this proposition only for the right cylinder, nevertheless you will readily adapt his demonstration to the scalene cylinder, provided you employ the description of the scalene cylinder, just as in Euclid's demonstration the description of the right cylinder was assumed; so that, just as in right cylinders [printed *conis* — "cones"; recte: *cylindris*] a rectangle is carried

around its stationary side, so too in scalene cylinders [printed *conis*; recte: *cylindris*] the radii LS, KR, EA, etc., are carried around, always equidistant, about the centers L, K, E, etc., together with the straight line SY joining the radii. For it is in this way that cylinders are always described, as is established from what Serenus [Serenus of Antinoöpolis] writes in Book 1 of *On the Section of the Cylinder*.

Diagram. The same side-view cylinder as before, its equal circular cross-sections drawn as tall vertical ovals set side by side, each named by the endpoints of its vertical diameter.

- Top edge, left to right: S, R, A, G, D, α , Z, V.
- Bottom edge, left to right: Q, P, B, H, C, T, V, X.
- Along the axis / section centers, left to right: L, K, E, I, F, M, N, O.

[Margin: xi.]

THEOR. 14. PROPOS. 14.

Cones and cylinders standing on equal bases are to one another as their heights.

Let there be, on the equal bases AB, CD, two cones ABE, CDF and two cylinders ABGH, CDIK, whose axes — or heights (for in right cones and cylinders the axes are themselves the heights) — are LE, ME. I say that cone ABE is to cone CDF, and cylinder ABGH to cylinder CDIK, as the height LE is to the height ME. For let the cylinder ABGH be extended on the side of GH, together with its axis LE and the rectangle AG; and let the axis EN be cut off equal to the axis ME, and about the center N let a circle OP be conceived, equal and parallel to the circle GH, so that a cylinder GHOP is formed, of the same height as the cylinder CDIK. Since, then, the cylinders HP, CI, having equal bases and heights, are equal (by the corollary to Prop. 11 of this book), [Margin: a — Elements V.7] the cylinder AG will have the same ratio to each of them. [Margin: b — Elements XII.13] But cylinder AG is to cylinder HP as the axis, or height, LE is to the axis, or height, EN — that is, to MF, which is equal to it. Therefore cylinder AG will be to cylinder CI also as the height LE to the height MF.

[Margin: c — Elements XII.10] And since the cones ABE, CDF are the third parts of the cylinders AG, CI, [Margin: d — Elements V.15] they will have the same ratio as the cylinders; and accordingly cone ABE will also be to cone CDF as the height LE to the height MF. Cones and cylinders, therefore, standing on equal bases, are to one another as their heights. Which was to be shown.

Diagram. Two upright solids, each a right cylinder with an inscribed right cone. On the left a tall solid, on the right a shorter one.

- Left solid — top circle O–N–P (N the center); middle circle H–E–G (E the center); bottom circle (base) A–L–B (L the center). The cone has its apex at E and its base AB (lines from E to A and to B).
- Right solid — top circle K–F–I (F the center); bottom circle (base) C–M–D (M the center). The cone has its apex at F and its base CD.

SCHOLIUM [to Proposition 14]

We shall demonstrate that this same result holds for scalene cones and cylinders in the following way. Let there be, on the equal bases AB, CD, the scalene cones ABE, CDF and the scalene cylinders ABGH, CDIK, whose heights are LM, NO. I say that cone ABE is to cone CDF, and cylinder ABGH to cylinder CDIK, as the height LM is to the height NO. For upon the base AB let a right cylinder ABPQ be constructed, under the height LM; likewise upon the base CD a cylinder CDRS, under the height NO.

Diagram. Two oblique (scalene) cylinder-and-cone assemblies, drawn slanted with construction lines converging to the cone apices.

- Left assembly — top: Q, M, P; an intermediate slanted circle H, E, G; base A, L, B. (Right cylinder ABPQ and scalene cylinder ABGH share the base AB; the cone apex is E.)
- Right assembly — top: S, O, R and K, F, I; base C, N, D (N the center). (Right cylinder CDRS and scalene cylinder CDIK share the base CD; the cone apex is F.)

[The crowded, foxed vertex-labels of these two oblique solids]

[Margin: c — Elements XII.14] Since, then, the right cylinder ABPQ is to the right cylinder CDRS as the height LM to the height NO; and since the cylinder ABGH is equal to the cylinder ABPQ, and likewise the cylinder CDIK is equal to the cylinder CDRS (by the corollary to Prop. 11 of this book): the cylinder ABGH will also be to the cylinder CDIK as the height LM to the height NO. Which was proposed.

[Margin: f — Elements XII.10] And since the cones, [Margin: g — Elements V.15] being the third parts of the cylinders, have the same ratio as the cylinders, cone ABE will also be to cone CDF as the height LM to the height NO. This, however, can be confirmed by the same argument that we used for the cylinders, as is

evident, if upon the bases AB, CD there are made right cones ABM, CDO, of the same heights as the cylinders ABPQ, CDRS.

The same conclusion will follow even if one of the cones and cylinders — namely cone ABE and cylinder ABGH — is scalene, while the other — namely cone CDO and cylinder CDRS — is right. For since the scalene cylinder ABGH is equal to the right cylinder ABPQ, and the right cylinder ABPQ is to the right cylinder CDRS as the height LM to the height NO (as Euclid demonstrated): the scalene cylinder ABGK [recte: ABGH] will also be to the right cylinder CDRS as the height LM to the height NO. Which was proposed.

[Margin: h — Elements V.15] And since the cones have the same ratio as the cylinders, [Margin: i — Elements XII.10] the former being the third parts of the latter: the scalene cone ABE will also be to the right cone CDO as the height LM to the height NO. This too will be shown by the same reasoning we used for the cylinders, as is manifest.

The same proposition holds also for prisms, parallelepipeds, and pyramids set upon equal bases. For let there be, upon the equal bases ABC, DEFG, the prisms ABCHIK, DEFGLMNO, and pyramids under the same heights. I say that

prism is to prism, and pyramid to pyramid, as height to height. For let the lines AI, CH, BK, standing upon the base ABC of the prism ABCHIK, be perpendicular to it, so that BK is the height of the prism ABCHIK. Then, the straight lines AI, CH, BK being produced, take IQ, HR, KP equal to the height of the prism EL; and, the straight lines PQ, QR, RP being drawn, let the prism HIKPQR be completed, of the same height as the prism EL. The prisms HIKPQR, EL, having equal bases and heights, will be equal to one another, by the corollary 2 of Prop. 7 of this book.

[Margin: k — Elements V.7] And therefore the prism ABCHIK will have the same ratio to each of them. And since, by the scholium of Prop. 25 of Book 11, the prism ABCHIK is to the prism HIKPQR as the plane BI to the plane KQ; [Margin: l — Elements VI.1] and as the plane BI is to the plane KQ, so is the straight line BK to KP: the prism ABCHIK will be to the prism HIKPQR — that is, to the prism EL — as BK, the height of the prism ABCHIK, is to KP, the height of the prism HIKPQR, that is, of the prism EL. Which was proposed.

Diagram. Two solids side by side: on the left a triangular prism surmounted by a pyramid; on the right a parallelepiped (box).

- Left solid — apex Q at the top; upper vertices P (left) and R (right); a central point I; the middle level K (left) and H (right); lower vertex A (front); base

vertices B (left) and C (right). This is the prism ABCHIK with the pyramid HIKPQR built upon it.

- Right solid (box) — top face M (left), L (right); N (front-left), O (front-right); D (center-front); base E (left), F (center), G (right). This is the prism/parallelepiped DEFGLMNO ("EL").

But if the lines AI, CH, BK are not perpendicular to the base ABC, then upon the base ABC one must construct another prism whose standing edges are perpendicular to the base and equal in height to the prism ABCHIK. For since, by the 2nd corollary of Prop. 7 of this book, these two prisms — having the same height and base — are equal, and since the constructed prism is to the prism EL as the height of the one to the height of the other: the prism ABCHIK will also be to the prism EL as the height of the one to the height of the other.

[Margin: m — Elements V.15] And since the pyramids have the same ratio to the prisms, [Margin: n — Elements XII.10] because they are the third parts of them: it is evident that pyramid is likewise to pyramid as height to height.

Exactly the same account is to be given of parallelepipeds, since parallelepipeds too are comprehended under the name of prisms, as has been said; and for this reason the same demonstration can be adapted to them.

All these results will be demonstrated in nearly the same way of cones and cylinders, and also of pyramids and prisms having the same base.

All these, moreover, can readily be converted, in this way:

Cones and cylinders, whether right or scalene; likewise prisms, parallelepipeds, and pyramids — if they have the same ratio as their heights, will have equal bases.

For otherwise a part would be equal to the whole, as was shown in the converse of Prop. 31 of Book 11.

[Margin: xii.]

THEOR. 15. PROPOS. 15.

In equal cones and cylinders the bases and heights are reciprocally proportional; and those cones and cylinders whose bases and heights are reciprocally proportional are equal.

Let there be equal cones ABC, DEF [printed *ABCD*, *DEF* — the first apparently for *ABC*, *DEF*] and equal cylinders ABGH, DEIK, whose bases are AB, DE and whose

axes or heights are LC, MF. I say that the bases and heights are reciprocally proportional — that is, that as AB is to DE, so is MF to LC. In the case of the cylinders the proposition will be shown thus. If the heights LC, MF are equal, then, since the cylinders are also supposed equal, the bases too will be equal, by the corollary to Prop. 11 of this book. Hence, as the base AB is to the equal base DE, so is the height MF to the equal height LC. And accordingly the bases and heights are reciprocally proportional.

Diagram. Two upright solids side by side, each a cylinder with an inscribed cone; the right-hand one carries a horizontal cutting-circle.

- Left solid — top circle H–C–G (C the center and cone apex); bottom circle (base) A–L–B (L the center); axis LC. Cylinder ABGH with inscribed cone ABC.
- Right solid — top circle K–F–I (F the center and cone apex); an intermediate cutting-circle through O and N (the plane ON, parallel to the base); bottom circle (base) D–M–E (M the center); axis MF. Cylinder DEIK with inscribed cone DEF, and the sub-cylinders DO, OK produced by the cutting-plane.

[Some interior vertex-labels of the right-hand solid (the cutting-circle points O, N) are crowded]

But if the heights LC, MF are unequal, let MF be the greater, and from it cut off MN equal to LC; and through N let a plane ON be drawn parallel to the base DE, as we taught in the scholium of Prop. 15 of Book 11, so that two cylinders DO, OK are formed. Since, then, the cylinders ABGH, DEIK are supposed equal, [Margin: a — Elements V.7] it follows that as the cylinder ABGH is to the cylinder DO, so is the cylinder DEIK to the same cylinder DO. [Margin: b — Elements XII.11] But as the cylinder ABGH is to the cylinder DO, so is the base AB to the base DE, since the heights are equal; [Margin: c — Elements XII.14] and likewise as the cylinder DEIK is to the cylinder DO, so is the height MF to the height MN, since the bases are equal — indeed one and the same, DE. Therefore it will also follow that as the base AB is to the base DE, so is the height MF to the height MN, that is, to LC, which is equal to it. And accordingly the bases and heights are reciprocally proportional.

In the case of cones we shall conclude the proposition thus. If the cones ABC, DEF are equal, then the cylinders ABGH, DEIK will also be equal, [Margin: d — Elements XII.10] since the cones are the third parts of the cylinders. Hence, as was shown, from the equality of the cylinders it will follow that the bases and heights are reciprocally proportional; and accordingly, from the equality of the

cones as well, it will follow that the bases and heights are reciprocally proportional. This, however, can be demonstrated in exactly the same way that we used for the cylinders, if only two cones are set up under the heights MN, NF, as in the figure.

But now let the bases and heights be reciprocally proportional. I say that the cones and cylinders are equal. This indeed will be confirmed for the cylinders by the following reasoning. If the heights LC, MF are equal, then, since as the base AB is to the base DE so is the height MF to the equal height LC, the bases AB, DE will also be equal; and accordingly the cylinders, standing on the equal bases AB, DE and under the equal heights LC, MF, will be equal, by the corollary to Prop. 11 of this book.

Euclid, *Elements*, Book XII (Clavius's commentary)

Proposition 15 (continued) — converse, case of unequal heights

**

But if the heights are unequal, let the construction be made as before. Since, then, it is assumed that as base AB is to base DE, so is the height MF to the height LC — that is, to MN, which is equal to LC — [Margin: e — XII.11] while, because the heights are equal, as base AB is to base DE, so is cylinder ABGH to cylinder DO; [Margin: f — XII.14] and likewise, because the bases are equal, as the height MF is to the height MN, so is cylinder DEIK to cylinder DO: it will follow that as cylinder ABGH is to cylinder DO, so is cylinder DEIK to that same cylinder DO. [Margin: g — V.9] Therefore cylinder ABGH will be equal to cylinder DEIK.

But in the case of cones the demonstration will run as follows. If in the cones ABC, DEF the bases and heights are reciprocally proportional, then the bases and heights of the cylinders ABGH, DEIK will likewise be reciprocally proportional, since the bases, and the heights, of the cones and of the cylinders are the same. For this reason, as was shown, the cylinders — and therefore the cones, which are their third parts — will be equal. It can nevertheless be demonstrated in the same way that the cones are equal, by which we showed the cylinders to be equal. Therefore in equal cones and cylinders the bases and heights are reciprocally proportional, and so on. Which was to be demonstrated.

Diagram. Two solid figures (cone-within-cylinder) drawn side by side, illustrating Proposition 15.

- Left solid: a cylinder with lower circle lettered **A** (left), **L** (front/bottom), **B** (right), and upper circle **H** (left), **C** (top), **G** (right); a cone with apex at **C** and generating lines drawn to the base points **A**, **L**, **B**.
- Right solid: a cylinder with lower circle **D** (left), **M** (front/bottom), **E** (right), upper circle **K** (left), **F** (top), **I** (right), and a mid-level circle lettered **N**; a cone with apex at **F** and generating lines to the base points **D**, **M**, **E**, with **O** marked on the right of the middle circle.

Scholium

It will not be difficult to show the same thing for scalene (oblique) cones and cylinders. For if scalene cones and cylinders are equal, then the right cones and cylinders that have the same bases and heights as they do will also be equal, by the corollary of Proposition 11 of this book. Hence, as has been shown, their bases and heights will be reciprocally proportional. And conversely, if the bases and heights of scalene cones and cylinders are reciprocally proportional, then the bases and heights of the right cones and cylinders will likewise be reciprocally proportional, provided the bases and heights of the right and of the scalene solids are the same. Hence, as has been demonstrated, the right cones and cylinders will be equal; and consequently the scalene cones and cylinders, being equal to the right ones, will — by the corollary of Proposition 11 of this book — also be equal to one another.

The same will easily be concluded even if one of the cones or cylinders be right and the other scalene, as is evident from what we taught in the scholium of Proposition 14 of this book.

PROBL. 1. PROPOS. 16.

[Margin: xiiij.]

Given two circles about the same centre, to inscribe in the greater circle an equilateral polygon, having an even number of sides, that does not touch the lesser circle.

Let **ABC** and **DE** be two circles about the same centre **F**, and let it be required to inscribe in the greater, **ABC**, an equilateral polygon whose sides are even in number and which does not touch the lesser circle **DE**. Let the straight line **AC** be extended through the centre **F**, cutting the circle **DE** at the point **E**; and through **E** let **GH** be drawn perpendicular to **AC**. This perpendicular will touch the circle **DE**

at E, by the corollary of Proposition 16 of Book 3. Now since the arc AGC is greater than the arc GC, if from AGC there be taken away the half AB, and from the remainder BC the half BI, and from the remainder IC the half Ik, and so on in succession, [Margin: a — X.1] there will at length be left an arc smaller than CG. Let arc Ck now be smaller than arc CG, and let the straight line Ck be drawn as its chord. I say that the straight line Ck is one side of the polygon to be inscribed.

For if the arc BI be divided into parts equal in number and magnitude to the parts of arc CI; and the quadrant AB be divided into as many equal parts as the quadrant BC has been divided into; and likewise the semicircle AHC into as many parts as the semicircle ABC contains; and if then straight lines be drawn as chords under all the arcs, [Margin: b — III.29] these chords will be equal to the straight line Ck, because they subtend arcs equal to arc Ck: there will then be described in the circle ABC a polygon that is both equilateral and of an even number of sides.

That this polygon does not touch the lesser circle DE is shown thus. From k let a perpendicular kL be dropped to AC, cutting AC at M. Since, then, the angles GEM and kME are right, [Margin: c — I.29] the straight lines GH and kL will be parallel. Hence, since the straight line GH touches the circle DE at the single point E, the straight line kL will lie wholly outside that circle and will never meet it, because it never meets the line GH. Much less, therefore, will the straight line Ck — which is farther from the circle DE than kL is — touch the circle DE. And for this reason neither will the other sides of the inscribed polygon touch the circle DE, since they are equal to the side Ck [Margin: d — III.14] and are therefore equally distant with Ck from the centre F. Thus, given two circles about the same centre, [an equilateral polygon of an even number of sides has been inscribed] in the greater circle, and so on. Which was to be done.

Diagram. Two concentric circles with a diameter drawn through the common centre, illustrating Proposition 16.

- A — left end of the horizontal diameter (on the greater circle).
- F — the common centre.
- E — the point where the diameter meets the lesser circle (on the right side); the tangent GH is drawn through E.
- C — right end of the diameter (on the greater circle) [partly off the tile].
- B — top of the greater circle.
- D — a point on the upper part of the lesser circle.
- M — foot of the perpendicular kL on the diameter.
- L — lower-right, on the perpendicular kL.

- **H** — lower part of the figure.
- Further points **G, I, k** lie around the periphery of the greater circle; several of these labels are faint under foxing. [?]

Corollarium

From this it is manifest that if, from the end of a side of the inscribed polygon which meets the diameter, a perpendicular be drawn to the diameter, this perpendicular can in no way touch the lesser circle, but falls wholly outside it. For of this kind is the line **kL**, which — being drawn from the end-point **k** of the side **Ck** (the side that meets the diameter **AC**) perpendicular to the diameter **AC** — has been shown not to touch the circle **DE**.

[Margin: kiv.]

PROBL. 2. PROPOS. 17.

Given two spheres about the same centre, to inscribe in the greater sphere a solid polyhedron that does not touch the surface of the lesser sphere.

Let **ABCD** and **EFGH** be two spheres about the same centre **I**, and let it be required to inscribe in the greater, **ABCD**, a solid polyhedron — that is, a many-sided solid — that does not touch the lesser sphere **EFGH**. Let both spheres be cut by some plane through the centre, and let the common sections made in the spheres be the planes **ABCD** and **EFGH**. These will be circles — by the description of the sphere — having the same centre **I** as the spheres. For the semicircles by whose revolution the spheres are described, when carried round, coincide with the sections **ABCD**, **EFGH**; hence the said sections will be circles. Or, more surely: because all the straight lines falling from **I** to the peripheries of the sections are equal — being drawn from the centre of the spheres to their surface — the sections themselves will be circles, by the definition of a circle.

[Margin: a — XII.16] In these circles let the diameters **AC**, **BD** be drawn, cutting one another at right angles at the centre **I**, so that **AB**, **BC**, **CD**, **DA** are quadrants, and so on. Next, in the greater circle **ABCD** let a polygon be inscribed that does not touch the lesser circle **EFGH**. That everything may be more easily demonstrated, let this be effected in the following manner. From **G** let **GF** be drawn perpendicular to **EG**, as far as the circumference of the circle **ABCD**; this perpendicular will touch the circle **EFGH** at **G**, by the corollary of Proposition 16 of Book 3. *[Margin: b — IV.1]* And equal to the straight line **GF** let a straight line **Aε** be fitted into the circle **ABCD**. *[Margin: c — I.19]* Now if a straight line be

imagined drawn as a chord under the arc CF , so as to form the triangle GCF , the side CF — being opposite the greater angle, namely the right angle — is greater than the side GF , which is opposite the lesser angle, namely the acute one. Hence the straight line CF will also be greater than the straight line $A\varepsilon$, and consequently the arc CF will be greater than the arc $A\varepsilon$, as is established from the scholium of Proposition 28 of Book 3. Let then an arc $C\delta$ be cut off equal to the arc $A\varepsilon$.

[Margin: d — X.1] And if from the quadrant CD the half DL be taken away, and from the remainder CL the half LK , and so on in succession, there will at length be left an arc smaller than the arc $C\delta$, that is, than the arc $A\varepsilon$. Let arc CK now be the smaller one; and the straight line CK , drawn as its chord, will be smaller than the straight line $A\varepsilon$ — that is, than GF — by the scholium of Proposition 29 of Book 3. I say, then, that the straight line CK [printed: GK ; recte CK] is one side of the equilateral polygon to be inscribed. For since the straight line subtending the arc $C\delta$ — which is less than the arc CF — does not touch the circle $EFGH$, as is clear from the demonstration of the preceding proposition, much less will the straight line CK , subtending an arc smaller than the arc $C\delta$, [Margin: e — XI.12] touch that same circle.

Again, the diameter KN having been drawn, [Margin: f — XI.18] let there be erected from the centre I a perpendicular IO to the planes of the circles $ABCD$, $EFGH$, meeting the surface of the greater sphere at O ; and through the straight lines OI , AC and OI , KN let planes be drawn, [Margin: g — schol. III.21] which will stand at right angles to the circle $ABCD$ and will produce, as common sections, circles — as has already been said — of which the semicircles are AOC , NOK . And since the angles OIC , OIK are right, by Definition 3 of Book 11, [Margin: h — III.29] OC and OK will be quadrants; and moreover, since the circles $ABCD$, AOC , NOK are equal — because their diameters are also diameters of the greater sphere — the quadrants CD , OC , OK will likewise be equal. If, then, the arc DL be distributed into as many equal parts as the arc CL was divided into, and the quadrants OC , OK into arcs equal in number and magnitude to the arcs of the quadrant CD , [Margin: i — XI.38 [?]] the straight lines subtended under all these equal arcs — namely CK , KL , LM , MD ; CP , PQ , QR , RO ; KS , ST , TV , VO — will be equal. And when the straight lines PS , QT , RV have been joined, let perpendiculars PX , SY be dropped from P and S to the plane of the circle $ABCD$, [Margin: k — XI.6] which will fall on the common sections AC , NK ; and these perpendiculars will be parallel to one another.

Diagram. A large schematic of two concentric spheres cut by great circles, showing the polyhedron inscribed in the greater sphere (Proposition 17). The same woodcut is reproduced on the facing page. The figure is crowded, foxed, and at the resolution floor; the labels below are those legible on the scan at high magnification, with uncertain ones flagged.

- **A** — left end of the horizontal diameter AC (great circle ABCD).
- **C** — right end of the horizontal diameter AC.
- **I** — the common centre of the spheres.
- **O** — top, the pole where perpendicular IO meets the greater sphere (on great circle AOC).
- **N** — upper-left, on the great circle (diameter KN).
- **B** — top region of the great circle ABCD [there appear to be two labels resembling "B", one near V at upper left and one near T in the middle — at least one is uncertain]. [?]
- **D** — lower-left, on the great circle.
- **M** — bottom-left, on the quadrant arc.
- **F** — a point on an inner (lesser-sphere) circle, upper-middle-left.
- **G, H** — points on the lesser circle EFGH (lower-middle region); **H** below centre, **G** faint. [?]
- **R, Q, P** — successive division points on the arc from O down to C (arcs RO, QR, PQ, CP).
- **V, T, S** — successive division points on the meridian arc from O down to K (arcs VO, TV, ST, KS).
- **K** — lower-right division point of the quadrant CD (arc CK); with **Y** beside it (foot of a perpendicular).
- **L** — lower division point (arcs KL, LM).
- **X** — foot of a perpendicular on AC, near C.
- **α** (alpha) — near X/C, foot of a perpendicular.
- **γ** (gamma), **δ** (delta), **ε** (epsilon) — auxiliary points near the lower-right and left periphery.

Proposition 17 (continued)

Since, then, in the triangles PCX, SKY the angles PXC, SYK are right, by Definition 3 of Book 11, [Margin: I — III.27] and the angles PCX, SKY are equal — because the arcs AOP, NOS on which they stand are equal (for if from the equal semicircles AOC, NOK the equal arcs CP, KS be taken away, the remaining arcs AOP, NOS will

also be equal) — the two angles PCX, PXC of triangle PCX will be equal to the two angles SKY, SYK of triangle SKY. And the sides PC, SK, opposite the right angles, are equal. [Margin: m — I.26] Therefore the remaining sides PX, XC will be equal to the remaining sides SY, YK.

Hence, since the straight lines PX and SY [text prints "PY"] are equal and parallel, if the straight line XY be joined, [Margin: n — I.33] PS and XY will also be equal and parallel to one another. [Margin: o — VI.2] But since the straight lines CK, XY are also parallel — because the sides IC, IK are cut proportionally (for if from the equal radii IC, IK the equal straight lines CX, KY be taken away, IX and IY will remain equal; and hence, as IX is to XC, so will IY be to YK) — [Margin: p — 9. duodecimi (?) — the parallels reasoning points to XI.9] the straight lines PS, CK will also be parallel to one another, since each is parallel to XY; [Margin: q — XI.7] and therefore the straight lines CP, KS that join them will lie in the same plane with them. The whole quadrilateral CKSP, therefore, will lie in one plane.

And if from Q and T perpendiculars be dropped to the plane of the circle ABCD, and the straight lines QC, TK be joined, we shall show in like manner that CK, QT are parallel, and further that PS, QT are parallel to one another (since they are both parallel to CK), and that the whole quadrilateral PSTQ lies in one plane. By the same reasoning the quadrilateral QTVR will lie in one plane; [Margin: r — 12. undecimi (?) — the "triangle in one plane" reasoning points to XI.2] and the triangle RVO too is in a plane. If, then, the same construction be carried out upon the remaining sides KL, LM, MD — namely by drawing the quadrants OL, MO, OD — and likewise in the other three quarters and in the remaining hemisphere, so that the whole greater sphere is filled with quadrilaterals and triangles similar to those constructed between the quadrants OC, OK upon the side CK, there will be inscribed in the greater sphere a solid polyhedron bounded by the said quadrilaterals and triangles. This, then, I say, does not touch the lesser sphere EFGH.

Diagram. The identical woodcut of the two concentric spheres and the inscribed polyhedron is reproduced on this page (see fig-p542-1 for the full label list). The reproduction is equally crowded and foxed; the same labels are readable (O at the top, A and C at the ends of the horizontal diameter, I at the centre, N upper-left, and the division points R, Q, P, V, T, S, K, L, M, D around the periphery, with the auxiliary Greek points α , γ , δ and points F, G, H, X, Y). More than two labels remain uncertain

Now let a perpendicular IZ be drawn from I to the plane CKSP, and let the straight lines ZC, ZK be joined. That the perpendicular IZ falls within the quadrilateral

CKSP we shall show in the following scholium. Since, then, by Definition 3 of Book 11 the angles IZC, IZK are right, [Margin: s — I.47] the square on the straight line IC will be equal to the squares on IZ, ZC, and the square on IK equal to the squares on IZ, ZK. Since, then, the squares on the equal straight lines IC, IK are equal, the squares on IZ, ZC will also be equal to the squares on IZ, ZK; and hence, the common square on IZ being taken away, the remaining squares on ZC, ZK will be equal, and therefore the straight lines ZC, ZK themselves are equal. In like manner we shall show that the straight lines drawn from Z to PS are equal both to one another and to the straight lines ZC, ZK. Hence a circle described from Z at the distance ZC will pass through the four points C, K, S, P. And by the same reasoning we shall demonstrate that circles can be described about the remaining quadrilaterals PSTQ, QTVR and the triangle RVO. And since, as we shall show later, the angle CZK is obtuse, [Margin: t — II.12] the square on the straight line CK will be greater than the squares on ZC, ZK; and therefore, since these squares are equal, the square on CK will be greater than twice the square on ZC.

Now let a perpendicular $K\alpha$ be drawn from K to the straight line AC. Since, then, AC is double AI, and $A\alpha$ is greater than AI, AC will be less than double $A\alpha$. For which reason, since [Margin: u — VI.1] as AC is to $A\alpha$, so is the rectangle contained by AC, αC to the rectangle contained by $A\alpha$, αC — because the bases of these rectangles are AC, $A\alpha$ and the height αC is the same — the rectangle contained by AC, αC will also be less than double the rectangle contained by $A\alpha$, αC . [Margin: x — VI.17] But the rectangle contained by AC, αC is equal to the square on the straight line CK, and the rectangle contained by $A\alpha$, αC is equal to the square on the straight line $K\alpha$; because CK is a mean proportional between AC and αC , and $K\alpha$ a mean proportional between $A\alpha$ and αC , by the corollary of Proposition 8 of Book 6. (For if the straight line...

(Continuation of the demonstration of Book XII, Proposition 17 — to inscribe in the greater of two concentric spheres a solid polyhedron that does not touch the lesser.)

(if the straight line AK were drawn, a right triangle ACK would be formed.) The square on CK, therefore, will also be less than twice the square on $K\alpha$. And since the square on CK has been shown to be greater than twice the square on ZC, the square on $K\alpha$ will be greater than the square on ZC. [Margin: y — 47.pri. (Elements I.47)] Now the square on IC equals the squares on IZ and ZC together, and the square on IK equals the squares on $I\alpha$ and αK together; and since the squares on the equal lines IC and IK are themselves equal, the squares on IZ and ZC together will also equal the squares on $I\alpha$ and αK together. If, then, from these

we take away the greater square — namely that on αK — and from the others the smaller — namely that on ZC — the remaining square on IZ will be greater than the remaining square on $I\alpha$, and therefore the line IZ is greater than the line $I\alpha$. Consequently, since the point α does not touch the smaller sphere $EFGH$ — because, by the corollary of the preceding proposition, the whole line $K\alpha$ lies outside that sphere — still less will the point Z , which lies farther off, touch that same sphere. And accordingly, since all the other points of the plane $CKSP$ lie farther from the sphere $EFGH$ than the point Z does, as we shall show shortly, the plane $CKSP$ will not touch the sphere $EFGH$.

But we may show more readily, almost from the very construction of the figure, that the plane $CKSP$ does not touch the smaller sphere $EFGH$, provided the line $I\gamma$ is first drawn, in this way. Since it was shown from the construction that the line CK is less than the line $C\gamma$ [Margin: z — 29.pri. (Elements I.29 [?] — printed digit ambiguous, possibly I.19)] — and CK is greater than ZC , because the angle CZK is obtuse, as will presently be demonstrated — $C\gamma$ will be much greater than ZC , and therefore the square on $C\gamma$ is greater than the square on ZC . [Margin: a — 47.pri. (Elements I.47)] Now since the square on $I\gamma$ equals the squares on IG and $C\gamma$ together, and the square on IC equals the squares on IZ and ZC together, and the squares on the equal lines $I\gamma$ and IC are themselves equal, the squares on IG and $C\gamma$ together will also equal the squares on IZ and ZC together. Taking away, then, on the one side the square on $C\gamma$ and on the other the square on ZC , the square on IG will be left less than the square on IZ , and therefore the line IG is less than IZ . For this reason, since IG is the semidiameter of the smaller sphere $EFGH$, the point Z will lie outside that same sphere; and hence, as before, the plane $CKSP$ will in no way touch the sphere $EFGH$.

Let the perpendicular $I\beta$ be drawn once more from I to the plane $PSTQ$; then β will be the center of the circle described about $PSTQ$, as has been demonstrated. And when the lines βP and IP are joined, since the angle $I\beta P$ is right, by definition 3 of Book 11, [Margin: b — 47.pri. (Elements I.47)] the square on IP will equal the squares on $I\beta$ and βP together. Now the square on IC also — which equals the square on IP , because the lines IC and IP are equal — equals the squares on IZ and ZC together; so the squares on $I\beta$ and βP together will equal the squares on IZ and ZC together. But the square on ZC is greater than the square on βP , because the line ZC is also greater than the line βP , as we shall show later. The remaining square on $I\beta$, therefore, will be greater than the remaining square on IZ , and hence the line $I\beta$ is greater than the line IZ . And accordingly the point β will lie even farther outside the sphere $EFGH$ than the point Z does; and therefore the plane $PSTQ$ will touch the larger sphere $EFGH$ even less than $CKSP$ does. In the

same way we shall demonstrate that the remaining planes cannot touch the said sphere either. Therefore, with two spheres existing about the same center, we have inscribed in the larger sphere a solid polyhedron that does not touch the surface of the smaller sphere. Which was to be done.

COROLLARIUM.

From what has been demonstrated it is manifest that, if in any other sphere a solid polyhedron similar to the aforesaid solid polyhedron be described, the ratio of the polyhedron in the one sphere to the polyhedron in the other sphere is the triplicate of the ratio that the diameters of the spheres have. For if from the centers of the spheres straight lines are drawn to all the angles of the bases of the said polyhedra, the polyhedra will be divided into pyramids equal in number and similar, whose homologous sides are the semidiameters of the spheres — as is clear if the smaller of these spheres is understood to be described within the larger about the same center. For the straight lines drawn from the centers to the angles of the bases will coincide with one another, because of the similarity of the bases, and therefore the pyramids will be made similar. Hence, since each pyramid in the one sphere has to each pyramid similar to it in the other sphere the triplicate ratio of the homologous sides — that is, of the semidiameters of the spheres, as is clear from the corollary of proposition 8 of this book — then, as one pyramid is to one pyramid, so are all the pyramids together (that is, the solid polyhedron composed of them) to all the pyramids together (that is, to the solid polyhedron constituted of them). The polyhedron of the one sphere, therefore, will also have to the polyhedron of the other sphere the triplicate ratio of the semidiameters, and thus of the diameters of the spheres, [Margin: c — 15.quin. (Elements V.15)] since the semidiameters and the diameters have the same ratio.

SCHOLIVM II.

Since, to keep the demonstration from becoming too long, certain things were assumed in it as true which nevertheless have not yet been demonstrated, we shall now demonstrate them briefly.

First, then, it must be shown that the point Z falls within the quadrilateral CKSP, and that the angle CZK within the quadrilateral CKSP is obtuse. To do this more conveniently, let a circle be described about the said quadrilateral from the center Z. Since, then, in the upper figure, as IK is to κC , so is IT to TX — because, by the corollary of proposition 4 of Book 6, the triangles ICK and IXT are similar — and since IK is greater than IT, [Margin: d — 14.quin. (Elements V.14)] KC will also be greater than TX. And since TX has been shown equal to SP, KC will also be

greater than SP; and therefore in this figure the arc CK will be greater than the arc SP, by the scholium of proposition 28 of Book 3. [Margin: e — 28.ter. (Elements III.28)] Hence, since the arcs CP and KS are equal to the arc CK — because the lines CP and KS were also shown equal to the line KC, for they are subtended by equal arcs of circles, as is clear from the construction of the upper figure — the arcs CP and KS will also be greater than the arc PS. And for that reason each of the arcs CK, CP, KS will exceed a quadrant of the circle CKSP, and will be surpassed by a semicircle; and accordingly the much greater segment SP will be less than a semicircle. From this it follows that the center Z is not in those segments, but outside them — namely, within the quadrilateral CKSP. By the same reasoning we shall show that the perpendiculars dropped from I to the planes of the other quadrilaterals — such as $\kappa\beta$ — fall within those quadrilaterals; and likewise that the perpendicular drawn from I to the triangle ORV falls within it. Since, then, the arc CK is greater than a quadrant, the angle CZK will be obtuse — that is, greater than a right angle — because a quadrant of the circle is subtended by a right angle at the center, as is clear from the scholium of proposition 27 of Book 3.

Diagram. A circle circumscribing the quadrilateral CKSP. The upper vertices S (left) and P (right) and the lower vertices K (left) and C (right) lie on the circle; the center Z is marked near the middle (with a monogram-like Z), and straight lines run from Z to K and to C, together with the chords forming the quadrilateral. The figure illustrates that the center Z lies within CKSP and that the angle CZK is obtuse.

- S (upper left): vertex of quadrilateral on the circle
- P (upper right): vertex on the circle
- K (lower left): vertex on the circle
- C (lower right): vertex on the circle
- Z (center): center of the circumscribed circle

Second, it must be demonstrated that all the other points of the quadrilateral CKSP are farther from the center I than the point Z is. For let any other point θ be taken in the quadrilateral CKSP, and let the lines $I\theta$ and $Z\theta$ be joined. Then, since the angle $IZ\theta$ is right, by definition 3 of Book 11, [Margin: f — 19.pri. (Elements I.19)] the side opposite to it, $I\theta$, will be greater than the side IZ , which is opposite the smaller angle $I\theta Z$ (an acute one); and therefore the point θ is farther from the center I than the point Z is. By a similar argument we shall conclude that all the other points lie farther off.

Diagram. An oblique (three-dimensional) view of the quadrilateral, drawn as SPKC — S (upper left) and P (upper right) above, K (lower left) and C (lower right) below — with the apex I placed far to the left and joined to the figure by long lines. The point β lies within, near the top, with a small square marking the right angle there. The figure illustrates the perpendicular $I\beta$ dropped from I to the plane of the quadrilateral.

- I (far left): apex from which the perpendicular is drawn
- S (upper left), P (upper right): upper vertices
- K (lower left), C (lower right): lower vertices
- β (interior, near top): foot of the perpendicular; marked with a right angle

Third and last, it must be proved that the line ZC is greater than the line βP . To do this more suitably, it must first be demonstrated that the line PS is greater than the line QT. Let the upper part of the figure be described — namely, that which is contained by the semidiameters IC, IK, IO and the quadrants OC, OK, and so on. Then let the perpendiculars $Q\mu$ and $T\xi$ be dropped from Q and T to the plane of the circle ABCD, in which lies the triangle ICK; [Margin: g — 38.undecimi (Elements XI.38)] these will fall on the common sections IC and IK, [Margin: h — 6.undecimi (Elements XI.6)] and will be parallel to one another, as was said of the lines PX and SY. And if the line $\mu\xi$ be joined, QT and $\mu\xi$ will be parallel and equal, just as PS and XY were shown to be parallel and equal. [Margin: i — 2.sex. (Elements VI.2)] Now since $\mu\xi$ is parallel to CK — because the sides IC and IK are cut proportionally at μ and ξ , as we said of the line XY — [Margin: k — 30.pri. (Elements I.30)] $\mu\xi$ and XY will also be parallel. Hence, by the corollary of proposition 4 of Book 6, as IY is to YX, so is I ξ to $\xi\mu$. But IY is greater than I ξ ; [Margin: l — 14.quin. (Elements V.14)] therefore YX will also be greater than $\xi\mu$, and accordingly PS, which is equal to XY, will be greater than QT, which is equal to $\xi\mu$.

Diagram. A quarter-sphere (spherical sector) with its apex O at the upper left (the center of the sphere); a curved quadrant arc sweeps from the top down to K at the bottom, bounded on the left by a radius descending to L and along the bottom by a radius to G. Within, a dense network of chords joins the vertices of the inscribed polyhedron. Readable labels, working around and inside the figure: O (apex), R, V, T, Q, P, S, X, L, G, K. The figure is heavily foxed.

- O (apex, upper left): center of the sphere / vertex of the quadrant
- R (on the arc, upper): polyhedron vertex
- V, T (interior): polyhedron vertices

- Q (on the arc, right): polyhedron vertex
- P (right), S, X (lower interior): polyhedron vertices
- L (lower left corner), G (lower right): ends of the bounding radii
- K (bottom): vertex of the quadrant

This being demonstrated, let circles be described about the quadrilaterals CKSP and PSTQ from the centers Z and β , and from the centers let the lines ZC, ZK, ZS, ZP and βP , βS , βT [?], βQ go out. Then if ZC be not believed greater than βP , it will be either equal or less. Let it first be equal. Since, then, the sides ZK and ZC are assumed equal to the sides βS and βP , and the base KC is greater than the base PS, [Margin: m — 25.pri. (Elements I.25)] the angle KZC will be greater than the angle $S\beta P$; and by the same reasoning the angle SZP will be greater than the angle $T\beta Q$. But since the bases KS and CP are equal to the bases ST and PQ, [Margin: n — 8.pri. (Elements I.8)] the angles KZS and CZP will be equal to the angles $S\beta T$ [?] and $T\beta Q$ [?]. Therefore the four angles at Z will be greater than the four angles at β . Yet they are also equal, since both these and those are equal to four right angles, by corollary 2 of proposition 15 of Book 1 — which is absurd. Therefore the line ZC is not equal to the line βP .

Diagram. The left of a pair of circles. A circle circumscribing the quadrilateral SPKC — S (upper left), P (upper right), K (lower left), C (lower right) — with both diagonals drawn so that they cross at the center Z (marked with a monogram), forming an X within the quadrilateral.

- S, P, K, C: vertices of the inscribed quadrilateral
- Z (center): center of the circle, intersection of the diagonals

Diagram. The right of the pair of circles, for the case ZC less than βP . A circle with center β ; the upper vertices are T (left) and Q (right); within it an inner quadrilateral is formed by the cut-off points ω , ϕ , ρ , π [individual assignments uncertain], with diagonals crossing at β . A lower vertex P is marked; the lower-left vertex is illegible. [?]

- T (upper left), Q (upper right): vertices on the circle
- β (center): center of the circle
- ω , ϕ , ρ , π (interior): the points $\beta\pi$, $\beta\rho$, $\beta\omega$, $\beta\phi$ cut off equal to ZC, ZK, ZS, ZP [positions uncertain]
- P (lower right): vertex on the circle

Next, let ZC be less than βP . And let $\beta\pi$, $\beta\rho$, $\beta\omega$, $\beta\phi$ be cut off equal to ZC, ZK, ZS, ZP, and let the lines $\pi\rho$, $\rho\omega$, $\omega\phi$, $\phi\pi$ be joined, [Margin: o — 2.sex. (Elements VI.2)]

which will be parallel to the lines PS, ST, TQ, QP, because the lines cut off from the centers are cut proportionally; and accordingly, by the corollary of proposition 4 of Book 6, as βS is to SP, so will $\beta\rho$ be to $\rho\pi$. Since, then, βS is greater than $\beta\rho$, [Margin: p — 14.quin. (Elements V.14)] SP will also be greater than $\rho\pi$. And by the same reasoning ST, TQ, QP will be greater than the lines $\rho\omega$, $\omega\phi$, $\phi\pi$. And therefore, since PS is less than CK, and ST, PQ are equal to the lines KS, CP, and TQ is less than PS, the lines $\pi\rho$, $\rho\omega$, $\omega\phi$, $\phi\pi$ will be less than the lines CK, KS, SP, PC. Hence, since the lines $\beta\pi$, $\beta\rho$, $\beta\omega$, $\beta\phi$ are equal to the lines ZC, ZK, ZS, ZP, [Margin: q — 25.pri. (Elements I.25)] the angles at Z will be greater than the angles at β . Yet they are also equal, because both those and these are equal to four right angles, by corollary 2 of proposition 15 of Book 1 — which is absurd. Therefore the line ZC is not less than βP ; and it has been shown to be not equal either; therefore it is greater. Which was to be demonstrated.

THEOR. 18. PROPOS. 18.

[Margin: XV. (vulgate proposition number)]

Spheres are to one another in the triplicate ratio of their diameters.

Let there be two spheres ABC and DEF whose diameters are AC and DF. I say that the sphere ABC has to the sphere DEF the triplicate ratio of the diameter AC to the diameter DF. For if this be not granted, the sphere ABC will have — to some other sphere GHI smaller, or KLM larger, than DEF — the triplicate ratio of the diameter AC to the diameter DF. First let the sphere ABC have to the sphere GHI, smaller than the sphere DEF, the triplicate ratio of the diameter AC to the diameter DF, and let the sphere GHI be understood as concentric with the sphere DEF. [Margin: a — 17.duodecimi (Elements XII.17)] Let there be inscribed in the larger sphere DEF a polyhedron DNEOFPQR not touching the smaller sphere GHI, and let a polyhedron ASBTCVXY similar to this be inscribed in the sphere ABC. Since, then, the ratio of the sphere ABC to the sphere GHI is assumed to be the triplicate of the ratio of the diameter AC to the diameter DF, and since also, by the corollary of the preceding proposition, the ratio of the polyhedron ASBTCVXY to the polyhedron DNEOFPQR is the triplicate of the ratio of the diameter AC to the diameter DF, it will follow that as the sphere ABC is to the sphere GHI, so is the polyhedron ASBTCVXY to the polyhedron DNEOFPQR. Hence, since the sphere ABC is greater than the polyhedron ASBTCVXY, [Margin: b — 14.quin. (Elements V.14)] the sphere GHI will also be greater than the polyhedron DNEOFPQR — the part greater than the whole. Which is absurd. Therefore the sphere ABC will not have to a sphere GHI smaller than the sphere DEF the triplicate ratio of the diameter AC to the diameter DF.

Diagram. The figure for Proposition 18 (foxed; several labels uncertain). At the left, a circle for the sphere ABC with an inscribed eight-sided polyhedron ASBTCVXY: A (left) and C (right) lie on the horizontal diameter; B is at the top, S upper-left, T upper-right, Y lower-left, V lower-right, X at the bottom. At the upper right, a circle for the sphere DEF with an inscribed eight-sided polyhedron DNEOFPQR: D (left) and F (right) on the horizontal diameter; E at the top, N upper-left, O upper-right, R lower-left, P lower-right, and a bottom vertex Q (the stray "L" belongs to the shaded KLM solid, not this octagon); a smaller concentric circle GHI is drawn inside it (G left, H top, I right), representing the smaller sphere GHI. Below these, two shaded solid spheres: a smaller one (the sphere GHI) and a larger one labeled M (the sphere KLM), with K marked near its upper left.

- Sphere ABC: A, C (diameter ends), B (top), S, T (upper), Y, V (lower), X (bottom) — vertices of polyhedron ASBTCVXY
- Sphere DEF: D, F (diameter ends), E (top), N, O (upper), R, P (lower), Q (bottom) — vertices of polyhedron DNEOFPQR
- Inner circle GHI: G (left), H (top), I (right) — the concentric smaller sphere
- Shaded solids: small sphere (GHI); large sphere M (KLM), with K marked

Next, let the sphere ABC have to the sphere KLM, larger than the sphere DEF, the triplicate ratio of the diameter AC to the diameter DF. Then, since by the corollary of the preceding proposition the polyhedron ASBTCVXY also has to the polyhedron DNEOFPQR the triplicate ratio of the diameter AC to the diameter DF, it will follow that as the sphere ABC is to the sphere KLM, so is the polyhedron ASBTCVXY to the polyhedron DNEOFPQR; and, inversely, as the sphere KLM is to the sphere ABC, so is the polyhedron DNEOFPQR to the polyhedron ASBTCVXY. But, by the said corollary of the preceding proposition, the polyhedron DNEOFPQR is to the polyhedron ASBTCVXY in the triplicate ratio of the diameter DF to the diameter AC. Therefore the sphere KLM will also be to the sphere ABC in the triplicate ratio of the diameter DF to the diameter AC. Let it be posited that, as the sphere KLM is to the sphere ABC, so is the sphere DEF to another sphere $Z\alpha\beta$. Then the sphere DEF will also have to the sphere $Z\alpha\beta$ the triplicate ratio of the diameter DF to the diameter AC. And since the sphere KLM is posited greater than the sphere DEF, the sphere ABC will also be greater than the sphere $Z\alpha\beta$. Wherefore the sphere DEF has to the sphere $Z\alpha\beta$ — which is smaller than the sphere ABC — the triplicate ratio of the diameter DF to the diameter AC. Which is absurd; for it has been shown that a sphere cannot have to another sphere smaller than it the triplicate ratio of the diameters. Therefore the sphere ABC will not have to the sphere KLM, larger than the sphere DEF, the triplicate ratio of the

diameter AC to the diameter DF. But neither does it have it to a smaller one, as has been demonstrated. Therefore it will have to the sphere DEF the triplicate ratio of the diameter AC to the diameter DF. And so spheres are to one another in the triplicate ratio of their diameters. Which was to be demonstrated.

Diagram. The figure for Proposition 18 repeated (identical to fig-p545-5; foxed). At the left, the circle for the sphere ABC with inscribed polyhedron ASBTCVXY: A (left) and C (right) on the horizontal diameter, B (top), S upper-left, T upper-right, Y lower-left, V lower-right, X (bottom). At the upper right, the circle for the sphere DEF with inscribed polyhedron DNEOFPQR (D and F on the diameter; E top; N, O upper; R, P lower; a bottom vertex Q, the stray "L" belonging to the shaded KLM solid) and the concentric inner circle GHI (G left, H top, I right). Below, two shaded solid spheres: a smaller one (GHI) and a larger one labeled M (KLM), with K marked.

- Sphere ABC: A, C, B, S, T, Y, V, X — vertices of polyhedron ASBTCVXY
- Sphere DEF: D, F, E, N, O, R, P, Q — vertices of polyhedron DNEOFPQR
- Inner circle GHI: G, H, I — the concentric smaller sphere
- Shaded solids: small sphere (GHI); large sphere M (KLM), with K marked

COROLLARIUM.

Hence it follows that a sphere is to a sphere as a polyhedron described in the one is to a similar polyhedron described in the other. For both the sphere to the sphere and the polyhedron to the polyhedron have the triplicate ratio of the diameters, as has been demonstrated.

FINIS ELEMENTI DVODECIMI.

(End of the Twelfth Element — i.e. of Book XII.)

EUCLID'S THIRTEENTH ELEMENT

And the Third [Book] of Solids

THEOR. 1. PROPOS. 1.

If a straight line is cut in extreme and mean ratio, then the greater segment, taking to itself half of the whole, is capable of five times the square described on half of the whole.

Let the straight line AB be cut in extreme and mean ratio at C, so that AC is the greater segment. Let BA be produced to D, so that AD is half of the whole AB. I say that the square on the line CD is five times the square on the line AD.

For let the square CE be described on CD. In it draw the diameter DF, and draw AG parallel to DE, cutting the diameter at the point H; and through H draw Ik parallel to CD. Then produce GA, complete the square AM, and produce FC to N. By the corollary of Proposition 4 of Book 2 [II.4], AI and kG will be the squares on the lines AD and AC.

[Margin: a — VI, definition 3] Now since, as AB is to AC, so is AC to CB, [Margin: b — VI.17] the rectangle CM contained by AB, CB will be equal to kG, the square on the line AC. Next, since AB is set equal to twice AD, and AL is set equal to AB, and AH equal to AD, then AL will also be double AH. [Margin: c — VI.1] And as AL is to AH, so is the rectangle AN to the rectangle Ak. Therefore AN is double Ak.

[Margin: d — I.43] But since Ak is equal to IG, AN will be equal to the two Ak and IG together. Therefore, adding the equal quantities CM and kG, the square AM will be equal to the gnomon OPQ. And since the square AM is four times the square AI — by the scholium of Proposition 4 of Book 2, because the line AB is set as double the line AD — the gnomon OPQ will likewise be four times the same square AI. Consequently, if to the gnomon OPQ the square AI itself is added, the square CE is made five times the square AI.

Diagram. A geometric construction for Proposition 1: an upper square (the square CE on CD) containing an inscribed circle and its diagonal, sitting above a lower rectangle (the square AM region on AB). Labelled points read from the figure:

- E (top-left corner of upper square)
- G (top edge, midpoint)
- F (top-right corner of upper square)
- P (upper part of the inscribed circle, on the diagonal)
- I (left edge, at the level of the horizontal Ik)
- O (left-of-centre, on line Ik)
- H (centre, intersection of diameter and Ik)
- K (right edge, on line Ik)
- D (left end of the middle horizontal line)
- A (just below D, left end of AB)
- Q (centre-low)
- C (centre-right, on line AB)

- B (right end of line AB)
- L (bottom-left corner of lower rectangle)
- N (bottom edge, midpoint)
- M (bottom-right corner of lower rectangle)

ALITER [Another way]. By the scholium of Proposition 4 of Book 2, the square on the line AB is four times the square on the line AD, since AB is set as double AD. [Margin: e — II.2] Now the square on the line AB is equal to the rectangles contained by AB, BC and by AB, AC together; and again, the rectangle contained by AB, AC is equal to twice the rectangle contained by AD, AC. [Margin: f — VI.1] (For just as AB is double AD, so too the rectangle under AB, AC will be double the rectangle under AD, AC, since both rectangles have the same height AC; and therefore twice the rectangle under AD, AC will be equal to the rectangle under AB, AC.) [Margin: g — VI.17] But the rectangle under AB, BC is equal to the square on the line AC, since, as AB is to AC, so is AC to CB. Therefore the square on the line AC, together with twice the rectangle under AD, AC, will likewise be four times the square on the line AD. And so the squares on the lines AC and AD, together with twice the rectangle under AD, AC, will be five times the square on the same line AD. [Margin: h — II.4] Hence, since the square on the line CD is equal to the squares on the lines AC and AD together with twice the rectangle under AD, AC, the square on the line CD will also be five times the square on the line AD. Therefore, if a straight line is cut in extreme and mean ratio, and so on. Which was to be demonstrated.

THEOR. 2. PROPOS. 2.

If a straight line is capable of five times the square on a segment of itself, then, when twice the said segment is cut in extreme and mean ratio, the greater segment is the remaining part of the original straight line.

Let the straight line AB be divided at C, and let it be capable of five times [the square on] the segment AC; and let CD be taken as double AC,

which — as we shall show below — will be greater than the remaining segment BC. I say that if CD is cut in extreme and mean ratio, its greater segment is CB, the remaining part of the original line.

For let the square BF be described on AB. In it draw the diameter AE, and from C draw CG parallel to AF, cutting the diameter at the point H; and through H let IK be drawn parallel to AB. Then produce GC and complete the square CM, producing EB to N. By the corollary of Proposition 4 of Book 2 [II.4], CI and KG will be the squares on the lines AC and CB. Now since the square BF is set as five

times the square CI, then, if the square CI is taken away, there remains the gnomon OPQ, which is four times the same square CI. But the square CM is also four times the square CI, by the scholium of Proposition 4 of Book 2, because the line CD is double AC. Therefore the gnomon OPQ will be equal to the square CM. Again, since CD is set as double AB, and LC is equal to CD, and CH equal to AC, then LC will also be double CH. [Margin: a — VI.1] Since therefore, as LC is to CH, so is the rectangle LB to the rectangle CK, that [rectangle LB] will likewise be double this [rectangle CK]. [Margin: b — I.43] But CK is equal to HF. Therefore the remaining square KG is also equal to the remaining rectangle BM. Now since the rectangle BM, contained under CD, BD, is equal to KG, the square on the line CB, [Margin: c — VI.17] it will be that, as CD is to CB, so is CB to BD. For which reason, by the definition, the line CD is cut at B in extreme and mean ratio, and CB is the greater segment.

Diagram. A geometric construction for Proposition 2, similar to the previous figure but reoriented: an upper square (BF on AB) with an inscribed circle and diagonal AE, above a lower rectangle. Labelled points read from the figure:

- F (top-left corner of upper square)
- G (top edge, midpoint)
- E (top-right corner of upper square)
- P (upper part of the inscribed circle, on the diagonal)
- J/I (left edge, at the level of line IK) [reads as "J"/"I"; the printed point-letter for line IK]
- O (left-of-centre, on line IK)
- H (centre)
- K (right edge, on line IK)
- A (left, at foot of upper square)
- Q (centre-low)
- C (just below Q, foot of the diagonal region)
- B (right, foot of upper square)
- L (bottom-left corner of lower rectangle)
- N (bottom edge, midpoint)
- M (bottom-right corner of lower rectangle)

ALITER [Another way]. Since the square on the line AB is set as five times [the square on the line AC], and the square on the line AB is equal to the squares on the lines CB, AC together with twice the rectangle under AC, CB, then the squares on the lines CB, AC, together with twice the rectangle under AC, CB, will be five

times the square on the line AC. [Margin: d — II.4] Therefore, if the square on the line AC is subtracted, there remains the square on the line CB together with the rectangle under AC, CB taken twice, [equal to] four times the same square on the line AC; and the square on the line CD is [set as] four times the same square on the line AC, [Margin: e — VI.1] and so the square on the line CB together with twice the rectangle under AC, CB is equal to the square on the line CD. (For, just as the line CD is double the line AC, so too twice the rectangle under CD, CB will be double the rectangle under AC, CB, since both rectangles have the same height CB; and therefore twice the rectangle under AC, CB will be equal to the rectangle under CD, CB.) Therefore the [rectangle] under CD, CB will be equal to the square on the line CD. [Margin: f — II.2] But the square on the line CD [together with the rectangle] under CD, BD [is equal to the squares under CD, CB and under CD, BD]; therefore the square on the line CB and the rectangle under CD, CB [together] will be equal to the rectangle under CD, BD. [Margin: g — VI.17] And so, when the common rectangle under CD, CB is taken away, there remains the square on the line CB equal to the rectangle under CD, BD; and therefore, as CD is to CB, so will CB be to DB. For which reason, by the definition, the line CD is cut at B in extreme and mean ratio, and CB is the greater segment. Therefore, if a straight line is capable of five times the square on a segment of itself, and so on. Which was to be demonstrated.

Diagram. A single divided straight line marking the points, in order: A, C, B, D (with the segments AC, CB, BD tick-marked).

LEMMA.

That the line CD is of necessity greater than CB we shall show from the hypothesis, as follows. Since the square on the line CD is four times [the square on the line AC], by the scholium of Proposition 4 of Book 2, then, adding the square on the line AC, the two squares on the lines CD, AC will be five times the same square on the line AC. But the square on the line AD is greater than the squares on the lines AC, CD together with twice the rectangle under AC, CD. Therefore the square on the line AD is also greater than five times the square on the line AC; and consequently greater than the square on the line AB, which is set as five times the same square on the line AC. Therefore the line AD will be greater than the line AB; and so, when the common part AC is taken away, the remaining CD will be greater than the remaining CB. Which was proposed.

THEOR. 3. PROPOS. 3.

If a straight line is cut in extreme and mean ratio, then the lesser segment, taking to itself half of the greater segment, is capable of five times the square described on half of the greater segment.

Let the straight line AB be cut in extreme and mean ratio at C, so that its greater segment is AC, and let AC be bisected at D. I say that the square on the line BD is four times the square on the line CD.

For let the square AE be described on AB, in which is the diameter BF. Then from C, D draw CG, DH parallel to AF, BE, cutting the diameter at the points I, K; through which draw LM, NO parallel to AB, EF, which cut the lines CG, DH at P, Q. By the corollary of Proposition 4 of Book 2 [II.4], LG, PQ, DO will be the squares on the lines AC, CD, BD. Now since the line AC is double the line CD, then, by the scholium of the same proposition, the square LG will be four times the square PQ. And the rectangle CE is equal to the square LG. (For since,

Diagram. A square (AE on AB) subdivided by two vertical and two horizontal lines into a grid, with the diagonal FB drawn and an inscribed arc. Labelled points read from the figure:

- F (top-left corner)
- H (top edge, left of centre)
- G (top edge, right of centre)
- E (top-right corner)
- N (left edge)
- K (interior, upper-left)
- Q (interior, upper-right)
- O (right edge)
- L (left edge, lower)
- D (interior, left of centre — repeated in the bottom row)
- R (interior, centre-left, on the arc)
- T (interior centre, small label)
- I (interior, centre-right)
- M (right edge, lower)
- A (bottom-left corner)
- D (bottom edge, left of centre)
- C (bottom edge, centre)
- B (bottom-right corner)

- S (bottom edge region, on the arc, centre)

[Continued on p. 549:] as AB is to AC, so is AC to CB, [Margin: a — VI.17] the rectangle under AB, CB — namely CE — will be equal to the square on the line AC, that is, to LG.) Therefore CE also will be four times the square PQ. And since the lines AD, CD are equal, their sides HK, IQ will be equal, [Margin: b — I.34] and the opposite [rectangles] will be equal. [Margin: c — I.36] Therefore the rectangles IO, QE will be equal. [Margin: d — I.43] And QE will be equal to the same ID; and consequently, adding the common rectangle CO, [the whole] CE will be equal. Since therefore CE has been shown to be four times the square PQ, [the gnomon is] four times the same square PQ; and consequently, adding the square PQ, [the square DO] is five times the square PQ, that is, [five times the square] on the line CD.

ALITER [Another way]. Since AC has been bisected at D, and CB added to it, [Margin: e — II.6] the [rectangle under AB, BC together with] the square on the line CD is equal to the square on the line BD. But the rectangle under AB, BC is four times the square on the line CD; ([Margin: f — VI.17 [?]] for since, as AB is to AC, so is AC to CB, the rectangle under AB, BC is equal to the square on the line AC; and since, by the scholium of Proposition 4 of Book 2, the square on the line AC is four times the square on the line CD, the rectangle under AB, BC will also be four times the same square on the line CD.) And therefore the rectangle under AB, BC, together with the square on the line CD, will be five times the square on the line CD; that is, the square on the line BD is five times the square on the line CD. Therefore, if a straight line is cut in extreme and mean ratio, and the lesser segment takes to itself half of the greater segment, it is capable of five times the square described on half of the greater segment. Which was to be shown.

Diagram. A single divided straight line marking the points, in order: A, D, C, B (with segments tick-marked).

SCHOLIUM.

The converse of this theorem we shall also demonstrate, with Campanus [Campanus of Novara], in this manner:

If a straight line is cut unequally, and the lesser segment, taking to itself half of the greater segment, is capable of five times the square described on half of the greater segment, then that straight line is cut in extreme and mean ratio.

Let the line AC be divided unequally at C, whose greater segment AC is bisected at D; and let the square on the line BD be five times the square on the line CD. I say that the line AB is divided at C in extreme [and mean ratio]. For, the figure of the previous proposition being repeated, the square LG will be four times the square PQ, [that is, four times the square] on the line CD, by the scholium of Proposition 4 of Book 2. And the gnomon RST is four times the same square PQ. (For since the square DO, on the line BD, is set as five times the square PQ, on the line CD, then, when the square PQ is taken away, there remains the gnomon RST, four times the same square [PQ]; and this will be equal to the square LG. But the rectangle CE is equal to the gnomon RST, as was shown above; [Margin: g — VI.17] and so the rectangle CE, contained under AB, BC, will be equal to the same [square] on the line AC.) And consequently, as AB is to AC, so is AC to CB. For which reason, by the definition, the line AB will be cut at C in extreme and mean ratio. Which was proposed.

ALITER [Another way]. Since AC has been bisected at D, and GB added to it, [Margin: h — II.6] the rectangle under AB, BC, together with the square on the line CD, is equal to the square on the line BD. And therefore, since the square described on the line BD is five times the square described on the line CD, the rectangle under the lines AB, BC, described together with the square on the line CD, will be five times the same square on the line CD. So, when the square on the line CD is taken away, the rectangle under AB, BC will be four times the square [on the line CD]. But four times the square on the line CD is equal to the square on the line AC; therefore the rectangle under AB, BC is equal to the square on the line AC. [Margin: i — VI.17] For which reason, as AB is to AC, so is AC to CB; and consequently the line AB is cut at C in extreme and mean ratio, by the definition.

THEOR. 4. PROPOS. 4.

If a straight line is cut in extreme and mean ratio, then the squares on the whole and on the lesser segment together are triple the square described on the greater segment.

Let the straight line AB be cut in extreme and mean ratio at C, so that the greater segment is AC and the lesser CB. I say that the squares on the lines AB, BC together are triple the square on the line AC.

For let the square AD be described on AB, in which is drawn the diameter BE; and from C draw CF parallel to BD, cutting the diameter at the point G; through which draw HI parallel to AB. By the corollary of Proposition 4 of Book 2 [II.4], CH, IF will be the squares on the lines BC, AC. Now since, as AB is to AC, so is AC to CB,

[Margin: a — VI.17] the rectangle under AB, BC — namely AH — will be equal to IF, the square on the line AC. Since therefore AH is equal to CD, the gnomon KLM, together with the square CH, will be double the square IF; and consequently, adding the square IF, the square AD, on the line AB, together with the square CH, on the line BC, will be triple the same square IF, on the line AC.

Diagram. A square (AD on AB) with the diagonal EB, a subdividing parallel CF and horizontal HI, and an inscribed quarter-arc. Labelled points read from the figure:

- E (top-left corner)
- F (top edge, centre)
- D (top-right corner)
- K (interior, upper, on the arc)
- I (left edge, at the level of line HI) [printed as a single point-letter on the left]
- M (interior, left of centre, on line HI)
- G (interior centre, intersection point)
- H (right edge, at the level of line HI) [the point-letter reads as a double stroke "II"; taken as H per the demonstration's "HI"]
- L (interior, lower-right, on the diagonal near B)
- A (bottom-left corner)
- C (bottom edge, centre)
- B (bottom-right corner)

ALITER [Another way]. Since, as AB is to AC, so is AC to CB, [Margin: b — VI.17] the rectangle under AB, BC will be equal to the square on the line AC; and so twice the rectangle under AB, BC, together with the square on the line AC, is triple the square on the line AC. [Margin: c — II.7] But the squares on the lines AB, BC together are equal to the square on the line AC together with twice the rectangle under AB, BC. Therefore the squares on the lines AB, BC together are triple the square on the line AC. Consequently, if a straight line is cut in extreme and mean ratio, then the squares on the whole and on the lesser segment together are triple the square described on the greater segment. Which was to be demonstrated.

SCHOLIUM.

The converse of this too is true, namely:

(Continuation of the Scholium to Proposition 4 — the converse.)

If a straight line is cut unequally, and the square on the whole together with the square on the lesser segment is three times the square described on the greater segment, then that line will be cut in extreme and mean ratio.

Let the line AB be cut unequally at C, so that the square on the whole AB and the square on the lesser segment BC, taken together, are three times the square on the greater segment AC. I say that AB is cut in extreme and mean ratio. For, with the figure constructed as before: since the squares AD and CH together are three times the square IF, then, once the square IF is subtracted, the gnomon KLM together with the square CH will be twice that same square IF. But the gnomon KLM together with the square CH is equal to the two rectangles AH and CD. Therefore the rectangles AH and CD together are twice the square IF; and since AH and CD are equal, AH will be equal to IF. Consequently, since the rectangle AH — that is, the rectangle contained by AB and BC — is equal to the square IF on AC, [Margin: d — *Elements* VI.17] as AB is to AC, so AC will be to CB; and therefore AB is divided at C in extreme and mean ratio.

ANOTHER WAY. Since the squares on the lines AB and BC are three times the square on AC, [Margin: e — *Elements* II.7] and since the squares on AB and BC are equal to the square on AC together with twice the rectangle contained by AB and BC, then the square on AC together with twice the rectangle contained by AB and BC will be three times the square on AC. Therefore, subtracting the square on AC, there remains twice the rectangle contained by AB and BC, equal to twice the square on AC. And so the rectangle contained by AB and BC, taken once, is equal to the square on AC. [Margin: f — *Elements* VI.17] Hence, as before, as AB is to AC, so AC will be to CB; and therefore AB is cut at C in extreme and mean ratio.

From Francesco Maurolico [Maurolico] we shall also demonstrate the theorem that follows.

If a straight line is cut in extreme and mean ratio, then the square on the whole line together with the lesser segment is five times the square described on the greater segment. And if a straight line is cut unequally, and the square on the whole line together with the lesser segment can be five times the square described on the greater segment, then that line is cut in extreme and mean ratio.

Let the line AB first be cut at C in extreme and mean ratio, and let the line BD, equal to the lesser segment BC, be added to it. I say that the square on AD is five times the square on AC, the greater segment. [Margin: g — *Elements* II.8] For since

the square on AD is equal to four times the rectangle contained by AB and BC, together with the square on AC, [Margin: h — *Elements* VI.17] and since the rectangle contained by AB and BC is equal to the square on AC (because AB, AC, CB are in continued proportion), the rectangle contained by AB and BC taken four times will be four times the square on AC; and so four times the rectangle contained by AB and BC, together with the square on AC, is five times the square on AC. Therefore the square on AD — which has been shown equal to four times the rectangle contained by AB and BC together with the square on AC — will likewise be five times that same square on AC. This is what was proposed.

Next, let the line AB be cut unequally at C, and let the line BD, equal to the lesser segment BC, be added to it, and let the square on AD be five times the square on AC, the greater segment. I say that AB is cut at C in extreme and mean ratio. [Margin: i — *Elements* II.8] For since the square on AD is equal to four times the rectangle contained by AB and BC, together with the square on AC, then four times the rectangle contained by AB and BC together with the square on AC will also be five times the square on AC; and consequently four times the rectangle contained by AB and BC will be four times the square on AC — that is, the rectangle contained by AB and BC is equal to the square on AC. [Margin: k — *Elements* VI.17] Therefore, as AB is to AC, so AC will be to CB; and hence AB is cut at C in extreme and mean ratio. This is what was proposed.

Diagram. A single straight line divided at one interior point, used for the "another way" argument of the Proposition 4 scholium.

- A (left end): endpoint of the whole line
- C (interior): point of division
- B (right end): endpoint, marking the lesser segment CB

Diagram. A straight line AB cut at C and extended beyond B to D, for the Maurolico theorem (BD added equal to the lesser segment BC).

- A (left end): endpoint of the whole line
- C (interior): point of division into greater segment AC and lesser BC
- B (right of C): endpoint of the original line AB
- D (right end): endpoint of the added segment BD (equal to BC)

THEOR. 5. PROPOS. 5.

[Margin: iv.]

If a straight line is cut in extreme and mean ratio, and there is added to it a line equal to the greater segment: then the whole line is cut in extreme and mean ratio, and the greater segment is the original straight line.

Let the line AB be cut at C in extreme and mean ratio, let the greater segment be AC, and let AD, equal to the greater segment AC, be added to it in a straight line. I say that the line BD is cut at A in extreme and mean ratio, and that the greater segment is AB. For let the square BF be described on AB. In it, when the diagonal AE has been drawn, let CG be drawn from C parallel to BE, cutting the diagonal at the point H; through H let IK be drawn parallel to BD; and from D let DL be drawn parallel to AF, meeting IK, produced, at L. Then, by the corollary of Proposition 4 of Book 2, CK and IG will be the squares on the lines AC and BC. Now since, as AB is to AC, so AC is to CB, [Margin: a — *Elements* VI.17] the rectangle IF, contained by AB and BC, will be equal to the square CK on the line AC. [Margin: b — *Elements* I.36] But CK is equal to AL. Therefore IF will be equal to AL; and, adding the common part BK, the rectangle BL, contained by BD and DA, will be equal to the square BF on the line AB. [Margin: c — *Elements* VI.17] Hence, as BD is to AB, so AB will be to DA. Therefore BD is cut at A in extreme and mean ratio, and the greater segment is AB.

ANOTHER WAY. Since, as AB is to AC — that is, to AD — so AC is to CB, then by conversion, as DA is to AB, so BC will be to CA; and by composition, as DB is to AB, so AB is to CA, that is, to AD. Therefore BD is cut at A in extreme and mean ratio. And so, if a straight line is cut in extreme and mean ratio, and there is added to it a line equal to the greater segment, the whole line, and so on. Which was to be demonstrated.

Diagram. The square construction for Proposition 5, erected on AB with its diagonal AE and the auxiliary parallels CG, IK, DL. The baseline (extended beyond A to D) carries D, A, C, B; the middle horizontal carries L, K, H, I; the top edge carries F, G, E. A diagonal runs from A up to E.

- F (top-left): upper-left vertex of the square on AB
- G (top-middle, above C): point on the top edge where CG meets it
- E (top-right): upper-right vertex of the square (end of the diagonal AE)
- L (middle-left): where DL meets the produced line IK
- K (middle, on the left side): point on side AF
- H (middle-centre): intersection of the diagonal AE with CG
- I (middle-right): point on side BE, on line IK
- D (baseline, far left): end of the added segment

- A (baseline): lower-left vertex of the square / point of section of BD
- C (baseline): point of section of AB
- B (baseline, right): lower-right vertex of the square

Diagram. The straight line for the "another way" of Proposition 5: line DB cut at A, with A cut again at C.

- D (left end): endpoint of the whole line DB
- A (interior): point where DB is cut in extreme and mean ratio
- C (interior, right of A): point dividing AB
- B (right end): endpoint

SCHOLIUM.

In the same way we shall demonstrate this too, along with Campanus [Campanus].

If a straight line is cut in extreme and mean ratio, and the lesser segment is subtracted from the greater, then the greater segment is itself cut in extreme and mean ratio, and its greater segment is that line which was the lesser segment of the original line.

Let the line BD be cut at A in extreme and mean ratio, its greater segment being AB and its lesser segment AD; and from the greater segment AB let the line AC, equal to the lesser segment AD, be subtracted. I say that the greater segment AB is divided at C in extreme and mean ratio, and that its greater segment is the line AC, which is equal to the lesser segment AD. For, with the construction of the earlier figure carried out: since, as BD is to AB, so AB is to AD, [Margin: d — *Elements* VI.17] the rectangle BL, contained by BD and AD, will be equal to the square BF on the line AB; and, removing the common part BK, AL will be equal to IF. [Margin: e — *Elements* I.36] But AL is equal to CK. Therefore IF, contained by AB and CB, is equal to the square CK on the line AC. [Margin: f — *Elements* VI.17] Hence, as AB is to AC, so AC is to CB. Therefore AB is cut at C in extreme and mean ratio, its greater segment being AC and its lesser segment the line CB. This is what was proposed.

ANOTHER WAY. Since, as the whole BD is to the whole AB, so is AB — that is, AC, subtracted from BD — to AD, that is, to AC, subtracted from AB; [Margin: g — *Elements* V.19] then also, as AD, the remainder of BD — that is, AC — is to CB, the remainder of AB, so is the whole DB to the whole AB, that is, as AB is to AC. Therefore AB is divided at C in extreme and mean ratio. This is what was proposed.

Again, we shall demonstrate the following too.

If a line is cut in extreme and mean ratio, and from its half is taken the half of the greater segment: then the half of the whole is likewise cut in extreme and mean ratio, and its greater segment is half of the greater segment of the whole line.

For let AB be divided at C in extreme and mean ratio, let DE be the half of the whole, and DF the half of the greater segment AC. I say that DE is cut at F in extreme and mean ratio, and that its greater segment is DF. For since, as the whole AB is to the whole DE, so is the subtracted AC to the subtracted DF — the ratio being double in each case — [Margin: h — *Elements* V.19] the remainder CB will also be to the remainder FE as the whole is to the whole; and hence CB will be double FE. [Margin: i — *Elements* V.15] Now since, as AB is to AC, so is DE, the half of AB, to DF, the half of AC — and since, by the definition of a line cut in extreme and mean ratio, AB is to AC as AC is to CB — then also DE will be to DF as DF is to FE. And so, by the same definition, DE will be cut at F in extreme and mean ratio. This is what was proposed.

Diagram. Two parallel straight lines, one above the other: the upper line carries A, C, B (the whole AB cut at C); the shorter lower line carries D, F, E (its half DE cut at F).

- A (upper line, left): endpoint of the whole line
- C (upper line, middle): point of section in extreme and mean ratio
- B (upper line, right): endpoint
- D (lower line, left): endpoint of the half-line DE
- F (lower line, middle): point of section of DE
- E (lower line, right): endpoint of the half-line

THEOR. 6. PROPOS. 6.

[Margin: vj.]

If a rational straight line is cut in extreme and mean ratio, each of the segments is an irrational line, of the kind called an apotome.

Let the rational line AB be cut at C in extreme and mean ratio. I say that each segment, AC and CB, is an irrational line, of the kind called an apotome. For let AD, equal to half the whole AB, be added to the greater segment AC. [Margin: a — *Elements* XIII.1] Since, then, the square on the line CD is five times the square on the line AD, the square on CD will have to the square on AD a ratio which a

number has to a number. [Margin: b — *Elements* X.6] Therefore the squares on the lines CD and AD are commensurable; and consequently the lines CD and AD themselves are also commensurable, at least in square. But AD is rational, since it is half the rational line AB. Therefore CD too will be rational. Now since the squares on the lines CD and AD do not have to one another a ratio which a square number has to a square number (as is clear from the corollary of Proposition 24 of Book 8 — for they have the ratio of 5 to 1, or of 25 to 5), [Margin: c — *Elements* X.9] the lines CD and AD will be incommensurable in length; and consequently they are rational lines commensurable in square only. Therefore, if from CD, which is rational, there is subtracted AD, which is rational and commensurable in square only, [Margin: d — *Elements* X.74] the remainder AC will be irrational, of the kind called an apotome.

Again, let the rectangle AE, contained by AB and CB, be applied to AB. [Margin: e — *Elements* VI.17] Then the rectangle AE will be equal to the square on the line AC. For this reason the square on the apotome AC — namely the rectangle AE, applied along the rational line AB — [Margin: f — *Elements* X.98] makes its other side BE, that is, the line CB equal to it, a first apotome. Therefore, if a rational straight line is cut in extreme and mean ratio, and so on. Which was to be demonstrated.

Diagram. The figure for Proposition 6: a horizontal line carrying D, A, C, B (with AD added to AC), and a rectangle erected on AB whose upper-right corner is E, with verticals dropped at A and at C (the applied rectangle AE).

- D (baseline, left): end of the added segment AD
- A (baseline): endpoint of the original line / point of section
- C (baseline): point of section in extreme and mean ratio
- B (baseline, right): endpoint of the line AB
- E (upper-right): upper vertex of the rectangle applied along AB

(Continuation of the Scholium to Proposition 6.)

But we shall also show this theorem along with Campanus [Campanus].

If a straight line is cut in extreme and mean ratio, and the greater segment is a rational line, then the lesser segment is an apotome.

Let the line AB be divided at C in extreme and mean ratio, and let the greater segment AC be a rational line. I say that the lesser segment CB is an apotome. For let AC be bisected at D. Then CD, half of the rational line AC, will be rational, since it is commensurable with AC. [Margin: g — *Elements* XIII.3] Now since the square

on the line BD is five times the square on the line CD, the squares on the lines BD and CD will have to one another a ratio which a number has to a number; [Margin: h — *Elements* X.6] and consequently they will be commensurable. Therefore the lines BD and CD themselves will also be commensurable, at least in square. Since, then, CD is rational, BD too will be rational. But since the squares on the lines BD and CD do not have to one another a ratio which a square number has to a square number (as is clear from the corollary of Proposition 24 of Book 8 — for they have the ratio of 5 to 1, or of 25 to 5), the lines BD and CD will be incommensurable in length. And hence, if from the rational line BD there is subtracted CD, which is rational and commensurable in square only, [Margin: i — *Elements* X.74] the remainder CB will be irrational, of the kind called an apotome.

Diagram. A single straight line for the continuation of the Proposition 6 scholium, carrying A, D, C, B.

- A (left end): endpoint of the greater segment AC
- D (interior): midpoint of AC
- C (interior): point of section of AB
- B (right end): endpoint (lesser segment CB)

THEOR. 7. PROPOS. 7.

[Margin: vij.]

If three angles of an equilateral pentagon are equal — whether they are consecutive or not consecutive — then the pentagon is equiangular.

First case. Let three angles of the equilateral pentagon ABCDE, which are first taken as consecutive, be equal: A, B, C. I say that the pentagon is equiangular. For let the lines BE, AC, BD be drawn subtending the said equal angles; and from the point F, where the lines BE and AC intersect, let ED be drawn. Since, then, the sides AB, AE of triangle ABE are equal to the sides CB, CD of triangle CBD, and the angles contained by them are also equal, by hypothesis, [Margin: a — *Elements* I.4] the bases BE, BD and the angles AEB, CDB will be equal. [Margin: b — *Elements* I.5] But the angles BED, BDE are also equal, since the sides BE, BD have been shown equal. Therefore the whole angles AED, CDE will be equal. Again, since the sides AB, AE of triangle ABE are equal to the sides BA, BC of triangle BAC, and the angles contained by them are also equal, by hypothesis, [Margin: c — *Elements* I.4] the base BE will be equal to the base AC, and the angles ABE, AEB equal to the angles BAC, BCA. Since, then, the angles ABE, BAC of triangle ABF are equal, [Margin: d — *Elements* I.6] the sides BF, AF will also be equal; and

consequently, if these are subtracted from the equal lines BE, AC, the remaining lines FE, FC will be equal. And so, since the sides FE, ED of triangle FED are equal to the sides FC, CD of triangle FCD, and the base FD is common, [Margin: e — *Elements* I.8] the angles contained by the said sides, FED and FCD, will also be equal. But the angles AEB, BCA have also been shown equal. Therefore the whole angles AED, BCD are equal. And hence, since the angle CDE has been shown equal to the angle AED, and the angles ABC, BAE are supposed equal to the angle BCD, all the angles of the pentagon will be equal, and so the pentagon will be equiangular.

Second case. Let three non-consecutive angles be equal: A, C, D. Since, then, the sides AB, AE of triangle ABE are equal to the sides CB, CD of triangle CBD, and the angles contained by them are also equal, by hypothesis, [Margin: f — *Elements* I.4] the bases BE, BD and the angles AEB, CDB will be equal; [Margin: g — *Elements* I.5] but the angles BED, BDE are also equal, since the sides BE, BD have been shown equal. Therefore the whole angles AED, CDE will be equal. And so, since the angles BAE, CDE are supposed equal, there will be three consecutive equal angles, A, E, D; and hence, as has already been demonstrated, the pentagon will be equiangular. Therefore, if three angles of an equilateral pentagon are equal, and so on. Which was to be demonstrated.

Diagram. The equilateral pentagon ABCDE with the diagonals BE, AC, BD drawn and ED joined, meeting the interior point F.

- A (top vertex): pentagon vertex
- B (upper-left vertex): pentagon vertex
- E (upper-right vertex): pentagon vertex
- C (lower-left vertex): pentagon vertex
- D (lower-right vertex): pentagon vertex
- F (interior): intersection of the diagonals BE and AC

THEOR. 8. PROPOS. 8.

[Margin: viij.]

If two straight lines subtend two consecutive angles of an equilateral and equiangular pentagon, then they cut one another in extreme and mean ratio, and their greater segments are equal to the side of the pentagon.

In the equilateral and equiangular pentagon ABCDE, let the lines DB, CE subtend the two consecutive angles C and D, cutting one another at F. I say that they are

cut in extreme and mean ratio, and that their greater segments FB, FE are equal to any side of the pentagon. [Margin: a — *Elements* IV.14] For let a circle be described about the pentagon; [Margin: b — *Elements* III.28] then the five arcs AB, BC, CD, DE, EA will be equal. Now since the sides CD, CB of triangle CDB are equal to the sides DC, DE of triangle DCE, and the angles contained by them are also equal, by hypothesis, [Margin: c — *Elements* I.4] the bases DB, CE and the angles CDB, DCE will be equal. And hence, in triangle CDF, the two angles FCD, FDC are equal; [Margin: d — *Elements* I.32] and the external angle BFC is equal to them together, so that the angle BFC is double the angle DCE. [Margin: e — *Elements* VI.33] But the angle BCE is also double that same angle DCE, since the arc BAE is likewise double the arc DE. Therefore the angles BFC, BCF are equal; [Margin: f — *Elements* I.6] and so the line BF is equal to the side BC of the pentagon. [Margin: g — *Elements* III.27] Now since the angles DBC, ECD, standing on equal arcs, are equal, the two angles DBC, CDB of triangle BCD will be equal to the two angles DCF, FDC of triangle CFD. [Margin: h — *Elements* I.32] And hence the triangles BCD, CFD will be equiangular. [Margin: i — *Elements* VI.4] Therefore, as BD is to DC, so DC — that is, its equal BF — is to FD; and consequently BD is cut at F in extreme and mean ratio, and its greater segment BF is equal to the side of the pentagon, as has been shown. By a like argument we shall show that the line CE is cut at F in extreme and mean ratio, and that its greater segment EF is equal to the side DE of the pentagon. Therefore, if two consecutive angles of an equilateral and equiangular pentagon, and so on. Which was to be shown.

Diagram. The equilateral, equiangular pentagon ABCDE inscribed in a circle, with the diagonals DB and CE drawn, cutting one another at the interior point F.

- A (top vertex): pentagon vertex on the circle
- B (upper-left vertex): pentagon vertex on the circle
- E (upper-right vertex): pentagon vertex on the circle
- C (lower-left vertex): pentagon vertex on the circle
- D (lower-right vertex): pentagon vertex on the circle
- F (interior): intersection of the diagonals DB and CE

SCHOLIUM.

We shall also easily demonstrate that the line which subtends an angle of an equilateral and equiangular pentagon is parallel to the opposite side — that is, that BD is parallel to the side AE, and CE to the side AE. For let a circle be described about the pentagon. [Margin: k — *Elements* X.22] Since [...]

(Continuing the Scholium to Proposition 8 — that a diagonal of an equilateral, equiangular pentagon is parallel to the opposite side.)

both the two angles A and BCE, and the two angles ABC and AEC, are equal to two right angles. But A and ABC are equal to one another in an equiangular pentagon; therefore the remaining angles BCE and AEC are also equal. Since, then, A and BCE are together equal to two right angles, A and AEC will likewise be equal to two right angles; and consequently AB and CE will be parallel. Which is what was proposed. [Margin: l — I.28]

THEOR. 9. PROPOS. 9.

[Margin: ix]

If the side of the hexagon and the side of the decagon inscribed in the same circle be joined together, the whole line is cut in extreme and mean ratio, and its greater segment is the side of the hexagon.

In the circle ABC let AB be the side of the decagon, and to it in a straight line let there be added BE, equal to the radius of the circle and therefore to the side of the hexagon inscribed in the same circle. I say that the line EA is cut at B in extreme and mean ratio, and that its greater segment is EB, the side of the hexagon.

For draw from A, through the centre D, the diameter AC, and join the lines DB and DE. Now since the arc AB is a tenth part of the whole circumference of the circle, the arc of the semicircle will contain the arc AB five times; and consequently the arc BC will be four times the arc AB. [Margin: a — VI.33] Therefore the angle BDC will also be four times the angle ADB. Again, since the sides BD and BE are equal — being sides of the hexagon (for BE, the side of the hexagon, is equal to the radius BD, by the corollary to Prop. 15 of Book IV) — [Margin: b — I.5] the angles BDE and BED will be equal; [Margin: c — I.32] and since the exterior angle ABD is equal to them together, the angle ABD will be double the angle BED. [Margin: d — I.5] But the angle BAD is equal to the angle ABD, because of the equality of the sides DA and DB. Therefore the angle BAD too will be double the angle BED; and consequently the two angles DAB and DBA together will be four times the angle BED. [Margin: e — I.32] Since, then, the exterior angle BDC is equal to the angles DAB and DBA together, the angle BDC will be four times that same angle BED. But this same angle BDC was shown to be four times the angle ADB. Therefore the angles AED and ADB are equal. And so, since the two angles AED and DAE of triangle ADE are equal to the two angles ADB and BAD of triangle ABD, [Margin: f — I.32] the triangles ADE and ABD will

be equiangular. [Margin: g — VI.4] Therefore, as EA is to AD — that is, to EB, which is equal to AD — so is AD — that is, EB, equal to itself — to BA. And therefore the line EA is cut at B in extreme and mean ratio, and its greater segment EB is the side of the hexagon. Thus, if the side of the hexagon and the side of the decagon inscribed in the same circle be joined together, the whole line is cut in extreme and mean ratio, and its greater segment is the side of the hexagon. Which was to be demonstrated.

Diagram. A circle with centre D and horizontal diameter running from C (at the left) through D to A (at the right). Point B lies on the upper-right of the circle; point E lies above and outside the circle, on the straight line AB produced (A–B–E collinear). AB is the side of the decagon; BE, its rectilinear extension, is the side of the hexagon (= radius). The radii DB and DE are drawn.

- C (left end of diameter): endpoint of diameter AC
- D (centre): centre of the circle
- A (right end of diameter): where the diameter meets the decagon side
- B (upper right, on circle): junction of decagon side AB and hexagon side BE
- E (top, outside circle): far end of the joined line, EB = hexagon side

SCHOLIUM.

Now since we demonstrated, at Prop. 5 of this book, that if the lesser segment of a line cut in extreme and mean ratio be subtracted from the greater segment, the greater segment too is cut in extreme and mean ratio, and that its greater segment is the lesser segment subtracted — and since here it has been shown that the side of the hexagon joined with the side of the decagon is cut in extreme and mean ratio — it follows that if the side of the hexagon be cut in extreme and mean ratio, its greater segment is the side of the decagon. For if this, which is the greater segment, be subtracted from the side of the hexagon, the side of the hexagon will have been divided in extreme and mean ratio, as we showed at Prop. 5 of this book. This, however, we shall demonstrate otherwise at Book XIV, Prop. 4.

The converse of this theorem too we shall easily demonstrate, along with Campanus. Namely:

If the greater segment of a line divided in extreme and mean ratio be the side of the hexagon of some circle, the lesser segment will be the side of the decagon of the same circle. And if the lesser segment be the side of the decagon of some circle, the greater segment will be the side of the hexagon of the same circle.

Let AB be divided in C in extreme and mean ratio, and let AC first be the side of the hexagon of the circle BCD. I say that the lesser segment CB is the side of the decagon of the same circle. For let BC be fitted into the circle, so that CA falls outside the circle. And, the diameter BED being drawn from B through the centre E, let the lines EA and EC be joined. Now since, as AB is to AC, so is AC to CB; and AC, the side of the hexagon, is equal to the radius EB, by the corollary to Prop. 15 of Book IV; it will also be that, as AB is to BE, so is EB to BC. [Margin: h — VI.6] Therefore the triangles ABE and EBC will be equiangular, since they have the sides about the common angle B proportional; and the angle A will be equal to the angle BEC. Again, since the sides CA and CE are equal — being sides of the hexagon — [Margin: i — I.5] the angles CAF [recte: CAE — no point F occurs in this figure] and CEA will be equal; [Margin: k — I.32] and since the exterior angle BCE is equal to them, BCE will be double the angle A. [Margin: l — I.5] But the angle CBE is equal to the angle BCE, because of the equality of the sides EB and EC. Therefore CBE too will be double the angle A; and consequently the two angles BCE and CBE together will be four times that same angle A. [Margin: m — I.32] Since, then, the exterior angle CED is equal to the angles BCE and CBE together, the angle CED too will be four times that same angle A, and therefore four times the angle BEC, which was shown equal to angle A. [Margin: n — VI.33] Therefore the arc CD will be four times the arc BC; and consequently the arc of the semicircle BCD will be five times that same arc BC, and so the whole circumference of the circle will be ten times the arc BC. Therefore the line BC is the side of the decagon.

Next, let CB, the lesser segment, be the side of the decagon of the circle BCD. I say that the greater segment AC is the side of the hexagon of the same circle. For let BC again be fitted into the circle BCD, and let the diameter BED be drawn. Then, with the interval AC, let the circle CFG be described, whose diameter is CAG; and by the corollary to Prop. 15 of Book IV, AC will be the side of the hexagon of the circle CF. Now since AB has been cut in C in extreme and mean ratio, and the greater segment AC is the side of the hexagon of the circle CFG, it will be, as has already been shown, that the lesser segment CB is the side of the decagon of the same circle CFG.

Diagram. A woodcut block holding two circles side by side, for the two cases of the converse. *Left circle:* centre E, horizontal diameter D (left)–E (centre)–B (right); a regular decagon is inscribed; C lies on the upper-right of the circle; A lies above and outside the circle (CA produced falls outside). The radii EA and EC are drawn, with the chord BC (decagon side). *Right circle* (circle CFG): centre A, horizontal diameter G (left)–A (centre)–C (right); a regular decagon is inscribed; F lies on the upper-right of the circle.

- Left circle — D (left): end of diameter; E (centre); B (right): end of diameter; C (upper right, on circle); A (top, outside): apex where CA is produced
- Right circle — G (left end of diameter); A (centre); C (right end of diameter); F (upper right, on circle)

(The demonstration of the second case concludes at the top of the following page, where this two-circle figure is repeated.)

(p. 542 — running header "EVCLID. GEOM." The two-circle converse figure of the preceding page is repeated at the top of this column.)

Diagram. A repetition of the two-circle converse figure from the previous page, placed here to accompany the concluding text. *Left circle:* centre E, diameter D–E–B, with C on the upper right and A above and outside; decagon inscribed. *Right circle:* centre A, diameter G–A–C, with F on the upper right; decagon inscribed.

- Left circle — D (left), E (centre), B (right), C (upper right), A (top, outside)
- Right circle — G (left), A (centre), C (right), F (upper right)

[Margin: o — III.31] [Margin: p — I.36 [?]] (sense: the sides BD, CG proved equal — perhaps I.26)]

...the lesser segment CB, the side of the decagon of the same circle CFG. Let there therefore be described, in the circle CFG, the equilateral and equiangular decagon CFG; likewise in the circle BCD, the equilateral and equiangular decagon BCD; and the sides BC, CF will be equal to one another. Now since, if the lines CD and FG be joined, the angles CDB and FGC are equal — as we showed in the scholium to Prop. 22 of Book III, because they stand on similar arcs BC, CF, namely tenth parts of the circumferences — and since the angles BCD and CFG in the semicircles are also equal, [Margin: o — III.31] being of course right angles: the sides BD and CG too will be equal to one another. [Margin: p — I.36 [?]] For this reason, since the diameters BD and CG are equal, the circles BCD and CFG will also be equal. And consequently, since the line AC is the side of the hexagon of the circle CFG, that same line AC will also be the side of the hexagon of the circle BCD. Which is what was proposed.

THEOR. 10. PROPOS. 10.

[Margin: x]

If an equilateral pentagon be inscribed in a circle, the side of the pentagon is equal in square [potest] to the side of the hexagon and the side of the decagon inscribed in the same circle.

In the circle $ABCDE$, whose centre is F , let an equilateral pentagon be inscribed. I say that the square on the side AB is equal to the square on the side of the hexagon together with the square on the side of the decagon of the same circle. For draw the diameter AFG , and join the line FB . Then, the arc AB being bisected at H , join the lines AH , BH , and FH , this last cutting the line AB at I ; and the line AH will be the side of the decagon, and BF the side of the hexagon. Again, the arc AH being bisected at K , join the line FK , cutting the line AH at L and the line AB at M , the point to which the line HM is drawn.

Now since the arcs AH and BH are equal, [Margin: a — III.27] the angles standing on them, AFH and BFH , will be equal. Therefore, since the sides AF and FI of triangle AFI are equal to the sides BF and FI of triangle BFI , and the angles contained by them are also equal, [Margin: b — I.4] the bases AI and BI will be equal, and the angles AIF and BIF will be right angles. And by the same reasoning the lines AL and HL will be equal, and the angles ALF and HLF right angles. Next, if from the equal semicircles $ABCG$ and $AEDG$ there be taken the equal arcs ABC and AED , the arcs CG and DG will be left equal to one another; and so the arc CG will be half the arc CD . But the arc AH is likewise half the arc AB . Therefore, since the arcs AB and CD are equal, their halves, the arcs CG and AH , will also be equal; and consequently, since the arc AH is double the arc HK , the arc CG will also be double that same arc HK . In the same way, since the arcs AB and BC are equal, and the arc AB is double the arc BH , the arc BC too will be double that same arc BH . Therefore the arcs CG and BC are equimultiples — namely doubles — of the arcs HK and BH ; [Margin: c — V.1] and hence the whole arc BG will also be double the whole arc BK , [Margin: d — VI.33] and therefore the angle BFG too will be double the angle BFK . [Margin: e — III.20] But since this same angle BFG , at the centre, is also double the angle FAB , the angles BFM and FAB will therefore be equal; [Margin: f — I.32] and consequently the triangles ABF and FBM , since they have the angles FAB and MFB equal and the angle ABF common, will be equiangular. [Margin: g — VI.4] Therefore, as AB is to BF , so is BF to BM ; [Margin: h — VI.17] and hence the rectangle contained by AB and BM will be equal to the square on the line BF .

Again, since the sides AL and LM of triangle ALM are equal to the sides HL and LM of triangle HLM , and the angles contained by them equal, namely right angles, [Margin: i — I.4] the bases AM and HM , and the angles LAM and LHM , will be equal. [Margin: k — I.5] But the angle LAM is equal to the angle HBA , because the sides HA and HB are equal. Therefore the angle LHM too will be equal to that same angle HBA ; [Margin: l — I.32] and consequently the triangles ABH and AHM , since they have the angles ABH and AHM equal and the angle HAM common, will

be equiangular. [Margin: m — VI.4] For this reason, as AB is to AH, so is AH to AM; [Margin: n — VI.17] and hence the rectangle contained by AB and AM will be equal to the square on the line AH. But it was also shown that the rectangle contained by AB and BM is equal to the square on the line BF. Therefore the rectangles contained by AB, BM and by AB, AM together are equal to the squares on the lines BF and AH. [Margin: o — II.2] But the rectangles contained by AB, BM and by AB, AM together are equal to the square on the line AB. Therefore the square on the line AB, namely the side of the pentagon, is equal to the squares on the lines BF and AH, that is, on the side of the hexagon and the side of the decagon. If, therefore, an equilateral pentagon be inscribed in a circle, and so on. Which was to be demonstrated.

Diagram. A circle with centre F in which is inscribed the equilateral pentagon ABCDE: A at the top, B at the upper left, E at the upper right, C at the lower left, D at the lower right. The vertical diameter runs A–F–G, G being the lowest point of the circle. The arc AB is bisected at H (on the circle, upper left); the arc AH is bisected at K (near A). The line FH cuts AB at I; the line FK cuts AH at L and AB at M. The construction lines AH, BH, FH, FK, HM and the radius FB are drawn.

- A (top): pentagon vertex, top of diameter
- B (upper left): pentagon vertex
- C (lower left): pentagon vertex
- D (lower right): pentagon vertex
- E (upper right): pentagon vertex
- F (centre): centre of the circle
- G (bottom): lower end of diameter AFG
- H (upper left, on circle): midpoint of arc AB
- K (near A, on circle): midpoint of arc AH
- I (on AB): foot of FH on AB
- L (on AH): foot of FK on AH
- M (on AB): foot of FK on AB

COROLLARIVM I.

From this it follows that a straight line which from the centre bisects any arc also bisects the line subtended to that arc, and at right angles. For it was shown that the line FH, because it bisects the arc AB, also bisects the line AB at I, and at right angles. And the demonstration is the same in the remaining cases.

COROLLARIUM II.

It is likewise clear that a diameter of the circle drawn from any angle of the pentagon bisects both the arc which the side of the pentagon opposite to that angle subtends, and the opposite side itself, and at right angles.

(p. 543 — running header "LIBER XIII.")

For it has been demonstrated that the diameter AG, drawn from the point A, bisects the arc CD which the opposite side CD subtends. Therefore, by the aforesaid corollary, since it passes through the centre, it will also cut the side CD in two, and at right angles. The same demonstration will hold in every equilateral polygon inscribed in a circle, whenever the number of sides is odd, as is evident.

SCHOLIUM.

Since, at the end of Book IV, we handed down a certain construction from Ptolemy by which, in one and the same operation, both an equilateral pentagon and an equilateral decagon are described in a given circle, and postponed its demonstration to this place, it must now be demonstrated by us in a few words. Let then a circle ABC be given, whose centre is D. The diameter BC being drawn, let DA be erected perpendicular to BC. Then, the radius CD being bisected at E, draw the line EA, and cut off EF equal to it, and join the line AF. I say that the line AF is the side of the pentagon, and DF the side of the decagon. For since CD is bisected at E, and DF is added to it, [Margin: p — II.6] the rectangle contained by CF and DF, together with the square on the line DE, will be equal to the square on the line EF, and therefore to the square on the line EA, which is equal to EF. [Margin: q — I.47] But the square on the line EA is equal to the squares on the lines AD and DE. Therefore the rectangle contained by CF and DF, together with the square on the line DE, is equal to the squares on the lines AD and DE; and consequently, the common square on the line DE being taken away, the rectangle contained by CF and DF will be left equal to the square on the line AD, that is, to the square on the line CD. [Margin: r — VI.17] For this reason it will be, as CF is to CD, so CD to DF; and therefore the line CF will be divided at D in extreme and mean ratio. Since, then, the greater segment CD is the side of the hexagon of the circle ABC, by the corollary to Prop. 15 of Book IV, the lesser segment DF will be the side of the decagon of the same circle, as we demonstrated at Prop. 9 of this book. [Margin: s — "X.10" as printed; recte XIII.10 (Prop. 10 of this book)] Again, since the square on the side of the pentagon of the same circle is equal to the square on the side of the hexagon AD together with the square on the side of the decagon DF; [Margin: t — I.47] and since the square on the line AF is equal to the squares on the lines AD and DF; the square

on the side of the pentagon will be equal to the square on the line AF; and therefore the line AF is equal to the side of the pentagon. Which is what was proposed.

Diagram. A circle with centre D and horizontal diameter B (left)–D–C (right). Point A is at the top of the circle; DA is the vertical radius, perpendicular to BC. On the diameter, between D and C, E is the midpoint of DC; F lies to the left of D (with EF = EA cut off). From A a fan of lines is drawn down to B, F, D, E and C. AF is the side of the pentagon; DF the side of the decagon.

- A (top, on circle): apex
- B (left end of diameter)
- C (right end of diameter)
- D (centre)
- E (between D and C): midpoint of the radius DC
- F (between B and D): point with EF = EA

THEOR. 11. PROPOS. 11.

[Margin: xij]

If an equilateral pentagon be inscribed in a circle having a rational diameter, the side of the pentagon is the irrational line called minor.

In the circle ABCDE, having a rational diameter, let the equilateral pentagon ABCDE be inscribed. I say that its side AB is the irrational line called minor. For draw, through the centre F, the diameters AG and BH; let AG cut the side CD at I; and join the lines AC and AH, of which AC cuts the diameter BH at K. From the radius FH cut off also its fourth part FL; likewise from the line AC its fourth part CM. Now since the diameter BH is set down as rational, FL and BF too will be commensurable with it — being rational aliquot parts of it; [Margin: a — X.16] and therefore the whole line BL, compounded of them, since it is commensurable with each of FL and BF, will be rational.

Again, since the line FB, drawn from the centre, cuts the arc AC in two at B, it will also cut the line AC in two at L, and at right angles, by Corollary 1 of the preceding proposition. And the line AG cuts the line CD at I in two, and at right angles, by Corollary 2 of the same proposition. Therefore the triangles ACI and AFK, since they have the angles AIC and AKF right angles and the angle CAI common, [Margin: b — I.32] will be equiangular with one another. [Margin: c — VI.4 ?] (printed digit reads "3") For this reason it will be, as CI is to CA, so FK to FA — or to FH, which is equal to FA. [Margin: d — V.15] But as CA is to FH, so is

CM, the fourth part of CA, to FL, the fourth part of FH. Therefore it will be, as CI is to FK, so CM to FL; and, alternately, as CI is to CM — that is, as CD, the double of CI, is to CK, the double of CM — so FK to FL; and, by composition, as CD and CK together are to CK, so FK and FL together — namely the line KL — to FL. [Margin: e — VI.22] And therefore it will be, as the square on the line compounded of CD and CK is to the square on the line CK, so the square on the line KL to the square on the line FL. Now since, if AC be cut in extreme and mean ratio (the line BD being drawn), [Margin: f — "III.8" as printed; recte XIII.8] its greater segment is equal to the side of the pentagon, namely to CD; the square on the line compounded of CD, the greater segment, and [Margin: g — "III.1" as printed; recte XIII.1] CK, the half of the whole, will be five times the square on the line CK, the half namely of the whole. Therefore the square on the line KL too will be five times the square on the line FL, [Margin: h — X.6] and hence the square on the line KL will be commensurable with the square on the line FL; and consequently the lines KL and FL too will be commensurable, at least in square. But FL was shown to be rational. Therefore KL too will be rational.

Next, since FL is one part of the four of which BF is four, and therefore FL is one part of the five of which BL is five, the square on the line BL will be twenty-five of the parts of which the square on the line FL is one (as is plain in these numbers, 1, 5, 25; [Margin: i — VI.20] for squares have the ratio of their sides doubled). But the square on the line KL was shown to be five of the parts of which the square on the line FL is one. Therefore, of the parts of which the square on the line BL is twenty-five, the square on the line KL is five; and consequently the square on the line BL will be five times the square on the line KL. Since, then, the squares on the lines BL and KL do not have to one another the ratio which a square number has to a square number (as is plain from the corollary to Prop. 24 of Book VIII, for their ratio is that of 25 to 5, or 5 to 1), [Margin: k — X.9] the lines BL and KL will be incommensurable in length; and so, since they have been shown to be rational, they will be rational commensurable in square only. Therefore, if from the rational line BL there be taken the rational line KL, commensurable in square only, [Margin: l — X.74] the remainder BK will be irrational — the irrational called an apotome — and KL will be congruent with it.

Book XIII (continued)

Proposition 11 (continued) — the side of the equilateral pentagon inscribed in a circle with rational diameter is the *minor*

Now the straight line BL is capable of more than the straight line KL by the square on N. Since, therefore, the square on BL equals the sum of the squares on KL and N, and since the square on BL was 5 such parts while the square on KL was 1 such part, the remaining square on N will be 4 of those parts of which the square on BL is 5. Consequently the squares on BL and N will have to one another the ratio of a number to a number, and so they will be commensurable [Margin: (m) X.6]; and therefore the lines BL and N are themselves commensurable — at least in square. But BL was shown to be rational. N, then, will also be rational.

Now the squares on BL and N do not have to one another the ratio of a square number to a square number — as is clear from the corollary to Proposition 24 of Book VIII, since their ratio is that of 5 to 4. It follows that the lines BL and N are incommensurable in length [Margin: (n) X.9], and hence rational and commensurable in square only. Therefore, since the whole line BL is commensurable in length with the rational line BH (for BH is 8 such parts of which BL is 5), and is capable of more than its commensurable KL by the square on a line, N, incommensurable with BL in length (as was shown), BK will be a fourth apotome, by the definition.

Finally, since the straight line AB is a mean proportional between BH and BK — by the corollary to Proposition 8 of Book VI, because the triangle ABH has the angle BAH right [Margin: (o) III.31], from which AK is drawn perpendicular to the base — the square on AB will equal the rectangle contained by BH and BK [Margin: (p) VI.17]. And so, since the straight line AB is capable of the area contained by the rational line BH and the fourth apotome BK, AB will be a *minor* line [Margin: (q) X.95], and so on. If, therefore, in a circle having a rational diameter, and so forth. Which was to be demonstrated.

[Margin: xij [?] (outer, vulgate proposition number)]

THEOR. 12. PROPOS. 12.

If an equilateral triangle be inscribed in a circle, the side of the triangle is in square triple the line drawn from the center of the circle [i.e. the radius].

Let an equilateral triangle ABC be inscribed in the circle ABC whose center is D. I say that the square on the side AB is triple the square on the radius.

Diagram. A circle with center D and an inscribed equilateral triangle ABC; the vertical diameter runs from the top vertex A through the center D and the foot F down to the point E at the bottom of the circle. Radii and connecting lines are drawn.

- A (top): upper vertex of the triangle
- B (lower left): left base vertex
- C (lower right): right base vertex
- D (center): center of the circle
- F (on BC, below D): foot where the diameter meets the base BC
- E (bottom): lower end of the diameter, diametrically opposite A

Draw the diameter AE, which — by the corollary to Proposition 10 of this Book — bisects the arc BC at E and also bisects the straight line BC at right angles in the point F. Since the arc BC is a third part of the whole circumference, the arc BE will be a sixth part of the same circumference; and therefore the chord BE, once joined, will be the side of a hexagon and equal to the radius of the circle, by the corollary to Proposition 15 of Book IV.

This can also be shown as follows. The semicircle ABE contains three sixth-parts of the whole circle; but the arc AB, being a third part of the whole circle, comprehends two sixth-parts of it. Therefore the arc BE is a sixth part of the same circle, and accordingly the chord BE is the side of a hexagon and equal to the radius DE, by the corollary to Proposition 15 of Book IV.

Since, then, the square on AE equals the sum of the squares on AB and BE [Margin: (a) I.47] — because the angle ABE, being in a semicircle, is right [Margin: (b) III.31] — and since the square on AE is quadruple the square on BE (by the scholium to Proposition 4 of Book II, seeing that the line AE is double the line BE), the squares on AB and BE together will also be quadruple the square on that same line BE. And therefore, of whatever 4 parts the squares on AB and BE together consist, the square on BE will be 1 such part; and accordingly the square on AB will be 3 such parts. Thus the square on AB is triple the square on BE, which is equal to the radius. If, therefore, an equilateral triangle be inscribed in a circle, and so forth. Which was to be shown.

COROLLARIVM I.

It follows from this that the diameter of the circle is in square sesquitertiate [4:3] to the side of the equilateral triangle inscribed in that circle. For since the side AB was shown to be in square triple the radius AD, then, setting the square on AB at 3, the square on AD will be 1; and so the square on AE — which, by the scholium to Proposition 4 of Book II, is quadruple the square on AD — will be 4. Thus the square on AE is sesquitertiate to the square on AB, that is, it has the ratio of 4 to 3.

Diagram. The same construction repeated for the corollary: a circle with center D and inscribed equilateral triangle ABC, with the vertical diameter through A, D, F, and E.

- A (top): upper vertex
- B (lower left): left base vertex
- C (lower right): right base vertex
- D (center): center of the circle
- F (on BC): foot of the perpendicular from A on BC
- E (bottom): lower end of the diameter

And since, by the corollary to Proposition 8 of Book VI, as AE is to AB, so is AB to AF, the perpendicular AF of the equilateral triangle AB — dropped from one angle to the base — will likewise be in square sesquitertiate [to the whole]; this, however, will be demonstrated otherwise by Euclid in Book XIV, Proposition 12.

COROLLARIVM II.

Hence it can also easily be gathered that the radius DE is bisected at F by the side BC of the triangle [Margin: (c) I.47]. For since the square on AB is triple the square on BE, if the square on AB be set at 12 parts, the square on BE will be 4 such parts. But the square on BE equals the sum of the squares on BF and FE.

Therefore the squares on BF and FE will also be 4 such parts. But — because the square on AB is quadruple the square on BF, by the scholium to Proposition 4 of Book II (for AB is double BF) — the square on BF is 3 of those parts. Hence the square on FE will be 1 such part; and accordingly the square on BE, that is, the square on DE (which equals it), will be quadruple the square on FE. Therefore, by the scholium to Proposition 4 of Book II, the line DE will be double FE. And so DE is bisected at F.

SCHOLIUM.

This corollary will nevertheless be demonstrated otherwise by the following theorem.

If an equilateral triangle be inscribed in a circle, the diameter drawn from one angle will bisect both the angle and the opposite side, and moreover at right angles; and the radius in turn will be bisected, and at right angles, by the opposite side.

Diagram. A circle with center D and inscribed equilateral triangle ABC; the vertical diameter AE passes through A, the center D, the foot F on BC, and E at the bottom. The chords BD, CD, BE, and CE are drawn as well.

- A (top): upper vertex
- B (left): left base vertex
- C (right): right base vertex
- D (center): center of the circle
- F (on BC, below D): intersection of the diameter with the base BC
- E (bottom): lower end of the diameter

Let an equilateral triangle ABC be inscribed in the circle ABC whose center is D. I say that the straight line AE drawn through the center D — that is, the diameter — bisects both the angle BAC and the side BC, and that at right angles; and that the radius DE is bisected by the side BC, and at right angles.

For, when the straight lines BD, CD, and BE have been drawn: since the sides AB and AD are equal to the sides AC and AD of the triangle CAD, and the bases BD and CD are equal as well, the angles BAD and CAD will be equal [Margin: (d) I.8]. Which is the first point.

Next, since the sides AB and AF of the triangle BAF are equal to the sides AC and AF of the triangle CAF, and the angles contained by them have been shown equal, the bases BF and CF, and the angle at F, will be equal [Margin: (e) I.4]; and therefore the angles at F are right. Which is the second point.

Finally, since the angles BAE and CAE have been demonstrated to be equal, the arcs BE and CE will also be equal [Margin: (f) III.26]. And so, since the arc BEC is a third part of the circumference, BE will be a sixth part of it; and accordingly the chord BE is the side of a hexagon, and thus equal to the radius BD, by the corollary to Proposition 15 of Book IV. Therefore the angles BDE and BED will be equal [Margin: (g) I.5]. Since, then, the angles BDF and BFD of the triangle BDF are

equal to the angles BEF and BFE of the triangle EBF, and the side BF is common, the sides DF and EF will also be equal [Margin: (h) I.26]. Which is the third point.

PROBL. 1. PROPOS. 13.

[Margin: xiiij (outer, vulgate proposition number)]

To construct a pyramid [regular tetrahedron], to comprehend it in a given sphere, and to demonstrate that the diameter of the sphere is in square sesquialterate [3:2] to the side of the pyramid itself.

Let AB be the diameter of the given sphere. About it describe the semicircle ABC, and from AB cut off a third part BD, so that AD is double DB, while AB is triple that same DB and sesquialterate [3:2] to AD. Then draw DC perpendicular to AB, and join the straight lines AC and BC. With the interval of the straight line HE, equal to DC itself, describe a circle EFG whose center is H, and in it inscribe an equilateral triangle EFG; and join the straight lines HF and HG, which will each be equal to DC, since they are equal to HE, at whose interval the circle was described. Next, erect from H, perpendicular to the plane of the circle, the line HI, which is set equal to the straight line AD; and from I let the straight lines IE, IF, and IG be drawn down. I say that the solid contained by the four triangles FEG, IEF, IFG, and IEG is a pyramid, or tetrahedron.

Diagram. Left-hand figure: a semicircle on the diameter AB with apex C. Along the diameter, from left to right, lie A, then L (the midpoint / sphere-center), then D, then B; the perpendicular CD rises from D to the arc at C, and the chords AC and CB are drawn.

- A (left): left end of the diameter
- L (between A and D): midpoint of AB (later identified as the center of the sphere)
- D (between L and B): point dividing AB so that AD is double DB
- B (right): right end of the diameter
- C (apex, on the arc): foot of the perpendicular DC produced to the semicircle

Diagram. Right-hand figure: a circle carrying the plane projection of the tetrahedron. E is at the top, I at the upper right, H at the center; F is at the lower left and G at the lower right (the base triangle EFG with center H); K is at the bottom of the circle. Straight lines join I to E, F, and G, and E, F, G to one another and to H.

- E (top): a base vertex of the tetrahedron
- I (upper right): the apex of the tetrahedron
- H (center): center of the base circle
- F (lower left): a base vertex
- G (lower right): a base vertex
- K (bottom): lower end of the axis IK on the circle

For since the sides DA and DC of the triangle ADC are equal to the sides HI and HE of the triangle IHE, and the angles contained by them are right, by the hypothesis and by Definition 3 of Book XI, the base AC will be equal to the base IE [Margin: (a) I.4]. In the same way we shall show that this same straight line AC is equal to the lines IF and IG.

Again, since the three lines AD, DC, and DB are proportional, by the corollary to Proposition 8 of Book VI, it will follow — by the corollary to Proposition 20 of Book VI — that as AD is to DB, so is the square on AD to the square on DC; and *componendo*, as AB is to DB, so are the squares on AD and DC together to the square on DC. But the squares on AD and DC together are equal to the square on AC [Margin: (b) I.47]. Therefore it will also hold that as AB is to DB, so is the square on AC to the square on DC. And hence, since AB is triple DB, the square on AC will be triple the square on HE (since HE is equal to DC). But the square on EF is likewise triple the same square on HE [Margin: (c) XIII.12]. The square on AC, therefore, will be equal to the square on EF, and consequently the line AC to the line EF.

And so, since the line AC has also been shown equal to the lines IE, IF, and IG, the four triangles EFG, EFI, FGI, and GEI will be equilateral and equal to one another. Therefore the pyramid, or tetrahedron, is now constructed, its base being EFG and its vertex I. I say that this pyramid is comprehended in the given sphere, whose diameter is AB, and that the diameter of the sphere AB is in square sesquialterate to the side EF, or AC.

Let the perpendicular IH be extended as far as K, so that HK is equal to DB and the whole IK equal to the whole AB. Since, then, the three lines AD, DC, and DB are proportional, and the three lines IH, HE, and HK are respectively equal to them, IH, HE, and HK will also be proportional. And so, since HE is perpendicular to IK and is a mean proportional between IH and HK, the semicircle described about IK in the plane of the lines IK and HE will pass through E — as will presently be shown. By the same reasoning, the semicircle described about the same line IK in the plane of the lines IK and HF will pass through F, and the

semicircle described about the same IK in the plane of the lines IK and HG will pass through G. Therefore any one of these semicircles, when carried around

Diagram. The Proposition-13 pair repeated at the head of the page. Left: the semicircle on AB with apex C, the points A, L, D, B along the diameter and the perpendicular CD. Right: the circle with the tetrahedron projection — E (top), I (upper right), H (center), F (lower left), G (lower right), K (bottom).

- Left figure — A (left), L (midpoint), D, B (right); C (apex)
- Right figure — E (top), I (upper right), H (center), F (lower left), G (lower right), K (bottom)

— carried around, the diameter remaining fixed, will describe a sphere that comprehends the constructed tetrahedron, since it passes through all its angles E, F, G, and I. Since, then, this described sphere is equal to the given sphere — because the diameter IK is set equal to the diameter AB — the same tetrahedron will be comprehended by the given sphere.

Now, since the three straight lines AB, AC, and AD are proportional, by the corollary to Proposition 8 of Book VI, it will follow — by the corollary to Proposition 20 of Book VI — that as AB is to AD, so is the square on AB to the square on AC. But the straight line AB is sesquialterate to the straight line AD, for of whatever 3 parts AB consists, AD is 2. Therefore the square on AB will also be sesquialterate to the square on AC; and accordingly, since AB is the diameter of the sphere and AC is equal to the side EF of the pyramid, it is clear that the diameter of the sphere is in square sesquialterate to the side of the tetrahedron, or pyramid. We have therefore constructed a pyramid and comprehended it in the given sphere, and so on. Which was to be done.

LEMMA.

That the semicircle described about IK, in the plane of the lines IK and HE, passes through E will easily be demonstrated by the following theorem.

If a perpendicular be drawn to a straight line such that it is a mean proportional between the segments of the line, the semicircle described about that straight line will pass through the extreme point of the perpendicular line.

Diagram. A semicircle on the diameter, with B, C, and A along it (B and C at the left, A at the right); the perpendicular at C rises to D on the arc. Two dotted arcs

are drawn to represent a semicircle passing short of, or beyond, D, meeting the perpendicular at the hypothetical point E (marked once above D and once below).

- B (left): one end of the diameter
- C (on the diameter, near B): foot of the perpendicular
- A (right): other end of the diameter
- D (on the true arc, above C): extreme point of the perpendicular
- E (upper and lower, dotted): hypothetical point where a wrongly-drawn semicircle would meet the perpendicular

Let CD be perpendicular to AB and a mean proportional between AC and CB. I say that the semicircle described about AB passes through D. For if it should pass short of D, or beyond it — namely through the point E — then CE too would be a mean proportional between AC and CB, by the corollary to Proposition 13 of Book VI. Hence both the square on CD and the square on CE would be equal to the rectangle contained by AC and CB [Margin: (d) VI.17]; and accordingly the squares on CD and CE would be equal to one another, and the very lines CD and CE would be equal — the part and the whole. Which is absurd. Therefore the semicircle passes through D.

Hence it is manifest that the semicircle described in the upper figure about IK, in the plane of the lines IK and HE, passes through E, since HE is perpendicular to IK and a mean proportional between IH and HK.

COROLLARIVM I.

Hence we shall easily gather that the diameter of the sphere is in square quadruple-and-sesquialterate [9:2] to the radius of the circle described about the base of the pyramid. For since the diameter of the sphere AB was shown to be in square sesquialterate to the side EF of the pyramid, the square on EF will be 6 such parts of which 9 is the square on AB. But of whatever 6 parts the square on EF consists, 2 is the square on HE [Margin: (e) XIII.12], because the square on EF is triple the square on HE. Therefore, of whatever 9 parts the square on AB consists, the square on HE will be 2 such parts; and accordingly the diameter of the sphere AB is in square quadruple-and-sesquialterate to the radius HE, since the ratio of the squares is that of 9 to 2.

Diagram. The Proposition-13 pair repeated once more for the corollaries. Left: the semicircle on AB with A, L, D, B along the diameter and apex C. Right: the circle with the tetrahedron projection — E (top), I (upper right), H (center), F (lower left), G (lower right), K (bottom).

- Left figure — A (left), L (midpoint), D, B (right); C (apex)
- Right figure — E (top), I (upper right), H (center), F (lower left), G (lower right), K (bottom)

COROLLARIUM II.

Again, the perpendicular dropped from the center of the sphere to the plane of the base of the pyramid will be a sixth part of the diameter of the sphere and a third part of the radius. For let L be the center of the semicircle ABC; then L will also be the center of the sphere. I say that the straight line LD — which is indeed drawn perpendicular from the center of the sphere to the base EFG, since AD is equal to IH and therefore the point D is the same as H, just as L is the same as the center of the sphere — is a sixth part of the diameter of the sphere, namely of the line AB, and a third part of the radius AL. For since AD is double DB: of whatever 4 parts AD consists, DB will be 2, and accordingly the whole AB will be 6 such parts, and the radius AL will be 3. And so, if AD is 4 and AL is 3, LD will be 1; and therefore LD will be a sixth part of the line AB, which was 6 of those same parts, and a third part of the line AL, which was 3 of those same parts.

Likewise, the altitude of the pyramid HI will contain two-thirds of the diameter itself. For HI is the same as AD, which contains two-thirds of the diameter, by the construction.

Again, if the square on the diameter be set at 9 parts, the square on the altitude of the pyramid will be 4 such parts. For the ratio of 9 to 4 is the duplicate of the ratio of 3 to 2, as here appears: 9, 6, 4.

[Scholium to Proposition 13, continued]

If from some material four equal equilateral triangles are made and arranged as this figure shows, then when they are properly folded together a tetrahedron is easily built up from them in its full solidity. This procedure — together with the four that follow, by which the remaining four regular solids are fashioned in any tangible material — we have taken from Albrecht Dürer [Alberto Durero], no undistinguished writer.

Diagram. The net (unfolding) of the tetrahedron: a strip of four equilateral triangles laid edge to edge in a band (alternately pointing up and down), which folds up into the solid. Uncaptioned; no letter-labels.

PROBL. 2. PROPOS. 14.

[Margin: xiv.]

To construct an octahedron and to comprehend it in a sphere (as was done for the pyramid too), and to demonstrate that the square on the sphere's diameter is double the square on the octahedron's own edge.

Let AB be the diameter of the sphere that comprehends the pyramid already constructed, and about it let the semicircle ACB be described; from the center D let the perpendicular DC be drawn to AB, and let the straight lines AC, BC be joined [a], which will be equal to one another. [Margin (a): Elements I.4] Now, taking a straight line EF equal to AC or BC, let the square EFGH be described upon it, and in it let the diagonals EG, FH be drawn, cutting one another at I; then the straight lines IE, IF, IG, IH will be equal, since they are radii of the circle described about the square, as was demonstrated in Book IV. Next, from I let a perpendicular KL be erected to the plane of the square on both sides, and let IK, IL be set equal to IE; and finally, from K and L let the straight lines KE, KF, KG, KH, LF, LE, LH, LG be drawn to the corners of the square. I say that the solid contained by the eight triangles KEF, KFG, KGH, KEH; LFE, LEH, LHG, LFG is the octahedron that is sought.

For the sides EI, IK of triangle EIK are equal to the sides EI, IH of triangle EIH, and the angles they contain are right — angle EIK by definition 3 of Book XI, and angle EIG [b] because it is equal to the successive angle GIH, on account of the equality of the sides EI, IH, GI, IH and of the bases EH, GH. [Margin (b): Elements I.8] Therefore [c] the base EK will be equal to the base EH; and in the same way KH will be equal to that same EH; and so triangle KEH will be equilateral. [Margin (c): Elements I.4] By the same reasoning we shall conclude that the remaining triangles are equilateral, and hence equal to triangle KEH, since their sides are equal, one by one, to the sides of the square EFGH. So now an octahedron has been constructed from eight equilateral and equal triangles. I say that this octahedron is comprehended by the sphere of diameter AB — the very sphere by which the constructed pyramid too is comprehended — and that the square on the sphere's diameter AB is double the square on the edge EF, or AC.

For since IE is perpendicular to KL (by definition 3 of Book XI) and is a mean proportional between the segments KI, IL — because the three straight lines KI, IE, IL are equal and therefore have the same ratio — the semicircle described about KL in the plane of the straight lines KL, IE will pass through the point E, by the lemma of the preceding proposition. By the same reasoning the semicircle described about the same KL in the plane of the straight lines KL, IF will pass through the point F; and the same must be said of the points G, H. Therefore each of these semicircles, carried around while the diameter KL remains fixed, will

describe a sphere that comprehends the constructed octahedron, since it passes through all of its vertices. Now this sphere is equal to the given sphere whose diameter is AB. (For since the straight lines AC, EF are equal, their squares are equal; and since the square on AC is double the square on AD, just as the square on FE is double the square on EI, the straight lines AD and EI — that is, KI — the radii, will be equal. Therefore the whole diameters AB, KL will also be equal, and consequently their spheres will be equal.) Hence the same octahedron will be comprehended by the given sphere.

And since the square on AB [d] is equal to the squares on AC, BC, which are equal, the square on AB will be double the square on AC. [Margin (d): Elements I.47] Therefore, since AB is the diameter of the sphere and AC is a straight line equal to the octahedron's edge EF, it is clear that the square on the sphere's diameter is double the squares on the octahedron's edges. And so we have constructed the octahedron and comprehended it in a sphere, as we did the pyramid, and so on. Which was to be done.

Diagram. The defining semicircle for the octahedron construction: a semicircle standing on the diameter, with the right-triangle construction inscribed. Uncaptioned.

- A (base, left endpoint of the diameter)
- D (base, foot of the perpendicular / center)
- B (base, right endpoint of the diameter)
- C (apex of the semicircle, joined to A and B; CD is the perpendicular to the base)

Diagram. The octahedron drawn as a solid (hexagonal outline with internal diagonals), showing its six vertices and center.

- K (top apex)
- E (upper-left square vertex)
- H (upper-right square vertex)
- I (center, where the three diameters meet)
- F (lower-left square vertex)
- G (lower-right square vertex)
- L (bottom apex)

COROLLARIVM. I.

From what has been said it is clear that in the octahedron three diameters cut one another at right angles at the center of the sphere: namely KL, EG, FH, cutting one another at I. For all the angles at I have been shown to be right. And accordingly, since KI is at right angles to the lines EG, HE [e], it will also be at right angles to the plane EFGH drawn through them; [Margin (e): Elements XI.4] and therefore [f] the planes KFLH, ELGK, drawn through KI, will be at right angles to the plane EFGH. [Margin (f): Elements XI.8] Indeed, by the same reasoning — to sum the matter up briefly — the three planes EFGH, KFLH, ELGK will cut one another at right angles. These planes are squares, since all their sides are edges of the octahedron itself and therefore are equal to one another and contain right angles; for (since the square on the sphere's diameter is double the square on the octahedron's edge, as we have shown) the square on the diameter HF turns out equal to the squares on the octahedron's edges Fk, kH. [g] For thus it comes about that the angle FkH is right. [Margin (g): Elements I.48] And the same reasoning holds for the remaining angles.

COROLLARIVM. II.

The octahedron is divided into two similar and equal pyramids whose common base is the square EFGH, whose apexes are k and L, and whose altitudes, finally, are the radii Ik, IL: such are the pyramids EFGH-k and EFGHL. For the faces of the one are similar and equal to the faces of the other, as is evident, since they are equilateral and equal triangles, apart from the common base, which is a square equal to half the square on the sphere's diameter, as has been demonstrated.

COROLLARIVM. III.

If a tetrahedron and an octahedron are described in the same sphere, the square on the tetrahedron's edge will be sesquitertian (4:3) to the square on the octahedron's edge. For let the square on the sphere's diameter be divided into six parts. [h] Since, then, the sphere's diameter is in square sesquialter (3:2) to the tetrahedron's edge, the square on the tetrahedron's edge will be 4 of such parts as the square on the sphere's diameter is 6. [Margin (h): Elements XIII.13] Again, since the sphere's diameter is in square double the octahedron's edge, [i] the square on the octahedron's edge will be 3 of such parts as the square on the sphere's diameter is 6. [Margin (i): Elements XIII.14] Therefore, since the square on the tetrahedron's edge is 4 such parts of which the square on the octahedron's edge is 3, it is clear that the tetrahedron's edge is in square sesquitertian (4:3) to the octahedron's edge.

COROLLARIVM. IV.

Finally, since in the square EFGH the straight line EH is parallel to FG, and in the square ELGk the straight line Ek is parallel to GL, [k] the plane EHk drawn through EH, Ek will be parallel to the plane FGL drawn through FG, GL. [Margin (k): Elements XI.15] And since the same reasoning holds for the other faces of the octahedron, it follows that the opposite faces of the octahedron are parallel to one another.

SCHOLIVM.

If from some material eight mutually equal equilateral triangles are made and arranged as this figure shows, then when they are properly folded together an octahedron is easily built up from them in its full solidity, as Albrecht Dürer [Albertus Durerus] says.

Diagram. The net (unfolding) of the octahedron: eight equilateral triangles arranged in a folding pattern (a row of triangles with additional triangles above and below), which folds up into the solid. Uncaptioned; no letter-labels.

PROBL. 3. PROPOS. 15.

[Margin: xiiii.]

To construct a cube and to comprehend it in a sphere (as was done for the previous figures too), and to demonstrate that the square on the sphere's diameter is triple the square on the cube's own edge.

Let AB be the diameter of the sphere that comprehends the aforesaid figures, and about it let the semicircle ACB be described; and from AB let the third part BD be cut off, so that AB is triple BD. Then, DC having been drawn perpendicular to AB, let the straight lines AC, BC be joined. Upon a straight line EF equal to BC let the square EFGH be constructed, and upon it let the perpendiculars EI, Fk, GL, HM be erected, equal likewise to BC, whose ends are joined by the straight lines Ik, kL, LM, MI. Now since the straight lines EI, Fk are at right angles to the plane EFGH, [a] the straight lines EI, Fk will be parallel; and they are also equal, since each has been set equal to BC, that is, to EF. [Margin (a): Elements XI.6] Therefore [b] EF and Ik are parallel and equal; and so EFkI is a parallelogram, in which, since the lines EI, Ik, kF are equal to EF, all four lines turn out equal. [Margin (b): Elements I.33] Moreover [c] all four angles are right, since Elk, FkI are equal to the opposite right angles kFE, IEF. [Margin (c): Elements I.34] Therefore EFkI is a square. By the same reasoning FkLG, GLMH, HMIE will be squares; and hence IkLM too will

be a square, [d] since it is similar and equal to the opposite square EFGH. [Margin (d): Elements XI.24] For EL is a solid parallelepiped, [e] because its opposite planes are parallel, being drawn through mutually tangent parallel lines. [Margin (e): Elements XI.15] Therefore EL will be a cube. I say that it is comprehended by the sphere of diameter AB. For let the diameters Ek, FI, GM, HL of the opposite planes EFkI, GHML be drawn, and through them let the planes EkLH, FIMG be drawn. [f] Now since both the plane EkLH and the plane FIMG bisect the cube, each will pass through the center of the cube — by the scholium to proposition 39 of Book XI — namely through the point P, at which, by the corollary of the same proposition, all the diameters of the cube also bisect one another; [Margin (f): Elements XI.28] and therefore the common section of the planes, namely the straight line NO, will pass through the same point P. And since the planes EkLH, FIMG are rectangular — for since HE is perpendicular to the two straight lines EI, EF, on account of the squares EM, Ek, [g] it will also be perpendicular to the plane Ek, and therefore to the straight line Ek, by definition 3 of Book XI; so the angle HEk is right, and by the same reasoning the remaining angles in the plane EkLH will be right, as also the angles in the plane FIMG; therefore the planes EkLH, FIMG are rectangular — [Margin (g): Elements XI.4] and equal, since the sides of the one are equal to the sides of the other, their diameters EL, Hk, FM, GI will also be equal, as...

Diagram. The defining semicircle for the cube construction: a semicircle on the diameter with the right-triangle construction inscribed, adjoining the cube figure at right. Uncaptioned.

- A (base, left endpoint of the diameter)
- D (base, foot of the perpendicular)
- B (base, right endpoint of the diameter)
- C (apex of the semicircle, joined to A and B) — the printed glyph is foxed and resembles a "G"; read as C from the construction (semicircle ACB, lines AC, BC)
- E (base, to the right of B — the corner where the square/cube EFGH begins)

Diagram. The cube drawn as a solid, with all its face-diagonals and body-diagonals inscribed, showing the base square EFGH, the opposite square IkLM, the center P, and the two face-centers N, O. It is a dense figure; the readable vertex labels are:

- M (top rear corner)
- L (top right corner)

- N (center of the upper face)
- H (upper-left front corner)
- G (upper-right corner)
- P (center of the cube, where the body-diagonals meet)
- I (lower-left corner)
- K (lower-right corner; the "k" of the text)
- O (center of the lower face)
- F (bottom front corner)
- E (bottom-left corner, adjoining the semicircle)

as is evident from the scholium to proposition 24 of Book I. And accordingly the radii PE, PL, PH, PK; PF, PM, PG, PI will be equal. Therefore the semicircle described about EL from the center P, and carried around while the diameter EL remains fixed, will describe a sphere passing through all the vertices of the cube; and that this sphere is equal to the sphere of diameter AB will be demonstrated thus.

Since the square on EK [h] is equal to the equal squares on EF, FK, and therefore double the square on EF — that is, the square on KL — the squares on EK, KL will be triple the square on KL. [Margin (h): Elements I.47] Now [i] the square on EL is equal to the squares on EK, KL, because the angle EKL is right in the right-angled figure EkLH. [Margin (i): Elements I.47] Therefore the square on EL too will be triple the square on kL, that is, the square on BC. But the square on AB is triple that same square on BC. (For the three straight lines AB, BC, BD are proportional, by the corollary of proposition 8 of Book VI; and accordingly, by the corollary of proposition 20 of Book VI, as AB is to BD, so will the square on AB be to the square on BC. Since, then, AB is triple BD, the square on AB will also be triple the square on BC.) Therefore the squares on EL, AB are equal; and consequently the straight lines EL, AB themselves — the diameters, that is, of the spheres — and therefore their spheres too are equal.

And since it has been shown that the square on the diameter EL is triple the square on the cube's edge kL, it is clear that the square on the sphere's diameter is triple the square on the cube's edge. Wherefore we have constructed the cube and comprehended it in a sphere, and so on. Which was to be done.

COROLLARIVM. I.

From the demonstration of this problem it is clear that all the diameters of the cube are equal to one another and bisect one another at the center of the sphere;

and that by the same reasoning the straight lines joining the centers of opposite squares are bisected at the same center. For it has been demonstrated that the diameters of the cube EL, Hk, FM, GI are equal and bisect one another at the center P. That the straight line NO too, joining the centers N and O of the opposite squares GHMI, EFkI, is bisected at the same center P, will be shown as follows. The two angles OFP, OPF of triangle OFP are equal to the two angles NMP, NPM of triangle NMP: ([k] for the angle OFP is equal to the alternate angle NMP, [Margin (k): Elements I.29] between the lines FI, GM [l] which are parallel, being the common sections of the parallel planes EFkI, GHML; [Margin (l): Elements XI.16] and [m] the angle OPF is equal to its vertical angle NPM.) [Margin (m): Elements I.15] And the sides PF, PM adjacent to them are equal, being radii. Therefore the sides PO, PN will also be equal; and consequently the straight line NO will be bisected at P. And the same reasoning holds for the other lines joining the centers of the other opposite squares.

COROLLARIVM. II.

Again, the square on the diameter of the sphere, or of the cube, is equal to the squares on the edges of the tetrahedron and the cube taken together. [n] For the square on the tetrahedron's edge is 6 of such parts as the square on the diameter is 9; [Margin (n): Elements XIII.13] while [o] the square on the cube's edge is 3 of such parts as that same square on the diameter is 9. [Margin (o): Elements XIII.15] Therefore the square on the tetrahedron's edge and the square on the cube's edge together are 9 parts, just as the square on the diameter is. And accordingly the square on the diameter is equal to the squares on the said edges. This is also evident from the semicircle described about the sphere's diameter, in which the tetrahedron's edge is AC and the cube's edge is BC. [p] For it is clear that the diameter of the sphere AB can sustain both the straight line AC and the straight line BC, since the angle ACB is right. [Margin (p): Elements I.47]

SCHOLIVM.

If from some material six mutually equal squares are made and arranged as this figure shows, a cube is easily put together in its full solidity, provided the said squares are properly folded together; for from them there will be produced a certain solid figure containing eight solid angles, like a die of some kind, as is evident.

Diagram. The net (unfolding) of the cube: six squares (drawn in dashed outline) laid out in a cross / T-shaped pattern — a horizontal row of squares with one square above and one below — which folds up into the solid. Uncaptioned; no letter-labels.

PROBL. 4. PROPOS. 16.

[Margin: xvi.]

To construct an icosahedron and to comprehend it in a sphere (as was done for the figures already described too), and to demonstrate that the edge of the icosahedron is the irrational line called *minor*.

Let AB be the diameter of the sphere that comprehends the aforesaid figures, and about it let the semicircle ACB be described; and from AB let the fifth part BD be cut off, so that AB is quintuple BD, and sesquiquartan (5:4) to AD, while AD is quadruple BD. Then, DC having been drawn perpendicular to AB, let the straight lines AC, BC be joined. At the interval of the straight line EF, which is equal to BC, let a circle be described from the center E, [a] in which let the equilateral pentagon FGHlk be inscribed. [Margin (a): Elements IV.11] After this, the arcs FG, GH, HI, Ik, kF having been bisected at the points L, M, N, O, P, let the straight lines FL, LG, GM, MH, and so on, be joined — namely the sides of a decagon. Then from the center E and from the points L, M, N, O, P let the perpendiculars EQ, LR, MS, NT, OV, PX be erected to the plane of the circle FGHlk, set equal to the radius EF, or the straight line BC. And they will all be equal to one another; [b] but they are also parallel. [Margin (b): Elements XI.6] Therefore the straight lines joining them — namely EL, QR; EM, QS; EN, QT; EO, QV; EP, QX (all of which, however, we have not drawn, in order to avoid confusion) — [c] will be equal two by two. [Margin (c): Elements I.33] And accordingly, since EL, EM, and so on, are equal to the radius EF, the lines QR, QS, QT, QV, QX will also be equal, both to one another and to the radius EF, or the straight line BC. [d] Now since the plane drawn through the straight lines QR, QS is parallel to the plane FGHlk drawn through the straight lines EL, EM; [Margin (d): Elements XI.15] and by the same reasoning the plane drawn through the straight lines QS, QT is parallel to the same plane FGHlk drawn through the straight lines EM, EN; and the plane through the straight lines QR, QS agrees with the plane through...

[Book XIII — Proposition 16, continued: the icosahedron comprehended in a sphere]

in the straight line QS. These two planes, therefore, will make a single plane, by what we demonstrated at Proposition 16 of Book XI. In the same way we shall show that the plane through QT, QV makes a single plane with this one; and likewise the plane through QV, QX, and the plane through QX, QR. There are, then, five lines QR, QS, QT, QV, QX in one and the same plane. And accordingly, if

from Q, at the distance QR, a circle is described in that plane, it will pass through the remaining points S, T, V, X, and will be equal to the circle FGHIK.

Let the points R, S, T, V, X be joined by the straight lines RS, ST, TV, VX, XR.
[Margin: e. Elements XI.6] Now since LR, PX are equal and parallel, then — if one imagines the line LP drawn — [Margin: f. Elements I.33] LP, RX will also be equal and parallel. [Margin: g. Elements III.28 (printed "28. tertijdec." = XIII.28, but Book XIII has only 18 propositions and the content is III.28: in equal circles equal chords cut off equal arcs) [recte: III.28]] And so, in equal circles they will cut off equal arcs. But LP cuts off a fifth part of the circle FGHIK, namely the two-tenths FL, FP. Therefore RX too will cut off a fifth part of the circle RSTVX. By exactly the same reasoning we shall conclude that the remaining lines RS, ST, TV, VX cut off fifth parts. For this reason RSTVX is an equilateral pentagon, all of whose sides are equal to the sides of the pentagon FGHIK.

From the angles of the pentagon RSTVX let there be dropped, to the angles of the pentagon FGHIK, the straight lines RF, RG, SG, SH, TH, TI, VI, VK, XK, XF. Now the perpendicular LR is set equal to the radius EF — that is, to the side of the hexagon of circle FGHIK — and LF is the side of the decagon of the same circle; [Margin: h. Elements XIII.10] therefore the square on the side of the pentagon of that same circle will be equal to the squares on the lines LR, LF. [Margin: i. Elements I.47] But the square on the line FR is also equal to those same squares, since the angle FLR is right, by definition 3 of Book XI. The square on FR is therefore equal to the square on the side of the pentagon; and hence the line FR is equal to the side of the pentagon LP, or FG — that is, to RX itself. By the same reasoning the remaining lines RG, SG, SH, and so on, will be equal to the remaining sides of both pentagons.

And therefore the ten triangles RFX, RFG, RGS, SGH, SHT, THI, TIV, VIK, VKX, XKF will be equilateral and equal to one another, since all their sides are equal to the sides of the equilateral pentagon.

After this, let the perpendicular EQ be extended in both directions, and let the lines QY, EZ each be made equal to the side of the decagon of circle FGHIK (or of RSTVX); and let there be joined the lines VQ, VY, XQ, XY, GE, GZ, HE, HZ. Now QX is the radius of circle RSTVX — that is, the side of its hexagon — and QY is the side of the decagon of the same circle; [Margin: k. Elements XIII.10] therefore the square on the side of the pentagon XV will be equal to the squares on the lines QX, QY. [Margin: l. Elements I.47] But the square on the line XY is also equal to those same squares, since the angle XQY is right, by definition 3 of Book XI. (For the planes of the circles FGHIK, RSTVX have been shown to be parallel, and EQ is

at right angles to the plane of the former by construction; so EQ will also be at right angles to the plane of the latter, by the scholium to Proposition 14 of Book XI, and therefore perpendicular to the line QX, in the same plane.) Therefore the square on the line XY is equal to the square on the line XV, and consequently the line XY is equal to XV.

In the same way VY will be shown equal to that same XV; and accordingly the triangle VYX is equilateral. In no other way — if the lines RY, SY, TY are drawn (which, to avoid confusion of lines, we did not mention) — shall we demonstrate that the four triangles RYX, RYS, SYT, TYV are equilateral and equal to the triangle VYX, that is, to the previous ten, since all their sides are equal to the sides of the pentagon. By similar reasoning the triangle GZH will be equilateral (since EG is the side of the hexagon, namely the radius, and EZ the side of the decagon, and the angle GEZ is right, by definition 3 of Book XI); and likewise the four triangles HZI, IZK, KZF, FZG (whose lines, again, to avoid producing confusion, we did not draw), which will all be equal to the previous fifteen, for the same reason. Since, then, all these twenty triangles are equilateral and equal, and each is joined to each by straight lines — namely by their sides — there will be constituted out of them an icosahedron. And I say that it is comprehended by the sphere of diameter AB.

For let the line EQ be bisected at α , and let the lines αF , αX , αV be drawn. The sides αQ , QV of the triangle αQV are equal to the sides αQ , QX of the triangle αQX — because QV, QX are radii of circle RSTVX — and the angles αQV , αQX are also equal, namely right, by definition 3 of Book XI; [Margin: m. Elements I.4] therefore the bases αV , αX will also be equal. And by the same reasoning, if lines are drawn from α and Q to R, S, T, the lines αR , αS , αT will be shown equal both to one another and to αV , αX themselves. Again, the sides αQ , QX of the triangle αQX are equal to the sides αE , EF of the triangle αEF — because QX, EF are radii of equal circles, and EQ has been bisected at α ; and the angles αQX ,

Diagram. An auxiliary semicircle standing on the horizontal diameter AB, used to obtain the mean proportional. The apex C lies on the arc; from C the chords CA and CB are drawn (the angle at C being right, in a semicircle), and CD is dropped perpendicular to the diameter at D, which divides AB (tick-marks on the diameter mark the division). This realizes $AB : BC = BC : BD$.

- A (left end of diameter): terminus of AB
- D (on the diameter, marked by ticks): foot of the perpendicular from C; the point of division
- B (right end of diameter): terminus of AB

- C (apex, on the arc): vertex of the right angle

Diagram. The plane projection of the icosahedron inscribed in its circumscribing circle. Within the great circle two interlocking equilateral pentagons are shown — the upper pentagon FGHIK and the pentagon RSTVX — together with their centers, the two apex ("pole") points Y and Z, the decagon-division points L, M, N, O, P, and the full mesh of chords forming the twenty triangular faces. All labels below are read directly from the plate.

- R (upper left, on the circle): vertex of pentagon RSTVX
- X (upper center, on the circle): vertex of pentagon RSTVX
- F (upper center, inner): vertex of pentagon FGHIK
- P (upper right): decagon-division point
- L (left of F): decagon-division point
- G (left, inner): vertex of pentagon FGHIK
- E (center): center of circle FGHIK
- α (just right of E; printed as the "œ"-like alpha): midpoint of EQ / sphere-center
- Q (right of center): center of circle RSTVX
- Y (right): apex (pole) point
- K (right, on/near circle): vertex of pentagon FGHIK
- Z (lower left of center): apex (pole) point
- M (left, lower): decagon-division point
- S (far left, on the circle): vertex of pentagon RSTVX
- H (lower left, inner): vertex of pentagon FGHIK
- N (lower center): decagon-division point
- I (lower right, inner): vertex of pentagon FGHIK
- V (right, on the circle): vertex of pentagon RSTVX
- O (lower right, small): decagon-division point
- T (bottom, on the circle): vertex of pentagon RSTVX

are equal, namely right, by definition 3 of Book XI. [Margin: n. Elements I.4] The bases αX , αF will also be equal; and in the same way, if lines are drawn from α and E to G, H, I, K, the lines αG , αH , αI , αK will be shown equal to that same αX . And accordingly the ten lines drawn to the ten angles F, G, H, I, K and R, S, T, V, X are equal.

Now QE is the radius — that is, the side of the hexagon of circle FGHIK — and EZ is the side of the decagon of the same circle; [Margin: o. Elements XIII.9] therefore

the line QZ, cut at E, is in extreme and mean ratio, and its greater segment will be QE. Hence the lesser segment ZE, taking to itself αE (the half of the greater segment) — that is, the line αZ — [Margin: p. Elements XIII.3] has as its square five times the square described on αE . But the line αF too has as its square five times that same square on αE . (For the square on the line EF is four times the square on αE , since EF is double αE , by the scholium to Proposition 4 of Book II; so the squares on the lines EF and αE together are five times the square on αE . [Margin: q. Elements I.47] Since, then, the square on the line αF is equal to the squares on the lines EF, αE , the square on the line αF will also be five times the square on αE .) Therefore the squares on the lines αZ , αF are equal, and hence the lines αZ , αF themselves will be equal. Since, then, ZY is bisected at α — because equal lines EZ, QY are added to the equal lines αE , αQ , so that the wholes $Z\alpha$, $Y\alpha$ become equal — all the lines drawn from α to all the angles of the icosahedron will be equal. Wherefore a sphere described about the diameter ZY, from the center α , will pass through all the angles of the constituted icosahedron. This sphere, then, I say is equal to the sphere of diameter AB.

For as αZ is to αE , so is YZ (which is double αZ) to QE (which is double αE); [Margin: s. Elements VI.22] and accordingly, as the square on the line αZ is to the square on the line αE , so is the square on the line YZ to the square on the line QE. But the square on the line αZ has been shown to be five times the square on the line αE ; therefore the square on the line YZ will also be five times the square on the line QE — that is, five times the square on the line BC, because the line QE is set equal to the radius EF, that is, to the line BC. But the square on the line AB is also five times that same square on the line BC. (For since, by the corollary of Proposition 8 of Book VI, as AB is to BC, so is BC to BD, it follows — by the corollary of Proposition 20 of Book VI — that as AB is to BD, so is the square on the line AB to the square on the line BC; and accordingly, AB being five times BD, the square on the line AB will also be five times the square on the line BC.) Therefore the squares on the lines YZ, AB are equal, and hence also the lines YZ, AB themselves; and consequently the spheres about them will be equal. [Margin: r. Elements V.15]

Finally, the diameter AB of the sphere is assumed rational — (for it is by comparison with it that the side of the icosahedron must be shown to be an irrational line, the one called minor) — and AB has, as its square, five times the square on the line BC, that is, on EF, which is equal to that line; therefore the radius EF of circle FGHK will also be rational. (For since the squares on the lines AB, EF have to one another the ratio of a number to a number — namely of 5 to 1, or 10 to 2 — [Margin: t. Elements X.6 (printed numeral read as "8" but the content

— commensurability of magnitudes with a numerical ratio — is X.6; glyph uncertain) [?]] they will be commensurable; and consequently the lines AB, EF will also be commensurable, at least in square. Since, then, AB is rational, EF will also be rational.) Wherefore the whole diameter of circle FGHIK will be rational; and consequently the line FG, the side of the pentagon — that is, the side of the icosahedron — [Margin: u. Elements XIII.11] will be an irrational line, the one called minor. We have therefore constituted an icosahedron, and comprehended it in a sphere — the same by which we comprehended the aforesaid figures — and we have demonstrated that the side of the icosahedron is an irrational line, the one called minor; which was to be done.

COROLLARIVM. I.

From what has been said it is inferred that the diameter of the sphere is, in square, five times the radius of the circle that encloses five sides of the icosahedron — the circle, that is, out of which the icosahedron was constituted, and which passes through five angles of the icosahedron. For it has been shown that the square on the diameter AB is five times the square on the line BC, that is, on the radius EF.

Diagram. A re-impression of the same auxiliary semicircle as on the preceding page (fig-p562-1): semicircle on diameter AB, apex C, foot D dividing the diameter, chords CA, CB and perpendicular CD, giving $AB : BC = BC : BD$.

- A (left end of diameter): terminus of AB
- D (on the diameter): foot of the perpendicular; point of division
- B (right end of diameter): terminus of AB
- C (apex, on the arc): vertex of the right angle

Diagram. A re-impression of the same icosahedron projection as on the preceding page (fig-p562-2): the great circle enclosing the two interlocking pentagons FGHIK and RSTVX, with center E, second center Q, sphere-center α , apex points Y and Z, decagon points L, M, N, O, P, and the full chord mesh. Labels as on fig-p562-2 (R, X, F, P, L, G, E, α , Q, Y, K, Z, M, S, H, N, I, V, O, T).

COROLLARIVM. II.

Likewise it is manifest that the diameter of the sphere is composed of the side of the hexagon — that is, of the radius — together with two sides of the decagon of the same circle. For YZ, the diameter of the sphere, is composed of EQ, the side of the hexagon, and of QY, EZ, the sides of the decagon of circle FGHIK.

COROLLARIVM. IIL [recte: III]

Finally, it is established that opposite sides of the icosahedron — such as RX, HI — are parallel. For the line RX has been shown to be parallel to the line that would be drawn from L to P. Since, then, the line HI is also parallel to that same line LP — as we demonstrated in the scholium to Proposition 27 of Book III (because the arcs HL, IP are equal, each containing three-tenths of the whole circumference) — it is clear that the lines RX, HI are parallel. And the same reasoning holds for the other opposite sides.

SCHOLIVM.

If out of some material twenty equilateral triangles, equal to one another, are made and arranged as this figure indicates, an icosahedron will be composed in its full solidity, provided they are properly fitted together.

Diagram. The net (unfolded development) of the icosahedron: a band of twenty equal equilateral triangles, arranged in alternating rows (points up / points down) so that, folded, they form the twenty faces of the solid. No letter-labels appear on the figure.

[Margin: xvii.]

PROBL. 5. PROPOS. 17.

To construct a dodecahedron; to comprehend it in a sphere — the same by which the aforesaid figures were comprehended; and to demonstrate that the side of the dodecahedron is an irrational line, the one called apotome [apotome].

Let one base of a cube comprehended in the given sphere be the square ABCD, and let another equal square ADEF stand upon it at right angles, so that their common section is the line AD. (For all the bases of a cube stand at right angles to one another. For in what preceded, in order that the cube might be constructed, there were erected from the angles of the base EFGH four lines EI, FK, GL, HM, perpendicular to the base EFGH itself — as can easily be seen from the figure of Proposition 15 of this book. [Margin: a. Elements XI.18] Wherefore the bases of the cube drawn through them will also be at right angles to the same base EFGH.)

Accordingly, all the sides having been bisected at the points G, H, I, K, L, M, N, let these be connected by lines cutting one another at O, P, the centers of the squares — as is established from the demonstration of the eighth problem of Book IV. Let OK, PL, PN be cut in extreme and mean ratio at the points Q, R, S; and from these

let there be raised, outside the cube, perpendicular to the planes of the squares, the lines QT, RV, SX, equal to the greater segments OQ, PR, PS. Then, the lines AT, AX, DT, DV, VX having been joined, I say that ATDVX is one of the twelve equilateral and equiangular pentagons of the dodecahedron to be constructed.

That the pentagon ATDVX lies in one plane will be shown thus. Since RV, SX are at right angles to the square ADEF, [Margin: b. Elements XI.6] they will be parallel; and accordingly, since they are equal — because the greater segments PR, PS, to which they are set equal, are equal, since the cut lines PL, PN are also equal — [Margin: c. Elements I.33] the connecting lines RS, VX will also be equal and parallel. [Margin: d. Elements I.33] But AD is also parallel to that same RS, or LN, because AN, DL are parallel and equal. [Margin: e. Elements XI.9] Therefore AD, VX are also parallel; [Margin: f. Elements XI.7] and consequently the lines joining them, AX, DV, will be in the same plane with them; and therefore

Diagram. A large perspective construction figure for Proposition 17 (the dodecahedron): a cube shown with its square faces (base ABCD and the upright square ADEF, with the further face EFGH), the edge-midpoints and square-centers, and the "roof" points T, V, X raised outside the cube on perpendiculars, generating the pentagon ATDVX and its companions. Besides the Roman-letter vertices, the plate carries a set of small Greek-letter and numeral labels marking the analogous construction on the other faces (added, as Clavius says, "to avoid confusion"); the page is heavily foxed and these small marks sit near the resolution floor. Roman-letter labels read with confidence are listed first; the Greek/numeral marks are flagged. (more than two labels uncertain).

- F (top left): angle of the cube face
- M (top center): edge-midpoint
- E (top right): angle of the cube face
- X (upper, left of center): raised "roof" vertex (SX perpendicular)
- Y (upper center): [?]
- V (upper, right of center): raised "roof" vertex (RV perpendicular)
- N (left, on the mid-line): edge-midpoint
- S (mid-line): extreme-and-mean division point (on PN)
- P (mid-line, center): center of a square face
- R (mid-line): extreme-and-mean division point (on PL)
- L (mid-line, right): edge-midpoint
- A (central node, upper): angle of the cube; pentagon vertex
- Z (center left): [?] construction point

- K (center left): edge-midpoint
- Q (below K): extreme-and-mean division point (on OK)
- T (lower left): raised "roof" vertex (QT perpendicular)
- O (lower left of center): center of a square face
- D (central node): angle of the cube; pentagon vertex
- I (lower center): edge-midpoint
- B (upper right): angle of the cube
- B (bottom center): angle of the cube
- H (bottom left of B): angle of the cube
- C (bottom center-right): angle of the cube
- α (right side; "œ"-like alpha): [?] construction point
- Small Greek-letter labels (δ , π , and others) and numeral labels (2, 3, 4, 7, 8, 9) on the left and lower parts of the figure marking the parallel construction on the other faces: [numerals/letters: ? — near the resolution floor]

(Continuing the demonstration of Proposition 17 — to construct a dodecahedron comprehended in a given sphere, and to show that its edge is the irrational line called the apotome.)

and therefore the trapezium AD, VX lies in one plane. But the triangle ATD too lies in one plane. I say that it lies in the same plane as the trapezium ADVX. For from P draw PY parallel to RV, [Margin: a — *Elem.* XI.8] which will be at right angles to the square ADEF, since RV is at right angles to that same square; and join the straight lines KT, KY. Now since, because the straight line OK has been cut [in extreme and mean ratio], as OK is to OQ, so is OQ to QK; and since KP equals OK (for they are halves of the cube's edge), and PY equals OQ (since it equals RV, which was set equal to OQ, the greater segment) — it follows that as KP is to PY, so is OQ, that is, QT, equal to them, to QK. Therefore the triangles PKY and QTK have two sides proportional and are set together at one angle K, [Margin: b — *Elem.* XI.6] so that their homologous sides are parallel — namely PK and QT to one another, each being at right angles to the square ABCD, and PY and QK to one another, each being at right angles to the square ADEF. [Margin: c — *Elem.* VI.32] Hence the remaining sides KY, KT will lie in one straight line; [Margin: d — *Elem.* XI.1] and therefore the straight line TY lies in one plane, so that the plane ATD and the plane ADVX constitute a single plane, drawn through TY, AD. Consequently the pentagon ATDVX lies in one plane.

That it is equilateral will be shown thus. Since the sides AN, NS of triangle ANS equal the sides AK, KQ of triangle AKQ (for AN, AK are halves of the cube's edge, and NS, KQ are the lesser segments of equal lines cut [in extreme and mean ratio]), and the angles ANS, AKQ are right, then, if AQ be drawn, [Margin: e — *Elem.* I.4] the bases AS, AQ will be equal. Again, since the sides AS, SX of triangle ASX equal the sides AQ, QT of triangle AQT (for the lines AS, AQ have been shown equal, and SX, QT are the greater segments of equal lines cut [in extreme and mean ratio]), and the angles ASX, AQT are right — by definition 3 of Book XI — [Margin: f — *Elem.* I.4] the bases AX, AT will likewise be equal. And if DQ be drawn, we shall show by the same argument that DT equals AT; and if DR be drawn, that DV equals DT. Therefore the four sides XA, AT, TD, DV of the pentagon are equal; and we shall show the remaining side VX equal to them in this way. [Margin: g — *Elem.* XIII.4] The squares of PN, NS together are triple the square of PS, that is, of SX. But the square of AN equals the square of PN. Therefore the squares of AN, NS together are triple the square of SX. [Margin: h — *Elem.* I.47] Since, then, the square of AS equals the squares of AN, NS, the square of AS will also be triple the square of SX; and hence the squares of AS, SX together are quadruple that same square of SX. [Margin: i — *Elem.* I.47] But the square of AX equals the squares of AS, SX. Therefore the square of AX will likewise be quadruple the square of SX, that is, of PS, that is, of XY, [Margin: k — *Elem.* I.34] for PS and XY are equal. Since, then, the square of VX is also quadruple that same square of XY (because VX is double XY, by the scholium of Proposition 4 of Book II), the squares of AX, VX will be equal, and hence the lines themselves equal. Therefore, since XA, AT, TD, DV have been shown equal, the pentagon ATDVX is equilateral.

Diagram. A large planar development of the dodecahedron-on-cube construction: the cube is drawn in oblique view, and pentagonal "roofs" are raised upon its faces, with construction lines radiating from the principal vertices A (upper-left hub) and D (central hub). It is the figure that accompanies the whole demonstration of Prop. 17.

- F, M: top edge of the development
- (top-right corner label): [?]
- X, Y, V: upper band of the net
- N, S, P, R, L: horizontal row beneath X, Y, V
- A: principal left-hand hub (many lines radiate from it)
- D: principal central hub
- T, K, G, O, I: interior construction points

- B (upper right), B, H, C (lower border): cube vertices of the development
- Numerous Greek-minuscule and numeral construction points (π , σ , ρ , ϕ , ω , and the digits 2, 3, 4, 6, 7, 8, 9) are distributed across the net but cannot be resolved cell-by-cell at this resolution. [?]*

That it is also equiangular will be made clear thus. Draw the straight lines AR, AV. Since PN has been cut at S in extreme and mean ratio, and PR, equal to the greater segment PS, has been added to it, [Margin: l — *Elem.* XIII.5] NR will also be cut at P in extreme and mean ratio, and NP will be the greater segment. [Margin: m — *Elem.* XIII.4] Therefore the squares of NR, RP together are triple the square of NP, that is, of AN. But the square of RV equals the square of RP. Therefore the squares of NR, RV are triple the square of AN; and hence the squares of NR, RV, AN are quadruple that same square of AN. [Margin: n — *Elem.* I.47] But the squares of NR, AN equal the square of AR. Therefore the squares of RV, AR are also quadruple the square of AN. [Margin: o — *Elem.* I.47] Since, then, the square of AV equals the squares of AR, RV, the square of AV will also be quadruple the square of AN. But quadruple that same square of AN is the square of AD, by the scholium of Proposition 4 of Book II, since AD is double AN. Therefore the squares of AV, AD are equal, and hence the lines AV, AD equal. By a similar argument, drawing DS, DX, we shall show that DX equals DA; and therefore the three straight lines AD, AV, AX,

AV, DX are equal to one another. Since, then, the sides TA, TD of triangle TAD equal the sides DV, VX of triangle DXV, and the bases AD, DX are equal as well, [Margin: a — *Elem.* I.8] the angles ATD, DVX will also be equal. In just the same way the angle DVX will be shown equal to the angle AXV; and hence, since in the equilateral pentagon ATDVX three angles ATD, DVX, AXV are equal, [Margin: b — *Elem.* XIII.7] it will be equiangular.

Now if two other squares of the cube, $C\alpha\beta D$ and $B\gamma\delta A$, at right angles to the square ABCD, be taken in the common sections CD, AB, and if each of their sides be bisected at the points ε , θ , λ , I, μ , ξ , π , G, which are joined by straight lines cutting one another at the centres of the squares, ρ , σ ; and if OH, OK, ρI , σG be then cut in extreme and mean ratio at the points ϕ , Q, 3, 4, from which perpendiculars ϕz , QT, 37, 48 [numeral: ?] be raised to the planes of the squares toward the outer parts of the cube, equal to the greater segments ϕO , QO, 3 ρ , 46; and if the straight lines T2, 2C, C7, 7D, DT, TA, A8, 8B, Bz be joined — we shall show by the same reasoning that the pentagons T2C7D and T2B8A are equilateral and equiangular and equal to the former ATDVX, on account of the common sides TD, TA. Three pentagons have therefore now been constructed, tangent to three

edges of the cube, AD, DC, AB, and cohering to one another along the common sides TD, TA. If, then, by the same method nine other similar pentagons be built, tangent to the remaining nine edges of the cube, a dodecahedron will be constituted. And that it is comprehended by the same sphere as the cube and the preceding figures, we shall demonstrate by this device.

Let the opposite faces of the cube be conceived as cut by planes drawn through the lines that bisect the sides of the faces — namely the plane ADEF and its opposite, by planes drawn through KM, LN; and likewise the plane ABCD and its opposite, by planes drawn through GI, HK, and so on. Since, then, the cutting planes are at right angles to the said planes of the cube (because they are parallel to the bases of the cube, which are at right angles to the planes ADEF, ABCD), [Margin: c — *Elem.* XI.19] their common sections will also be at right angles to those same planes of the cube. Hence, since YP is at right angles to the plane ADEF, when produced toward Z it will be the common section of the planes drawn through KM, LN. In the same way, if from O a perpendicular ZO be erected to the plane ABCD, it will be the common section of the planes drawn through GI, HK. [Margin: d — *Elem.* XI.39] Now since the said common sections and the diameters of the cube bisect one another, let them bisect one another at Z. But the diameters of the cube bisect one another at the centre of the sphere enclosing the cube, by Corollary 1 of Proposition 15 of this book. Therefore the point Z will be the centre of the sphere described about the cube; and hence all the straight lines drawn from Z to all the angles of the cube will be equal to one another. [Margin: e — *Elem.* XI.6] That all the straight lines drawn from Z to all the angles of the dodecahedron are equal to these as well will be made clear thus, by drawing the line ZX. Since KP, OZ are parallel, because each is at right angles to the plane ABCD,

Diagram. The same dodecahedron-on-cube construction net, reprinted on the verso to accompany the continuation of the proof. The cube is shown in oblique view with pentagonal roofs on its faces; the principal hubs are A (upper-left) and D (centre), with the sphere's centre Z among the interior points.

- F, M (and top-right corner label [?]): top edge of the development
- X, Y, V: upper band
- N, S, P, R, L: horizontal row beneath
- A: left-hand hub; D: central hub; Z: interior centre point
- K, T, G, O, I: interior construction points
- B (upper right), B, H, C (lower border): cube vertices

- Greek-minuscle and numeral construction points (π , σ , ρ , ϕ , ω ; digits 2, 3, 4, 6, 7, 8, 9) distributed over the net, not reliably legible at this resolution. [?]*

and likewise OK, ZP are parallel, because each is at right angles to the plane ADEF, PKOZ will be a parallelogram (for the two points Z must be understood as joined), and therefore PZ will equal KO, that is, NP, half the cube's edge. But PZ also equals PR. Therefore the whole ZY will equal the whole NR. [Margin: f — *Elem.* XIII.4] Hence, since the squares of NR, PR together are triple the square of NP (because NR has been cut at P in extreme and mean ratio), the squares of ZY, PY — that is, the squares of ZY, YX — will also together be triple the square of ZX. (For XY equals PS, that is, PY.) [Margin: g — *Elem.* I.47] But the square of ZX equals the squares of ZY, YX. Therefore the square of ZX will be triple the square of ZP, that is, of half the cube's edge. And since triple that same square of half the cube's edge is also the square of the radius of the sphere enclosing the cube [Margin: h — *Elem.* V.15] (for as the diameter is to the cube's edge, so is the radius to half the edge; [Margin: i — *Elem.* VI.22] and hence, as the square of the diameter is to the square of the edge, so is the square of the radius to the square of half the edge; [Margin: k — *Elem.* XIII.15] since, then, the square of the diameter is triple the square of the edge, the square of the radius will also be triple the square of half the edge) — the square of ZX is therefore equal to the square of the radius of the sphere, or of the cube; and hence ZX equals the radius. By the same reasoning the line ZV, when drawn, will equal the radius. Or indeed, since the sides ZY, YX of triangle ZYX equal the sides ZY, YV of triangle ZYV, and the contained angles are right (for since PY is perpendicular to LN, it will also be perpendicular to VX, which is parallel to it), [Margin: l — *Elem.* I.4] the bases ZX, ZV

ZX, ZV are equal. But ZA, ZD, drawn to the angles of the cube, are equal to the radius of the sphere, as was said. There are therefore now four straight lines drawn from Z, the centre of the sphere, to the four angles A, D, V, X of the pentagon, equal to the radius of the sphere. And to show that the line ZT drawn to the fifth angle T is likewise equal to the radius, let a circle ATDVX be described about the pentagon (since it is equilateral and equiangular), and from Z, the centre of the sphere, let a perpendicular $Z\omega$ be dropped onto its plane, and let the lines ωA , ωT , ωD , ωV , ωX be joined. [Margin: a — *Elem.* I.47] Now since the square of ZA equals the squares of $Z\omega$, ωA , and the square of ZX equals the squares of $Z\omega$, ωX — while the squares of ZA, ZX are equal, because the lines ZA, ZX have been shown equal — the squares of $Z\omega$, ωA will equal the squares of $Z\omega$, ωX ; and hence, the common square of $Z\omega$ being taken away, the remaining squares of ωA , ωX will be equal, and therefore the lines ωA , ωX themselves equal. Likewise the

lines ωV , ωD will be shown equal both to one another and to ωA , ωX , since the lines ZV , ZD have been shown equal to ZA , ZX . Since, then, from ω there fall upon the circumference more than two equal straight lines, [Margin: b — *Elem.* XIII.9] the point ω will be the centre of the circle; and hence ωT is equal to the others drawn from the centre, and therefore the squares of $Z\omega$, ωA will equal the squares of $Z\omega$, ωT . [Margin: c — *Elem.* I.47] Since, then, the square of ZA equals those two, and the square of ZT equals these, the squares of ZA , ZT will be equal, and therefore the lines ZA , ZT equal. Hence the sphere enclosing the cube, described from the centre Z , will comprehend the pentagon $ATDVX$ as well; and in the same way the remaining pentagons of the dodecahedron. Therefore the dodecahedron will be comprehended by the same sphere as the cube.

Finally, since as NP is to PS , so is PS to SN , [Margin: d — *Elem.* V.15] it follows that as NL , double NP , is to RS , double PS , so is RS , double PS , to the line composed of SN , RL — that is, to double SN . If, then, the cube's edge NL be cut in extreme and mean ratio, the greater segment will be SR . But NL , thus cut as a whole, is rational (for its square is commensurable with the square of the diameter of the sphere, which is set as rational, since it is a third part of it). [Margin: e — *Elem.* XIII.6] Therefore the greater segment SR — that is, the dodecahedron's edge XV , equal to it — is an irrational line, the one called the apotome. We have therefore constituted the dodecahedron and enclosed it in the same sphere by which the aforesaid figures are enclosed; and we have demonstrated that the edge of the dodecahedron is the irrational line called the apotome. Which was to be done.

Diagram. A circle (a great circle of the sphere passing through the pentagon) with the regular pentagon inscribed and its centre marked; straight lines join the centre to every vertex and the pentagon's diagonals are drawn.

- X: upper-left vertex
- V: upper-right vertex
- A: left vertex
- D: right vertex
- T: lower vertex
- ω : centre of the circle
- z (small label near D): construction point [?]

COROLLARY I.

Since, then, it has been demonstrated that the line LN , cut in extreme and mean ratio, has for its greater segment the line RS — and LN was set equal to the cube's edge, and RS to the dodecahedron's edge — it is clear that, if the cube's edge be

cut in extreme and mean ratio, the greater segment is the edge of the dodecahedron described in the same sphere.

COROLLARY II.

Likewise, the cube's edge equals the straight line subtending an angle of the pentagon of the dodecahedron comprehended in the same sphere. For AD, the cube's edge, subtends the angle T of the pentagon, and it has been shown equal to the lines AV, DX, which subtend the angles X and V of the same pentagon. And since the line subtending the said angle of the pentagon, cut in extreme and mean ratio, [Margin: f — *Elem.* XIII.8] makes the edge of the pentagon its greater segment, [Margin: g — *Elem.* XIII.5] therefore the line composed of the said line subtending the pentagon's angle — that is, of the cube's edge — together with the greater segment — that is, the dodecahedron's edge — is divided in like manner, and makes the dodecahedron's edge its lesser segment and the cube's edge its greater segment. It follows that, if a straight line be cut in extreme and mean ratio such that its lesser segment is the dodecahedron's edge, the greater segment is the cube's edge in the same sphere.

COROLLARY III.

It is also established from what has been said that in the dodecahedron there are six [cross-]lines, of which two opposite ones are parallel and are bisected — and cut at right angles — by three equal straight lines that likewise bisect one another at the centre of the dodecahedron and cut at right angles. For two of the said lines are VX, T2, which are plainly parallel to the cube's edges EF, AB; and four others are found upon the remaining four faces of the cube. Of these, the ones standing upon opposite faces of the cube are parallel (since they are parallel to parallel and opposite edges of the cube), and they are cut, by what has been demonstrated, in half and at right angles by the straight lines that join the centres of opposite faces of the cube — such as the lines YZ, 9Z, and so on, which, as has been demonstrated, intersect one another at the centre Z. I say that they cut one another in half and at right angles. In half, because, since the lines PZ, OZ equal half the cube's edge, and YP, 9O equal the equal greater segments PS, OQ, the wholes YZ, 9Z will be equal; and by the same reasoning the remaining lines drawn from Z to the sections of the opposite sides will be equal. Hence all three lines are equal and bisect one another at the centre Z. And at right angles, because in the parallelogram KOZP the angle OZP is right, being opposite to the right angle OKP. Likewise the remaining straight lines joining the centres of the faces will be shown to make right angles; for all are parallel to the edges of the cube that constitute right angles, as is clear.

COROLLARY IV.

Finally, it is established that, if the line dividing opposite lines of the dodecahedron in half and at right angles, as was said, be cut in extreme and mean ratio, the greater segment is the cube's edge, and the lesser the edge of the dodecahedron comprehended in the same sphere. For since the squares of ZY, YP have been shown triple the square of PZ, ZY will be cut at P in extreme and mean ratio, and the greater segment will be PZ, by the scholium of Proposition 4 of this book. And hence, as ZY is to ZP, so is ZP to PY; [Margin: h — *Elem.* V.15] therefore, as double ZY — namely the line dividing the opposite lines of the dodecahedron in half — is to double ZP — namely the cube's edge — so is that same double of ZP to double PY, that is, to the dodecahedron's edge. Therefore the line dividing the opposite lines of the dodecahedron in half, cut in extreme and mean ratio, makes its greater segment the cube's edge, and its lesser the edge of the dodecahedron of the same sphere.

[Continuation of the Scholium to Proposition 17 — the dodecahedron.]

Diagram. An unfolded net for the dodecahedron: twelve equal, regular pentagons drawn edge to edge in the plane. One central pentagon is ringed by five others, with a further cluster of pentagons continuing to the right, arranged so that, when folded up, the twelve faces close into the solid. No vertices are lettered.

But if twelve equilateral and equiangular pentagons, all equal to one another, are made out of some material and laid out as this figure shows, then — provided they are correctly folded together — a dodecahedron will be built up as a complete solid.

PROBL. 6. PROPOS. 18.

[Margin: xviii]

To set out the sides of the five figures and to compare them with one another.

Let AB be the diameter of the sphere. Bisect it at C; cut it at D so that BD is a third part of it; and finally cut it at E so that BE is a fifth part of it. Then describe a semicircle on AB, and from C, D, and E draw the perpendiculars CF, DG, EH to its circumference; and join the straight lines AF, BF, AG, BG, AH, BH. Next, from HA cut off HI equal to the side of the decagon inscribed in the circle whose radius — that is, whose side of the hexagon — is BH; and join the straight line BI. Finally, cut BG in extreme and mean ratio at K, and let BK be the greater segment.

Diagram. The construction for the proposition. A semicircle stands on the diameter AB (A at the left end, B at the right). On the diameter, between A and B, lie the points C, D, E (in that order from A toward B). From C, D, E the perpendiculars CF, DG, EH rise to meet the circumference at F, G, H respectively (F highest, near the top; G to its right; H lower and further right). The chords AF, BF, AG, BG, AH, BH are all drawn. The point I lies on the chord AH (with HI cut off equal to the decagon's side), and the point K lies on BG (dividing it in extreme and mean ratio).

- A (left end of diameter): endpoint
- C, D, E (on the diameter, A→B): division points (C bisects; BD = one third; BE = one fifth)
- B (right end of diameter): endpoint
- F, G, H (on the circumference): feet of the perpendiculars from C, D, E
- I (on chord AH): HI = side of the decagon
- K (on BG): point of section in extreme and mean ratio

[Clavius's demonstration.] As to setting out the sides of the said figures, I say that AG is the edge of the pyramid, or tetrahedron; AF the edge of the octahedron; BG the edge of the cube; BI the edge of the icosahedron; and finally BK the edge of the dodecahedron.

For since AG is the mean proportional between BA and AD, by the corollary of Proposition 8 of Book 6, it follows from the corollary of Proposition 20 of the same book [Margin: a. Elements XIII.13] that as BA is to AD, so is the square on BA to the square on AG. But BA is one-and-a-half times AD, since AD is 2 of the parts of which BA is 3. Therefore the square on BA is likewise one-and-a-half times the square on AG. And since the diameter of the sphere is in square one-and-a-half times the edge of the tetrahedron, AG is the edge of the tetrahedron.

Again, since AF is the mean proportional between BA and AC, by the same corollary of Proposition 8 of Book 6 — and consequently, by the adduced corollary of Proposition 20 of Book 6, as BA is to AC, so is the square on BA to the square on AF — and since BA is double AC, the square on BA is double the square on AF. [Margin: b. Elements XIV.14 — recte XIII.14] Therefore, since the diameter of the sphere is in square double the edge of the octahedron, AF is the edge of the octahedron. In the same way BF too will be the edge of the octahedron.

Next, since BG is the mean proportional between AB and BD — so that, by the two aforesaid corollaries of Book 6, as AB is to BD, so is the square on AB to the square on BG — and since AB is triple BD by construction, the square on AB is triple the

square on BG. [Margin: c. Elements XIII.15] Hence, since the diameter of the sphere is in square triple the edge of the cube, BG is the edge of the cube.

Furthermore, since by the same corollaries BH is the mean proportional between AB and BE, so that as AB is to BE, so is the square on AB to the square on BH — and since by construction AB is five times BE — the square on AB is five times the square on BH. Therefore, since the diameter of the sphere is in square five times the radius of the circle enclosing the five sides of the icosahedron, by the corollary of Proposition 1 of Proposition 16 of this book, BH will be the radius, that is, the side of the hexagon of the said circle; while HI is the side of the decagon of the same circle, by construction. [Margin: d. Elements XIII.10] Therefore the side of the pentagon has as its square the squares on BH and HI together; and consequently, [Margin: e. Elements I.47] since the square on BI is equal to the squares on BH and HI, BI is the side of the pentagon of that same circle. [Margin: f. Elements XIII.16] For which reason BI is the edge of the icosahedron.

Finally, since the edge BG of the cube is cut at K in extreme and mean ratio, BK is the greater segment — the edge of the dodecahedron inscribed in the same sphere, by the corollary of Proposition 2 of Proposition 17 of this book. The sides, then, of the five regular figures inscribed in the same sphere have been set out.

[The comparison of the sides.] As for the comparison of the sides now found: since it has been demonstrated that the diameter of the sphere is, in square, one-and-a-half times the edge of the pyramid, double the edge of the octahedron, and triple the edge of the cube, it is plain that, of whatever parts the square on the diameter is 6, of such parts the square on the edge of the tetrahedron is 4, the square on the edge of the octahedron is 3, and finally the square on the edge of the cube is 2:

Figure	Square (parts)
Diameter [Diamet.]	6
Tetrahedron [Tetraed.]	4
Octahedron [Octaed.]	3
Cube [Cub.]	2

And therefore the edge of the pyramid, or tetrahedron, is in square one-and-a-third times the edge of the cube. Moreover, the edge of the octahedron is in square one-and-a-half times [the edge of the cube]. Since, then, the squares on the sides of these three figures — namely the tetrahedron, octahedron, and cube —

together with the square described on the diameter of the sphere, have among themselves the proportion which number has to number, [Margin: g. Elements X.6] the diameter of the sphere and the sides of the said figures are commensurable lines; and hence, since the diameter of the sphere is taken to be rational, the said sides will also be rational. But because the said squares do not have the proportion which a square number has to a square number, as is established from the corollary of Proposition 24 of Book 8, [Margin: h. Elements X.9] these lines — namely the diameter of the sphere, the edge of the tetrahedron, the edge of the octahedron, and the edge of the cube — will be incommensurable in length, and hence rational and commensurable in square only.

But the sides of the icosahedron and of the dodecahedron, since they are irrational lines — the former a minor, the latter an apotome, as has been demonstrated — are in no way, that is, neither in length nor in square, commensurable with the sides just mentioned. For if the sides of these latter figures were commensurable with the sides of the former, then, since the sides of the former are commensurable with the diameter of the sphere, the sides of the latter would also be commensurable with the same diameter; and consequently, the diameter being rational, those sides too would be rational. But this is absurd, for it has been shown that they are irrational.

Diagram. The same construction as on the preceding page, repeated at the head of this page. A semicircle stands on the diameter AB, with C, D, E marked on the diameter between A and B; the perpendiculars CF, DG, EH rise to F, G, H on the circumference; the chords AF, BF, AG, BG, AH, BH are drawn; I lies on AH and K on BG.

- A, B: ends of the diameter
- C, D, E: division points on the diameter
- F, G, H: feet of the perpendiculars on the circumference
- I (on AH): with HI equal to the decagon's side
- K (on BG): section in extreme and mean ratio

Nor, moreover, are the said sides of the icosahedron and dodecahedron commensurable with each other in any way whatever. For if they were believed to be commensurable in length with each other, then, since the side of the dodecahedron is an apotome, [Margin: i. Elements X.104] the side of the icosahedron too would be an apotome. Likewise, since the side of the icosahedron is a minor line, [Margin: k. Elements X.106] the side of the dodecahedron too would be a minor line. And if they were said to be commensurable in square

only, it would follow in the same way that the side of the dodecahedron is a minor line, from the fact [Margin: l. Elements XIII.16] that the side of the icosahedron is a minor. But this is absurd. For, by the scholium to Proposition 112 of Book 10, the apotome and the minor are irrational lines wholly different from each other, so that neither is the same as the other.

Therefore, although among the sides of the five regular figures thus set out the first three are commensurable both with one another and with the diameter of the sphere, and so are rational, while the two last are incommensurable both with each other and with the diameter and the remaining sides, and so are irrational — nevertheless there is discerned among all the said sides this order: the edge of the tetrahedron is greater than the edge of the octahedron; the edge of the octahedron greater than the edge of the cube; the edge of the cube greater than the edge of the icosahedron; and finally the edge of the icosahedron greater than the edge of the dodecahedron. That is, the said sides exceed one another in turn in the very order in which we constructed the aforesaid figures. This was perhaps the reason why Euclid did not construct these figures in the same order in which he defined them.

For that the edge of the tetrahedron is greater than the edge of the octahedron is plain. Since AG, the edge of the tetrahedron, subtends a greater arc than AF, the edge of the octahedron, the straight line AG will be greater than AF, by the scholium to Proposition 28 of Book 3.

In the same way we shall show that the edge of the octahedron is greater than the edge of the cube. For BF, the edge of the octahedron, subtends a greater arc than BG, the edge of the cube.

That the edge of the cube BG is greater than the edge of the icosahedron BI we shall demonstrate by this device. Since AB is set as triple BD, [Margin: m. Elements VI.20] and squares have the doubled proportion of their sides, the square on AB will be nine times the square on BD — for the ninefold proportion is the double of the triple proportion, as appears here: 1, 3, 9. And therefore, of whatever parts the square on AB is set as 9, of such parts the square on BD will be 1. But of the parts of which the square on AB is 9, the square on BG is 3, [Margin: n. Elements XIII.15] because the former is triple the latter; and consequently the squares on GD and BD together are likewise 3 of those parts, since they are equal to the square on BG. Therefore, if the square on BD — 1 of those parts — is taken away, there will remain 2 of those parts as the square on GD. Now since the square on AB is five times the square on BH, the square on BH will be less than 2 of those parts. For since 9 to 2 has a proportion less than fivefold — because 10 to

2 has the fivefold proportion — [Margin: o. Elements V.10 [?]] the number to which 9 has the fivefold proportion will be less than 2. Hence the square on GD is greater than the square on BH, and therefore the straight line GD is greater than the straight line BH.

Now GD is composed of BH, the radius of the circle from which the icosahedron is constituted, and of the line HI, double the side of the decagon of the same circle — by the corollary of Proposition 2 of Proposition 16 of this book — for it is established that the side of the decagon [is HI, and that HI] is less than the straight line BD. For if HI were believed equal to a third part of the straight line AB, then its double would be equal to two-thirds, namely to AD; and consequently BH would be equal to the remaining third part, namely to BD. And so, since the square on AB has been shown to be nine times the square on BD, the same square on AB would be nine times the square on BH. But this is absurd, for it has been demonstrated to be five times it. Therefore HI is not a third part of AB. Nor yet is it greater than a third part. For if HI were said to be greater than a third part of AB, and consequently its double greater than two-thirds — that is, than AD — then BH would be less than a third part of BD. Hence the square on AB would be greater than nine times the square on BH; which is even more absurd. Therefore HI is not greater than a third part of AB; nor is it equal to a third part, as we have shown. It is therefore less than a third part, that is, less than BD. And so, since the straight lines GD and DB are greater than the straight lines BH and HI, as has been demonstrated, the squares on the former will also be greater than the squares on the latter. [Margin: p. Elements I.47] But the square on BG is equal to the squares on GD and DB, and the square on BI is equal to the squares on BH and HI. Therefore the square on BG is also greater than the square on BI; and consequently the straight line BG, namely the edge of the cube, is greater than the straight line BI, that is, the edge of the icosahedron.

Finally, that the edge of the icosahedron BI is greater than the edge of the dodecahedron BK will become clear by this reasoning. Since, the straight line BG being divided in extreme and mean ratio, BK is the greater segment and GK the less, the rectangle contained by BG and BK is greater than the rectangle contained by BG and GK; and so the rectangles under BG, BK and under BG, GK together are greater than twice the rectangle under BG, GK. [Margin: q. Elements II.2] But the square on BG is equal to the rectangles under BG, BK and under BG, GK; and the square on BK is equal to two rectangles under BG, GK — for since as BG is to BK, so is BK to GK, [Margin: r. Elements VI.17] the square on BK is equal to the rectangle under BG, GK, and therefore two squares on BK are equal to two rectangles under BG, GK. Therefore the square on BG is also greater than two

squares on BK; and accordingly three squares on BG will be greater than six squares on BK. But three squares on BG are equal to three squares on BH — for since the square on AB is triple the square on BG and five times the square on BH, both the three squares on BG and the five squares on BH are equal to the same square on AB, and therefore those three are equal to these five. Therefore the five squares on BH will also be greater than the six squares on BK; and consequently one square on BH will be greater than one square on BK. For if the square on BH were equal to the square on BK, or less, then five squares on BH would be less than six squares on BK — which is absurd, since those five have been shown to be greater than these six. For which reason the straight line BH too will be greater than the straight line BK; and therefore, since BI is greater than BH, BI — namely the edge of the icosahedron — will be much greater than BK, the edge of the dodecahedron. We have therefore set out the sides of the five figures and compared them with one another. Which was to be done.

SCHOLIVM.

The commentators at this point prove that, besides the said five figures, no other solid figure can be constructed that is contained by plane figures which are equilateral, equiangular, and equal to one another. Their reasoning is roughly as follows.

Out of two triangles, or out of two other figures, a solid angle cannot be constructed, since at least three plane angles are required for the construction of a solid angle.

Out of three equilateral triangles the angle of the pyramid is composed;

out of four, the angle of the octahedron;

out of five, the angle of the icosahedron.

But out of six such triangles meeting at the same point a solid angle cannot be made. For since the angle of an equilateral triangle contains two-thirds of one right angle, by the corollary of Proposition 3 of Proposition 32 of Book 1, six such angles will be twelve thirds of one right angle — that is, they will be equal to four right angles. Hence out of them no solid angle will be constructed. [Margin: 20. undecimi — Elements XI.21 [?]] For every solid angle is contained by angles less than four right angles. Much less, then, will a solid angle be made out of more than six such plane angles.

Next, out of three squares the angle of the cube is made. But out of four squares no solid angle can be constructed, for again there will be four right angles. Much less, then, will a solid angle be made out of more than four such angles.

Finally, out of three equilateral and equiangular pentagons the angle of the dodecahedron is composed. For since the five angles of a pentagon are equal to six right angles, as we showed at Proposition 32 of Book 1, and consequently one angle contains one right angle plus a fifth part of one right angle, three such angles will contain three right angles and, in addition, three-fifths of one right angle. They will therefore be less than four right angles, and so out of them a solid angle will be constructed. But out of four such angles no solid angle can be made; for since one of them, as has been said, is greater than a right angle — because it contains one right angle and, in addition, a fifth part of a right angle — four of them will be greater than four right angles. Hence no solid angle will be constructed out of them; and therefore much less can a solid angle be constructed out of more than four such angles. Nor indeed can a solid angle be constructed out of any other equilateral and equiangular figures. For since the six angles of a hexagon are equal to eight right angles, as was demonstrated at Proposition 32 of Book 1 — and consequently one angle contains one right angle plus two-sixth parts of one right angle — three such angles will contain three right angles and six sixth-parts, that is, four right angles. Hence out of them a solid angle can in no way be composed; and therefore much less will a solid angle be made out of more than three, nor for that reason out of any other polygons. For any three of them are greater than four right angles. For which reason it is plain that, besides the said five figures, no other solid figure can be constructed that is contained by plane figures equilateral, equiangular, and equal to one another.

For if three equilateral and equal triangles meet at one point to make up a solid angle, there will be constructed a pyramid contained by four such triangles.

But if four such triangles come together to compose a solid angle, an octahedron will be made, comprehended by eight such triangles.

If, finally, five such triangles are taken to form a solid angle, an icosahedron will be made, contained by twenty such triangles.

But if three equal squares meet at one point so as to constitute a solid angle, a cube will be built up out of six such squares.

Lastly, if three equilateral and equiangular pentagons, equal to one another, come together to make up a solid angle, a dodecahedron will be composed, containing twelve such pentagons.

It is also fitting to add here a certain method, taken from Hypsicles of Alexandria, by which the number of the sides and angles of any regular figure may readily be had.

[The method of Hypsicles.] If anyone asks how many edges any of the regular solids has: all its faces are to be multiplied by the number of sides of one face, so that the sides of all the faces are obtained. But since two sides of the faces always come together to make up one edge of the regular solid, half of all the sides is to be taken, so that the number of edges of the regular solid appears.

Example. If the four triangles of the pyramid, or tetrahedron, are multiplied by the number of sides of one triangle — as by 3 — they make 12, of which the half is 6, namely the number of edges of the pyramid. And likewise for the rest. For if the six squares composing the cube are multiplied by the sides of one square — namely by 4 — they make 24, of which the half is 12, namely the number of edges of the cube. And if the eight triangles of the octahedron are multiplied by 3, the number of sides of one triangle, they make 24, of which the half is again 12, the number of edges of the octahedron. If again the 20 triangles of the icosahedron are multiplied by 3, the number of sides of one triangle, they make 60, of which the half is 30, the number of edges of the icosahedron. If, finally, the 12 pentagons constituting the dodecahedron are multiplied by 5, the number of sides of one pentagon, they make 60, of which the half is again 30, the number of edges of the dodecahedron.

But if anyone should further ask how many solid angles any of the regular solids has: all its faces are to be multiplied by the angles of one face, so that the angles of all the faces are obtained. But since three, four, or five angles of the faces always come together to make up one solid angle, all the angles are to be divided by the number of angles that meet together to constitute a solid angle, so that the number of solid angles of the regular solid appears.

Example. If the four triangles of the pyramid are multiplied by 3, the number of angles of one triangle, they make 12 angles, which, if divided by 3 (for that many angles constitute one solid angle of the pyramid), yield 4 solid angles of the pyramid. If instead the 6 squares of the cube are multiplied by 4, the number of angles of one square, they make 24 angles, which, divided by 3 (for that many angles make up one solid angle of the cube), yield 8 solid angles of the cube. If also the eight triangles of the octahedron are multiplied by 3, the number of angles of one triangle, they make 24 angles, which, divided by 4 (for that many angles compose a solid angle of the octahedron), yield 6 solid angles of the octahedron. If again the 20 triangles of the icosahedron are multiplied by 3, the

number of angles of one triangle, they make 60 angles, which, divided by 5 (for that many angles constitute a solid angle of the icosahedron), produce 12 solid angles of the icosahedron. If, finally, the 12 pentagons of the dodecahedron are multiplied by 5, the number of angles of one pentagon, they make 60 angles, which, divided by 3 (for that many angles come together to constitute a solid angle of the dodecahedron), yield 20 solid angles of the dodecahedron.

Finally, it is also worth considering that the cube and the octahedron are reciprocal with respect to their faces and their solid angles. For as many faces as are contained in the one, so many solid angles are found in the other, and conversely. Thus the cube has six faces and eight solid angles, while the octahedron contains eight faces and six solid angles. In the same way the icosahedron and the dodecahedron are reciprocal. For the icosahedron has twenty faces and twelve solid angles, while the dodecahedron possesses twelve faces and twenty solid angles. The tetrahedron alone (that is, the pyramid) is reciprocal with none but itself; for it has four faces and just as many solid angles.

FINIS ELEMENTI TERTIIDECIMI.

EUCLID'S FOURTEENTH ELEMENT

And the fourth of the Solids — as some think; but, as others hold, the work of **Hypsicles of Alexandria on the five bodies.**

BOOK ONE

PREFACE of Hypsicles of Alexandria to Protarchus

Basilides of Tyre, Protarchus, having come to Alexandria and been commended to my father for their shared devotion to the discipline, spent a very long stretch of his travels in his company. On one occasion the two of them were discussing the treatment that Apollonius [of Perga] had written on the comparison of the dodecahedron and the icosahedron inscribed in the same sphere — what ratio these bear to one another — and they concluded that Apollonius had not set the matter out correctly. And so, as my father used to say, they published in writing the corrected account that they had worked out for themselves. Later, however, I came upon another book issued by Apollonius, which contained an accurate demonstration of the question proposed; and from investigating its problem I took great pleasure indeed. What Apollonius wrote can certainly be examined by everyone, since it is in everyone's hands. But what we ourselves seem to have

written afterward — with as much careful study as one may conjecture — this I have thought fit to set down in a memoir and dedicate to you, so that you, who are happily versed in all the disciplines and above all in geometry, may judge skillfully and prudently the things we are about to say. And for the sake of the intimacy of life you shared with my father, and the goodwill with which you embrace us, may you gladly give the treatment itself a hearing. But it is now time to put an end to the preface and take up what has been proposed.

[Clavius's introduction to the book.]

Thirteen books, expounded up to this point, are ascribed by everyone to Euclid; but the two that follow are commonly — and rightly, in my judgment — attributed by most to Hypsicles of Alexandria. In the earlier of the two, the icosahedron and the dodecahedron are compared with one another, both with respect to their surfaces and to their solidity; in the later, several descriptions are set out of one regular figure inside another — so that these two books are, as it were, a kind of appendix to Euclid's *Elements*. But since Hypsicles omits a great deal that is not unpleasant to know concerning the comparisons of the said five regular bodies and their mutual inscription of one within another — for this fourteenth book is completed in just four propositions, and the fifteenth in five problems — it seemed good, in order to bring some light to this treatment, to discuss the said regular figures somewhat more fully in general, proposing various theorems and problems by which the aforesaid regular bodies are both compared with one another and inscribed within one another. In this matter we do not deny that Campanus and François de Foix, Count of Candale [Franciscus Flussas Candalla], were of the greatest help to us, for he bestowed diligent and careful labor on this business. He not only enriched the two following books with a great many theorems and problems, but also added a whole book, sixteenth in order, in which he sets out notable properties of the said bodies. Moreover, that Hypsicles's chief aim in this fourteenth book was to discuss the mutual comparison of the icosahedron and the dodecahedron can readily be understood from his preface to Protarchus, set out above.

Now since in these two books Hypsicles and Campanus differ, both in the number of the propositions and in their order — differing both from one another and from François de Foix, and consequently from us as well — we have noted the order of each in the margin by two numbers: the upper indicates Hypsicles's order, the lower Campanus's series. When only a single number appears repeated in the margin, it signifies that there is then no difference between Campanus and Hypsicles. But if at any point the mark **O** is placed in the position of the upper

number, it is to be understood that the said proposition is missing in Hypsicles; and understand the same for Campanus if that mark should happen to be placed in the position of the lower number. Finally, when nothing at all is found in the margin, it is a sign that the proposition is absent from both Hypsicles and Campanus.

THEOR. 1. PROPOS. 1.

[Margin: i.]

The perpendicular drawn from the center of any circle to the side of a pentagon inscribed in that same circle is half of both lines together — namely of the side of the hexagon and the side of the decagon inscribed in the same circle.

[Demonstration.] From the center A of circle BCD let AE be drawn perpendicular to BC, the side of a pentagon described in the said circle. I say that AE is half of the line compounded of the side of the hexagon and the side of the decagon of the same circle. For let AE be extended in both directions so as to complete the whole diameter DF, and let EG be taken equal to EF; and let the straight lines AC, CG, CF, BF be joined. [Margin: a. Elem. III.3] Since, then, the perpendicular AE cuts the straight line BC into two equal parts, the sides BE, EF of triangle BEF will be equal to the sides CE, EF of triangle CEF. And since the contained angles are also equal — namely right angles — [Margin: b. Elem. I.4] the bases CF, BF will be equal; [Margin: c. Elem. III.28] and therefore the arcs CF, BF will be equal as well. Accordingly, since the arc BC is a fifth part of the whole circumference, CF will be a tenth part of it; and so the straight line CF will be the side of the decagon.

Again, since the sides CE, EF of triangle CEF are equal to the sides CE, EG of triangle CEG, and the contained angles are right angles, [Margin: d. Elem. I.4] the bases CF, CG and the angles CFG, CGF will also be equal. And since the arc CF is a fifth part of the semicircumference — being a tenth part of the whole circumference — the arc CF will be 1 such part, while the arc DC is 4 such parts; that is, the arc DC will be four times the arc CF. [Margin: e. Elem. VI.33] Hence the angle DAC will also be four times the angle CAF. [Margin: f. Elem. III.20] But the angle CFD is half the angle DAC, since the one stands at the circumference and the other at the center on the same base DC. Therefore the angle CFD — and consequently CGF, which was shown equal to it — will be double the angle CAF. [Margin: g. Elem. I.32] But the angle CGF, being exterior, equals the interior angles GAC, GCA in triangle GAC. Since, then, GAC is half of the angle CGF, as was shown, GCA will be the remaining half of that same angle CGF; and so the angles

GAC, GCA will be equal, [Margin: h. Elem. I.6] and consequently the sides AG, CG equal as well. But CG was shown equal to the straight line CF; therefore AG too will equal that same CF. Hence, adding the equal segments GE, EF, AE becomes equal to the two EF, FC together; and accordingly the three together AE, EF, FC — that is, the two together AF, FC — will be double AE. Therefore, conversely, AE — namely the perpendicular drawn to the side of the pentagon — will be half of the two AF, FC together, that is, of the side of the hexagon and the side of the decagon. Therefore the perpendicular drawn from the center of any circle to the side of a pentagon inscribed in that same circle, etc. Which was to be demonstrated.

Diagram. A circle with center A. The perpendicular from the center rises through the chord BC to the top of the circle; a vertical diameter runs through the figure.

- A (center of circle): the center from which the perpendicular is drawn
- B (upper-left, on circle): left endpoint of the chord BC (side of the pentagon)
- C (upper-right, on circle): right endpoint of the chord BC
- D (bottom, on circle): lower end of the vertical diameter DF
- F (top, on circle): upper end of the vertical diameter DF
- E (on chord BC): foot of the perpendicular AE, bisecting BC
- G (on the vertical between E and A): point taken so that EG = EF

Lines drawn: chord BC; diameter DF (through center A); and the straight lines AC, CG, CF, BF.

COROLLARY I.

From this it follows that a perpendicular drawn from the center of a circle to a straight line fitted within the same circle bisects the arc which that straight line subtends. For it has been demonstrated that the straight line AE, which is perpendicular from the center to BC, divides the arc BC into two equal parts at F.

COROLLARY II.

Hence it also follows that the perpendicular drawn from the center to the side of the pentagon equals both lines together — namely the perpendicular drawn from the center to the side of the equilateral triangle inscribed in the same circle, together with half the side of the decagon. For, as was shown in this proposition, the perpendicular to the side of the pentagon equals half the side of the hexagon and half the side of the decagon together; and the perpendicular to the side of the equilateral triangle, by the corollary of Prop. 12 of Book XIII, equals half the

radius — that is, half the side of the hexagon. It is therefore clear that this same perpendicular to the side of the pentagon equals the perpendicular to the side of the triangle together with half the side of the decagon.

THEOR. 2. PROPOS. 2.

[Margin: iv. / ij.]

If two straight lines are each cut in extreme and mean ratio, they will be cut similarly — that is, into the same proportions.

[Demonstration.] Let the two straight lines AB and CD be cut in extreme and mean ratio at E and F. I say that they are cut into the same proportions — that is, that as AB is to CD, so is AE to CF and EB to FD; and likewise, as AE is to EB, so is CF to FD, and so on. For since AB is to AE as AE is to EB, and CD is to CF as CF is to FD, [Margin: a. Elem. VI.17] the rectangle contained by AB, EB equals the square on AE, and the rectangle contained by CD, FD equals the square on CF. For this reason, as the rectangle under AB, EB is to the square on AE, so is the rectangle under CD, FD to the square on CF (for in both cases it is a ratio of equality). And therefore, by what we demonstrated in the scholium to Prop. 22 of Book V, as four times the rectangle under AB, EB is to the square on AE, so four times the rectangle under CD, FD is to the square on CF; and, compounding, as four times the rectangle under AB, EB together with the square on AE is to the square on AE, so four times the rectangle under CD, FD together with the square on CF is to the square on CF.

[Margin: b. Elem. II.8] But four times the rectangle under AB, EB together with the square on AE equals the square on the line compounded of AB, EB; and four times the rectangle under CD, FD together with the square on CF equals the square on the line compounded of CD, FD. Therefore also, as the square on the line compounded of AB, EB is to the square on AE, so is the square on the line compounded of CD, FD to the square on CF; [Margin: c. Elem. II.2] and accordingly, as the line compounded of AB, EB is to AE, so is the line compounded of CD, FD to CF. And, compounding, as the compound line of AB, EB together with AE — that is, twice AB — is to AE, so is the compound line of CD, FD together with CF — that is, twice CD — to CF. And, alternating, as twice AB is to twice CD, so is AE to CF. [Margin: d. Elem. V.15] But as twice AB is to twice CD, so is AB to CD. Therefore, as AB is to CD, so is AE to CF. And since, as the whole AB is to the whole CD, so is AE (removed) to CF (removed), [Margin: e. Elem. V.19] the remainder EB will also be to the remainder FD as whole to whole. Hence, as AE is to CF, so will EB be to FD, since each ratio is the same as the ratio of AB to CD. And, alternating,

as AE is to EB, so will CF be to FD. And so, finally, by all the modes of arguing from proportionality, we shall show that both the whole lines and their parts are proportional to one another. Therefore, if two straight lines are cut in extreme and mean ratio, etc. Which was to be demonstrated.

Diagram. Two horizontal straight segments, each divided at an interior point.

- Upper segment: A (left end) — E (division point) — B (right end)
- Lower segment: C (left end) — F (division point) — D (right end)

LIBER XIV.

[The demonstration of Prop. 2 continues from the previous page; the figure is repeated at the head of the page.]

Diagram. The same two divided segments, repeated on the continuation page.

- Upper segment: A (left end) — E (division point) — B (right end)
- Lower segment: C (left end) — F (division point) — D (right end)

SCHOLIUM.

Here we shall easily demonstrate the following theorem.

The side of the hexagon has the same ratio to the side of the decagon described in the same circle as the straight line subtending an angle of an equilateral and equiangular pentagon has to the side of that pentagon.

[Demonstration.] For since [Margin: f. Elem. XIII.9] the side of the hexagon together with the side of the decagon makes a line cut in mean and extreme ratio, and [Margin: g. Elem. XIII.8] the straight line subtending an angle of the pentagon, when cut in extreme and mean ratio, has for its greater segment the side of the pentagon itself, [Margin: h. Elem. XIV.2] it comes about that the ratio of the side of the hexagon to the side of the decagon — that of the greater segment to the greater — is the same as the ratio which the greater segment of the straight line subtending the pentagon's angle has to the lesser. Since, then, as the greater segment is to the lesser, so is the whole to the greater segment, the side of the hexagon will also be to the side of the decagon as the straight line subtending an angle of the equilateral and equiangular pentagon is to the side of that pentagon. Which is what was proposed.

From this proposition, together with the preceding one, we shall also infer this further theorem.

If the perpendicular drawn from the center of a circle to the side of a pentagon described in the same circle is cut in extreme and mean ratio, the greater segment equals the perpendicular drawn from the same center to the side of the equilateral triangle, while the lesser segment equals half the side of the decagon inscribed in the same circle.

[Demonstration.] For from the center A of circle BC let the perpendicular AD be drawn to BC, the side of a pentagon inscribed in the same circle, and let it be cut at E in extreme and mean ratio. I say that the greater segment AE equals the perpendicular drawn from A to the side of the equilateral triangle of the same circle, while the lesser segment ED equals half the side of the decagon of the same circle. For let the straight line AD be extended to F, and let the straight line FC be joined; then FC will be the side of the decagon, since — by Corollary 1 of Prop. 1 of this book — the arc BC (namely a fifth part of the whole circumference) is bisected at F. [Margin: i. Elem. XIII.9] And accordingly the line compounded of AF (the side of the hexagon) and FC (the side of the decagon) is cut in extreme and mean ratio, its greater segment being AF, the side of the hexagon, and its lesser segment FC, the side of the decagon. Therefore, as was shown in this proposition, as the whole AD, divided according to extreme and mean ratio, is to the whole compounded of AF, FC, similarly divided, so is AE (the greater segment) to AF (the greater segment), and ED (the lesser segment) to FC (the lesser segment). [Margin: k. Elem. XIV.1] But the whole AD is half of the whole compounded of AF, FC. Therefore AE will be half of AF (the radius), and ED half of FC (the side of the decagon). Since, then, by the corollary of Prop. 12 of Book XIII, the perpendicular drawn from the center to the side of the equilateral triangle equals half the radius, it is clear that the greater segment AE equals the said perpendicular, while the lesser segment ED equals half the side of the decagon. Which is what was proposed.

Diagram. A circle with center A near the lower middle; a short chord BC near the top, with the perpendicular from the center rising through it to the circumference.

- A (lower center, center of circle): center from which the perpendicular AD is drawn
- B (upper-left, on circle): left endpoint of the chord BC (side of the pentagon)
- C (upper-right, on circle): right endpoint of the chord BC
- D (on chord BC): foot of the perpendicular AD
- F (top, on circle): point where AD, extended, meets the circle (bisecting arc BC)

- E (on AD, between D and A): point cutting AD in extreme and mean ratio

Lines drawn: chord BC; the perpendicular AD extended to F; and the straight line FC.

PROBL. 3. PROPOS. 3.

[Margin: ij. / iv.]

If an equilateral pentagon is described in a circle, then the square on the side of the pentagon and the square on the line that subtends two sides of the pentagon, both together, are five times the square described on the radius.

[Demonstration.] In the circle ABCDE, whose center is F, let a pentagon ABCDE be described, and let the straight line AC subtend the two sides AB, BC. I say that the squares on the straight lines CD, AC together are five times the square on the radius of the described circle. For let the straight line AFG be drawn from A through the center; by Corollary 2 of Prop. 10 of Book XIII it bisects both the arc CD and the side CD — at G and H respectively. Then the arc CG will be a tenth part of the whole circumference, and so the straight line CG subtending it will be the side of the decagon. Now since the straight line AG is double the straight line FG, by the scholium to Prop. 4 of Book II the square on AG will be four times the square on FG. [Margin: a. Elem. I.47] But the squares on the straight lines AC, CG together equal the square on AG. Therefore the squares on the straight lines AC, CG together are also four times the square on the [side FG] ...

Diagram. A circle with an inscribed equilateral pentagon ABCDE and center F; a vertical diameter runs from the top vertex A down through the center to the bottom of the circle, with diagonals drawn.

- A (top vertex, on circle): apex of the pentagon; upper end of the diameter AFG
- B (upper-left vertex, on circle)
- C (lower-left vertex, on circle)
- D (lower-right vertex, on circle)
- E (upper-right vertex, on circle)
- F (center of circle)
- G (bottom, on circle): lower end of the diameter AFG (bisecting arc CD)
- H (on the chord CD): point where AG meets and bisects CD

Lines drawn: the pentagon ABCDE; the diagonal AC (subtending sides AB, BC); the diameter AFG; and the chords CG and CD.

Book XIV (Hypsicles) — continued

Proposition 3 (continued) — Demonstration

[to] FG. And so the three squares on the lines AC, CG, and FG are five times the square on that same line FG. [Margin: b — *Elements* XIII.10] Since, therefore, the squares on the lines FG and CG — that is, on the side of the hexagon and on the side of the decagon — together equal the square on the line CD, namely on the side of the pentagon, the squares on the lines AC and CD will likewise be five times the square on the line FG, that is, on the radius. If, then, an equilateral pentagon is inscribed in a circle, the square on the side of the pentagon together with the square on the line that subtends two sides of the pentagon are, taken together, five times the square described on the radius. This was to be demonstrated.

Diagram. A circle with center F, in which the equilateral pentagon ABCDE is inscribed. The diameter AFG runs vertically from A (top) through the center F to G (bottom); it bisects the pentagon's base CD at the point H. The diagonal AC is drawn, together with the chords CG and CD.

- A (top of circle): pentagon vertex; upper end of the diameter
- B (upper left): pentagon vertex
- C (lower left): pentagon vertex
- D (lower right): pentagon vertex
- E (upper right): pentagon vertex
- F (center): center of the circle
- G (bottom): lower end of the diameter, diametrically opposite A
- H (on CD, below F): point where the diameter AG meets and bisects the base CD

Corollary

There follows from these results a theorem of this kind.

If a cube and a dodecahedron are inscribed in the same sphere, the square on the side of the cube and the square on the side of the dodecahedron, taken together, are five times the square on the radius of the circle that circumscribes a pentagon of the dodecahedron.

For let the circle ABCDE circumscribe one pentagon of a dodecahedron inscribed in some sphere. Then, by the second corollary to Proposition 17 of Book 13, the line AC, which subtends the pentagon's angle B, will be the side of the cube inscribed in the same sphere. [Margin: c — *Elements* XIV.3] And it is clear that the squares on the lines AC and CD — that is, on the side of the cube and on the side of the dodecahedron — are five times the square on the radius FG, and so on.

THEOR. 4. PROPOS. 4.

[Margin: outer dual proposition numbers "[?]. / iij." — Hypsicles-upper / Campanus-lower; foxed]

If the side of the hexagon of some circle is cut in extreme and mean ratio, its greater segment will be the side of the decagon of the same circle.

Demonstration. For let AB, the side of the hexagon of some circle, be cut in extreme and mean ratio at C, and let AC be the greater segment. I say that AC is the side of the decagon of the same circle. Let AD be the side of the decagon of the same circle, and let it be joined in a straight line to AB. [Margin: a — *Elements* XIII.9] Then the whole line BD, composed of the side of the hexagon and the side of the decagon, will have been divided at A in extreme and mean ratio, its greater segment being BA. But AB too has been cut at C in the same way. [Margin: b — *Elements* XIV.2] Therefore, as BD, the whole, is to its greater segment AB, so is AB, the whole, to its greater segment AC. But it is also the case that, as BD, the whole, is to its greater segment AB, so is AB, the greater segment, to AD, the lesser segment, by the definition of a line cut in extreme and mean ratio. Therefore, as AB is to AC, so is the same AB to AD; and consequently AC and AD are equal. Since, then, AD is taken to be the side of the decagon of that circle whose side of the hexagon is AB, AC too will be the side of the decagon of the same circle.

Alternatively. Let there be joined to AB, in a straight line, the line AD equal to the greater segment AC. [Margin: c — *Elements* XIII.5] Then the whole line BD will have been divided at A in extreme and mean ratio. Since, therefore, AB, its greater segment, is taken to be the side of the hexagon of some circle, then — by what we demonstrated at Proposition 9 of Book 13 — AD, the lesser segment, will be the side of the decagon of the same circle; and consequently AC too, being equal to AD, will be the side of the decagon inscribed in the same circle whose side of the hexagon is taken to be AB. If, then, the side of the hexagon of some circle, and so on. This was to be shown.

Diagram. A single straight line marked, from left to right, with the points B, C, A, D. AB is the side of the hexagon (cut at C in extreme and mean ratio, greater segment AC); AD, joined on in a straight line, is the side of the decagon.

- B (left end): endpoint of the hexagon side
- C (between B and A): point of section in extreme and mean ratio
- A (between C and D): common point where the hexagon side AB meets the decagon side AD
- D (right end): outer endpoint of the decagon side

THEOR. 5. PROPOS. 5.

[Margin: outer dual proposition numbers "ij. / v."]

The same circle circumscribes both a pentagon of the dodecahedron and a triangle of the icosahedron inscribed in the same sphere.

Demonstration. In a sphere whose diameter is A, let there be understood to be described both a dodecahedron, one of whose pentagons is BCDEF, and an icosahedron, one of whose triangles is GHI. I say that the same circle circumscribes both the pentagon BCDEF and the triangle GHI — that is, that the circles BCDEF and GHI, which circumscribe them, are equal. For let the circle KLMNO circumscribe the pentagon KLMNO [Margin: a — *Elements* XIII.13] from which the icosahedron is inscribed in the sphere, in such a way that the five sides of this pentagon are sides of the icosahedron, and therefore each of them is equal to a side of the triangle GHI. Next, from the centers P, Q, R let the lines PE, QH, RM be drawn; and let RM be cut in extreme and mean ratio at S. [Margin: b — *Elements* XIV.4] Then the greater segment RS will be the side of the decagon in the circle KLMNO, in which RM is the side of the hexagon. Let the line CF also subtend the angle B of the pentagon of the dodecahedron; by the second corollary to Proposition 17 of Book 13, this will be the side of the cube described in the same sphere. Since, therefore, by the first corollary to the same Proposition, BF, the side of the dodecahedron, is the greater segment of the side of the cube CF cut in extreme and mean ratio, [Margin: c — *Elements* XIV.2] it will follow that, as the whole CF is to the whole RM, so is BF, the greater segment, to RS, the greater segment; and consequently, as the square on the line CF is to the square on the line RM, so is the square on the line BF to the square on the line RS. And so, by Proposition 22 of Book 5, as three times the square on the line CF is to five times the square on the line...

[The three-circle figure for this proposition is printed here at the foot of the page; it is reproduced again at the head of the following page. See fig-p574-3 below.]

Diagram. Three circles. At the very top a separate short straight segment is marked A — the diameter of the sphere. **Upper-left circle:** the pentagon BCDEF inscribed, with center P; B is at the top, C at the upper left, F at the upper right, D at the lower left, E at the lower right; the line PE is drawn from the center to E. **Upper-right circle:** the triangle GHI inscribed, with center Q; G is at the top, H at the lower left, I at the lower right; the line QH is drawn from the center to H. **Lower circle:** the pentagon KLMNO inscribed, with center R; K is at the top, L at the upper left, O at the upper right, M at the lower left, N at the lower right; the line RM is drawn from the center to M and is cut at S (between R and M).

- A (top segment): the sphere's diameter, shown apart
- B, C, D, E, F: vertices of the dodecahedron's pentagon (upper-left circle); P its center; PE drawn
- G, H, I: vertices of the icosahedron's triangle (upper-right circle); Q its center; QH drawn
- K, L, M, N, O: vertices of the pentagon circumscribing the icosahedron (lower circle); R its center; RM drawn and cut at S

[The three-circle figure of Proposition 5 (fig-p574-3) is reprinted here in the outer part of the page, alongside the continuing demonstration.]

Proposition 5 — Demonstration (continued)

on the line RM, so is three times the square on the line BF to five times the square on the line CF, [and this is] equal to five times the square on the line RM — because both three times the square on the side of the cube CF (by Proposition 15 of Book 13) and five times the square on the radius RM (by the first corollary to Proposition 16 of Book 13) are equal to the square on A, the diameter of the sphere. Therefore three times the square on the line BF too will be equal to five times the square on the line RS. [Margin: e — *Elements* XIII.10] Now since the square on the line MN, the side of the pentagon, is equal to the square on the side RM of the hexagon together with the square on the line RS, the side of the decagon, five squares on the line MN will likewise equal five squares on the line RM together with five squares on the line RS. Therefore five squares on the line MN are also equal to three squares on the line CF (since these have been shown to be equal to five squares on the line RM) together with three squares on the line BF (which were equal to five squares on the line RS). But since [Margin: f — *Elements* XIV.3] the squares on the lines CF and BF are together equal to five

squares on the radius PE, three squares on the line CF together with three squares on the line BF are equal to fifteen squares on the line PE; and five squares on the line MN are equal to fifteen squares on the radius QH. [Margin: g — *Elements* XIII.12] (For since MN has been shown to be equal to GH, the side of the icosahedron, that is, of the equilateral triangle, and since the square on that GH is equal to three squares on the radius QH, five squares on the line GH — that is, on the line MN — will be equal to fifteen squares on the line QH.) Fifteen squares on the line PE will therefore also be equal to fifteen squares on the line QH; and consequently one square on the line PE is equal to one square on the line QH, and therefore the radius PE is equal to the radius QH. The circles BCDEF and GHI are therefore equal. The same circle, then, circumscribes both, and so on. This was to be demonstrated.

THEOR. 6. PROPOS. 6.

[Margin: outer dual proposition numbers "iij. / vi."]

If, from the center of the circle circumscribing a pentagon of the dodecahedron, a perpendicular is drawn to one side of the pentagon, then the rectangle contained by the said side and the perpendicular, taken thirty times, will be equal to the surface of the dodecahedron.

Demonstration. From F, the center of the circle circumscribing the pentagon ABCDE of the dodecahedron, let the perpendicular FG be drawn to the side CD. I say that the rectangle under FG and CD, taken thirty times, is equal to the whole surface of the dodecahedron. For if lines are drawn from F to all the angles of the pentagon, the pentagon will be resolved into five equal triangles, by the corollary to Proposition 8 of Book 1, since two sides of each are equal to two sides of the others, and the base of each is likewise equal to the bases of the others. Let there also be drawn through F the line HI, parallel to CD, meeting the two perpendiculars CH and DI — drawn from C and D to CD — at the points H and I. Then the rectangle CI will be that contained by HC, that is, by FG and CD. [Margin: a — *Elements* I.41] Now since the rectangle CI is also double the triangle FCD — and so double each of the remaining four triangles equal to FCD — the rectangle CI, taken five times, will be equal to ten such triangles, that is, to two pentagons of the dodecahedron, since five equal triangles make up one pentagon. And therefore six times the rectangle CI taken five times — that is, the rectangle CI taken thirty times — will likewise be equal to six times the two pentagons, that is, to twelve pentagons, that is, to the whole surface of the dodecahedron. If, then, from the center of the circle circumscribing a pentagon of the dodecahedron, and so on. This was to be demonstrated.

Alternatively. With the same construction of the figure made, [Margin: b — *Elements* I.41] since the rectangle CI is double the triangle FCD — that is, equal to the triangle FCD taken twice — the rectangle CI taken thirty times will be equal to the triangle FCD taken twice thirty times, that is, taken sixty times; and hence to the whole surface of the dodecahedron. For since one pentagon of the dodecahedron consists of five such triangles, the twelve pentagons — namely the whole surface of the dodecahedron — will consist of sixty such triangles. And therefore the triangle FCD, taken sixty times, will be equal to the surface of the dodecahedron.

Diagram. A circle with center F, in which the pentagon ABCDE is inscribed: A at the top, B at the upper left, E at the upper right, C at the lower left, D at the lower right. Lines are drawn from F to all five vertices. The perpendicular FG drops from F to the base CD, meeting it at G. Through F the line HI is drawn parallel to CD, meeting the perpendiculars CH (at H, left) and DI (at I, right); the rectangle HC DI (the rectangle CI) is thereby formed.

- A (top): pentagon vertex
- B (upper left), E (upper right): pentagon vertices
- C (lower left), D (lower right): endpoints of the base
- F (center): center of the circle
- G (on CD): foot of the perpendicular FG
- H (left, on HI): corner of the rectangle, foot of perpendicular from C
- I (right, on HI): corner of the rectangle, foot of perpendicular from D

THEOR. 7. PROPOS. 7.

[Margin: outer dual proposition numbers "iij. / vij."]

If, from the center of the circle circumscribing a triangle of the icosahedron, a perpendicular is drawn to one side of the triangle, then the rectangle contained by the said side and the perpendicular, taken thirty times, will be equal to the surface of the icosahedron.

Demonstration. From D, the center of the circle circumscribing the triangle ABC of the icosahedron, let the perpendicular DE be drawn to the side. I say that the rectangle contained by DE and BC, taken thirty times, is equal to the whole surface of the icosahedron. For if lines are drawn from D to all the angles of the triangle, the triangle will be resolved into three equal triangles, by the corollary to Proposition 8 of Book 1; for two sides of each are equal to two sides of the others, and the base of each is equal to the bases of the others.

Proposition 7 — Demonstration (continued)

Let there also be drawn through D the line FG, parallel to BC, meeting at F and G the two perpendiculars BF and CG drawn from B and C to DE. Then the rectangle BG will be that contained by BF, that is, by DE and BC. [Margin: a — *Elements* I.41] Now since the rectangle BG is double the triangle BDC — and so double each of the remaining two triangles equal to BDC — the rectangle BG, taken three times, will be equal to six such triangles, that is, to two triangles of the icosahedron, since three such equal triangles make up one triangle of the icosahedron. And therefore the same rectangle taken thirty times — namely ten times the rectangle BG taken three times — will be equal to twenty triangles of the icosahedron, that is, to the whole surface of the icosahedron, namely to ten times the two triangles of the icosahedron. If, then, from the center of the circle circumscribing a triangle of the icosahedron, and so on. This was to be shown.

Alternatively. With the same construction of the figure made, [Margin: b — *Elements* I.41] since the rectangle BG is double the triangle DBC — that is, equal to the said triangle taken twice — the rectangle BG taken thirty times will be equal to the triangle DBC taken sixty times. But sixty such triangles make up twenty triangles of the icosahedron, since three make one. Therefore the rectangle BG, taken thirty times, is equal to the whole surface of the icosahedron.

Diagram. A circle with center D, in which the equilateral triangle ABC is inscribed: A at the top, B at the lower left, C at the lower right. Lines are drawn from D to the three vertices. The perpendicular DE drops from D to the base BC at E. Through D the line FG is drawn parallel to BC, meeting the perpendiculars BF (at F, left) and CG (at G, right); the rectangle FBCG (the rectangle BG) is thereby formed, with its diagonals drawn.

- A (top): triangle vertex
- B (lower left), C (lower right): endpoints of the base
- D (center): center of the circle
- E (on BC): foot of the perpendicular DE
- F (left, on FG): corner of the rectangle, foot of perpendicular from B
- G (right, on FG): corner of the rectangle, foot of perpendicular from C

Corollary

And so the surface of any dodecahedron is to the surface of any icosahedron whatever — even if the two figures are not described in the same sphere — as the rectangle contained by the side of the dodecahedron and the perpendicular

drawn from the center of the pentagon of the dodecahedron to the said side, is to the rectangle contained by the side of the icosahedron and the perpendicular drawn from the center of the triangle of the icosahedron to its said side. [Margin: c — *Elements* XIV.6] For since the surface of the dodecahedron is equal to the first rectangle taken thirty times, [Margin: d — *Elements* XIV.7] and likewise the surface of the icosahedron is equal to the second rectangle taken thirty times as well, that first rectangle will be a thirtieth part of the surface of the dodecahedron, just as this second rectangle is a thirtieth part of the surface of the icosahedron. [Margin: e — *Elements* V.15] Therefore the surface of the dodecahedron will be to the surface of the icosahedron as that first rectangle — the thirtieth part of the dodecahedron — is to this second rectangle, the thirtieth part of the icosahedron. This is what was proposed.

Scholium

By an argument not unlike this we shall investigate the surfaces of the remaining three regular solids, namely of the tetrahedron, the octahedron, and the cube. For if, from the center of the circle circumscribing the triangle of the tetrahedron, or of the octahedron, or the square of the cube, a perpendicular is drawn to any one of their sides, then the rectangle contained by the said side and the perpendicular will — in the case of the tetrahedron, taken six times — be equal to the surface of the tetrahedron; while in the case of the octahedron and the cube, taken twelve times, it will be equal to the surface of the octahedron and of the cube. For let there be, first, the triangle ABC of the tetrahedron, circumscribed by a circle whose center is D, from which let the perpendicular DE be drawn to the side BC, and let the rest be done as was said of the triangle of the icosahedron. I say that the rectangle BG, taken six times, is equal to the surface of the tetrahedron. [Margin: f — *Elements* I.41] For since the rectangle BG is double the triangle DBC — that is, equal to two such triangles — it, taken six times, will be equal to twelve such triangles. Since, then, twelve such triangles make up the whole surface of the tetrahedron — namely four equilateral triangles equal to ABC, because three make one — the rectangle BG, taken six times, will be equal to the surface of the tetrahedron, or pyramid.

Next let the same triangle ABC be of the octahedron, and so on. I say that the rectangle BG, taken twelve times, is equal to the surface of the octahedron. [Margin: g — *Elements* I.41] For since the rectangle BG is double the triangle DBC — that is, equal to two such triangles — the same rectangle BG, taken twelve times, will be equal to twenty-four such triangles. But twenty-four such triangles make up the eight triangles of the octahedron, that is, the whole surface of the

octahedron, because three make one. Therefore the rectangle BG, taken twelve times, is equal to the surface of the octahedron.

Let there be, thirdly, the square HIKL of the cube, about which a circle is described from the center M; and, MN having been drawn as a perpendicular to IK, let lines be drawn from M to all the angles of the square, so that the square is resolved into four equal triangles. Then through M let OP be drawn parallel to IK. I say that the rectangle IP, contained by OI — that is, by MN — and IK, taken twelve times, is equal to the surface of the cube. [Margin: h — *Elements* I.41] For since the rectangle IP is double the triangle MIK — that is, equal to two such triangles — it, taken twelve times, will be equal to twenty-four such triangles. Therefore, since twenty-four such triangles make up six squares — that is, the whole surface of the cube, because four such triangles make one square — the rectangle IP, taken twelve times, will be equal to the surface of the cube. This is what was proposed.

Alternatively. Since two rectangles IP make up the square HIKL — for IP and PH are equal, since the line OI is half the side HI, being equal to MN — twelve such rectangles will constitute six such squares, that is, the whole surface of the cube.

Diagram. Two circles side by side. **Left circle:** the equilateral triangle ABC inscribed, center D, as in Proposition 7 — A at the top, B at the lower left, C at the lower right; perpendicular DE to the base BC at E; the line FG through D parallel to BC, meeting BF (at F) and CG (at G), forming the rectangle FBCG (the rectangle BG) with its diagonals. **Right circle:** the square HIKL of the cube inscribed, center M — H at the upper left, L at the upper right, I at the lower left, K at the lower right; lines drawn from M to the four corners; MN dropped perpendicular from M to the side IK at N (lower middle); OP drawn through M parallel to IK, meeting the left side at O and the right side at P; the rectangle OIKP (the rectangle IP), the lower half of the square, is thereby formed.

- Left circle — A (top), B (lower left), C (lower right): triangle vertices; D (center); E (foot of perpendicular on BC); F (left), G (right): rectangle corners on FG
- Right circle — H (upper left), L (upper right), I (lower left), K (lower right): square vertices; M (center); N (foot of perpendicular MN on IK); O (left, on OP), P (right, on OP): endpoints of the parallel through the center

PROBL. 8. PROPOS. 8.

[Margin: outer dual proposition numbers "iv. / viij."]

The rectangle contained by three-quarters of the diameter of some circle and by five-sixths of the line subtending the angle of an equilateral pentagon described in the same circle is equal to the said pentagon.

Book XIV (the Hypsicles supplement)

Proposition 8 (demonstration, continued)

In the circle whose center is F, let the equilateral pentagon ABCDE be inscribed, and let the diameter AG be drawn and divided into four equal parts, of which three make up AH. Let the straight line BE, which subtends the angle A, be cut into six equal parts, of which five make up BI and one makes up IE, so that BI is five times IE. I say that the rectangle contained by AH and BI is equal to the pentagon ABCDE.

For let the straight line BF be drawn. Since AH contains three-quarters of the diameter, and AF two-quarters (being the radius), AH will be to AF as three to two. And since the straight line FA, which bisects the arc BAE, also bisects the straight line BE at the point K (by corollary 1 of *Elements* XIII.10), KB and KE will each contain three-sixths of BE; and so, since BI contains five of them, KI will contain two. Consequently KB is to KI also as three to two. Therefore AH is to AF as KB is to KI [Margin: a — *Elements* VI.16]; and hence the rectangle under the extremes AH, KI will be equal to the rectangle under the means AF, KB [Margin: b — *Elements* I.41]. Now the rectangle under AF, KB is double the triangle ABF, since the rectangle and the triangle have the same base AF and the same height BK—that is, they stand between the same parallels. Therefore the rectangle under AH, KI is likewise double the same triangle ABF. But the rectangle under AH, KI is also double the rectangle under AH, IE [Margin: c — *Elements* VI.1], because the base KI is double the base IE (for KI contains two sixths of BE, and IE one), the height AH remaining the same. Therefore the triangle ABF and the rectangle under AH, IE are equal; and so their quintuples are equal as well. But the pentagon ABCDE is five times the triangle ABF, since the pentagon is resolved into five such triangles by drawing straight lines from the center F, as we showed in Proposition 6 of this book. And the rectangle under AH, BI is five times the rectangle under AH, IE [Margin: d — *Elements* VI.1], since the base BI is five times the base IE, the height AH remaining the same. Therefore the pentagon ABCDE and the rectangle under AH, BI are equal. The rectangle, then, contained by the three-quarters [of the diameter and the five-sixths of the subtending line], and so forth. Which was to be demonstrated.

Diagram. A circle with center F and inscribed equilateral pentagon ABCDE. The vertical diameter runs from A (top) down through the interior to G (bottom). The chord BE is horizontal, subtending the pentagon's apex angle at A, and is marked with tick-strokes dividing it into equal parts.

- A (top of circle): apex vertex of the pentagon; upper end of the diameter AG
- B (left, on chord BE): pentagon vertex
- E (right, on chord BE): pentagon vertex
- C (lower left): pentagon vertex
- D (lower right): pentagon vertex
- G (bottom of circle): lower end of the diameter
- K (interior, on the vertical diameter where it meets chord BE): bisection point of BE
- I (on chord BE, right of K): division point marking the segment IE
- F (interior, below K on the vertical line): center of the circle
- H (interior, below F on the vertical line): point dividing the diameter (AH = three-quarters of the diameter)

Corollary

Moreover, if the pentagon ABCDE is set up as the base of a dodecahedron inscribed in some sphere, then BE is the side of the cube inscribed in the same sphere (by corollary 2 of *Elements* XIII.17), and it is likewise one side of the triangle of the icosahedron set up in the same sphere [Margin: e — *Elements* XIV.5]—being, that is, comprehended by the same circle as the pentagon of the dodecahedron. This line cuts the diameter AG at H (since, by the corollary to *Elements* XIII.12, it bisects the semidiameter FG, just as the point H likewise bisects the same semidiameter FG). It is therefore manifest that the rectangle contained by the perpendicular drawn from an angle of the icosahedron's triangle to one of its sides—namely AH—and by five-sixths of the side of the cube belonging to the same sphere in which the icosahedron is inscribed—namely BI—is equal to the pentagon of the dodecahedron set up in the same sphere.

THEOR. 9. PROPOS. 9.

[Margin: iv. / viij.]

The surface of the dodecahedron has to the surface of the icosahedron inscribed in the same sphere the same ratio as the side of the cube to the side of the icosahedron.

In the circle ABCD [Margin: a — *Elements* XIV.5]—which circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron—let BD be the side of the pentagon and AD the side of the triangle; and from the center E let the perpendiculars EF, EG be drawn to AD and BD; and finally let H be the side of the cube inscribed in the same sphere as the dodecahedron and the icosahedron. I say that as H, the side of the cube, is to AD, the side of the icosahedron, so is the surface of the dodecahedron to the surface of the icosahedron. For since, when the side of the cube H is cut in extreme and mean ratio, the greater segment is BD, the side of the dodecahedron (by corollary 2 of *Elements* XIII.17); and since, when the straight line EG is likewise cut in extreme and mean ratio, the greater segment is EF (by the scholium to Proposition 2 of this book) [Margin: b — *Elements* XIV.2]: as H, the whole, is to BD, its greater segment, so will EG, the whole, be to EF, its greater segment. And accordingly the rectangle under the extremes H, EF will be equal to the rectangle under the means BD, EG [Margin: c — *Elements* VI.16]. Now since, as H is to AD, so is the rectangle under H, EF to the rectangle under AD, EF [Margin: d — *Elements* VI.1] (their bases being H and AD, while the width remains the same, EF): as H is to AD, so also will the rectangle under BD, EG (which is equal to the rectangle under H, EF) be to the same rectangle under AD, EF. Therefore, since by the corollary to Proposition 7 of this book, as the rectangle under BD, EG is to the rectangle under AD, EF, so is the surface of the dodecahedron to the surface of the icosahedron: as H, the side of the cube, is to AD, the side of the icosahedron, so also will the surface of the dodecahedron be to the surface of the icosahedron. Therefore the surface of the dodecahedron is to the surface of the icosahedron, and so forth. Which was to be shown.

Diagram. A circle with center E, in which a pentagon-side and a triangle-side are drawn, with perpendiculars from the center to those sides; the cube-side H is shown separately as a vertical measuring-segment (marked with tick-strokes) to the right of the circle.

- A (left, on the circle): triangle vertex (endpoint of the icosahedron side AD)
- D (right, on the circle): common endpoint of the pentagon side BD and the triangle side AD
- E (interior, upper): center of the circle
- F (interior, below E): foot of the perpendicular EF onto AD
- G (interior, below F): foot of the perpendicular EG onto BD
- B (lower left, on the circle): endpoint of the pentagon side BD
- C (bottom, on the circle): circle point

- H (right of the circle, vertical segment): the side of the cube, drawn as a separate measuring line

Otherwise

[Margin: e — *Elements* XIV.5] Let the same circle, whose center is H, circumscribe the pentagon ABCDE of the dodecahedron and the triangle AFG of the icosahedron. Then let the straight line AI be drawn from A through the center H; by corollary 2 of *Elements* XIII.10 it will bisect the side FG at right angles. Finally let the straight line BE be joined, which will be the side of the cube (by corollary 2 of *Elements* XIII.17); and let BK be five-sixths of it. I say again that as BE, the side of the cube, is to FG, the side of the icosahedron, so is the surface of the dodecahedron to the surface of the icosahedron. For since the rectangle under AI, BK is equal to the pentagon ABCDE (by the corollary to Proposition 8 of this book), and the rectangle under AI, IF is equal to the triangle AFG (by the scholium to *Elements* I.41, since FG, the base of the triangle, is double IF, the base of the rectangle): as the rectangle under AI, BK is to the rectangle under AI,

Diagram. A circle with center H in which the pentagon ABCDE and the triangle AFG are inscribed, with connecting chords forming a pentagram-like figure.

- A (top): common apex of the pentagon and the triangle
- B (upper left): pentagon vertex
- E (upper right): pentagon vertex
- K (interior, near E): intersection point of connecting lines
- H (center): center of the circle
- F (left): triangle vertex
- I (bottom center): midpoint of the triangle base FG (foot of AI)
- G (right): triangle vertex
- C (lower left): pentagon vertex
- D (lower right): pentagon vertex

THEOR. 9. PROPOS. 9. — "Otherwise" (continued)

[Margin: f — *Elements* VI.1] IF, so is the pentagon ABCDE to the triangle AFG. But as the rectangle under AI, BK is to the rectangle under AI, IF, so is BK to IF, since their bases are BK and IF while the height AI remains ever the same. Therefore, as the pentagon ABCDE is to the triangle AFG, so also is BK to IF. And accordingly—if the first and third, namely the pentagon ABCDE and the straight line BK, are equally multiplied by the number 12, and likewise the second and fourth, namely

the triangle AFG and the straight line IF, are equally multiplied by the number 20 [Margin: g — scholium to *Elements* V.22]—then also, as the pentagon ABCDE taken twelve times (that is, the surface of the dodecahedron) is to the triangle AFG taken twenty times (namely, to the surface of the icosahedron), so is the straight line BK taken twelve times to the straight line IF taken twenty times. But BK taken twelve times equals the side of the cube BE taken ten times (for since BK contains five-sixths of BE, BK taken twelve times will contain sixty sixths of BE, which make up ten whole lines BE). And IF taken twenty times equals the side of the icosahedron FG taken ten times (because FG is double IF). Therefore the surface of the dodecahedron will be to the surface of the icosahedron as the side of the cube BE taken ten times is to the side of the icosahedron FG taken ten times. And accordingly, since ten sides of the cube have to ten sides of the icosahedron the same ratio as one side of the cube to one side of the icosahedron [Margin: h — *Elements* V.15], it is clear that the surface of the dodecahedron is to the surface of the icosahedron as the side of the cube to the side of the icosahedron. Which is what was proposed.

Diagram. The figure for the "Otherwise" proof, reproduced: a circle with center H, in which the pentagon ABCDE and the triangle AFG are inscribed with their connecting chords.

- A (top): common apex of the pentagon and the triangle
- B (upper left): pentagon vertex
- E (upper right): pentagon vertex
- K (interior, near E): intersection point of connecting lines
- I (interior, right of center): point on the horizontal line
- F (interior, below K): triangle vertex / midpoint region
- H (center): center of the circle
- C (lower left): pentagon vertex
- D (lower right): pentagon vertex
- G (bottom): triangle vertex

THEOR. 10. PROPOS. 10.

[Margin: iv. / ix.]

If a straight line is cut in extreme and mean ratio, then the line whose square equals the [squares] on the whole and on the greater segment will be to the line whose square equals the [squares] on the whole and on the

lesser segment as the side of the cube to the side of the icosahedron inscribed in the same sphere as the cube.

Let the straight line AB be cut at C in extreme and mean ratio, and let AC be the greater segment. Let D be the side of the cube and E the side of the icosahedron inscribed in the same sphere as the cube. With A as center and radius AB let a circle be described, and in it let the equilateral pentagon BFGHI and the equilateral triangle BKL be constructed. If, then, BF is taken as the side of the dodecahedron of some sphere, the straight line BG, when drawn, will be the side of the cube inscribed in the same sphere (by corollary 2 of *Elements* XIII.17). Now since AC, the greater segment of the semidiameter AB cut in extreme and mean ratio, is the side of the decagon [Margin: a — *Elements* XIV.4] in the circle BFGHI—in which AB is the side of the hexagon—BF will be able to produce, as its square, the [square on the] whole AB (that is, the side of the hexagon) together with the [square on the] greater segment AC (that is, the side of the decagon) in the same circle [Margin: b — *Elements* XIII.10]. And now let the straight line M be able to produce, as its square, the [squares on the] whole AB and the lesser segment BC, by what we demonstrated at *Elements* I.47. I say that the proposed side D of the cube is to the proposed side E of the icosahedron as BF—whose square equals the [squares on the] whole AB and the greater segment AC—is to M—whose square equals the [squares on the] whole AB and the lesser segment BC. For since the square on the side of the equilateral triangle BK [Margin: c — *Elements* XIII.12] is triple the square on the semidiameter AB; and likewise the squares on AB, BC are triple the square on AC [Margin: d — *Elements* XIII.4]: as the square on BK is to the square on AB, so will the squares on AB, BC—that is, the square on M, which is equal to them—be to the square on AC. For in both cases the ratio is triple. Therefore, by permutation, as the square on BK is to the square on M, so is the square on AB to the square on AC. But as the square on AB is to the square on AC, so is the square on BG to the square on BF. (For since BF is the greater segment of BG cut in extreme and mean ratio—by corollary 1 of *Elements* XIII.17, because BG is the side of the cube and BF the side of the dodecahedron of the same sphere—it will follow [Margin: e — *Elements* XIV.2] that, as the whole AB is to its greater segment [AC, so the whole BG is to] its greater segment BF [?]; and accordingly [Margin: f — *Elements* VI.22], as the square on AB is to the square on AC, so is the square on BG to the square on BF.) Therefore, as the square on BG is to the square on BF, so is the square on BK to the square on M; and, by permutation, as the square on BG is to the square on BK, so is the square on BF to the square on M. And on this account [Margin: g — *Elements* VI.22], as the straight line BG is to the straight line BK, so is BF—whose square equals the [squares on the] whole AB and the greater segment AC—to M—whose square equals the

[squares on the] whole AB and the lesser segment BC. Since, then, the proposed sides D, E of the cube and icosahedron have the same ratio as the straight lines BG, BK (as we shall show shortly), it is clear that, as the straight line BF (whose square equals the whole AB and the greater segment AC) is to the straight line M (whose square equals the whole AB and the lesser segment BC), so also the proposed side D of the cube is to the proposed side E of the icosahedron. If, therefore, a straight line is cut in extreme and mean ratio, and so forth. Which was to be shown.

Diagram. A circle with center A, in which an equilateral pentagon BFGHI and an equilateral triangle BKL are inscribed; below the circle three separate measuring-segments are drawn, labelled D and E (bracketed together) and M.

- B (top, on the circle): common vertex of the pentagon and the triangle
- F (upper left): pentagon vertex
- I (upper right): pentagon vertex
- C (interior, mid-left): division point on the diameter (segment AC)
- A (center): center of the circle
- L (interior, mid-right): triangle vertex region
- K (lower left, on the circle): triangle vertex
- G (bottom, on the circle): pentagon vertex
- H (lower right, on the circle): pentagon vertex
- D, E (segments below the circle, bracketed together): the side of the cube (D) and the side of the icosahedron (E)
- M (segment below the circle): the constructed line whose square equals the squares on AB and BC

SCHOLIUM

That the proposed sides D and E of the cube and icosahedron of the same sphere have the same ratio as the straight lines BG, BK, we shall demonstrate by the following theorem.

The ratio that the sides of the cube and the icosahedron of the same sphere have, the sides of the cube and the icosahedron described in any other sphere also have.

For let A be the side of the cube and BC the side of the icosahedron of the same sphere; and let D be the side of the cube and EF the side of the icosahedron in any other sphere. I say that as A is to BC, so is D to EF. For when the triangles of the

icosahedra, BCG and EFH, have been completed, and circles have been described about them, let there be inscribed in these circles the equilateral pentagons BIKLM, ENOPQ; and let the straight lines BK,

SCHOLIUM (continued)

CK, EO, FO be joined. [Margin: h — Elements XIV.5] Since, then, the same circle comprehends both the pentagon of the dodecahedron and the triangle of the icosahedron of the same sphere; and since BCG is taken as the triangle of the icosahedron described in the same sphere in which the cube of side A [is inscribed]: BIKLM will be the pentagon of the dodecahedron described in the same sphere as they are. And therefore, by corollary 2 of Elements XIII.17, the straight line BK will be the side of the cube of the same sphere, and hence equal to A itself. By a similar argument the straight line EO will be equal to D, the side of the cube. Again, since the arcs BIK, ENO are similar (for each contains two-fifths of its own circumference), and likewise the arcs BGC, EHF (for each comprehends two-thirds of the whole circumference): the angles BCK, EFO will be equal to one another, by the definition of similar segments, as will the angles BKC, EOF to one another, since both the former and the latter stand in similar segments of circles. And accordingly, by corollary 1 of Elements I.32, the remaining angle CBK of the triangle BCK will be equal to the remaining angle FEO of the triangle EFO, and the whole triangle BCK will be equiangular with the whole triangle EFO. [Margin: i — Elements VI.4] Therefore, as BK—that is, the straight line A equal to it, namely the side of the cube—is to BC, the side of the icosahedron, so is EO—that is, the straight line D equal to it, namely the side of the other cube—to EF, the side of the icosahedron. Which is what was proposed.

It is therefore clear what was assumed in the demonstration, namely that D and E, the sides of the cube and icosahedron in one sphere, have the same ratio as BG, BK, the sides of the cube and icosahedron in another sphere.

From this it is also inferred that the surface of the dodecahedron has to the surface of the icosahedron not only the same ratio as the side of the cube to the side of the icosahedron in the same sphere as they are (as Proposition 9 of this book maintains), but also the ratio that the side of the cube has to the side of the icosahedron in any other sphere whatever.

Diagram. Two circles side by side. In each, an icosahedron triangle and an inscribed equilateral pentagon are drawn, with connecting chords; beneath each circle a measuring-segment gives the side of the cube of that sphere.

Left circle (triangle BCG, pentagon BIKLM):

- B (top): common vertex of triangle and pentagon
- I (left): pentagon vertex
- C (lower left): triangle vertex
- G (lower right): triangle vertex
- M (right): pentagon vertex
- K (bottom left): pentagon vertex
- L (bottom right): pentagon vertex
- A (segment beneath the circle): the side of the cube of this sphere

Right circle (triangle EFH, pentagon ENOPQ):

- E (top): common vertex of triangle and pentagon
- N (left): pentagon vertex
- F (lower left): triangle vertex
- H (lower right): triangle vertex
- Q (right): pentagon vertex
- O (bottom left): pentagon vertex
- P (bottom right): pentagon vertex
- D (segment beneath the circle): the side of the cube of this sphere

LEMMA I

If a sphere is cut by any plane, the common section will be a circle.

Let the sphere ABC, whose center is D, be cut by some plane producing the plane common section BEFCG. I say that BEFCG is a circle. First, let the cutting plane pass through the center of the sphere, so that the center D lies in the plane—that is, in the common section BEFCG; and from D, the center of the sphere, let any number of straight lines DE, DF, DG be drawn to the boundary of the common section. Since, then, all these lines that have been drawn, however many there be, are equal—by the definition of the sphere, since they fall from its center to its surface—the plane figure BEFCG, contained by a single line, will (by the definition of a circle) be a circle, and its center D will be the same as the center of the sphere.

Secondly, let the cutting plane not pass through the center of the sphere. From D, the center of the sphere, let the perpendicular DH be dropped to the cutting plane; and from H let straight lines HE, HF be drawn anywhere, as far as the boundary of the common section; and let the straight lines DE, DF be joined. Since, then, the angles DHE, DHF are right angles (by definition 3 of Book XI), [Margin: k —

Elements I.47] the square on DE will be equal to the squares on DH, HE, and the square on DF equal to the squares on DH, HF. But the squares on DE, DF—which are equal, falling from the center of the sphere to its surface—are equal. Therefore the squares on DH, HE will also be equal to the squares on DH, HF; and accordingly, when the common square on DH is taken away, the remaining square on HE will be equal to the remaining square on HF. Hence HE and HF will also be equal. By the same argument we shall show that all lines falling from H to the boundary of the common section are equal both to one another and to the two named lines HE, HF. Therefore the plane figure BEFCG, contained by a single line, will (by the definition of a circle) be a circle; and its center is the point H, upon which the perpendicular DH falls. Which is what was proposed.

Diagram. Two shaded globe (sphere) figures illustrating the two cases. In the left globe the cutting plane passes through the center; in the right globe it does not, and the perpendicular from the center to the cutting plane is drawn.

Left globe (plane through the center):

- A (top): pole/point of the sphere
- G (upper left, on the section): boundary point of the common section
- B (left): boundary point
- D (center): center of the sphere and of the section-circle
- C (right): boundary point
- E (lower left, on the section): boundary point
- F (lower center, on the section): boundary point

Right globe (plane not through the center):

- A (top): pole/point of the sphere
- B (left): boundary point of the common section
- D (interior, upper): center of the sphere
- G (interior, just below D): point on the perpendicular / section
- C (right): boundary point
- E (lower left, on the section): boundary point (endpoint of HE)
- H (bottom center): foot of the perpendicular DH, center of the section-circle
- F (bottom, right of H, on the section): boundary point (endpoint of HF)

COROLLARY

Therefore, if the cutting plane passes through the center of the sphere, the circle produced will have the same center as the sphere. But if the cutting plane does

not pass through the center of the sphere, the circle produced will have a center different from the sphere's—namely, that point upon which the perpendicular drawn from the center of the sphere to the cutting plane falls. For it can always be demonstrated that the straight lines falling from this point to the circumference of the circle are equal, as is clear.

LEMMA II

Equal circles in a sphere are equally distant from the center of the sphere; and circles equally distant from the center of the sphere are equal.

In the sphere ABCD, whose center is E, let there be equal circles AB, DC. I say that they are equally distant from the center E. For let the perpendiculars EF, EG be drawn from E to their planes; which, by the corollary of the preceding lemma,

Euclid's *Elements*, Book XIV (the Hypsicles supplement) — continued

Lemma II (concluded)

(Continuing the proof that equal circles in a sphere are equally distant from the center. Sphere ABCD, center E; equal circles AB, DC; perpendiculars EF, EG.)

of the lemma, will fall on the centers of those circles. From these centers draw the lines FB, GC to the circumferences of the circles at any points you like, and join the lines EB, EC. Now since the angles EFB and EGC are right angles, by definition 3 of Book XI, [Margin: l 47.pri. (Elements I.47)] the square on EB will equal the squares on EF and FB together, and likewise the square on EC will equal the squares on EG and GC together. But the squares on the equal lines EB and EC are themselves equal, since these lines fall from the center of the sphere to its surface. Therefore the squares on EF and FB together equal the squares on EG and GC together. And so, when the equal squares on the equal lines FB and GC are taken away — for these are the radii of equal circles — the remaining square on EF equals the remaining square on EG. The perpendiculars EF and EG are therefore equal. For this reason the circles AB and DC are equally distant from the center E.

Diagram. A circle representing the sphere, with center E, and two lens-shaped chord-figures (the two circles seen nearly edge-on) drawn inside it toward the left and right.

- A (upper left of the sphere): endpoint of the left circle
- D (upper right of the sphere): endpoint of the right circle
- B (lower left): endpoint of the left circle
- C (lower right): endpoint of the right circle
- E (center of the sphere): the center from which the perpendiculars are drawn
- F (left of center): foot of the perpendicular EF on the plane of circle AB
- G (right of center): foot of the perpendicular EG on the plane of circle DC

Conversely, let the circles AB and DC now be equally distant from the center E of the sphere. I say that they are equal. For, once the figure has been constructed as before, we shall show in the same way that the squares on EF and FB together equal the squares on EG and GC together. And so, when the equal squares on the equal lines EF and EG are taken away — the circles being assumed equally distant from the center — the remaining square on FB equals the remaining square on GC; hence the lines FB and GC are equal, that is, the radii of the circles AB and DC. Therefore the circles AB and DC are equal. This is what was proposed.

THEOR. 11. PROPOS. 11.

[Margin: iv. / x. (Hypsicles / Campanus proposition numbers)]

The dodecahedron is to the icosahedron described in the same sphere with it as the edge of the cube is to the edge of the icosahedron — in one and the same sphere.

Let a dodecahedron, one of whose pentagons is FGHK, be inscribed in the sphere ABCD, whose center is E; likewise an icosahedron, one of whose triangles is LMN. I say that the dodecahedron is to the icosahedron as the edge of the cube is to the edge of the icosahedron, in one and the same sphere — whether this sphere be the same as sphere ABCD or not. Since all the vertices, both of the pentagon and of the triangle, touch the surface of the sphere, then if planes are passed through the pentagon and through the triangle, there will be produced, by the preceding Lemma 1, two circles in the sphere, FGHK and LMN, containing the pentagon and the triangle respectively.

[Margin: a 5.quartidec. (Elements XIV.5)] Since one and the same circle contains both the pentagon of the dodecahedron and the triangle of the icosahedron, the circles FGHK and LMN will therefore be equal; and consequently, by the preceding Lemma 2, they will be equally distant from the center E of the sphere. Hence the perpendiculars drawn from E to the planes of the circles, and falling — by the corollary of Lemma 1 — on their centers O and P, will be equal, namely EO and EP. In just the same way it will be shown that the perpendiculars drawn from E to all the pentagons of the dodecahedron, and to all the triangles of the icosahedron, are equal. Therefore, if straight lines are drawn from all the vertices of the dodecahedron to E, the dodecahedron will be resolved into twelve pyramids of equal height; and since their bases too are equal — namely the pentagons of the dodecahedron — the pyramids will be equal to one another, by the corollary of Proposition 6 of Book XII. In the same way, if straight lines are drawn from all the vertices of the icosahedron to E, the icosahedron will be resolved into 20 pyramids equal to one another, whose heights will equal the heights of the pyramids of the dodecahedron.

[Margin: b 6.duodecim. (Elements XII.6)] Now since the base FGHK is to the base LMN as the pyramid FGHKE is to the pyramid LMNE — for the two have the same height — then also, by what we demonstrated in the scholium to Proposition 22 of Book V, as the twelve bases of the dodecahedron, that is, the whole surface of the dodecahedron, is to the base LMN, so are the twelve pyramids of the dodecahedron, that is, the whole solid of the dodecahedron, to the pyramid LMNE. Again, since (as we have just said) the whole surface of the dodecahedron is to the icosahedron's base LMN as the dodecahedron is to the pyramid LMNE, it will follow, by the same scholium, that as the whole surface of the dodecahedron is to the 20 bases of the icosahedron — that is, to the whole surface of the icosahedron — so is the dodecahedron to the 20 pyramids of the icosahedron, namely to the whole solid of the icosahedron.

[Margin: c 9.quartidec. (Elements XIV.9)] And thus the surface of the dodecahedron is to the surface of the icosahedron as the edge of the cube is to the edge of the icosahedron inscribed in the same sphere with them; and moreover, by the scholium to Proposition 10 of this book, as the edge of the cube is to the edge of the icosahedron of any single other sphere. Therefore the dodecahedron will also be to the icosahedron as the edge of the cube is to the edge of the icosahedron of any sphere. And so the dodecahedron is to the icosahedron, and so forth. Which was to be demonstrated.

Diagram. A spherical dome ABCD (center E) above two circles: a left circle containing the dodecahedron's pentagon FGHIK (center O) and a right circle containing the icosahedron's triangle LMN (center P).

- A (top of the dome): vertex of the sphere
- B (upper left of the dome): vertex of the sphere
- D (upper right of the dome): vertex of the sphere
- C (bottom, between the two circles): vertex of the sphere
- E (near the top, between the circles): center of the sphere
- F (top of left circle): vertex of the pentagon
- G (left of left circle): vertex of the pentagon
- H (lower left of left circle): vertex of the pentagon
- I (lower right of left circle): vertex of the pentagon
- K (right of left circle): vertex of the pentagon
- O (center of left circle): center of the circle circumscribing the pentagon
- L (top of right circle): vertex of the triangle
- M (lower left of right circle): vertex of the triangle
- N (lower right of right circle): vertex of the triangle
- P (center of right circle): center of the circle circumscribing the triangle

COROLLARIVM.

[Margin: d 9.quartidec. (Elements XIV.9); e 10.quartidec. (Elements XIV.10); f 11.quartidec. (Elements XIV.11)] Since it has also been shown [d] that the surface of the dodecahedron is to the surface of the icosahedron of the same sphere as the edge of the cube is to the edge of the icosahedron inscribed in the same sphere with them; [e] likewise that the edge of the cube is to the edge of the icosahedron of one and the same sphere as the line whose square equals the whole line cut in extreme and mean ratio together with its greater segment is to the line whose square equals that same whole line together with its lesser segment; [f] likewise that the dodecahedron is to the icosahedron of the same sphere as the edge of the cube is to the edge of the icosahedron in one and the same sphere — it is clear that these four ratios, namely: the edge of the cube to the edge of the icosahedron; the surface of the dodecahedron to the surface of the icosahedron; the line whose square equals any whole line cut in extreme and mean ratio together with its greater segment, to the line whose square equals the same whole line together with its lesser segment; and the dodecahedron to the icosahedron of the same sphere — are all equal to one another.

THEOR. 12. PROPOS. 12.

[Margin: o. / xj.]

The side of an equilateral triangle is, in square, one and a third times (in the ratio 4 : 3) the perpendicular drawn from one angle to the opposite side.

In the equilateral triangle ABC let the perpendicular AD be drawn from the angle A to the opposite side BC. I say that the square on the side AB is one and a third times the square on the perpendicular AD. Since, on account of the equality of the sides AB and AC, [Margin: a 5.pri. (Elements I.5)] the angles B and C are equal, and the angles at D are right angles, the two angles B and D of triangle ADB equal the two angles C and D of triangle ADC; and moreover they have the side AD in common, and AB, AC equal. [Margin: b 6.pri. (Elements I.6)] Therefore the remaining sides BD and CD are also equal; and consequently the side BC — that is, the side AB — is double the line BD, and therefore the square on the side AB is quadruple the square on the line BD, by the scholium to Proposition 4 of Book II. [Margin: c 47.pri. (Elements I.47)] Since, then, the squares on the lines AD and BD together equal the square on AB, the squares on AD and BD together will also be quadruple the square on BD. And so, of whatever parts the square on AB (or the squares on AD and BD together) is 4, of such parts the square on AD will be 1 [i.e. the squares on AD and BD are as 4 : 1, whence the square on BD is 1 part and the square on AD is 3 parts]. Therefore the square on AB, of 4 parts, is one and a third times the square on AD, of 3 parts. Hence the side of an equilateral triangle is, in square, one and a third times [the perpendicular], and so forth. Which was to be shown.

Diagram. An equilateral triangle with apex A and base BC, with the perpendicular AD dropped from A to the base.

- A (apex): the angle from which the perpendicular is drawn
- B (lower left): base vertex
- D (foot of the perpendicular, midpoint of the base): where AD meets BC
- C (lower right): base vertex

COROLLARIVM.

From this it is clear that the perpendicular dropped from one angle of an equilateral triangle to the opposite side bisects both the angle and the side. For it was demonstrated that the lines BD and CD are equal — that is, that the side BC is bisected by the perpendicular AD. Hence, since the sides AB and AD of triangle

ABD equal the sides AC and AD of triangle ACD, and the base BD equals the base CD, [Margin: d 8.pri. (Elements I.8)] the angles BAD and CAD will also be equal; that is, the angle BAC will be bisected by the same perpendicular AD.

THEOR. 13. PROPOS. 13.

[Margin: o. / xiiij.]

If the diameter of a sphere is rational, then the surface both of the tetrahedron and of the octahedron in that sphere is a medial area.

Let A be the rational diameter of a sphere. I say that the surface both of the tetrahedron and of the octahedron in the said sphere is a medial area. For let BCD be one triangle of the tetrahedron, and EFG one triangle of the octahedron of that sphere; and from the angles B and E let the perpendiculars BH and EI be drawn to the opposite sides CD and FG. [Margin: a 13.tertijdec. (Elements XIII.13)] Now since the square on the diameter A is one and a half times the square on the tetrahedron's edge BC, [Margin: b 14.tertijdec. (Elements XIII.14)] and double the square on the octahedron's edge EF, the squares on the lines A, BC, EF will have the ratio of the numbers 6, 4, 3; [Margin: c 6.dec. (Elements X.6)] and therefore they are commensurable. Hence the lines A, BC, EF themselves are commensurable at least in square; and moreover, since A is assumed rational, BC and EF will also be rational, and therefore so too will their halves CH and FI.

[Margin: d 12.quartidec. (Elements XIV.12)] Again, since the square on the line BC is one and a third times the square on BH, and likewise the square on EF one and a third times the square on EI, the squares on BC and BH, and likewise the squares on EF and EI, will have to one another the ratio of the numbers 4, 3; [Margin: e 6.dec. (Elements X.6)] and therefore both pairs are commensurable with one another. Hence the lines BC and BH, and likewise the lines EF and EI, are commensurable. But CH is commensurable with BC, and FI with EF — being their halves, that is, commensurable with the whole. [Margin: f 12.dec. (Elements X.12)] Therefore BH and CH, and likewise EI and FI, are commensurable with one another; and consequently, since CH and FI have been shown rational, BH and EI will also be rational. [Margin: g 9.dec. (Elements X.9)] But since CH and BH, and likewise FI and EI, are incommensurable in length — because their squares do not have the ratio of a square number to a square number (as is clear from the corollary of Proposition 24 of Book VIII, for they have the ratio of 1 to 3, [Margin: h 12.quartidec. (Elements XIV.12)] since it has been shown that of whatever parts the square on CH, or the square on FI, is 1, of such parts the square on BH, or the square on EI, is 3) — [Margin: i 22.dec. (Elements X.22)] CH and BH, and likewise

FI and EI, will be rational commensurable in square only. And consequently the rectangle contained by CH and BH — rational and commensurable in square only — and the rectangle contained by FI and EI — likewise rational and commensurable in square only — will each be a medial area. But the triangles BCD and EFG are equal to the rectangles under CH, BH and under FI, EI respectively, by the scholium to Proposition 41 of Book I, since the bases CD and FG of the triangles are double the bases CH and FI of the rectangles. Therefore the triangles BCD and EFG will also be medial areas. And so, since the whole surface of the tetrahedron is commensurable with the triangle BCD (for the triangle BCD is a quarter of the whole surface), and likewise the whole surface of the octahedron is commensurable with the triangle EFG (for the triangle EFG is an eighth of the whole surface), the surfaces of the tetrahedron and of the octahedron will also, by the corollary of Proposition 24 of Book X, be medial areas. Therefore, if the diameter of a sphere is rational, the surface both of the tetrahedron and of the octahedron in that sphere is a medial area. Which was to be shown.

Diagram. At the top, a straight segment A (the sphere's diameter, marked off with ticks). Below it, two triangles: the tetrahedron's face BCD (apex B, with perpendicular BH to the base) and the octahedron's face EFG (apex E, with perpendicular EI to the base).

- A (top segment): the diameter of the sphere
- B (apex of left triangle): angle of the tetrahedron's face
- C (lower left of left triangle): base vertex
- H (foot of perpendicular on left base): where BH meets CD
- D (lower right of left triangle): base vertex
- E (apex of right triangle): angle of the octahedron's face
- F (lower left of right triangle): base vertex
- I (foot of perpendicular on right base): where EI meets FG
- G (lower right of right triangle): base vertex

COROLLARIVM.

It follows from the demonstration of this proposition that every equilateral triangle whose side is rational is a medial area. For from the fact that the sides BC and EF, and hence their halves CH and FI, were shown to be rational, it was demonstrated that CH and BH, and likewise FI and EI, are rational commensurable in square only; and therefore the rectangles under CH, BH and under FI, EI — that is, the triangles BCD and EFG, which are equal to them — are

medial areas. Since, then, the same reasoning holds for every equilateral triangle whose one side is rational, what is inferred is clear.

THEOR. 14. PROPOS. 14.

[Margin: o. / xiv.]

If a tetrahedron and an octahedron are inscribed in the same sphere, the base of the tetrahedron will be one and a third times ($4 : 3$) the base of the octahedron; but the surface of the octahedron will be one and a half times ($3 : 2$) the surface of the tetrahedron.

Let A be the diameter of a sphere in which are described a tetrahedron, whose base is the triangle BCD, and an octahedron, whose base is the triangle EFG. I say that the triangle BCD is one and a third times the triangle EFG, but that the surface of the octahedron is one and a half times the surface of the tetrahedron. [Margin: a 13.tertijdec. (Elements XIII.13)] For since the square on the diameter A is one and a half times the square on the tetrahedron's edge CD, [Margin: b 14.tertijdec. (Elements XIII.14)] and double the square on the octahedron's edge FG, it follows that of whatever parts the square on the diameter A is 6, of such parts the square on the edge CD is 4, and the square on the edge FG is 3. Therefore the square on the line CD is one and a third times the square on the line FG. But as the square on CD is to the square on FG, so is the triangle BCD to the triangle EFG, [Margin: c 20.& 19.sex. (Elements VI.20 & VI.19)] since both the squares and the triangles have to one another the doubled ratio of the sides CD and FG, being similar. Therefore the triangle BCD is also one and a third times the triangle EFG.

Again, since it has been shown that the triangle BCD is one and a third times the triangle EFG, and hence that of whatever parts the triangle BCD is 4, of such parts the triangle EFG is 3, it follows that the triangle EFG taken eight times — that is, the whole surface of the octahedron — is 24 of those same parts, while the triangle BCD taken four times — that is, the whole surface of the tetrahedron — is 16 of those same parts. For this reason the surface of the octahedron is one and a half times the surface of the tetrahedron of the same sphere. Therefore, if a tetrahedron and an octahedron are inscribed in the same sphere, and so forth. Which was to be shown.

Diagram. At the top, a horizontal segment A (the sphere's diameter). Below it, two triangles: the tetrahedron's base BCD (apex B) at the left and the octahedron's base EFG (apex E) at the right.

- A (top segment): the diameter of the sphere
- B (apex of left triangle): angle of the tetrahedron's base
- C (lower left of left triangle): base vertex
- D (lower right of left triangle): base vertex
- E (apex of right triangle): angle of the octahedron's base
- F (lower left of right triangle): base vertex
- G (lower right of right triangle): base vertex

THEOR. 15. PROPOS. 15.

[Margin: o. / xv.]

A straight line drawn from any angle of a tetrahedron described in a sphere, passing through the center of the sphere, falls on the center of the opposite base and is perpendicular to that base.

Let ABCD be a tetrahedron, contained by four equilateral and equal triangles ABC, DAB, DBC, DCA, in a sphere whose center is E; and from the angle D, through E the center of the sphere, let the straight line DEF be drawn, meeting the base ABC — inscribed in a circle — at the point F. I say that F is the center of the base ABC, and that DF is perpendicular to that base. For draw from E and from F to the angles of the said base the straight lines EA, EB, EC and FA, FB, FC. Since the lines EA, EB, EC, drawn from the center of the sphere to its surface, are equal, the two sides EA, ED of triangle EAD equal the two sides EC, ED of triangle CED; and the bases AD, CD are also equal, being sides of equal equilateral triangles. [Margin: a 8.pri. (Elements I.8)] Therefore the angles AED and CED will be equal; and consequently the remaining angles AEF and CEF will also be equal, [Margin: b 13.pri. (Elements I.13)] because both AED with AEF, and CED with CEF, are equivalent to two right angles. By the same reasoning the angles BEF and CEF will be equal, if the triangles BED and CED are considered. Since, then, the sides EA, EF of triangle EAF equal the sides EC, EF of triangle ECF, and the angles contained by them have been shown equal, [Margin: c 4.pri. (Elements I.4)] the bases FA and FC will be equal. In the same way FB and FC will be equal, if the triangles EBF and ECF are considered. Hence, since the three lines FA, FB, FC falling from F onto the circumference of the circle are equal, [Margin: d 9.ter. (Elements III.9)] F will be the center of the base ABC, that is, of the circle circumscribing it.

And since, by the corollary of Lemma 1 of Proposition 10 of this book, the perpendicular dropped from E, the center of the sphere, to the plane of the circle ABC falls on the center of the circle, it follows that the line DE, drawn through the

center and falling on the center F of the circle, is perpendicular to the plane of the circle, that is, to the base ABC of the tetrahedron. For if another perpendicular were drawn, it would fall on the center of the circle, by the said corollary. Hence, since F has also been demonstrated to be the center of that same circle, one and the same circle would have two centers — which is absurd. Therefore a straight line from any angle, and so forth. Which was to be demonstrated.

Diagram. A tetrahedron ABCD inscribed in a circle (the sphere), with E the center of the sphere and F the center of the base ABC. The lines EA, EB, EC, ED and FA, FB, FC are drawn, together with the axis DEF.

- A (upper left, on the circle): vertex of the tetrahedron
- D (upper right, on the circle): vertex of the tetrahedron (top of the axis DEF)
- B (lower left, on the circle): vertex of the tetrahedron
- C (lower right, on the circle): vertex of the tetrahedron
- E (near center): center of the sphere
- F (below E): center of the base ABC (foot of the axis DEF)

THEOR. 16. PROPOS. 16.

[Margin: o. / xvj.]

An octahedron described in a sphere is divided into two equal and similar pyramids of equal height; and the base of each is a square equal to half the square on the diameter of the sphere.

Let ABCDEF be an octahedron, contained by eight equilateral and equal triangles EAD, EAB, EBC, ECD, FBC, FBA, FAD, FDC, and described in a sphere whose center is G. I say that it is divided into two equal and similar pyramids of equal height, whose common base is a square equal to half the square on the diameter of the sphere. For draw from G, the center of the sphere, to all the angles the straight lines GA, GB, GC, GD, GE, GF, which are equal...

(The demonstration of Proposition 16 begins here and continues on the following page.)

Book XIV

Proposition 16 (continued) — demonstration

for they are semidiameters of the sphere, drawn, that is, from the center to the surface of the sphere. Now the square on the sphere's diameter is double the

square on the octahedron's edge AB, and quadruple the square on the semidiameter AG (from the scholium to *Elements* II.4) [Margin: (a) XIII.14]. So if the square on the sphere's diameter is set at 4 such parts, the square on the line AB will be 2 of those parts and the square on the line AG will be 1; and therefore the square on AB is double the square on AG. Since, then, the square on AG is equal to the square on BG, the square on AB will be equal to the two squares on AG and BG together; and therefore the angle AGB is a right angle [Margin: (b) I.48]. In the same way the remaining angles AGD, BGC, and CGD are shown to be right angles [Margin: (c) I.14]. For this reason AG together with CG, and BG together with DG, will lie in a straight line [Margin: (d) XI.2]; and consequently the lines AC and BD, cutting each other at G, will lie in one plane [Margin: (e) XI.7]; and therefore the lines that join their extremities — AB, BC, CD, DA — lie in the same plane with them. So the quadrilateral ABCD lies in one plane.

The octahedron, then, has been divided into two pyramids ABCDE and ABCDF, whose common base is ABCD and whose apexes are E and F. These pyramids are equal and similar (by *Elements* XI, def. 10), since each is made up of four equilateral triangles of the octahedron, which are equal and similar, together with the common base ABCD.

I now say that their heights are equal, and that the base ABCD is a square equal to half the square on the sphere's diameter. For it has been shown that the square on the octahedron's edge AB is equal to the two squares on the semidiameters AG and BG; likewise the square on the edge AE is equal to the two squares on the semidiameters AG and EG; and therefore the angle AGE is a right angle [Margin: (f) I.48]. By the same reasoning the angle DGE is a right angle. Since, then, the line EG stands at right angles to the two lines AG and DG that cut each other at G, EG itself is perpendicular to the plane ABCD drawn through the lines AG and DG — that is, to the base of the pyramids [Margin: (g) XI.4]. So EG is the height of the pyramid ABCDE. In the same way we shall show that FG is the height of the pyramid ABCDF. Since, then, EG and FG are equal semidiameters of the sphere, it is clear that the said pyramids are of equal height.

Again, since AC is a diameter of the sphere, its square is double the square on the edge AD [Margin: (h) XIII.14], and therefore equal to the two squares on the edges AD and DC. Hence the angle ADC is a right angle [Margin: (i) I.48]. In the same way the angles DAB, ABC, and BCD will be right angles. Therefore the quadrilateral ABCD is a rectangle; but it is also equilateral, since its sides are edges of the octahedron. So it is a square; and since it is described on the octahedron's edge AB, it follows that it is half the square on the sphere's diameter

[Margin: (k) XIII.12]. Therefore an octahedron described in a sphere, and so on. Which was to be shown.

Diagram. The regular octahedron ABCDEF of Proposition 16, drawn with all its face-edges and internal diagonals meeting at the center G. The square "equatorial" section ABCD is shown with its two apexes E and F above and below.

- E (top): upper apex of the octahedron
- A (upper left): vertex of the equatorial square
- D (upper right): vertex of the equatorial square
- G (center): center of the octahedron (and of the circumscribing sphere)
- B (lower left): vertex of the equatorial square
- C (lower right): vertex of the equatorial square
- F (bottom): lower apex of the octahedron

SCHOLIUM

We demonstrated this proposition far more briefly in the second corollary to *Elements* XIII.14, namely from the very construction of the octahedron. Nevertheless, since it is here proved by Campanus without reference to the construction of the octahedron, it seemed good to set down his demonstration in this place as well.

THEOR. 17. PROPOS. 17.

[Margin: outer dual-numbering "o. xvij."]

A tetrahedron inscribed in a sphere is to an octahedron described in the same sphere as the rectangle contained by a line whose square is twenty-seven sixty-fourths ($27/64$) of the square on the tetrahedron's edge, and by a line containing eight ninths ($8/9$) of the same edge, is to the square on the sphere's diameter.

Or, as Campanus puts it:

A tetrahedron inscribed in a sphere is to an octahedron described in the same sphere as the rectangle under a line whose square is subsestertian [3:4] to three quarters of the tetrahedron's edge, and under a line that is superquintupartient-of-twenty-sevenths [in the ratio 32:27] to those same three quarters of the tetrahedron's edge, is to the square on the sphere's diameter.

Demonstration. Let there be a tetrahedron ABCD and an octahedron EFGHIK in one and the same sphere, whose center is L. From the vertex D, through the center L, draw the sphere's diameter DM, meeting the base ABC at the point N. Then N will be the center of the base ABC, and DN will be perpendicular to that base [Margin: (a) XIV.15]. Now the part of the diameter LN, lying between the sphere's center and the base of the tetrahedron, is a sixth part of the diameter and a third of the semidiameter (by *Elements* XIII.13, cor. 2). So DL, the semidiameter, will be 3 such parts as LN is 1; and therefore DN will be 4 such parts. Next, by the scholium to *Elements* XI.15, draw through the center L a plane OPQ parallel to the plane ABC; then the lines QO, OP, PQ will be parallel to the lines AB, BC, CA [Margin: (b) XI.16]; and therefore the triangle QOP will be equiangular with the equilateral triangle ABC [Margin: (c) XI.10], since the angles of the one are equal to the angles of the other, each to each. But ABC is equiangular (by *Elements* I.5, cor.). Therefore QOP too will be equiangular, and consequently equilateral (by *Elements* I.6, cor.), and so similar to ABC itself. Moreover, the triangles DQO, DOP, DQP are similar to the equilateral triangles DAB, DBC, DAC (by *Elements* VI.4, cor.). Therefore the pyramids ABCD and QOPD are similar (by *Elements* XI, def. 9) [Margin: (d) XII.17]. Again, since the parallel planes ABC and QOP cut the lines DN and DC proportionally, DN will be to DL as DC is to DP. But DN was to DL as 4 to 3 (since

Diagram. The composite figure for Proposition 17, comprising three parts: (1) the tetrahedron ABCD with its inscribed parallel triangle QOP; (2) the octahedron EFGHIK; and (3) an auxiliary triangle-and-baseline construction with apex R. (The same composite figure is repeated at the top of the following page.)

Tetrahedron (left):

- D (top): apex, from which the diameter DM is drawn
- A (left), B (bottom), C (right): vertices of the base triangle ABC
- N (center of base): center of ABC, foot of the perpendicular DN
- L (mid-height): center of the sphere, lying on DN
- Q, O, P (mid-height triangle): vertices of the triangle QOP, parallel to base ABC and passing through L (Q upper-left, O center, P right)
- M (below B): the far end of the diameter, opposite D

Octahedron (center):

- I (top): upper apex
- E (upper left), H (upper right): two vertices of the equatorial square
- L (center): center

- F (lower left), G (lower right): two vertices of the equatorial square
- K (bottom): lower apex
- R (right of K): auxiliary apex point (belongs to the baseline construction below)

Auxiliary construction (baseline, left to right): X, Y, S, V, T, with apex R above; the equilateral triangle RST has perpendicular RV, TS is extended to X, and TX is bisected at Y.

- R (apex): apex of the auxiliary triangle
- X (far left): extension of TS beyond S
- Y: midpoint of TX
- S: base vertex of triangle RST
- V: foot of the perpendicular RV
- T (far right): base vertex of triangle RST

Diagram. Repeat of the Proposition 17 composite figure from the preceding page: the tetrahedron ABCD with inscribed parallel triangle QOP (labels D, A, Q, L, P, N, O, C, B, M), the octahedron EFGHIK with center L and auxiliary point R, and the baseline construction with apex R over the points X, Y, S, V, T. (Labels as described at fig-p583-2.)

Proposition 17 (continued) — demonstration

it has been shown that DN is 4 such parts as DL is 3.) Therefore DC will be to DP as 4 to 3. And consequently, since similar pyramids have the triplicate ratio of their edges [Margin: (e) XII.8], the pyramid ABCD will be to the pyramid QOPD as 64 to 27; for this is the triplicate of the ratio 4 to 3, as is clear from the numbers 64, 48, 36, 27. For these are the least numbers in the ratio 4 to 3 (as is clear from *Elements* VIII.2). Again, since squares have the duplicate ratio of their sides [Margin: (f) VI.20], the square on DC will be to the square on DP as 64 to 36, as is likewise clear from those same numbers.

Now let there be an equilateral triangle RST equal to the triangle DQP, and from R let the perpendicular RV be dropped to ST. Then the side RT — that is, DP — will be equal to it, and in square four-thirds (sesquitertian) of the perpendicular RV [Margin: (g) XIV.12]. And accordingly, as the square on DP was 36 such parts, the square on RV will be 27 of them. But the square on DC was 64 of those same parts. Therefore the line RV is capable of [has for its square] twenty-seven sixty-fourths of the square on the tetrahedron's edge.

Again, let TS be extended so that TX is to TS as 64 to 27 (by *Elements* X.6, cor.), and let TX be bisected at Y, so that TY is 32 such parts as TS — that is, its equal DP — is 27. Now since the ratio of DC to DP was as 4 to 3, that is, sesquitertian, DC will be 36 such parts as DP is 27 (for the ratio 36 to 27 is sesquitertian), and therefore as TY is 32. So the line TY contains eight ninths of the tetrahedron's edge DC. I say, then, that the tetrahedron ABCD is to the octahedron EFGHIK as the rectangle contained by the line RV (which is capable of twenty-seven sixty-fourths of the square on the edge DC) and by the line TY (which contains eight ninths of the same edge) is to the square on the sphere's diameter.

For when the lines RX, RY are drawn: since the triangle RTX is double the triangle RTY (because the base TX is double the base TY) [Margin: (h) VI.1], and the rectangle under RV, TY is likewise double the same triangle RTY [Margin: (i) I.41], the triangle RTX will be equal to the rectangle under RV, TY. And therefore, as the triangle RTX is to the triangle RST, so the rectangle under RV, TY is to that same triangle RST [Margin: (k) V.7]. But the triangle RTX is to the triangle RST as the base TX is to the base ST [Margin: (l) VI.1], namely as 64 to 27. Therefore the rectangle under RV, TY is to the triangle RST — that is, to the triangle QDP, or its equal QOP — as 64 to 27. But the pyramid ABCD was also to the pyramid QOPD as 64 to 27. Therefore, as the rectangle under RV, TY is to the triangle QOP, so the pyramid ABCD is to the pyramid QOPD. And as the triangle QOP is to the square on the sphere's diameter DM, so is the pyramid QOPD to the octahedron EFGHIK.

(For since DN is perpendicular to the plane ABC, it will also be perpendicular to the plane QOP, by the scholium to *Elements* XI.14; and therefore the sphere's semidiameter DL will be the height of the pyramid, or tetrahedron, QOPD. But the octahedron EFGHIK is divided into two equal pyramids EFGHI and EFGHK [Margin: (m) XIV.16], whose common base is the square EFGH — half the square on the sphere's diameter — and whose heights are the equal semidiameters of the sphere LI and LK. So the pyramids QOPD and EFGHI have equal heights. Therefore, by the scholium to *Elements* XII.5, as the base QOP is to the base EFGH, so is the pyramid QOPD to the pyramid EFGHI; and consequently, by what we showed in the scholium to *Elements* V.22, as the base QOP is to double the base EFGH — that is, to the square on the sphere's diameter DM — so is the pyramid QOPD to double the pyramid EFGHI, that is, to the octahedron EFGHIK.) Ex aequo, then, as the rectangle contained by RV (capable of twenty-seven sixty-fourths of the square on the tetrahedron's edge DC) and by TY (eight ninths of the same edge DC) is to the square on the sphere's diameter DM, so is the pyramid, or tetrahedron, ABCD to the octahedron EFGHIK. Therefore a tetrahedron inscribed in a sphere is to an octahedron, and so on. Which was to be demonstrated.

SCHOLIUM

Now since DC was 4 such parts as DP was 3, DP — or its equal RT — will contain three quarters of the tetrahedron's edge DC. And accordingly, since the line RT is in square sesquitertian [4:3] to the line RV [Margin: (n) XIV.12], RV will be in square subsesquitertian [3:4] to RT — that is, to three quarters of the tetrahedron's edge. Again, since TY was 32 such parts as ST — or DP — was 27 (that is, three quarters of the tetrahedron's edge), TY will be superquintupartient-of-twenty-sevenths [in the ratio 32:27] to ST, namely to three quarters of the tetrahedron's edge. Hence it is established that the tetrahedron ABCD is to the octahedron EFGHIK as the rectangle under RV — which is in square subsesquitertian to three quarters of the tetrahedron's edge — and under TY — which is superquintupartient-of-twenty-sevenths to those same three quarters of the tetrahedron's edge — just as Campanus maintained.

THEOR. 18. PROPOS. 18.

[Margin: outer dual-numbering "o. xvij." [?] — possibly "xviiij"; small and foxed]

A perpendicular dropped from any angle of an equilateral triangle to the opposite base is triple the perpendicular that is drawn from the center of the triangle to the same base.

Demonstration. Let there be an equilateral triangle ABC whose center is D. From A, through D, draw the line AE, which will be perpendicular to BC and will bisect it (by *Elements* XIII.10, cor. 2). I say that AE is triple the line DE — the perpendicular, that is, drawn from the center to the base. For when the lines BD, CD are drawn: since the sides DA, DB of the triangle ADB are equal to the sides DB, DC of the triangle BDC (for they are drawn from the center), and the base AB is also equal to the base BC, the triangles ABD and BDC will be equal (by *Elements* I.8, cor.). And by the same

Diagram. The equilateral triangle ABC of Proposition 18, with its center D and the perpendicular AE dropped from the apex A to the base BC; lines BD and CD are also drawn.

- A (top): apex
- D (center): center of the triangle
- B (bottom left): base vertex
- E (bottom middle): foot of the perpendicular AE on BC (midpoint of BC)
- C (bottom right): base vertex

Proposition 18 (continued) — demonstration

reasoning the triangles ADB and ADC will be equal. Therefore, because of the equality of the three triangles ADB, BDC, CDA, the triangle ABC is triple the triangle BDC. Now since the rectangle under AE, EB is equal to the triangle BDC (by the scholium to *Elements* I.41), because the base of the triangle BDC is double the base BE, which belongs to each rectangle, the rectangle under AE, EB will also be triple the rectangle under DE, EB. But as the rectangle under AE, EB is to the rectangle under DE, EB — of the same height EB — so is the base AE to the base DE [Margin: (a) VI.1]. Therefore AE too is triple DE. So a perpendicular from any angle, and so on. Which was to be shown.

COROLLARIUM

And so the same perpendicular AE is one-and-a-half times (sesquialter) the remaining line AD, contained between the angle and the center. For since AE is triple DE, if AE is set at 3 such parts, DE will be 1; the remaining line AD will therefore contain 2 of those same parts; and consequently AE will be sesquialter to AD.

From which it further follows that the line AD is double the line DE.

THEOR. 19. PROPOS. 19.

[Margin: outer dual-numbering "o. xvij." [?] — small and foxed]

If an octahedron is inscribed in a sphere, the semidiameter of the sphere will be in square triple the perpendicular that is drawn from the center of the sphere to any base of the octahedron.

Demonstration. Let there be some base of an octahedron inscribed in a sphere whose center is D — the equilateral triangle ABC — and from the center D of the sphere let a perpendicular DE be dropped onto it; and draw the semidiameter of the sphere DB. I say that the semidiameter DB is in square triple the perpendicular DE. For since, by the corollary to Lemma 1 of Proposition 10 of this book, the perpendicular DE falls on the center of the circle of the sphere that circumscribes the triangle ABC, E will be the center of the triangle. So when the line BE is drawn: since the square on the sphere's diameter is double the square on the octahedron's edge AB [Margin: (a) XIII.14], and quadruple the square on the sphere's semidiameter DB (by the scholium to *Elements* II.4) — if the square on the diameter is set at 12 parts, the square on the edge AB will be 6, and the square on the sphere's semidiameter DB will be 3. Now since the square on the

edge AB is triple the square on the semidiameter of the circle enclosing the triangle — namely the line BE [Margin: (b) XIII.12] — the square on the line BE will be 2 such parts as the square on the edge AB was 6. Therefore, since the square on the line DB is equal to the squares on the lines BE and DE [Margin: (c) I.47], and the square on DB has been shown to be 3 parts, the remaining square on the perpendicular DE will be 1 such part. And accordingly the square on the line DB is triple the square on the line DE. So if an octahedron is inscribed in a sphere, and so on. Which was to be shown.

Diagram. The figure for Proposition 19: the equilateral triangle ABC (one base of the octahedron), with the sphere's center D drawn to the left, the perpendicular DE falling on the triangle's center E, and the semidiameter DB drawn to the base vertex B.

- A (top): apex of the triangle
- D (left): center of the sphere
- E (interior/center): center of the triangle ABC, foot of the perpendicular DE
- B (bottom left): base vertex; DB is the drawn semidiameter
- C (bottom right): base vertex

THEOR. 20. PROPOS. 20.

[Margin: outer dual-numbering "o. xviiij." [?]]

Double the square described on the diameter of any sphere is equal to the surface of a cube placed in that sphere; and the perpendicular dropped from the center of the sphere onto any base of the cube is equal to half the cube's edge.

Demonstration. Let there be a cube ABCDEFGH described in a sphere whose center is I, and from I let the perpendicular IK be drawn to one base of the cube, namely to ADEF. I say that double the square on the sphere's diameter is equal to the surface of the cube, and that the perpendicular IK is equal to half the cube's edge. For since the square on the sphere's diameter is triple the square on the cube's edge — that is, equal to three squares of the cube [Margin: (a) XIII.15] — double that same square described on the sphere's diameter will be equal to six squares of the cube, that is, to the whole surface of the cube.

Now let KI be produced to the opposite base BCHG, as far as the point L. Since KL is perpendicular to the plane AE, it will also be perpendicular to the plane BH, which is parallel to it (by the scholium to *Elements* XI.14) [Margin: (b) XI.6]. Again, since AB and KL are parallel — because they are both perpendicular to the plane

AE — let a plane be drawn through them, making common sections in the opposite parallel planes AE, BH: the straight lines AK, BL, which will also be parallel to each other [Margin: (c) XI.16]; and therefore ABLK will be a parallelogram, and so the line KL will be equal to the cube's edge AB. Now since the perpendicular KL, from the sphere's center I, falls on the bases AE, BH at the points K and L, the points K and L will be the centers of the circles enclosing those bases in the sphere (by the corollary to Lemma 1 of Proposition 10 of this book). Therefore KL will be bisected at the center I by the said diameter CF, as we showed in the first corollary to *Elements* XIII.15; and consequently, since KL has been shown equal to the cube's edge, IK will be half of that same edge of the cube. Therefore double the square described on the diameter of any sphere, and so on. Which was to be shown.

Diagram. The cube ABCDEFGH of Proposition 20, drawn in perspective, with center I, the perpendicular IK to the top face ADEF, and IK produced to L, the center of the opposite face BCHG. The internal lines AK, BL and the diameter CF are shown.

- F (top left): vertex of the top face
- E (top right): vertex of the top face
- K (upper interior): center of the top face ADEF, foot of the perpendicular IK
- A (left): front-upper vertex
- D (right): vertex of the top face
- I (center): center of the cube (and of the sphere)
- G (lower left): vertex of the bottom face
- H (lower right): vertex of the bottom face
- L (lower interior): center of the bottom face BCHG
- B (bottom left): front-lower vertex
- C (bottom right): front-lower vertex

COROLLARIUM

From what has been said, it follows that the solid formed from half the cube's edge multiplied by two thirds of the square on the diameter of the sphere enclosing the said cube is equal to the cube. For since the square on the sphere's diameter is triple the square on the cube's edge, the two squares AE and BH will be two thirds of the square on the sphere's diameter [Margin: (d) XIII.15]. And it is evident that the solid formed from IK (half the cube's edge) multiplied by AE (a third of the square on the sphere's diameter) is equal to the solid formed from IL (half the cube's edge) multiplied by BH (a third of the square on the sphere's

diameter) [Margin: (e) XI.31], because the said solids have equal bases and equal heights. Therefore, since the said two solids fill out the whole cube, it is manifest that what is formed from half the cube's edge multiplied by two thirds of the square on the sphere's diameter is equal to the whole cube. Which was what was proposed.

Book XIV (continued)

Proposition 20 — Alternative proof of the Corollary

This corollary can also be stated in another way, namely: the solid produced by the edge of the cube times a third of the square of the diameter of the sphere is equal to the cube. For the cube BE is produced from BA — the edge of the cube — times the square AE, which is a third of the square of the diameter of the sphere. As is evident.

SCHOLIVM

More generally: the solid produced by the perpendicular drawn from the center of any regular body to one of its bases, times a third part of the surface of the body, is equal to the given regular body.

Let ABCD be any regular body, contained by the planes ABC, ABD, ACD, BCD; and let EFGH be a solid contained by EI — a third part of the surface of the body ABCD — and by the height FG, which is equal to the perpendicular KL drawn from the center K to the base ABC. I say that the solid EFGH is equal to the given regular body ABCD.

For when the straight lines KA, KB, KC, KD are drawn from the center K to all the vertices of the body, the regular body is resolved into equal pyramids, since they have equal bases and equal heights. For since the circles surrounding the said bases are equal, they are equally distant from the center, by lemma 2 of the [relevant] proposition of this book. And therefore the perpendiculars drawn from the center K to their planes — namely the heights of the pyramids — will be equal.

Now the prism contained by the base ABC and the height KL is triple the pyramid ABCK, by corollary 1 of proposition 6 of book 12; and by the same reasoning the prisms contained by the remaining bases, at the same height KL, are triple their own pyramids. Therefore the prism contained by the whole surface of the regular body, at the height KL (namely the prism composed of all those prisms), will be triple the regular body ABCD. Consequently, since this same prism is triple the

prism EFGH — because the base of the whole-surface prism is set as triple the base of EFGH — by what we demonstrated at proposition 7 of book 12, the solids EFGH and ABCD will be equal. Which is what was proposed.

Diagram. Two solids side by side. On the left, a tetrahedron with an interior center; on the right, a rectangular solid (right prism) representing the solid EFGH.

- D (apex, top): top vertex of the tetrahedron
- A (left): left base vertex
- B (bottom front): front base vertex
- C (right): right base vertex
- K (interior center): center of the regular body, from which the radii KA, KB, KC, KD are drawn
- L (below K): foot of the perpendicular KL to the base ABC
- H (top-left rear of the box): upper vertex of the rectangular solid
- I (top-right front of the box): upper vertex of the rectangular solid
- E (bottom-left front): lower vertex
- F (bottom-right front): lower vertex
- G (right side, mid): vertex of the rectangular solid EFGH

THEOR. 21. PROPOS. 21.

[Margin: o. / xj. (dual Hypsicles/Campanus proposition number)]

The same circle comprehends both the square [face] of the cube and the triangle [face] of the octahedron of the same sphere.

In a sphere whose diameter is A, let a cube be imagined described, of which one square face is BCDE, and an octahedron, of which one triangular face is FGH. I say that the circles BCDE, FGH circumscribing them are equal. For let there be radii from the centers, IB and KF. [Margin: a 15.tertij dec. (Elements XIII.15)]

Now the square of the diameter A is triple the square of the edge of the cube BC, and the square of the edge BC is double the square of the line BI. (For when the line IC is drawn, the angle BIC becomes a right angle, since it subtends the square [Margin: b 47.pri. (Elements I.47)]; and therefore the square on BC will be equal to the squares on BI, IC. And since the lines BI, IC are equal, their squares [?] are equal; so the square on BC will be double the square on BI.) Therefore, if the square of the diameter A be set at 6 such parts, the square of the edge BC will be 2 of them, and the square of the line BI will be 1.

Again [Margin: c 14.tertij dec. (Elements XIII.14)], since the square of the diameter A is double the square of the edge of the octahedron FG, and the square of the edge FG is triple the square of the line FK drawn from the center [Margin: d 12.tertij dec. (Elements XIII.12)]: if the square of the diameter A be set at 6 such parts, the square of the edge FG will be 3, and the square of the line FK will be 1. Since therefore the square on the line BI and the square on the line FK have each been shown to be 1 of the same parts, the squares on BI, FK are equal; and therefore the lines themselves, and the circles BCDE, FGH described from them, are equal. Therefore the same circle comprehends [both], etc. Which was to be shown.

Diagram. Two circles side by side sharing a common diameter drawn below. In the left circle a square face BCDE is inscribed; in the right circle an equilateral triangular face FGH is inscribed.

- B (top-left), E (top-right), C (bottom-left), D (bottom-right): the four vertices of the inscribed square BCDE
- I (center of left circle): center of the circumscribing circle
- F (top), G (bottom-left), H (bottom-right): the three vertices of the inscribed triangle FGH
- K (center of right circle): center of the circumscribing circle
- A (line drawn beneath both circles): the diameter of the sphere

COROLLARIVM

From this it follows that the perpendicular lines joining the centers of the circles which circumscribe the opposite bases — both of the cube and of the octahedron — are equal.

For if perpendiculars are drawn from the center of the sphere to the bases of the cube and of the octahedron, they fall on the centers of the circles circumscribing those bases, by the corollary of lemma 1 of proposition 10 of this book. And since these have been shown equal, they are equally distant from the center of the sphere, by lemma 2 of the same proposition. [Margin: e 21.quartidec. (Elements XIV.21)] And therefore the said perpendiculars will be equal. Now these are the halves of the lines that join the centers of the opposite bases. (For if they are produced, they will also be perpendicular to the opposite bases, by the scholium of proposition 14 of book 11, because the opposite bases in the octahedron are parallel, by corollary 4 of proposition 14 of book 13, just as the opposite bases of the cube are parallel.) Therefore the perpendiculars joining the centers of the

opposite bases will be equal, both in the cube and in the octahedron. Which is what was proposed.

THEOR. 22. PROPOS. 22.

If an octahedron and a tetrahedron are inscribed in the same sphere, the octahedron will be to triple the tetrahedron as the edge of the octahedron is to the edge of the tetrahedron.

In the same sphere let both an octahedron $ABCDEF$ and a tetrahedron $GHIK$ be inscribed. I say that the octahedron is to triple the tetrahedron as the edge BF is to the edge HI . For, when the diameter AD is drawn in the octahedron, the octahedron is divided into four pyramids whose common edge is the diameter AD itself and whose common vertex is A , namely: $BCDA$, whose base is BCD ; $BDFA$, whose base is BDF ; $CDEA$, whose base is CDE ; and $DEFA$, whose base is

DEF . All of these bases lie below the square $BCEF$, and together with it contain half of the octahedron. And since each of these pyramids consists of two equilateral triangles of the octahedron together with two other triangles which — because their common side is the diameter AD , while their remaining sides are edges of the octahedron itself — are equal to one another [Margin: a 8.pri. (Elements I.8)], it follows that these four pyramids are equal to one another, since they are contained by planes equal in both number and magnitude.

Let a prism $GHIKLM$ be set up on GHI , the base of the tetrahedron, of the same height as the tetrahedron; this will be triple the tetrahedron itself, by corollary 1 of proposition 7 of book 12. Again, on the base of the octahedron NOP , and on TVX (equal to the base of the tetrahedron), let two prisms $NOPQRS$ and $TVXYZ\alpha$ be set up, of the same height as the octahedron, or the pyramid $ACDE$. [Margin: b 14.quartidec. (Elements XIV.14)] Since therefore GHI — that is, TVX — the base of the tetrahedron, is 4:3 [sesquitertia] of NOP , the base of the octahedron, the prism $TVXYZ\alpha$ will also be 4:3 of the prism $NOPQRS$, since prisms of the same height are as their bases, by what we demonstrated at proposition 7 of book 12. But the octahedron $ABCDEF$ is likewise 4:3 of such a prism $NOPQRS$ (for a prism is equal to three pyramids, by corollary 1 of proposition 7 of book 12, four of which we said the octahedron contains). Therefore the octahedron is equal to the prism $TVXYZ\alpha$.

But since the prism $TVXYZ\alpha$ has to the prism $GHIKLM$ the proportion which the height of the former has to the height of the latter, by the scholium of proposition 14 of book 12; and the height of the prism $TVXYZ\alpha$ (which is equal to the octahedron) is the same as the height of the octahedron, by construction: the

octahedron will also be to the prism GHIKLM — that is, to triple the tetrahedron GHIk — as the height of the octahedron is to the height of the tetrahedron. Now the height of the octahedron — namely the perpendicular joining the centers of the opposite bases — is equal to the edge of the cube, that is, to the perpendicular connecting the centers of the opposite bases, by the corollary of proposition 21 of this book. And as the edge of the cube is to the height of the tetrahedron, so is the edge of the octahedron to the edge of the tetrahedron.

(For both the square of the edge of the cube to the square of the height of the tetrahedron, and the square of the edge of the octahedron to the square of the edge of the tetrahedron, is as 3 to 4. For if the square of the diameter of the sphere be set at 9, the square of the edge of the cube will be 3, by proposition 15 of book 13, and the square of the height of the tetrahedron 4, by corollary 2 of proposition 13 of book 13; and therefore the square of the edge of the cube to the square of the height of the tetrahedron is as 3 to 4. Again, if the square of the diameter of the sphere be set at 6, the square of the edge of the octahedron will be 3, and the square of the edge of the tetrahedron 4, and so on.)

Therefore the octahedron is to triple the tetrahedron as the edge of the octahedron is to the edge of the tetrahedron. If therefore an octahedron and a tetrahedron are inscribed in the same sphere, etc. Which was to be demonstrated.

Diagram. A group of figures for Proposition 22 : an octahedron (upper left), a prism shown as a solid (upper right), and two triangular prisms (lower left and lower right).

- Octahedron ABCDEF: A (top apex), B (upper-left), F (upper-right), C (lower-left), E (lower-right), D (bottom apex)
- Prism GHIKLM: L (top apex), M (left), K (right), G (interior center), H (bottom-left), I (bottom-right)
- Lower-left triangular prism NOPQRS: R (top apex), S (upper-left), N (interior), O (bottom-left), P (bottom-center); the sixth vertex letter (Q) is not individually resolvable [?]
- Lower-right triangular prism TVXYZα: Z (top apex), Y (upper-right), T (interior), V (bottom-center), X (bottom-right); the vertex α (and one further letter) not individually resolvable [?]

COROLLARIVM

From what has been said it is inferred that the height of the octahedron is to the height of the tetrahedron as the edge of the octahedron is to the edge of the

tetrahedron. For it was shown that the edge of the cube — which is equal to the height of the octahedron — is to the height of the tetrahedron as the edge of the octahedron is to the edge of the tetrahedron. Likewise it is gathered that the diameter of the sphere is to the edge of the tetrahedron as the edge of the octahedron is to the edge of the cube. [Margin: c 18.tertij dec. (Elements XIII.18)] For both the square of the diameter to the square of the edge of the tetrahedron, and the square of the edge of the octahedron to the square of the edge of the cube, has the ratio 3:2 [sesquialter].

THEOR. 23. PROPOS. 23.

If a given straight line is such that its square equals [the squares on] a whole line cut in extreme and mean ratio together with its greater segment; and likewise [equals the squares on] another whole line similarly cut together with its lesser segment: then the greater segment of the first line will be the edge of the icosahedron, and the lesser segment of the latter line the edge of the dodecahedron, of that sphere whose diameter the given straight line is.

Let the straight line AB be equal in square to the line CD together with its greater segment CE; and likewise to the line FG together with its lesser segment HG. I say that CE is the edge of the icosahedron, and HG the edge of the dodecahedron, of the sphere whose diameter AB is.

For let there be imagined, in the sphere whose diameter is AB, an icosahedron AIBk described, and a dodecahedron ANBO. Then let the opposite edges of the icosahedron, AI and Bk, be joined by the straight line IB, which will subtend one angle of the pentagon composed of the edges of the icosahedron, as is established from proposition 16 of book 13. (For in that figure, when the line ZF is drawn, RSHZF will be a pentagon, on account of the five equal edges of the icosahedron GR, GS, GH, GZ, GF falling on the surface of the sphere; and in it the line RH, joining the opposite sides RX, HI, subtends the angle RSH.) And it will appear more clearly from the construction of the icosahedron which we shall bring forward at the end of book 16 from Pappus; and it may also be seen in some material icosahedron.

But since AI, Bk are opposite parallel sides, by corollary [of] proposition 16 of book 3 [recte: book 13] [Margin: a 9.pri. (Elements I.29 — the "2" appears dropped)], the angles AIB, IBk will be equal to two right angles; and therefore, since they are equal to one another —

— (for when another diameter IK is drawn: since the sides AI, IB of the triangle AIB are equal to the sides KB, BI, and the bases AB, IK also equal [Margin: b 8.pri. (Elements I.8)], the angles AIB, IBK will be equal) — they will be right angles; and therefore [Margin: c 47.pri. (Elements I.47)] the square on the proposed line AB will be equal to the squares on the lines AI, IB. But the same square on AB was, by hypothesis, equal to the squares on the lines CD, CE. Therefore the squares on the lines IB, IA are equal to the squares on the lines CD, CE.

And therefore [Margin: d 8.tertij dec. (Elements XIII.8)], since CE is the greater segment of CD cut in extreme and mean ratio, and likewise IA is the greater segment of IB similarly cut (for IB subtends an angle of the pentagon whose side is IA, an edge of the icosahedron; and it is established that when a line subtending an angle of a pentagon is cut in extreme and mean ratio, its greater segment is equal to the side of that same pentagon) [Margin: e 2.quartidec. (Elements XIV.2)], CE, the greater segment, will be equal to IA, the edge of the icosahedron. [Margin: f 22.sex. (Elements VI.22)] For since as CD is to CE, so is IB to IA, it will also be as the square on CD to the square on CE, so the square on IB to the square on IA; and, componendo, as the squares on CD, CE to the square on CE, so the squares on IB, IA to the square on IA. But the squares on IB, IA were shown equal to the squares on CD, CE. [Margin: g 14.quin. (Elements V.14)] Therefore the square on IA is also equal to the square on CE; and therefore the lines CE, IA are equal. Which is what was proposed.

Diagram. Two projected polyhedra with two divided reference segments drawn above them. On the left, an icosahedron with its interior diagonals; on the right, a dodecahedron. (The same figure is repeated on the following page.)

- Reference segment CD divided at E: C (left), E (middle), D (right)
- Reference segment FG divided at H: F (left), H (middle), G (right)
- Icosahedron AIBK: I (top-left), A (top-right), B (bottom-left), K (bottom-right); the interior pentagon labels (R, S, H, Z, F, X, G) are not individually resolvable on the tile [?]
- Dodecahedron ANBO: N (top-left), L (top-center), A (top-right), B (bottom-left), M (bottom-center), O (bottom-right)

Diagram. The same icosahedron-and-dodecahedron figure of Proposition 23, repeated at the head of this page, with the two divided reference segments above.

- Reference segments: CD (divided at E) and FG (divided at H) above
- Icosahedron AIBK: I (top-left), A (top-right), B (bottom-left), K (bottom-right); interior pentagon labels not individually resolvable [?]

- Dodecahedron ANBO: N (top-left), L (top-center), A (top-right), B (bottom-left), M (bottom-center), O (bottom-right)

Again, let the opposite edges of the dodecahedron, AN and BO, be bisected at the points L and M; let the line LM join these points, and let the edges themselves be joined by the straight line NB. Since therefore NL, BM are equal and parallel, by corollary 3 of proposition 17 of book 13 [Margin: h 33.pri. (Elements I.33)], LM and NB will also be equal and parallel. But the line LM, bisecting the opposite edges, is perpendicular to those opposite edges, by the same corollary. Therefore NB will also be perpendicular to them. [Margin: i 47.pri. (Elements I.47)] And therefore the square on the proposed line AB will be equal to the squares on the lines NB, NA. And since the same square on AB was set equal to the squares on the lines FG, HG, the squares on NB, NA will be equal to the squares on FG, HG.

And therefore, since HG is the lesser segment of FG cut in extreme and mean ratio, and likewise NA is the lesser segment of NB — that is, of LM — similarly divided, by corollary 4 of proposition 17 of book 13: HG, the lesser segment, will be equal to NA, the edge of the dodecahedron. This we shall show in no other way than we demonstrated above, that CE, the greater segment, is equal to IA, the edge of the icosahedron. If therefore a proposed straight line is equal in square to a whole line cut in extreme and mean ratio together with its greater segment, and likewise to another line similarly cut together with its lesser segment: then the greater segment of the first line will be the edge of the icosahedron, and the lesser segment of the latter line the edge of the dodecahedron of that sphere whose diameter the proposed line is. Which was to be shown.

THEOR. 24. PROPOS. 24.

If the edge of the octahedron is equal in square to the greater and lesser segments of a straight line cut in extreme and mean ratio: then the edge of the icosahedron described in the same sphere will be equal in square to double the [square on the] lesser segment.

Let the edge AB of the octahedron ABC be equal in square to the greater segment DE and the lesser segment EF of the line DF cut in extreme and mean ratio. Let there also be inscribed in the same sphere an icosahedron GHI whose edge is GH; let GI subtend an angle of the pentagon composed of the edges of the icosahedron, and let HI be the diameter. I say that the square of the edge GH is double the square of the lesser segment EF.

For since GH is the side of the pentagon one of whose angles the line GI subtends [Margin: a 8.tertij dec. (Elements XIII.8)], GH will be the greater segment of GI cut

in extreme and mean ratio; and therefore [Margin: b 5.tertij dec. (Elements XIII.5)] the line composed of GI, GH will likewise be cut at the point G in extreme and mean ratio. [Margin: c 47.pri. (Elements I.47)] Now since the square of the diameter HI is equal to the squares on the lines GI, GH — because the angle IGH is right, as was shown in the preceding proposition — and the square of the edge AB is set equal to the squares on the lines DE, EF: it follows that as the square of the diameter HI is to the square of the edge AB, so are the squares on the lines GI, GH to the squares on the lines DE, EF. [Margin: d 14.tertij dec. (Elements XIII.14)] But the square of the diameter HI is double the square of the edge of the octahedron AB. Therefore the squares on the lines GI, GH are also double the squares on the lines DE, EF.

And as the squares on the lines GI, GH are to the squares on the lines DE, EF, so is the square on the line GH to the square on the line EF. ([Margin: e 2.quartidec. (Elements XIV.2)] For since as GI is to GH, so is DE to EF, it will also be, as the square on GI to the square on GH, so the square on DE to the square on EF; and, componendo, as the squares on GI, GH to the square on GH, so the squares on DE, EF to the square on EF; and, permutando, as the squares on GI, GH to the squares on DE, EF, so the square on GH to the square on EF.) [Margin: f 22.sex. (Elements VI.22)] Therefore the square on the line GH is also double the square on the line EF. Wherefore, if the edge of the octahedron is equal in square to the greater and lesser segments, etc. Which was to be shown.

Diagram. An octahedron (left) and an icosahedron (right), with a divided reference segment drawn above.

- Reference segment DF divided at E: D (left), E (middle), F (right)
- Octahedron ABC: A (top apex), B (left), C (bottom apex)
- Icosahedron GHI: G (top-left), H (top-right), I (bottom apex)

THEOR. 25. PROPOS. 25.

If a straight line cut in extreme and mean ratio forms a right angle with its lesser segment, and a straight line is subtended to that right angle: then the straight line whose square is half of the [square on the] subtending line will be the edge of the octahedron of that sphere in which the said lesser segment is the edge of the dodecahedron.

(continuation of the demonstration of Theorem 25 / Proposition 25, whose enunciation was given at the end of the preceding pages)

Let the straight line AB, cut at C, be set in extreme and mean ratio with the straight line AD (which is equal to the lesser segment AC), forming a right angle at A; let BD be subtended to that right angle, and let E be a line whose square is half the square on BD. I say that E is the octahedron edge of that sphere in which AD, or AC, is the dodecahedron edge.

For since — by Corollary 2 of Proposition 17, Book 13 — the greater segment BC is the edge of the cube of that sphere whose lesser segment AC (or AD) is the dodecahedron edge, the diameter of that sphere will be, in square, triple the greater segment BC, that is, triple the cube edge. [Margin (a): 4. tertijdec. — *Elements* XIII.4] But the squares on the straight lines AB and AD, namely on the whole line and on the lesser segment, are also together triple the same square on the greater segment BC. Therefore the square on the diameter of the sphere will be equal to the squares on AB and AD, that is, to the square on BD; and consequently BD is the diameter of the sphere. [Margin (b): 14. tertijdec. — *Elements* XIII.14] Now since the square on the diameter of the sphere is double the square on the octahedron edge, and since the square on BD is double the square on E — because E is posited to be half of BD in square — E will be the octahedron edge of that same sphere in which AD is set as the dodecahedron edge. If, therefore, a straight line be cut, and so on. Which was to be demonstrated.

COROLLARIUM.

[Margin (c): 18. tertijdec. — *Elements* XIII.18]

Since the octahedron edge E is, in square, the sesquialter (3:2) of the cube edge BC — of which cube edge the greater segment [when BC is itself cut in extreme and mean ratio] is AD, the dodecahedron edge, as is established from what we showed at Proposition 4, Book 13, seeing that BC is the greater segment and AD the lesser of the line AB cut in extreme and mean ratio — it is evident that: if any straight line whatever (namely BC) be cut in extreme and mean ratio, and another line (for example E) be such that its square taken in the ratio 3:2 gives the octahedron edge, then the greater segment (namely AD) is the dodecahedron edge of that same sphere in which the octahedron will have been inscribed.

THEOR. 26. PROPOS. 26.

If the edge of the tetrahedron has for its square the greater and the lesser segment together of a straight line cut in extreme and mean ratio, then

the edge of the icosahedron inscribed in the same sphere is, in square, the sesquialter (3:2) of the lesser segment.

Let the edge BC of the tetrahedron ABCD have for its square the greater segment EF together with the lesser segment FG of the straight line EG cut in extreme and mean ratio; and let HI likewise be the edge of the icosahedron HIK of the same sphere. I say that the straight line HI is, in square, the sesquialter of the lesser segment FG.

For let the straight line HK subtend one angle of the pentagon formed from the edges of the icosahedron, and let the straight line IK, a diameter, be joined. [Margin (a): 8. tertijd. — *Elements* XIII.8] Since, then, HI is the greater segment of the straight line HK cut in extreme and mean ratio, [Margin (b): 5. tertijd. — *Elements* XIII.5] the composite line KHI will also be cut in the same way at H, its greater segment being HK and its lesser HI. [Margin (c): 22. prim. — *Elements* I.22] From the straight lines BC, EF, FG let the triangle BCL be constructed, so that BL is equal to EF and CL to FG; [Margin (d): 48. pri. — *Elements* I.48] then the angle L will be right, because BC is posited to have for its square the lines BL and CL, that is, the lines EF and FG. But the angle IHK is also right, as we showed at Proposition 23 of this book; [Margin (e): 2. quartidec. — *Elements* XIV.2] and as BL is to LC, so is KH to HI. [Margin (f): 6. sexti. — *Elements* VI.6] Therefore the triangles BLC and KHI are equiangular. [Margin (g): 4. sexti. — *Elements* VI.4] And consequently, as KI is to IH, so is BC to CL; and, alternately, as KI is to BC, so is HI to CL. [Margin (h): 13. tertijdec. — *Elements* XIII.13] But the diameter IK is, in square, the sesquialter of the tetrahedron edge BC. Therefore HI, the icosahedron edge, will also be, in square, the sesquialter of CL, that is, of the lesser segment FG. If, then, the tetrahedron edge have for its square the greater and the lesser segment, and so on. Which had to be shown.

Diagram. The figure for Proposition 25 (top right of the page): a right-angled figure standing on a vertical line, with a separate short segment beside it.

- A (top left): right-angle vertex; the vertical line AB descends from here, and the horizontal line AD runs right.
- D (top right): far end of the horizontal top line; the hypotenuse BD is drawn from B up to D.
- C (on line AB, below A): the point cutting AB in extreme and mean ratio.
- B (bottom left): foot of the vertical line; endpoint of the subtending line BD.
- E (center right, a short standalone vertical segment): the line whose square is half the square on BD.

Diagram. The figure for Proposition 26: at the left a tetrahedron with an appended triangle, at the right an icosahedron shown in outline, and below a divided straight line.

- A (apex, upper left): top vertex of the tetrahedron ABCD.
- D (interior of the tetrahedron): the far/interior vertex.
- B (lower left): base vertex.
- C (lower right): base vertex.
- L (below the base, near E): apex of the constructed triangle BCL.
- H (icosahedron, top left) and I (icosahedron, top right): upper vertices; HI is the icosahedron edge.
- K (icosahedron, bottom vertex): the point to which HK is subtended.
- E, F, G (horizontal line along the bottom, left to right): the straight line EG cut at F, with greater segment EF and lesser segment FG.

THEOR. 27. PROPOS. 27.

The cube is to the octahedron inscribed in the same sphere with it as the surface of the cube is to the surface of the octahedron; and likewise as the edge of the cube is to the semidiameter of the sphere.

Let there be, in the same sphere, the cube ABCDEFGH and the octahedron IKLMNO, whose centers are P and Q. I say that the ratio of the cube to the octahedron is the same as the ratio of the surface of the cube to the surface of the octahedron; and likewise the same as the ratio of the cube's edge to the semidiameter of the sphere.

For if straight lines are drawn from the centers P and Q to all the vertices, both of the cube and of the octahedron, the cube will be resolved into six equal pyramids, since these have equal bases and equal heights. (For since the circles described about the pyramids' bases — that is, about the cube's faces — are equal, they are equally distant from the center of the sphere; and therefore the perpendiculars drawn from the center P to the planes of the circles, or of the said bases — namely the heights of the pyramids — are equal, by Lemma 2 of Proposition 10, Book 14.) And, for the same reason, the octahedron is resolved into eight equal pyramids.

[Margin (a): 21. quartidec. — *Elements* XIV.21]

Since, moreover, one and the same circle encloses both the cube's square face and the octahedron's triangular face, the bases of the cube's pyramids and of the

octahedron's pyramids are equally distant from the center of the sphere, by the aforesaid lemma; and consequently the heights of the cube's pyramids and of the octahedron's pyramids will be equal. For this reason, as one pyramid of the cube is to one pyramid of the octahedron, so is the base of the cube's pyramid to the base of the octahedron's pyramid, by the scholium to Proposition 6, Book 12. And accordingly — from what we demonstrated in the scholium to Proposition 22, Book 5 — as six times the cube's pyramid (that is, the cube itself) is to the octahedron's pyramid, so is six times the base of the cube's pyramid (that is, the whole surface of the cube) to the base of the octahedron's pyramid. And so again, from the same scholium, as the cube is to eight times the octahedron's pyramid (that is, to the octahedron itself), so is the surface of the cube to eight times the base of the octahedron's pyramid (that is, to the surface of the octahedron itself). Which was the point proposed to be shown in the first place.

Let there be again, in the same sphere, the cube ABCD and the octahedron EFGHIK, let EG be the diameter of the sphere (or of the octahedron), and let L be the center. From AB, the cube's edge, let a third part AM be taken away; likewise from the semidiameter EL let a third part EN be taken away. Now since the square IFKH, the octahedron's edge, being the base of the two pyramids into which we said the octahedron is divided at Proposition 16 of this book, [Margin (b): 18. tertijdecim. — *Elements* XIII.18] is, in the ratio 3:2, the square BD on the cube's edge BC; and since the straight line AB is, by construction, three-to-two of the straight line BM; the bases IFKH and BD, and the heights AB and BM, will be reciprocally proportional. [Margin (c): 34. vnd. — *Elements* XI.34] And consequently the parallelepiped on the base IFKH and under the height BM will be equal to the parallelepiped on the base BD and under the height AB, that is, to the cube ABCD.

Again, since the straight line EL is triple the straight line EN, the parallelepiped on the base IFKH and under the height EL will be triple the parallelepiped on the same base and under the height EN, as we showed in the scholium to Proposition 14, Book 12. But this same parallelepiped on the base IFKH and under the height EL is also triple the pyramid IFKHE, by Corollary 1 of Proposition 7, Book 12. Therefore the pyramid IFKHE will be equal to the parallelepiped on the base IFKH and under the height EN. But of this parallelepiped the double is the parallelepiped on the same base and under the height LN, by the scholium to Proposition 14, Book 12, since the height LN is also double the height EN. [Margin (d): 16. quartidecim. — *Elements* XIV.16] Likewise the octahedron EFGHIK is double the pyramid IFKHE. Therefore the parallelepiped on the base IFKH and under the height LN will be equal to the octahedron EFGHIK.

Furthermore, since the parallelepiped on the base IFKH and under the height BM is to the parallelepiped on the same base and under the height LN as the height BM is to the height LN, by the scholium to Proposition 14, Book 12 — and since, as BM is to LN, so is AB to EL (for as AM is to MB, so is EN to NL; and by composition, as AB is to BM, so is EL to LN; and therefore, alternately, as AB is to EL, so is BM to LN) — and since the parallelepiped on the base IFKH and under the height BM was shown to be equal to the cube ABCD, while the parallelepiped on the same base and under the height LN is equal to the octahedron EFGHIK: it follows that, as the cube ABCD is to the octahedron EFGHIK, so is AB, the cube's edge, to EL, the semidiameter of the sphere. Which was the point proposed to be demonstrated in the second place. Therefore the cube is to the octahedron inscribed in the same sphere with it, and so on. Which was to be shown.

Diagram. The first figure for Proposition 27: a cube with all its diagonals drawn to its center, beside an octahedron with all its diagonals drawn to its center. (The cube also appears, partly, at the foot)

- Cube ABCDEFGH, center P:
- F (top left), E (top right): upper front vertices.
- A (upper-left interior), D (upper-right interior): rear upper vertices.
- P (center): center of the cube, to which all vertices are joined.
- G (lower left), B (bottom left), C (bottom right): lower vertices.
- H (lower-right interior): rear lower vertex.
- Octahedron IKLMNO, center Q:
- I (top vertex), L (bottom vertex).
- K (left vertex), M (right vertex).
- N (upper interior), O (lower interior): the remaining two vertices.
- Q (center): center of the octahedron, to which all vertices are joined.

Diagram. The second figure for Proposition 27: a cube beside an octahedron drawn with interior lines.

- Cube ABCD:
- A (upper left), D (upper-right interior): upper vertices.
- M (mid left, marked with a tick): the point cutting off the third part AM of the edge AB.
- B (bottom left), C (bottom right): lower vertices.
- Octahedron EFGHIK:
- E (top vertex), G (bottom vertex): EG is the diameter.

- F (left vertex), H (right vertex).
- I (upper interior), K (lower interior): the two vertices flanking the axis.
- L (center, marked with a tick): the center of the octahedron and of the sphere.

SCHOLIUM.

[Margin (e): 13. tertijdec. — *Elements* XIII.13]

Since it has been demonstrated that the diameter of the sphere in which all five regular solids are inscribed is, in square, the sesquialter (3:2) of the tetrahedron edge — so that, if the square on the diameter be set at 3 parts, 2 of the same parts are the square on the tetrahedron edge — it is evident that, if from the square on a given diameter one takes away a square containing two-thirds of it (as we showed in the scholium to Proposition 33, Book 6), the side of this remaining square is the tetrahedron edge. Or again: if two equal squares be set up on [i.e. summing to] the square on the diameter, standing to one another in the ratio 2:1 (by the same scholium), the side of the larger square is the tetrahedron edge. For if from the square on the diameter, of 3 parts, one subtracts the square on the tetrahedron edge, of 2 parts, there necessarily remains a square of 1 part, to which the square on the tetrahedron edge has the ratio 2:1. So, if the diameter of the sphere be set as AB, from whose square one subtracts the square on the straight line AC, containing two-thirds of the square on the diameter AB; or if two squares on the straight lines AC and BC be set up equal to the square on the diameter AB, so that the square on AC is double the square on BC: then AC will be the tetrahedron edge.

[Margin (f): 14. tertijdec. — *Elements* XIII.14]

Again, since it has been shown that the diameter of the sphere is, in square, double the octahedron edge, it comes about that — if from the square on the diameter AB one removes the square on the straight line AD, which is half the square on the diameter AB; or if two squares on the straight lines AD and BD be set up equal to the square on the diameter AB and standing in a ratio of equality to one another, according to the teaching of the scholium to Proposition 33, Book 6 — the straight line AD is the octahedron edge.

[Margin (g): 15. tertijd. — *Elements* XIII.15]

Next, since we taught that the diameter of the sphere is, in square, triple the cube edge, it results that — if from the square on the diameter AB one takes away the square on the straight line BC, which is a third part of the square on the diameter

AB; or if on the square on the diameter AB two equal squares on the straight lines AC and BC be constructed, so that the square on AC is double the square on BC, according to the teaching of the same scholium to Proposition 33, Book 6 — the straight line BC is the cube edge. Or indeed, since the diameter AB has for its square both the tetrahedron edge and the cube edge, by Corollary 2 of Proposition 15, Book 13, it is manifest that if from the square on the diameter AB one takes away the square on the tetrahedron edge AC, the square on the cube edge BC is left; just as, conversely, if from the square on the diameter AB one removes the square on the cube edge BC, the square on the tetrahedron edge AC remains.

[Margin (h): 23. quartidecim. — *Elements* XIV.23]

Furthermore, since — if some straight line have for its square a whole straight line cut in extreme and mean ratio together with the greater segment of that line — the greater segment is the icosahedron edge of that sphere whose diameter is the former line: it is established that, if on the square on the diameter AB there be exhibited (according to what we demonstrated in the scholium to Proposition 33, Book 6) two equal squares on the straight lines AE and BE,

standing in the proportion that any straight line cut in extreme and mean ratio has to its greater segment: then the lesser line BE is the icosahedron edge.

[Margin (i): 23. quartidec. — *Elements* XIV.23]

Lastly, since — if some straight line have for its square a whole straight line cut in extreme and mean ratio together with the lesser segment of that line — the lesser segment is the dodecahedron edge of that sphere whose diameter is the former line: it is clear that, if on the square on the diameter AB there be set up two equal squares on the straight lines AF and BF, standing in the proportion that any straight line cut in extreme and mean ratio has to its lesser segment: then the lesser line BF is the dodecahedron edge.

Thus far, then, have Hypsicles, Campanus, and François de Foix, Count of Candale [Franciscus Candalla], written on the comparison of the regular solids inscribed in the same sphere. The following five propositions, bearing on the same matter, I have gathered from Cardano [Gerolamo Cardano], Book 5 of *On Proportions*, in the manner that here follows. Then, at the end of Book 16, I shall add some further matters that bear on the inscribing of these same five regular solids in a sphere and on their comparison.

Diagram. The figure for the Scholium: a fan of straight lines rising from a base line, illustrating the several squares compared in the text.

- A (bottom left) and B (bottom right): endpoints of the base line AB (the diameter).
- D, C, E, F (upper points, left to right): the upper endpoints of the lines drawn up from A (and toward B), marking the edges of the various regular solids compared (tetrahedron, cube, octahedron, icosahedron, dodecahedron).

THEOR. 28. PROPOS. 28.

If there are four straight lines in continued proportion, and also another four, such that the first term of all is the same: then the ratio of the third to the third will be the duplicate of the ratio of the second to the second; and the ratio of the fourth to the fourth will be the triplicate of that same ratio of the second to the second.

Let there be four straight lines A, B, C, D in continued proportion, and also four, A, E, F, G, such that A is the first term of all. I say that the ratio of C to F — of the third to the third — is the duplicate of the ratio of B to E, of the second to the second; and that the ratio of D to G — of the fourth to the fourth — is the triplicate of that same ratio of B to E, of the second to the second.

For since the straight line C, by Definition 10, Book 5, has to the straight line A the duplicate of the ratio of the straight line B to the straight line A; [Margin (a): 20. sexti. — *Elements* VI.20] and since the square on B also has to the square on A the duplicate of that same ratio of B to A: then, as C is to A, so is the square on B to the square on A. By the same reasoning, as F is to A, so is the square on E to the square on A; and, by inversion, as A is to F, so is the square on A to the square on E. Since, then, the square on B is to the square on A as C is to A, and the square on A is to the square on E as A is to F, it follows *ex aequo* that the square on B is to the square on E as C is to F.

square on B,
square on A.
square on E.

line C,
line A.
line F.

[Margin (b): 20. sexti. — *Elements* VI.20] But the ratio of the square on B to the square on E is the duplicate of the ratio of B to E. Therefore the ratio of C to F is likewise the duplicate of the ratio of B to E. Which is the first point.

Again, since the straight line D, by Definition 10, Book 5, has to the straight line A the triplicate of the ratio of B to A; [Margin (c): 33. vnde. — *Elements* XI.33] and since the cube on B also has to the cube on A the triplicate of that same ratio of B to A: then, as D is to A, so is the cube on B to the cube on A. By the same reasoning, as G is to A, so is the cube on E to the cube on A; and, by inversion, as A is to G, so is the cube on A to the cube on E. Since, then, the cube on B is to the cube on A as D is to A, and the cube on A is to the cube on E as A is to G, it follows *ex aequo* that the cube on B is to the cube on E as D is to G.

cube on B.	line D.
cube on A.	line A.
cube on E.	line G.

[Margin (d): 33. vnde. — *Elements* XI.33] But the ratio of the cube on B to the cube on E is the triplicate of the ratio of B to E. Therefore the ratio of D to G is likewise the triplicate of the ratio of B to E. Which is the second point. And so, if there are four straight lines, and so on. Which had to be demonstrated.

Diagram. The figure for Proposition 28: two sets of straight line-segments of differing lengths, laid out for the continued-proportion argument.

- A (short segment at the top center): the common first term of both series.
- B, C, D (left column, top to bottom): the remaining terms of the first series A, B, C, D.
- E, F, G (right column, top to bottom): the remaining terms of the second series A, E, F, G.

SCHOLIVM.

We demonstrated this same thing at Proposition 10, Book 8, concerning any number of numbers in two series of continued proportionals beginning from some one and the same number.

THEOR. 29. PROPOS. 29.

The square on the side of an equilateral triangle has to the triangle itself a ratio which is the duplicate of the ratio of the side of the triangle to the

mean proportional between the perpendicular drawn from one angle to the opposite side, and half of that side.

In the equilateral triangle ABC, from the angle A to the opposite side BC let the perpendicular straight line AD be drawn, which will bisect the side BC, as was demonstrated in the corollary to Proposition 12 of this book; and consequently BD is half of the side AB. [Margin (a): 13. sexti. — *Elements* VI.13] Let, then, the straight line E be found as the mean proportional between AD and BD. I say that the square on the side AB has to the triangle ABC a ratio which is the duplicate of the ratio of AB to the line E.

For since the triangle ABC is equal to the rectangle contained by AD and BD, as is evident from the scholium to Proposition 41, Book 1; [Margin (b): 17. sexti. — *Elements* VI.17] and since the square on E is equal to that same rectangle under AD and BD: the triangle ABC and the square on E will be equal. [Margin (c): 20. sexti. — *Elements* VI.20] And since the square on the side AB has to the square on E a ratio which is the duplicate of the ratio of AB to E, it is established that the square on the side AB has also to the triangle ABC a ratio which is the duplicate of the ratio of AB to E. For which reason, the square on the side of an equilateral triangle, and so on. Which was to be demonstrated.

Diagram. The figure for Proposition 29: an equilateral triangle with its altitude drawn, and a separate segment to the right.

- A (apex, top): the angle from which the perpendicular is drawn.
- B (bottom left), C (bottom right): the base vertices.
- D (midpoint of BC): the foot of the perpendicular AD, which bisects BC.
- E (a standalone vertical segment to the right): the mean proportional between AD and BD.

Book XIV — conclusion of Proposition 29 (Corollary)

COROLLARIUM

Therefore, if in an equilateral triangle a perpendicular is drawn from one angle to the opposite side, then the square on the straight line that is the mean proportional between that perpendicular and half the base is equal to the triangle itself. For it has been shown that the square on the straight line E — the mean proportional between AD and DB — is equal to the triangle ABC.

THEOR. 30. PROPOS. 30.

If a cube and a tetrahedron are described in the same sphere, then the square of the cube will be to the triangle of the tetrahedron as the side of the tetrahedron is to the perpendicular drawn from one angle of the tetrahedron's triangle to the opposite side.

Diagram. Two figures set side by side. On the left, a square ABCD — A upper-left, D upper-right, B lower-left, C lower-right — representing the square on the cube's side. Between the two figures stands a short vertical segment marked I (top) and K (below it), the mean proportional IK. On the right, a triangle EFG with apex E and base FG, and the perpendicular EH dropped from E to the base, meeting it at H (so the base reads F–H–G left to right).

- A (square, upper-left)
- D (square, upper-right)
- B (square, lower-left)
- C (square, lower-right)
- I / K (mean-proportional segment between the figures)
- E (triangle apex)
- F (base, left)
- H (foot of the perpendicular EH)
- G (base, right)

Let ABCD be the square on the side of any cube, and let EFG be the triangle of a tetrahedron described in the same sphere, in which the perpendicular EH is drawn from the angle E to the opposite side FG, bisecting the side FG (by the corollary to Proposition 12 of this book). I say that the square AC is to the triangle EFG as the side EF of the triangle is to the perpendicular EH.

For let IK be placed as the mean proportional between EH and FH. [Margin: a — *Elements* XIII.28] Since the square on the tetrahedron's side EF is double the square on the cube's side AB, [Margin: b — *Elements* VI.1] and is also double the rectangle contained by the side EF and half of it, FH — because that square and rectangle have the same height, namely EF — the rectangle contained by EF and FH will be equal to the square on the side AB. [Margin: c — *Elements* VI.17] Consequently AB will be the mean proportional between EF and FH; that is, the three straight lines FH, AB, EF are in continued proportion. But by construction the three straight lines FH, IK, EH are also in continued proportion. Therefore, since FH is the antecedent of both series, [Margin: d — *Elements* XIV.28] the ratio

of the straight line EF to the straight line EH — third term to third term — is the duplicate of the ratio of the straight line AB to the straight line IK, second term to second term. And since the square AC on the straight line AB likewise has, to the square on the straight line IK — that is, to the triangle EFG (for by the corollary to Proposition 29 of this book the square on IK is equal to the triangle EFG) — [Margin: e — *Elements* VI.20] a ratio that is the duplicate of the ratio of AB to IK, it follows that the square of the cube AC is to the triangle of the tetrahedron EFG as the side EF is to the perpendicular EH. And so, if a cube and a tetrahedron are described in the same sphere, and so on. Which was to be demonstrated.

THEOR. 31. PROPOS. 31.

The side of the tetrahedron, taken in square, is sesquialter to its own altitude; and the axis, or altitude, of the tetrahedron, taken in square, is sesquitercius to the side of the cube described in the same sphere.

Diagram. A circle with the inscribed triangle ABC — A at the top, B lower-left, C lower-right. D is the centre of the circle; from A the radius AD is drawn to the centre, and AE is drawn to the point E lying to the right (E, D and the base being joined so as to display the axis DE). Below the circle, set apart, is a short horizontal segment labelled F, standing for the side of the cube.

- A (triangle apex, top of circle)
- B (triangle, lower-left)
- C (triangle, lower-right)
- D (centre of the circle; foot of the axis)
- E (point to the right; end of AE / the axis DE)
- F (separate segment below — the side of the cube)

Let ABC be the triangle of a tetrahedron, DE its altitude or axis, and F the side of the cube of the same sphere. I say that the side AB of the tetrahedron is sesquialter in square to the axis DE, and that the axis DE is sesquitercius in square to the side F of the cube. For let a circle ABC be circumscribed about the triangle; [Margin: a — *Elements* IV.5] to its centre D — into which, as is clear from Proposition 15 of this book, the axis DE falls — let the radius AD be drawn, and AE, the side of the tetrahedron. Now since the square on the side AE — that is, on the side AB — [Margin: b — *Elements* I.47] is equal to the squares on the straight lines AD and DE, [Margin: c — *Elements* XIII.12] and is triple the square on the radius AD, it follows that, if the square on the side AB (or AE) is set at 6 parts, the square on the straight line AD is 2 of those same parts. Hence the remaining

square, that on the axis DE, is 4 such parts. Therefore the square on the tetrahedron's side is sesquialter to the square on the tetrahedron's axis.

And since the side AB of the tetrahedron [Margin: d — *Elements* XIII.18] is double in square to the side F of the cube, the square on the side F will be 3 parts of the kind of which the square on side AB was set at 6; but it has been shown that the square on the axis DE is 4 of the same. Therefore the square on the axis DE is sesquitercius to the square on the side F of the cube. And so the side of the tetrahedron is sesquialter in square, and so on. Which was to be demonstrated.

COROLLARIUM I.

Now since, when the square on the side AB is set at 6 parts, the square on the diameter of the sphere is 9 such parts — [Margin: e — *Elements* XIII.13] because the diameter is sesquialter in square to the side of the tetrahedron — while the square on the axis DE has been shown to be 4 of those same parts, it is evident that the diameter of the sphere is, in square, dupla-sesquiquarta (9 to 4) to the axis of the tetrahedron. And accordingly, [Margin: f — *Elements* VI.20] since squares have the duplicate ratio of their sides, it is plain that the diameter is sesquialter to the axis of the tetrahedron. For the duplicate of the sesquialter ratio is the dupla-sesquiquarta ratio, as appears in these numbers: 4, 6, 9. This too we taught in the second corollary to Proposition 13 of Book XIII.

COROLLARIUM II.

It is likewise established that the axis, or altitude, of the tetrahedron is to the side of the cube inscribed in the same sphere as the side of the tetrahedron is to the perpendicular drawn from one angle of the base to the opposite side. This is because [Margin: g — *Elements* XIV.12] the side of the equilateral triangle — that is, the side of the tetrahedron — is sesquitercius in square to the perpendicular drawn from one angle to the opposite side, just as, by this proposition, the axis of the tetrahedron itself is sesquitercius in square to the side of the cube.

LIBER XIV — THEOR. 32. PROPOS. 32.

The cube is triple the tetrahedron inscribed in the same sphere.

Diagram. Two solids drawn in oblique projection. On the left, a cube: its visible near corners are labelled B (upper-left) and C (upper-right, at the back), with A at the lower front; the remaining vertices are unlettered. On the right, a triangular prism DEFGHI standing on the triangular base DEF, of the same height as the tetrahedron: H is the top apex, I the upper-left corner and G the upper-right corner of the top triangle, F lies at the front centre, and D (lower-left) and E

(lower-right) mark the base, with the several internal diagonals of the prism drawn in.

- B (cube, upper-left)
- C (cube, upper-right/back)
- A (cube, lower front)
- H (prism, top apex)
- I (prism, upper-left)
- G (prism, upper-right)
- F (prism, front centre)
- D (prism base, lower-left)
- E (prism base, lower-right)

Let a cube ABC and a tetrahedron DEFG be described in the same sphere. I say that the cube is triple the tetrahedron. Upon DEF, the base of the tetrahedron, let a prism DEFGHI be set up, having the same height as the tetrahedron. [Margin: a — *Elements* XIV.30] Now since the base of the prism (that is, of the cube) ABC is to the base of the prism (that is, of the tetrahedron) DEF as the side of the tetrahedron is to the perpendicular drawn from any angle of the base DEF to the opposite side — and as the side of the tetrahedron is to that perpendicular, so too, by the second corollary to Proposition 31 of this book, is the altitude of the tetrahedron (and hence of the prism DEFGHI) to the side of the cube, and therefore to the altitude of the cube, that is, of the prism ABC — it follows that the base of the prism ABC is to the base of the prism DEFGHI as the altitude of the latter is to the altitude of the former. Consequently, by what we demonstrated at Proposition 9 of Book XII, the prisms ABC and DEFGHI are equal, since their bases and altitudes are reciprocally proportional. But the prism DEFGHI is triple the pyramid, that is, the tetrahedron DEFG, by the first corollary to Proposition 7 of Book XII. Therefore the cube ABC is likewise triple the same tetrahedron DEFG. And so the cube is triple the tetrahedron inscribed in the same sphere. Which was to be shown.

COROLLARIUM.

Hence a prism having the same base and altitude as the tetrahedron is equal to the cube described in the same sphere in which the tetrahedron is described; for it has been shown that the cube ABC is equal to the prism DEFGHI, and so on.

FINIS ELEMENTI DECIMIQUARTI. [End of the Fourteenth Element.]

EVCLIDIS ELEMENTVM QVINTVMDECIMVM

Euclid's Fifteenth Element,

And the Fifth of the Solids, as some think; but, as others hold, **the Fifth Element of Hypsicles of Alexandria, On the Five Bodies.**

LIBER SECVNDVS. [Second Book.]

PROBL. 1. PROPOS. 1.

To describe a pyramid in a given cube.

Diagram. A cube ABCDEFGH drawn in oblique projection, with the inscribed tetrahedron (pyramid) ACGE traced by its face-diagonals. The vertices read: E at the upper-left (back), F at the upper-right (back), C at the right, D as the upper-left inner corner, G near the lower centre, H at the lower-left, A at the lower front, and B at the lower right. The diagonals EA, EG, EC and AG, GC, CA are drawn, forming the four equilateral triangles of the inscribed tetrahedron.

- E (cube, upper-left back; apex from which the three diagonals issue)
- F (cube, upper-right back)
- C (cube, right)
- D (cube, upper-left inner corner)
- G (cube, lower centre)
- H (cube, lower-left)
- A (cube, lower front)
- B (cube, lower right)

(Enunciation:) In a given cube to describe a pyramid, that is, a tetrahedron.

In the given cube ABCDEFGH it is required to describe a pyramid, that is, a tetrahedron. From one of its angles — namely from E — let three diagonals EA, EG, EC be drawn in the three bases that form it; and from the extremities A, G, C of these let diagonals AG, GC, CA likewise be drawn in the remaining three bases, connecting the ends of the three former diagonals. Now since the diagonals of equal squares are equal — because they are double in square to the sides of the equal squares, as we demonstrated in the scholium to Proposition 47 of Book I — it is evident that the four triangles ACE, GAC, GAE, GCE, composed of the said diagonals, are equilateral and equal to one another; and consequently a pyramid, that is, a tetrahedron, is formed from them. And since all the angles of the tetrahedron are set at the angles of the cube, it is manifest that it is inscribed in

the cube, by Definition 31 of Book XI. Therefore in the given cube we have described a pyramid. Which was to be done.

PROBL. 2. PROPOS. 2.

To describe an octahedron in a given pyramid.

Diagram. An equilateral triangle ABC — A at the apex, B lower-left, C lower-right — with the inscribed octahedron drawn from the midpoints of the tetrahedron's edges. The labelled points are: A (apex), E (mid-left), G (mid-right), H (upper centre), D (centre), I (lower-left), K (lower-right), and F (bottom centre), with B and C at the base corners. The twelve connecting straight lines are drawn, marking out the eight triangular faces of the octahedron.

- A (triangle apex)
- E (mid-left)
- G (mid-right)
- H (upper centre)
- D (centre)
- I (lower-left)
- K (lower-right)
- F (bottom centre)
- B (base, lower-left)
- C (base, lower-right)

(Enunciation:) In a given pyramid to describe an octahedron.

Let an octahedron be described in the pyramid, that is, the tetrahedron ABCD. When all the edges have been bisected at E, F, G, H, I, K, and the twelve straight lines EF, FG, GE, HI, IK, KH, EI, IF, FK, KG, GH, HE have been drawn, eight triangles will be formed: four of them — EHI, IHK, KHG, HGE — standing above the plane EIKG, and four others — IFE, EFG, GFK, KFI — below the same plane. Now since the sides AE, AH of the triangle AEH are equal to the sides AG, AH of the triangle AGH — being halves of the equal straight lines AB, AD, AC — and the contained angles EAH, GAH are likewise equal, being angles of the equilateral triangles ABD, ACD, [Margin: a — *Elements* I.4] the bases EH, GH will be equal. And by the same reasoning the straight lines EH, HI will be equal, and the rest in the same way. Therefore the said eight triangles are equilateral and equal to one another; and so they compose an octahedron EIGHF, which by Definition 31 of Book XI is described within the tetrahedron, since all its angles touch all the edges

of the tetrahedron, by construction. Therefore in the given pyramid we have described an octahedron. Which was to be done.

COROLLARY [to Proposition 2, continued]

Now the three planes EIKG, GHIF, FKHE are equal squares, cutting one another mutually at right angles, as we demonstrated in the first corollary to Proposition 14 of Book 13; and each of them cuts the pyramid ABCD in half. (For the square EIKG divides it into the pyramid HIKD and the prism HIKGAE, and again into the pyramid EBFH and the prism EFCGKI; and it is established [Margin: b — "8. duod." (Elements XII.8)] that the one pyramid is equal to the other, and the one prism to the other. In the same way the remaining squares GHIF and FKHE each divide the same pyramid ABCD in half.) It is therefore clear that, if an octahedron is inscribed in a tetrahedron, the tetrahedron is cut in half by three equal squares, which likewise cut the octahedron in half and intersect one another at right angles.

PROBL. 3. PROPOS. 3.

[Margin: iii.]

To inscribe an octahedron in a given cube.

Let the octahedron to be described be in the given cube AH. Let the sides of the base ABCD be bisected at the points I, K, L, M, and let these be joined by the straight lines IL and KM, which cut one another at N — a point that is the center of the square ABCD, as is clear from the demonstration of Proposition 8 of Book 4. Next, by the same reasoning let the centers O, P, Q, R, S of the remaining bases be found. Then all the straight lines drawn from the said centers to the midpoints of the bases — such as NI, RI, NK, SK, and so on — will be equal to the halves of the sides of the cube, that is, of its squares, as is evident from the aforesaid demonstration of Proposition 8 of Book 4.

Finally, if the said centers are joined by the twelve straight lines NO, OP, PQ, QR, RS, SN, NP, PR, RN, SO, OQ, QS, eight triangles will be constituted: four of them — NSR, RSQ, QSO, OSN — lie above the plane NOQR, while the other four — OPQ, QPR, RPN, NPO — lie below the same plane. Now the sides IN, IR of triangle INR are equal to the sides KN, KS of triangle KNS, since all are halves of equal sides of the cube, as has been said; and likewise the angles they contain are equal. For, since NI and IR are parallel to the straight lines BA and AF, [Margin: a — "10. vnde." (Elements XI.10)] the angle NIR is equal to the right angle KAF in the square AG; and by the same reasoning the angle NKS is equal to the right angle

IAF in the square AE. [Margin: b — "4. primi" (Elements I.4)] Therefore the bases NR and NS will be equal.

In no other way shall we show that all the remaining lines are equal both to one another and to these two — namely, if from the centers straight lines are drawn to the midpoints of the sides, under this rule: that any two are drawn to the midpoint of that side which is common to two squares of the cube, the two points from which the straight lines issue being the centers of those squares. For thus you see that the two straight lines NI, RI are drawn to I, the midpoint of the side AD, which is common to the squares AC and AE, whose centers are the points N and R; and likewise the two straight lines NK, SK are drawn to K, the midpoint of the side AB, which is common to the squares AC and AG, whose centers are the points N and S; and so on. For this reason the eight triangles, once constituted, are equilateral and equal to one another, and therefore they constitute the octahedron NOQRSP. And this, by Definition 31 of Book 11, is described within the cube, since all its six angles touch all six bases of the cube at their centers. We have therefore inscribed an octahedron in the given cube. Which was to be done.

Diagram. An oblique (cavalier) projection of the cube AH with the octahedron NOQRSP inscribed within it, its six vertices resting at the centers of the cube's six faces. The cube is drawn with its front-top-left corner at A and its network of face-diagonals and octahedron edges shown. Readable labels:

- A (top-left cube corner)
- F (top-center cube corner)
- E (right cube corner, mid-right)
- B (left cube corner, mid-left)
- C (bottom-left cube corner)
- H (bottom-right cube corner)
- G (bottom-center cube corner)
- D (interior, center — rear cube corner)
- I (upper interior, below A — base-edge midpoint)
- K (left interior — base-edge midpoint)
- L (lower-left interior — base-edge midpoint)
- M (bottom-center interior — base-edge midpoint)
- N (left-center interior — octahedron vertex / face-center)
- O (bottom-center interior — octahedron vertex)
- P (right-center interior — octahedron vertex)
- Q (right interior — octahedron vertex)

- R (upper-center interior, below F — octahedron vertex)
- S (center-left interior — octahedron vertex)

ALITER [Another way]

[Margin: c — "2. quintidec." (Elements XV.2)] Let a pyramid be inscribed in the given cube; [Margin: d — "1. quintidec." (Elements XV.1)] then let an octahedron be figured within this pyramid, and what was proposed will have been accomplished. For since, by the demonstration of Proposition 2 of this book, the angles of the octahedron touch the edges of the pyramid, while the edges of the pyramid, by the demonstration of Proposition 1 of this book, lie in the planes of the cube's bases (being the diagonals of those bases), it is evident that the angles of the octahedron also touch the bases of the cube; and thus the octahedron is inscribed in the cube.

COROLLARY

Now the diameters of the octahedron described in the cube, NQ and RO (were they to be drawn), cut one another mutually at right angles, by the first corollary to Proposition 14 of Book 13 — these being the lines that join the centers of the cube's opposite bases. It is therefore clear that the straight lines connecting the centers of the cube's opposite bases not only bisect one another mutually, as we taught in the first corollary to Proposition 15 of Book 13, but also divide one another at right angles, as we here show.

PROBL. 4. PROPOS. 4.

[Margin: iv.]

To inscribe a cube in a given octahedron.

Let the given octahedron be ABCDEF, in which a cube is to be described. Since an octahedron contains six quadrangular pyramids — whose vertices are the six solid angles of the octahedron, and any two of which constitute the octahedron — let one of them be the pyramid ABCDE, whose base ABCD is a square and whose triangles ABE, EBC, CED, DEA are equilateral; and let the centers of these triangles be G, H, I, K, through which are drawn the straight lines LM, MN, NO, OL parallel to the sides AB, BC, CD, DA. Then the triangles LME, EMN, NEO, OEL will also be equilateral, since they are similar to the equilateral triangles ABE, EBC, CED, DEA, by the corollary to Proposition 4 of Book 6,

and hence equal to one another, because they have their sides in common. For EM is the side common to the triangles LME and EMN; EN is common to the triangles EMN and NEO; and lastly EO is common to the triangles NEO and OEL. Hence it comes about that triangle LME is equal to triangle EMN, this to NEO, and this to OEL. For the straight lines LM, MN, NO, OL meet at the points L, M, N, O, as we shall show presently. Therefore the four straight lines LM, MN, NO, OL are also equal to one another. [Margin: a — "18. quartidec." (Elements XIV.18)] Now since the straight line EK, drawn through P, is triple the line KP, [Margin: b — "2. sexti" (Elements VI.2)] and as EP is to KP, so is EA to LA, EA too will be triple the line LA. By the same reasoning EB, EC, ED will be triple the lines MB, NC, OD. Therefore the straight lines LM, MN, NO, OL will meet at the points L, M, N, O, at which the sides are cut in that proportion, and they will constitute the quadrilateral LMNO.

Again, since LO and LM are parallel to the straight lines AD and AB, though not lying in the same plane with them, [Margin: c — "10. vn de." (Elements XI.10)] the angle MLO will be equal to the angle BAD, which is a right angle (since ABCD is a square, as we showed in the first corollary to Proposition 14 of Book 13 and in Proposition 16 of Book 14). And therefore the angle MLO too will be a right angle. In just the same way we shall demonstrate that the remaining angles LMN, MNO, NOL are right angles. Now since both the plane drawn through LO, LM and that drawn through NM, NO [Margin: d — "15. vn de." (Elements XI.15)] are parallel to the plane ABCD, [Margin: e — "15. vn de." (Elements XI.15)] they themselves, since they meet at the points M and O, will make one plane, by the scholium to Proposition 16 of Book 11. Therefore LMNO will be a square, since it has both equal sides and right angles.

Next, since the sides EL, EK of triangle ELK are equal to the sides EO, EK of triangle EOK, and the angles contained by them are likewise equal — because the straight line EK, drawn from the angle E through the center K of the equilateral triangle EAD, bisects the angle AED, by the corollary to Proposition 12 of Book 13 — [Margin: f — "4. prim." (Elements I.4)] the bases KL, KO will also be equal; that is, the straight line LO will be bisected at K. By the same reasoning LM, MN, NO will be bisected at G, H, I. And therefore the straight lines GL, GM, HM, HN, IN, IO, KO, KL, being halves of the sides of the square LN, will be equal to one another; and hence, since they also contain equal angles — namely right angles — [Margin: g — "4. prim." (Elements I.4)] the straight lines GH, HI, IK, kG connecting the centers G, H, I, k will likewise be equal. Now since these straight lines lie in the plane of the square LN and comprehend right angles — (for since the angles LkG, GkI, IkO are together equal to two right angles, [Margin: h — "13. prim."])

(Elements I.13)] while LkG and IkO are each half a right angle, by the second corollary to Proposition 32 of Book 1, the angle GkI will be a right angle; and by the same reasoning the angles kGH, HIk will be right angles) — GHIk will be a square, since it has both equal sides and right angles.

And if by the same art the centers of the triangles are joined by straight lines in the remaining five pyramids of the octahedron, squares will likewise be described; and since these have their sides in common, they will be equal to one another. Therefore six squares of this kind will compose a cube, which, by Definition 31 of Book 11, will be described within the octahedron, since its eight angles touch the eight bases of the octahedron at their centers. Therefore we have described a cube in the given octahedron. Which was to be done.

Diagram. The octahedron ABCDEF drawn upright, with the apex E at top and F at bottom and the square ABCD across the middle; the cube (whose upper face is the square GHIk) is inscribed with its eight vertices at the centers of the octahedron's eight triangular faces. Numerous face-medians and cube edges are drawn.

Readable labels:

- E (top apex)
- F (bottom apex)
- A (left vertex of the middle square)
- D (right vertex of the middle square)
- B (lower-left vertex)
- C (lower-right vertex)
- I (upper-left interior — triangle center)
- K (upper-center interior — triangle center)
- O [?] (upper-right interior — triangle center; glyph could be O or Q)
- G (interior, left of center)
- P (center)
- I [?] (interior, right of center — a second I-glyph; reading uncertain)
- M (lower-left interior)
- H (lower-center interior)
- N (lower-right interior)

PROBL. 5. PROPOS. 5.

[Margin: v.]

To inscribe a dodecahedron in a given icosahedron.

Of the twelve pentagonal pyramids contained in the icosahedron — whose bases are pentagons composed of five sides of the icosahedron, and whose vertices are the twelve solid angles of the icosahedron — let one pyramid be ABCDEF, whose base is the pentagon ABCDE and whose equilateral triangles are ABF, BCF, CDF, DEF, EAF; and let the centers of these triangles be G, H, I, K, L, which are connected by the straight lines GH, HI, IK, KL, LG. Again, from the vertex F, let the straight lines FM, FN, FO, FP, FQ be dropped through the centers of the triangles; these, by the scholium to Proposition 12 of Book 13, will bisect the sides AB, BC, CD, DE, EA at M, N, O, P, Q, so that the ten straight lines MA, MB, NB, NC, OC, OD, PD, PE, QE, QA are equal; and since they comprehend equal angles — namely the angles of the pentagon — [Margin: a — "4. prim." (Elements I.4)] the straight lines MN, NO, OP, PQ, QM connecting the points M, N, O, P, Q will also be equal. [Margin: b — "8. primi" (Elements I.8)] And accordingly the angles MFN, NFO, OFP, PFQ, QFM will be equal, because the sides FM, FN, FO, FP, FQ are also equal — being the perpendiculars dropped from the angles of equal equilateral triangles to the opposite bases. For since the sides FB, BM of triangle FBM are equal to the sides FB, BM of triangle FBM, and contain equal angles — namely those of equilateral triangles — [Margin: c — "4. primi" (Elements I.4)] the bases FM, FN will be equal; and so with the rest. And since the straight lines FG, FH, FI, FK, FL — namely the radii of the equal circles that circumscribe the equal equilateral triangles — are equal and comprehend equal angles, as has been demonstrated,

[Margin: d — "4. prim." (Elements I.4)] the straight lines GH, HI, IK, KL, LG will also be equal. [Margin: e — "13. prim." (Elements I.13)] Again, since both the angles AMQ, QMN, NMB and the angles BNM, MNO, ONC are equal to two right angles, [Margin: f — "4. primi" (Elements I.4)] and the angles AMQ, NMB are equal to the angles BNM, ONC, the remaining angles QMN, MNO will be equal; and by the same reasoning the remaining angles NOP, OPQ, PQM will be equal both to one another and to these. Therefore the pentagon MNOPQ, lying in the plane of the pentagon ABCDE, will be equilateral and equiangular. Finally, since the straight lines FM, FN, FO, FP, FQ are cut proportionally at G, H, I, K, L — because FG, FH, FI, FK, FL, drawn from the centers, are equal; or indeed because FG, FH, FI, FK, FL are double GM, HN, IO, KP, LQ, by the corollary to Proposition 18 of Book 14 — [Margin: g — "2. sexti" (Elements VI.2)] GH, HI, IK, KL, LG will be parallel to MN, NO, OP, PQ, QM; [Margin: h — "10. vnde." (Elements XI.10)] and therefore the angles LGH, GHI, HIK, IKL, KLG will be equal to the equal angles QMN, MNO, NOP, OPQ, PQM, and hence equal to one another. [Margin: i — "15. vnde." (Elements XI.15)] Now since the planes drawn through GH, HI and through HI, IK are parallel to the plane of the pentagon MNOPQ drawn through MN, NO,

OP, and meet in the straight line HI, they will make one plane, by the scholium to Proposition 16 of Book 11. In the same way we shall show that the planes drawn through Ik, kL and through kL, LG, as well as through LG, GH, constitute the same plane with the plane drawn through GH, HI, Ik. Therefore GHIkL is an equilateral and equiangular pentagon, since it has both equal sides and equal angles.

And if by the same art the centers of the triangles are connected by straight lines in the remaining eleven pyramids of the icosahedron, equilateral and equiangular pentagons will likewise be described; and since these have their sides in common, they will be equal to one another. For this reason twelve pentagons of this kind will constitute a dodecahedron, which, by Definition 31 of Book 11, will be described in the icosahedron, since the twenty angles of the dodecahedron rest at the centers of the icosahedron's twenty bases. Therefore we have described a dodecahedron in the given icosahedron. Which was to be done.

Diagram. The same icosahedron figure that is fully described at fig-p597-1 below, printed again at the foot of this page: the pentagon ABCDE with its five triangles meeting at the central vertex F, the triangle-centers G, H, I, k, L, and the inner pentagon MNOPQ that forms one face of the inscribed dodecahedron. (Labels as in fig-p597-1.)

COROLLARY

Therefore, since all the straight lines from the center of the icosahedron to the centers of the bases — that is, to the angles of the dodecahedron inscribed in it — are equal (for, by the corollary to the first lemma of Proposition 10 of Book 14, they are perpendicular to the planes of the equal circles that circumscribe the bases of the icosahedron, and so their distances stand from the center of the sphere, or of the icosahedron; and they will be equal to one another, by the second lemma of the same Proposition 10 of Book 14) — the center of the icosahedron and of the dodecahedron inscribed in it will be one and the same.

Diagram. A plane projection of one pentagonal pyramid of the icosahedron seen from above its apex: the outer regular pentagon ABCDE with apex-vertex A at the top, the central point F (the projected apex of the pyramid), the five triangle-centers G, H, I, k, L around F, and the inner pentagon MNOPQ (with M, N, O, P, Q at the midpoints of the outer pentagon's sides) which is one face of the inscribed dodecahedron. The pentagon is filled with the connecting radii and diagonals. Readable labels:

- A (top apex of the outer pentagon)
- M (upper-left vertex, outer pentagon)

- Q (upper-right vertex, outer pentagon)
- B (left vertex, outer pentagon)
- E (right vertex, outer pentagon)
- C (bottom-left vertex, outer pentagon)
- O (bottom-center vertex, outer pentagon)
- D (bottom-right vertex, outer pentagon)
- F (center — projected pyramid apex)
- G (interior, upper-left — triangle center)
- L (interior, upper-right — triangle center)
- H (interior, lower-left — triangle center)
- k (interior, lower-right — triangle center)
- I (interior, lower-center — triangle center)
- N (lower-left interior — inner-pentagon vertex)
- P (lower-right interior — inner-pentagon vertex)

PROBL. 6. PROPOS. 6.

[Margin: vi.]

To describe a pyramid in a given octahedron.

Let the pyramid to be described be in the octahedron ABCDEF. In the upper pyramid of the octahedron, ABCDE, take G and H, the centers of the mutually opposite corresponding triangles AED, BEC; likewise in the lower pyramid ABCDF take I and k, the centers of the triangles CFD, AFB, which also correspond to one another oppositely. Let these centers be joined by the six straight lines GH, GI, Gk, HI, Hk, Ik, so that the pyramid HIkG is formed. Next, from A let the straight lines AL, AM be dropped through the centers G, k, cutting the opposite edges of the octahedron DE, BF in half at L and M, by the scholium to Proposition 12 of Book 13; and let the straight line LM be joined. [Margin: a — "33. primi" (Elements I.33)] This will be equal and parallel to BE, or to FD, since EL, BM, and likewise DL, FM, are equal and parallel straight lines — namely the halves of the sides DE, BF, which lie opposite in the square EF. Now since the straight lines AL, AM are cut proportionally at the points G and k — because AG, Ak are double GL, kM, by the corollary to Proposition 18 of Book 14 — [Margin: b — "2. sexti" (Elements VI.2)] Gk and LM will be parallel; and therefore, by the corollary to Proposition 4 of Book 6, the triangles ALM, AGk are similar. [Margin: c — "4. sexti" (Elements VI.4)] Hence, as AL is to LM, so is AG to Gk; and, alternately, as AL is to AG, so is LM to Gk. But AL is one and a half times AG, [Margin: d — "18. quartidec."]

(Elements XIV.18)] because GL is a third part of AL, and so AG is two thirds of the same AL. Therefore LM also — that is, its equal DF — will be one and a half times Gk.

By this same art we shall show that the remaining lines joining the centers G, H, I, k are one and a half times the edges of the octahedron; and accordingly that the straight lines composing the pyramid HIKG are equal to one another — that is, that equilateral and equal triangles are formed from them, and so a tetrahedron is constituted. And this, since it has all its angles at the centers of the four bases of the octahedron, will be described in the octahedron, by Definition 31 of Book 11. Therefore we have described a pyramid in the given octahedron. Which was to be done.

Diagram. A projection of the octahedron ABCDEF (drawn as the square ABCD with the apexes E and F shown among the interior points) with the pyramid — the inscribed tetrahedron HIKG whose vertices lie at the centers of four of the octahedron's faces. The square is crossed by the full set of face-diagonals and tetrahedron edges. Readable labels:

- A (top-left square corner)
- D (top-right square corner)
- B (bottom-left square corner)
- C (bottom-right square corner)
- G (upper-center interior — triangle center / tetrahedron vertex)
- L (right interior, upper — edge midpoint)
- E (right-center interior — octahedron apex)
- I (right-center interior, near E — triangle center)
- k (left interior — triangle center)
- F (center interior — octahedron apex)
- M (lower-left interior — edge midpoint)
- H (bottom-center interior — triangle center)

ALITER [Another way]

[Margin: e — Elements XV.4] Let a cube be inscribed in the given octahedron, [Margin: f — "1. quintidecim." (Elements XV.1)] and in the cube a pyramid, and what is proposed will have been accomplished. For since the angles of the cube rest at the centers of the octahedron's bases, and the angles of the pyramid are placed at the angles of the cube, it is evident that the angles of the pyramid touch

the bases of the octahedron at their centers; and therefore the pyramid is inscribed in the octahedron.

COROLLARIVM I.

(continuation — Corollary I to Proposition 6 of Book XV)

Since it has been shown that both GK and DF are parallel to LM [Margin: g — Elements XI.9], GK is therefore parallel to DF, and so also to BE. By the same reasoning KH is parallel to FC and to EA [Margin: h — Elements XI.15].

Consequently the plane GKH, drawn through GK and KH, is parallel to the plane DFC through DF and FC, and likewise to the plane BEA through BE and EA. In the same way the remaining triangular faces of the pyramid are parallel to the remaining triangular faces of the octahedron, each single face to two of them. It is therefore established that, if a tetrahedron is inscribed in an octahedron, the four faces of the tetrahedron are parallel to the eight faces of the octahedron — each single face parallel to two opposite ones.

COROLLARIVM II.

Moreover, since the octahedron circumscribes both the cube and the tetrahedron inscribed in the cube — because the tetrahedron's angles are set at the cube's angles, and the cube's angles at the centers of the octahedron's faces — the straight line joining the centers of the octahedron's opposite faces is the diameter of the cube; that is, the diameter of the sphere that circumscribes both the cube and the tetrahedron inscribed in it, as is clear from Prop. 4 of this book. Now the diameter of the sphere is one and a half times [*sesquialter*] the axis of the tetrahedron inscribed in that sphere, by Coroll. 1 of Prop. 31, Book 14, or by Coroll. 2 of Prop. 13, Book 13. It is therefore manifest that, if a tetrahedron is inscribed in an octahedron, the straight line joining the centers of the octahedron's opposite faces is one and a half times the axis of the tetrahedron — that is, of the perpendicular drawn from an angle of the tetrahedron to the opposite face.

PROBL. 7. PROPOS. 7. [Margin: o. vii.]

In a given dodecahedron to inscribe an icosahedron.

Let a half of the dodecahedron, in which the icosahedron is to be inscribed, be contained by the six pentagons ABLPK, PKIHO, OHGFN, NFEDM, MDCBL, LMNOP, whose centers are Q, R, S, T, V, X. Let each two neighbouring centers be joined by

the straight lines QR, RS, ST, TV, VQ, QX, RX, SX, TX, VX. From the angles K and D, which the straight lines AP and CM subtend, let the lines KY and DY be drawn through the centers Q and R; by Coroll. 2 of Prop. 10, Book 13, these will bisect the opposite side BL at right angles, and will therefore meet at the point Y. The same lines YK and YD also bisect the lines AP and CM at right angles, at α and β , by Coroll. 1 of the same proposition — and they likewise bisect the arcs AKP and CDM of the circles described about the pentagons [Margin: a — Elements III.28] (for the straight lines AK and PK, and likewise CD and MD, subtend equal arcs). Therefore $A\alpha$ and $C\beta$ are parallel to the line BY [Margin: b — Elements I.28], and so also parallel to each other [Margin: c — Elements XI.9]. They are moreover equal, being the halves of the lines AP and CM [Margin: d — Elements I.4], which are equal, since the sides KA, KP are equal to the sides DC, DM and contain equal angles. The straight lines AC and $\alpha\beta$ that connect them will therefore be equal and parallel [Margin: e — Elements I.33].

We shall show it no differently if, from the angles B and F, which the lines CL and EN subtend, the lines FZ and EZ are drawn through the centers R and S, and the lines CE and $\gamma\delta$ are connected: these CE and $\gamma\delta$ will be equal. Now AC and CE are equal to one another, since they contain equal angles enclosed by equal lines. (For both the dodecahedron's sides AB, BC and the sides CD, DE contain the angle of an equiangular pentagon, as is clear from any material dodecahedron. These angles, however, do not incline toward X, as the figure seems to show, but outward, as is evident from the same material dodecahedron.) Therefore $\alpha\beta$ and $\gamma\delta$ will also be equal.

Consequently, since the lines $Y\alpha$, $Y\beta$, $Z\gamma$, $Z\delta$ are equal [Margin: f — Elements I.4] — for YQ, YR, ZR, ZS, being perpendiculars, are the distances of the equal lines BL, DM from the centers Q, R, S of equal circles (which of course circumscribe equal pentagons), and are therefore equal [Margin: g — Elements III.14]; likewise $Q\alpha$, $R\beta$, $R\gamma$, $S\delta$ are the distances of the equal lines AP, CM, CL, EN from the centers of those same equal circles, and are therefore equal [Margin: h — Elements III.14]; and hence the whole lines $Y\alpha$, $Y\beta$, $Z\gamma$, $Z\delta$ will be equal — the angles $\alpha Y\beta$ and $\gamma Z\delta$ will also be equal [Margin: i — Elements I.8]. And therefore, since the lines YQ, YR, ZR, ZS are also equal, the lines QR and RS joining the centers Q, R, S will likewise be equal [Margin: k — Elements I.4]. By exactly the same method all the lines joining the centers of the pentagons will be shown equal. Hence the triangles QXR, RXS, SXT, TXV, VXQ, formed beneath the five solid angles L, M, N, O, P of the dodecahedron, will be equilateral and equal. And if in like manner triangles of the same sort are formed beneath the remaining fifteen solid angles of the dodecahedron (for the dodecahedron has twenty solid angles), triangles

which share their sides in common, then an icosahedron will have been inscribed within the dodecahedron, since the twelve solid angles of the icosahedron rest at the centers of the twelve faces of the dodecahedron. Thus in the given dodecahedron we have inscribed an icosahedron. Which was to be done.

Diagram. A dense plane projection of an icosahedron inscribed in a dodecahedron. The outer figure is a dodecahedral outline; a lattice of chords marks the inscribed icosahedron and the auxiliary construction lines of the proof. The labelling is very crowded and many interior letters overlap the ruled lines.

- A (top): apex vertex
- B (upper left): dodecahedron vertex
- K (upper right): dodecahedron vertex
- C (left): dodecahedron vertex
- D (lower left): dodecahedron vertex
- E (bottom left): dodecahedron vertex
- F (bottom): dodecahedron vertex
- G (lower right): dodecahedron vertex
- H (right, lower): dodecahedron vertex
- I (right): dodecahedron vertex
- V (right): dodecahedron vertex
- O (right of center): pentagon center / icosahedron vertex
- T (lower right): pentagon center / icosahedron vertex
- P (center, right of L): pentagon center / icosahedron vertex
- L (center): pentagon center / icosahedron vertex
- M (center left): pentagon center / icosahedron vertex
- N (center, lower): pentagon center / icosahedron vertex
- S (bottom center): pentagon center / icosahedron vertex
- X (center): pentagon center / icosahedron vertex
- R (center left): pentagon center / icosahedron vertex
- Q (upper center, near A): pentagon center / icosahedron vertex
- Y (upper center): construction point (meeting of KY, DY)
- Z (center left, near R): construction point (meeting of FZ, EZ)
- α (near top, right of Q): bisection point of AP
- β (center left, near R): bisection point of CM
- γ (center left): bisection point

- δ (center, near S): bisection point

[Several interior labels overlap the construction lines and cannot be resolved with confidence]

PROBL. 8. POPROS. 8. [Margin: o. viii.]

In a given dodecahedron to inscribe a cube.

Let four pentagons of the dodecahedron in which the cube is to be inscribed be set out — ABCDE, EFGHA, AHIKB, BKLMC — meeting at the side AB: two of them, ABCDE and ABKIH, along the same common side AB; the other two, AEFGH and BCMLK, along the angles EAH and CBK. In these pentagons let the straight lines EH, CK, CE, HK subtend the angles A, B, D, I, forming the quadrilateral EHKC, which will be equilateral. For since all the sides AE, AH, DE, DC, BC, BK, IH, IK are equal and contain equal angles, by hypothesis, the bases EH, HK, KC, CE will be equal [Margin: a — Elements I.4]. Now since the equal straight lines AE and BC cut off equal arcs from the circle circumscribing the pentagon ABCDE [Margin: b — Elements III.28], the lines AB and CE will be parallel, by the scholium to Prop. 27, Book 3. By the same reasoning AB and HK will be parallel. Therefore EC and HK will also be parallel to each other [Margin: c — Elements XI.9]; and consequently the lines EH and CK that join them will be parallel [Margin: d — Elements I.33], and lie in the same plane with them [Margin: e — Elements XI.7]. So EHKC will be a parallelogram, which we have shown to be equilateral.

And since the plane ECKH, produced, makes its common section with the sphere a circle — by Lemma 1 of Prop. 10, Book 14 — described about the quadrilateral EK, both the opposite angles E, K and the opposite angles C, H will be equal to two right angles [Margin: f — Elements III.22]. Therefore, since the opposite angles E and K are equal, and likewise the opposite C and H [Margin: g — Elements I.34], all four angles E, C, K, H will be right. Hence EK will be a square, placing its angles at four angles of the dodecahedron and its sides in the planes of four pentagons of the dodecahedron. If then in the remaining eight pentagons eight other straight lines are similarly drawn to subtend eight angles in the same order, a solid contained by six squares will be constituted. And since these squares are equal — because their sides, which subtend equal angles of the pentagons contained by equal straight lines, are equal [Margin: h — Elements I.4] — they will constitute a cube inscribed within the dodecahedron, since its twelve edges lie in the planes of twelve pentagons of the dodecahedron, and its eight angles

rest at eight angles of the dodecahedron. Thus in the given dodecahedron we have inscribed a cube. Which was to be done.

Diagram. A plane projection of a cube inscribed in a dodecahedron. The outer hexagonal outline represents the dodecahedron; the inner heavy square-faced figure is the inscribed cube EHKC with its further vertices.

- F (top left): dodecahedron vertex
- G (top right): dodecahedron vertex
- E (upper left): cube vertex
- H (upper right): cube vertex
- A (center, upper): cube vertex
- D (left): dodecahedron vertex
- B (center, lower): cube vertex
- C (lower left): cube vertex
- K (lower right): cube vertex
- I (right): dodecahedron vertex
- M (bottom left): dodecahedron vertex
- L (bottom right): dodecahedron vertex

COROLLARIUM.

From this it follows that the straight line subtending one angle of an equilateral and equiangular pentagon is parallel to the opposite side. For it has been shown that CE is parallel to AB.

PROBL. 9. PROPOS. 9. [Margin: o. ix.]

In a given dodecahedron to inscribe an octahedron.

In the dodecahedron in which the octahedron is to be inscribed, take those six edges that we said are opposite in Coroll. 3 of Prop. 17, Book 13 — namely AB, CD, EF, GH, IK, LM — and let them be bisected at N, O, P, Q, R, S. Then the straight lines NO, PQ, RS, which join the bisections of the opposite edges, will be equal, and will bisect one another at T, the center of the sphere, at right angles, by Coroll. 3 of Prop. 17, Book 13. Their halves NT, OT, PT, QT, RT, ST will therefore also be equal. Consequently the twelve straight lines that subtend the twelve right angles NTP, PTO, OTQ, QTN, NTR, RTP, PTS, STQ, QTR, RTO, OTS, STN — namely NP, PO, OQ, QN, NR, RP, PS, SQ, QR, RO, OS, SN — will be equal, since the said angles are contained by equal straight lines [Margin: a — Elements I.4]. Hence the

eight triangles NRP, PRO, ORQ, QRN, OSP, PSN, NSQ, QSO will be equilateral and equal, and will constitute an octahedron inscribed within the dodecahedron, since the six angles of the octahedron touch six opposite edges of the dodecahedron, by construction. Thus in the given dodecahedron we have inscribed an octahedron. Which was to be done.

Diagram. A plane projection of an octahedron inscribed in a dodecahedron. The octahedron NOPQRS has its center at T, with the six vertices resting on the midpoints of six opposite edges of the surrounding dodecahedron ABCDE...LM.

- A (top left), N (top center), B (top right): N is the top octahedron vertex
- E (left), R (upper interior): dodecahedron / octahedron vertices
- P (left interior): octahedron vertex
- I (right): dodecahedron vertex
- T (center): center of the octahedron/sphere
- M (center interior): octahedron vertex
- G (right), Q (right interior): Q is a side octahedron vertex
- F (lower left): dodecahedron vertex
- S (lower interior): octahedron vertex
- L (lower interior): dodecahedron vertex
- C (bottom left), O (bottom center), D (bottom right): O is the bottom octahedron vertex
- H (right): dodecahedron vertex

[A small interior label near the top (read as "k") could not be resolved; the octahedron/dodecahedron letters are otherwise legible]

PROBB. 10. PROPOS. 10. [Margin: o. x.]

In a given dodecahedron to inscribe a pyramid.

Let the dodecahedron be contained by the twelve pentagons ABCNL, CDEPN, EFGRP, GHITR, IKALT, MOQSV, BCDOM, DEFQO, FGHSQ, HIKVS, KABMV, LNPRT, in which a pyramid, or tetrahedron, is to be inscribed. Take the three pentagons ABCNL, LAKIT, TLNPR that make up the solid angle L, in which are the three angles C, R, K — of these, C is opposite the side AL common to the pentagons CAL and ALI; K is opposite the side LT common to the pentagons KLT and LTP; and finally R is opposite the side LN common to the pentagons RLN and LNB. Let these angles be connected by the straight lines CK, CR, KR; and from the points C, K, R let the straight lines CQ, KQ, RQ be drawn to the angle Q opposite L, so that the

pyramid CKRQ is constructed. Now let the three diameters of the dodecahedron, or of the sphere, CH, KE, RM be drawn, and let the straight lines KH, RH, RE, QE, QM, QH be connected. Since the dodecahedron is described in a sphere [Margin: a — Elements XIII.17], if the plane of the triangle CHK is extended, its common section with the sphere will be a circle, by Lemma 1 of Prop. 10, Book 14. And therefore the angle CKH, standing in a semicircle, will be right [Margin: b — Elements III.31]. We shall show in the same way that the angles CRH, KRE, KQE, RQM, CQH are right.

Consequently — since in every right-angled triangle the square on the side opposite the right angle is equal to the squares on the sides about the right angle [Margin: c — Elements I.47], and since the squares of the equal diameters CH, KE, RM are equal — the squares of the lines CK, KH will be equal to the squares of the lines CR, RH, and to the squares of the lines kR, RE, and to the squares of the lines kQ, QE, and to the squares of the lines RQ, QM, and also to the squares of the lines CQ, QH. Therefore, when the equal squares of the equal lines kH, RH, RE, QE, QM, QH are taken away (for these lines, since they subtend equal angles of the pentagons contained by equal straight lines, are equal [Margin: d — Elements I.4]), the squares of the lines Ck, CR, Rk, kQ, RQ, CQ will remain equal; and hence the lines themselves will be equal. Therefore the four triangles CkR, CkQ, kRQ, RCQ are equilateral and equal, and so make up the tetrahedron CkRQ; and since this has its four angles C, k, R, Q at four angles of the dodecahedron, it will be inscribed within the dodecahedron.

ALITER. In the proposed dodecahedron let a cube be inscribed [Margin: e — Elements XV.8]; and in this cube a pyramid [Margin: f — Elements XV.1]; and what is proposed will be accomplished. For since the angles of the pyramid rest at the angles of the cube, and the angles of the cube at the angles of the dodecahedron, it is evident that the angles of the pyramid have their seat at the angles of the dodecahedron. Therefore the pyramid will be inscribed in the dodecahedron. And so in the given dodecahedron we have inscribed a pyramid. Which was to be done.

Diagram. A very dense plane projection of a pyramid (tetrahedron) inscribed in a dodecahedron. The outer figure is the dodecahedral outline; a heavy web of chords marks the inscribed tetrahedron CkRQ together with the diameters CH, KE, RM and the numerous auxiliary connecting lines of the proof. Many interior labels overlap the ruled lines.

- A (top): dodecahedron vertex
- B (upper left): dodecahedron vertex

- K (upper right): tetrahedron vertex
- C (left): tetrahedron vertex
- D (lower left): dodecahedron vertex
- E (bottom, left of center): dodecahedron vertex
- F (bottom): dodecahedron vertex
- G (lower right): dodecahedron vertex
- H (right): dodecahedron vertex
- I (right): dodecahedron vertex
- T (upper right, interior): dodecahedron vertex
- S (right of center): dodecahedron vertex
- M (center, upper): interior label / pentagon center
- N (center left): interior label
- O (center): interior label
- P (center, lower): interior label
- L (center, upper): solid-angle vertex of the three chosen pentagons
- V (upper right, interior): interior label
- R (center, lower right): tetrahedron vertex
- Q (center, lower): tetrahedron vertex (opposite L)

[The interior of this figure is heavily overwritten by construction lines and several labels (M, N, O, P, L, V, R) cannot be fixed with confidence]

PROBB. 11. PROPOS. 11. [Margin: o. xi.]

In a given icosahedron to inscribe a cube.

In the given icosahedron let a dodecahedron be inscribed [Margin: a — Elements XV.5]; and in this dodecahedron a cube [Margin: b — Elements XV.8]; and what is proposed will be accomplished. For since the angles of the cube rest at the angles of the dodecahedron, and the angles of the dodecahedron are set at the centers of the faces of the icosahedron, the angles of the cube will touch those same centers of the icosahedron's faces. Therefore the cube will be inscribed in the icosahedron. Accordingly, in the given icosahedron we have inscribed a cube. Which was to be done.

PROBL. 12. PROPOS. 12. [Margin: o. xii.]

In a given icosahedron to inscribe a pyramid.

In the given icosahedron let a cube be inscribed [Margin: a — Elements XV.11], and in this cube a pyramid [Margin: b — Elements XV.1]; and what is proposed will be accomplished. For since the angles of the cube touch the centers of the icosahedron's faces [Margin: c — Elements XV.11], and the angles of the pyramid coincide with the angles of the cube, it is manifest that the angles of the pyramid reach the centers of those same faces of the icosahedron [Margin: d — Elements XV.1]; and consequently the pyramid is inscribed in the icosahedron. Thus in the given icosahedron we have inscribed a pyramid. Which was to be done.

THEOR. 13. PROPOS. 13.

In a given cube to inscribe a dodecahedron.

[Margin: Ducatur in figura recta HaC — "Let the straight line HaC be drawn in the figure"]

Let a dodecahedron be inscribed in the proposed cube AF. Let each of its edges be bisected at I, k, L, M, N, O, P, R, π , ξ , Z, Y. Draw the connecting straight lines IL, YZ, ξ O, M π , kP, NR, nowhere meeting one another, so that those drawn on opposite faces are parallel to one another, and from no one point are two lines drawn, but one only. Let these in turn be bisected at S, V, T, ς , Q, X, and let the straight lines ST, VX, ς Q be connected, which bisect one another at the center α , at right angles, as we taught in the corollary to Prop. 3 of this book; and, like the former six, they are equal to the edges of the cube. Now let the halves of the lines YZ and M π be cut in extreme and mean ratio at β , γ , θ , χ , so that the greater segments are Y β , Z γ , M θ , $\pi\chi$; and draw the straight lines Y ς , Y χ , $\beta\theta$, $\beta\chi$. Since, then, Y ς and YV are equal to half the edge of the cube [Margin: a — Elements XIII.4], and $\rho\chi$, βV , the lesser segments of the equal lines YV, $\rho\pi$, are likewise equal, the squares of the lines...

(Continuation of THEOR. 13. PROPOS. 13 — to describe a dodecahedron in a given cube AF. The demonstration begun in the previous pages continues without a break; the catchword there was "[re]ctarum.")

of the straight lines YV, βV — three times the square on the straight line Y ς . And the squares on the straight lines Y ρ , $\rho\kappa$ will likewise be three times the square on that same straight line Y β . Therefore [Margin: b — Elements I.47], since the square on the straight line Yk equals the squares on the straight lines Y ρ , $\rho\kappa$ [Margin: c — Elements I.29] — because the angle Y $\rho\kappa$ is right — the square on Yk will also be triple the square on Y β . Again, since VY is perpendicular to AY, Y ρ [Margin: d — Elements XI.4], and so stands at right angles to the plane AE drawn through the straight lines AY, Y ρ , the angle ς Yk will be right, by definition 3 of

Book 11. [Margin: e — *Elements* I.47] And for this reason the square on $\zeta\kappa$, being equal to the squares on $Y\kappa$, $Y\zeta$, will be four times the square on $Y\beta$.

(For the squares on $Y\kappa$, $Y\beta$ are likewise four times the square on $Y\zeta$, since the square on $Y\kappa$ was shown to be triple the square on $Y\beta$.) [Margin: f — *Elements* VI.20] And so, since squares stand in the doubled ratio of their sides, the straight line $\zeta\kappa$ will be double $Y\beta$. For a quadruple ratio is the double of a double ratio, as appears in the numbers 1, 2, 4. But $\theta\kappa$ is also double $\rho\kappa$, since the smaller segments $\theta\rho$, $\kappa\rho$ of the equal straight lines $M\rho$, $\pi\zeta$ are equal. [Margin: g — *Elements* V.15] Therefore, as $Y\beta$ is to $\rho\kappa$, so $\beta\kappa$ (double $Y\beta$) is to $\theta\kappa$ (double $\rho\kappa$). But $Y\beta$, the greater segment, is to $\rho\kappa$ — a straight line equal to the smaller segment $\beta\rho$ — as the whole to its greater segment. Hence $\beta\kappa$ too will be to $\theta\kappa$ as the whole to its greater segment; and consequently $\theta\kappa$ will be the greater segment of the straight line $\beta\kappa$ cut in extreme and mean ratio.

By the same reasoning we shall show that $\theta\kappa$ is the greater segment of the straight line $\beta\theta$ cut in extreme and mean ratio — provided the straight line $Y\theta$ is drawn. Hence $\zeta\theta$, $\beta\kappa$ will be equal, forming an isosceles triangle $\zeta\theta\kappa$ similar to the one Euclid constructed in proposition 10 of Book 4, as we shall demonstrate later. And accordingly $\theta\kappa$ will be the side of the pentagon described in the circle that circumscribes the triangle $\zeta\theta\kappa$, as is clear from proposition 11 of Book 4.

Now suppose the plane of the triangle $\beta\theta\kappa$ to be extended — the straight line $\beta\phi$ being drawn through β parallel to $M\pi$ — along the parallels $M\pi$, $\phi\theta$, so that the extended plane $\phi\pi$ meets the diameter AC of the square $ABCD$ at point 3, and the diameter AF of the cube at point 4. Through 3 let 37 be drawn parallel to $A\phi$, and let MS be joined [Margin: h — *Elements* I.33], which is parallel to both $A\phi$ and 73. Since, then, by the corollary to proposition 4 of Book 6 the triangle $A73$ is similar to triangle AMS [Margin: i — *Elements* VI.4], it follows that as AM is to MS , so $A7$ is to 73. But AM , MS are equal, being halves of the cube's edges; therefore $A7$, 73 are equal as well. By the same reasoning, as MA is to $A\phi$, so $M7$ is to 73. But $A\phi$ is the greater segment of the straight line MA divided in extreme and mean ratio [Margin: k — *Elements* I.34] (since $A\phi$, $Y\zeta$ are equal, and likewise MA , YV , the halves of the cube's edge). Therefore 73 — that is, 7A — will be the greater segment of the straight line $M7$ cut in extreme and mean ratio [Margin: l — *Elements* XIII.5]; and consequently the whole MA will be divided in extreme and mean ratio at 7, its greater segment being $M7$. [Margin: m — *Elements* VI.2] But as $M7$ is to 7A, so $S3$ is to 3A. Therefore SA too is cut in extreme and mean ratio at 3, its greater segment being $S3$. [Margin: n — *Elements* I.33] And since $S\alpha T$ is parallel to $M\pi$ — and since the straight line πT that is drawn is equal and parallel

to MS — if a plane parallel to plane $\varphi\pi$ (drawn through $M\pi$) is passed through $S\alpha T$ [Margin: o — *Elements* XI.17], the straight lines AS, $A\alpha$ will be cut in the same ratios at points 3 and 4: for plane $\varphi\pi$ cuts the straight lines AS, $A\alpha$ at 3 and 4, while the plane through $S\alpha T$, parallel to $\varphi\pi$, cuts them at S and α . And therefore, since AS is cut at 3 in extreme and mean ratio, $A\alpha$, the semidiameter of the cube, will be cut in the same way at 4, its greater segment being $\alpha 4$.

By the same arguments we shall show that, when the diameter HF of the cube — the diameter of square EFGH — and the diameter HC of the cube are drawn, the semidiameter of the cube is cut at point 9 in extreme and mean ratio by the plane $\varphi\pi$, as is evident if the cube is inverted, as appears in this second figure. For its lines are entirely similar to those of the first figure, even though the cube's bases have exchanged places, so that the square AC takes the position of square HF, and square HF the position of square AC, in this second figure. Yet this can also be proved in the first figure, without such inversion, in the following way. Let the diameter HC of the cube be drawn in that figure — it is missing from the figure, but anyone can draw it — and let the plane $\varphi\pi$ cut it at 9, near point H; and through the straight line AH conceive a plane parallel to plane $\varphi\pi$. Since, then, the straight lines $A\alpha$, $H\alpha$ meet at the center α and are cut by the parallel planes at the points A, H and 4, 9 — for the plane $\varphi\pi$ cuts them at 4 and 9, while the plane drawn through AH parallel to $\varphi\pi$ cuts them at A and H — [Margin: p — *Elements* XI.15] it follows that as $\alpha 4$ is to 4A, so $\alpha 9$ is to 9H. And since αA is cut at 4 in extreme and mean ratio, αH will likewise be so cut — a demonstration that holds also for the cube's inversion in the second figure. Let us now show that, if the straight lines $\theta 4$, $\beta 4$, $\beta 9$, $\kappa 9$ are joined, the rectilinear figure $\beta 4\theta\kappa 9$, lying in plane $\varphi\pi$, is an equilateral and equiangular pentagon.

Diagram. The first figure (*prima figura*) of the dodecahedron-in-cube construction: a cube drawn in oblique projection, its faces gridded and crossed by many diagonals, with the twelve dodecahedron pentagon-vertices and the extreme-and-mean-ratio division points marked by Greek minuscules and by numerals; the cube's center is α . The figure is extremely crowded and many of the small interior Greek-minuscule labels sit at the resolution floor. Labels I can read with confidence:

- Cube corners and principal points: A, B, G, F, E, D
- Upper/edge points: φ (a small dotted circle, top edge), I, N, H, M, P
- Interior construction/division points (read): Y, V, γ , θ , ς , S, K, π , α (center), δ , X, R, O
- Numerals used as points: 3, 4, 9, 10 (a 7 and an 8 also appear but are faint)

- Further interior labels (a μ -like, an Ω /Q-like, an ε -like, and additional Greek minuscules) are present but not securely resolvable at this resolution: [?]

Diagram. The second figure (*secunda figura*): the same cube-and-dodecahedron construction with the cube inverted (its bases exchanged), so that square AC and square HF swap places. Structure and label-set are those of the first figure; readable labels:

- Top edge: H, 10, ς [?], G
- Upper interior: Y, β , V, γ , Z
- Mid: A, ϕ , I, B; ρ [?], θ , α (center); P, μ [?], K
- Lower: κ [?], M, F, S; O, δ ; N, X, ς [?], R
- Bottom edge: D, L, G [?] (bottom-right corner, could be C)
- Numerals: 4, 8, 9

Let the cube again be inverted, so that the squares ADEH, ABGH — which of course contain the triangle $\beta\theta\kappa$, that is, the plane $\phi\pi$ — take the position shown in this third figure. Then in the plane $\phi\pi$ let the straight line joining $\theta 4$ be extended as far as the straight line $\phi 10$, to the point 13. Now since, when the straight line KP is drawn and the straight lines MK are joined [Margin: a — *Elements* XI.15], the plane MP is parallel to the plane AG — because the straight lines PK, KM are parallel to the straight lines GB, BA — [Margin: b — *Elements* XI.32] and it cuts the cube in half, and therefore passes through the center α (by what was shown in the scholium to proposition 39 of Book 11): if a plane is extended through the straight lines $\alpha 4$, A, $\theta 4$ 13 [Margin: c — *Elements* XI.16], then $\alpha\theta$, A13 become the common sections of the parallel planes MP, AG, and are therefore parallel. Consequently the triangles $\alpha\theta 4$ and $\alpha 34$ will be equiangular [Margin: d — *Elements* VI.4]; and as $\alpha 4$ is to 4A, so $\theta 4$ is to 413. Therefore, since αA is cut at 4 in extreme and mean ratio, as we showed, $\theta 13$ will be cut in the same way at 4. Again, let a plane pass through the center α parallel to plane NZ, and let another plane be drawn through the point 4 (by the scholium to proposition 15 of Book 11) [Margin: e — *Elements* XI.15], cutting $\phi\pi$ in the straight line 4·14. But the plane DB is parallel to that same plane NZ. [Margin: f — *Elements* XI.17] Therefore the straight lines αA , ρM will be cut in the same ratios at the points 4 and 14. Hence ρM too will be cut at 14 in extreme and mean ratio, and the segment 14M will be the smaller, and thus equal to the smaller segment $\rho\theta$ of Mp. Therefore, when from $\rho 14$ — the greater segment of ρM — the smaller segment $\rho\theta$ is taken away, then (as we demonstrated at proposition 5 of Book 13) $\rho 14$, divided at θ in extreme and mean ratio, will have $\rho\theta$ as its greater segment. Finally, when the straight line 13·15 is drawn through 13 parallel to 4·14 [Margin: g — *Elements*

VI.2], the straight lines $\theta 13$, $\theta 15$ will be cut proportionally; and hence $\theta 15$ too will be cut at 14 in extreme and mean ratio, its greater segment being $\theta 14$. [Margin: h — *Elements* XIV.2] Therefore, as $\rho\theta$, the greater segment, is to $\theta 14$, the smaller, so $\theta 14$, the greater, is to 14·15, the smaller; and as $\theta 14$ is to 14·15, so is the whole $\theta 15$ to $\theta 14$, the greater segment. Therefore, as $\rho\theta$ is to $\theta 14$, so $\theta 15$ is to that same $\theta 14$. And for this reason the straight lines $\rho\theta$ (or $\rho\kappa$) and $\theta 15$ will be equal.

Now since the plane NZ and the plane parallel to it drawn through 4·14 are both cut by the plane $\varphi\pi$ [Margin: i — *Elements* XI.16], the common sections $\beta\rho$, 4·14 will be parallel, and therefore 13·15 will also be parallel to $\beta\rho$ [Margin: k — *Elements* I.29], and the exterior angle $\beta\rho\kappa$ will be equal to the interior angle 13·15· θ . Since, then, the sides $\beta\rho$, $\rho\kappa$ are equal to the sides 13·15, 15· θ [Margin: l — *Elements* I.4], the straight lines $\beta\kappa$, 13· θ will also be equal. And accordingly their greater segments $\theta\kappa$, $\theta 4$ (for $\theta\kappa$ was shown to be the greater segment of $\beta\kappa$) will be equal. By the same technique $\zeta 4$ will be shown, in the first figure, to be equal to $\beta\gamma$, if an isosceles triangle is formed in a similar position on the base $\beta\gamma$, its vertex resting at the point that divides the straight line IS in extreme and mean ratio. And in the same way $\kappa 9$, $\zeta 9$ will be equal to those same $\theta\kappa$, $\zeta\gamma$. Therefore the pentagon $\beta 4\theta\kappa 9$ is equilateral, seeing that $\beta\gamma$, $\theta\kappa$ are equal straight lines — for since they are composed of the smaller segments of equal straight lines cut in extreme and mean ratio, the lines YV, ZV, $\pi\rho$, $M\rho$ are equal. But it is equiangular too, as we shall show presently. If, then, by this technique twelve triangles — two upon each of the straight lines $M\pi$, YZ, NR, IL, ξO , KP — are constructed similar and equal to $\beta\theta\kappa$ (for at $M\pi$ you see two such triangles, $\beta\theta\kappa$, $\delta\theta\kappa$), and if by their means twelve pentagons similar and equal to $\beta 4\theta\kappa 9$ are fabricated, then a Dodecahedron will be constituted. Its twelve angles lie on the six straight lines $M\pi$, YZ, NR, IL, ξO , KP — two on each — while the remaining eight rest on the eight semidiameters of the cube, namely at the points that divide them in extreme and mean ratio, such as the points 4 and 9 on the semidiameters $A\alpha$, $H\alpha$ of the second figure. And so such a Dodecahedron will be said to be in a certain manner inscribed in the cube — though not strictly, since not all its angles are set at the cube's angles, or on its edges, or finally in its faces, but twelve of them lie in six faces of the cube, while eight rest at the points that cut the eight semidiameters of the cube in extreme and mean ratio. Therefore we have described a Dodecahedron in a given cube. Which was to be done.

Diagram. The third figure (*tertia figura*): again the cube-and-dodecahedron construction, in the orientation described in the text (squares ADEH, ABGH containing the triangle $\beta\theta\kappa$). Densely lined; readable labels:

- Top edge: B, Z, G
- Upper interior: φ , 13, β , K, 10, P
- Mid: A, H; α (center); C, R
- Lower interior: M, 15, 14, θ , ς [?], κ [?], π
- Bottom edge: D, N, E
- Numerals as points: 4, 13, 14, 15, 10

LEMMA.

That the triangle $\beta\theta\kappa$ — whose base $\theta\kappa$ is the greater segment of each of the sides $\beta\theta$, $\beta\kappa$ divided in extreme and mean ratio — is similar to the one Euclid constructed in proposition 10 of Book 4, we shall demonstrate as follows.

About the isosceles triangle ABC, such as Euclid teaches how to construct in proposition 10 of Book 4, let the circle ADBCE be described; and when the angles ABC, ACB have been bisected by the straight lines BE, CD, let the equilateral and equiangular pentagon ADBCE be completed, as we set out in proposition 11 of Book 4; and let the straight lines AB, CD cut one another at F. [Margin: a — *Elements* XIII.8] Since, then, AB is cut at F in extreme and mean ratio, and its greater segment AF is equal to the side BC [Margin: b — *Elements* XIV.2], it will follow that, as the whole AB is to its greater segment BC, so the whole $\beta\theta$ is to its greater segment $\theta\kappa$ [the source here repeats "to its greater segment $\theta\kappa$ "; the doubled clause appears to be a compositor's dittography — [?]]. And in the same way, as the whole AC is to its greater segment CB, so the whole $\varsigma\kappa$ is to its greater segment $\kappa\theta$. But also, as AB is to AC, so $\beta\theta$ is to $\beta\kappa$, for in both cases the ratio is one of equality. Therefore the triangles $\beta\theta\kappa$, ABC, having their sides proportional, are equiangular; and the angles θ and B will be equal, likewise κ and C, and β and A; and for that reason each of the angles θ , κ will be double the angle β ; and the triangles $\beta\theta\kappa$, ABC are similar.

Diagram. Two figures side by side for the Lemma. On the left, a circle with an inscribed equilateral pentagon ADBCE and its diagonals forming a pentagram; vertices: A (top), D (upper left), E (upper right), B (lower left), C (lower right), with F marked at the interior intersection of the diagonals. On the right, a separate isosceles triangle $\beta\theta\kappa$ with apex β (top) and base endpoints θ (lower left) and κ (lower right). (Labels all read cleanly.)

That the pentagon $\beta\theta\kappa\varsigma\theta$ is equilateral, and also equiangular, we shall show by this reasoning. About the triangle $\beta\theta\kappa$ let the circle $\beta A\theta\kappa B$ be described. If it passes through the points 4 and 9, then it is clear — from what we taught at

proposition 16 of Book 4 — that the pentagon is equiangular. But if it does not pass through the points 4 and 9, let there be described in the said circle, following the doctrine of proposition 11 of Book 4, an equilateral and equiangular pentagon $\beta A \theta \kappa B$. Now since the straight lines $A\theta$, $A\varsigma$, as well as the straight lines 4θ , 4β , are equal to the straight line $\theta\kappa$ — because both $\beta 4 \theta \kappa 9$ and $\beta A \theta \kappa B$ are equilateral pentagons — the two straight lines θA , $\theta 4$ will then be equal, as will the two ςA , 4ς ; which was demonstrated to be impossible in proposition 7 of Book 1. Therefore the circle circumscribed about the triangle $\beta \theta \kappa$ passes through the points 4 and 9. And accordingly the pentagon $\beta 4 \theta \kappa 9$ is equiangular.

Diagram. A circle with an inscribed pentagon and its diagonals drawn from the apex. Vertices: β (top), A (upper left), B (upper right), θ (lower left), κ (lower right). The numerals 4 (left) and 9 (right) mark interior triangular cells formed by the diagonals from β . (Labels read cleanly.)

SCHOLIUM.

From these results the following propositions may be gathered.

I.

The diameter of the Dodecahedron is capable of two sides — namely of the Dodecahedron itself, and of the cube in which the Dodecahedron is described.

For, in the first figure, let the diameter of the Dodecahedron $\beta\epsilon$ be drawn, and let the straight line $\gamma\epsilon$ be understood to be joined [Margin: a — *Elements* I.33], which is equal and parallel to the cube's side VX (because $V\gamma$, $X\epsilon$ are equal and parallel). Since, then, the angle βVX is right [Margin: b — *Elements* I.29], the angle $\beta\gamma\epsilon$ constructed at γ will be right as well; [Margin: c — *Elements* I.47] and consequently the square on the diameter $\beta\epsilon$ will equal the squares on the sides $\varsigma\gamma$ [?: β/ς are confusable here — possibly $\beta\gamma$], $\gamma\epsilon$ — namely of the Dodecahedron and of the cube.

II.

If the side of the cube is cut in extreme and mean ratio, the smaller segment is the side of the Dodecahedron described in the cube; but the greater segment is the side of the cube described in this dodecahedron.

For, in the same first figure [Margin: d — *Elements* V.15], since as YX is to βV , so YZ (double YV) is to $\beta\gamma$ (double ςV); and let ςV be the smaller segment of YV cut in extreme and mean ratio: then $\varsigma\gamma$ too will be the smaller segment of the cube's

side YZ cut in extreme and mean ratio, while the segment composed of $Y\beta$, γZ — that is, the straight line $\zeta\kappa$ (which was shown to be double $Y\zeta$, and therefore equal to the sum of $Y\zeta$, γZ) — will be the greater segment. But it is established from the construction that $\zeta\gamma$ is the side of the Dodecahedron [Margin: e — *Elements* XV.8], and that $\zeta\kappa$ (or $\zeta\theta$) is the side of the cube described in the said Dodecahedron, since it subtends the angle of the pentagon $\beta 9\kappa$, or $\beta 4\theta$.

III.

The side of the cube is equal to two sides — namely of the Dodecahedron described in it, and of the Dodecahedron described about the same cube.

For, in the first figure again, the side of the cube YZ is composed of the side of the Dodecahedron $\beta\gamma$ described in it, and of the straight line composed of $Y\beta$, γZ , which is established to be the side of the Dodecahedron circumscribed about the same cube. For since YZ , the cube's side, subtends the angle of a pentagon of the Dodecahedron described about the cube, as is clear from proposition 8 of this book [Margin: f — *Elements* XIII.8], and since the side of that pentagon is the greater segment of the side YZ cut in extreme and mean ratio: then $\beta\theta$, being likewise the greater segment of that same side YZ and equal to the two $Y\beta$, γZ together (as we have already demonstrated), will be the side of the Dodecahedron circumscribed about the cube of side YZ .

IV.

The straight line joining the two angles of the pentagons of the Dodecahedron that lie opposite the common side is equal to the side of the cube in which the Dodecahedron is inscribed.

For, in the first figure, the straight line joining the points β , δ — at which are constituted the two angles of the pentagons opposite the common side $\theta\kappa$ [Margin: g — *Elements* I.33] — is equal to the straight line VX (because βV , δX are equal and parallel), that is, to the side of the cube in which the Dodecahedron is described.

Diagram. The first figure of the construction, reproduced here to serve the Scholium (the same cube-and-dodecahedron figure as fig-p601-1). Readable labels: H, 10, ζ [?], G (top edge); Y, β , V, γ , Z (upper); A, ϕ , I, B; κ [?], ρ [?], θ , α (center); P, μ [?], K; M, F, S; O, δ ; N, X, ζ [?], R (lower); D, L, G [?] (bottom edge). Numerals 4, 8, 9.

PROBL. 14. PROPOS. 14.

To inscribe an icosahedron in a given cube.

Let AB be the given cube in which the icosahedron is to be inscribed. Through the centres C, D, E, F, G, H of its faces, draw straight lines parallel to the edges of the cube, none of them meeting one another; and cut the halves of these lines in extreme and mean ratio, so that the greater segments — those near the centres — end at the points I, K; L, M; N, O; P, Q; R, S; T, V. Join each of these points to its four nearest neighbours: L to I, K, T, and R; R to L, M, I, and N; and so on. In this way twenty triangles are formed: IKL, IKQ, LMR, LMT, NOM, NOP, PQV, PQS, RSI, RSN; TVR, TVO, ILR, ISQ, KTL, KQV, OPV, OTM, NSP, NMR. The first twelve of these lie under the eleven [recte: twelve] edges of the cube, while the latter eight are set beneath its eight corners.

Now join the straight lines XC, XI, XK. Since the sides XC, CI of triangle XCI are equal to the sides XC, CK of triangle XCK, and [Margin: a — *Elements* I.29] the angles they contain are right — because XC is parallel to AY — [Margin: b — *Elements* I.4] the bases XI, XK will also be equal. Next, since the line LX is perpendicular to the plane AZ (by definition 4 of Book XI), it is also perpendicular to the lines XI, XK. Therefore, since the sides LX, XI of triangle LXI are equal to the sides LX, XK of triangle L XK, and the angles they contain are right, [Margin: c — *Elements* I.4] the bases LI, LK will be equal.

Again, [Margin: d — *Elements* I.47] the square on LI equals the squares on LX and XI; the square on LX equals the square on IY; and the square on XL equals the squares on XC and CI. Hence the square on LI equals the three squares on XC, CI, and IY. But the squares on XC and CY are also equal, because their sides are halves of the cube's edges. Therefore the square on LI equals the three squares on CY, CI, and IY. Now these three squares are quadruple the square on CI. [Margin: e — *Elements* XIII.4] For the squares on CY and IY — on the whole line and on the lesser segment — are triple the square on CI, the greater segment; and when the square on CI is added to those two, the three squares on CY, CI, IY become quadruple that same square on CI. So the square on LI too is quadruple the square on CI. And since the square on KI is quadruple the same square on CI (by the scholium to proposition 4 of Book 2), the squares on LI and IK are equal, and hence the lines themselves are equal. Therefore triangle IKL, laid under the cube's face AG, is equilateral.

In just the same way we shall show that the remaining eleven triangles, placed under the eleven remaining edges of the cube, are equilateral; and moreover that

they are equal to one another, since their sides IK, LM, NO, PQ, RS, TV are equal. Now the sides of the remaining eight triangles, set beneath the cube's eight corners, are shared with the twelve triangles already named — for instance, in triangle KTL the side KL is common to triangle IKL, the side KT to triangle TVK, and the side TL to triangle LMT. It is therefore clear that these too are equal both to one another and to the twelve triangles already named. So all twenty such triangles compose an icosahedron inscribed in the cube, since its twelve vertices all rest on the six faces of the cube, two on each. Thus we have inscribed an icosahedron in the given cube. Which was to be done.

Diagram. A large three-dimensional cube shown in oblique projection with an icosahedron inscribed inside it; a dense web of interior lines joins the construction points on the cube's faces. The cube's diagonal is AB. The six face-centres are C, D, E, F, G, H, and the icosahedron's twelve vertices (I, K, L, M, N, O, P, Q, R, S, T, V) are marked as golden-section points on the segments through the face-centres, with additional auxiliary points X, Y, Z, and the cube's centre α [Greek: ἄλφα (alpha)]. Readable labels, by approximate position :

- A (top-left rear corner)
- Y (top edge)
- C (upper region, near top face)
- K (upper right)
- X (upper left)
- L (left)
- B (upper-right region) [?]
- "2" (right-hand face) [?] — a construction point marked with a numeral, or a compositor artifact
- Z (right; printed as a reversed N)
- R (left-middle)
- G (middle) [?]
- S (middle)
- α [Greek: alpha] (centre of the cube)
- D (left, lower)
- F (right-middle)
- T (lower centre)
- H (lower centre-right)
- V (lower right)
- P (lower right)

- M (lower left)
- N (bottom left)
- E (bottom)
- O (bottom)
- B (bottom-right corner, opposite end of the diagonal AB)

Several of the small interior labels are crowded and abraded; the readings marked [?] are uncertain.

SCHOLIVM.

We gather from this the following propositions.

I.

The diameter of the icosahedron is equal in square [potest — Greek δύναται, "is equal in power/square to"] to both the side of the icosahedron and the side of the cube that surrounds the icosahedron.

For draw the straight line LQ, [Margin: f — *Elements* I.33] which is parallel and equal to the cube's edge DF, since DL and FQ are parallel and equal. [Margin: g — *Elements* I.47] Then it is clear that the square on the diameter QM equals the squares on the lines LM, the side of the icosahedron, and LQ, the side of the cube in which the icosahedron is inscribed, [Margin: h — *Elements* I.29] since the angle MLQ is right — because the angle LDF is also right.

II.

The bisection points of six opposite sides of the icosahedron are joined by three equal straight lines, which cut one another in half at the centre of the icosahedron, and also at right angles.

For three such lines — namely DF, CE, GH — since they join the centres of opposite faces of the cube, are equal to the edges of the cube; and by the corollary to proposition 3 of this book they cut one another in half, and at right angles, at the centre of the cube, which is the point α [Greek: ἄλφα (alpha)]. That this point is also the centre of the icosahedron is shown as follows. Draw the line αP . Since the sides αF , FQ of triangle αFQ are equal to the sides αF , FP of triangle αFP , and they contain right angles, [Margin: i — *Elements* I.4] the lines αQ , αP will be equal. By the same reasoning all the lines drawn from α to the vertices of the icosahedron will be equal. Therefore α is the centre of the icosahedron.

III.

If the edge of the cube is cut in extreme and mean ratio, the greater segment is the side of the icosahedron inscribed in the said cube.

For since CI is the greater segment of the line CY, [Margin: k — *Elements* V.15] and as CI is to CY, so KI — double CI — is to βY [Greek: $\beta\eta\tau\alpha$ (beta)], double CY, [Margin: l — *Elements* XIV.2] then KI, the side of the icosahedron, will also be the greater segment of βY , the side of the cube. For if some straight line other than KI were the greater segment of the line βY , then βY would be to that line as CY is to CI — that is, as βY is to KI — making it different from that line: which is absurd.

IV.

In the icosahedron both the sides and the opposite triangular faces are parallel to one another.

For if the extreme points of any two opposite sides are connected by two diameters, two triangles are formed that are equal to one another and have their alternate angles equal (by proposition 4 of Book 1); and these diameters meet at the centre. Therefore any opposite sides will be parallel. [Margin: m — *Elements* XI.15] And for that reason the planes of the triangles as well, drawn through the parallel sides themselves, will be parallel to one another.

PROBL. 15. PROPOS. 15.

To inscribe an octahedron in a given icosahedron.

Let there be set out the six opposite sides AB, CD, EF, GH, IK, LM of the given icosahedron in which the octahedron is to be inscribed, whose bisection points N, O, P, Q, R, S are joined by three equal straight lines NO, PQ, RS, cutting one another in half at the centre T and at right angles — as was shown in the scholium to the preceding proposition. Draw the lines NP, NR, NS, NQ; OP, OR, OS, OQ; PR, PS; QR, QS, forming the eight triangles SNP, SPO, SOQ, SQN; RNP, RPO, ROQ, RQN. Now since the sides TN, TS of triangle TNS are equal to the sides TN, TP of triangle TNP — being halves of equal lines — and they contain right angles, as was said, [Margin: a — *Elements* I.4] the bases NS, NP will be equal. In the same way all the remaining lines will be equal both to these two and to one another. Therefore the eight named triangles are equilateral and equal to one another, and so constitute an octahedron. This octahedron is inscribed in the icosahedron, since its vertices rest on the bisection points of the six opposite sides of the icosahedron. Thus we have inscribed an octahedron in the given icosahedron. Which was to be done.

Diagram. A rhombus-shaped figure representing an icosahedron seen in projection, with an octahedron inscribed. Six "diameters" of the icosahedron — AB, CD, EF, GH, IK, LM — are drawn; their midpoints N, O, P, Q, R, S are the vertices of the inscribed octahedron, whose faces are traced by the interior lines meeting at the centre T. Readable labels, by position :

- A, N, B (top horizontal line; N is the midpoint of AB)
- K (interior, just below N)
- E (upper left end), F (lower left end) — endpoints of the left "diameter"
- G (upper right end), H (lower right end) — endpoints of the right "diameter"
- P (left-centre; midpoint of EF)
- Q (right-centre; midpoint of GH)
- R (upper-left interior), I (interior left), T (centre), M (interior right)
- S (lower-centre interior)
- L (interior, just below S)
- C, O, D (bottom horizontal line; O is the midpoint of CD)

All labels read cleanly.

COROLLARIVM.

It follows from this that the icosahedron and the octahedron inscribed in it have the same centre. For the lines NO, PQ, RS, passing through T, the centre of the icosahedron, are the diameters of the octahedron NPOQRS inscribed in it. Therefore T, the centre of the icosahedron where they intersect, is also the centre of the octahedron.

PROBL. 16. PROPOS. 16.

To inscribe an icosahedron in a given octahedron.

Let ABCDEF be the given octahedron in which the icosahedron is to be inscribed. Since four edges issue from each vertex, take from each vertex two that are not adjacent but lie opposite one another — that is, the ones which fall at the opposite corners of the octahedron's square sections. Always take the two beside opposite corners of the same square: thus BA, BC and DA, DC beside the opposite corners B, D of the square ABCD; AF, AE and CE, CF beside the opposite corners A, C of the square AECF (for only two edges are to be taken from each vertex, whereas the edges AB, AD and CB, CD of the square ABCD, beside the same corners A and C, have already been taken); and EB, ED and FD, FB beside the opposite corners E, F

of the square EBFD (for the edges EA, EC and FC, FA of the square AECF, beside the same corners E and F, have already been taken). Let all of these be cut in extreme and mean ratio, under this rule: that the greater segments lie near the corners from which they issue, at the twelve points G, H, I, K, L, M, N, O, P, Q, R, S. From each of these draw five straight lines to the five nearest section-points — for example, from R the lines RQ, RI, RL, RO, RK; and from I the lines IQ, IR, IM, IL, IG; and so with the rest — although not

Diagram. A rhombus-shaped figure of the octahedron ABCDEF drawn in projection, with the icosahedron inscribed; a dense web of interior lines joins the twelve golden-section points on the octahedron's edges. (The same figure is repeated at the top of the following page.) The octahedron's four "equatorial" vertices are A (top), B (left), C (bottom), D (right); E and F are the remaining opposite pair, shown at the interior; X marks the centre. Readable labels, by position :

- A (top vertex), C (bottom vertex), B (left vertex), D (right vertex)
- G (upper-left edge), I (upper-right edge), H (lower-left edge), K (lower-right edge)
- E (upper-interior apex), F (lower-interior apex)
- Y (near top interior), M, L (upper interior), P (left interior), S (lower-left interior)
- X (centre), T, V (right of centre), Q (right interior)
- R (right-of-centre interior), N (lower interior), O, Z (lower interior)

Most labels read cleanly; Q (printed as a rounded form) and Z are the least certain.

all are drawn in the figure, so as to avoid a confusion of lines. Which these five nearest points are, the solid octahedron shows beautifully. Three of them lie on the three remaining edges which, together with the edge on which the point was taken, form a pyramid [tetrahedron]; the other two lie on two edges of two octahedron-faces that share the edge on which the point was taken. For instance, since the point taken is R, on the edge DF — which, with the three edges DE, DA, DC, forms a pyramid at the vertex D — this edge DF is common to the octahedron's triangles DFA and DFC; and so from R five straight lines are drawn to the five points of these five edges; and likewise for the rest.

After this, draw the three diameters of the octahedron AC, BD, EF, cutting one another in half and at right angles at the centre X (by the corollary to proposition 14 of Book 13); of these, AC and BD cut the lines GI and RQ at Y and V. Join the line

IV, and draw IT parallel to YX. Now since the sides DQ, DI of triangle DQI are equal to the sides DR, DI of triangle DRI (for DQ and DR are the lesser segments of the equal lines DE, DF), and they contain equal angles — namely those of the octahedron's triangles ADE, ADF — [Margin: a — *Elements* I.4] the bases IQ, IR will be equal. Next, since [Margin: b — *Elements* XIV.2] as BG is to GA, so DI is to IA, [Margin: c — *Elements* VI.2] GI will be parallel to the line BD. By the same reasoning QR is parallel to EF; [Margin: d — *Elements* VI.4] and consequently, as DI is to IA, so XY is to YA. Therefore the line XA is cut at Y in extreme and mean ratio.

Again, since as AX is to XD, so AY is to YI (because triangle AYI is similar to triangle AXD, by the corollary to proposition 4 of Book 6), and AX is equal to XD, AY too will be equal to YI. Therefore YI is the lesser segment of the line XA. In just the same way we shall show that AY is equal to YG, and hence that GI is cut in half at Y, and likewise QR at V.

Now since the sides AG, AI of triangle AGI are equal to the sides DQ, DR of triangle DQR (being, of course, lesser segments of the lesser segments of equal lines), and they contain right angles — namely those of the squares ABCD, DEBF — [Margin: e — *Elements* I.4] the bases GI, QR will be equal. By the same reasoning the lines HK, PS, MN, LO will be shown equal to these — the lines which subtend the remaining corners of the squares ABCD, DEBF, AECF (at each square, two opposite corners) and which embrace two lesser segments of the sides.

Next, since YI, the lesser segment of XA, has been shown to be greater than the greater segment of the greater segment XY when XY is divided in extreme and mean ratio (as we demonstrated at proposition 5 of Book 13); and since XY equals XV, [Margin: f — *Elements* I.34] and YI equals XT (both being the greater segments of the equal lines XA, XD); XT will be the greater segment of XV, and therefore TV the lesser segment of that same XV. Hence the squares on XV — the whole, that is, on IT itself, [Margin: g — *Elements* I.34] since IT is equal to XY — and on TV, the lesser segment, [Margin: h — *Elements* XIII.4] are triple the square on XT, the greater segment. And so the square on IV, [Margin: i — *Elements* I.47] being equal to the squares on IT and VT, is triple that same square on XT — that is, on RV, which is equal to XT, or to YI, from what has been shown, since RV and YI are halves of the equal lines RQ, GI. Therefore, adding the square on RV to the square on IV, the squares on IV and VR will be quadruple the square on RV. [Margin: k — *Elements* I.47] But the square on IR is equal to the squares on IV and VR, [Margin: l — *Elements* I.8] because the angles IVR, IVQ — contained by the equal sides IV, VR and IV, VQ, and subtended by the equal bases IR, IQ — are equal, and

therefore right. So the square on IR is quadruple the same square on VR; and the square on QR is likewise quadruple it (by the scholium to proposition 4 of Book 2). The squares on IR and QR will therefore be equal, and hence the lines themselves equal. But IQ was equal to IR. Therefore triangle IQR is equilateral.

In the same way we shall show that the triangles similarly situated to IQR — namely KQR; GIL, GIM; GPS; HPS, NHK, OHK; RLO, SLO; PMN, QMN — are equilateral and equal to triangle IQR, since the sides of all of them are equal to one another. This we shall also demonstrate by the following reasoning. In each of these triangles one side subtends the right angle of a square and is contained by two lesser segments of the octahedron's edges; while the other two sides subtend two equal angles of the equilateral triangles, each contained by two segments of the octahedron's edges, a greater and a lesser. Therefore the sides of any one triangle are equal to the sides of the others, and consequently the triangles themselves are equal (by the corollary to proposition 8 of Book 1). For example, in the triangles IQR, GIL, the sides RQ, GI — which subtend the right angles QDR, GAI of the squares DEBF, ABCD, contained by the lesser segments DQ, DR and AG, AI — are equal. In the same way the sides IQ, IL will be equal, since they subtend the equal angles IDQ, IAL of the equilateral triangles ADE, ADF; the former angle is contained by the octahedron's edge-segments DI, DQ, a greater and a lesser, the latter by the segments AL, AI, likewise a greater and a lesser. In the same way IR, GL will be equal. Therefore the triangles IQR, GIL are equal (by the corollary to proposition 8 of Book 1). But IQR has been shown to be equilateral; therefore GIL is equilateral too. And the same reasoning holds for the rest. These twelve triangles, then, are set beneath the six vertices of the octahedron, two at each. From this it follows that the remaining eight triangles, set upon the eight faces of the octahedron, namely

Diagram. The same rhombus-shaped figure of the octahedron ABCDEF with inscribed icosahedron that appears at the foot of the preceding page, here repeated at the top of this page beside the demonstration. The octahedron's vertices are A (top), B (left), C (bottom), D (right), with E and F the remaining opposite pair shown in the interior; X is the centre; the twelve icosahedron points G, H, I, K, L, M, N, O, P, Q, R, S lie at golden-section points on the edges, with auxiliary points Y, T, V, Z. Readable labels, by position :

- A (top vertex), C (bottom vertex), B (left vertex), D (right vertex)
- G (upper-left edge), I (upper-right edge), H (lower-left edge), K (lower-right edge)
- E (upper-interior apex), F (lower-interior apex)

- Y, M, L (upper interior), P (left interior), S (lower-left interior)
- X (centre), T, V (right of centre), Q (right interior)
- R (right-of-centre interior), N (lower interior), O, Z (lower interior)

The interior labels agree with those of fig-p605-2; Q and Z are the least certain readings.

namely GMP in AEB, HNP in BEC, KNQ in CED, IMQ in AED, GLS in AFB, HOS in BFC, KOR in CFD, and ILR in AFD — are likewise equilateral and equal to the aforesaid triangles, since their sides are shared with the sides of those. Therefore these twenty triangles — five of which meet at each section-point of the octahedron's edges to form a solid angle (as you see in the five RQI, RIL, RLO, ROK, RKQ, which compose the solid angle R) — make up an icosahedron inscribed in the octahedron, since the icosahedron's twelve vertices occupy the twelve section-points of the octahedron's edges. Therefore in a given octahedron, and so on. Which was to be done.

COROLLARIVM I. [Corollary I]

It is gathered from what has been demonstrated that, if two sides of an equilateral triangle are cut in extreme and mean ratio in such a way that the greater segment of the one and the lesser segment of the other lie next to the angle they contain, then the square on the straight line joining these two section-points is double the square on the lesser segment. For in the equilateral triangle ADE, let DI be the greater segment of side AD, and DQ the lesser segment of side DE; and let the straight line IQ be drawn. It was shown to be equal to GI, the square on which is double the square on the lesser segment AI [Margin: l 47.primi (Elements I.47)], since the square on GI equals the two equal squares on the equal straight lines AG and AI. It is therefore evident that the square on IQ also is double the square on the same lesser segment AI.

COROLLARIVM II. [Corollary II]

Again, when eight bases of the icosahedron have been placed on eight bases of the octahedron, [it follows] that the center of a base of the octahedron and of the icosahedron is one and the same. For take the triangle AED of the upper part of the octahedron, in which the icosahedron's triangle IMQ has been placed, and let C be the center of triangle AED, from which the straight lines CA, CE, CD and CI, CM, CQ are drawn. Now the sides CA, AM of triangle CAM are equal to the sides CE, EQ of triangle CEQ (because these sides run from the center, and are the greater segments of equal straight lines) and contain equal angles — namely

halves of the angles of an equilateral triangle, as we showed in the scholium of prop. 12, book 13. The bases CM, CQ will therefore be equal. In the same way we shall demonstrate that CI is equal to the lines CM, CQ. Therefore C will be the center of triangle IMQ.

Diagram. An equilateral triangle AED with an inscribed smaller triangle IMQ and a common center C, illustrating Corollary II. Labels read on the leaf:

- A (apex, top)
- E (bottom-left vertex)
- D (bottom-right vertex)
- Q (bottom, center of base ED)
- M (left, on side AE)
- I (upper right, on side AD)
- C (center, joined by lines to A, E, D and to I, M, Q)

COROLLARIUM III. [Corollary III]

Finally, it is not difficult to gather from what has been said that the center of the octahedron and that of the icosahedron inscribed in it are the same. For draw from X, the center of the octahedron, two straight lines to any two vertices of the icosahedron — namely the straight lines XG, XO to the vertices G and O. The sides XA, AG of triangle XAG are equal to the sides XF, FO of triangle XFO (for XA and XF are semidiameters of the octahedron, while AG and FO are the lesser segments of the equal edges of the octahedron cut in extreme and mean ratio), and they contain equal angles, namely half-right angles — since AC, EF are the diagonals of the squares ABCD, EAFC, and hence, by what was demonstrated in the scholium of prop. 34, book 1, they bisect the angles of the squares. The bases XG, XO will therefore be equal [Margin: m 4.primi (Elements I.4)]. By the same argument all the other straight lines drawn from X to the vertices of the icosahedron will be shown equal. Therefore X, the center of the octahedron, is also the center of the icosahedron.

PROBL. 17. PROPOS. 17. [Problem 17, Proposition 17]

To inscribe a dodecahedron in a given octahedron.

Inscribe an icosahedron in the given octahedron [Margin: a 18[?].quintidec. (Elements XV.18[?]; referent is the icosahedron-in-octahedron construction, prop. 16 of this book)], and in this icosahedron a dodecahedron [Margin: b 5.quintidecim (Elements XV.5[?])], and what is required will be done. For since

the vertices of the dodecahedron are set at the centers of the icosahedron's bases, eight of which are concentric with the bases of the octahedron (by cor. 2 of prop. 16 of this book), it is manifest that eight vertices of the dodecahedron rest in those same centers of the eight bases of the octahedron, while the remaining twelve rest in the centers of the twelve bases of the icosahedron, of which pairs lie under each of the six vertices of the octahedron. For this reason the dodecahedron will be said to be inscribed in the octahedron — though not properly so, as can be seen from def. 31, book 11. We have therefore described a dodecahedron in a given octahedron. Which was to be done.

PROBL. 18. PROPOS. 18. [Problem 18, Proposition 18]

To inscribe a cube in a given pyramid [tetrahedron].

Let there be given a pyramid ABCD in which a cube is to be described, and let its center be E. Through E draw from the four vertices of the pyramid the straight lines AEF, BEG, CEH, DEI, which fall on the centers F, G, H, I of the opposite bases and are perpendicular to those same bases BCD, ACD, ABD, ABC [Margin: a 13[?].quartidec. (Elements XIV.13[?])]. Then through these centers draw from the vertices A, B, C the straight lines AHK, AGL, AIT, BFL, BHS, CIM, CFK, CGS, cutting the opposite bases BD, CD, AB, AD, BC in half and at right angles (by the scholium of prop. 12, book 13) at the points K, L, M, S, T. Let the straight lines AF, BG, CH, DI, dropped from the solid angles through the center E, also be bisected at the points N, O, P, Q, each of which is joined to its three nearest

centers (by the centers, I mean, the line itself of the base on which the point lies, standing around it, so that there remains the center of that base on which the perpendicular falls) by straight lines: thus N with G, H, I; O with F, H, I; P with F, G, I; and Q with F, G, H. I say that the solid NF, contained by the quadrilaterals NO, NP, NQ, FG, FH, FI, is a cube. For the sides AF, FK of triangle AFK are equal to the sides AF, FL of triangle AFL (because FK, FL, being drawn perpendicular from the center F to BD, CD, show equal distances of the equal straight lines BD, CD from the center F) [Margin: b 4.primi (Elements I.4)], and they contain right angles, by def. 3, book 11. The bases AK, AL will therefore be equal, and the angles KAF, LAF. Again, since the sides AN, AH of triangle ANH are equal to the sides AN, AG of triangle ANG (for AH, AG are semidiameters of the equal equilateral triangles ABD, ACD) [Margin: c 4.primi (Elements I.4)], and they likewise contain equal angles, as we have demonstrated, the bases NH, NG will be equal. In just the same way it will be shown that the straight lines QG, QH are equal, if the straight lines DG, DH are drawn — and likewise all the rest, since they all subtend equal angles contained by equal sides, as is evident. For if the

straight line FT be drawn, since the sides AF, FT are equal to the sides AF, FK (for FT, FK are the distances of the equal straight lines BC, BD from the center) and the contained angles are right (by def. 3, book 11), the angles TAF, KAF will be equal. And accordingly, since the sides AN, AI are equal to the sides AN, AH (for AI, AH are semidiameters of the equal equilateral triangles ABC, ABD) and they contain equal angles [Margin: d 4.prim. (Elements I.4)], the bases NI, NH will be equal, and so on. Therefore the six quadrilaterals mentioned are equilateral. Let us now show that they are plane and rectangular, so that we may at last conclude that they are squares and thus compose the cube NF.

Take the quadrilateral NQ, and draw the straight lines GH, NQ. Since in the pyramid the sides BC, AD are opposite — that is, not meeting at any point — if their midpoints S and T are joined by the straight line ST, this will be the diameter of the octahedron described in the pyramid, as is clear from the figure of prop. 2 of this book. For the point S of this figure corresponds to the point H of that one, and the point T to the point F, since in that figure too BC, AD are opposite, or non-meeting, sides. Therefore, since the pyramid and the octahedron described in it have the same center (as we shall soon demonstrate), T will pass through E, the center of the pyramid, and ES, ET will be equal, namely the semidiameters of the octahedron [Margin: e 18.quartidec. (Elements XIV.18)]. Next, since BS, CS are perpendicular to AD (by the scholium of prop. 12, book 13), HS, GS, from the centers H and G, will be third parts of the perpendiculars BS, CS. Hence the sides BS, CS of triangle BSC are cut proportionally [Margin: f 2.sexti (Elements VI.2)], and therefore HG is parallel to the side BC, and cuts the sides TS, CS of triangle TSC (which is indeed a part of triangle BSC) proportionally; and so RS too is a third part of ST, and accordingly two thirds [of it are] the half SE.

Again, since DI is perpendicular to triangle ABC [Margin: g 15.quartidec. (Elements XIV.15)], EI will be a third part of the semidiameter DE (by cor. 2 of prop. 13, book 13); that is, of whatever parts DE is reckoned as 3, EI will be 1, and accordingly the straight line DI will be 4 of the same, and so its half DQ will be 2 of those parts. For this reason the remainder EQ will be 1 of those parts, and therefore EQ will be a third part of DE. By the same reasoning EN will be a third part of AE. Accordingly, since the sides DE, AE of triangle AED are cut proportionally [Margin: h 2.sexti (Elements VI.2)], the straight line NQ will be parallel to the side AD, and will cut the sides ES, ED of triangle DES (which is indeed a part of triangle AED) proportionally; and so, since EQ is a third part of DE, ER will be a third part of SE, and accordingly SR will be two thirds of the same SE. Since therefore each of HG, NQ has taken from SE two thirds, at the same point R, SE will be cut by the straight lines HG, NQ [Margin: i 2.vndec. (Elements

XI.2)]; and therefore HG, NQ will intersect one another at the same point. Hence they lie in the same plane, and for that reason the straight lines HQ, QG, GN, NH also lie in the same plane with them. The quadrilateral NQ is therefore plane; and the same reasoning holds for the other five. Let us now demonstrate that it is also rectangular.

Since HG has been shown parallel to BC, and hence triangle SHG similar to triangle SBC (by the corollary of prop. 4, book 6) [Margin: k 4.sexti (Elements VI.4, cor.)], it will follow that as SH is to HG, so SB is to BC; and, by alternation, as SH is to SB, so HG is to BC. But SH is a third part of SB; therefore HG too will be a third part of the side BC. In just the same way we shall show NQ to be a third part of the side AD. Since, then, the sides BC, AD are equal, their third parts HG, NQ will also be equal. And accordingly, since they are subtended by the equal sides HN, GN; GQ, HQ; NG, QG; NH, QH [Margin: l 8.primi (Elements I.8)], the four angles HNG, NGQ, GQH, QHN will be equal. Since therefore these are together equal to four right angles, as we showed at prop. 32, book 1, they will be right angles, and so NQ will be a square. By the same reasoning the remaining quadrilaterals will be squares; and for this reason they will constitute a cube, which will therefore be said to be inscribed in the pyramid — though improperly — because four of its vertices F, G, H, I rest in the centers of the four bases of the pyramid, while the remaining four N, O, P, Q rest at the bisection points of the perpendiculars drawn from the vertices of the pyramid to its bases, as is clear.

Alternatively. In the given pyramid let an octahedron be described, and in this octahedron a cube; and what is required will be done. For since the eight vertices of the cube are set at the centers of the eight bases of the octahedron, four of which lie in the four bases of the pyramid (as is clear from prop. 2 of this book) and have the same centers as they (as we shall soon show), it comes about that four vertices of the cube rest in the centers of the four bases of the pyramid, just as in the previous

Diagram. A large, densely ruled triangular pyramid (tetrahedron) ABCD with an inscribed cube and its many auxiliary lines, illustrating Proposition 18. It is one of the crowded three-dimensional figures of Book XV; the interior labels are cramped and foxed. Labels read on the leaf (positions as visible; roles as defined in the text):

- A (apex, top)
- B (bottom-left vertex)
- C (right vertex)
- D (front vertex, lowest point of base)

- M (upper-left, on edge AB)
- N (upper-center, a cube vertex — midpoint of AF)
- S (just below N, center — midpoint of edge BC)
- H (mid-left, base-center / cube vertex)
- G (mid-right, base-center / cube vertex)
- R (center)
- E (center, just below R — center of the pyramid)
- O (left of center, cube vertex)
- P (right of center, cube vertex)
- T (lower-center — midpoint of edge AD)
- Q (just right of T, cube vertex)
- K (bottom, between B and F)
- F (bottom-center, base-center / cube vertex)
- L (bottom-right)
- I [?] (cube vertex / base-center per text; not confidently locatable among the interior lines)

previous demonstration it was stated. Again, since the remaining four bases of the octahedron are parallel to the bases of the pyramid, as is clear from the figure of prop. 2 of this book (for since the straight lines EG, EH are parallel to the straight lines BC, BD [Margin: m 2.sexti (Elements VI.2)], the triangles EGH, BCD drawn through those lines will also be parallel to one another [Margin: n 15.vnde. (Elements XI.15)], and so with the rest) — it comes about that the perpendiculars dropped from the vertices of the pyramid to the opposite bases are also perpendicular to those four remaining bases of the octahedron, by the scholium of prop. 14, book 11. Hence, since the aforesaid bases cut off from the pyramid smaller pyramids similar to the whole, those perpendiculars will fall on the centers of the said bases, just as they fall on the centers of the bases of the whole pyramid [Margin: o 17.vnde. (Elements XI.17)]. And since the said four bases of the octahedron divide those perpendiculars proportionally with the sides of the pyramid — that is, in half — it is brought about that the remaining four vertices of the cube are set at the midpoints of the four perpendiculars, just as the earlier demonstration taught. It is therefore established that the cube is described in the pyramid, as before. And so we have described a cube in a given pyramid. Which was to be done.

COROLLARIVM I. [Corollary I]

It is established from these things that the center of the pyramid and of the cube described in it is the same. For the diagonals of the cube NF, OG, PH, QI, since they are the halves of the perpendiculars dropped from the vertices of the pyramid through its center to the bases, pass through E, the center of the pyramid.

COROLLARIVM II. [Corollary II]

It is also inferred that the straight line joining the bisection points of the opposite edges of the pyramid passes through the center of the pyramid, and is triple the side of the cube described in the pyramid. For since ER, drawn from E, the center of the cube, to R, the center of the base GNHQ, is half the side of the cube (as is clear), the straight line RS will be equal to the side of the cube. Since therefore RS was shown to be a third part of ST, it is evident that the straight line ST, joining S and T, the bisection points of the opposite edges AD, BC of the pyramid, passes through the center of the pyramid, and is triple the cube's side RS.

LEMMA.

(The Lemma is set entirely in italic type in the source.)

But we shall demonstrate as follows that the center of the pyramid and of the octahedron described in it is one and the same, and that the center of a base of the pyramid and of a base of the octahedron described in it is likewise the same.

Let there first be a pyramid ABCD, all six of whose edges are bisected at E, F, G, H, I, K; and from L, the center of the pyramid, let perpendiculars LM, LN be dropped to the bases ABD, BCD, which will fall on the centers of the circles circumscribing the said bases, by the corollary of lemma 1 of prop. 10, book 14. Now, having joined the straight lines LE, EM, LF, FN: since the sides LM, ME of triangle LME are equal to the sides LN, NF of triangle LNF (for since the circles circumscribing the bases ABD, BCD are equal, they are equidistant from the center L, by lemma 2 of prop. 10, book 14, and therefore their distances LM, LN will be equal; likewise ME, NF, the distances of the equal straight lines AB, BD from the centers M and N of the equal circles, will be equal), and they contain right angles by def. 3, book 11 [Margin: p 4.primi (Elements I.4)]: the straight lines LE, LF will be equal. By the same reasoning the straight lines LG, LH, LI, LK will be equal to these and to one another. Hence, since the vertices of the octahedron described in the pyramid are

set at the points E, F, G, H, I, K (as is clear from prop. 2[?] of this book), L will be the center of the octahedron, since all the lines falling from it to its vertices are equal. Since therefore L is also the center of the pyramid, the center of the pyramid and of the octahedron described in it will be the same.

Diagram. A triangular pyramid ABCD with edge-midpoints and the inscribed octahedron's centers, illustrating the first part of the Lemma. Labels read on the leaf:

- A (apex, top)
- E (left, on edge AB)
- H (upper right, on edge AC)
- L (center, the center of the pyramid)
- I (right of L)
- M (left of L)
- K (center-low)
- N (right of K)
- B (bottom-left vertex)
- C (right vertex)
- F (bottom-center)
- G (bottom-right)
- D (bottom, front vertex)

Next, let the base DEF of the octahedron lie in the base ABC of the pyramid, cutting the sides of that base in half, as is evident from prop. 2 of this book; and let G be the center of triangle ABC, from which straight lines are drawn to the vertices of each base. Now the sides AG, AD are equal to the sides BG, BF, and they contain equal angles — namely halves of the equal angles of an equilateral triangle [Margin: q 4.sexiti (Elements VI.4)]; the bases GD, GF will therefore be equal. By the same reasoning GE will be equal to each of GD, GF. Therefore G is the center of triangle DEF. Which was proposed.

Diagram. An equilateral triangle ABC with the inscribed triangle DEF and common center G, the interior lines forming a star, illustrating the second part of the Lemma. Labels read on the leaf:

- A (apex, top)
- D (upper-left, on side AB)
- E (upper-right, on side AC)
- G (center)

- B (bottom-left vertex)
- F (bottom-center, on side BC)
- C (bottom-right vertex)

PROBL. 19. PROPOS. 19. [Problem 19, Proposition 19]

To inscribe an icosahedron in a given pyramid.

In the given pyramid ABCD let an octahedron EFGHIK be inscribed [Margin: a 2.quartidecim (Elements XIV.2)], all twelve edges of which are cut in extreme and mean ratio, on this condition: that from each edge two corresponding points be taken — that is, points falling toward the opposite vertices of the octahedron's squares, as we stated in prop. 16 of this book. Such are EG, EK, HK, HG; KF, KI; GI, GF; FH, FE; IE, IH, [cut] at the points L, M, N, O, P, Q, R, S, T, V, X, Y. For if we join each of these points to its five nearest by straight lines — as M with V, L, X, P, and Q, and so on — an icosahedron will be described in the said

This batch completes Book XV of the *Elements* (with Propositions 20–21 and the concluding Scholium), reaches the end of Euclid's *Elements* (FINIS ELEMENTI QVINTIDECIMI), and opens **Book XVI** — the treatise added by François de Foix, Count of Candale (*Franciscus Flussas Candalla*), on how the regular solids inscribe one within another — through its preface and first two theorems.

[Euclid, Book XV — continuation of PROBL. 19 / PROPOS. 19: to inscribe an icosahedron in a given pyramid]

Diagram. A triangular projection of the pyramid (tetrahedron) ABCD with an inscribed icosahedron built upon an inscribed octahedron.

- A (top apex): a vertex of the pyramid
- B (bottom-left corner): a vertex of the pyramid [reads as a B with a long descender]
- C (bottom-right corner): a vertex of the pyramid
- D (interior center): the fourth vertex of the pyramid
- E (upper center): octahedron vertex (on edge AD)
- F (left): octahedron vertex (on edge AB)
- G (lower left): octahedron vertex (on edge BD)
- H (bottom center): octahedron vertex (on edge BC)
- I (right): octahedron vertex (on edge CD)
- K (upper right): octahedron vertex (on edge AC)

- V, M, P, X, S, L, N, T, R, O, Y: the extreme-and-mean section points on the octahedron's edges, joined to form the icosahedron
- [one further interior minuscule label, to the right of N near I, is not resolvable — possibly q [?]]

in the said octahedron, as is clear from Prop. 16 of this book, since it has all twelve of its angles in the twelve aforementioned sections of the twelve edges of the octahedron. Since, then, these very edges of the octahedron have been drawn in the planes of the pyramid's bases, it is evident that the twelve angles of the icosahedron are established in the planes of the pyramid's bases — three in each, inasmuch as three edges of the octahedron lie in each base of the pyramid. For in the base ABC lie the edges FH, HK, KF; in ABD, the edges EF, FG, GE; in BDC, the edges GH, HI, IG; and in ADC, finally, the edges IK, KE, EI. Therefore the said icosahedron will have been inscribed in the pyramid, by Def. 31 of Book XI — its four bases MXQ, ROY, LSV, NPT lying in the four bases of the pyramid, as is evident. And so we have inscribed an icosahedron in the given pyramid. Which was to be done.

PROBL. 20. PROPOS. 20.

To inscribe a dodecahedron in a given pyramid.

In the given pyramid inscribe an icosahedron [Margin: a. *Elements* XV.19], and in this icosahedron a dodecahedron [Margin: b. *Elements* XV.5] having its twenty angles at the centers of the twenty bases of the icosahedron; and what is required will have been done. For since four bases of the icosahedron lie in the four bases of the pyramid, as we said in the preceding proposition, four angles of the dodecahedron inscribed in that icosahedron will fall in the four bases of the pyramid — namely at their centers. For those four bases of the icosahedron are concentric with the four bases of the pyramid, as is clear from the second corollary of Prop. 16 of this book.

Next, since the cube inscribed in the dodecahedron places its eight angles at the eight angles of the dodecahedron [Margin: c. *Elements* XV.8]; and likewise the cube inscribed in the pyramid sets four of its angles at the centers of the pyramid's bases (of which we have already spoken), and its remaining four at the bisection-points of the perpendiculars dropped from the pyramid's angles onto the bases [Margin: d. *Elements* XV.18] — the four corresponding angles of the cube will be placed at those same bisection-points of the said perpendiculars. There remain, moreover, twelve angles of the dodecahedron, of which pairs will be assigned to the several edges of the pyramid. And thus the said dodecahedron

will be said to have been inscribed in the proposed pyramid, even if not in the strict sense. For which reason we have inscribed a dodecahedron in the given pyramid. Which was to be done.

PROBL. 21. PROPOS. 21.

To inscribe a sphere in a given regular solid.

From the center of the sphere enclosing the given solid let perpendiculars be dropped onto the several bases. By the corollary of Lemma 1 of Prop. 10 of Book XIV, these will fall at the centers of the circles circumscribing the bases, and will therefore be equal — since those circles, being equal, are equally distant from the center, by Lemma 2 of the same proposition. Consequently a sphere whose radii are the aforesaid equal perpendiculars will have been inscribed in the solid, for its convex surface will touch each of the bases at their centers.

SCHOLIVM.

Since, then, each of the five regular solids is inscribed in the four remaining ones, so that there are twenty inscriptions in all, only the dodecahedron alone is inscribed improperly — in the cube, the octahedron, and the pyramid, as is plain from Props. 13, 17, and 20 of this book. Likewise the cube alone is inscribed not properly in the pyramid, as is manifest from Prop. 19 of this book. For, that some solid be said to be inscribed in a solid in the proper sense, it is necessary that all the angles of the inscribed solid be established either at the angles, or on the edges, or finally in the planes (that is, the bases) of the solid in which it is inscribed — as we set forth in Def. 31 of Book XI. And this condition does not hold for the dodecahedron if it be inscribed in the cube, the octahedron, and the pyramid; nor yet for the cube, if it be inscribed in the pyramid.

Furthermore, among all five regular solids, one — the octahedron — receives the remaining four mutually inscribed within itself. [Margin: a. Elements XV.16] For in the octahedron an icosahedron is inscribed, setting its twelve angles at the twelve points at which the twelve edges of the octahedron are cut in extreme and mean ratio. Next, [Margin: b. Elements XV.5] upon the icosahedron is set a dodecahedron, which is inscribed in that same octahedron, as is clear from Prop. 17 of this book. After this, [Margin: c. Elements XV.8] in the dodecahedron a cube is inscribed, whose eight angles, coinciding with the eight angles of the dodecahedron, occupy the centers of the octahedron's bases; and for that reason the cube itself too is inscribed in the octahedron. [Margin: d. Elements XV.1] Finally, a pyramid is enclosed in the cube, which will therefore be said to be inscribed in the octahedron,

because its four angles, coinciding with four angles of the cube, rest at the centers of four bases of the octahedron. And that this privilege is denied to the remaining regular solids is plain from what has been said.

Nor, indeed, is it to be passed over that a dodecahedron, the cube inscribed within it, and the pyramid inscribed within that cube are all comprehended by the same sphere. [Margin: e. Elements XV.1] For since the four angles of the pyramid occupy four of the eight angles of the cube, [Margin: f. Elements XV.8] and since all the angles of the cube coincide with eight angles of the dodecahedron, it is clearly established that the sphere circumscribed about the dodecahedron encloses also the cube and the pyramid, since the surface of that sphere passes through the angles of these solids.

By the same reasoning it is manifest from what has been said that the dodecahedron, the cube, and the pyramid are likewise inscribed in the icosahedron, the octahedron, and the pyramid. For their angles are established at the centers of the bases of these, as has been demonstrated: at the centers, indeed, of the bases of the icosahedron, by Props. 5, 11, and 12; at the centers of the bases of the octahedron, by Props. 17, 4, and 6; and finally at the centers of the bases of the pyramid, by Props. 20 and 18 of this book.

In the twenty preceding propositions of this book, moreover, we have completed all the delineations, that is inscriptions, of the five regular solids one within another, as many as can be devised, since in each one we have marked out the remaining four. Namely: in the pyramid, the octahedron, cube, icosahedron, and dodecahedron, by Props. 2, 18, 19, 20; in the octahedron, the pyramid, cube, icosahedron, and dodecahedron, by Props. 6, 4, 16, 17; next, in the cube, the pyramid, octahedron, icosahedron, and dodecahedron, by Props. 5, 1, 3, 14, 13 [?]; in the icosahedron, the pyramid, octahedron, cube, and dodecahedron, by Props. 12, 15, 11 [?]; and finally, in the dodecahedron, the pyramid, octahedron, cube, and icosahedron, by Props. 10, 9, 8, 7 — these we have described. And in the last, single proposition we have taught by what art a sphere may be drawn in any given regular solid.

FINIS ELEMENTI QVINTIDECIMI. [End of the Fifteenth Element — and of Euclid's *Elements*.]

ELEMENTVM SEXTVMDECIMVM.

In which the various comparisons of the regular solids mutually inscribed one within another, and of their edges, are explained — added by François de Foix,

Count of Candale [Franciscus Flussas Candalla] — together with matter concerning the five bodies.

LIBER TERTIVS.

Just as in Book 14 there were handed down very many comparisons of the regular figures and of their edges described within the same sphere — whose description Euclid set forth in Book 13 — so also Franciscus Flussas Candalla, in this Book 16, compares those same regular figures inscribed one within another, and likewise their edges, the inscription of which we ourselves handed down in the preceding Book 15. And so this Book 16 bears to the preceding Book 15 the same connection that the fourteenth bore to the thirteenth, so that the treatment of these five regular bodies might in some way appear incomplete were this sixteenth book not appended.

THEOR. 1. PROPOS. 1.

If in a dodecahedron a cube be inscribed, and in this cube another dodecahedron, then the ratio of the outer dodecahedron to the inner dodecahedron will be the triplicate of the ratio which the greater segment of a straight line divided in extreme and mean ratio has to the lesser.

Let the edge of the cube inscribed in the dodecahedron whose edge is C be AB, and let the edge of the dodecahedron inscribed in the cube of edge AB be D; and let AB, the edge of the cube, be divided at E in extreme and mean ratio. I say that the dodecahedron of edge C has to the dodecahedron of edge D the triplicate of the ratio which AB — the greater segment — has to EB — the lesser.

For since, as we showed in the scholium to the last proposition of Book 15, the dodecahedron and the cube inscribed in it are comprehended by the same sphere; and since, when the edge of the cube is divided in extreme and mean ratio, the greater segment is the edge of the dodecahedron described in the same sphere as the cube, by the first corollary of Prop. 17 of Book 13 — AE, the greater segment of the cube's edge, will be equal to the edge C of the dodecahedron described in the same sphere and about that cube. Again, since it has been demonstrated in the scholium to Prop. 13 of Book 15 that, if the edge of the cube be divided in extreme and mean ratio, the lesser segment is the edge of the dodecahedron inscribed in that cube — EB, the lesser segment of the cube's edge, will be equal to the edge D of the dodecahedron which is inscribed in that cube. And therefore C, the edge of the dodecahedron circumscribed about the cube, is to D, the edge of the dodecahedron inscribed in the same cube, as AE, the greater

segment, is to EB, the lesser. But the dodecahedron of edge C has to the dodecahedron of edge D the triplicate of the ratio of edge C to edge D, by the corollary of Prop. 17 of Book 12. Therefore the same dodecahedron of edge C also has to the dodecahedron of edge D the triplicate of the ratio of the greater segment AE to the lesser segment EB. And so, if in a dodecahedron a cube be inscribed, etc. Which was to be demonstrated.

Diagram. A simple lettered line-figure.

- A — E — B: a straight segment AB with the division-point E marking its section in extreme and mean ratio (E lies toward B, so AE is the greater segment)
- C: a shorter reference segment (drawn below, at the left; the edge of the outer dodecahedron)
- D: a still shorter reference segment (drawn below, at the right; the edge of the inner dodecahedron)

THEOR. 2. PROPOS. 2.

A perpendicular dropped from any angle of an equilateral and equiangular pentagon onto the opposite side is cut, by the straight line subtending that angle, in extreme and mean ratio.

Opera Mathematica, Tomus I — Book XVI (Candalla's treatise on the regular solids)

(This batch covers global source- i.e. printed pages 611–613. It completes the demonstration of Theorem 2 / Proposition 2 and continues through Candalla's propositions on the mutual inscriptions of the five regular solids, Theorems 3–11.)

Demonstration of THEOR. 2. PROPOS. 2. (conclusion)

(Enunciation given in the preceding batch: "A perpendicular dropped from any angle of an equilateral and equiangular pentagon onto the opposite side is cut, by the straight line subtending that angle, in extreme and mean ratio.")

In the equilateral and equiangular pentagon ABCDE, let a perpendicular AF be dropped from angle A onto the opposite side CD. I say that AF is cut at G in extreme and mean ratio [ratio extrema ac media] by the straight line BE, which subtends angle A. For draw the straight line AD, and let it cut BE at H. Since BE is parallel to the side CD (by the scholium to Prop. 8, Book 13), the straight lines AF

and AD will be cut proportionally [Margin: a — 2.sex. = *Elements* VI.2]. But AD is cut at H in extreme and mean ratio, and DH is the greater segment [Margin: b — 18[?].tertiidecimi = *Elements* XIII.8 (printed "18"; the sense requires XIII.8)]. Therefore AF too will be cut in the same way at G. And so the perpendicular from any angle, etc. Which was to be demonstrated.

Diagram. An equilateral, equiangular pentagon with apex A at top, "shoulders" B (left) and E (right), and lower vertices C (left) and D (right). A horizontal chord BE runs across the figure and subtends angle A. From A a perpendicular AF drops to the midpoint F of the base CD. The diagonal AD is drawn from A to the lower-right vertex D. The chord BE is met by the perpendicular AF at G and by the diagonal AD at H.

- A (top apex): vertex of the pentagon from which the perpendicular AF and the diagonal AD are drawn
- B (upper-left vertex): left endpoint of the subtending chord BE
- E (upper-right vertex): right endpoint of the subtending chord BE
- G (on BE, left of center): point where the perpendicular AF crosses BE — the point of extreme-and-mean-ratio division on AF
- H (on BE, right of center): point where the diagonal AD crosses BE
- C (lower-left vertex)
- F (midpoint of base CD): foot of the perpendicular from A
- D (lower-right vertex)

THEOR. 3. PROPOS. 3.

If from the angles of the triangular face of a tetrahedron [pyramis] straight lines are drawn cutting the opposite sides in extreme and mean ratio, in such a way that near each angle there is the greater segment of one side and the lesser segment of the other, then these lines, by their own intersections, will produce in the middle the base of the icosahedron inscribed in that tetrahedron. This base is itself inscribed in another equilateral triangle, whose angles cut the sides of the tetrahedron's triangle in extreme and mean ratio, and whose sides are themselves bisected by the angles of the base of the icosahedron.

Let the triangular face of the tetrahedron be ABC, whose sides are bisected at D, E, F; and joining the straight lines DE, EF, FD, let these in turn be cut in extreme and mean ratio at G, H, I. Joining GH, HI, IG then forms GHI, the triangle of the icosahedron inscribed in that tetrahedron, as is established from Prop. 19, Book 15. Next, let the sides of triangle ABC be cut in extreme and mean ratio at K, L, M,

and join BK, CL, AM. I say that these straight lines, by their sections, produce the triangle GHI of the icosahedron — that is, that BK passes through the points G, I; CL through the points H, I; and AM through the points G, H.

For, drawing through H the straight line NO parallel to BC, the triangle FHO will be similar to triangle FEC (by the corollary to Prop. 4, Book 6), and therefore also equilateral, since FEC too is equilateral — being similar to triangle ABC by the same corollary. Hence HO, a straight line equal to FH, is equal to the greater segment of FE (that is, of DF). (For DEF too is an equilateral triangle, since its sides subtend equal angles of the equilateral triangle, contained by equal — namely half — of the sides of that same triangle; and therefore they are equal [Margin: a — 4.pri. = *Elements* I.4].) Also NH is equal to the whole of DF, in the parallelogram DH [Margin: b — 34.pri. = *Elements* I.34], because FH and DF are parallel to AB and BC (that is, to NO) [Margin: c — 2.sexti = *Elements* VI.2].

Therefore the straight line NO, composed of NH (the whole) and HO (the greater segment), will be cut at H in extreme and mean ratio [Margin: d — 5.tertiidecimi = *Elements* XIII.5]. And accordingly, since the three parallels BC, NO, DF are similarly cut at M, H, G — namely in extreme and mean ratio — and since the straight line AM cuts them similarly (from what we demonstrated at Prop. 4, Book 6), it clearly follows that AM passes through the points G, H. In just the same way we shall show that BK passes through the points G, I, and CL through the points H, I, if through G and I straight lines are drawn parallel to the sides AC and AB. And therefore the straight lines BK, CL, AM, by their sections, produce the triangle GHI of the icosahedron.

Now let PQ be drawn through G parallel to CL. Since therefore AF is equal to DF (that is, to FE), whose greater segment was FH — that is FO, equal to it — FO will also be the greater segment of AF [Margin: e — 5.tertiidecimi = *Elements* XIII.5]; and accordingly the whole AO is cut at F in extreme and mean ratio. Now, as AF is to FO, so is AG to GH (because of triangle AHO); and as AG is to GH, so is AQ to QC, and AP to PL (because of the triangles AHC, AHL) [Margin: f — 2.sexti = *Elements* VI.2]. Therefore the straight lines AH, AC, AL are cut at G, Q, P in extreme and mean ratio, and AG, AQ, AP are the greater segments. Therefore AQ and AL — the greater segments of the equal straight lines AC, AB — will be equal. And since, as AQ, the greater segment, is to QC, the lesser, so is AL, the whole, to AP, the greater segment [Margin: g — 2.quartidecimi = *Elements* XIV.2]; and AQ is equal to AL: then QC too will be equal to AP. Therefore AP too will be the lesser segment of the side AB, and the remainder BP the greater; and consequently the straight line PQ will cut the sides BA, AC in extreme and mean ratio. By the same reasoning the

straight line PR, which is drawn through I parallel to AM, will cut the sides CB, BA in extreme and mean ratio, and therefore will fall on the point P. Not otherwise, the straight line RQ, drawn through H parallel to BK, will cut the sides AC, CB in extreme and mean ratio, and so will fall on the points R, Q.

And since the individual sides of triangle PQR — subtending equal angles of the equilateral triangle, contained by equal lines, namely each by the greater and the lesser segment — are equal to one another [Margin: h — 4.pri. = *Elements* I.4], the triangle PQR composed of them will be equilateral, and its sides are indeed bisected by the angles of the triangle GHI of the icosahedron. (For since in triangle ACL the straight line PQ is parallel to the side CL, PQ and LC will be cut in the same ratios, from what we demonstrated at Prop. 4, Book 6. Now CL is bisected at H, on account of triangle ACL, in which FH is parallel to the side AL, cutting the sides CA, CL proportionally [Margin: i — 2.sexti = *Elements* VI.2]. Therefore PQ too is bisected at G. And the same reasoning holds for the remaining ones, QR, RP.) And therefore the triangle GHI of the icosahedron is inscribed

Diagram. A large equilateral triangle ABC (apex A, base BC) subdivided by a dense network of interior lines into the construction of the inscribed icosahedron triangle. Down the left edge from A to B lie the points P, D, L, N; down the right edge from A to C lie K, F, Q, O; along the base from B to C lie R, E, M, S. Interior points are G (upper center), I (center-left), H (lower center), and T (center-right). The central triangle PQR encloses the smaller icosahedron triangle GHI.

- A (top apex): vertex of the tetrahedron's triangular face
- B (lower-left vertex); C (lower-right vertex)
- P (left edge, upper): angle of the inscribed equilateral triangle PQR
- K (right edge, upper)
- D (left edge, midpoint of AB); F (right edge, midpoint of AC); E (base, midpoint of BC): bisection points of the three sides
- L (left edge, lower); Q (right edge, lower)
- N (left edge, near B); O (right edge, near C): endpoints of the parallel NO drawn through H
- R, M, S (base points): sections of the base
- G (upper center); H (lower center); I (center-left): vertices of the icosahedron triangle GHI
- T (center-right): point on the parallel NO

(continuation of Prop. 3) ...in the equilateral triangle PQR, whose angles divide the sides of the base of the tetrahedron in extreme and mean ratio, and whose sides are themselves bisected at G, H, I by the angles of the base GHI of the icosahedron. Therefore, if from the angles of the triangular face of a tetrahedron, etc. Which was to be shown.

COROLLARIVM.

From these things it can easily be gathered that the side of the icosahedron inscribed in an octahedron is the greater segment of a straight line divided in extreme and mean ratio — the line which, drawn from one angle of the base of the octahedron, cuts the opposite side likewise in extreme and mean ratio. For since, as is established from Prop. 16, Book 15, GHI is the triangle of the icosahedron inscribed in the octahedron, whose base is DEF: let there be drawn from angle F, of triangle FEC — which is equal to triangle DEF — the straight line FS parallel to AM, cutting MC and HO at S and T. Since therefore MC is bisected at S, on account of triangle AMC [Margin: k — 2.sex. = *Elements* VI.2], and MS is equal to GF, on account of the parallelogram GS [Margin: l — 34.pri. = *Elements* I.34], SC too will be equal to that same GF, the lesser segment of DF. And consequently, since the wholes DF, EC are equal, the remainder ES too will be equal to the remainder DG, the greater segment. Therefore EC is cut at S in extreme and mean ratio. Then, since FE and FS are cut similarly [Margin: m — 2.sex. = *Elements* VI.2], and FE is cut in extreme and mean ratio, FS too will be cut at T in extreme and mean ratio; and the greater segment will be FT, a straight line equal to the side GH of the icosahedron, on account of the parallelogram GT [Margin: n — 34.pri. = *Elements* I.34]. Therefore FT, the side of the icosahedron inscribed in the octahedron, is the greater segment of the straight line FS — which divides EC, the side of the octahedron, in extreme and mean ratio, being dropped from the angle F. Which was proposed.

Diagram. The same large subdivided equilateral triangle ABC as on the preceding page, reprinted here for the corollary. Apex A; left edge A→B carries P, D, L, N; right edge A→C carries K, F, Q, O; base B→C carries R, E, M, S. Interior points G (upper center), I (center-left), H (lower center), T (center-right).

- A (apex); B (lower-left); C (lower-right)
- P, D, L, N (left edge, top to bottom)
- K, F, Q, O (right edge, top to bottom)
- R, E, M, S (base, left to right)
- G (upper center); H (lower center); I (center-left): icosahedron triangle GHI

- T (center-right)

THEOR. 4. PROPOS. 4.

The lesser segment of the side of a tetrahedron cut in extreme and mean ratio is, in square [potential], double the side of the icosahedron inscribed in that tetrahedron.

Let the base of the tetrahedron be ABC, whose sides are cut in extreme and mean ratio at K, L, M. Drawing the straight lines BK, CL, AM, which produce GHI, the base of the icosahedron inscribed in that tetrahedron, inscribed in turn in the equilateral triangle PQR — whose angles cut the sides of triangle ABC in extreme and mean ratio at P, Q, R, and whose sides PQ, QR, RP are bisected at G, H, I, as was demonstrated in the preceding proposition. I say that AP, the lesser segment of the side of the tetrahedron, is in square double the side HI of the icosahedron. For since PQ is in square double the lesser segment AP (by Corollary 1 to Prop. 16, Book 15), but quadruple the square of HI (by the scholium to Prop. 4, Book 2 — because the straight line PQ is double PG, that is, double HI equal to it, on account of the parallelogram PH): PQ will be, in square, 4; AP, 2; and HI, 1. For which reason AP will be in square double HI. Therefore the lesser segment of the side, etc. Which was to be demonstrated.

Diagram. A smaller equilateral triangle ABC (apex A, base BC) with the inscribed construction. Down from A on the left: P then L; near the top center G; on the right near the top K; interior center-left I, center H; right side Q; base points R and M.

- A (apex); B (lower-left); C (lower-right)
- P (left, upper); G (center, upper); K (right, upper)
- L (left, lower); I (interior, center-left); H (interior, center); Q (right, lower)
- R, M (base points between B and C)

COROLLARIVM.

It follows from these things that the side of the icosahedron inscribed in the tetrahedron is an apotome [*apotome*, from Greek ἀποτομή — a species of irrational straight line]. For since the diameter of the sphere is, in square, one-and-a-half times [sesquialtera] the side of the tetrahedron [Margin: a — 13.tertiidecimi = *Elements* XIII.13]: if the diameter is set as rational, the side of the tetrahedron too will be rational, since it is commensurable in square with the diameter [Margin: b — 6.dec. = *Elements* X.6]. Therefore the lesser segment of the side of the tetrahedron cut in extreme and mean ratio will be an apotome

[Margin: c — 6.tertiidecimi = *Elements* XIII.6]. And accordingly, since the side of the icosahedron is commensurable in square with the lesser segment of the side of the tetrahedron [Margin: d — 6.dec. = *Elements* X.6] — for we showed that the lesser segment is, in square, double the side of the icosahedron — the side of the icosahedron too will be an apotome, by the scholium to Prop. 104, Book 10.

THEOR. 5. PROPOS. 5.

The side of a cube is, in square, half the side of the tetrahedron inscribed in it. The side of the tetrahedron, in turn, is double, in length, the side of the octahedron inscribed in it. And finally, the side of the cube is, in square, double the side of the octahedron inscribed in it.

For since the side of the tetrahedron inscribed in a cube is the diameter of a face of the cube, as is established from Prop. 1, Book 15; and since the square of this diameter is double a face of the cube — that is, double the square of the side of the cube — by the scholium to Prop. 47, Book 1: it is manifest that the side of the cube is, in square, half the side of the tetrahedron inscribed in that cube. Again, since the straight lines joining the bisection-points of the sides of the tetrahedron constitute the octahedron inscribed in the tetrahedron, as is clear from Prop. 2, Book 15; and since the side of an equilateral triangle is double the straight line joining the bisections of its sides — that line being equal to half the side of the triangle, as is established from the corollary to Prop. 4, Book 6 (for since it is parallel to the third side, it cuts off an equilateral triangle, etc.) [Margin: a — 2.sex. = *Elements* VI.2]: it is evident that the side of the tetrahedron is double, in length, the side of the octahedron inscribed in it. Finally, since the diameter of the sphere, or of the octahedron, is, in square, double the side of the octahedron [Margin: b — 14.tertiidecimi = *Elements* XIII.14]; and since the side of the cube is equal to the diameter of the octahedron inscribed in it — because the diameter of the octahedron joins the centers of opposite faces, as

(*continuation of Prop. 5*) ...we said in the corollary to Prop. 3, Book 15: it is clear that the side of the cube is, in square, double the side of the octahedron inscribed in it. And therefore the side of the cube is, in square, half the side of the tetrahedron, etc. Which was to be shown.

THEOR. 6. PROPOS. 6.

The side of a dodecahedron is the greater segment of a straight line which is, in square, half the side of the tetrahedron inscribed in it.

For since the tetrahedron inscribed in the cube is also inscribed in the dodecahedron in which that cube is inscribed, as we taught at Prop. 10, Book 15; and since the side of the dodecahedron is the greater segment of the side of the cube inscribed in it (by Corollary 1 to Prop. 17, Book 13); and finally, since the side of the cube is, in square, half the side of the tetrahedron inscribed in it [Margin: $a — 5.\text{sextidecimi} = \textit{Elements XVI.5}$, i.e. Prop. 5 of this book]: it is plainly gathered that the side of the dodecahedron is the greater segment of a straight line which is, in square, half the side of the tetrahedron inscribed in it. Which was to be demonstrated.

THEOR. 7. PROPOS. 7.

If both an icosahedron and a dodecahedron are inscribed in a cube, the side of the icosahedron will be the mean proportional between the side of the cube and the side of the dodecahedron.

Let the side of the cube be A, the side of the icosahedron inscribed in it B, and finally the side of the dodecahedron inscribed in the same cube C. I say that B, the side of the icosahedron, is in the mean position, proportional between A, the side of the cube, and C, the side of the dodecahedron. For since, from what was demonstrated at Prop. 14, Book 15, B, the side of the icosahedron, is the greater segment of the side of the cube A; and C, the side of the dodecahedron, is the lesser segment of that same side of the cube, by what we showed in the scholium to Prop. 13 of the same book: it becomes plain that A, the whole, is to B, the greater segment, as B, the greater segment, is to C, the lesser. Therefore, if both an icosahedron and a dodecahedron are inscribed in a cube, etc. Which was to be shown.

Diagram. Three horizontal straight-line segments drawn one above another, decreasing in length, illustrating the mean-proportional relation $A : B :: B : C$.

- A: the longest segment (side of the cube)
- B: the middle segment (side of the icosahedron — the mean proportional)
- C: the shortest segment (side of the dodecahedron)

THEOR. 8. PROPOS. 8.

The side of a tetrahedron is, in square, eighteen times [octodecuplum] the side of the cube inscribed in it.

For since, by what was demonstrated at Prop. 18, Book 15, the side of the tetrahedron is triple, in length, the diameter of a face of the cube inscribed in it —

for it was shown there that GH, the diameter of the face of the inscribed cube, is a third part of the side BC of the tetrahedron — and since squares have the duplicate ratio of their sides [Margin: a — 20.sex. = *Elements* VI.20]: the square of the side of the tetrahedron will be nine times the square of the diameter of the cube's face inscribed in it, since the ninefold ratio is the duplicate of the triple, as appears in these numbers 1, 3, 9. Therefore, since the square of the diameter of the cube's face is double the face of the cube itself (by the scholium to Prop. 47, Book 1), it comes about that, if the face of the cube — that is, the square of the side of the cube — is set as 1, the square of the diameter of that face is 2, and the square of the side of the tetrahedron is 18. For which reason the side of the tetrahedron is, in square, eighteen times the side of the cube inscribed in it. Which was to be shown.

THEOR. 9. PROPOS. 9.

The side of a tetrahedron is, in square, eighteen times a straight line cut in extreme and mean ratio, whose greater segment is the side of the dodecahedron inscribed in the tetrahedron.

For since the cube inscribed in the dodecahedron is also inscribed in the tetrahedron in which that dodecahedron is inscribed, as we taught in the scholium to the last proposition of Book 15: the side of the tetrahedron will be, in square, eighteen times the side of the cube inscribed both in it and in the dodecahedron [Margin: a — 8.sex tidecimi = *Elements* XVI.8, i.e. Prop. 8 of this book]. Now the greater segment of this side of the cube, cut in extreme and mean ratio, is (by Corollary 1 to Prop. 17, Book 13) the side of the dodecahedron inscribed in the tetrahedron, in which of course the cube is inscribed. Therefore the side of the tetrahedron is, in square, eighteen times a straight line cut in extreme and mean ratio, whose greater segment is the side of the dodecahedron inscribed in the tetrahedron. Which was to be demonstrated.

THEOR. 10. PROPOS. 10.

If an icosahedron is inscribed in an octahedron, the side of the icosahedron will be, in square, double the lesser segment of the side of the octahedron divided in extreme and mean ratio.

For, as is established from Prop. 16, Book 15, the straight line joining the two sections — those near the lesser segments — of two sides of the octahedron that contain a right angle (such as was there the straight line GI, subtending the right angle GAI, contained by the lesser segments AG, AI) is the side of the icosahedron inscribed in the octahedron. Therefore, since this very side of the icosahedron is,

in square, equal to those two equal lesser segments, it will be, in square, double one lesser segment [Margin: a — 47.pri. = *Elements* I.47]. Therefore, if an icosahedron is inscribed in an octahedron, etc. Which was to be demonstrated.

THEOR. 11. PROPOS. 11.

The side of an octahedron is, in square, four-and-a-half times [quadruplum sesquialterum] the side of the cube inscribed in it.

Let ABCDE be one of the pyramids of the octahedron [vna ex(?) pyramidum octaedri — printed "vna lex"], whose base is the square ABCD, and whose triangular faces at the vertex E are ABE, EBC, CED, DEA; and let the centers of these triangles be F, G, H, I, through which the straight lines kL, LM, MN, Nk are drawn...

(Continuation of Book XVI, Proposition 11: the side of an octahedron is, in square, four-and-a-half times the side of the cube inscribed in it. The demonstration begun in the prior pages continues.)

MN and Nk are parallel to the straight lines AB, BC, CD, DA. And if the straight lines FG, GH, HI, IF are joined, FGHI will be the square of the cube inscribed in the octahedron, as we demonstrated in proposition 4 of book 15. Now when the straight line EHO is drawn, it is one-and-a-half times [sesquialterate] the straight line EH, by the corollary to proposition 18 of book 14; so that if EO is set at 3 such parts, EH will be 2 of them. [Margin: a — *Elements* VI.2 ("2. sexti")] But as EO is to EH, so is EB to EL. Therefore, of whatever parts EB contains 3, EL — or LM, which equals it — will be 2.

And so the square on the octahedron's side EB will be 9 parts, while the square on the straight line LM — namely kLMN — will be 4 parts; and hence also the square on the cube's side GH — that is, FGHI — will be 2 such parts, since the square LN is double the square GI inscribed in it, by what we demonstrated in the scholium to proposition 47 of book 1. For LN is the square described on the diagonal of the square HI, [Margin: b — *Elements* I.33 ("33. pri.")] since GI, the diagonal (if it be drawn), equals the side LM. Therefore the octahedron's side EB is, in square, four-and-a-half times [quadruple-sesquialterate] the cube's side GH, since their squares have the ratio of 9 to 2. Therefore the side of an octahedron is, in square, four-and-a-half times the side of the cube inscribed in it. Q.E.D.

Diagram. The octahedral pyramid figure for Proposition 11, drawn as a square with an inscribed cross-hatched figure. The outer square has A at the top-left corner and D at the top-right; B at the bottom-left and C at the bottom-right, with

O marked at the middle of the base BC. Within it are the diagonals and the inscribed square FGHI.

- A (top-left corner)
- D (top-right corner)
- B (bottom-left corner)
- C (bottom-right corner)
- O (midpoint of the base BC)
- K (upper-left interior)
- F (upper-center interior)
- N (upper-right interior)
- E (center)
- G (left interior, mid-height)
- I (right interior, mid-height)
- L (lower-left interior)
- H (bottom-center interior)
- M (lower-center-right interior)

COROLLARY

Since the cube's side, divided in extreme and mean ratio, makes its greater segment the side of the dodecahedron in which the cube is inscribed, by the second corollary to proposition 13 of book 15, and since both the dodecahedron and the cube inscribed in it are inscribed in the same octahedron, as we taught in the scholium to the last proposition of book 15, it becomes clear that the octahedron's side is, in square, four-and-a-half times that straight line — divided in extreme and mean ratio — whose greater segment is the side of the dodecahedron inscribed in the octahedron. For such a straight line is the side of the cube inscribed in the same octahedron, of which we showed the octahedron's side to be, in square, four-and-a-half times.

THEOR. 12. PROPOS. 12.

The side of an icosahedron is the greater segment of that straight line — cut in extreme and mean ratio — which is, in square, double the side of the octahedron inscribed in the icosahedron.

Since the six opposite sides of the icosahedron inscribed in a cube are set upon the six faces of the cube, and three straight lines equal to the cube's side join their bisection points, as is clear from proposition 14 of book 15 and its scholium; and

since the six vertices of the octahedron inscribed in the icosahedron lie at the bisection points of those six sides of the icosahedron, by proposition 15 of the same book 15; it follows that the diameter of the octahedron inscribed in the said icosahedron equals the side of the cube in which the icosahedron is inscribed.

Therefore, since by what we showed at proposition 14 of book 15 the icosahedron's side is the greater segment of the aforesaid cube's side when it is divided in extreme and mean ratio, the same icosahedron's side will also be the greater segment of the diameter of the octahedron inscribed in that icosahedron. [Margin: a — *Elements* XIII.14 ("14. tertijdec.")] But the diameter of the sphere — that is, of the octahedron — is, in square, double the octahedron's side. Therefore the icosahedron's side is the greater segment of that straight line, cut in extreme and mean ratio, which is, in square, double the side of the octahedron inscribed in the icosahedron. Q.E.D.

THEOR. 13. PROPOS. 13.

The side of a cube has to the side of the dodecahedron inscribed in it the duplicate ratio of that which the greater segment has to the lesser of a straight line divided in extreme and mean ratio. But the side of the dodecahedron has to the side of the cube inscribed in it the ratio which the lesser segment has to the greater of the same straight line.

It was shown in the scholium to proposition 13 of book 15 that the lesser segment of the cube's side, cut in extreme and mean ratio, is the side of the dodecahedron inscribed in that cube. Since, then, the whole line divided in extreme and mean ratio has to the lesser segment the duplicate of the ratio of the whole line to the greater segment (or of the greater segment to the lesser) — because the whole line, the greater segment, and the lesser are three straight lines in continued proportion — it is clearly established that the ratio of the cube's side to the side of the dodecahedron inscribed in it is likewise the duplicate of the ratio of the greater segment to the lesser.

Moreover, since it is established from the first corollary to proposition 17 of book 13 that the greater segment of the cube's side, divided in extreme and mean ratio, is the side of the dodecahedron in which the cube is inscribed, it is evident that, as the side of the dodecahedron — that is, the greater segment — is to the side of the cube inscribed in it — that is, to the whole line cut in extreme and mean ratio — so is the lesser segment to the greater. For since, as the whole line is to the greater segment, so the greater segment is to the lesser, it will also be, by conversion, that as the greater segment is to the whole line, so the lesser segment

is to the greater. Therefore the side of the cube to the side of the dodecahedron, and so forth. Q.E.D.

THEOR. 14. PROPOS. 14.

The side of an octahedron is one-and-a-half times [sesquialterate] the side of the tetrahedron inscribed in it.

Let ABCDE be one of the six pyramids of the octahedron, whose base is the square ABCD and whose triangles at the apex E are ABE, EBC, CED, DEA; and let the centers of these triangles be F, G, H, I, which are joined by the straight lines FG, GH, HI, IF, forming the square of the cube inscribed in the octahedron, as we taught in proposition 4 of book 15. Now since the tetrahedron inscribed in the cube is also inscribed in the octahedron on which the cube is set — as we explained in the scholium to the last proposition of book 15 — and since the side of the tetrahedron inscribed in the cube is the diagonal of the cube's face, as is clear from proposition 1 of book 15, the diagonal GI, when drawn, will be the side of the tetrahedron inscribed in the said octahedron. I say, then, that the octahedron's side BC is one-and-a-half times the tetrahedron's side GI.

For when the straight lines EGK, EIL are drawn, the sides AB, CD are bisected at K, L, by the scholium to proposition 12 of book 13, and EK, EL themselves will be one-and-a-half times the straight lines EG, EI, by the corollary to proposition 18 of book 14. [Margin: a — *Elements* VI.2 ("2. sex."); the text itself cites the corollary to VI.4] And therefore GI will be parallel to the straight line KL, cutting off — by the corollary to proposition 4 of book 6 — the triangle EGI similar to the triangle EKL. Hence, as EK is to KL, so EG is to GI; and alternately, as EK is to EG, so KL is to GI. And so, since EK is one-and-a-half times EG, KL too will be one-and-a-half times GI. [Margin: b — *Elements* I.33 ("33. pri.")] Since, then, BC equals KL — because BK and CL are equal and parallel — BC, the octahedron's side, will be one-and-a-half times GI, the side of the tetrahedron inscribed in it. Q.E.D.

Diagram. The octahedral pyramid ABCDE for Proposition 14, drawn in oblique view with apex E uppermost and the square base ABCD below, with the inscribed square GI–KL of face-centers.

- E (apex, top)
- F (just below the apex, a face-center)
- A (upper-left base corner)
- D (upper-right base corner)
- G (left interior face-center)

- I (right interior face-center)
- K (lower-left, midpoint of AB)
- L (lower-right, midpoint of CD)
- H (bottom-center interior)
- B (bottom-left base corner)
- C (bottom-right base corner)

THEOR. 15. PROPOS. 15.

If from the square on the icosahedron's diameter there is taken away three times the square on the side of the cube inscribed in it, there remains a square that is four-thirds [sesquitercian] of the square on the icosahedron's side.

Let there be an icosahedron ABCDEFGH, whose diameter is BE and whose center is I; and let BCG, EFH be opposite and parallel triangles, by the fourth corollary to proposition 14 of book 13, whose centers are K, L. From the center I let a perpendicular IK be drawn to the plane BCG, which — by the corollary to lemma 1 of proposition 10 of book 14 — falls upon the center K. KI produced will therefore also be perpendicular to the plane EFH, which is parallel to the plane BCG, by the scholium to proposition 14 of book 11, and will accordingly fall upon the center L, by the corollary to the aforesaid lemma. For this reason, since the cube inscribed in the icosahedron sets its vertices at the centers of the icosahedron's faces, as is clear from proposition 11 of book 15 — and hence its opposite vertices at the centers of opposite faces — KL will be the diameter of the cube inscribed in the icosahedron, passing through I, the icosahedron's center.

Since, then, by the same reasoning the remaining diameters of the cube pass through the same center I, I will also be the center of the inscribed cube, and therefore the half-diameters IK, IL are equal. Now if KB — the radius of the circle circumscribing the triangle BCG — is joined, the angle BKI will be a right angle, by definition 3 of book 11. [Margin: a — *Elements* I.47 ("47. pri.")] And accordingly the square on the straight line BI equals the squares on the straight lines BK, KI. Therefore the square on the diameter BE also — being four times the square on the half-diameter BI, as we showed in the scholium to proposition 4 of book 2 — will equal the squares on two straight lines, namely on the diameter of the circle circumscribing the triangle BCG, and on KL. For these squares too, by the same scholium, are four times the squares described on the half-diameters BK, KI. So when the square on the cube's diameter KL — [Margin: b — *Elements* XIII.15 ("15. tertijdec.")] which is three times the square on the cube's side — is taken away

from the square on the icosahedron's diameter BE, there remains the square on the diameter of the circle circumscribing the triangle BCG.

This, I say, is four-thirds of the square on the icosahedron's side BC. For let a circle be described about the equilateral triangle BCG from the center K, and let Bk be extended to M — cutting CG at N — and let the straight line CM also be drawn. Now since BN, drawn through the center k, is perpendicular to CG, by what we showed in the scholium to proposition 12 of book 13, [Margin: c — *Elements* XIV.12 ("12. quartidec.")] the straight line BC will be, in square, four-thirds of the straight line BN; and as BC is to BN, so is BM to BC, since BC is the mean proportional between BM and BN, by the corollary to proposition 8 of book 6, [Margin: d — *Elements* III.31 ("31. ter.")] the angle BCM being right, standing in a semicircle. Therefore the straight line BM too is, in square, four-thirds of the straight line BC; and so the square on the diameter BM is four-thirds of the square on the side BC. Wherefore, if from the square on the icosahedron's diameter, and so forth. Q.E.D.

Diagram. The icosahedron figure for Proposition 15, drawn as a hexagonal projection ABCDEFGH with interior points and the two parallel triangles BCG and EFH.

- A (top-left)
- B (top-right)
- F (far left)
- G (upper-center interior)
- K (right, between B and C)
- I (center)
- L (left-center interior)
- H (lower-center interior)
- C (right)
- E (bottom-left)
- D (bottom-center)

Diagram. The circle figure for the latter part of Proposition 15: a circle circumscribing the equilateral triangle BCG, with the diameter BM drawn vertically through the center K, and CG cut at N.

- B (top of circle)
- K (center)
- C (left, on the circle)

- G (right, on the circle)
- N (on CG, below center, where BM meets CG)
- M (bottom of circle, opposite B)

COROLLARY

From this it follows that the icosahedron's diameter is, in square, [equal to the sum of] two straight lines: namely the diameter of the cube inscribed in it, and the diameter of the circle surrounding the icosahedron's triangle. For it was shown that the square on the icosahedron's diameter BE equals the squares on the cube's diameter kL and on the diameter of the circle circumscribing the triangle BCG.

THEOR. 16. PROPOS. 16.

The side of a dodecahedron is the lesser segment of a straight line divided in extreme and mean ratio, which is, in square, double the side of the octahedron inscribed in the dodecahedron.

Since the diameter of the octahedron inscribed in the dodecahedron — being, in square, double the octahedron's side — [Margin: a — *Elements* XIII.14 ("14. tertijdec.")] joins the bisection points of opposite sides of the dodecahedron, as is evident from proposition 9 of book 15; and since the side of the dodecahedron is the lesser segment of the straight line joining those bisection points when it is divided in extreme and mean ratio, by the fourth corollary to proposition 17 of book 13; it is clear that the side of the dodecahedron is the lesser segment of that straight line — divided in extreme and mean ratio — which is, in square, double the side of the octahedron inscribed in it. For such a straight line joins the bisection points of opposite sides of the dodecahedron, and its lesser segment is indeed the side of the dodecahedron, by the aforesaid corollary. And so the side of the dodecahedron is the lesser segment of a straight line [divided] in extreme and mean ratio, and so forth. Q.E.D.

THEOR. 17. PROPOS. 17.

The diameter of an icosahedron is, in square, both four-thirds of its own side and one-and-a-half times the side of the tetrahedron inscribed in it.

Since the cube and the tetrahedron inscribed in it are inscribed in the same icosahedron, as we taught in the scholium to the last proposition of book 15, and are contained by the same sphere, the diameter of the cube and the diameter of the sphere surrounding the tetrahedron will be the same. [Margin: a — *Elements*

XIII.13 ("13. tertijdec.")] Now the sphere's diameter is, in square, one-and-a-half times the tetrahedron's side. Therefore the diameter of the cube inscribed in the icosahedron — in which the tetrahedron too is inscribed — will also be, in square, one-and-a-half times the side of the tetrahedron inscribed in that cube and in the icosahedron. But the diameter of the circle circumscribing the icosahedron's face is, in square, four-thirds of the icosahedron's side, as we demonstrated in the proof of proposition 15 of this book. Therefore the icosahedron's diameter — which is, in square, [equal to the sum of] both the diameter of the cube inscribed in it and the diameter of the circle surrounding the icosahedron's face, by the corollary to proposition 15 of this book — is also, in square, both four-thirds of its own side (namely the diameter of the circle described about the icosahedron's face) and one-and-a-half times the side of the tetrahedron inscribed in it (namely the diameter of the cube). Q.E.D.

THEOR. 18. PROPOS. 18.

The side of a dodecahedron is to the side of the icosahedron inscribed in it as the lesser segment of the perpendicular line — drawn from one vertex of a pentagon to the opposite side and divided in extreme and mean ratio — is to that part of the same line lying between the pentagon's center and its side.

Let there be half a dodecahedron contained by six pentagons ABMLK, LKIHP, PHGFO, OFEDN, NDCBM, LMNOP, whose centers Q, R, S, T, V, X (each two neighboring ones, that is) are joined by the straight lines QR, RS, ST, TV, VQ, QX, RX, SX, TX, VX. Thus there are formed five triangles of the icosahedron inscribed in the dodecahedron, composing the solid angle X at the center of the pentagon LMNOP, as was shown in the proof of proposition 7 of book 15. From the two pentagon-vertices D, K, let straight lines DY, KY be dropped to the common base BM through the centers R, Q; and since these cut the side BM in half and at right angles, by the second corollary to proposition 10 of book 13, they will meet at the point Y. And if the straight line AL is drawn subtending the angle K, the perpendicular KY will be cut at Z in extreme and mean ratio. [Margin: a — *Elements* XVI.2 ("2. sextidec.")] I say that the dodecahedron's side LM is to the side QR of the icosahedron inscribed in it as KZ — the lesser segment of the perpendicular KY — is to the straight line QY intercepted between the center Q and the side BM.

For the straight line DK, drawn as a subtense — which, by what was demonstrated in the scholium to proposition 13 of book 15, is the side of the cube in which the given dodecahedron is inscribed — makes LM, the dodecahedron's

side, the lesser segment of DK, if DK is cut in extreme and mean ratio, by the same scholium to proposition 13 of book 15. Now since the sides YD, YK of the triangle YDK are cut proportionally at R and Q — for RD, QK are equal, being radii of equal circles, and likewise YR, YQ, being the distances of the straight line BM from the centers R and Q of those same equal circles — [Margin: b — *Elements* VI.2 ("2. sex.")] QR will be parallel to DK; and therefore the triangle YRQ is similar to the triangle YDK, by the corollary to proposition 4 of book 6. [Margin: c — *Elements* VI.4 ("4. sex.")] For this reason, as DK is to KY, so QR will be to QY; [Margin: d — *Elements* XIV.2 ("2. quartidec.")] but as the whole DK is to the whole KY, so is LM, the lesser segment, to the lesser [segment] KZ. Therefore, as LM is to KZ, so will QR be to QY; and alternately, as LM, the dodecahedron's side, is to QR, the side of the icosahedron inscribed in it, so is KZ, the lesser segment of the perpendicular KY, to QY, the part set between the center Q and the side BM. Therefore the side of the dodecahedron to the side of the icosahedron inscribed in it, and so forth. Q.E.D.

Diagram. The projection of half a dodecahedron with its inscribed icosahedron, for Proposition 18. The outer boundary is a ten-vertex figure (the six pentagons seen in plane), with a central pentagon LMNOP (center X) surrounded by the pentagon-centers Q, R, S, T, V that form the vertices of the inscribed icosahedron. This is a dense figure; positions are read from the plate as follows.

- A (top outer vertex)
- B (upper-left outer, below A)
- C (left outer)
- D (lower-left outer)
- E (bottom-left outer)
- F (bottom outer)
- G (bottom-right outer)
- H (right outer, lower)
- I (right outer)
- K (upper-right outer)
- L (central pentagon, upper-right)
- M (central pentagon, upper-left)
- N (central pentagon, lower-left)
- O (central pentagon, bottom)
- P (central pentagon, lower-right)
- Q (center of the top pentagon ABMLK; upper-center interior)

- R (center of a pentagon; left-center interior)
- S (center of a pentagon; lower-left-center interior)
- T (center of a pentagon; lower-right-center interior)
- V (center of a pentagon; right-center interior)
- X (center of the whole figure; center of pentagon LMNOP)
- Y (constructed point, upper-center, near Q)
- Z (constructed point on KY, near Q) [?]

THEOR. 19. PROPOS. 19.

If half the side of an icosahedron is cut in extreme and mean ratio, and its lesser segment is taken away from the whole side of the icosahedron, and from the remaining straight line a third part is again subtracted: there will be left the side of the dodecahedron inscribed in the icosahedron.

About the pentagon ABCDE, composed of five sides of the icosahedron, let a circle be described, whose center is F; and upon the five sides of the pentagon let five equilateral triangles be raised, forming the solid angle G of the icosahedron, by what was demonstrated in proposition 16 of book 13. Then in the said circle let an equilateral triangle AHI be inscribed. Now since the arcs ABC, AED are equal — each containing two-fifths of the whole circumference — and likewise the two arcs ABH, AEI, since they are thirds of the same circumference, the remaining arcs HC, ID will be equal; and therefore the straight lines HI, CD are parallel, by the scholium to proposition 27 of book 3. To these let a perpendicular FKL be drawn from the center F, [Margin: a — *Elements* III.3 ("3. ter.")] cutting the straight lines HI, CD in half at K and L; and let the straight lines FC, FD be joined. Let there further be taken the centers N and O of the equilateral triangles GBC, GCD, through which let the straight lines GNP, GOL be drawn, which — by the scholium to proposition 12 of book 13 — divide the opposite sides BC, CD in half and at right angles. And so the straight line GO will fall upon the point L, at which CD is bisected. Now let the centers N, O be joined by the straight line NO, which — as is established from what was demonstrated in proposition 5 of book 15 — will be the side of the dodeca

(Conclusion of the demonstration of Proposition 19: the side of the dodecahedron inscribed in the icosahedron.)

the dodecahedron inscribed in the icosahedron. Finally, let BP — half the side of the icosahedron — be cut at Q in extreme and mean ratio; and when the lesser segment BQ has been removed from the whole side BC, let a third part CR be

subtracted from the remainder CQ. I say that the remainder QR is equal to NO, the side of the dodecahedron.

For FL, the perpendicular from the side HI of the equilateral triangle, is cut at K in extreme and mean ratio — since FL cut in this way makes the greater segment equal to the perpendicular FK, by the corollary of proposition 2 of book 14. And KM is set equal to the greater segment FK, by the corollary of proposition 12 of book 13. It follows, from what we demonstrated at proposition 5 of book 13, that the greater segment KM is likewise cut at L, in extreme and mean ratio, by the lesser segment KL. Therefore the line FM exceeds the line FL by the segment LM, which is less than half of FM — that is, by a segment less than the line KM, as we showed.

Now PL, when joined, is the side of the equilateral and equiangular pentagon inscribed within the pentagon ABCDE, having the same center F, by the last scholium of book 4; and so it is similar to ABCDE. Hence the triangles FCD and FPL — if FP were drawn — will be similar, by the scholium of proposition 20 of book 6. Therefore, as FC is to CD, so FL is to LP; and, alternately, as FC is to FL, so CD is to LP. But FC — that is, FM, which is equal to it — exceeds the line FL by a segment less than half of FM, namely by a segment less than KM, as we showed. Therefore the line CD, or BC which is equal to it, will also exceed the line LP by a segment less than half of BC — namely by a segment less than the line BP, which was BQ; and consequently the remainder CQ will be equal to the line PL.

Again, since GP and GL are cut proportionally at the centers N, O — because GN and GO are double NP and OL respectively, by the corollary of proposition 18 of book 14 — [Margin: b — Elements VI.2] PL and NO will be parallel, and consequently the triangle GNO is similar to GPL, by the corollary of proposition 4 of book 6. [Margin: c — Elements VI.4] Therefore, as GP is to PL, so GN is to NO; and, alternately, as GP is to GN, so PL is to NO. But GP is one-and-a-half times GN, by the corollary of proposition 18 of book 14. Therefore PL — that is, CQ, which is equal to it — will also be one-and-a-half times NO. And since CQ is likewise one-and-a-half times the line QR — for of whatever parts CQ is 3, CR is 2, by construction, and therefore QR is 2 of those same parts — QR will be equal to NO, the side of the dodecahedron inscribed in the icosahedron. So then, if half the side of the icosahedron has been cut in extreme and mean ratio, and so on. Which was to be demonstrated.

Diagram. The figure for Proposition 19: a circle with an inscribed pentagon ABCDE and, within it, the construction that yields the dodecahedron side.

- A (top of circle): apex of the pentagon
- B (upper left, on circle): pentagon vertex
- E (upper right, on circle): pentagon vertex
- G (upper center, interior): solid angle / apex of the inner triangle
- N (left of center, interior): center of triangle GBC
- F (center): center of the circle
- O (right of center, below F): center of triangle GCD
- Q (left interior, below B): division point on the icosahedron side
- P (lower left, interior): division point
- H (left, on circle): base vertex of the inscribed triangle AHI
- R (below H, interior): division point
- K (low center, interior): foot on HI of the perpendicular FKL
- I (right, on circle): base vertex of the inscribed triangle AHI
- L (below K, interior): foot on CD of the perpendicular
- C (lower left, on circle): pentagon vertex
- D (lower right, on circle): pentagon vertex
- M (bottom, on circle): endpoint of the perpendicular extended

THEOR. 20. POPROS. 20.

(Heading printed "POPROS." — evidently a compositor error for "PROPOS." — see the anomalies note.)

The cube is triple the pyramid inscribed in it.

Let the pyramid ACEG be inscribed in the cube AF. I say that the cube is triple the pyramid. For the inscribed equilateral pyramid is surrounded by four pyramids, founded upon its four faces and having as their apexes the remaining four angles of the cube — those in which the four angles of the inscribed equilateral pyramid do not sit. (For upon the base ACE is founded the pyramid ACED; upon the base AEG, the pyramid AEGH; upon the base CEG, the pyramid CEGF; and upon the base ACG, the pyramid ACGB. These four pyramids are equal, by definition 10 of book 11, since they are contained by planes similar and equal both in number and in magnitude. For each consists of three equal and similar triangles — namely halves of the squares of the cube, as is clear — together with one triangle of the inscribed equilateral pyramid.)

Now let the cube be six times any one of these pyramids. (For example, the pyramid ABCG, having ABC as its base — half of BD, the base of the cube — and

the same height as the cube, namely the line BG, is a third part, by corollary 1 of proposition 7 of book 12, of the prism that has the same base and height. [Margin: a — Elements XI.32] Since, then, the cube is double this prism — because the base of the cube is double the base of the prism — it is manifest that the pyramid ABCG is a sixth part of the cube, and consequently that the cube is six times that pyramid.) It follows that the four surrounding pyramids are four sixth-parts, that is, two-thirds of the cube; and consequently that the remaining equilateral pyramid ACEG, inscribed in the cube, is a third part of the same cube. Therefore the cube is triple the pyramid inscribed in it. Which was to be demonstrated.

ANOTHER PROOF. Since the cube and the pyramid described within it are comprehended by the same sphere — as we taught in the scholium of the last proposition of book 15 — [Margin: b — Elements XIV.32] and since the cube is triple the pyramid described in the same sphere, as has been demonstrated, it is clearly gathered that the cube is triple the pyramid inscribed in it, seeing that this pyramid is contained in the same sphere as the cube.

Diagram. The figure for Proposition 20: the cube AF with the inscribed equilateral pyramid (tetrahedron) ACEG, its faces cut by the diagonals of the cube's faces.

- E (top left): cube vertex
- F (top right): cube vertex
- D (upper interior): cube vertex
- C (right): cube vertex
- G (center, interior): cube vertex
- H (left): cube vertex
- A (bottom left): cube vertex
- B (bottom right): cube vertex

THEOR. 21. PROPOS. 21.

The pyramid is double the octahedron inscribed in it.

In the pyramid ABCD let the octahedron EIKGHF be described. I say that the pyramid is double the octahedron. For the octahedron is surrounded by four pyramids similar to the pyramid ABCD — as was demonstrated at proposition 3 of book 15 — founded upon the four faces of the octahedron EGH, EFI, FGK, HIK, namely the pyramids EGHA, EFIB, FGKC, HIKD. (These are equal to one another, by definition 10 of book 11, since they are contained by triangles similar, and equal in number and in magnitude — being indeed similar to the bases of the

equilateral pyramid ABCD, by the corollary of proposition 4 of book 6, and having equal sides, namely the halves of the equal sides of the same pyramid ABCD, by construction —

and having the triplicate ratio of homologous sides, by proposition 2 of book 15. [Margin: a — Elements XII.8] Since, then, similar pyramids have the triplicate ratio of their homologous sides, the pyramid ABCD will be eight times any one of those four pyramids set around the octahedron — since an eightfold ratio is the duplicate of a double ratio, such as the sides of the pyramid ABCD have to the sides of the said four pyramids when tripled, as is plainly seen in these numbers 1. 2. 4. 8. That is, each of those four pyramids will be an eighth part of the pyramid ABCD; and therefore all four of the said pyramids will make four eighths, that is, half of the pyramid ABCD. Hence the remaining octahedron EIKGHF will be the other half of the same pyramid ABCD.) Therefore the pyramid is double the octahedron inscribed in it. Which was to be demonstrated.

Diagram. The figure for Proposition 21: the equilateral pyramid (tetrahedron) ABCD with the inscribed octahedron EIKGHF, drawn as a triangle ABC with the internal edges of the octahedron.

- A (top apex): pyramid vertex
- H (upper interior): octahedron vertex
- E (mid left): octahedron vertex
- G (mid right): octahedron vertex
- D (center, interior): pyramid vertex
- I (lower left, interior): octahedron vertex
- K (lower right, interior): octahedron vertex
- B (bottom left): pyramid vertex
- F (bottom center): octahedron vertex
- C (bottom right): pyramid vertex

THEOR. 22. PROPOS. 22.

The cube is six times the octahedron inscribed in it.

Since, if a pyramid be described in a cube, and an octahedron in the pyramid, the octahedron is likewise inscribed in the cube — as is clear from the second demonstration of proposition 3 of book 15 — and since the cube is triple the pyramid described in it [Margin: a — Elements XVI.20] while the pyramid is double the octahedron inscribed in it, [Margin: b — Elements XVI.21] it comes about that, of whatever parts the cube is 6, the pyramid is 2 and the octahedron 1;

and thus the cube is six times the octahedron. Therefore the cube is six times the octahedron inscribed in it. Which was to be demonstrated.

THEOR. 23. PROPOS. 23.

The octahedron is four-and-a-half times the cube inscribed in it.

In the octahedron $ABCDEF$ let the cube $GHikLMNO$ be described, setting its angles at the centers of the faces of the octahedron. I say that the octahedron is four-and-a-half times the cube. For, taking the pyramid $BEDFA$ of the octahedron, let there be drawn through G, k, I, H — the centers of the four faces of the octahedron — the lines PQ, QR, RS, SP , parallel to the sides BE, ED, DF, FB , forming the parallelogram-square $PQRS$, as was demonstrated at proposition 4 of book 15.

[Margin: a] This square will be parallel to the square $BEDF$. Let there also be drawn the diameter AC of the octahedron, perpendicular to the square $BEDF$, at the center T , by what we showed in corollary 1 of proposition 14 of book 13; and hence it is perpendicular to the faces GI, MO of the cube, at the points V, X , by the scholium of proposition 14 of book 11. Now V is the center of the square $PQRS$; [Margin: b] (for since the parallel planes $BEDF, PQRS$ cut the equal lines AB, AE, AD, AF proportionally, AP, AQ, AR, AS will also be equal. Let the lines VP, VQ, VR, VS be understood as drawn. [Margin: c] Then, since the equal squares of the equal lines AP, AQ are equal to the squares of the lines AV, VP and of AV, VQ — because the angles AVP, AVQ are right, by definition 3 of book 11 — the common square of the line AV being taken away, the squares of the lines VP, VQ will be equal, and therefore the lines VP, VQ themselves equal. In no other way we shall show that VR, VS are equal, and that these are equal to one another. Therefore V is the center of the square $PQRS$.) So then V will be the center of the square $GHik$ described in that plane, by the last scholium of book 4; for the sides PQ, QR, RS, SP are bisected at G, k, I, H , as is clear from what was demonstrated at proposition 4 of book 15.

By the same argument we shall conclude that X is the center of the base $LMNO$; and likewise that the other diameters drawn, BD and EF , pass through Y, Z, α, β , the centers of the remaining faces. Now, the line $A\beta$ having been drawn through G , the center of the equilateral triangle ABE , AG will be double $G\beta$, by the corollary of proposition 18 of book 14. [Margin: d] Since, then, the parallel planes PR and BD cut the lines $A\beta, AT$ proportionally, AV will also be double VT . In the same way it will be proved that CX is double XT ; and hence, since AT, CT are equal, so likewise are VT, XT . Therefore the line VX , joining the centers of the opposite faces of the cube, is bisected at T . And so, since in the same way YZ and $\alpha\beta$ are bisected at T , T will be the center of the cube, by corollary 1 of proposition

15 of book 13 — for the lines joining the centers of the opposite faces of the cube bisect one another at the center. Now since VX is double VT , and AV was likewise double VT — that is, of the height of the cube — the pyramid $GHkA$, when the lines AI , Ak , AH have been drawn, is (since the base GI is common to both and the height of $PQRSA$ is double that of the pyramid $GHkA$, and the base $PQRS$ is double the base GHk) double the pyramid $GHkA$, by corollary 1 of proposition 7 of book 12. [Margin: e] And the pyramid $PQRSA$ contains two-thirds of the cube GO . But since the pyramids $BEDFA$, $PQRSA$ are similar — by definition 9 of book 11, their triangles being similar, by the corollary of proposition 4 of book 6 — [Margin: f] they have the triplicate ratio of their homologous sides. [Margin: g] And the side AB is one-and-a-half times the side AP , because the line $A\beta$, cut proportionally, is one-and-a-half times AG .

Therefore the pyramid $BEDFA$ will be to the pyramid $PQRSA$ as 27 to 8; for this triplicate ratio is one-and-a-half — as appears in these numbers 8. 12. 18. 27. And consequently the whole octahedron $ABCDEF$, being double the pyramid $BEDFA$, will have to the same pyramid $PQRSA$ the ratio of 54 to 8. But it has already been demonstrated that, of whatever parts the pyramid $PQRSA$ is 8, the cube GO is 12 (for the number 8 contains two-thirds of the number 12). Therefore the octahedron $ABCDEF$ has to the cube GO the ratio of 54 to 12 — that is, in the least numbers, found by proposition 35 of book 7, the ratio of 9 to 2, which is the quadruple-plus-one-half ratio. And so the octahedron is four-and-a-half times the cube inscribed in it. Which was to be shown.

Diagram. The figure for Proposition 23: the octahedron $ABCDEF$ (drawn as a large rhombus, apexes A above, C below, B left, D right, with E and F the front/back apexes) enclosing the inscribed cube $GHkLMNO$, together with the auxiliary square $PQRS$ and the centers/diameters. The interior is very densely lettered and foxed; several labels cannot be placed with confidence. — more than two interior labels flagged [?].**

- A (top apex): octahedron vertex
- B (left apex): octahedron vertex
- D (right apex): octahedron vertex
- C (bottom apex): octahedron vertex
- E, F : front and back apexes of the octahedron (E interior upper-left region; F interior lower region) [?]
- P (upper left, interior): vertex of the auxiliary square $PQRS$
- R (upper right, interior): vertex of the auxiliary square

- Q, S: further vertices of the square PQRS (positions among the upper-center labels) [?]
- G, H, I, k: cube-face centers / cube vertices in the upper tier [?]
- L, M, N, O: cube vertices in the lower tier (M lower left, N bottom-center-left, O bottom right, L center-right) [?]
- T (center): center of the cube
- V (upper interior): center of square PQRS / GHlk
- X (lower interior): center of base LMNO
- Y, Z (interior, along the diameters BD/EF) [?]
- α , β , γ (Greek labels, interior — face-centers) [?]

COROLLARIVM I.

Therefore the octahedron and the cube inscribed in it have the same center; for we showed that T is the center of both.

COROLLARIVM II.

From this too we shall plainly infer that the octahedron has to the cube inscribed in it the same ratio that the squares of their sides have. [Margin: h — Elements XVI.11] For the squares of their sides have the quadruple-plus-one-half ratio, which is precisely the ratio we demonstrated the octahedron to have to the cube inscribed in it.

THEOR. 24. PROPOS. 24.

The octahedron is thirteen-and-a-half times the pyramid inscribed in it.

Since, as we showed at the end of book 15, a cube and a pyramid inscribed in it are described within the same octahedron, and since the octahedron is four-and-a-half times the cube described in it [Margin: a — Elements XVI.23] while the cube is triple the pyramid inscribed in it, and hence in the octahedron, [Margin: b — Elements XVI.20] it follows that, of whatever parts the pyramid is set as 2, the cube will be 6 and the octahedron 27. For 27 has to 6 the quadruple-plus-one-half ratio, and 6 to 2 the triple ratio. Therefore the octahedron has to the pyramid inscribed in it the ratio of 27 to 2 — that is, thirteen-and-a-half. And so the octahedron is thirteen-and-a-half times the pyramid inscribed in it, and so on. Which was to be shown.

THEOR. 25. PROPOS. 25.

The pyramid is nine times the cube inscribed in it.

In the pyramid ABCD let a cube be understood as described. I say that the pyramid is nine times the cube. For let G, H be taken, the centers of the two faces ACD, ABD; and to the common opposite side AD let the lines CGS, BHS be drawn, dividing the side AD in half and at right angles, by the scholium of proposition 12 of book 13, and therefore meeting at the midpoint S. Now the line GH having been joined — since BH, CG are double the lines HS, GS, by the corollary of proposition 18 of book 14, and are therefore cut proportionally at BS, CS — [Margin: a — Elements VI.2] BC and GS will be parallel, and, by the corollary of proposition 4 of book 6, the triangle HGS is similar to the triangle BCS. Therefore, as BC is to CS, so HG is to GS; and, alternately, as BC is to HG, so CS is to GS. [Margin: b — Elements VI.4] But CS is triple GS. Therefore BC will also be triple HG. [Margin: c — Elements XIV.18]

Now since the side of the pyramid described in the cube is the diameter of the base of the cube — as was demonstrated at proposition 1 of book 15 — while the line joining the two centers of the faces of the pyramid that have a common side is the diameter of the base of the cube described in the pyramid, as is clear from what was demonstrated at proposition 18 of book 15 (for there the line GH, joining the centers G, H of the bases ACD, ABD which have the common side AD, was the diameter of GNHQ, the base of the cube described in the pyramid): the side of the pyramid BC will be the diameter of the base of the cube circumscribing the pyramid ABCD, and GH the diameter of the base of the cube described in the same pyramid. Since, then, the diameters of the squares have the same ratio to one another as their sides — (for as the diameter of one square is to its side, so is the diameter of the other square to its side, because each ratio is a duplicate in power; alternately, then, as diameter is to diameter, so side is to side. Or thus: the square of the diameter of one square is to that square as the square of the diameter of the other is to that square, since in both the ratio is double. [Margin: d — Elements VI.22] Therefore diameter will also be to side as diameter to side; and so, alternately, as diameter to diameter, so side to side) — and since BC, the diameter of the base of the circumscribed cube, has to GH, the diameter of the base of the inscribed cube, a triple ratio, as we showed, the side of the circumscribed cube will likewise have to the side of the inscribed cube a triple ratio.

But cubes, being similar parallelepipeds, have the triplicate ratio of their homologous sides. [Margin: e — Elements XI.33] Therefore the circumscribed cube has to the cube inscribed in the pyramid ABDC the ratio of 27 to 1; for this triplicate ratio is the triplicate of a triple ratio, as may be seen here: 1. 3. 9. 27. For which reason, since the cube has to the inscribed pyramid the ratio of 27 to 9 —

that is, triple — [Margin: f — Elements XVI.20] the pyramid ABDC will have to the cube inscribed in it the ratio of 9 to 1, that is, ninefold; and so the pyramid is nine times the cube inscribed in it. Which was to be demonstrated.

Diagram. The figure for Proposition 25: the equilateral pyramid (tetrahedron) with apex B and base A, D, C, showing the inscribed cube and the auxiliary lines CGS, BHS meeting at S.

- B (top apex): pyramid vertex
- A (bottom left): pyramid vertex
- D (bottom right): pyramid vertex
- C (right): pyramid vertex
- H (mid-left, interior): center of face ABD
- G (center-low, interior): center of face ACD
- S (bottom center, on AD): midpoint of AD, meeting-point of CGS and BHS

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Therefore the cube has to the cube described in the pyramid — the one that is described in the prior cube — the ratio of 27 to 1. For this was demonstrated from proposition 33 of book 11 in the demonstration of the present proposition.

THEOR. 26. PROPOS. 26.

The octahedron has to the icosahedron inscribed in it the ratio that two bases of the octahedron have to five bases of the icosahedron.

In the octahedron ABCDEF imagine an icosahedron described. I say that the octahedron is to the icosahedron as two bases of the octahedron are to five bases of the icosahedron. For, as is clear from what was demonstrated in (Elements XV.16), the eight bases of the icosahedron are set upon the eight bases of the octahedron. Let two bases of the icosahedron, GHI and kLM, lie on two opposite bases of the octahedron, ADE and BCF. From N, the center of the octahedron — which is also the center of the icosahedron, since the center of both is the same, by (XV.16 coroll. 3) — let a perpendicular NO be drawn to the base ADE, falling on the center O of the triangle ADE, that is, of the circle circumscribing it, by (XIV.10 lemma 1 coroll.). This O is also the center of the triangle GHI, by (XV.16 coroll. 2).

Now since the opposite planes ADE and BCF are parallel, by (XIII.14 coroll. 4), the straight line ON, produced, will likewise be perpendicular to the plane BCF, by the scholium of (XI.14), and will therefore fall on its center P, which is also

common to the triangle kLM . Hence OP is the true altitude both of the octahedron and of the icosahedron inscribed in it.

Since the octahedron is divided into eight equal pyramids by straight lines drawn from the center N to all the vertices of the octahedron — pyramids whose altitude is NO , half the altitude of the octahedron — and likewise the icosahedron into 20 equal pyramids of the same altitude NO ; and since a prism is triple a pyramid having the same base and altitude as itself, by (XII.7 coroll. 1): the prism whose base is ADE and altitude ON will be equal to three pyramids of the octahedron. But a prism whose base is the same ADE , but whose altitude is OP , double the altitude ON , is double the prism whose base is likewise ADE but altitude ON , by what we showed in the scholium of (XII.14). Therefore the prism whose base is ADE and altitude OP is equal to six pyramids of the octahedron.

Consequently, to the remaining two pyramids of the octahedron, which make up a third part of those six, will be equal a third part of the prism set upon the base ADE and under the altitude OP ; namely the prism whose base is a third part of the triangle ADE and whose altitude is the same OP . For this prism is a third part of the prism upon the base ADE and under the altitude OP , since — by what we taught at (XII.7) — prisms of the same altitude are to one another as their bases. Therefore two prisms, one having base ADE and altitude OP , the other having as base a third part of the base ADE and the same altitude OP , are together equal to the octahedron $ABCDEF$.

In just the same way we shall demonstrate that the prism whose base is GHI and altitude OP is equal to six pyramids of the icosahedron, and therefore that triple this prism — namely the prism whose base is triple the base GHI and whose altitude is the same OP — is equal to 18 pyramids of the icosahedron. To the remaining two pyramids of the icosahedron, which make up a third part of six pyramids of that same icosahedron, will therefore be equal a third part of the prism set upon the base GHI and under the altitude OP , whose base is a third part of the triangle GHI and whose altitude is the same OP . And accordingly two prisms — one whose base is triple the triangle GHI and altitude OP , the other whose base is a third part of the base GHI and the same altitude OP — will together be equal to the icosahedron described in the said octahedron.

Now, since those two prisms equal to the octahedron are to these two prisms equal to the icosahedron — by what we showed at (XII.7) — as the bases of the prisms equal to the octahedron, containing four thirds of the triangle ADE (since one has for base ADE , that is, three thirds of ADE , and the other has for base a third part of ADE), are to the bases of the prisms equal to the icosahedron,

comprehending ten thirds of the triangle GHI (since one has as base triple the triangle GHI, namely nine thirds of GHI, and the other a third part of GHI): let it be that, as four thirds of the triangle ADE are [Margin: a — Elements V.15] to two thirds of ADE (namely half of the four thirds) so are five thirds of GHI to half of the ten thirds. And as two thirds of ADE are to five thirds of GHI, so also, by what we demonstrated in the scholium of (V.22), let six thirds of ADE (that is, two bases of the octahedron, making up triple the two thirds of one base) be to six thirds of GHI [recte: fifteen thirds of GHI] (that is, five bases of the icosahedron, comprehending triple the five thirds of one base, since they comprehend fifteen thirds). Then those two prisms equal to the octahedron — that is, the octahedron itself — will be to these two prisms equal to the icosahedron — namely to the icosahedron itself — as two bases of the octahedron are to five bases of the icosahedron. Therefore the octahedron has to the icosahedron inscribed in it the ratio that two bases of the octahedron have to five bases of the icosahedron. Which was to be demonstrated.

Diagram. A square figure representing the octahedron ABCDEF with an inscribed icosahedron, drawn with many diagonal chords.

- A (top-left corner): vertex of the octahedron
- D (top-right corner): vertex
- B (bottom-left corner): vertex
- C (bottom-right corner): vertex
- G (top edge, between A and D): vertex/point
- K (bottom edge, between B and C): point
- I (upper-left interior): icosahedron vertex
- O (upper interior, below G): center of triangle ADE
- H (upper interior, right of O): point
- E (upper-left interior, below D): vertex
- N (center): center of octahedron and icosahedron
- F (interior, right of center): vertex
- L (lower-left interior): point
- P/M (bottom-center interior, P and M coincident): center P of triangle BCF and vertex M

THEOR. 27. PROPOS. 27.

The icosahedron has to the dodecahedron inscribed in it a ratio compounded of the ratio of the edge of the icosahedron to the edge of the

cube described in the same sphere as the icosahedron, and of the triplicate ratio of that which the diameter of the icosahedron has to the straight line joining the centers of the opposite bases of the icosahedron.

Let the icosahedron ABC be presented, in which the straight line DE joins the centers of the opposite bases, and let its diameter be AC; let F be the edge of the icosahedron, and G the edge of the cube described in the same sphere as the icosahedron. I say that the ratio of the icosahedron ABC to the dodecahedron inscribed in it is compounded of the ratio of F, the edge of the icosahedron, to G, the edge of the cube, and of the triplicate ratio of the diameter AC to the straight line DE.

For since, by what was demonstrated at (XV.5), the dodecahedron described in the icosahedron sets its vertices at the centers of the bases of the icosahedron, and hence its opposite vertices at the centers of the opposite bases, DE will be the diameter of the dodecahedron described in the icosahedron ABC, since it joins its opposite vertices residing at D and E, the centers of the opposite bases of the icosahedron.

Now that three magnitudes have been posited — namely the icosahedron ABC; the dodecahedron of diameter AC, that is, described in the same sphere in which the icosahedron ABC is described (since the diameter of both is the same); and the dodecahedron of diameter DE, constructed within the icosahedron ABC — the ratio of the first magnitude, namely the icosahedron ABC, to the third, namely the dodecahedron of diameter DE inscribed in it, will be compounded of the ratios of the first to the second and of the second to the third, by what we have demonstrated on (Elements VI, def. 5); that is, of the ratios of the icosahedron ABC to the dodecahedron of diameter AC described in the same sphere as it, and of the dodecahedron of that same diameter AC to the dodecahedron of diameter DE described in the icosahedron ABC.

Since, then, the icosahedron ABC is to the dodecahedron of diameter AC comprehended in the same sphere as F, the edge of the icosahedron, is to G, the edge of the cube — [Margin: a — Elements XIV.11] (for the dodecahedron of diameter AC is to the icosahedron ABC of the same diameter as G, the edge of the cube, is to F, the edge of the icosahedron; therefore by inversion the icosahedron ABC will be to the dodecahedron of diameter AC as F, the edge of the icosahedron, is to G, the edge of the cube) — and since the dodecahedron of that same diameter AC has to the dodecahedron of diameter DE, described in the icosahedron ABC, a ratio triplicate of that of diameter AC to diameter DE, that is, to the straight line joining the centers of the opposite bases of the icosahedron, by (XII.17 coroll.): it

is manifest that the ratio of the icosahedron ABC to the dodecahedron of diameter DE inscribed in it is likewise compounded of the ratio of F, the edge of the icosahedron, to G, the edge of the cube, and of the triplicate of the ratio of diameter AC to the straight line DE. Therefore the icosahedron has to the dodecahedron inscribed in it a compounded ratio, and so on. Which was to be demonstrated.

Diagram. A hexagonal projection of the icosahedron ABC with the inscribed dodecahedron, drawn with internal chords, accompanied to its right by two vertical reference segments marked F and G (the edge of the icosahedron and the edge of the cube).

- B (top-left, on top edge): vertex of the icosahedron
- A (top, right of B): vertex
- D (interior, upper-left): center of one base (endpoint of the dodecahedron diameter)
- E (interior, lower-right): center of the opposite base (other endpoint of the diameter)
- Q (bottom-left corner): vertex
- F (right, vertical segment): the edge of the icosahedron
- G (right, vertical segment): the edge of the cube

THEOR. 28. PROPOS. 28.

The dodecahedron exceeds the cube inscribed in it by a parallelepiped, whose base falls short of the square of the cube by a rectangle contained by the edge of the cube and the third part of the minor segment of the same edge of the cube; but whose altitude falls short of the altitude — that is, of the edge of the cube — by the minor segment of that line which is the minor segment of the halved edge of the cube.

Since, by what was demonstrated at (XV.8), the four sides of the base of the cube described in the dodecahedron subtend four angles of four bases of the dodecahedron that meet at one edge: let there be pentagons of this kind — ABCDE, EFGHA, AHIKB, BKLMC — meeting at the edge AB; two of them, ABCDE and AHIKB, along that same common edge AB, and the other two, AEFGH and BCMLK, along the angles EAH and CBK. In these pentagons let there be inscribed the square ECKH of the cube, upon which stands a part of the dodecahedron that circumscribes the cube — namely the solid CBKHAE, consisting of five planes: CBK, EAH, ABCE, ABKH, and ECKH. Of these, the two triangles CBK and EAH [Margin: a — Elements I.4] are equal, since the sides BC, BK of the pentagon are

equal to the sides AE, AH of the pentagon, and they contain equal angles of the pentagons EAH and CBK. The two trapezia ABCE and ABKH are likewise equal: for since the pentagons ABCDE and ABKIH are equal by hypothesis, and the triangles CDE and HIK cut off from them are [Margin: b — Elements I.4] equal, the remaining trapezia ABCE and ABKH will also be equal. The fifth plane, finally, is ECKH, the base of the inscribed cube. In the same way five other solids like and equal to this — by (XI def. 10) — will be found upon the remaining five bases of the cube, so that the dodecahedron exceeds the cube inscribed in it by six such solids.

I say, then, that this whole excess, made up of those six solids standing around the inscribed cube, is equal to a parallelepiped whose base falls short of CEHK, the base of the cube, by a rectangle contained by the edge of the cube CK and the third part of the minor segment of the same edge of the cube; while its altitude falls short of the altitude — that is, of the edge of the cube — by the minor segment of that line which is the minor segment of the halved edge of the cube.

For, the sides EH and CK being bisected at N and O, and the straight line NO being joined — which is parallel to the side AB of the pentagon, and onto which fall the perpendiculars AP, BQ dropped into the plane EK, as is clear from what was demonstrated at (XIII.17) (for the lines NO, AB, AP, BQ of this figure express the lines LN, VX, VR, XS of the figure of XIII.17) — each of AB and NO will be the major segment of the edge of the cube [recte: NO or HK will be the major segment], and each of AP, BQ will be equal to half the side AB of the pentagon, that is, half the major segment of the edge of the cube, as is clear from the same (XIII.17). And therefore, if through P and Q straight lines RS, TV be drawn parallel to the lines EH, CK, then RT and SV will also be the major segments of the sides EC, HK; and thus ER, CT together, and HS, KV together, are the minor segments of those same sides. Then, straight lines AR, AS, BT, BV being drawn, let two pyramids ERS^{HA} and CTV^{KB} be constructed, which by (XII.6 coroll.) will be equal to one another, since their bases ERS^H, TCK^V and their altitudes PA, QB are equal; and they will stand outside the solid ABTVSR.

Now since AS is equal and parallel to BV, and AR to BT [Margin: c — Elements I.33] — since they join the equal and parallel straight lines AB, SV, RT — the planes of the triangles ARS, BTV will be parallel [Margin: d — Elements XI.15], and thus equal to one another, by (I.8 coroll.), since the sides of one are equal to the sides of the other. Therefore the solid ABTVSR, contained by two opposite equal, similar, and parallel triangles ARS, BTV, and by three parallelograms RV, AT, AV, is a prism, by definition.

Now, since it has been shown that the straight lines HS, KV make up the minor segment of the side HK, let VX be made equal to them — namely double each of HS and KV — and let VY contain two thirds of VX, the minor segment of the side HK, while XY is its third part; and let XZ, Y α be drawn parallel to the line TV. Since, then, the pyramids ERS α HA, CTVKB have been shown to be equal, and the pyramid set upon the base TX and under the same altitude PA (or QB) is double each of them [Margin: e — Elements XII.6], because the base TX is [Margin: f — Elements VI.1] double the base CV (or RH), the pyramid TVXZB will be equal to both pyramids ERS α HA, CTVKB together. But the pyramid TVXZB contains two thirds of the prism whose base is the same TX, whose altitude is the same QB, but whose vertex is a straight line — namely the part of the straight line AB equal to VX — as we demonstrated in the scholium of (XII.7). Likewise, of that same prism whose base is TX, altitude QB, and vertex the part of AB equal to VX, the prism whose base is TY and altitude the same QB, with vertex the part of AB equal to VY, contains two thirds, by the scholium of (XI.25) [Margin: g — Elements VI.1], because the base TY comprehends two thirds of the base TX.

Therefore the prism whose base is TY and altitude QB, with vertex the part of AB equal to VY, will be equal to both pyramids ERS α HA, CTVKB. And accordingly this prism, added to the prism ABTVSR, will make a prism equal to the solid CB α HAE, having as base the rectangle contained by the edge of the cube TV and by the straight line composed of VY (comprehending two thirds of the minor segment VX) and VS (the major segment); and therefore falling short of the base of the cube by the rectangle α X, contained by the edge of the cube α Y and by XY, the third part of the minor segment VX. By a not dissimilar procedure we shall investigate five other prisms, similar and equal to the one just named, and to the remaining five solids set upon the remaining five bases of the cube. Six such prisms, then, will be equal to those six solids — that is, to the excess of the dodecahedron over the cube inscribed in it.

But since any two prisms of that kind, fitted to one another, compose a parallelepiped of the same altitude, from those six prisms three parallelepipeds will be constructed under the same altitude QB — which is equal to half the side AB of the pentagon, that is, half the major segment of the edge of the cube — and upon bases each falling short of the base of the cube by a rectangle contained by the edge of the cube and by the third part of the minor segment of that same edge of the cube. And accordingly, from these three parallelepipeds composed together, there will arise a parallelepiped equal to the same excess of the dodecahedron over the cube inscribed in it, whose base falls short of the base of the cube by a rectangle contained by the edge of the cube and by the third part of

the minor segment of that same edge of the cube, and whose altitude is equal to the three altitudes of the aforesaid three parallelepipeds — that is, to the altitude QB, which was equal to half the major segment of the edge of the cube, taken three times.

I say, then, that this altitude falls short of the altitude — that is, of the edge of the cube — by the minor segment of that line which is the minor segment of the halved edge of the cube. For let Ck, the edge of the cube, be divided in extreme and mean ratio at β ; likewise let C β , the major segment, and β k, the minor, be bisected at γ and δ ; and from β k let $\beta\epsilon$ be cut off equal to $\beta\gamma$ (or C γ), so that C γ (or $\gamma\beta$, or $\beta\epsilon$), half the major segment C β , is equal to QB, and C ϵ to the whole altitude of the aforesaid parallelepiped. Now a straight line equal to C γ , half the major segment, can be cut off from the minor segment β k, because β k is greater than half the major segment C β . For since Ck is to C β as C β is to β k, and Ck is less than double C β (because C β is its major segment), C β will also be less than double β k — that is, β k will be greater than half of C β .

Since, then, as the whole Ck is to the major segment C β , so is the major segment C β to the minor β k, it will also be [Margin: h — Elements V.15] that as CO, half of Ck, is to C γ (or $\beta\epsilon$), so is $\beta\delta$ [recte: $\beta\epsilon$ to $\beta\delta$, half of β k]. And accordingly, if CO be cut in extreme and mean ratio, its major segment will be C γ (or $\beta\epsilon$) and its minor $\beta\delta$. Therefore $\beta\delta$ too is the minor segment in extreme and mean ratio, as we demonstrated at (XIII.5); and the major segment will be $\beta\delta$, the minor $\delta\epsilon$. And therefore again the minor segment $\delta\epsilon$ will cut the straight line δ k in extreme and mean ratio, its major segment being $\beta\delta$ [recte: the major segment $\delta\epsilon$ equal to $\beta\delta$], and its minor segment ϵ k. Therefore C ϵ , the altitude of the aforesaid parallelepiped, falls short of the altitude — that is, of the edge of the cube Ck — by the minor segment ϵ k of the straight line δ k (or of the straight line $\beta\delta$ equal to it), which was shown to be the minor segment of the halved edge of the cube.

It is established, therefore, that the base of the parallelepiped by which the dodecahedron exceeds the cube inscribed in it falls short of the base of the cube by a rectangle contained by the edge of the cube and the third part of the minor segment of the same edge of the cube; while its altitude falls short of the altitude of the cube by the minor segment of that line which is the minor segment of the halved edge of the cube. Therefore the dodecahedron exceeds the cube inscribed in it by a parallelepiped, and so on. Which was to be demonstrated.

Diagram. A large hexagonal figure showing the cube ECKH inscribed in the dodecahedron, with the construction lines, pyramids, and the extreme-and-mean-

ratio division points of Prop. 28. The same figure is repeated at the top of the following page.

- F (top-left) and G (top-right): upper vertices of the outer hexagon
- D (far left) and I (far right): side vertices of the outer hexagon
- M (bottom-left) and L (bottom-right): lower vertices of the outer hexagon
- E, H (upper corners) and C, K (lower corners): the four corners of the inscribed cube-face ECKH
- N (midpoint of EH) and O (midpoint of CK): bisection points
- A (upper apex, below N) and B (lower apex): pyramid apices
- P (center, below A) and Q (below B): feet of the perpendiculars AP, BQ
- R (left) and S (right): points on line RS through P
- T (lower-left) and V (lower-right): points on line TV through Q
- X (right, below S) and Y (right, below X): construction points on the right
- Z (left) and α (left, below Z): construction points (endpoints of XZ, $Y\alpha$)
- γ , O, β , δ , ϵ (along the bottom side CK): division points of the edge Ck in extreme and mean ratio

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From this it is elicited that the dodecahedron falls short of double the inscribed cube by two parallelepipeds. Of one, the length is equal to the edge of the cube, the breadth is a third part of the minor segment of that same edge of the cube, and finally the altitude falls short of the edge of the cube by the minor segment of that line which is the minor segment of the halved edge of the cube. Of the other, both the length and the breadth are equal to the edge of the cube, while the altitude is the minor segment of that line which is the minor segment of the halved edge of the cube — so that the two altitudes together are equal to the altitude, that is, to the edge of the cube. For if the first parallelepiped be added to that parallelepiped by which we showed the cube to be surpassed by the dodecahedron, there will be made up a parallelepiped whose length and breadth are equal to the edge of the cube — having, indeed, the same base as the cube — while its altitude falls short of the edge of the cube by the minor segment of that line which is the minor segment of the halved [edge of the cube].

Diagram. A repetition of the large cube-in-dodecahedron construction figure from the previous page, printed again at the head of this page for reference while the demonstration and corollary continue. Same labeling as fig-p621-2.

- Outer hexagon: F, G (top), D, I (sides), M, L (bottom)

- Inscribed cube-face: E, H (upper), C, K (lower)
- Interior construction: N, A, R, P, S, Z, α , X, Y, B, T, V, Q, and the Greek division points γ , O, β , δ , ε along CK

is less than the edge of the cube by the [minor] segment — that is, by the height of the latter parallelepiped. Therefore, if to the parallelepiped now completed the latter parallelepiped is joined, the composite will be a parallelepiped exactly equal to the cube described within the dodecahedron, since both the base and the height of the parallelepiped are equal to the base and the height of the cube. And accordingly, in order that the dodecahedron may turn out to be double its inscribed cube, the two aforesaid parallelepipeds are lacking; and when these are added to the excess of the dodecahedron over the inscribed cube, a parallelepiped is constituted equal to the inscribed cube — indeed, a second cube — as has been demonstrated.

THEOR. 29. PROPOS. 29.

The dodecahedron has to the icosahedron inscribed in it a ratio compounded of the triplicate of the ratio which the diameter of the dodecahedron has to the straight line joining the centres of opposite bases of the dodecahedron, and of the ratio of the edge of the cube to the edge of the icosahedron described in the same sphere with the cube.

In the dodecahedron ABC let the diameter be AC, and let DE be the straight line joining the centres of the opposite bases; and let F be the edge of the cube, and G the edge of the icosahedron described in the same sphere with the cube. I say that the ratio of the dodecahedron ABC to the icosahedron inscribed in it is compounded of the triplicate of the ratio of the diameter AC to the straight line DE, and of the ratio of F, the edge of the cube, to G, the edge of the icosahedron. For since the angles of the icosahedron described in the dodecahedron are set at the centres of the bases of the dodecahedron — and hence the opposite angles at the centres of the opposite bases — the straight line DE joining the centres of the opposite bases of the dodecahedron will be the diameter of the inscribed icosahedron. And on that account a dodecahedron whose diameter is DE will be described in the same sphere with the icosahedron placed within the dodecahedron ABC, since they have the same diameter DE.

Now that the three magnitudes are grasped — namely the dodecahedron ABC; the dodecahedron of diameter DE; and the icosahedron of the same diameter DE described within the dodecahedron ABC — the ratio of the dodecahedron ABC, the first magnitude, to the icosahedron inscribed in it, the third magnitude, will

be compounded (by what we demonstrated at Definition 5 of Book 6) out of the ratios: of the dodecahedron ABC, the first magnitude, to the dodecahedron of diameter DE, the second magnitude; and of the dodecahedron described from the diameter DE, the second magnitude, to the icosahedron described within the dodecahedron ABC, the third magnitude. Since, then, the dodecahedron ABC has to the dodecahedron of diameter DE the triplicate ratio of the diameter AC to the diameter DE, by the corollary of Prop. 17 of Book 12 [Margin: a 11.quartidec.]; and since the dodecahedron of diameter DE has to the icosahedron of the same diameter DE — which is of course described within the dodecahedron ABC — the ratio which F, the edge of the cube, has to G, the edge of the icosahedron: it is plainly established that the ratio of the dodecahedron ABC to the icosahedron inscribed in it, whose diameter is DE, is likewise compounded of the triplicate of the ratio of the diameter AC to the diameter DE — namely the straight line joining the centres of the opposite bases of the dodecahedron — and of the ratio of F, the edge of the cube, to G, the edge of the icosahedron described in the same sphere with the cube. The dodecahedron, therefore, has to the icosahedron inscribed in it a compounded ratio, etc. Which was to be demonstrated.

Diagram. A roughly hexagonal outline representing the dodecahedron ABC seen along its axis, with an inner pentagon and radiating edges (the projection of the solid). To the right of it stand two separate short vertical parallel line-segments, representing the edge of the cube and the edge of the icosahedron.

- A (top): a vertex of the dodecahedron
- B (upper left): a vertex
- D (right): a vertex
- E (left): a vertex
- C (bottom): a vertex
- F (right, left-hand vertical segment): the edge of the cube
- G (right, right-hand vertical segment): the edge of the icosahedron

THEOR. 30. PROPOS. 30.

The dodecahedron contains two ninth parts of the pyramid [tetrahedron] in which it is described, less two parallelepipeds — of which the length of the one is equal to the edge of the cube described in the same pyramid, its breadth is equal to the third part of the minor segment of the edge of the same cube, and finally its height falls short of the edge of the same cube by the minor segment of that line which is the minor segment of the half-edge of the same cube; while of the other, both the length and the

breadth are equal to the edge of the aforesaid cube, but the height is the minor segment of that line which is the minor segment of the half-edge of the same cube — so that the heights of both together are equal to the height, or the edge, of the same cube.

Since, if a cube is described in the dodecahedron inscribed in the pyramid, this same cube is described in the same pyramid, as we set forth at the end of Book 15; and since the cube is a ninth part of the pyramid in which it is described [Margin: a 25.sextidecim.], the pyramid being ninefold its inscribed cube: it comes about that the doubled magnitude of the said cube contains two ninth parts of the pyramid surrounding the dodecahedron, or the cube. Therefore, since the dodecahedron is double its inscribed cube — by the corollary of Prop. 28 of this book — less those two parallelepipeds mentioned in the words of this proposition, it is not obscurely gathered that the dodecahedron inscribed in the pyramid contains two ninth parts of the pyramid itself, less the said two parallelepipeds; so that, if the said two parallelepipeds are added to the dodecahedron, then at last the magnitude arises that embraces two ninth parts of the pyramid. Wherefore the dodecahedron contains two ninth parts of the pyramid in which it is described, less two parallelepipeds — of which the length of the one is equal to the edge of the cube described in the same pyramid, etc. Which was to be demonstrated.

THEOR. 31. PROPOS. 31.

The octahedron exceeds the icosahedron inscribed in it by a parallelepiped whose base is the square of the edge of the icosahedron, and whose height is the greater segment of the semidiameter of the octahedron cut in extreme and mean ratio.

Let the octahedron in which the icosahedron is inscribed be ABCDEF — the figure of Prop. 16 of Book 15 being repeated, where it was demonstrated that the twenty triangles composing the inscribed icosahedron are IQR, KQR, GIL, GIM, GPS, HPS, NHK, OHK, RLO, SLO, PMN, QMN, GMP, HNP, KNQ, IMQ, GLS, HOS, KOR, ILR, of which the last eight are set upon the eight faces of the octahedron, cutting namely all its edges in extreme and mean ratio; and likewise that the straight line IT, perpendicular to BD, is equal to the greater segment XY of the semidiameter AX. I say that the octahedron ABCDEF exceeds the icosahedron inscribed in it by a parallelepiped whose base is the square of the edge QR of the icosahedron, and whose height is IT, the greater segment of the semidiameter AX.

For since upon the four faces of the octahedron DEA, DFA, DEC, DFC — those constituting the solid angle D of the octahedron — are set the four bases of the icosahedron IMQ, ILR, KNQ, KOR, there will remain, next to the solid angle D, in those same four faces of the octahedron, four triangles IQD, IRD, KQD, KRD; and if to these are added the two triangles of the icosahedron QIR, KQR, together with the common triangle DQR, there will be constituted two pyramids upon the common base DQR — namely IDQR, KDQR — equal to each other, by the corollary of Prop. 5 of Book 12, since their height is the same straight line IT, namely the greater segment of the semidiameter of the octahedron. (For just as AX is cut by the straight line GI at Y in extreme and mean ratio, so too is CX cut by the straight line HK at Z. Therefore, if from K a perpendicular is drawn to BD [Margin: a 34.prim.] — the height, namely, of the pyramid KDQR — it will be equal to the straight line ZX, the greater segment of the semidiameter CX, that is, to IT itself. These pyramids, indeed, stand outside the icosahedron described in the octahedron.) In like manner, near almost each of the remaining five angles of the octahedron, two pyramids are constituted, equal both to each other and to the two just described, outside the aforesaid icosahedron, so that the octahedron exceeds the icosahedron inscribed in it by those twelve equal pyramids.

Now since the triangle DQR is half of the square of the edge DQ — for DQ, DR are equal straight lines, namely the minor segments of the edges of the octahedron, and contain a right angle — the straight line QR will be the diagonal of that square; and accordingly the square of QR will be double the square of the edge DQ, and therefore quadruple the triangle DQR. Therefore the prism, or parallelepiped, whose base is the square of the straight line QR and whose height is IT, will likewise be quadruple the prism whose base is the triangle DQR and whose height is the same IT, by what we demonstrated at Prop. 7 of Book 12. But the prism whose base is DQR and whose height is IT, being triple the pyramid IDQR — by the first corollary of Prop. 7 of Book 12 — is equal to three of those twelve pyramids which we said stand around the icosahedron. Therefore the prism, or parallelepiped, whose base is the square of the straight line QR and whose height is IT, is likewise quadruple those three pyramids. Wherefore, since all twelve pyramids together surrounding the icosahedron are quadruple those same three pyramids, the prism, or parallelepiped, whose base is the square of the straight line QR — namely the edge of the icosahedron — and whose height is IT, the greater segment of the semidiameter AX, will also be equal to those twelve pyramids, that is, to the excess of the octahedron over the icosahedron inscribed in it. And accordingly the octahedron will exceed the icosahedron by the said prism, or parallelepiped. Therefore the octahedron exceeds the icosahedron

inscribed in it by a parallelepiped whose base is the square of the edge of the icosahedron, etc. Which was to be shown.

Diagram. A large octahedron drawn as a rhombus — with vertices B (left), D (right), A (top apex) and C (bottom apex), together with the two mid-plane vertices — inside which the inscribed icosahedron and its construction lines are drawn (the figure of Prop. 16 of Book 15, repeated). The interior is very densely lettered with the icosahedron's vertices and the division points; several of these small labels are crowded and foxed.

- A (top apex): vertex of the octahedron
- B (left): vertex of the octahedron
- D (right): vertex of the octahedron
- C [?] (bottom apex): vertex of the octahedron (the drawn glyph resembles "G"; read as C from the text, where CX is a semidiameter)
- G (upper-left, on edge BA): construction / icosahedron point
- I (upper-right): icosahedron vertex
- Y (upper-middle, on axis AX): division point (extreme-and-mean section of AX)
- M (middle, below Y): icosahedron vertex
- L (centre-left): icosahedron vertex
- E (centre): mid-plane vertex of the octahedron
- P (left of L): icosahedron vertex
- O (right, toward D): icosahedron vertex
- X (centre): centre of the octahedron
- T (centre-right): foot of the perpendicular IT
- N (below centre): icosahedron vertex [?]
- S (lower-left): icosahedron vertex
- F (lower-middle): mid-plane vertex of the octahedron
- R (lower-right): icosahedron vertex
- H (lower-left, toward B): icosahedron vertex
- K (lower-right): icosahedron vertex
- Z (bottom-middle): division point (extreme-and-mean section of CX)
- V [?] (near R): crowded — not securely read

COROLLARIUM

Hence it is inferred that the pyramid exceeds double the icosahedron inscribed in it by a parallelepiped whose base is the square of the edge of the icosahedron,

and whose height is the straight line joining the bisection-points of opposite edges of the icosahedron. For since it is established from what was demonstrated in Prop. 19 of Book 15 that, if the octahedron is inscribed in the pyramid and the icosahedron in this octahedron [Margin: b 31.sexidec.], and the octahedron exceeds the icosahedron inscribed in it by a parallelepiped whose base is the square of the edge of the icosahedron — namely of the straight line QR — and whose height is the greater segment of the semidiameter of the octahedron, namely the straight line IT, or YX: it comes about that the octahedron taken twice — namely the pyramid, in which it is inscribed [Margin: c 21.sexidec.] (since the pyramid is double the octahedron inscribed in it) — exceeds double the icosahedron in the octahedron, and hence also [double the icosahedron] described in the pyramid, by double that parallelepiped: namely by a parallelepiped whose base is the same square of the edge of the icosahedron, and whose height is the straight line YZ joining the bisection-points of the opposite edges GI, HK of the icosahedron. For this parallelepiped is double the former, by what we demonstrated in the scholium of Prop. 14 of Book 12, since the heights have a double ratio. Wherefore by this same parallelepiped — whose base is the square of the edge of the icosahedron, and whose height is the straight line joining the bisection-points of opposite edges of the icosahedron — the pyramid exceeds double the icosahedron inscribed in it.

ON THE DESCRIPTION OF THE FIVE REGULAR SOLIDS IN A GIVEN SPHERE, FROM PAPPUS OF ALEXANDRIA

Since Euclid in Book 13 constructed the five regular solids separately, outside a sphere, and then confirmed by a new demonstration that these same solids are comprehended by a sphere, it will not be beside the point to teach in what way Pappus of Alexandria describes those same five solids in a given sphere — especially since his method appears much clearer and easier than that construction prescribed by Euclid. But first, from the same Pappus, certain lemmas must be set out beforehand — some of which, even though they are demonstrated

by means of the spherical elements of Theodosius, yet not on that account can we be judged to pervert the order of Euclid, since the propositions of the spherical elements which we need here depend only on the first 19 propositions of Book 11 of Euclid, so that they can conveniently be understood before Book 12, and much more so before this construction of the regular solids. But in citing the propositions of Theodosius we shall follow the order of our own commentaries,

which we wrote upon those same elements. Let us therefore make our beginning here.

LEMMA I.

Given two parallel circles in a sphere, and a straight line given in one of them, to draw in the other a diameter parallel to this given straight line.

In a sphere let there be two parallel circles ABCD, EFG; and first, in ABCD, let the diameter AD be given, to which a parallel diameter is to be drawn in EFG. Let there be drawn — as we taught in the scholium of Prop. 18 of Book 11 — through the diameter AD a plane perpendicular to the circle ABCD [Margin: a 1.Theod.], making on the surface of the sphere the circle AEGD. Since, then, the circle AEGD cuts the circle ABCD in two and at right angles, the circle AEGD will be a great circle, and it will pass through the poles of the circle ABCD, by the scholium of Prop. 15 of Book 1 of Theodosius; and accordingly also through the poles of the circle EFG [Margin: b 1.2.Theod.], since parallel circles have the same poles [Margin: c 15.1.Theod.]; and therefore it will cut it in two. Hence the common section EG will be its diameter [Margin: d 16.vnde.]; and since this is parallel to AD, the proposition is evident.

Next, in the circle ABCD let the given straight line BC not be its diameter, to which a parallel diameter is to be drawn in the circle EFG. [Margin: e 31.primi.] Let there be drawn parallel to BC, through the centre of the circle ABCD, the diameter AD; to which, in the circle EFG, let the diameter EG be drawn parallel, as we have just shown. Since, then, each of BC, EG is parallel to the same AD, by construction [Margin: f 9.vndeci.], BC and EG will also be parallel to each other. Which was proposed.

By the same reasoning a straight line HI will be drawn in the circle EFG parallel to BC and distinct from the diameter EG, equal to BC — provided only that BC is not greater than the diameter EG. For if in the circle EFG there is drawn HI parallel and equal to BG [recte: BC?], as we set forth at Prop. 1 of Book 4, what is required will have been done.

Diagram. A circle representing the sphere in outline, containing two horizontal lens-shapes — the two parallel circles ABCD and EFG seen nearly edge-on.

- A (upper lens, left) and D (upper lens, right): endpoints of the diameter AD
- B (upper lens, inner left) and C (upper lens, inner right): endpoints of the chord BC
- E (lower lens, left) and G (lower lens, right): endpoints of the diameter EG

- H (lower lens, inner left) and I (lower lens, inner right): endpoints of the chord HI
- F (bottom of the circle): point on the outline

LEMMA II.

If in parallel planes two circles are described, in which two parallel straight lines cut off similar arcs, then the two straight lines joining the extremities of the one straight line with the centre will be parallel to the two straight lines which join the extremities of the other straight line with the centre.

In parallel planes AB, CD let two circles be described, in which let the straight lines EF, GH be first parallel, toward the same parts of the centres I, K; and let there be cut off the similar arcs EF, GH; and finally let there be joined to the centres the straight lines EI, FI, GK, HK. I say that both EI, GK and FI, HK are parallel. For since the arcs EF, GH are similar, the angles I, K will be equal, by the scholium of Prop. 22 of Book 3 [Margin: a 32.prim.]. Therefore the remaining angles E, F will be equal to the remaining angles G, H [Margin: b 5.primi.]. Since, then, both those and these are equal to each other, the angle E will be equal to the angle G, and the angle F to the angle H. Since, therefore, in parallel planes the straight lines EF, GH are parallel, and the straight lines EI, GK, drawn toward the same parts of the plane through EF, GH, make with them equal angles E, G, it will follow — from what we demonstrated in the scholium of Prop. 16 of Book 11 — that EI, GK are parallel; and by the same reasoning FI, HK will be parallel.

Next, let EF [printed DF] and GH be parallel, but not toward the same parts of the centres I, K. I say that both EI, HK and FI, GK are parallel. For, GK and HK being produced, let the straight line LM be joined. And since the sides GK, HK are equal to the sides LK, MK [Margin: c 14.prim.], and contain equal vertical angles at K [Margin: d 4.primi.], the bases GH, LM will be equal, and the angles H, L equal, and the angles G, M [Margin: e 28.tertij.]. Therefore the arcs GH, LM are equal [Margin: f 27.prim.], and the straight lines GH, LM parallel to each other; and accordingly, since each of EF, LM is parallel to GH [Margin: g 9.vndec.], EF and LM will also be parallel to each other. Since, then, EF, LM are parallel, cutting off toward the same parts of the centres the similar arcs EF, LM — for since the arc GH is similar to the arc EF, the arc LM also, which is equal to the arc GH, will be similar to the same arc EF — both EI, LK (that is, EI, HK) and FI, MK (that is, FI, GK) will be parallel. Which was proposed.

Diagram. Two stacked rectangles representing the two parallel planes, each containing lens-shaped circles with their radii drawn to the centre.

- A (upper rectangle, top-left corner) and B (upper rectangle, bottom-right corner): the plane AB
- Upper rectangle, left ellipse: E (left), F (right), with I at the centre (radii EI, FI drawn)
- Upper rectangle, right ellipse: E (left), F (right), with I at the centre
- C (lower rectangle, top-left corner) and D (lower rectangle, bottom-right corner): the plane CD
- Lower rectangle, left ellipse: G (left), H (right), with K at the centre (radii GK, HK drawn)
- Lower rectangle, right ellipse: L (top-left), M (top-right), G (bottom-left), H (bottom-right), with K at the centre (the crossing radii of the second construction)

LEMMA III.

If in a sphere there are two parallel and equal circles, in which are drawn two parallel and equal straight lines toward the same parts of the centres, then the straight lines joining the extreme points of these parallels toward the same parts are also equal and parallel, and perpendicular to the planes of the circles.

In a sphere let there be two parallel and equal circles AB, CD, in which let there be two equal and parallel straight lines AB, CD, toward the same parts of the centres, whose extremities are joined toward the same parts by the straight lines AC, BD. I say that the straight lines AC, BD are also equal, and parallel, and perpendicular to the planes of the circles. For since AB, CD are parallel [Margin: a 7.vnde.], AC, BD will be in the same plane with them. Wherefore, since AB, CD are also equal [Margin: b 33.primi.], AC, BD will likewise be equal and parallel — which is the first [part].

(This page concludes the demonstration of the preceding lemma — that in a sphere two equal parallel circles AB, CD, whose equal parallel chords AB, CD lie toward the same side of the centers, are joined at their ends by AC, BD, which are equal, parallel, and perpendicular to the planes of the circles.)

Diagram. A sphere (large circle) enclosing a rectangular box. The top face carries the chord/diameter AB, with A at the upper left and B at the upper right, and its center E marked just below the middle of that face. The lower face carries the

chord/diameter CD, with C at the lower left and D at the lower right, and its center F marked at the bottom. AC and BD join the corresponding ends; E and F are the centers of the two parallel equal circles.

- A (upper left): end of the upper chord AB
- B (upper right): end of the upper chord AB
- E (upper center): center of the upper circle
- C (lower left): end of the lower chord CD
- D (lower right): end of the lower chord CD
- F (bottom center): center of the lower circle

Next, from the centers E and F draw the lines EA, EB, FC, FD, and join the line EF. Then AE and CF, and likewise BE and DF, are parallel (by the second lemma), and they are also equal because the circles are equal. [Margin c: Elements I.33] Therefore AC and EF, and BD and EF, are parallel and equal. [Margin d: Theodosius, *Sphaerica* I.2 [?]] Now since the circles AB, CD are parallel about the same poles, [Margin e: Theodosius, *Sphaerica* I.10 [?]] the line drawn through their poles will pass through their centers and through the center of the sphere, and will stand perpendicular to their planes. Consequently the line EF, which joins their centers, will run through their poles and through the center of the sphere — otherwise two distinct straight lines would be said to pass through their centers, namely EF and the line extended through their poles, which is absurd, for then those two lines would have the segment EF in common. [Margin f: Theodosius, *Sphaerica* I.10 [?]] Therefore EF is perpendicular to the planes of the circles AB, CD; [Margin g: Elements XI.8] and for that reason each of AC, BD, being parallel to EF, will also be perpendicular to those same planes. This is the second point.

This demonstration also holds when the parallels AB, CD are diameters of the circles; but in that case EA, EB, FC, FD will be parts of the diameters, as is plain.

LEMMA IV.

If in a sphere there are two parallel and equal circles in which two parallel and equal straight lines are drawn, not toward the same sides of the centers, then: the straight lines that join the end points of these parallels **not** toward the same sides intersect at the center of the sphere, and are therefore diameters of the sphere and equal to one another. But the straight lines that connect the end points of the same parallels toward the same sides are both equal and parallel to one another, and make right angles with the parallels.

Let there be in a sphere two parallel and equal circles AB, CD, and in them two parallel and equal straight lines AB, CD, not toward the same sides of the centers. First let their end points be joined **not** toward the same sides, by the lines AD, BC. I say that AD, BC intersect at the center of the sphere, and are therefore diameters of the sphere and equal to one another.

Diagram. A sphere (circle) containing two parallel chords: the upper chord AB (with center E) near the top and the lower chord CD (with center F) near the bottom. The lines AD and BC are drawn as an X crossing at the point G in the middle, which is the center of the sphere. Radii and connecting lines EA, EG, FD, FG appear about the axis E–G–F.

- A (top left): end of upper chord
- E (top center): center of the upper circle
- B (top right): end of upper chord
- G (center): intersection of AD and BC, the center of the sphere
- C (bottom left): end of lower chord
- F (bottom center): center of the lower circle
- D (bottom right): end of lower chord

Since AB, CD are parallel, [Margin a: Elements XI.7] the lines AD, BC lie in the same plane with them; they will therefore cut one another, at the point H [recte: G] — which I say is the center of the sphere. For let the centers E, F of the circles be connected by the lines EA, EG and FD, FG. [Margin b: Elements I.15] Since the vertical angles AGB, CGD are equal, [Margin c: Elements I.29] and likewise both the alternate angles BAD, CDA and the alternate angles ABC, DCB are equal, the triangles ABG, CDG are equiangular. [Margin d: Elements VI.4 [?]] Therefore AB is to BG as DC is to CG, and likewise AB is to AG as DC is to DG. And since AB, CD are supposed equal, [Margin e: Elements V.14 [?]] both BG, CG and AG, DG are equal to one another. Again, since AE, DF, the semidiameters of equal circles, are equal, and are also parallel (by the second lemma), [Margin f: Elements I.29] the alternate angles EAD, FDA are equal; hence the two sides EA, AG equal the two sides FD, DG, and they contain equal angles. [Margin g: Elements I.4] Therefore the bases EG, FG, and the angles AGE, DGF, are equal. [Margin h: Elements XI.7] And since AD lies in the same plane in which the parallels AE, DF lie, [Margin i: Elements XI.2] EG and FG are set in one straight and continuous line, on account of the equal angles AGE, DGF, as we showed from Proclus on Proposition 15 of Book I. Hence the line EF, which joins the centers of the parallel circles, will pass through the center of the sphere, [Margin k: Theodosius, *Sphaerica* I.10] and will be perpendicular to their planes. Since EG, FG have been shown equal, G will be

the center of the sphere, [Margin l: Theodosius, *Sphaerica* I.6] because the equal circles AB, CD are equally distant from the center of the sphere; whereas they would be unequally distant if some point on the line EF other than G were said to be the center of the sphere. Therefore AD, BC are diameters of the sphere, and for that reason equal to one another. This is the first point.

But now let the end points of the equal parallels AB, CD be joined toward the same sides, by the lines AC, BD. I say that AC, BD are both equal and parallel to one another, and that the angles CAB, ABD, BDC, DCA are right angles.

Diagram. A sphere (circle) enclosing a rectangular figure. The upper chord AB has A at the upper left and B at the upper right; the lower chord CD has C at the lower left and D at the lower right. The perpendiculars CE, DF rise from C, D, with E and F marked in the middle band, and EF, AE, BF are joined, forming the right-angled figure.

- A (upper left): end of upper chord
- B (upper right): end of upper chord
- E (middle left): foot of the perpendicular from C's line
- F (middle right): foot of the perpendicular from D's line
- C (lower left): end of lower chord
- D (lower right): end of lower chord

That they are parallel and equal [Margin m: Elements I.33] is clear, since they join the end points of equal parallels. That they are also perpendicular to AB, CD is shown thus. In the circle CD, erect on the line CD the perpendiculars CE, DF, and join EF, AE, BF. Since the angles ECD, FDC are right, the arcs ECD, FDC are semicircles (by the scholium to Elements III.31), and consequently the remaining arcs EFD, CEF are also semicircles, [Margin n: Elements III.31] and the angles EFD, CEF are right. Hence in the right-angled parallelogram DE the lines CD, EF are equal, [Margin o: Elements XI.9] and therefore AB, EF are also parallel and equal. Therefore, by the third lemma, AE, BF are perpendicular to the planes of the circles, and consequently, by Elements XI, definition 3, the angles AEC, BFD are right. The line EF is therefore perpendicular to both EC and EA, and hence [Margin p: Elements XI.4] perpendicular to the plane of the triangle AEC drawn through EC, EA, and to the plane of the triangle BFD drawn through FD, FB; [Margin q: Elements XI.8] and for that reason CD, AB too, being parallel to EF, are perpendicular to those same planes of the triangles. Therefore, by Elements XI, definition 3, the angles CAB, ABD, BDC, DCA are right. This is the second point.

LEMMA V.

(printed "LEMA V." — single M)

If in a sphere there are two parallel straight lines, the straight lines joining their end points toward the same sides will be equal to one another. And if the parallels are equal, the joining lines will be not only equal but also parallel, and will make right angles with them.

Let there be in a sphere two parallels AB, CD, whose end points are connected toward the same sides by the lines AC, BD. I say that AC, BD are equal. For since AB, CD are parallel, [Margin a: Elements XI.7] the lines AC, BD lie in the same plane with them. Therefore, passing a plane through them, [Margin b: Theodosius, *Sphaerica* I.1] let there be made in the sphere the circle ABDC; and since AB, CD are parallel in it, the arcs AC, BD are equal (by the scholium to Elements III.27), [Margin c: Elements III.29] and therefore the lines AC, BD are equal as well. But if the parallels AB, CD are equal — so that the quadrilateral ABCD has in the circle two opposite sides AB, CD that are parallel and equal — then, by the scholium to Elements III.31, ABDC will be a right-angled parallelogram, and consequently the angles A, B, D, C will be right. This is what was proposed.

Diagram. Two circles side by side, illustrating the two cases. In the left circle an inscribed quadrilateral ABDC with A upper left, B upper right, C lower left, D lower right, forming a trapezium (the parallels unequal). In the right circle the same lettering A, B, C, D forms a rectangle/square (the parallels equal).

- Left circle — A (upper left), B (upper right), C (lower left), D (lower right): quadrilateral with parallel sides AB, CD
- Right circle — A (upper left), B (upper right), C (lower left), D (lower right): right-angled parallelogram

LEMMA VI.

In a given sphere, to describe two equal and parallel circles such that the diameter of the sphere is, in square, sesquialter to the diameter of each — that is, such that the square on the sphere's diameter is one-and-a-half times (the ratio 3 : 2) the square on each circle's diameter.

Let A be the center of the sphere, through which let a plane be drawn making the great circle BCDE, whose diameter is BD (by the corollary to Theodosius, *Sphaerica* I.1). [Margin: corollary, propos. 1, lib. 1, Theod.] Let BD be cut at F so that BF is double FD; then BD will be, of such parts as BF is 2, equal to 3, and

therefore BD is sesquialter to BF. Draw also FC perpendicular to BD, join the lines BC, DC, draw DE parallel to BC, and connect the line BE. [Margin a: Elements I.29] Since the two angles BCD, EDC together, and likewise the two BED, CBE together, equal two right angles, [Margin b: Elements III.31] and since each of BCD and BED is right, both EDC and CBE will also be right. Therefore, by the scholium to Elements I.34, the right-angled quadrilateral BCDE is a parallelogram, [Margin c: Elements I.34] and hence the lines BC, DE are equal. Now, the arc BC having been bisected at G, the arc EGD will likewise be bisected at G — since, by the scholium to Elements III.27, the arcs BE, CD are equal. Then, with pole G and radii GB, GE, let circles be described on the surface of the sphere passing through C, D, on account of the equality both of the arcs GB, GC and of the arcs GE, GD; [Margin d: Theodosius, *Sphaerica* II.2 [?]] and these circles will be parallel, since they have the same poles. [Margin e: Theodosius, *Sphaerica* I.15] And because the great circle BCDE cuts the circles BC, ED through their poles, bisecting them — and their diameters, having been shown equal, make the circles BC, ED equal as well. I say that the diameter of the sphere BD is, in square, sesquialter to each of the diameters BC, DE. For since, by the corollary to Elements VI.8, the three lines DB, BC, BF are continually proportional, then, by the corollary to Elements VI.20, as BD is to BF, so is the square on BD to the square on BC. But BD is sesquialter to BF, as was shown. Therefore the square on BD is sesquialter to the square on BC, and hence also to the square on DE. Accordingly the diameter of the sphere BD is, in square, sesquialter to each of the diameters BC, DE. This is what was proposed.

Diagram. The great circle BCDE, with C at the top, B at the left, D at the right, and E at the bottom. A is the center of the sphere, marked near the middle; F lies on the diameter BD just below A. G is marked at the upper left (the pole). The chords BC, CD, DE, EB form the inscribed quadrilateral, with the perpendicular FC and the lines BE, DC drawn.

- C (top): point on the great circle
- B (left): end of the diameter BD
- D (right): end of the diameter BD
- E (bottom): point on the great circle
- A (center): center of the sphere
- F (below center, on BD): point where BD is cut (BF double FD)
- G (upper left): pole of the described circles

PROBL. 1. PROPOS. 1.

In a given sphere, to describe a pyramid [tetrahedron].

By the sixth lemma, describe in the given sphere two equal and parallel circles AD, BC, the square of whose diameters is sesquialter to [is two-thirds of the square on] the sphere's diameter; and through their poles draw the great circle ABCD, [Margin a: Theodosius, *Sphaerica* I.15] which will cut them at right angles and bisect them, so that its common sections with them are the diameters AD, BC. In the circle AD draw another diameter EF, cutting AD at right angles at the center, and draw the lines EB, EC, FB, FC. I say that EBCF is a pyramid — that is, that the four triangles EBC, BCF, CFE, EFB are equilateral and equal.

For, joining the lines AE, ED, DF, FA, AB, CD, CA: AC will be a diameter of the sphere (by the fourth lemma); AB, DC will be equal and perpendicular to the planes of the circles (by the third lemma); and AEDF will be a square inscribed in the circle AD, as is established by Elements IV.6. Now, since by construction the square on the sphere's diameter AC is sesquialter to the square on AD, the diameter of the circle, then, setting the square on AC equal to 3, the square on AD will be 2. And since the square on AC [Margin b: Elements I.47] equals the squares on AD, DC (because the angle ADC is right, by Elements XI, definition 3), the square on DC will be 1; and consequently the square on AD is double the square on DF (by the scholium to Elements I.47). Therefore the squares on DC, DF are equal; [Margin c: Elements I.47] and since the square on CF equals them (because the angle CDF is right, by Elements XI, definition 3), the square on CF is likewise double the square on DF, and therefore equal to the square on AD. Hence the line CF equals the line AD — that is, equals the line BC; and similarly it will be shown that BF equals that same BC. Therefore the triangle BCF is equilateral.

Diagram. A circle with an inscribed pentagon-and-tetrahedron figure. E is at the top, A at the left, D at the right; F is marked at the center just below E; B is at the lower left and C at the lower right. The chords EA, ED, AD, and the lines EB, EC, FB, FC, BC (and AB, CD) are drawn, giving the four triangular faces of the tetrahedron EBCF.

- E (top): apex vertex of the tetrahedron
- A (left): point of the circle AD (end of diameter AD)
- D (right): end of diameter AD
- F (center, below E): center of the circle / point of diameter EF
- B (lower left): vertex of the tetrahedron
- C (lower right): vertex of the tetrahedron

Diagram. The same tetrahedron figure repeated at the head of the page for the continuation of the demonstration: a circle with E at the top, A at the left, D at the

right, F at the center below E, B at the lower left, and C at the lower right, with the edges of the tetrahedron EBCF drawn.

- E (top): apex vertex
- A (left): end of diameter AD
- D (right): end of diameter AD
- F (center, below E): center of the circle
- B (lower left): vertex
- C (lower right): vertex

Again, since the square on AD was shown to be double the square on DC, [Margin d: Elements I.47] and since (by the scholium to Elements I.47) it is also double the square on DE, the squares on DC, DE will be equal. And since the square on CE equals them (because the angle CDE is right, by Elements XI, definition 3), the square on CE will likewise be double the square on DE. Therefore the squares on CE, AD are equal, and for that reason the line CE equals the line AD — that is, equals the line BC, EF — and so also equals the line CF, which was shown equal to BC. The triangle CEF is therefore equilateral as well, and equal to BCF because of the equality of their sides. In the same way we shall demonstrate that the triangle BEF is equilateral and equal to BCF and FCE. From this it follows that the triangle BCE is also equilateral and equal to the other three, since its sides BE, CE were shown equal to BC. Therefore EBCF is a pyramid, by Elements XI, definition 26, and it is described in the given sphere, since its solid angles E, B, C, F lie on the surface of the sphere. This is what was proposed.

COROLLARIUM

From this it follows that the diameter of the sphere is, in square, sesquialter to the edge of the pyramid described in that sphere. For the edge of the pyramid equals the diameter AD, as we showed, and by construction the diameter of the sphere is, in square, sesquialter to the diameter AD.

PROBL. 2. PROPOS. 2.

In a given sphere, to describe an octahedron.

By the sixth lemma, describe in the given sphere two equal and parallel circles ABC, DEF, such that the sphere's diameter is, in square, sesquialter to the diameter of either of them. [Margin a: Elements IV.2] Then, an equilateral triangle ABC having been described in the circle ABC, draw (by the first lemma) in the circle DEF the line DE, parallel and equal to AB, toward the opposite sides of the centers; DE will also be a side of the equilateral triangle described in the

circle DEF, because the circles are equal. Therefore, having completed the equilateral triangle DEF, its several sides will be equal and parallel to the several sides of the triangle ABC — equal, on account of the equality of AB, DE; and parallel, by the scholium to Elements XI.16. For since AB, DE are parallel and the angles ABC, DEF are equal, then, producing the lines DE, FE to G, H, the angles ABC, GEH will also be equal. And since AB, GE are parallel, and BC, EH lie toward the same sides of the plane drawn through AB, GE, then BC, HE — that is, BC, EF — will also be parallel, by that scholium to Elements XI.16. By the same reasoning DF will be parallel to AC. Finally, connect AE, EC, CD, DB, BF, FA, as in the figure.

I say that AD is an octahedron — that is, that the eight triangles ABC, CAE, ECD, DCB, BFA, AFE, EFD, DFB are equilateral and equal. For since in the parallel and equal circles of the sphere the lines AB, DE are equal and parallel toward opposite sides of the centers, then, by the fourth lemma, the joined lines AD, BE (which, however, we have not drawn, to avoid confusion) are diameters of the sphere, and the lines AE, BD will also be equal and perpendicular to AB, DE themselves. [Margin b: Elements I.47] Therefore the square on BE equals the two squares on BD, DE. Now, setting the square on the sphere's diameter BE equal to 6 (for, since by construction the sphere's diameter is, in square, sesquialter to the diameter of the circle DEF, then, setting the square on BE equal to 6, the square on the diameter of the circle DEF will be 4; and, the square on the diameter of the circle DEF being 4, the square on DE, the side of the equilateral triangle, is 3 — because, by the corollary 1 to Elements XIII.12, the diameter of the circle is, in square, sesquitertian to [is four-thirds of the square on] the side of the equilateral triangle): therefore the square on BD is also 3; and consequently the squares on DE, BD are equal, and the lines DE, BD themselves are equal. But DE equals AB (by construction), and BD equals AE (by the fourth lemma). Therefore the four lines AB, BD, DE, EA are equal, and they constitute the square ABDE, since the angles at A, B, D, E are right, by the same fourth lemma.

Again, since BC, EF are parallel and equal toward opposite sides of the centers, then the joined lines BE, CF (not toward the same sides) will be diameters of the sphere, by the fourth lemma, and the lines BF, CE will be equal to one another and perpendicular to BC, EF; and again, as before, BCEF will be a square. In no other way we shall show that ACDF is a square, since the joined lines AD, CF are likewise diameters of the sphere by the fourth lemma, and so forth. Since then the three squares AD, BE, CF are constructed upon the equal sides AB, BC, CA — namely upon the sides of the equilateral triangle, AB being the side of the square AD, BC of the square BE, and CA of the square CF — all eight triangles ABC, CAE, ECD, DCB, BFA, AFE, EFD, DFB will be equilateral and equal to one another.

Therefore AD is an octahedron, by Elements XI, definition 27, and it is described in the given sphere, since its solid angles A, B, D, E, C, F touch the surface of the sphere. This is what was proposed.

Diagram. A circle enclosing an octahedron. A is at the upper left, B at the upper right, C in the upper-middle; D is at the lower right, E at the lower left, and F near the center-right. G and H are marked at the far left beside E (the points to which DE and FE are produced). The connecting lines AE, EC, CD, DB, BF, FA and the diagonals are drawn, giving the eight triangular faces.

- A (upper left): vertex of the octahedron
- B (upper right): vertex
- C (upper middle): vertex
- D (lower right): vertex
- E (lower left): vertex
- F (center right): vertex
- G (far left, above H): point to which DE is produced
- H (far left, below G): point to which FE is produced

COROLLARIVM

It is gathered from these things that the diameter of the sphere is, in square, double the edge of the octahedron described in that sphere. For it was demonstrated that the square on BE, the diameter of the sphere, is double the square on DE, the edge of the octahedron.

PROBL. 3. PROPOS. 3.

In a given sphere, to describe a cube.

BOOK XVI

(Continuation of the demonstration of PROBL. 3 / PROPOS. 3 — to describe a cube in a given sphere.)

By Lemma 6, let two equal and parallel circles ABCD and EFGH be described in the given sphere, the squares on whose diameters are in sesquialter ratio [3:2] to the square on the diameter of the sphere. [Margin: a — Elements IV.6] Now, with the square ABCD described in circle ABCD, let the straight line AD be drawn (by Lemma 1) in circle EFGH, parallel to EH and equal to it on the same side of the centers; because the circles are equal, this too will be the side of the square described in circle EFGH. When, therefore, the square EFGH has been completed,

its individual sides will be equal and parallel to the individual sides of square ABCD — equal because the sides AD and EH are equal, and parallel by the scholium to Elements XI.16. For since AD and EH are parallel, and the angles ADC and EHG are equal (both being right angles), DC and HG will likewise be parallel, since they are drawn toward the same parts of the plane through AD and EH; and so with the rest. Finally, let the straight lines AE, BF, CG, DH be joined toward the same parts of the centers; by Lemma 3 these will be perpendicular to the planes of the circles and parallel to one another.

I say that AG is a cube — that is, that the six quadrilaterals ABCD, EFGH, ADHE, BCGF, ABFE, CDHG are squares and equal to one another. For when the straight lines BH and BD are drawn, BH will (by Lemma 4) be the diameter of the sphere, while BD is the diameter of circle ABCD, being the diameter of the square inscribed in the circle. Since, then, by the construction, the square on BH is sesquialter to the square on BD, while the square on BD is double the square on BC (by the scholium to Elements I.47), [Margin: b — Elements I.47] if the square on BH is set at 3, the square on BD will be 2 and the square on DC will be 1; and so the square on BH will be triple the square on DC. Again, since the square on BH is 3 and the square on BD is 2, and since the square on BH is equal to the squares on BD and DH together (because the angle BDH is right, by Elements XI, def. 3), the square on DH will be 1. Therefore the square on BH is triple the square on DH. But the same square on BH was also triple the square on DC. Hence the squares on DC and DH are equal, and therefore the straight lines DC and DH are equal.

Since, then, these are equal to the opposite sides GH and CG, all four sides of the parallelogram CH will be equal; and since its angles are also right (by Elements XI, def. 3), CH will be a square. By the same reasoning BE, AH, and BG will be shown to be squares; and AC and EC are squares by construction. Therefore all six quadrilaterals are squares, and equal to one another because their sides are equal. So AG is a cube (by Elements XI, def. 25), described in the given sphere, since its solid angles A, B, C, D, E, F, G, H touch the surface of the sphere. Which was what was proposed.

Diagram. A cube inscribed in a circle (representing the sphere), drawn in oblique projection. The top face is bounded by the upper vertices and the bottom face by the lower vertices; the diagonals BD (of the top square) and BH (the sphere's diameter) are drawn.

- A (top-left): upper-back vertex
- D (top-right): upper-back vertex
- B (upper-left): upper-front vertex

- C (upper-right): upper-front vertex
- E (mid-left): lower-back vertex
- H (mid-right): lower-back vertex
- F (bottom-left): lower-front vertex
- G (bottom-right): lower-front vertex

COROLLARY

It is established from these results that the diameter of the sphere is triple in square [potentia tripla] to the side of the cube described in that sphere. For it has been demonstrated that the square on the diameter of the sphere BH is triple the square on the side of the cube DC, or DH, or AD, etc.

It is likewise manifest that one and the same circle contains both the square of the cube and the triangle of the octahedron described in the same sphere: namely, circle ABCD of the present figure is equal to circle ABC of the preceding proposition. This too was demonstrated in Elements XIV.21.

Diagram. An octahedron inscribed in a circle (the sphere), with its edges and internal diagonals drawn; carried over from the preceding proposition for the comparison stated in the corollary. The letters G and H are set close together at the left, marking one endpoint of a diameter/axis.

- A (upper-left): vertex
- B (upper-right): vertex
- C (upper-center): vertex
- E (lower-left): vertex
- F (lower-center): vertex
- D (lower-right): vertex
- G (left, upper of the pair): axis/diameter point
- H (left, lower of the pair): axis/diameter point

PROBL. 4. PROPOS. 4.

In a given sphere, to describe an icosahedron.

Let a straight line be found [Margin: a — Elements VI.12] to which the diameter of the given sphere has the same ratio as the side of a pentagon has to the side of a hexagon described in the same circle with the pentagon. And by the corollary to Elements X.6, let it be brought about that, as the number 3 is to 1, so the square on the straight line found is to the square on some other line; and about an interval equal to this other line, taken as a radius, let a circle be described in the

given sphere. This will be done easily, as follows. In the given sphere let a great circle be described, and in it [Margin: b — Elements V.1 [?] ("1.quinti"); context suggests IV.1] let a straight line be fitted that is double the radius found — for this straight line is less than the diameter of the sphere, or of that great circle, as will be proved below. For if through this straight line a plane is drawn at right angles to that great circle, it will make [Margin: c — Theodosius I.1] the circle ABC in the sphere, whose radius, by the construction, is subtriple in square to the straight line found at the outset.

Next, let another straight line be found [Margin: d — Elements VI.12] to which the diameter of the given sphere has the same ratio as the side of a pentagon has to the side of a decagon described in the same circle with the pentagon; and, by the corollary to Elements X.6, let it be brought about that, as 3 is to 1, so the square on the straight line found is to the square on another line; and about an interval equal to this let another circle be described in the sphere, parallel to the former circle ABC. This will come about if in that same great circle a straight line is fitted equal to double the radius found and parallel to the diameter of the former circle — from what we demonstrated at Elements IV.1 — and if through that straight line a plane is extended parallel to the former circle: that is,

that is, at right angles to the said great circle, from the scholium to Elements XI.18. For by the same scholium this plane will be parallel to circle ABC, and will make [Margin: e — Theodosius I.1] in the sphere a circle DEF, whose radius is equal to the radius found. On the other side of the center of the sphere let two other circles GHI and KLM be described, equal and parallel to these two; this will easily be done if straight lines equal and parallel to their diameters are drawn in that same great circle, etc. [Margin: f — Elements IV.2]

Now, with an equilateral triangle DEF described in circle DEF, let a straight line AB be drawn (by the first lemma) in the parallel circle ABC, on the other side of the center, parallel to the side EF and equal to one side of the equilateral triangle described in circle ABC. Likewise, parallel and equal to this AB, let KL be drawn on the other side of the center in circle KLM. And finally let HI be drawn, in circle GHI, on the other side of the center, parallel to this KL and equal to EF — that is, to the side of the equilateral triangle described in circle GHI. When, therefore, the equilateral triangles ABC, KLM, GHI have been completed, their individual sides will be parallel to the individual sides of triangle DEF — namely AC, ML, GH to DF, and BC, MK, GI to DE — as will be proved from the scholium to Elements XI.16, just as in the second problem we showed that BC and EF, drawn toward different parts of the centers, are parallel. In the figure we have not drawn those

triangles in circles ABC and KLM, lest the multitude of lines cause confusion, but have marked only the endpoints of their sides.

Finally, from each of the 12 points of the four equilateral triangles, let five straight lines be drawn to the five nearest points — namely, from point D the lines DE, DA, DM, DB, DF; from G the lines GH, GK, GC, GL, GI; and so on, as appears in the figure. For you see that from each point five straight lines are sent out. I say that DG is an icosahedron — that is, that the twenty triangles DEF, DEA, DAM, DMB, DFB, etc. are equilateral and equal. To demonstrate this, it must first be shown that the icosahedron described in the sphere (that an icosahedron can be described in a given sphere, Euclid taught at Elements XIII.16) sets its 12 angles in the four parallel circles which we described — namely three angles in each circle, and in that same position in which we marked the points.

Diagram. An icosahedron inscribed in a circle (the sphere), redrawn from the foot of the preceding page, densely crossed by the edges and the internal chords joining each vertex to its five neighbors. The twelve vertices named in the text are D, E, A, K, H, G, I, L, B, F, M, C. Legible label positions: D (top), a point at upper-left [K or E, ?], a point at upper-right [F or B, ?], A (left), B (right), C (center-left), M (center), a point at lower-left [?], L (lower-right), H (bottom-left), I (bottom-right), G (bottom). Several peripheral labels cannot be resolved with certainty against the crossing lines and foxing.**

Let there be, then, an icosahedron DG described in the sphere, setting its twelve angles at the points D, E, A, K, H, G, I, L, B, F, M, C. And since the five straight lines DE, DA, DM, DB, DF are equal, being sides of equal equilateral triangles, if from the pole D through E a circle is described in the sphere, it will pass through the remaining points A, M, B, F; and therefore the pentagon E, A, M, B, F will lie in one plane — namely in the plane of that circle. Since its sides are equal, being sides of the icosahedron, that pentagon will also be equiangular, from what we demonstrated at the end of Elements Book IV.

By the same reasoning the pentagon AKCFD will be equilateral and equiangular, because of the five equal straight lines EA, EK, EC, EF, ED; and KHMDE, because of the five equal sides of the icosahedron AK, AH, AM, AD, AE; and HGCEA, because of the five equal sides of the icosahedron KH, KG, KC, KE, KA; and GIMAK, because of the five equal sides of the icosahedron HG, HI, HM, HA, HK; and ILCKH, because of the five equal straight lines GI, GL, GC, GK, GH; and LBMHG, because of the five equal straight lines IL, IB, IM, IH, IG; and BFCGI, because of the five equal straight lines LB, LF, LC, LG, LI; and FDMIL, because of the five equal straight lines BF, BD, BM, BI, BL; and DECLB, because of the five equal straight

lines FD, FE, FC, FL, FB; and DAHIB, because of the five equal sides of the icosahedron MD, MA, MH, MI, MB; and finally GLFEK, because of the five equal sides of the icosahedron CG, CL, CF, CE, CK — so that 12 equilateral and equiangular pentagons, equal to one another, are constituted.

Now, since a straight line subtending an angle of an equilateral and equiangular pentagon is equidistant from [i.e., parallel to] the opposite side — as we demonstrated in the scholium to Elements XIII.8 — the line AB joined will be parallel to EF in the pentagon EAMBF; and AC joined will be parallel to DF in the pentagon AKCFD; and BC joined will be parallel to DE in the pentagon DECLB. [Margin: g — Elements XI.15] Therefore the planes drawn through the straight lines DE, EF and through CB, BA are parallel, [Margin: h — Theodosius I.1] and they make parallel circles in the sphere, DEF and ABC. In the same way KL joined will be parallel to HI in the pentagon ILCKH; and KM joined to GI in the pentagon GIMAK; and LM joined to GH in the pentagon LBMHG. Not otherwise, AB joined will be parallel to HI in the pentagon DAHIB, and KL joined to EF in the pentagon GLFEK. [Margin: i — Elements XI.9] Therefore EF and HI, since each is parallel to the same joined line AB or KL, are parallel to each other. And by the same reasoning DE and GI, since each is parallel to the same joined line BC or KM; and because of this the planes of the circles DEF and GHI, drawn through the straight lines DE, EF and HI, GI, will be parallel. Therefore all four circles in the sphere — DEF, GHI, ABC, KLM — are parallel to one another; and the circles DEF and GHI are equal to each other, because of the equal equilateral triangles described in them, just as the circles ABC and KLM are also equal to each other, because of the equal equilateral triangles in them, which are constituted from the joined lines AB, BC, CA and KL, LM, MK — since these sides are straight lines subtending the angles of equal pentagons.

Now, since the joined straight line KB — which connects the endpoints k and B of the parallels kL and AB (drawn, in parallel circles, not toward the same parts, and not toward the same parts of the centers) — is a diameter of the sphere, and the angle BAK is right (by Lemma 4, because kA connects the endpoints k and A of those same parallels toward the same parts), [Margin: k — Elements I.47] the square on the diameter kB will be equal to the two squares on AB and Ak together. Again, since in the pentagon EAMBF the joined straight line AB subtends the angle AMB — and since a straight line subtending an angle of a pentagon has to the side of the pentagon the same ratio as the side of a hexagon has to the side of a decagon described in the same circle, by the scholium to Elements XIII.2 [Margin: l — Elements XIII.10] — AB joined will be to EF (or to Ak, which equals EF) as the side of a hexagon is to the side of a decagon. And therefore, since the

side of a pentagon has the power of both the side of a hexagon and the side of a decagon described in the same circle, the diameter of the sphere kB (which has the power of both AB and Ak together) will be the side of a pentagon in the circle in which AB is the side of a hexagon and Ak , or EF , the side of a decagon. Hence the sphere's diameter has to the side of a pentagon the ratio that the side of a pentagon has to the side of a hexagon, but to EF the ratio that the side of a pentagon has to the side of a decagon. [Margin: m — Elements XIII.12] And accordingly, since AB — the side of the equilateral triangle in circle DEF — is triple in square to the radius of its own circle, the circles DEF , ABC , and therefore also GHI and kLM (which are equal to them), will be those which we described at the beginning of the proposition. From this it is clear that the straight line which is double the radius found at the outset is less than the diameter of the sphere, as we said above.

Having demonstrated these things, we shall easily conclude that the solid figure DG described by us is an icosahedron. For, taking the icosahedron described in the sphere whose base is the triangle DEF , all its solid angles fall on the three-point sets of the other three parallel circles from which we drew the sides of the triangles; and accordingly the individual sides of these coincide with the individual sides of that icosahedron.

COROLLARY I

It is established from these results that the diameter of the sphere has to the side of the icosahedron described in it the ratio that

the side of a pentagon has to the side of a decagon described in the same circle; so that, if the diameter of the sphere is the side of a pentagon in some circle, the side of the icosahedron will be the side of a decagon in that same circle. For, with the icosahedron DG described in the sphere, it was shown that the diameter of the sphere KB is to EF , the side of the icosahedron, as the side of a pentagon is to the side of a decagon.

From this it follows that the side of the icosahedron is greater than the radius of the sphere — just as the side of a decagon is greater than half the side of a pentagon, since two sides of a decagon together with one side of a pentagon constitute an isosceles triangle, [Margin: n — Elements I.20] and therefore two sides of a decagon are greater than the side of a pentagon, and hence one side of a decagon is greater than half the side of a pentagon.

COROLLARY

It is also gathered from what has been demonstrated that the diameter of the sphere is triple in square to the side of the pentagon described in circle ABC. For since, as the side of a pentagon is to the side of a hexagon described in the same circle ABC, so is the diameter of the sphere KB to AB joined, by the construction, it will be, alternately, as the side of a pentagon is to the diameter of the sphere, so the side of a hexagon to AB joined. [Margin: o — Elements XIII.12] But the side of a hexagon in circle ABC (that is, the radius) is subtriple in square to the joined straight line AB, because AB is the side of an equilateral triangle. Therefore the side of a pentagon in the same circle ABC is likewise subtriple in square to the diameter of the sphere — that is, the diameter of the sphere is triple in square to the side of the pentagon described in circle ABC.

PROBL. 5. PROPOS. 5.

In a given sphere, to describe a dodecahedron.

In the given sphere let four parallel circles be described, as in the preceding proposition — ABCDE, FGHIK, LMNOP, QRSTV — such that the radii of the circles ABCDE and QRSTV are subtriple in square to two straight lines to which the diameter of the sphere has the same ratio as the side of a pentagon has to the side of a decagon; while the radii of the circles FGHIK and LMNOP are subtriple in square to two straight lines to which the diameter of the sphere has the same ratio as the side of a pentagon has to the side of a hexagon. Now, with the equilateral and equiangular pentagon ABCDE described in circle ABCDE, let a line HI be drawn parallel to its side CD, toward the same parts of the centers, in circle FGHIK, equal to the side of the pentagon of that same circle, by the first lemma. In the same way let the straight line HI be drawn parallel toward the different parts of the centers as LP, in circle LMNOP; and finally let QV be drawn parallel to this LP, toward the same parts of the centers, in circle QRSTV. When the pentagons in these three circles have been completed (whose sides, however, we have not drawn, to avoid confusion, but have marked only their endpoints), their individual sides will be parallel to the individual sides of the pentagon ABCDE, in the same order in which the parallels to side CD were drawn — as will be demonstrated, just as in Problem 2 we showed BC and EF to be parallel; namely, the joined sides HG, PO, TV will be parallel to BC, and so on. Finally, from each of the 20 points of the four pentagons, let three straight lines be drawn to the three nearest points, as appears in the figure. For from each point you see three straight lines sent out.

I say that AS is a dodecahedron — that is, that the twelve pentagons ABCDE, DEKOI, OINST, STVQR, QRMGL, GLFAB, AFPKE, KPVTO, RSNHM, FLQVP, CDINH, BCHMG are equilateral, equiangular, and equal. To show this, it must first be demonstrated that the dodecahedron described in the sphere (that a dodecahedron can be described in a given sphere, Euclid taught at Elements XIII.17) sets its 20 angles in the four parallel circles which we described — five in each circle — and in that same position in which we marked the 20 points.

Let there be, then, a dodecahedron AS described in the sphere, setting its twenty angles at the points A, B, C, D, E, F, G, H, I, K, L, M, N, O, P, Q, R, S, T, V. [Margin: a — Elements XI.13 [?] ("13.unde.")] And since a straight line subtending an angle of an equilateral and equiangular pentagon is equidistant from [parallel to] the opposite side — as we demonstrated in the scholium to Elements XIII.8 — GH joined will be parallel to the side BC in the pentagon BCHMG; and HI joined to the side CD in the pentagon CDINH; and Ik to the side DE in the pentagon DEKOI; and kF to AE in the pentagon AEkPF; and FG to AB in the pentagon ABGLF; and so with the rest.

Since, then, the straight lines AB, AE are parallel to the straight lines FG, FK, [Margin: c — Elements [reference illegible, ?]] the plane drawn through FG, FK will also be parallel to the plane of the pentagon ABCDE drawn through AB, AE. By the same reasoning the plane drawn through FG, GH will be parallel to that same plane of the pentagon, because these lines are parallel to the lines AB, BC. Therefore, by the scholium to Elements XI.16, the planes drawn through FG, FK and through FG, GH are parallel to each other, since they are parallel to the same plane of the pentagon ABCDE; and accordingly, by the same scholium, since they meet in the straight line FG, they will make one plane. In just the same way we shall show that the plane through FK, KI and the plane through IH, HG make one plane with the plane drawn through FK, FG. Therefore the planes drawn through the points A, B, C, D, E and through the points F, G, H, I, K will make [Margin: d — Theodosius I.1] two parallel circles in the sphere, ABCDE and FGHIK. And in the same way the planes drawn through the points Q, R, S, T, V and through the points L, M, N, O, P will make parallel circles QRSTV and LMNOP; [Margin: e — Elements I.4] and both the two circles ABCDE, QRSTV and the two FGHIK, LMNOP will be equal to one another, because of the equal pentagons ABCDE, QRSTV, and because of the straight lines FG, GH, HI, IK, KF; LM, MN, NO, OP, PL, which are all equal, since they subtend equal angles in equal pentagons.

Next, since by the scholium to Elements XIII.8 the joined straight lines ND, MB are parallel to the side CH in the pentagons CHNID and BGMHC, [Margin: f —

Elements XI.9] the joined lines ND, MB will themselves also be parallel; and they are also equal, because they subtend equal angles in equal pentagons. [Margin: g — Elements I.33] Therefore the joined lines NM, BD will also be equal and parallel. In the same way the joined lines NO, CE will be equal and parallel, since they connect the two lines NC, OE, which are equal and parallel because each is parallel to the side DI. [Margin: h — Elements XI.15] Therefore circle LMNOP is parallel to circle ABCDE, since the former is drawn through the straight lines NM, NO, and the latter through the straight lines BD, CE, to which those have been shown parallel. Not otherwise will the circles QRSTV, FGHIK be shown parallel. Therefore all four circles are parallel.

Diagram. A dodecahedron inscribed in a sphere, drawn in oblique projection with the sphere's outline and the four parallel circles indicated; the twenty vertices are named A–V and connected by the edges and chords of the twelve pentagonal faces. Legible label positions:

- B (top-left), A (top-center), E (top-right)
- C (upper-left), D (upper-right)
- G (left), F (center), K (right)
- H (below G, left), I (below K, right)
- L (lower-left), P (lower-right)
- M (lower-left), O (lower-right)
- N (bottom-center, upper)
- Q (bottom-left), V (bottom-center)
- R (bottom-left, lowest), S (bottom-center, lowest), T (bottom-right, lowest)

(continuation of PROBL. 5 / PROPOS. 5 — inscribing a dodecahedron in a given sphere; conclusion of the demonstration)

Again, since the joined lines EF and OV, being (by the scholium of prop. 8 of book 13) equidistant from the side PK in the pentagons AEKPF and TVPKO, are parallel [Margin: a — Elements XI.9], and are moreover equal to one another — because they subtend equal angles in equal pentagons — the joined lines EO and FV will likewise be parallel and equal [Margin: b — Elements I.33]. Therefore the quadrilateral EFVO lies in one plane [Margin: c — Elements XI.7], since EF and OV are in the same plane as the parallels EO and FV. It also lies in a circle [Margin: d — Theodosius, *Sphaerica* I.1], because the plane in which it lies produces a circle in the sphere. Therefore, from what we demonstrated on prop. 31 of book 3, since the opposite sides of this quadrilateral are equal, the quadrilateral EFVO is a

rectangle; and consequently, since all its sides are equal — subtending equal angles in equal pentagons — it will be a square.

So, drawing its diagonal FO, the square on FO is double the square on the side FE (by the scholium of prop. 47 of book 1) — that is, double the square on the joined line FG, since FG and FE are equal, subtending angles of equal pentagons. And so, taking the square on FG to be 1 of some parts, the square on FO will be 2 of those parts, and both squares together 3 of the same parts. Now the square on the joined line OG — which is the diameter of the sphere — is equal to these [Margin: e — Elements I.47]. (For since the joined lines FG and ON are equal, subtending equal angles of the pentagons, and are parallel to one another — being parallel to the two parallels AB, ST, by the scholium of prop. 8 of book 13 — it follows, by lemma 4, that the joined line OF is perpendicular to FG, and that OG is the diameter of the sphere.) Hence the square on the diameter of the sphere is also 3 of the parts of which the square on FG is 1; that is, the square on the sphere's diameter is triple the square on FG.

But the square on the side of an equilateral triangle is likewise triple the square on the radius — that is, on the side of the hexagon [Margin: f — Elements XIII.12]. Therefore, as the square on the diameter of the sphere is to the square on FG, so is the square on the side of the triangle to the square on the side of the hexagon; and hence, as the diameter of the sphere is to FG, so is the side of the equilateral triangle to the side of the hexagon [Margin: g — Elements VI.22]. But as FG is to the side of the triangle in the same circle FGHIK, so is the side of the pentagon of that circle to the side of the triangle — since FG is the side of the pentagon. Therefore, by perturbed proportion, as the diameter of the sphere is to the side of the equilateral triangle in circle FGHIK, so is the side of the pentagon to the side of the hexagon, as appears in the adjoining scheme:

Diameter of the sphere	Side of the pentagon
Line FG	Side of the triangle
Side of the triangle	Side of the hexagon

Since, then, in the figure of the preceding proposition it is likewise the case that, as the diameter of the sphere is to the side of the triangle — that is, to the joined line AB in that figure — so is the side of the pentagon to the side of the hexagon, each of the circles ABC and KLM of that figure will be equal to each of the circles FGHIK and LMNOP of this figure.

Furthermore, since by the scholium of prop. 2 of book 14 it holds that, as the joined line FG (subtending the angle of the pentagon ABGLF) is to AB, the side of

the pentagon, so is the side of the hexagon to the side of the decagon; and as FG, the side of the pentagon in circle FGHIK, is to AB, the side of the pentagon in circle ABCDE, so is the side of the triangle in circle FGHIK to the side of the triangle in circle BCDE [recte: ABCDE?] — it will follow that, as the side of the triangle in circle FGHIK is to the side of the triangle in circle ABCDE, so is the side of the hexagon to the side of the decagon described in the same circle with the hexagon. But as the diameter of the sphere is to the side of the triangle in circle FGHIK, so is the side of the pentagon to the side of the hexagon described in the same circle with the pentagon, as was just demonstrated. Therefore, *ex aequali*, as the diameter of the sphere is to the side of the triangle in circle ABCDE, so is the side of the pentagon to the side of the decagon described in the same circle with the pentagon, as appears in this second scheme:

Diameter of the sphere	Side of the pentagon
Side of the triangle in circle FGHIK	Side of the hexagon
Side of the triangle in circle ABCDE	Side of the decagon

Since, then, in the figure of the preceding proposition it likewise holds that, as the diameter of the sphere is to EF, the side of the triangle in circle DEF, so is the side of the pentagon to the side of the decagon, each of the circles DEF and GHI of the preceding figure will be equal to each of the circles ABCDE and QRSTV of this figure. And so the circles ABCDE, FGHIK, LMNOP, QRSTV are those which we described in the preceding proposition and in this one.

With these things demonstrated, we shall prove without difficulty that the solid figure AS, which we have described, is a dodecahedron. For, once a dodecahedron has been described in the sphere with the pentagon ABCDE as its base, all its solid angles fall on the five points of each of the other three parallel circles, from which we drew out the sides of the pentagons; and consequently each of these sides coincides with each side of that dodecahedron.

COROLLARIUM.

It is clearly gathered from these results that one and the same circle encompasses both the triangle of the icosahedron and the pentagon of the dodecahedron, when they are described in the same sphere — namely, because the circle DEF of the preceding proposition's figure is equal to the circle ABCDE of this proposition's figure. This too has been demonstrated in book 14, prop. 5.

It is also manifest that the very same circles support both the solid angles of the icosahedron and the solid angles of the dodecahedron.

SCHOLIUM.

From what we have demonstrated here, and in books 13 and 14, we shall also easily show that, if all five regular solids are described in one and the same sphere, the greatest of all is the dodecahedron; next, the icosahedron is greater than the remaining three; third, the cube is greater than the remaining two; and finally the octahedron is greater than the tetrahedron. From this it will be established that Euclid constructed these five solids in the correct order, since after the tetrahedron he set down the octahedron immediately, and not the cube — for in this way he always advanced from the smaller to the greater. But first some other things must be demonstrated. Let us therefore begin with these.

I.

If a square and an equilateral triangle have equal sides, then twenty triangles together will be greater than eight squares together.

Diagram. A schematic proportion figure for the theorem. At the left stand two reference figures: a square labelled **A**, and an equilateral triangle labelled **B** with its altitude drawn from the apex to the base. To the right are two long horizontal rectangles, stacked one above the other:

- **A** — the square (side equal to the side of triangle **B**).
- **B** — the equilateral triangle, with an internal altitude.
- **C** — lower-left corner of the upper rectangle (rectangle **CE**).
- **D** — lower-right corner of the upper rectangle; **CD** is the base (eight times the side of square **A**).
- **E** — upper-right corner of the upper rectangle; **DE** is its height (equal to the side of square **A**).
- **F** — lower-left corner of the lower rectangle (rectangle **FH**).
- **G** — lower-right corner of the lower rectangle; **FG** is its base (ten times the side of triangle **B**).
- **H** — upper-right corner of the lower rectangle; **GH** is its height (equal to the altitude of triangle **B**).
- **I** — upper-left corner of the lower rectangle.
- **K** — a point on the right edge between **H** and **G**; the line **KI** is drawn parallel to the base **FG**.

Let there be a square **A** and an equilateral triangle **B** with equal sides. I say that twenty triangles are greater than eight squares. For let there be a rectangle **CE**, one side of which, **CD**, is eight times the side of the square **A**, and the side **DE**

equal to the side of the same square. Then the rectangle CE will be eight times the square A [Margin: h — Elements VI.1], since, on account of their equal height, as CD is to the side of the square A, so is the rectangle CE to the square A. Let there also be a rectangle FH, one side of which, FG, is ten times the side of the triangle B, and the side GH equal to the perpendicular dropped from one angle of the triangle to its base — that is,

— that is, equal to the altitude of the triangle B. And since a right-angled triangle, one side of which is twenty times the side of the triangle B, and the other side equal to GH itself (that is, to the altitude of the triangle B), is twenty times the triangle B [Margin: a — Elements VI.1]; and since the rectangle FH is equal to that right-angled triangle (by the scholium of prop. 41 of book 1), inasmuch as the base FG is half the base of that triangle; the rectangle FH will also be twenty times the triangle B. It must therefore be proved that the rectangle FH is greater than the rectangle CE [Margin: b — Elements XIV.12].

Since the side of the triangle B — that is, the side DE — is, in square [*potential*], four-thirds [*sesquitertium*] of its perpendicular, that is, of the side GH; while the side FG has to the side CD, in square, the ratio of 25 to 16 (since FG is 10 and CD is 8, that is, FG is 5 and CD is 4): the ratio of FG to CD will be greater than the ratio of DE to GH. For the ratio 25 to 16 — a ratio of one plus nine-sixteenths [*supernonupartiens sedecimas*] — is greater than 4 to 3, the sesquitertian ratio, since the fraction $9/16$ is greater than $1/3$. Let it be arranged that, as FG is to CD, so is DE to HK [Margin: c — Elements VI.12]; then the ratio of DE to HK will also be greater than that of DE to GH; and consequently GH will be greater than HK [Margin: d — Elements V.10]. So, drawing KI parallel to FG, it will hold that, as IK (equal to FG) is to CD, so is DE to HK. Therefore the rectangle IH will be equal to the rectangle CE [Margin: e — Elements VI.14 or VI.16]; and so EH [recte: FH?] is greater than CE. Which is what was proposed.

II.

Eight squares described on the radius of a sphere are equal to the surface of a cube described in that sphere.

Since two squares described on the diameter of a sphere are equal to the surface of a cube described in that sphere [Margin: f — Elements XIV.20]; and since these same two squares are equal to eight squares described on the radius of the sphere — because the square on the diameter is quadruple the square on the radius — it follows, by the scholium of prop. 4 of book 2, that eight squares on the

radius of the sphere are likewise equal to the surface of the cube of the same sphere. Which is what was proposed.

III.

If an octahedron, a cube, and an icosahedron are described in the same sphere, then the perpendicular drawn from the center of the sphere to one face of the icosahedron will be greater than the perpendicular drawn from the same center of the sphere to one face of the octahedron and of the cube.

For, since the edge of the octahedron is greater than the edge of the icosahedron in the same sphere [Margin: g — Elements XIII.18], one triangle of the octahedron will also be greater than one triangle of the icosahedron. Therefore, if both solids are so set within the sphere that a triangle of the one is parallel to a triangle of the other, the icosahedron's triangle will lie farther from the center than the octahedron's triangle, since each touches the surface of the sphere with its three angles. And consequently the perpendicular drawn from the center of the sphere to a face of the icosahedron will be greater than the perpendicular from the same center to a face of the octahedron. Since, then, the perpendicular of the octahedron is equal to the perpendicular of the cube — because the face of the cube and the face of the octahedron are circumscribed by the same circle, and so are equally distant from the center [Margin: h — Elements XIV.21] — the perpendicular of the icosahedron will also be greater than the perpendicular of the cube. Which is what was proposed.

IV.

If a cube and an icosahedron are described in the same sphere, the surface of the icosahedron is greater than the surface of the cube.

Since twenty equilateral triangles [constructed] on the radius of the sphere are greater than eight squares described on the radius of the same sphere, as we demonstrated in theorem 1; and since the twenty triangles of the icosahedron are greater than twenty equilateral triangles constructed on the radius of the sphere — because the edge of the icosahedron is greater than the radius of the sphere, by corollary 1 of the fourth problem above — the twenty triangles of the icosahedron are therefore much greater than eight squares described on the radius of the sphere. For this reason, since the twenty triangles of the icosahedron make up the whole surface of the icosahedron, and since eight squares described on the radius of the sphere are equal to the surface of the cube

(by theorem 2, demonstrated just above), the surface of the icosahedron will also be greater than the surface of the cube. Which is what was proposed.

DE COMPARATIONE SOLIDITATVM QVINQVE CORPORVM Regularium in eadem sphaera descriptorum.

(On the comparison of the solidities of the five regular bodies described in the same sphere.)

With these premises laid down, let us now demonstrate that the dodecahedron is greater than the icosahedron, the icosahedron greater than the cube, the cube greater than the octahedron, and finally the octahedron greater than the tetrahedron — which will be done as follows.

I.

DODECAEDRVM Icosaedro maius est.

The dodecahedron is greater than the icosahedron.

Since, as the edge of the cube is to the edge of the icosahedron, so is the dodecahedron to the icosahedron [Margin: i — Elements XIV.11]; and since the edge of the cube is greater than the edge of the icosahedron [Margin: k — Elements XIII.18]; the dodecahedron will also be greater than the icosahedron. Which is what was proposed.

II.

ICOSAEDRVM cubo maius est.

The icosahedron is greater than the cube.

Since the perpendicular falling from the center of the sphere onto one face of the icosahedron is greater than the perpendicular dropped from the same center onto one face of the cube (as was shown in the third theorem above); and likewise, by the fourth theorem, the surface of the icosahedron is also greater than the surface of the cube: the solid contained by the perpendicular falling from the center of the sphere onto one face of the icosahedron and by a third part of the surface of the icosahedron will be greater than the solid contained by the perpendicular falling from the same center of the sphere onto one face of the cube and by a third part of the surface of the cube. Since, then — as we demonstrated in the scholium of prop. 20 of book 14 — the former solid is equal to the icosahedron and the latter to the cube, the icosahedron will also be greater than the cube. Which is what was proposed.

III.

CVBVS Octaedro maius est.

The cube is greater than the octahedron.

Since, as the edge of the cube is to the radius of the sphere, so is the cube to the octahedron [Margin: a — Elements XIV.27]; and since the edge of the cube is greater than the radius of the sphere — (for, since the edge of the cube is greater than the edge of the icosahedron [Margin: b — Elements XIII.18], and the edge of the icosahedron, by corollary 1 of the fourth problem above, is greater than the radius of the sphere, the edge of the cube will be much greater than the radius of the sphere) — the cube will therefore also be greater than the octahedron. Which is what was proposed.

IV.

OCTAEDRVM Tetraedro maius est.

The octahedron is greater than the tetrahedron.

Since the face of the octahedron is smaller than the face of the tetrahedron — because the face of the tetrahedron has to the face of the octahedron the sesquitercian ratio [4:3] [Margin: a — Elements XIV.14] — the perpendicular dropped from the center of the sphere to one face of the octahedron will be greater than the perpendicular falling from the same center to one face of the tetrahedron; which will be proved in no other way than we showed above, in theorem 3, that the perpendicular in the icosahedron is greater than the perpendicular in the octahedron and in the cube. Moreover, the surface of the octahedron is also greater than the surface of the tetrahedron, since the former is one-and-a-half times [*sesquialtera*] the latter [Margin: b — Elements XIV.14]. Therefore the solid contained by the perpendicular falling from the center of the sphere onto one face of the octahedron and by a third part of the surface of the octahedron is greater than the solid contained by the perpendicular drawn from the same center of the sphere onto one face of the tetrahedron and by a third part of the surface of the tetrahedron. Wherefore, since — by the scholium of prop. 20 of book 14 — the former solid is equal to the octahedron and the latter to the tetrahedron, the octahedron will also be greater than the tetrahedron. Which is what was proposed.

ALITER. (*Otherwise.*) Since, as the edge of the octahedron is to the edge of the tetrahedron, so is the octahedron to triple the tetrahedron [Margin: c — Elements XIV.22]; and since, as the edge of the tetrahedron is to its third part, so is triple the tetrahedron to a third part of it — that is, to the tetrahedron itself: therefore,

ex aequali, it will also hold that, as the edge of the octahedron is to the third part of the edge of the tetrahedron, so is the octahedron to the tetrahedron. Since, then, the edge of the octahedron is greater than the third part of the edge of the tetrahedron — (for, because the square on the edge of the tetrahedron is four-thirds of the square on the edge of the octahedron [Margin: d — Elements XIII.18], the edge of the tetrahedron will have to the edge of the octahedron a ratio less than sesquitercian [4:3], and therefore much less than triple; and consequently the third part of the edge of the tetrahedron will be less than the edge of the octahedron [Margin: e — Elements V.10]) — the octahedron will also be greater than the tetrahedron. Which is what was proposed.

ALITER. (*Otherwise.*) Since the tetrahedron is to the octahedron as the rectangle — contained by a line whose square is $27/64$ of the square on the edge of the tetrahedron, and by a line containing [numeral: ?] of the edge of the tetrahedron [Margin: f — Elements XIV.17] — is to the square on the diameter of the sphere; and since that rectangle is less than the square on the diameter of the sphere, inasmuch as each of its lines is less than the edge of the tetrahedron (which is itself less than the diameter of the sphere): therefore the tetrahedron will also be less than the octahedron — that is, the octahedron will be greater than the tetrahedron. Which is what was proposed.

ALITER. (*Otherwise.*) Since the cube is triple the tetrahedron, but less than triple the octahedron [Margin: g — Elements XIV.32] — (for the cube is to the octahedron as the edge of the cube is to the radius of the sphere; and the edge of the cube is less than triple the radius of the sphere [Margin: h — Elements XIV.27], because even the whole diameter, which is greater than the edge of the cube, has to the radius a ratio less than triple) — the octahedron will be greater than the tetrahedron. Which is what was proposed.

In exactly the same order the convex surfaces of these same regular bodies exceed one another: that is, the convex surface of the dodecahedron is greater than the convex surface of the icosahedron; the convex surface of the icosahedron is greater than the convex surface of the cube; the convex surface of the cube is greater than that of the octahedron; and finally the convex surface of the octahedron is greater than the convex surface of the tetrahedron.

For since the surface of the dodecahedron is to the surface of the icosahedron as the edge of the cube is to the edge of the icosahedron [Margin: i — Elements XIV.9], and since the edge of the cube is greater than that of the icosahedron [Margin: k — Elements XIII.18] (as was demonstrated above), the surface of the dodecahedron will also be greater than the surface of the icosahedron. That the

surface of the icosahedron is greater than the surface of the cube was demonstrated a little earlier, in the fourth theorem before the comparison of the solidities of the five regular bodies. Next, since the surface of the cube is to the surface of the octahedron as the edge of the cube is to the radius of the sphere [Margin: l — Elements XIV.27], and since the edge of the cube is greater than the radius of the sphere (as was demonstrated in the third theorem, in which we showed the cube to be greater than the octahedron), the surface of the cube will also be greater than the surface of the octahedron. Finally, since the surface of the octahedron is one-and-a-half times the surface of the tetrahedron [Margin: m — Elements XIV.14], it is plain that the surface of the octahedron is greater than the surface of the tetrahedron.

You see, then, that the surfaces and solidities of the regular bodies correspond to their edges in the opposite order: so that the larger a body is, the greater indeed is its surface, but the smaller its edge. For, as was demonstrated above, the edge of the tetrahedron is the largest of all [Margin: n — Elements XIII.18], whereas its surface and solidity, as we have demonstrated here, are the smallest of all; likewise the edge of the dodecahedron is the smallest of all, whereas its convex surface and solidity are the greatest of all. Finally, just as — if the tetrahedron, octahedron, cube, icosahedron, and dodecahedron are described in the same sphere — their edges exceed one another in the order in which they have been set out and described by Euclid [Margin: o — Elements XIII.18] (as was shown above), so, in the opposite order, their convex surfaces and their solidities exceed one another, as we have demonstrated here.

For the rest, we must not pass over here what Pappus demonstrated in book 4 of the *Mathematical Collections*: namely that the solidities of the five regular bodies, if they are isoperimetric — and therefore not constructed in the same sphere — stand in a far different relation than the one we have demonstrated. For a body that has more faces is always greater than a body with fewer faces: so that the icosahedron exceeds the dodecahedron, the dodecahedron is greater than the octahedron, the octahedron exceeds the cube, and finally the cube is greater than the tetrahedron.

INVENTIO ANGVLI INCLINATIONIS DVARVM basium cuiuscunque solidi regularis, vnus ad alteram, ex Isidoro Hypsiclis praeceptore.

(The finding of the angle of inclination of two faces of any regular solid, of one to the other, from Isidore [Isidorus], the teacher of Hypsicles [Hypsicles of Alexandria].)

I.

To find the angle of inclination of the two faces of a pyramid [tetrahedron], of one to the other.

Let there be a pyramid ABCD. Its edge BD being bisected at E, from the angles A and C — which are opposite to the side BD — let the straight lines AE and CE be drawn. Since these are perpendicular to BD, the common section of the planes ABD and CBD in which they lie — (for in the triangles AEB and AED the angles at E are equal, and so on [Margin: a — Elements I.8]) — they will contain the angle of inclination AEC of the faces ABD and CBD, by definition 6 of book 11. We shall therefore display an angle equal to this in any plane, in the following way. Let FGH be taken: an equilateral triangle constructed on the straight line FG, equal to the edge of the pyramid, and therefore a triangle

[an equilateral triangle FGH] equal to any face of the proposed pyramid — namely to ABD — in which HI has been drawn perpendicular to FG. On FG let there be constructed the triangle FGK, having each of its sides FK, GK equal to the perpendicular HI — that is, equal to each of the perpendiculars AE, CE — since the perpendiculars HI, AE, CE are equal, lying as they do in equal triangles. I say that the angle FKG equals the inclination angle AEC of the faces ABD, CBD. For since the sides KF, KG equal the sides EA, EC, and the base FG equals the base AC, [Margin: a — Elements I.8] the angles FKG, AEC will be equal.

I say further that the inclination angle ACE is acute. [Margin: b — Elements XIV.12] For since the side AD, or its equal AC, is such that the square on it is one and a third [sesquitercian, 4:3] of the square on each of the perpendiculars AE, CE: of such parts as the square on the straight line AC is taken to be 4, the square on each of the straight lines AE, CE will be 3; and therefore both squares on the straight lines AE, CE together will be 6 of the same parts. Since then in triangle ABC [recte: AEC?] the square on the side AC is less than the squares on the sides AE, CE together, the angle AEC will be acute, as we demonstrated in the scholium to Proposition 13 of Book 2.

COROLLARIUM [Corollary]

We shall easily gather from this that all the inclinations of the faces of a pyramid are equal to one another. For since each of the inclination angles is contained by two perpendiculars drawn from the angles of the triangles to the opposite bases — such as AE, CE — and these are subtended by the edges of the pyramid, such as AC was, all the angles will be equal to one another, because all those

perpendiculars that contain them are equal, and likewise the edges of the pyramid that subtend the same angles are equal.

Diagram. The tetrahedron (pyramid) of the first problem. A triangular pyramid seen in oblique view.

- D (top): apex
- A (left): base vertex
- C (right): base vertex
- B (bottom): base vertex
- E (interior, center): midpoint of edge BD, foot of the perpendiculars AE, CE

Diagram. The plane construction (a rhombus formed of two triangles sharing base FG) displaying the inclination angle.

- K (top): apex of the isosceles triangle FGK (with $FK = GK = HI$)
- F (left): base vertex
- G (right): base vertex
- I (center): midpoint of FG; on the segment KI–HI
- H (bottom): apex of the lower equilateral triangle FGH (= a face of the pyramid), with altitude HI

II.

To find the angle of inclination of two faces of an octahedron, one to the other.

Let there be exhibited an octahedron ABCDEF, whose diameter is BD. From the angles B and D let straight lines BG, DG be drawn to the side AE — common to the triangles ABE, ADE and bisected at G. Since these are perpendicular to AE, the common section of the planes ABE, ADE in which they lie, [Margin: c — Elements I.8[?]] (because in the triangles BGA, BGE the angles at G are equal, and so on), they will contain the angle BGD, the inclination angle of the faces ABE, ADE, by Definition 6 of Book 11.

To this angle, therefore, we shall in any plane whatever exhibit an equal one, in this way. Take HIK, an equilateral triangle built upon the straight line HI equal to the edge of the octahedron — that is, equal to any triangle of the proposed octahedron, namely to ABE — in which the straight line KL is drawn perpendicular to HI. Upon a straight line MN equal to the diameter BD of the octahedron (that is, equal to the diameter of the square ABCD described on AB, the side of the octahedron) construct the triangle OMN, having each of its sides

MO, NO equal to the perpendicular KL — that is, equal to each of the perpendiculars BG, DG — since the perpendiculars KL, BG, DG are equal in equal triangles. I say that the angle MON equals the inclination angle BGD of the faces ABE, ADE. For since the sides OM, ON equal the sides GB, GD, and the base MN equals the base BD, [Margin: d — Elements I.8] the angles MON, BGD will be equal.

I say moreover that the inclination angle BGD is obtuse. [Margin: e — Elements XIII.14] For since the square on the diameter BD of the octahedron is double the square on the side AB, [Margin: f — Elements XIV.12[?]] while the square on the side AB is one and a third [sesquitertian, 4:3] of the square on the straight line BG: of such parts as the square on the diameter BD is taken to be 8, the square on the side AB will be 4, and the square on the straight line BG will be 3 of the same; and therefore both squares on the straight lines BG, DG together will be 6 of those parts. Hence the square on the side BD, in triangle GBD, is greater than the squares on the sides GB, GD together; and for that reason the angle BGD will be obtuse, as we showed in the scholium to Proposition 12 of Book 2.

Diagram. The octahedron of the second problem, drawn in oblique view with its interior diagonals.

- A (top): vertex
- C (bottom): vertex
- B (left): vertex
- D (right): vertex
- E, F (near center): the two remaining vertices (front and back)
- G (upper interior): point on the edge AE

Diagram. The plane construction for the octahedron: an inverted triangle HI with apex K, and, below it, the triangle MON.

- H (upper left): base vertex of the equilateral triangle
- L (upper center): foot of the perpendicular KL on HI
- I (upper right): base vertex
- K (below, apex): apex of the inverted triangle HIK
- O (lower, apex): apex of triangle MON
- M (lower left): base vertex
- N (lower right): base vertex

COROLLARIVM [Corollary]

Hence it follows that all the inclinations of the faces of an octahedron are equal. For since each of the inclination angles is contained by two perpendiculars drawn from the angles of the triangles to the opposite bases — such as BG, DG — and these are subtended by the diameters of the octahedron, such as BD is, [Margin: g — Elements I.8] all will be equal to one another, because all those perpendiculars that contain them are equal, and likewise the diameters of the octahedron that subtend the same angles are equal.

III.

To find the angle of inclination of two faces of a cube, one to the other.

Let there be a cube ABCDEFGH, in which the side DE is bisected at I, and likewise the sides opposite it — in the squares AE, EC, which have that side DE in common — are bisected at the points K, L. Then let the straight lines IK, IL be drawn. [Margin: h — Elements I.33] Since these are equal and parallel to the sides EF, EH (because both KF, IE are equal and parallel to one another, and likewise IE, LH are equal and parallel to one another), [Margin: i — Elements I.29] they therefore make right angles with the side DE, and so will contain the angle KIL, the inclination angle of the faces AE, EC, by Definition 6 of Book 11. And since the straight lines IK, IL, being parallel to the straight lines EF, EH but not in the same plane with them, [Margin: k — Elements XI.10] make the angle KIL e-

Diagram. The cube of the third problem, drawn in oblique view with the bisection points of three edges.

- F (top left): vertex
- E (top right): vertex
- A (upper left): vertex
- H (right): vertex
- D (interior, upper): vertex
- G (interior, lower): vertex
- B (bottom left): vertex
- C (bottom center): vertex
- K (upper edge, left): midpoint of an edge
- I (upper edge, right): midpoint of edge DE
- L (right edge, lower): midpoint of an edge

---equal to the right angle FEH, the inclination angle of the faces of the cube will be a right angle. Therefore, in order that an angle equal to the inclination angle of the faces of the cube may be exhibited in some plane, a right angle must be constructed. For this, as we have just demonstrated, will be equal to the inclination angle of the faces of the cube.

COROLLARIUM. I. [Corollary I]

Since by the same argument it can be demonstrated that the inclination angle of any two faces of a cube is a right angle, it is plainly concluded that all the inclinations of the faces of a cube are equal to one another.

IV.

To find the angle of inclination of two faces of an icosahedron, one to the other.

Let there be a pyramid ABCDEF, one of the 12 pyramids of the icosahedron, whose base is the pentagon ABCDE — composed of five edges of the icosahedron — and whose vertex is F. So, with the edge AF bisected at G, let straight lines BG, EG be drawn from the angles B and E (opposite to the side AF) of the triangles ABF, AEF, which have the side AF in common. Since, as follows from what was demonstrated in the second problem of this scholium, these are perpendicular to AF — the common section of the planes ABF, AEF in which they are drawn — they will contain the angle BGE, the inclination angle of the faces ABF, AEF, by Definition 6 of Book 11.

To this angle we shall in any plane whatever exhibit an equal one, in this way. Take an equilateral triangle HIK built upon the straight line HI equal to the edge of the icosahedron — that is, equal to any triangle of the icosahedron, namely to ABF itself — in which the straight line KL is drawn perpendicular to HI. Upon a straight line MN equal to the straight line BE (which subtends the angle A of the pentagon) construct the triangle OMN, having each of its sides MO, NO equal to the perpendicular KL — that is, equal to each of the perpendiculars BG, EG — since the perpendiculars KL, BG, EG are equal in equal triangles. I say that the angle MON equals the angle BGE, the inclination angle of the faces ABF, AEF. For since the sides OM, ON equal the sides GB, GE, and the base MN equals the base BE, [Margin: a — Elements I.8] the angles MON, BGE will be equal.

I say too that this inclination angle BGE is obtuse. For since BG makes the right angle AGB, [Margin: b — Elements I.19] the straight line AB will be greater than the straight line BG; and in the same way AE will be greater than EG. Therefore

the straight lines AB, AE will also be greater than the straight lines OM, ON, since these last were set equal to GB, GE. So if the straight line BE be supposed placed upon a straight line MN equal to it, the angle A will fall outside the triangle MON, in such a way that the straight lines AB, AE entirely enclose the straight lines OM, ON. [Margin: c — Elements I.21] The angle O, therefore, will be greater than the angle BAE. But this last is obtuse — for since the five angles of the pentagon equal six right angles, by what we demonstrated in the scholium to Proposition 32 of Book 1, each of them will comprise one right angle and, over and above, one fifth part of a right angle; being greater than a right angle, it is therefore obtuse. Much more, then, will the angle MON — that is, its equal BGE — be obtuse.

Diagram. The pentagonal pyramid (one of the twelve pyramids of the icosahedron).

- A (top): pentagon vertex
- B (left): pentagon vertex
- E (right): pentagon vertex
- C (lower left): pentagon vertex
- D (lower right): pentagon vertex
- F (center): apex of the pyramid
- G (upper interior): midpoint of edge AF

Diagram. The plane construction for the icosahedron: an inverted triangle HI with apex K, and, below it, the triangle MON.

- H (upper left): base vertex
- L (upper center): foot of the perpendicular KL
- I (upper right): base vertex
- K (below, apex): apex of the inverted triangle HIK
- O (lower, apex): apex of triangle MON
- M (lower left): base vertex
- N (lower right): base vertex

COROLLARIVM [Corollary]

It also follows, not obscurely, from these things that all the inclinations of the faces of an icosahedron are equal to one another. For since each of the inclination angles is contained by two perpendiculars drawn from the angles of the triangles to the opposite bases — such as BG, EG — and these are subtended by the straight lines that subtend the angles of the pentagons, such as BE is: [Margin: d —

Elements I.8] all will be equal to one another, because all those perpendiculars that contain them are equal, and likewise all those straight lines that subtend the angles of the pentagons are equal, [Margin: e — Elements I.4] being comprised by equal straight lines equal to equal ones; [Margin: e — Elements I.4] and so they will be equal.

V.

To find the angle of inclination of two faces of a dodecahedron, one to the other.

Since, by what we showed in Proposition 8 of Book 15, the four edges of the cube described in the dodecahedron subtend four angles of four faces of the dodecahedron meeting at one edge: let there be pentagons of this kind — ABCDE, EFGHA, AHIKB, BKLMC — meeting at the edge AB; two of them, namely ABCDE, AHIKB, along the same common edge AB; and the other two, AEF GH, BCMLK, along the angles EAH, CBK. In these pentagons, moreover, let there be inscribed the square ECKH of the cube, whose four edges subtend four angles of the pentagons, namely D, B, I, A. Now, with the edge AB bisected at N, let perpendiculars NO, NP be drawn from N, in the planes of the pentagons ABCDE, ABKIH, to AB, cutting the sides EC, HK at O and P at right angles as well — since the straight lines EC, HK are parallel to AB itself, by the scholium to Proposition 8 of Book 13. Therefore NO, NP contain the angle ONP, the inclination angle of the faces ABCDE, ABKIH.

To this angle we shall in any plane construct an equal one, in this way. Upon a straight line QR equal to the edge AB of the dodecahedron, let there be constructed an equilateral and equiangular pentagon QRSTV — that is, equal to any pentagon of the dodecahedron, namely to ABCDE itself — whose angle T is subtended by the straight line VS, [Margin: f — Elements I.4] which is equal to the straight line EC,

Diagram. The composite dodecahedron construction: four pentagons meeting at edge AB, with the inscribed cube-square, drawn as a hexagonal cluster crossed by diagonals.

- F (top left), G (top right): outer pentagon vertices
- E (upper left), H (upper right): vertices of the inscribed square ECKH
- A (center): shared vertex
- N (center): midpoint of edge AB

- D (left), O (left interior): D a pentagon vertex; O foot of perpendicular NO on EC
- α (left interior, below O): the point α (Greek alpha) used in the demonstration
- B (lower center): shared vertex
- P (right interior), I (far right): P foot of perpendicular NP on HK; I a pentagon vertex
- C (lower left), K (lower right): vertices of the inscribed square
- M (lower left), L (lower center): outer pentagon vertices

Diagram. The plane pentagon QRSTV of the construction, with interior points Y and Z.

- T (top): apex
- V (left): vertex
- S (right): vertex
- Q (lower left): vertex
- R (lower right): vertex
- X (bottom center): between Q and R (midpoint of QR)
- Y (center): interior point
- Z (upper center): apex of triangle VZS

---since the sides TV, TS equal the sides DC, DE and contain equal angles. Then, with the straight line QR bisected at X, let a perpendicular XY be drawn from the point X to QR; and this will also be perpendicular to VS, since VS is parallel to QR itself, by the scholium to Proposition 8 of Book 13. Finally, upon the straight line VS let a triangle VZS be built, having each of its sides ZV, ZS equal to the perpendicular XY.

I say that the angle VZS equals the inclination angle ONP. For when the straight lines NE, NH, VX have been drawn: since the sides EA, AN equal the sides HA, AN, and likewise the sides VQ, QX, and they contain equal angles, [Margin: a — Elements I.4] the bases NE, NH, VX will be equal, and so too the angles ANE, ANH, QXV; and consequently the remaining angles made with the perpendiculars — namely ENO, HNP, VXY — will be equal. Since, then, the angles ENO, EON of triangle ENO equal the angles HNP, HPN of triangle HNP, and also the angles VXY, VYX of triangle VXY (for EON, HPN, VYX have been shown to be right angles), and the side NE has been demonstrated equal to the sides NH, VX, [Margin: b — Elements I.26] the remaining sides NO, OE will be equal to the remaining sides

NP, PH and likewise to the sides XY, YV. Therefore the straight lines ZV, ZS, being equal to XY, will be equal to the straight lines NO, NP.

Now, since OE, PH are equal and parallel, [Margin: c — Elements I.33] the joining straight line OP will also be equal and parallel to EH, and therefore equal to EC — that is, to the straight line VS. For this reason, since the sides ZV, ZS equal the sides NO, NP, and the base VS equals the base OP, [Margin: d — Elements I.8] the angle VZS will be equal to the inclination angle ONP.

I say also that this inclination angle ONP is obtuse. For, a straight line $B\alpha$ having been drawn parallel to NO — since NB is likewise parallel to $O\alpha$, by the scholium to Proposition 8 of Book 13 — the figure $NO\alpha B$ will be a parallelogram, [Margin: e — Elements I.34] and therefore the straight line $B\alpha$ will be equal to NO. But since BC, the side of the pentagon opposite the angle $C\alpha B$ [Margin: f — Elements I.29] (which is equal, indeed, to the interior angle CON), is greater than the side $B\alpha$ — that is, than the straight line NO, or than each of ZV, ZS, which were shown to be equal to NO — the two straight lines TV, TS will be greater than the two straight lines ZV, ZS. Therefore the point Z falls inside the triangle TVS, in such a way that the straight lines TV, TS entirely enclose the straight lines ZV, ZS. And accordingly the angle VZS — and so its equal ONP — is greater than the angle T of the pentagon. But this last, being an angle of the pentagon, was shown a little before (when I was treating of the inclination of the faces of the icosahedron) to be greater than a right angle. Much more, then, will the inclination angle ONP be greater than a right angle, and so obtuse.

Diagram. The dodecahedron construction figure, repeated on this page: four pentagons meeting at edge AB with the inscribed cube-square, drawn as a hexagonal cluster crossed by diagonals.

- F (top left), G (top right): outer pentagon vertices
- E (upper left), H (upper right): vertices of the inscribed square ECKH
- A (center): shared vertex
- N (center): midpoint of edge AB
- D (left), O (left interior): D a pentagon vertex; O foot of perpendicular NO
- α (left interior, below O): the point α (Greek alpha)
- B (lower center): shared vertex
- P (right interior), I (far right): P foot of perpendicular NP; I a pentagon vertex
- C (lower left), K (lower right): vertices of the inscribed square
- M (lower left), L (lower center): outer pentagon vertices

Diagram. The plane pentagon QRSTV, repeated on this page, with interior points Y and Z.

- T (top): apex
- V (left): vertex
- S (right): vertex
- Q (lower left): vertex
- R (lower right): vertex
- X (bottom center): midpoint of QR
- Y (center): interior point
- Z (upper center): apex of triangle VZS

COROLLARIUM [Corollary]

It is therefore clear from these things that all the inclinations of the faces of a dodecahedron are equal to one another. For since each of the inclination angles is contained by two perpendiculars [Margin: g — Elements I.8] drawn from the midpoint of an edge of the dodecahedron to two edges of the cube described in the dodecahedron — such as NO, NP — and these are subtended by straight lines equal to the edges of the inscribed cube, or at any rate equal to the straight lines subtending the angles of the pentagons (of which sort is the straight line OP, equal to EH, the edge of the inscribed cube, which subtends the angle EAH of the pentagon): all will be equal to one another, since all those perpendiculars that contain them are equal, and likewise all those straight lines that subtend the same angles — namely the edges of the cube — are also equal to one another.

ELEMENTI DECIMISEXTI FINIS. [The End of the Sixteenth Element.]

QVID IN SINGVLIS XV. LIBRIS AGAT EVCLIDES. [What Euclid Treats in Each of the Fifteen Books]

EX FEDERICO COMMANDINO. [From Federico Commandino]

In the first book he discusses rectilinear figures — namely triangles and parallelograms. First he sets out the generation and properties of triangles, comparing them both with respect to their angles and with respect to their sides. Then, interposing the properties of parallels, he passes to parallelograms and explains their generation, and demonstrates the attributes belonging to them. Afterward he shows the mutual relation of triangles and parallelograms, and by

what means a parallelogram may be made equal to a triangle. Finally, concerning the squares described on the sides of right-angled triangles, he shows what proportion the square described on the side subtending the right angle has to those made on the sides comprising it.

In the second book the rectangular parallelogram and the gnomon are defined. Then the proportions of rectangular parallelograms and squares that arise from the sections of straight lines are set forth. Afterward, concerning the squares described on the sides of obtuse-angled and acute-angled triangles, he shows what proportion those made on the sides subtending the obtuse and the acute angle have to those described on the sides comprising them. Finally, by what method a square equal to a given rectilinear figure may be constructed.

In

(Commandino's synopsis, continued — "What Euclid treats in each of the fifteen books," from Federico Commandino. The previous portion summarized Books I–II; the account resumes here with Book III.)

In the **third** book he treats of the things that belong to circles, and of the straight lines drawn within a circle or to a circle; likewise of the angles that stand at the centers of circles or at their circumferences.

In the **fourth** book, of the inscription and circumscription of plane figures.

In the **fifth**, of proportions [analogiae].

In the **sixth**, of the proportions of figures to one another; of similar and reciprocal figures; of proportional straight lines; of the application of parallelograms to straight lines, such that they either fall short of the similar parallelograms or exceed them; how a given finite straight line may be cut in extreme and mean ratio; and of the proportions of arcs, of angles, and likewise of sectors, in equal circles.

In the **seventh** he treats of prime and composite numbers, and by what method the greatest common measure of numbers that are not prime is found; of a part and of parts of numbers; of multiple numbers; of proportionals; and whatever was demonstrated generally of magnitudes in the fifth book is here demonstrated in particular, and almost in the same way, of numbers.

In the **eighth**, of numbers continuously proportional; of plane numbers; of squares; of cubes and of solid numbers; of similar plane and similar solid numbers.

In the **ninth**, likewise of similar plane numbers; of cubes and solids; and of numbers continuously proportional, whether from unity or without qualification; of prime numbers; of even numbers; of odd; of even-times-even, of even-times-odd, and of odd-times-odd; and of perfect numbers.

In the **tenth**, of commensurable and incommensurable magnitudes, and likewise of rational and irrational ones.

In the **eleventh** he treats first of straight lines insofar as they bear on solid bodies: namely, when they lie in one plane; when they are erect, that is, perpendicular to a plane; when they are parallel; and how, from a given point set on high, a perpendicular is drawn to a plane. Then he discusses planes taken together, and next solid angles. Finally, of solid parallelepipeds, and some things about prisms.

In the **twelfth**, of pyramids and prisms; then of cones and cylinders; and lastly of spheres.

In the **thirteenth**, of the construction of the five cosmic figures, which they call the regular solids: namely the tetrahedron (or pyramid), the hexahedron (or cube), the octahedron, the dodecahedron, and the icosahedron. To make these clear he sets out first some things about what belongs to a straight line cut in extreme and mean ratio; about the equilateral pentagon; about the sides of the hexagon and the decagon; and about the equilateral triangle.

In the **fourteenth**, of the comparison of the dodecahedron and the icosahedron described in the same sphere.

In the **fifteenth** and last, of the inscription of the five figures already named, and of their sides and angles.

This figure was omitted at Book One, Proposition 24, Number 2.

Diagram. Two triangles standing side by side on a common base line, illustrating Euclid *Elements* I.24 (of two triangles with two sides respectively equal, the one with the greater included angle has the greater base). The left triangle is narrow and steep; the right triangle is broader, with an internal line drawn from its apex down to the base.

- A (apex of the left triangle)
- B (left foot of the left triangle's base)
- C (right foot of the left triangle's base)
- D (apex of the right triangle)

- E (left foot of the right triangle's base, adjacent to C)
- G (right foot of the right triangle's base)

[The right triangle has an internal segment descending from its apex D toward a point on the base near G; a further label at that lower point is too faint on the foxed sheet to resolve with certainty [?].]

THEODOSII TRIPOLITAE SPHAERICORVM LIBRI III

(Beginning of Clavius's commentary on Theodosius's Spherics — display title page of the Theodosius section.)

**THEODOSIUS OF TRIPOLI [Theodosius Tripolites] THE THREE BOOKS
OF THE SPHERICS [*Sphaericorum Libri III*]**

by

**CHRISTOPHER CLAVIUS [Christophorus Clavius] of Bamberg, of the
Society of Jesus,**

illustrated with lucid demonstrations and scholia.

And by the same author corrected and enlarged in this second edition.

MAINZ [Moguntiae], At the expense of **Antonius Hierat**; printed by
Reinhard Eltz [Reinhardus Eltz]. With the grace and privilege of His
Sacred Imperial Majesty.

In the year **1611 [M.DC.XI.]**

[Margin: two handwritten ownership inscriptions across the title page, in an old hand — at left "Monasterij [?]" ("of the monastery of..."), at right an inscription too cursive to resolve [?]. Not part of the printed text.]

(Verso of the Theodosius title page — blank. The leaf carries no printed text; only faint mirror-image show-through of the title-page setting and the IHS device is visible through the foxed parchment.)

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THEODOSIUS OF TRIPOLI, *SPHERICS*, IN THREE BOOKS

[Printed page number: 3.]

PREFACE

That there is a city named Tripolis both in Phoenicia and in Africa is attested by geographers and historians alike; but the sources give us no certain word as to which of these two cities was Theodosius's homeland. Neither is there sufficient agreement among them about the period in which he flourished. Still, it is no slight conjecture that he lived around the time of Pompey the Great [Pompeius Magnus], since Strabo writes that he flourished in Bithynia together with the physician Asclepiades [Asclepiades] — who, if we may believe Pliny [Pliny the Elder], flourished in Pompey the Great's day.

He wrote a number of mathematical treatises — *On Habitations* [*De Habitationibus*], *On Nights and Days* [*De Noctibus et Diebus*], and also these three books of *Spherics*, packed with the greatest learning. In them he demonstrates various properties of the sphere, knowledge of which is exceedingly necessary for mastering the doctrine of the heavenly bodies. Indeed, without these results astronomy has no way at all of maintaining its own dignity; and gnomonics too — the science of laying out sundials [horologia Solaria] — depends on them above all. Add to this that they are of no small importance for rightly understanding geography and perspective [optics] as well — and this is to pass over in silence, for now, the other uses of the elements of the sphere.

Now, since two versions of Theodosius's *Spherics* are in circulation — the one genuine and faithful, that of Joannes Pena [Jean Pena], corresponding word for word to the Greek exemplar; the other that of Francesco Maurolico [Francesco Maurolico], Abbot of Messina, from the tradition of the Arabs — we have followed the former, which is completed in fifty-nine propositions. We have inserted various scholia in which we demonstrate many theorems, necessary and pleasant to know, that Theodosius indeed omitted but the Arabs added. In the demonstrations, however, we have not followed the wording of the Greek codex but its sense, so that the demonstrations themselves might come out clearer. Here and there we have also added certain corollaries and scholia, as well as lemmas,

[Printed page number: 4.]

so that we may use them whenever the matter requires. In the margin, moreover, we have added numbers indicating the sequence of the propositions according to Maurolico's version, so that anyone may easily locate the propositions of Theodosius that some cite according to the order of the Arabs. As for the figures found in the Greek exemplar, we have for the most part passed them over, because those that Maurolico drew are more convenient, and much easier for grasping matters of the sphere. Finally, so that the flow of the demonstrations should not be interrupted, we have cited the propositions of Euclid and of these books in the margin — a practice we have observed in the following works as well. The citations are to be understood in the following way.

The table below pairs each marginal abbreviation (verbatim) with its full expansion.

Marginal form	Full expansion (Latin, verbatim)	Meaning
1. primi.	Prima propositio lib. 1. Euclid.	First proposition, Book 1 of Euclid
18. vndec.	Decimaoctaua propositio lib. 11. Euclid.	Eighteenth proposition, Book 11 of Euclid
Coroll. 16. tertij.	Corollarium propositionis sextædecimæ lib. 3. Euclid.	Corollary of the sixteenth proposition, Book 3 of Euclid
Coroll. 2. 33. sexti.	Corollarium secundum propositionis trigesimætertiae sexti libr. Euclid.	Second corollary of the thirty-third proposition, Book 6 of Euclid
Schol. 1. 2. octauu.	Scholium primum propositionis secundæ lib. 8. Euclid.	First scholium of the second proposition, Book 8 of Euclid
4. huius.	Propositio quarta huius lib.	Fourth proposition of this book
12. 2. huius.	Propositio duodecima libr. 2. huius operis.	Twelfth proposition, Book 2 of this work
Coroll. 10. huius.	Corollarium propositionis decimæ huius lib.	Corollary of the tenth proposition of this book
Coroll. 1. 1. huius.	Corollarium propositionis primæ lib. 1. huius operis.	Corollary of the first proposition, Book 1 of this

Marginal form	Full expansion (Latin, verbatim)	Meaning
		work
Schol. 15. huius.	Scholium propositionis quintædecimæ huius lib.	Scholium of the fifteenth proposition of this book
Schol. 15. 1. huius.	Scholium propositionis quintædecimæ lib. 1. huius operis.	Scholium of the fifteenth proposition, Book 1 of this work
20. 1. Theod.	Propositio vigesima lib. 1. Theod.	Twentieth proposition, Book 1 of Theodosius
Coroll. 16. 1. Theod.	Corollarium propositionis sextædecimæ lib. 1. Theodosii.	Corollary of the sixteenth proposition, Book 1 of Theodosius
Schol. 19. 1. Theod.	Scholium propositionis decimænonæ lib. 1. Theodosii.	Scholium of the nineteenth proposition, Book 1 of Theodosius

From these the other citations will be readily understood, since in all of them the principle is the same.

THEODOSIUS, *SPHERICS*, BOOK ONE

[Printed page number: 5.]

DEFINITIONS

I.

A sphere is a solid figure contained by a single surface, such that all straight lines drawn to it from one of the points that lie within the figure are equal to one another.

II.

That point, moreover, is the center of the sphere.

III.

The axis of the sphere is a certain straight line drawn through the center and terminated at both ends on the surface of the sphere, about which — remaining at rest — the sphere revolves.

IV.

The poles of the sphere are the end points of that axis.

V.

The pole of a circle on the sphere is a point on the surface of the sphere from which all straight lines running to the circumference of the circle are equal to one another.

SCHOLIUM

In the Greek exemplar there is added yet another definition, by which it is explained what it is for one plane to be similarly inclined to another plane, and one thing to another. But since the inclination of a plane to a plane has been explained by Euclid in Book 11, definition 6; and since what it is for one plane to be said similarly inclined to another plane, and one thing to another, has been set out in the same book, definition 7 — I have decided to omit that definition altogether in this place, and to put in its stead the next one, which is not unlike Euclid Book 3, definition 4, so that it occupies the sixth place.

VI.

Circles are said to be equally distant from the center of the sphere when the perpendiculars drawn from the center of the sphere to their planes are equal. And that circle is said to be farther off onto whose plane the greater perpendicular falls.

THEOREM 1. PROPOSITION 1.

If a spherical surface is cut by some plane, the line that is formed on the surface of the sphere is the circumference of a circle.

[Margin: 1.]

Let the spherical surface ABC be cut, its center being D, by some plane producing on the surface of the sphere the line BEFCG. I say that BEFCG is the circumference of a circle. For let the cutting plane first pass through the center D of the sphere, so that D lies in the cutting plane; and in that plane let there be drawn from D, to the line BEFCG that has been formed, any number of straight lines DE, DF, DG. Since, then, all these lines that have been drawn — however many there be — are equal to one another (because they fall from the center of the sphere to its surface), the line BEFCG will, by Euclid, Book 1, definition 15, be the circumference of a circle whose center D is the same as the center of the sphere.

[Margin: 15. primi — *Elements* I, def. 15]

Diagram. The first of two figures for Proposition 1: the case in which the cutting plane passes through the center. A circle (the outline of the sphere) with a great circle drawn through it as a lens/almond shape. Labelled points, read off the scan under magnification:

- A (top of the sphere's outline)
- B (left extremity, where the section-circle meets the outline)
- C (right extremity, where the section-circle meets the outline)
- G (upper point of the section-circle, above the center)
- D (center of the sphere and of the circle)
- E (lower-left point of the section-circle)
- F (lower-right point of the section-circle)

The straight lines DE, DF, DG are drawn from center D to the section curve; B–D–C runs across as a diameter of the section. The section line is lettered BEFCG.

Diagram. The second figure for Proposition 1: the case in which the cutting plane does *not* pass through the center. A full circle (the sphere's outline) with a small circle drawn near the bottom, and a perpendicular dropped from the center to that plane. Labelled points, read off the scan under magnification:

- A (top of the sphere's outline)
- D (the center, from which the construction lines descend)
- B (lower-left, on the sphere's outline where the small circle meets it)
- E (lower point on the small circle, left of center)
- H (foot of the perpendicular from D, at the center of the small circle)
- F (lower point on the small circle, right of center)
- G (right point of the small circle, above C)
- C (lower-right, on the sphere's outline)

Lines run from D down to the rim of the small circle (B–E–H–F–G–C), forming a cone-like figure with apex D and the perpendicular D–H to the center of the section.

(Continuation of the demonstration of Proposition 1 — that a plane section of a sphere is a circle — begun on the preceding page. This second case treats the plane that does not pass through the center.)

Next, let the cutting plane not pass through the center of the sphere. [Margin: a. Elements XI.11] From D, the center of the sphere, drop a perpendicular DH to the cutting plane, and from H draw any lines HE, HF to the line BEFCG, and join DE, DF. Since the angles DHE, DHF are right angles (by Elements XI, def. 3), [Margin: b. Elements I.47] the square on DE equals the squares on DH, HE together, and the square on DF equals the squares on DH, HF together. But the squares on DE, DF are equal to one another, because the lines DE, DF, falling from the center of the sphere onto its surface, are equal. Therefore the squares on DH, HE together equal the squares on DH, HF together. Take away the common square on DH, and the remaining squares on HE, HF are equal to one another; and therefore the lines HE, HF are equal. By the same argument we shall show that all the lines drawn from H to the line BEFCG are equal both to one another and to the two lines HE, HF already named. Therefore the line BEFCG is the circumference of a circle (by Elements I, def. 15), whose center is the point H, on which the perpendicular DH falls. Therefore, if a spherical surface be cut by any plane, and so on. Which was to be demonstrated.

Diagram. Two panels illustrating Proposition 1. The **left panel** shows a sphere (outer circle) with a section taken through the center: A at the top, B at the left, C at the right, G between A and the center, D at the center, and E, F at the lower rim of the section — the plane through the center yields a great circle.

- A (top): pole of the sphere
- B (left) / C (right): ends of the horizontal section
- G (upper middle): point on the section
- D (center): center of the sphere and of the great circle
- E (lower left) / F (lower right): rim points of the section

The **right panel** shows the cone-like figure for the off-center section: a sphere with A at the top, D as the apex within, and the base rim of the section carrying B, E, H, F, C (with H at the base center) and G to the right. Lines DE, DF and the perpendicular DH are drawn from D.

- A (top): pole of the sphere
- D (apex): center of the sphere
- H (base center): foot of the perpendicular DH, center of the section
- B (left) / C (right): ends of the section rim
- E, F (base, either side of H): points on the rim to which HE, HF are drawn
- G (right of H): point on the rim

COROLLARIVM.

Thus, if the cutting plane passes through the center of the sphere, the circle produced has the same center as the sphere. But if it does not pass through the center, the circle produced has a center different from the sphere's — namely, the point on which the perpendicular drawn from the center of the sphere to the cutting plane falls. For it will always be demonstrated that the straight lines falling from this point to the circumference of the circle are equal.

HOC EST.

That is: the center of the sphere and the center of a circle drawn through the center of the sphere are the same. And the perpendicular drawn from the center of the sphere onto the plane of a circle not drawn through the center of the sphere falls on the center of the circle — for the point H, on which the perpendicular DH falls, has been demonstrated to be the center of the circle.

PROBL. 1. PROPOS. 2.

[Margin: ij.]

To find the center of a given sphere.

Let the center of the sphere ABCD be sought. Let its surface be cut by some plane, producing in it the line BDE, [Margin: a. Sphaerica I.1 (this book)] which will be the circumference of a circle. [Margin: b. Elements III.1] Let the center of this circle be F. If, then, the circle BDE passes through the center of the sphere, [Margin: c. Corollary 1 of this book] the point F will also be the center of the sphere. But if it does not pass through the center of the sphere, [Margin: d. Elements XI.12] erect from F a perpendicular FG to the plane of the circle BDE; and this, produced both ways to the surface of the sphere at the points A, C, let it be bisected at G. I say that G is the center of the sphere. For if it is not, let the center, if it be possible, be H, bisecting all the diameters — which point will indeed not lie on the line AC (since AC is bisected only at the point G), but outside it. [Margin: e. Elements XI.11] From H, the supposed center of the sphere, drop a perpendicular HI to the plane of the circle BDE, [Margin: f. Elements XI.6] which will be parallel to the line FG; and therefore it will not fall on the point F, for then the two parallels HI, GF would meet at F — which cannot happen. Now since a perpendicular dropped from the center of the sphere onto the plane of the circle BDE [Margin: g. Corollary 1 of this book] falls on its center, I will be the center of the circle BDE. But F also, by construction, is the center of the same circle. This is

absurd, for the same circle must have only one center. Therefore no point other than G will be the center of the sphere. Thus we have found the center of the given sphere. Which was to be done.

Diagram. A sphere (outer circle) ABCD with a vertical diameter A–C and a horizontal section BDE shown as a flattened ellipse. On the vertical axis lie G and H (the true and the hypothetical center); on the section lie F (its center) and I (the foot of the perpendicular HI).

- A (top): upper end of the axis AC
- C (bottom): lower end of the axis AC
- B (left): left end of the section
- right end of the section [?] (label formed like a reversed C; on geometric grounds the point D of the section BDE)
- E (front, left of center): point on the section
- F (center of section): center of the circle BDE
- G (on axis, center): the center of the sphere
- H (on axis, beside G): the hypothetical alternative center
- I (beside F): foot of the perpendicular HI

COROLLARIVM.

Hence it is clear that, if there is in a sphere a circle not drawn through the center of the sphere, and from its center a perpendicular is raised to its plane, the center of the sphere lies on that perpendicular. For it has been shown that the point G, which bisects the perpendicular AC, is the center of the sphere.

THEOR. 2. PROPOS. 3.

[Margin: iij.]

A sphere does not touch a plane, by which it is not cut, in more than one point.

For if it be possible, let a sphere touch a plane, by which it is not cut, in more than one point — say at A and B. [Margin: a. Prop. 2 of this book] Having found C, the center of the sphere, draw the lines CA, CB; and through CA, CB pass a plane, which produces on the surface of the sphere [Margin: b. Prop. 1 of this book] the circumference of a circle ABD, and in the tangent plane [Margin: c. Elements XI.3] the straight line EABF. Now since the tangent plane, in which the line EABF lies, does not cut the sphere — and therefore does not cut the circle ABD lying on the

sphere's surface either — it follows that the line EABF does not cut the circle ABD. Therefore the straight line AB falls wholly outside the circle. But since the two points A, B are taken on the circumference of the circle ABD, [Margin: d. Elements III.2] the same straight line AB, drawn from point A to point B, falls wholly inside the circle ABD. Which is absurd. Therefore a sphere does not touch a plane, by which it is not cut, in more than one point. Which was to be demonstrated.

Diagram. A circle (the great circle ABD of the sphere) with D at the top and C at the center. A horizontal line EABF runs across the bottom, tangent to the plane; A and B are the two supposed points of contact on the circle, with the lines CA, CB drawn from the center to form a triangle.

- D (top): point on the circle
- C (center): center of the sphere
- E (far left) / F (far right): ends of the line EABF in the tangent plane
- A (left of center) / B (right of center): the two supposed contact points on the circle

COROLLARIUM.

Hence it follows that, if two points are marked on the surface of a sphere, the straight line joining them falls within the sphere — [Margin: e. Elements III.2] namely because it falls within the circle whose circumference lies on the surface of the sphere.

THEOR. 3. PROPOS. 4.

[Margin: iv.]

If a sphere touch a plane which does not cut it, the straight line drawn from the center of the sphere to the point of contact will be perpendicular to the plane.

Let a sphere touch a plane, which does not cut it, at the point A. [Margin: a. Prop. 2 of this book] Having found B, the center of the sphere, draw from it the line BA to the point of contact A. I say that the line BA is perpendicular to the said plane. For through the line AB draw any two planes cutting one another, which produce on the surface of the sphere [Margin: b. Prop. 1 of this book] the circumferences of the circles ACDE, AFDG, and in the tangent plane [Margin: c. Elements XI.3] the straight lines HAI, KAL. Now since each circle ACDE, AFDG passes through B, the center of the sphere, [Margin: d. Corollary 1 of this book] B will also be the center of both. Again, since the tangent plane does not cut the sphere, it follows that the

lines HAI, KAL lying in it do not cut it either — and therefore neither do they cut the circles ACDE, AFDG lying on the sphere's surface. Therefore the line HAI touches the circle ACDE at the point A, and the line KAL touches the circle AFDG at the same point A. [Margin: e. Elements III.18] Therefore the line BA is perpendicular both to the line HAI and to the line KAL. [Margin: f. Elements XI.4] Hence the same line BA will be perpendicular to the tangent plane, which is drawn through the lines HAI, KAL. Therefore, if a sphere touch a plane which does not cut it, and so on. Which was to be shown.

Diagram. A sphere (outer circle) with D at the top and A at the bottom as poles, B at the center. Two great circles are drawn as vertical ellipses through the poles D and A, carrying C, F on the left and G, E on the right. At the bottom, in the tangent plane at A, run the two straight lines HAI (H left, A center, I right) and KAL (K lower-left, L at the base).

- D (top) / A (bottom): poles; A is the point of contact
- B (center): center of the sphere and of both circles
- C, F (left) / G, E (right): points on the two circles ACDE, AFDG
- H (left) / I (right): ends of the tangent line HAI
- K (lower left) / L (base): ends of the tangent line KAL

THEOR. 4. PROPOS. 5.

[Margin: v.]

If a sphere touch a plane which does not cut it, and from the point of contact a straight line be raised at right angles to that plane, the center of the sphere will lie on the line so raised.

Let the sphere ABCD touch the plane EF, which does not cut it, at the point C; and from the point C [Margin: a. Elements XI.12] let a perpendicular CA be raised to the plane EF. I say that the center of the sphere lies on AC. For if it does not, let G be the center of the sphere, outside the line AC, if that be possible, and from G to C draw the line GC, [Margin: b. Prop. 4 of this book] which will be perpendicular to the plane EF. But AC too was perpendicular to the same plane. Therefore from the same point C two perpendiculars are drawn to the same plane EF. Which is absurd; for [Margin: c. Elements XI.13] from a given point in a given plane two straight lines cannot be raised at right angles to it. Therefore, if a sphere touch a plane which does not cut it, and so on. Which was to be shown.

Diagram. A sphere (circle) ABCD resting on a plane (drawn as a parallelogram) which it touches at C below. A vertical line runs A (top) through the center G down to C (contact); the plane EF extends to either side.

- A (top): upper end of the axis AC
- B (left) / D (right): points on the sphere
- G (center, marked by a dot): supposed center of the sphere
- C (bottom): point of contact with the plane
- E (left) / F (right): ends of the tangent plane EF

THEOR. 5. PROPOS. 6.

[Margin: vj.] [Margin: vij.]

Of the circles that are in a sphere, the greatest are those drawn through the center of the sphere; of the rest, those are equal to one another which are equally distant from the center, while those farther from the center are smaller. And the greatest circles in a sphere pass through the center of the sphere; of the rest, the equal ones are equally distant from the center, while the smaller ones are farther from the center.

In the sphere ABCDEF, whose center is G, let the circle AD pass through the center G, and let others, BC, FE, not pass through the center. I say that the circle AD is the greatest of all, and so on. [Margin: a. Elements XI.11] From the center G draw perpendiculars GH, GI to the planes of the circles BC, FE, [Margin: b. Corollary 1 of this book] which will fall on their centers, so that H, I are the centers of the circles BC, FE. [Margin: c. Corollary 1 of this book] Now G, the center of the sphere, is also the center of the circle AD, which is drawn through the center of the sphere. If, then, from G, H, I the lines GD, HC, IE be drawn to the surface of the sphere, these will be the radii of the circles AD, BC, FE. Let the lines GC, GE be joined. Now since in the triangle GHC the angle H is a right angle (by Elements XI, def. 3), [Margin: d. Elements I.47] the square on GC equals the squares on GH, HC. Taking away the square on GH, the square on GC is greater than the square on HC; and therefore the line GC — that is, its equal GD (for GC, GD are drawn from the center of the sphere to the surface) — is greater than the line HC. Hence the circle AD, having a greater radius than the circle BC, is greater than the circle BC. In the same way we shall show that the circle AD is greater than any other that does not pass through the center G. Therefore the circle AD is the greatest.

Diagram. A sphere (outer circle) ABCDEF with center G. The circle AD is drawn vertically through the center (A at the top, D at the bottom); two off-center circles

BC and FE are drawn to the left and right (B upper-left, C lower-left; F upper-right, E lower-right). The perpendiculars GH (to the plane of BC, H being its center) and GI (to the plane of FE, I being its center) are drawn, together with the radii GC, GE, GD, HC, IE.

- A (top) / D (bottom): ends of the great circle AD through the center
- G (center): center of the sphere
- B (left) / F (right): upper points of the off-center circles
- C (lower left) / E (lower right): lower points of the off-center circles
- H (inner left): center of circle BC
- I (inner right): center of circle FE

(The demonstration of Proposition 6 continues.)

Now let the circles BC, FE be equally distant from the center G — that is, let the perpendiculars GH, GI be equal (by definition 6 of this book). I say that the circles BC, FE are equal. For since the lines GC, GE, falling from the center of the sphere onto its surface, are equal, their squares too are equal; [Margin: e. Elements I.47] and since the square on GC equals the squares on GH, HC, and the square on GE equals the squares on GI, IE, the squares on GH, HC together will equal the squares on GI, IE together. Taking away the equal squares on GH, GI (for these lines are assumed equal), the remaining squares on HC, IE will be equal, and therefore the lines HC, IE will be equal; and since these are the radii of the circles BC, FE, the circles BC, FE themselves will be equal.

But if one of these circles, namely BC, be supposed to be farther from the center G than the other, FE — that is, the perpendicular GH be supposed greater than the perpendicular GI — we shall show in almost the same way that the circle BC is smaller than the circle FE. For since the squares on GH, HC have been demonstrated equal to the squares on GI, IE, if the unequal squares on the unequal lines GH, GI be taken away — of which the former is the greater (since the line GH is supposed greater than the line GI) — the remaining square on HC will be less than the remaining square on IE; and therefore the line HC will be less than the line IE. Therefore the circle BC will also be smaller than the circle FE.

Now let AD be the greatest circle of all. I say that it passes through G, the center of the sphere. For if it did not pass through the center, there would be some other circle, passing through the center G, greater than the circle AD not passing through the center, as has been demonstrated in this proposition. Hence AD

would not be the greatest circle. Which is absurd, for it is assumed to be the greatest. Therefore it passes through G, the center of the sphere.

Next, let the circles BC, FE be equal. I say that they are equally distant from the center G. For, the figure being constructed as before, the radii HC, IE will be equal. And since the squares on GH, HC are equal to the squares on GI, IE, [Margin: f. Elements I.47] as has been demonstrated, then, on taking away the equal squares on the equal lines HC, IE, the remaining squares on the lines GH, GI will be equal; and therefore the lines GH, GI will be equal. Since these are, by construction, perpendicular to the planes of the circles BC, FE, the circles BC, FE will be equally distant from the center G (by definition 6 of this book).

But if one of the circles BC, FE — namely the circle BC — be supposed smaller than the other circle FE, we shall show in almost the same way that the perpendicular GH is greater than the perpendicular GI. For since the squares on GH, HC have been shown equal to the squares on GI, IE, and since the square on HC is less than the square on IE (because the radius HC of the smaller circle is less than the radius IE of the larger circle), the remaining square on the line GH will be greater than the remaining square on the line GI; and therefore the line GH will be greater than GI. Hence, since GH, GI are by construction perpendicular to the planes of the circles, the smaller circle BC will be farther from the center G (by definition 6 of this book) than the larger circle FE. And so, of the circles that are in a sphere, the greatest are those drawn through the center of the sphere, and so on. Which was to be demonstrated.

Diagram. A repetition of the figure for Proposition 6 (identical to fig-p647-3). A sphere ABCDEF with center G: the great circle AD is drawn vertically through the center (A top, D bottom); the off-center circles BC and FE lie to the left and right (B upper-left, C lower-left; F upper-right, E lower-right). The perpendiculars GH, GI and the radii GC, GE, GD, HC, IE are drawn.

- A (top) / D (bottom): ends of the great circle AD
- G (center): center of the sphere
- B (left) / F (right): upper points of the off-center circles
- C (lower left) / E (lower right): lower points of the off-center circles
- H (inner left): center of circle BC
- I (inner right): center of circle FE

THEOR. 6. PROPOS. 7.

[Margin: viij.]

If there be a circle in a sphere, and a straight line be joined from the center of the sphere to the center of the circle, the joining line will be perpendicular to the plane of the circle.

In the sphere ABC, whose center is D, let there be a circle BFCG, whose center is E; and let the line DE join the two centers D, E. I say that DE is perpendicular to the plane of the circle BFCG. For, drawing any two diameters BC, FG in the circle, draw from their extremities to D, the center of the sphere, the lines BD, CD, FD, GD, which will all be equal to one another, since they fall from the center of the sphere to its surface. But BE, CE, FE, GE, the radii of the circle BFCG, are also equal. Therefore the two triangles DEB, DEC have the two sides DE, EB equal to the two sides DE, EC, and the base DB equal to the base DC; from which it follows [Margin: a. Elements I.8] that the angles DEB, DEC are equal, and therefore right angles. Therefore the line DE stands at right angles to the line BC. In the same way we shall show that the line DE stands at right angles to the line FG. [Margin: b. Elements XI.4] Therefore it also stands at right angles to the plane of the circle BFCG, drawn through the lines BC, FG. If, then, there be a circle in a sphere, and so on. Which was to be shown.

Diagram. A sphere (circle) ABC with center D, containing a circle BFCG with center E, shown as an inscribed flattened figure. From D and E the lines BD, CD, FD, GD and the joining line DE are drawn to the base rim.

- A (top): point on the sphere
- D (upper center): center of the sphere
- E (center): center of the circle BFCG
- B (left) / C (right): ends of the diameter BC
- F (bottom) / G (inner): ends of the diameter FG

THEOR. 7. PROPOS. 8.

[Margin: ix.]

If there be a circle in a sphere, and from the center of the sphere a perpendicular be drawn to the circle and produced both ways, it will fall on the poles of that circle.

In the sphere ABCD, whose center is E, let there be a circle BGDH, [Margin: a. Elements XI.11] to whose plane, from the center of the sphere E, let a perpendicular EF be drawn; and let this, produced both ways, fall on the surface of the sphere at the points A, C. I say that A, C are the poles of the circle BGDH. For the perpendicular EF will fall on the center of the circle BGDH, and therefore F

will be the center of the circle. But if the circle BGDH be drawn through the center of the sphere, [Margin: b. Corollary 1 of this book] it will be the very center...

Diagram. Two panels for Proposition 8. In each, a sphere (outer circle) ABCD with center E contains a circle BGDH shown in perspective; the polar axis runs from A (top) through E (center) and F to C (bottom).

Left panel:

- A (top) / C (bottom): the poles
- E (center): center of the sphere
- F (below E): foot of the perpendicular / center of the circle BGDH
- B (left) / D (right): ends of a diameter of the circle
- H (upper inner) / G (lower inner): points on the circle

Right panel (same construction, more fully drawn):

- A (top) / C (bottom): the poles
- E (center): center of the sphere
- F (below E): center of the circle BGDH
- B (left) / D (right): ends of a diameter
- H (upper inner) / G (lower inner): points on the circle

the center of the sphere, E, is the same point as F, the center of the circle, from which the perpendicular AC has been raised to the plane of the circle [Margin: c — *Elements* XI.12].

So let the diameters BD and GH be drawn in any way, and from their endpoints let straight lines be drawn to the points A and C. Since AF is perpendicular to the plane of the circle BGDH, all the angles it makes at F are right angles, by *Elements* XI, def. 3. The two triangles AFB and AFH therefore have two sides, AF and FB, equal to the two sides AF and FH, and these contain equal angles — namely right angles. [Margin: d — *Elements* I.4] So the bases AB and AH will be equal. In the same way we shall show that AD, AG, and all other lines drawn from A to the circumference of the circle BGDH are equal both to one another and to the lines AB, AH. Therefore the point A is the pole of the circle BGDH, by def. 5 of this book. In just the same way we shall prove that C too is a pole of the same circle. So, if in a sphere there is a circle, and from its center, and so on. Which was to be shown.

Diagram. The left of a pair of sphere figures illustrating Proposition 8. The great outer circle is the sphere; the inner ellipse is the circle BGDH seen edge-on.

Straight lines run from the apex A and the lowest point C to the endpoints of the diameters and to the inner points.

- A (top apex): pole of the circle BGDH
- C (bottom): the other pole of the circle BGDH
- B (left): endpoint of a diameter of the circle
- D (right): endpoint of a diameter of the circle
- E (center, marked with a dot): center of the sphere
- F (just below E): center of the circle BGDH
- G (lower, toward C): point on the circle
- H (upper left): point on the circle

Diagram. The right of the pair — the same configuration redrawn with a fuller set of connecting lines, again illustrating Proposition 8.

- A (top apex): pole of the circle BGDH
- C (bottom): the other pole
- B (left): endpoint of a diameter
- D (right): endpoint of a diameter
- E (center): center of the sphere
- F (below E): center of the circle BGDH
- G (lower): point on the circle
- H (upper left): point on the circle

SCHOLIVM

In Maurolico's [Maurolico] version the following two theorems are added, which the Arabs appended.

I.

[Margin: x.]

If in a sphere there is a circle, and from its center a perpendicular is drawn to the plane of the circle, and this is produced in both directions, it will fall on both poles of the circle.

In the same figure, from F, the center of the circle BGDH, [Margin: a — Elements XI.12] let the straight line FA be erected perpendicular to the plane of the circle, meeting the surface of the sphere at the points A and C. I say that A and C are the poles of the circle BGDH. For once again, by Elements XI, def. 3, all the angles that

the line AF makes at F will be right angles. [Margin: b — Elements I.4] Hence, as before, the lines AB, AD, AG, AH, and so on, will be equal to one another, and so forth.

ANOTHER PROOF. [Margin: c — Coroll. 2 of this book] Since the perpendicular FA passes through E, the center of the sphere, the straight line EF, drawn from E, the center of the sphere, to the plane of the circle BGDH, will be perpendicular. [Margin: d — Sphaerica I.8] Therefore, as has been demonstrated, it falls on the poles of that same circle. Which is what was proposed.

II.

[Margin: xj.]

If in a sphere there is a circle, and from one of its poles a straight line is drawn through its center, this line will be perpendicular to the plane of the circle, and produced it will fall on the remaining pole.

Still in the same figure, from A, a pole of the circle BGDH, through its center F, let the straight line AF be dropped, meeting the surface of the sphere at C. I say that the line AF is perpendicular to the plane of the circle BGDH, and that C is the other pole of the same circle. For the two triangles AFB and AFD have two sides, AF and FB, equal to the two sides AF and FD, and the base AB equal to the base AD, by the definition of pole; [Margin: a — Elements I.8] so they will also have the two angles AFB and AFD equal, and hence right angles. Therefore AF stands at right angles on the straight line BD. In the same way we shall show that this same AF stands at right angles on the straight line GH. [Margin: b — Elements XI.4] So this same line AF will also stand at right angles on the plane of the circle BGDH, which passes through the straight lines BD and GH. Which is the first point proposed. Since, then, AF is at right angles to the plane of the circle BGDH — being perpendicular from the center of the circle, F, to the plane of the circle — therefore, as has just been demonstrated in this scholium, when produced in both directions it will fall on both poles of the circle; and consequently C will be the remaining pole of the circle BGDH. Which is the second point proposed.

THEOR. 8. PROPOS. 9.

[Margin: xij.]

If in a sphere there is a circle, and from one of its poles a perpendicular straight line is drawn to it, this line will fall on the center of the circle, and produced from there it will fall on the remaining pole of that circle.

In the sphere ABCD let there be a circle BFDG, [Margin: a — *Elements* XI.11] and from its pole A let a perpendicular AE be drawn to its plane, meeting the surface of the sphere at C. I say that E is the center of the circle BFDG, and C the other pole. For let two straight lines BD and FG be drawn through E in any way, and let their endpoints be joined to the pole A by the straight lines AB, AD, AF, AG, which will all be equal to one another, by the definition of pole. Likewise all the angles that the line AE makes at E are right angles, by *Elements* XI, def. 3. [Margin: b — *Elements* I.47] The square on AB will therefore equal the squares on AE and EB together, and the square on AG will equal the squares on AE and EG together. And since the squares on the equal lines AB and AG are equal, the squares on AE and EB together will equal the squares on AE and GE together. So, removing the common square on the line AE, the remaining squares on EB and EG will be equal, and consequently the lines EB and EG themselves will be equal. In the same way we shall show that the lines EG and ED are equal. [Margin: c — *Elements* III.9] Therefore E is the center of the circle BFDG — which is what was proposed. Now since the perpendicular EA has been drawn from E, the center of the circle BFDG, to its plane, [Margin: d — Coroll. 2 of this book] it will pass through H, the center of the sphere; and so this same line HE, drawn from H, the center of the sphere, will be perpendicular to the plane of the circle BFDG. [Margin: e — *Sphaerica* I.8] Therefore HE, produced in both directions, will fall on the poles of that same circle; and consequently C will be the remaining pole of the circle BFDG. So, if in a sphere there is a circle, and from one of its poles, and so on. Which was to be shown.

Diagram. A sphere figure for Proposition 9. The outer circle is the sphere; an inner ellipse is the circle BFDG seen edge-on, with its horizontal diameter BD. Straight lines run from the apex A down to B, D, and to the inner points E, F, G, H, and the vertical line AEC (through the center) reaches the lowest point C.

- A (top apex): pole of the circle BFDG
- G (inner, left of center): point on the circle
- H (inner, middle): point on the circle
- B (left): endpoint of the diameter BD
- D (right): endpoint of the diameter BD
- E (center): center of the circle BFDG
- F (below E, near bottom): point on the circle
- C (bottom): the other pole of the circle BFDG

THEOR. 9. PROPOS. 10.

[Margin: xiiij.]

If in a sphere there is a circle, a straight line drawn through its poles is perpendicular to the circle, and passes through the center of the circle and of the sphere.

In the sphere ABCD let there be a circle BFDG, and through its poles A and C let the straight line AC be drawn, meeting the plane of the circle at E. I say that the line AC is perpendicular to the plane of the circle, and passes through its center (that is, that E is its center), as well as through the center of the sphere. For let two straight lines BD and FG be drawn through E in any way, and let their endpoints be joined to the poles A and C by straight lines, as in the figure. Then AB, AG, AF, AD will be equal to one another, and likewise CB, CG, CF, CD to one another, by the definition of pole. Therefore the two triangles ABC and ADC have two sides, AB and AC, equal to the two sides AD and AC, and the base BC equal to the base DC. [Margin: a — *Elements* I.8] So they will also have the angles BAC and DAC equal. And since the two triangles ABE and ADE have two sides, AB and AE, equal to the two sides AD and AE, and the angles contained by them, BAE and DAE, equal (as has just been demonstrated), [Margin: b — *Elements* I.4] the angles AEB and AED will also be equal, and therefore right angles. In just the same way we shall show that the angles AEG and AEF are right angles. So the line AE stands at right angles on the two straight lines BD and FG. [Margin: c — *Elements* XI.4] It will therefore be perpendicular to the plane of the circle BFDG, which is drawn through the straight lines BD and FG. Which is the first point proposed. Now since the perpendicular AE has been drawn from A, a pole of the circle BFDG, to its plane, [Margin: d — *Sphaerica* I.9] AE will fall on its center. So E is the center of the circle BFDG. Again, since the perpendicular EA has been drawn from E, the center of the circle BFDG, to its plane, [Margin: e — Coroll. 2 of this book] this line will also pass through the center of the sphere. Therefore the line AC is perpendicular to the plane of the circle BFDG, and passes through its center and through the center of the sphere. Which is what was proposed. So, if in a sphere there is a circle, and a straight line is drawn through its poles, and so on. Which was to be demonstrated.

Diagram. A sphere figure for Proposition 10. The outer circle is the sphere; the inner ellipse is the circle BFDG seen edge-on. Straight lines connect the apex A and the lowest point C to the endpoints B and D of the diameter and to the inner points, and the vertical line AC passes through the center E.

- A (top apex): pole of the circle
- G (inner, upper left): point on the circle
- E (center): center of the circle BFDG / where AC meets the plane
- B (left): endpoint of the diameter
- D (right): endpoint of the diameter
- F (lower, inner): point on the circle
- C (bottom): the other pole

SCHOLIUM

Two further theorems of this kind are added here.

I.

[Margin: xiv.]

If in a sphere there is a circle, and from one of its poles a straight line is drawn through the center of the sphere, this line will be perpendicular to the plane of the circle, and produced it will fall on its center and on the remaining pole.

In the sphere ABCD, whose center is E, let there be a circle BGDH; and from its pole A, through E, the center of the sphere, let the straight line AE be drawn, meeting the plane of the circle at F and the surface of the sphere at C. I say that AE is perpendicular to the plane of the circle, and passes through its center as it does to the remaining pole — that is, that F is its center and C the remaining pole. For let two straight lines BD and GH be drawn through F in any way, and let their endpoints be joined to the points A and E, as in the figure. Then AB, AH, AD, AG will be equal to one another, by the definition of pole; and likewise EB, EH, ED, EG, the radii of the sphere, will be equal to one another. Since, then, the two triangles ABE and ADE have two sides, AB and AE, equal to the two sides AD and AE, and the base EB equal to the base ED, [Margin: a — Elements I.8] the angles BAE and DAE will be equal. And so the two triangles ABF and ADF have two sides, AB and AF, equal to the two sides AD and AF, and the angles contained by them, BAF and DAF, equal, as was just shown. [Margin: b — Elements I.4] So the angles AFB and AFD will be equal, and hence right angles. In the same way we shall demonstrate that the angles AFH and AFG are right angles. Therefore the line AF stands at right angles on the two straight lines BD and GH. [Margin: c — Elements XI.4] So it will be perpendicular to the plane of the circle BGDH, which is drawn through the straight lines BD and GH. [Margin: d — Sphaerica I.9] And so, produced, it will fall on both the center of the circle and the remaining pole; and consequently F will be the center

of the circle, and C the remaining pole. Which is what was proposed. So, if in a sphere there is a circle, and so on. Which was to be shown.

Diagram. A sphere figure for the added Scholium theorem. The outer circle is the sphere; an inner ellipse is the circle BGDH seen edge-on, with center F. Lines run from the apex A to B, D and the inner points, and the vertical line AC passes through E (center of the sphere) and F.

- A (top apex): pole of the circle BGDH
- E (center, upper): center of the sphere
- H (inner, left of center): point on the circle
- B (left): endpoint of the diameter BD
- D (right): endpoint of the diameter BD
- F (lower, inner): center of the circle BGDH
- G (bottom, inner near C): point on the circle
- C (bottom): the other pole

COROLLARIVM

From this it follows that a great circle which passes through one of the poles of any circle in the sphere also passes through the other pole. For if from one pole, through the center of the sphere, the diameter of the great circle passing through that pole is drawn, it will fall on the other pole, as has been demonstrated. So the same great circle will pass through the remaining pole.

And since the diameter of a great circle is also a diameter of the sphere, it is clear that the two poles of any circle in the sphere are diametrically opposite; and so a semicircle of a great circle lies between them.

II.

[Margin: xv.]

If in a sphere there is a circle, and from the center of the sphere a straight line is drawn through the center of the circle, it will fall on both poles of the circle.

In the same figure, through E, the center of the sphere, and F, the center of the circle BGDH, let the straight line EF be drawn in both directions. I say that EF falls on both poles of the circle BGDH. For since the straight line EF, joining the center of the sphere and the center of the circle BGDH, [Margin: e — Sphaerica I.7] is perpendicular to the plane of that same circle, [Margin: f — Sphaerica I.8] this same

line EF, produced in both directions, will fall on both poles of that same circle. Which is what was proposed.

COROLLARIVM

From all of this it is established that in a sphere these four points — namely the two poles of any circle, its center, and the center of the sphere — always lie on one straight line, namely a diameter of the sphere; and that this diameter is perpendicular to the plane of that same circle. So a straight line drawn through any two of these points passes through the remaining two, and is perpendicular to the plane of the circle; and a straight line drawn through any one of them perpendicular to the plane of the circle also passes through the other three points.

THEOR. 10. PROPOS. 11.

[Margin: xvj.]

In a sphere, great circles bisect each other.

In the sphere ABCD let two great circles AC and BD cut each other at the points E and F. I say that they bisect each other. [Margin: a — *Sphaerica* I.6] Since great circles in a sphere pass through the center of the sphere, the circles AC and BD will pass through the center of the sphere, which let be G. [Margin: b — Corollary 1 of this book] And since the center of the sphere and the center of a circle carried through the center of the sphere are the same, the point G — which is taken to be the center of the sphere — will also be the center of each circle AC and BD, so that it lies in the same plane of each of the circles AC and BD. But the points E and F also lie in each of the same planes. Therefore the three points E, G, F lie in each plane of the circles AC and BD, and so they will lie on the common section of the two, since only their common section lies in both planes. [Margin: c — *Elements* XI.3] Now their common section is a straight line. So the three points E, G, F lie on a straight line drawn from E through G to F; and since this line passes through G, the center of each circle and of the sphere, as has been shown, it will be a diameter both of the sphere and of each circle, and will bisect each of them — so that EAF, FCE, EBF, FDE are semicircles. Therefore in a sphere great circles bisect each other. Which was to be demonstrated.

Diagram. A sphere figure for Proposition 11. Within the great outer circle (the sphere) two great circles AC and BD are drawn, crossing to form a star-like pattern; the diameters AC and BD meet at the center G.

- A (upper left): endpoint of the great circle AC
- D (upper right): endpoint of the great circle BD
- E (left, on the line EF): intersection point of the two great circles
- G (center, marked with a dot): center of the sphere
- F (right, on the line EF): intersection point of the two great circles
- B (lower left): endpoint of the great circle BD
- C (lower right): endpoint of the great circle AC

THEOR. 11. PROPOS. 12.

[Margin: 17.]

In a sphere, circles that bisect each other are great circles.

In the sphere ABCD let the circles AE [recte: AC] and BD bisect each other at the points E and F. I say that the circles AC and BD are great circles. For since they bisect each other at E and F, the line EF, when drawn, will be a diameter of each — since only a diameter divides any circle in half. And so, when the line EF is itself divided in half at G, G will be the center of each circle; and this, I say, is also the center of the sphere, and hence each circle is drawn through the center of the sphere. For suppose G is said not to be the center of the sphere, and so the circles AC and BD not to be drawn through the center of the sphere; by this very supposition we shall show that G is the center of the sphere, and therefore that each circle is drawn through the center of the sphere. [Margin: a — *Elements* XI.12] For from G let the perpendicular GH be erected to the plane of the circle AC, and likewise GI perpendicular to the plane of the circle BD. Now since the circles AC and BD are supposed not to pass through the center of the sphere, [Margin: b — Coroll. 2 of this book] each perpendicular GH and GI will pass through the center of the sphere. So the point G, where they meet, will be the center of the sphere — otherwise the center would not lie in both. And consequently each circle will be carried through the center of the sphere. [Margin: c — *Sphaerica* I.6] Therefore the circles AC and BD, carried through the center of the sphere, are great circles. So in a sphere, circles that bisect each other are great circles. Which was to be shown.

Diagram. A sphere figure for Proposition 12. Within the outer circle (the sphere) two circles AC and BD cross in a star-like pattern; their common line EF passes through the center G, with the perpendiculars meeting at the inner points I and H above G.

- A (upper left): endpoint of a circle

- D (upper right): endpoint of a circle
- I (inner, upper left of center): foot of a perpendicular / point on a circle
- H (inner, upper right of center): foot of a perpendicular / point on a circle
- E (left): intersection point of the two circles
- F (right): intersection point of the two circles
- G (center): common center of the two circles / center of the sphere
- B (lower left): endpoint of a circle
- C (lower right): endpoint of a circle

SCHOLIVM

Here you see a truly remarkable manner of argument. For from the fact that G is said *not* to be the center of the sphere, it is proved — by an affirmative demonstration — that G *is* the center of the sphere. Euclid [Euclid] also used this manner of argument in Book 9, Prop. 12, and Cardano [Cardano] in Book 5 of *De Proportionibus*, Prop. 201, as we noted in the scholium to that same proposition.

THEOR. 12. PROPOS. 13.

[Margin: 18.]

If in a sphere a great circle cuts some circle at right angles, it also cuts it in half and through the poles.

In a sphere let the great circle ABCD cut the circle BED at the points B and D at right angles — that is, let the plane of the circle ABCD be at right angles to the plane of the circle BED, and let their common section be the straight line BD. I say that the circle ABCD cuts the circle BED both in half and through the poles.

[Margin: a — *Elements* III.1] For let F be taken as the center of the great circle ABCD, which [Margin: b — *Sphaerica* I.6] will also be the center of the sphere [Margin: c — Coroll. 1 of this book] (for since a great circle is drawn through the center of the sphere, its center is the same as that of the sphere). [Margin: d — *Elements* XI.11] From F let the perpendicular FG be drawn to the plane of the circle BED, [Margin: e — *Elements* XI.38] which will fall on BD, their common section. Let it fall on the point G. [Margin: f — Coroll. 1 of this book] And since this same perpendicular also falls on the center of the circle BED, G will be the center of the circle BED; and so BD, drawn through G, is a diameter of that circle. Since this diameter divides the circle BED in half, the great circle ABCD, drawn through the straight line BD, will also divide the same circle in half. Which is the first point proposed. Now since the line FG lies in the plane of the circle ABCD, when produced it will fall on the circumference at the points A and C, which lie on the

surface of the sphere. [Margin: g — *Sphaerica* I.8] But it also falls on each pole of the circle BED, since the perpendicular is drawn from F, the center of the sphere, to the plane of the circle. Therefore A and C are the poles of the circle BED; and consequently the great circle ABCD passes through the poles of the circle BED — which is the second point that was proposed for demonstration. So, if in a sphere a great circle cuts some circle, and so on. Which was to be demonstrated.

Diagram. A sphere figure for Proposition 13. The outer circle is the great circle ABCD (the sphere); the vertical diameter AC passes through F, marked with a dot. A smaller circle BED is drawn low in the figure, with its diameter BD horizontal and its center G on the line AC; E lies to the left.

- A (top): pole of the circle BED / endpoint of the vertical diameter
- F (center, marked with a dot): center of the great circle ABCD and of the sphere
- E (lower left, inner): point on the small circle BED
- B (left): endpoint of the diameter BD of the small circle
- G (center, below F): center of the small circle BED
- D (right): endpoint of the diameter BD
- C (bottom): the other pole of the circle BED

SCHOLIVM

Moreover, this proposition, together with 8, 9, 10 and their scholia, is also to be understood of the case where the circle BD is a great circle and passes through the center of the sphere. For the demonstration is nearly always the same, as is evident.

THEOR. 13. PROPOS. 14.

[Margin: 19.]

If in a sphere a great circle bisects a circle that is not a great circle, it cuts it at right angles and through the poles.

(Continuation of the demonstration of THEOR. 13 / PROPOS. 14, whose enunciation stood at the end of the previous section: "If in a sphere a great circle bisects a circle that is not a great circle, it cuts it at right angles and through the poles.")

Diagram. A sphere shown as the great circle ABCD (the outer circle), with a lens-shaped small circle BED across its upper part.

- A (top of the sphere): pole

- B (upper left, on the small circle): one end of the common section
- D (upper right, on the small circle): other end of the common section
- F (on the chord BD, below A): center of small circle BED
- E (just below F, on the near/far arc of the small circle): point of the small circle's circumference
- G (center of the sphere, marked with a dot): center of the sphere and of the great circle ABCD
- C (bottom of the sphere): pole

In the sphere let the great circle ABCD bisect the circle BED, which is not a great circle, at the points B and D, and let their common section be the straight line BD. I say that the circle ABCD cuts the circle BED at right angles and through its poles. For since the circle BED is bisected at B and D — that is, into semicircles — BD, their common section, will be its diameter. So when BD has been bisected at F, F will be the center of circle BED.[Margin: a. 2. huius (*Sphaerica* I.2)] Now let G be taken as the center of the sphere, which will also be the center of the great circle ABCD, and from G to F let the straight line FG be drawn; it will be perpendicular to the plane of circle BED.[Margin: b. 7. huius (*Sphaerica* I.7)] Therefore the plane of the great circle ABCD, drawn through the straight line FG onto that same plane of circle BED, will also be at right angles to it.[Margin: c. 18. unde. (*Elements* XI.18)] Hence the great circle ABCD cuts the circle BED, which is not a great circle, at right angles — which was the first thing proposed.

And since it has been shown that the straight line FG, drawn from the sphere's center G to the plane of circle BED, is perpendicular to it, FG produced in both directions will fall on the poles of circle BED.[Margin: d. 8. huius (*Sphaerica* I.8)] Therefore, since GF, lying in the plane of circle ABCD, when produced falls upon that circle's circumference at the points A and C, which also lie on the surface of the sphere, A and C will be the poles of circle BED. And so the great circle ABCD will cut the circle BED, which is not a great circle, through the poles A and C — which was the second thing proposed. If, therefore, in a sphere a great circle cuts a circle that is not a great circle, and so on. Which was to be shown.

THEOR. 14. PROPOS. 15. [Margin: 20.]

If in a sphere a great circle cuts through the poles any one of the circles that are in the sphere, it cuts it both in half and at right angles.

Diagram. A sphere shown as the great circle ABCD, with a lens-shaped small circle BED across its lower part.

- A (top): pole
- B (lower left, on the small circle): one end of the common section
- D (lower right, on the small circle): other end of the common section
- F (on chord BD): meeting point of AC with the plane of the small circle
- E (just above F): point of the small circle's circumference
- C (bottom): pole (the vertical line AC is the axis through both poles)

In the sphere let the great circle ABCD cut the circle BED through the poles A and C. I say that the circle ABCD cuts the circle BED both in half and at right angles. For let the straight line AC join the poles A and C, meeting the plane of circle BED at the point F.[Margin: a. 10. huius (*Sphaerica* I.10)] Now since the straight line AC is perpendicular to the plane of circle BED, and passes through the center of the sphere and of circle BED, F will be the center of circle BED. Therefore, since the great circle ABCD, cutting circle BED, passes through the straight line AC and hence through the center F, the common section BFD will be a diameter of circle BED. So circle BED is bisected. I say also that it is cut at right angles. For since the straight line AC has been shown to be perpendicular to the plane of circle BED, [Margin: b. 18. unde. (*Elements* XI.18)] the plane of the great circle ABCD, drawn through the straight line AC onto that same plane of circle BED, will also be at right angles to it. Therefore, if in a sphere a great circle, and so on. Which was to be demonstrated.

SCHOLIVM.

Four other theorems are added at this place in another version, in this order.

I. [Margin: 21.]

If in a sphere a great circle passes through the poles of some other great circle, this latter in turn passes through the poles of the former.

Diagram. A sphere shown as two intersecting great circles.

- A (top): pole
- B (left): one intersection point / end of common section BD
- D (right): other intersection point / end of common section BD
- G (bottom): pole

In the sphere let the great circle ABCD pass through A and C, the poles of the great circle BD. I say that the great circle BD also passes through the poles of the great circle ABCD. For since the great circle ABCD cuts the circle BD through its poles, it will cut it at right angles.[Margin: a. 15. huius (*Sphaerica* I.15)] Therefore, in turn,

the great circle BD will cut the circle ABCD at right angles,[Margin: b. 13. huius (*Sphaerica* I.13)] and so it will cut it through its poles. Which is what was proposed.

II. [Margin: 22.]

If in a sphere a circle cuts another circle through its poles, it is a great circle, and it cuts it both in half and at right angles.

Diagram. A sphere with a vertical diameter and a circle BD across the middle.

- A (top): pole
- B (left): one end of the diameter of circle BD
- D (right): other end of the diameter of circle BD
- C (bottom): pole (the vertical line AC passes through the two poles and the center)

In the sphere let the circle ABCD cut the circle BD through the poles A and C. I say that it is a great circle, and that it cuts the circle BD both in half and at right angles. For let the straight line AC join the poles A and C; it will necessarily lie in the plane of circle ABCD, since that circle's circumference is assumed to pass through the same poles A and C. Now since the straight line AC, drawn through A and C, the poles of circle BD, passes through the center of the sphere,[Margin: a. 10. huius (*Sphaerica* I.10)] the circle ABCD (since it is drawn through the straight line AC) will also pass through the center of the sphere,[Margin: b. 6. huius (*Sphaerica* I.6)] and so it will be a great circle. Therefore, since it is assumed to pass through A and C, the poles of circle BD, it will cut it both in half and at right angles.[Margin: c. 15. huius (*Sphaerica* I.15)] Which is what was proposed.

III. [Margin: 23.]

If in a sphere a circle cuts another circle both in half and at right angles, it is a great circle, and it cuts it through the poles.

(Continuation of the demonstration of Scholium theorem III.)

In the sphere let the circle ABCD cut the circle BD both in half and at right angles. I say that it is a great circle, and that it passes through the poles of circle BD. Let the straight line BD be the common section of the circles. Since, then, the circle ABCD cuts the circle BD in half, the straight line BD — namely the common section of the circles — will be a diameter of circle BD; and so, when the straight line BD has been bisected at E, E will be the center of that circle. In the plane of circle ABCD let the straight line EA be drawn perpendicular to the straight line

BD. And since the circle ABCD is assumed to cut the circle BD at right angles, EA will, by *Elements* XI, definition 4, be at right angles to the plane of circle BD; and hence, since it is drawn from E, the center of that circle, it falls on each of its poles.[Margin: h. schol. 8. huius (Scholium on *Sphaerica* I.8)] But it falls on the circumference of circle ABCD, lying on the surface of the sphere, at the points A and C. Therefore A and C are the poles of circle BD; and so the circle ABCD cuts circle BD through the poles A and C. Hence, by the preceding theorem, it is a great circle. And it has been proved that it also cuts circle BD through the poles. So what was proposed is established.

I V. [Margin: xxiv.]

If in a sphere there is a circle, and a straight line falling from one of its poles onto its plane at right angles is equal to its radius, the circle is a great circle.

Diagram. A sphere with a vertical diameter AC and a horizontal chord BD meeting at the center E.

- A (top): pole
- B (left): endpoint of the chord
- D (right): endpoint of the chord
- C (bottom): pole
- E (center): center of the sphere / meeting point

In the sphere let there be a circle AB, and from one of its poles C let the perpendicular straight line CD, falling onto its plane, be equal to its radius. I say that AB is a great circle. For since CD is perpendicular to circle AB, it will fall on the circle's center,[Margin: i. 9. huius (*Sphaerica* I.9)] and when produced it will fall on the other pole, which let be E. Therefore D is the center of circle AB, [Margin: k. coroll. 2. huius (Corollary 2, *Sphaerica*)] and so the perpendicular CD passes through the center of the sphere. Through the straight line CE let a plane be drawn in the sphere in any way whatever,[Margin: l. 1. huius (*Sphaerica* I.1)] making in the sphere the circle AEBC; since this passes through the center of the sphere, it will be a great circle,[Margin: m. 6. huius (*Sphaerica* I.6)] and it cuts the circle AB at the points A and B. Let the radius DB be joined, to which, by hypothesis, CD is equal. Now since CD is assumed perpendicular to circle AB, the angle CDB will, by *Elements* XI, definition 3, be a right angle.[Margin: n. schol. 13. sexti (Scholium on *Elements* VI.13)] Therefore BD is the mean proportional between CD and DE — that is, as CD is to BD, so BD is to DE. But CD is equal to BD. Therefore DE too will be equal to that same BD; and so CD and DE will be equal to

each other. Since, then, CE has been shown to pass through the center of the sphere, and passes through the center of circle AB, D will be the center of the sphere. But it was also the center of circle AB. Therefore the center of the sphere and of circle AB is the same,[Margin: o. 6. huius (*Sphaerica* I.6)] and hence the circle AB is a great circle. Which is what was proposed.

THEOR. 15. PROPOS. 16. [Margin: xxv.]

If in a sphere there is a great circle, a straight line drawn from a pole of that circle to its circumference is equal to the side of a square inscribed in a great circle.

Diagram. A sphere with vertical diameter CD (through the poles) and horizontal diameter AB, meeting at the center E; a great circle ADBC is shown.

- C (top): pole
- A (left): endpoint of diameter AB
- B (right): endpoint of diameter AB
- D (bottom): other pole
- E (center): center of the sphere / meeting of the diameters

In the sphere let there be a great circle AB, and from its pole C let the straight line CB be drawn to its circumference. I say that CB is equal to the side of a square inscribed in circle AB, or in any other great circle.[Margin: a. 11. undec. (*Elements* XI.11)] From C let the perpendicular CE be drawn to circle AB;[Margin: b. 9. huius (*Sphaerica* I.9)] it will fall on the circle's center, which let be E, and produced it will fall on the remaining pole, which let be D. Now through the straight lines CB and CD let a plane be drawn,[Margin: c. 1. huius (*Sphaerica* I.1)] making in the sphere the circle ADBC; since this passes through E, the center of the sphere (for E, the center of the great circle AB, is the same as the center of the sphere, because AB passes through the sphere's center),[Margin: d. coroll. 1. huius (Corollary 1, *Sphaerica*)] it will be a great circle,[Margin: e. 6. huius (*Sphaerica* I.6)] and so it will bisect the great circle AB.[Margin: f. 11. huius (*Sphaerica* I.11)] This is also clear from the fact that it passes through its poles;[Margin: g. 15. huius (*Sphaerica* I.15)] for from this it follows that it bisects it. Let the common section, then, be the diameter BEA. And since the perpendicular CE has been drawn to circle AB, it will be the same perpendicular to the straight line AB, by *Elements* XI, definition 3. Therefore the two diameters AB and CD, in the great circle ADBC, cut each other at right angles;[Margin: h. 6. quart. (*Elements* IV.6)] and on that account, as is demonstrated in *Elements* IV, CB is the side of a square inscribed in the great circle ADBC, and so also in the great circle AB. If, therefore, in a sphere

there is a great circle, and a straight line is drawn, and so on. Which was to be demonstrated.

COROLLARIVM.

Now since the four right angles at the center E are equal, and hence the four arcs BC, CA, AD, DB, on which they stand, are equal — namely quadrants — it is clear that in a sphere the pole of a great circle is distant from the circumference of a great circle by a quadrant of a great circle. For C, the pole of the great circle AB, is distant from that circle's circumference by the quadrant CB, and the same holds of the rest.[Margin: i. 26. tertij (*Elements* III.26)] For a straight line drawn from the circumference of a great circle to its pole is always equal to the side of a square inscribed in a great circle,[Margin: k. 16. huius (*Sphaerica* I.16)] and so it subtends a quadrant in a great circle.

SCHOLIVM.

The converse of this is also demonstrated in another version, in this theorem.

If in a sphere there is a circle, and a straight line drawn from its pole to the circumference is equal to the side of a square inscribed in it, the circle itself is a great circle.

(Continuation of the converse-theorem demonstration.)

In the same figure, from C, the pole, let a straight line CB be drawn to the circumference of circle AB, equal to the side of a square inscribed in circle AB. I say that AB is a great circle. For from C let the perpendicular CE be drawn to circle AB;[Margin: l. 11. undec. (*Elements* XI.11)] it will fall on its center, which let be E.[Margin: m. 9. huius (*Sphaerica* I.9)] Now when the radius EB has been drawn, the angle E will, by *Elements* XI, definition 3, be a right angle.[Margin: n. 47. pri. (*Elements* I.47)] Therefore the square on CB — that is, the square inscribed in circle AB — is equal to the squares on BE and CE. But the square on the radius BE is half the square inscribed in circle AB, as we shall show presently. Therefore the square on CE too will be half of that same square inscribed in circle AB; and so the squares on BE and CE will be equal to each other, and consequently the lines BE and CE will also be equal. Therefore, since CE has been drawn from C, the pole of circle AB, perpendicular to that circle, and has been shown to be equal to the radius BE, the circle AB will be a great circle.[Margin: o. schol. 15. huius (Scholium on *Sphaerica* I.15)]

LEMMA.

In every circle the square on the radius is half the square inscribed in that circle.

Diagram. A circle with a square ABCD inscribed in it, its two diameters AC and BD drawn through the center E.

- A (top): vertex of the inscribed square
- B (left): vertex of the inscribed square
- D (right): vertex of the inscribed square
- C (bottom): vertex of the inscribed square
- E (center): center of the circle (intersection of the diameters)

In a circle whose center is E, let two diameters AC and BD be drawn, cutting each other at right angles at the center E. When, therefore, the straight lines AB, BC, CD, DA have been joined, ABCD will be a square inscribed in the circle, as is established from *Elements* IV.6. Now since the squares on the equal radii EA and EB, being equal to each other, [Margin: p. 47. pri. (*Elements* I.47)] are together equal to the square on AB, the square on the radius EA will be half the square on AB, which is inscribed in the circle. Which is what was proposed. From this it is established that, in the upper figure, the square on the radius BE is half the square on CB, which is assumed equal to the one inscribed in circle AB.

THEOR. 16. PROPOS. 17. [Margin: xxvij.]

If in a sphere there is a circle, and a straight line drawn from its pole to its circumference is equal to the side of a square inscribed in a great circle, that circle will be a great circle.

Diagram. A sphere with a great circle ACB inscribed, and lines drawn from C to A and B.

- C (top): pole of circle AB
- A (left): endpoint of the diameter AB
- B (right): endpoint of the diameter AB
- (center, marked with a dot): center of the sphere

In the sphere let there be a circle AB, and from its pole C let the straight line CA be drawn to its circumference, equal to the side of a square inscribed in a great circle of the sphere. I say that the circle AB is a great circle. For through the straight line AC and the center of the sphere let a plane be drawn, [Margin: a. 1.

huius (*Sphaerica* I.1)] making in the sphere the circle ACB,[Margin: b. 6. huius (*Sphaerica* I.6)] which will be a great circle, since it is drawn through the sphere's center. Also from C let the straight line CB be drawn to B, the point where the great circle ACB cuts the circle AB; and, by the definition of the pole, the straight line CB will be equal to the straight line CA. Since, then, AC is assumed to be the side of a square inscribed in the great circle ACB, CB too will be the side of the same square; and so the two arcs AC and CB will be quadrants making up the semicircle ACB, since the four equal sides of the square[Margin: c. 28. ter. (*Elements* III.28)] subtend four equal arcs of the circle. Therefore the straight line AB, the common section of the circles, will be a diameter of the great circle ACB, and hence of the sphere. Now since the great circle ACB, cutting the circle AB through the poles, bisects it,[Margin: d. 15. huius (*Sphaerica* I.15)] AB too will be the common section, a diameter of circle AB; and hence, since it is also a diameter of the sphere, AB will be a great circle. If, therefore, in a sphere there is a circle, from whose pole, and so on. Which was to be demonstrated.

PROBL. 2. PROPOS. 18. [Margin: xxvii.]

To describe a straight line equal to the diameter of any given circle in a sphere.

Diagram. Two figures side by side. On the left, a circle ABCD with the inscribed triangle ABD, the diameter AC, and the chord DC. On the right, a kite-shaped quadrilateral EFGH.

- A (top of circle): point taken on the circumference
- B (left of circle): point taken on the circumference
- D (right of circle): point taken on the circumference
- C (bottom of circle): opposite end of diameter AC
- (center of circle, marked with a dot): center
- E (top of the kite): apex, corresponding to A
- F (left of the kite): foot of perpendicular, corresponding to B
- G (right of the kite): corresponding to D
- H (bottom of the kite): meeting point of the perpendiculars FH and GH

In the sphere let there be any given circle ABCD, a straight line equal to whose diameter is to be described. Taking any three points A, B, D on the circle's circumference, and joining the straight lines AB, AD, BD,[Margin: a. schol. 22. primi (Scholium on *Elements* I.22)] let there be constructed a triangle EFG equal to the triangle ABD, in such a way that the side EF is equal to AB, EG to AD, and

FG to BD. For the three intervals AB, AD, BD, taken with a compass on the surface of the sphere, can be transferred onto a plane, and thus a triangle can be constructed whose three sides are equal to those three distances. Then from G and F let the perpendiculars FH and GH be drawn to the straight lines EF and EG, meeting at H, and let the straight line EH be joined. I say that EH is equal to the diameter of circle ABCD. For when the diameter AC has been drawn, let the straight line DC be joined.[Margin: b. schol. 32. primi (Scholium on *Elements* I.32)] Now since the four angles of the quadrilateral EFHG are equal to four right angles, and the angles EFH and EGH are right angles, the angles FEG and FHG will be equal to two right angles; and so in the quadrilateral EFHG any two opposite angles will be equal to two right angles.[Margin: c. schol. 22. tertij (Scholium on *Elements* III.22)] Therefore a circle can be described about it;[Margin: d. 27. tert. (*Elements* III.27)] and when it has been described, the angles EFG and EHG will stand on the same segment, whose chord is EG, insist-

(The demonstration of Prop. 18 breaks off mid-word at the foot of the page and continues onto the following page.)

PROBL. 2. PROPOS. 18 (continued)

[Continuation of the demonstration begun in the previous proposition — construct a straight line equal to the diameter of a given circle.]

standing [on equal arcs], are equal. ^e Now the angle EFG is equal to the angle ABD, because the two sides EF, FG are equal to the two sides AB, BD, and the base EG to the base AD, by construction; ^f and the angle ABD is equal to the angle ACD. Therefore the angle EHG too will be equal to the angle ACD. Moreover the right angle EGH is equal to the angle ADC, ^g since this angle too is a right angle, standing in the semicircle ADC. Therefore the triangles EHG and ACD have two angles equal to two angles, and also the side EG equal to the side AD, which subtends one of the equal angles. ^h Hence the side EH too will be equal to the side AC. Therefore we have constructed the straight line EH equal to the diameter AC of the circle ABCD. Which was to be done.

[Margin: e — *Elements* I.8 (8. primi)] [Margin: f — *Elements* III.27 (27. tertij)]

[Margin: g — *Elements* III.31 (31. tertij)] [Margin: h — *Elements* I.26 (26. pri.)]

Diagram. The figure for Proposition 18. On the left, a circle with a vertical diameter and interior lines; on the right, a kite-shaped quadrilateral.

- A (top of circle): endpoint of the diameter
- B (left of circle): vertex

- D (upper right, on circle): vertex, with chords to A and C
- C (bottom of circle): lower endpoint of the diameter
- E (apex, right figure): vertex; EH is the long axis
- F (left, right figure): vertex
- G (right, right figure): vertex
- H (bottom, right figure): vertex where the perpendiculars meet

PROBL. 3. PROPOS. 19.

[Margin: xxix.]

To construct a straight line equal to the diameter of a given sphere.

In the given sphere let two points A, B be taken at random, and from A as pole, with radius AB, ^a let a circle BD be described. ^b Let a straight line FG be drawn equal to the diameter of this circle, and on FG let a triangle EFG be built, having each of its remaining sides EF, EG equal to the drawn line AB — that is, by taking the interval AB with the compass, and so on. Then from F and G let perpendiculars FH, GH be drawn to EF, EG, meeting at H; and let the straight line EH be joined. I say that EH is equal to the diameter of the given sphere.

For let the diameter AC of the sphere be drawn, and through the straight lines AB, AC let a plane be passed, ^c making in the sphere the circle ABCD, ^d which will be a great circle, since it is drawn through the diameter of the sphere and hence through its center. Therefore this same plane, passing through A, the pole of the circle BD, ^e will bisect the circle BD; and consequently the common section BD will be a diameter of the circle BD. Now when the straight lines AD, DC are joined, the two sides AB, BD will be equal to the two sides EF, FG, and likewise the bases AD, EG will be equal. For FG is equal to the diameter BD by construction, and each of EF, EG is equal to AB, or to AD. ^f Therefore the angles ABD, EFG will be equal. ^g Now the angle ACD is equal to the angle ABD, and the angle EHG to the angle EFG, as was demonstrated in the preceding proposition. Therefore the angles ACD, EHG are also equal. But the right angles ADC, EGH are equal too, and the side AD, which is opposite one of the equal angles, is equal to the side EG. ^h Therefore the straight line EH too will be equal to the straight line AC. So we have constructed the straight line EH equal to the diameter AC of the given sphere. Which was to be done.

[Margin: a — Prop. 18 of this book (18. huius)] [Margin: b — Scholium on *Elements* I.22 (schol. 22. primi)] [Margin: c — Prop. 1 of this book (1. huius)] [Margin: d — Prop. 6 of this book (6. huius)] [Margin: e — Prop. 15 of this book

(15. huius)] [Margin: f — *Elements* I.8 (8. primi)] [Margin: g — *Elements* III.27 (27. tert.)] [Margin: h — *Elements* I.26 (26. pri.)]

Diagram. The figure for Proposition 19, of the same form as the preceding: a circle with a vertical diameter on the left, and a kite-shaped quadrilateral on the right.

- A (top of circle): endpoint of the sphere's diameter AC
- B (left of circle): point on the circle BD
- D (right, on circle): endpoint of the diameter BD of circle BD
- C (bottom of circle): lower endpoint of the diameter AC
- E (apex, right figure): vertex; EH is the constructed line equal to the diameter
- F (left, right figure): vertex, with perpendicular FH
- G (right, right figure): vertex, with perpendicular GH
- H (bottom, right figure): meeting point of the perpendiculars

SCHOLIVM.

[Margin: xxx.]

In another version the following theorem is added.

A straight line drawn from the pole of any circle in a sphere to the surface of the sphere, being equal to the straight line drawn from the same pole to the circumference of the circle, falls upon the circumference of the circle.

In the sphere, from A, the pole of the circle BC, let a straight line AD be drawn at random to its circumference. This line will be less than the diameter of the sphere, and hence less than the diameter of a great circle in the sphere, since the diameter of the sphere is the greatest of all straight lines drawn in the sphere. Now from the same pole A let a straight line AE be drawn to the surface of the sphere, equal to AD. I say that the straight line AE falls upon the circumference of the circle BC. For if this is not so, suppose it does not fall on that circumference. Through the straight line AE and the center of the sphere let a plane be passed,ⁱ making in the sphere the circle ABC,^k which will be a great circle, since it passes through the center of the sphere. Let the circle ABC cut the circle BC at the points B, C. Then the straight line AE does not fall on the points B, C, since it is supposed not to fall on the circumference of the circle BC. So when the straight line AB is drawn, it will, by the definition of the pole, be equal to AD, and hence to AE. And

since each of AB, AE is less than the diameter of the great circle ABC, as has been said, ¹ the arcs AB, AE — being segments less than a semicircle — will be equal, the part and the whole. Which is absurd. Therefore the straight line AE falls upon the circumference of the circle BC. Which was the thing proposed.

[Margin: i — Prop. 1 of this book (1. huius)] [Margin: k — Prop. 6 of this book (6. huius)] [Margin: l — *Elements* III.28 (28. tertij)]

Diagram. The figure for this scholium theorem : a sphere (large circle) with an arc across its upper part, the pole A above, and lines drawn from A to points on the circle at the left.

- A (top): the pole of the circle BC
- F (left, on the arc): point on the circumference
- B (left): point where the arc meets the great circle
- E (lower left): point on the sphere's surface
- D (upper middle, on the arc): point on the circumference of circle BC
- C (right): point on the circumference

PROBL. 4. PROPOS. 20.

[Margin: xxxj.]

Through two given points on a spherical surface to describe a great circle.

On the given spherical surface let there be two points A, B, through which a great circle is to be described. Now if the points A, B are diametrically opposite on the sphere, it is certain that infinitely many great circles can be drawn through them, namely by passing infinitely many planes through the diameter of the sphere that joins those points. But if the points A, B are not on a diameter of the sphere, then from A as pole, with an interval equal to the side of the square inscribed in a great circle,

let a circle CD be described, ^a which will be a great circle, since the straight line drawn from A, the pole, to its circumference is equal to the side of the square inscribed in a great circle, on account of the interval by which the circle CD was described. And this interval is found as follows. When the diameter of the sphere, or of a great circle, has been found — as was shown in the preceding proposition — the side of the square inscribed in the circle of that diameter will be the interval sought, as is plain. Similarly, from B as pole and with the same interval as before, let a circle EF be described, ^b which again will be a great circle. Let this

cut the former in the point G, and from G let the straight lines GA, GB be drawn to the poles A, B; each of these, by construction, will be equal to the side of the square inscribed in a great circle, for the circles CD, EF were described from the poles A, B with just such an interval. Therefore GA, GB are equal. Now from G as pole and with radius GA let a circle AEDFCB be described, ^c which will be a great circle, since the straight line GA, drawn from G, the pole, to its circumference, is equal to the side of the square inscribed in a great circle, as has been demonstrated. And since the straight line GB, equal to GA, drawn to the surface of the sphere, ^d falls on the circumference of the circle AEDFCB, the described great circle AEDFCB will therefore pass through the two given points A, B on the surface of the sphere. So through the two given points on the spherical surface we have described a great circle. Which was to be done.

[Margin: a — Prop. 17 of this book (17. huius)] [Margin: b — Prop. 17 of this book (17. huius)] [Margin: c — Prop. 17 of this book (17. huius)] [Margin: d — Scholium on Prop. 19 of this book (schol. 19. huius)]

Diagram. The figure for Proposition 20 : a circle within which two great circles cross, meeting at the central point G.

- A (top left, on circle): a given point / pole
- B (top right, on circle): a given point / pole
- E (left, on circle): point on the circumference
- C (right, on circle): point on the circumference
- G (center): intersection of the two described circles
- D (lower left, on circle): point on the circumference
- F (lower right, on circle): point on the circumference

PROBL. 5. PROPOS. 21.

[Margin: xxxij.]

To find the pole of any given circle in a sphere.

Let the pole of the circle AB, given in a sphere, be sought, and let the circle AB first be not a great circle. Taking two points C, D at random on the circumference, ^a let each of the arcs CAD, CBD be bisected at the points A and B, ^b through which let a great circle AEB be described; and let the arc AEB be bisected at E. I say that E is a pole of the circle AB. For since the arcs AC, AD are equal, and likewise BC, BD, the whole arcs ACB, ADB will be equal. Therefore, since the great circle AEB bisects the circle AB, which is not a great circle, at A and B, ^c it will cut it through

the poles. So the point E, being equally distant from the circumference of the circle AB, is the pole of the circle AB. In the same way, if the remaining arc AFB is bisected at F, then F will be the other pole of the circle AB.

But now let the given circle AB be a great circle. Taking the points C, D again at random, ^d and bisecting the arcs CAD, CBD at A, B, we shall show, as before, that the whole arcs ACB, ADB are equal, and therefore that each is a semicircle of a great circle. So when one of the semicircles, namely ACB, is bisected at G, the straight line GA, subtending a quadrant of the circle, will be the side of the square inscribed in the great circle AB — as is clear from Proposition 6 of Book IV of Euclid. Accordingly, from the pole G, with radius GA, let a circle AEB be described, ^e which will be a great circle, since the straight line drawn from G, the pole, to its circumference — namely to the point A — is equal to the side of the square inscribed in the great circle AB. Finally let the arc AEB be bisected at E. I say that E is a pole of the circle AB. For since the great circle ACB passes through G, the pole of the great circle AEB, ^f conversely the great circle AEB will pass through the poles of the great circle ACB. Therefore the point E, equally removed from the circumference of the circle ACB, is a pole of the circle ACB. In the same way, when the arc AFB is bisected at F, F will be the other pole of the circle ACB. Therefore we have found the pole of any given circle in a sphere. Which was to be done.

[Margin: a — *Elements* III.30 (30. tert.)] [Margin: b — Prop. 20 of this book (20. huius)] [Margin: c — Prop. 14 of this book (14. huius)] [Margin: d — *Elements* III.30 (30. tert.)] [Margin: e — Prop. 17 of this book (17. huius)] [Margin: f — Scholium on Prop. 15 of this book (schol. 15. huius)]

Diagram. The two figures for Proposition 21, side by side, each a sphere (circle) with a lens formed by two arcs crossing its middle. Left figure (circle AB not a great circle); right figure (circle AB a great circle).

- Left — E (top): a pole of circle AB; D (upper, on arc): point on the described circle; A (left) and B (right): endpoints where the arcs meet; C (center, below): point on the circle AB; F (bottom): the other pole
- Right — E (top): a pole; D (upper, on arc): point; A (left) and B (right): endpoints; C and G (center, below): points on the bisected semicircle; F (bottom): the other pole

SCHOLIVM.

[Margin: xxxiiij.]

In another version the following two theorems are demonstrated.

I.

If some point is taken on the surface of a sphere, and from that point more than two equal straight lines fall to the circumference of some circle given in the sphere, the point taken is the pole of that circle.

On the surface of the sphere ABC let a point A be taken, from which more than two equal straight lines AD, AE, AF fall to the circumference of the circle BC. I say that A is the pole of the circle BC. ^a For let a perpendicular AG be dropped from A to the plane of the circle BC, and let the straight lines DG, EG, FG be joined; then, by the third definition of Book XI of Euclid, all three angles at G are right angles. ^b Hence the square on AD equals the squares on AG, GD; likewise the square on AE equals the squares on AG, GE; and the square on AF equals the squares on AG, GF. Since, then, the squares of the equal straight lines AD, AE, AF are equal, the squares on AG, GD together will be equal to the squares on AG, GE together, and likewise to the squares on AG, GF together; and when the common square of the line AG is taken away, the remaining squares of the lines GD, GE, GF will be equal, and hence the straight lines GD, GE, GF themselves will be equal. ^c Therefore G will be the center of the circle BC; ^d and consequently the straight line GA, drawn from the center G perpendicular to the circle BC, will fall on the pole of the circle BC. So the point A is the pole of the circle BC. Which was the thing proposed.

[Margin: a — *Elements* XI.11 (11. undec.)] [Margin: b — *Elements* I.47 (47. pri.)]
[Margin: c — *Elements* III.9 (9. tertij)] [Margin: d — Scholium on Prop. 8 of this book (schol. 8. huius)]

Diagram. The figure for this first scholium theorem : a circle with the apex A above, from which lines descend to points along the base, meeting at the central foot G.

- A (top): the point taken on the surface, shown to be the pole
- B (left): point on the circle
- C (right): point on the circle
- G (center, at the base): foot of the perpendicular, the center of circle BC
- D, E, F (along the base): points on the circumference of circle BC to which the equal lines are drawn

II.

[Margin: xxxiv.]

In a sphere, circles from whose poles the straight lines drawn to their circumferences are equal, are equal to one another. And of equal circles, the straight lines drawn from their poles to the circumferences are equal.

In the sphere ABCDEF, whose center is G, let there be two circles BF, CE, from whose poles A, D the straight lines AF, DE drawn to their circumferences are equal. I say that the circles BF, CE are equal. ^a From the poles A, D let perpendiculars AH, DI be drawn to the planes of the circles; ^b these will fall on their centers H, I, and, produced further, on the remaining poles, ^c and hence also on G, the center of the sphere. So when the radii of the sphere FG, EG are drawn, and the radii of the circles FH, EI, since the sides AG, GF are equal to the sides DG, GE, and the base AF to the base DE, ^d the angles AGF, DGE will be equal. Now the angles H, I are right angles, by the third definition of Book XI of Euclid. Therefore the triangles FGH, EGI have two angles equal to two angles; and they have also the side FG, opposite the right angle, equal to the side EG. ^e Therefore the radii FH, EI too will be equal, and hence the circles BF, CE are equal. Which was proposed in the first place.

Now let the circles BF, CE be equal. I say that the straight lines AF, DE drawn from their poles to the circumferences are also equal. For when the same construction is made, the radii FH, EI will be equal, ^f and the circles themselves will be equally distant from the center of the sphere. Therefore the perpendiculars GH, GI will be equal, and hence the remaining lines AH, DI will be equal too. Since, then, the sides AH, HF are equal to the sides DI, IE, and they contain the equal angles H, I — these being right angles, by the third definition of Book XI of Euclid — ^g the bases AF, DE will be equal. Which was proposed in the second place.

[Margin: a — *Elements* XI.11 (11. undec.)] [Margin: b — Prop. 9 of this book (9. huius)] [Margin: c — Prop. 10 of this book (10. huius)] [Margin: d — *Elements* I.8 (8. primi)] [Margin: e — *Elements* I.26 (26. pri.)] [Margin: f — Prop. 6 of this book (6. huius)] [Margin: g — *Elements* I.4 (4. pri.)]

Diagram. The figure for the second scholium theorem : a circle (the sphere ABCDEF) with center G, showing two circles BF and CE seen edge-on as arcs, with perpendiculars from the poles.

- A (top left): pole of the circle BF
- B (top right): point on the circle BF
- F (left): point on the circumference of circle BF
- H (upper interior): center of circle BF, foot of AH
- G (center): center of the sphere

- E (lower left): point on circle CE
- I (lower interior): center of circle CE, foot of DI
- C (right): point on circle CE
- D (bottom): pole of the circle CE

THEOR. 17. PROPOS. 22.

[Margin: o.[?]]

If in a sphere a straight line drawn through the center bisects some straight line not drawn through the center, it will cut it at right angles. And if it cuts it at right angles, it will also bisect it.

In the sphere whose center is A, let the straight line AB, drawn through the center, bisect at B the straight line CD, which is not drawn through the center. I say that CD is cut at right angles. For when a plane is passed through the straight lines AB, CD, ^a it makes the circle CD, ^b which will be a great circle, since it passes through the center of the sphere. Now in the circle CD, the straight line AB, passing through its center A, bisects at B the straight line CD, which is not drawn through the center; ^c therefore it cuts it at right angles. And if it cuts it at right angles, it will also bisect it. So if in a sphere a straight line, and so on. Which was to be demonstrated.

[Margin: a — Prop. 1 of this book (1. huius)] [Margin: b — Prop. 6 of this book (6. huius)] [Margin: c — *Elements* III.3 (3. tertij)]

Diagram. The figure for Proposition 22 : a circle with center A, and a chord CD near the bottom cut by the radius AB.

- A (center): the center of the sphere / great circle
- B (bottom, on the chord): point where AB meets CD
- C (bottom left): endpoint of the chord
- D (bottom right): endpoint of the chord

SCHOLIVM.

Here in the Greek exemplar another theorem is added, which is exactly the same as what was demonstrated in Proposition 7. Hence we have thought it superfluous to repeat it here.

FINIS LIBRI I. THEODOSII.

THEODOSIUS'S *SPHAERICA*, BOOK TWO

DEFINITION

In a sphere, circles are said to be tangent to one another when the common section of their planes touches each of the two circles.

For a straight line tangent to some circle in a sphere is tangent also to the surface of the sphere at the very point where it touches the circle. (Otherwise, if it did not touch the surface but cut it, it would necessarily cut the circle as well, since it lies in the circle's plane and would join two points on the sphere's surface — the two points at which it is said to cut it; and these two points would lie on the circumference of the circle too, since the plane of the circle is drawn through that straight line and is therefore cut by it at those two points.) It follows that the circumferences of two circles whose common section — the section produced by extending their planes — is tangent to both circles have in common only that single point at which the section touches the sphere. For at that point, and at no other, can that common section be tangent to both circles, since all its other points lie outside the surface of the sphere and hence outside each circle. Theodosius, then, was right to say that circles in a sphere are tangent to one another when the common section of their planes has touched each circle.

THEOR. 1. PROPOS. 1. [Margin series: j.]

In a sphere, parallel circles are around the same poles.

In the sphere ABCDEF, let BF and CE be parallel circles. I say that they are around the same poles. [Margin: a — 21. 1. huius (*Sphaerica* I.21)] Let A and D be the poles of circle BF, and let the straight line AD be joined; [Margin: b — 10. 1. huius (*Sphaerica* I.10)] this line will be perpendicular to circle BF and will pass through the center of the sphere. Since, then, the straight line AD is perpendicular to circle BF, [Margin: c — schol. 14. undec. (scholium on *Elements* XI.14)] it will also be perpendicular to the parallel circle CE. And since it passes through the center of the sphere, as has been shown, [Margin: d — 8. 1. huius (*Sphaerica* I.8)] it falls on the poles of circle CE. Therefore A and D are the poles of circle CE; but they are also the poles of circle BF. In the sphere, then, the parallel circles BF and CE are around the same poles A and D, and so on. Which was to be demonstrated.

Diagram. A sphere shown as a circle with a vertical axis. The two parallel circles are drawn as horizontal ellipses.

- A (top): pole; upper end of the axis

- D (bottom): pole; lower end of the axis, with a center dot marked on the axis AD
- B (left, mid-height): point on the upper parallel circle BF
- F (right, mid-height): point on the upper parallel circle BF
- C (lower left): point on the lower parallel circle CE
- E (lower right): point on the lower parallel circle CE

THEOR. 2. PROPOS. 2. [Margin series: ij.]

In a sphere, circles that are around the same poles are parallel.

In the same sphere ABCDEF, let BF and CE be circles around the same poles A and D. I say that they are parallel. When the straight line AD is joined, [Margin: a — 10. 1. huius (*Sphaerica* I.10)] it will be perpendicular to each of the two circles. [Margin: b — 24. undec. (*Elements* XI.24)] Therefore the planes of circles BF and CE are parallel. In the sphere, then, circles that are around the same poles are parallel. Which was to be shown.

SCHOLIUM

This theorem and the one that follows are demonstrated in another version as well.

In a sphere there are not more than two circles that are equal and parallel.

In any sphere, let there be, if possible, more than two circles that are equal and parallel — say three, AB, CD, EF — [Margin: c — 1. huius (*Sphaerica* II.1)] which will be around the same poles. Let their poles then be G and H, and let the straight line GH be joined; [Margin: d — 10. 1. huius (*Sphaerica* I.10)] it will pass through I, the center of the sphere, and through K, L, M, the centers of the circles, and it will be perpendicular to circles AB, CD, EF. Since, then, the circles AB, CD, EF are equal, [Margin: e — 6. 1. huius (definition 6 of *Sphaerica* Book I)] they will be equally distant from the center of the sphere I. By definition 6 of Book I of this work, therefore, the perpendiculars

Diagram. A sphere shown as a circle with a vertical axis GH; three parallel circles are drawn as horizontal ellipses, with their centers K, L, M marked on the axis.

- G (top): pole; upper end of axis GH
- H (bottom): pole; lower end of axis GH
- I (center): center of the sphere, on the axis

- A (upper left) / K (on axis) / B (upper right): the topmost parallel circle AB, with center K
- C (mid left) / L (on axis) / D (mid right): the middle parallel circle CD, with center L
- E (lower left) / M (on axis) / F (lower right): the lowest parallel circle EF, with center M

the perpendiculars IK, IL, IM will be equal — namely a part, IL, and the whole, IM. Which is absurd. In a sphere, therefore, there are not more than two circles that are equal and parallel. Which was to be demonstrated.

THEOR. 3. PROPOS. 3. [Margin series: iv.]

If in a sphere two circles cut, at one and the same point, the circumference of that great circle on which they have their poles, then those circles are tangent to one another.

In a sphere let two circles AB, AC cut, at the point A, the circumference of the great circle ABC that passes through their poles. I say that circles AB, AC are tangent to one another at A. Since the great circle ABC cuts circles AB, AC through their poles, [Margin: a — 15. 1. huius (*Sphaerica* I.15)] it will bisect them and cut them at right angles. Therefore the common sections of circle ABC with circles AB, AC — namely the straight lines AB, AC — are diameters of circles AB, AC. Let DE also be the common section of the planes in which circles AB, AC lie; it will pass through the point A, because the planes of the circles at A are supposed to cut circle ABC. And since the plane of circle ABC has been shown to be at right angles to the planes of circles AB, AC, conversely the planes of circles AB, AC will be at right angles to the plane of circle ABC; [Margin: b — 19. undeci. (*Elements* XI.19)] and hence DE, their common section, will also be perpendicular to that same plane of circle ABC. It will therefore also be perpendicular to the diameters AB, AC lying in the same plane, by definition 3 of Book XI of Euclid. [Margin: c — coroll. 16 tertij (corollary of *Elements* III.16)] Consequently DE is tangent to each of the circles AB, AC at A; and therefore, by the definition of this book, circles AB, AC are tangent to one another at the point A. If, then, in a sphere two circles cut, and so on. Which was to be shown.

Diagram. A great circle ABC drawn as a circle; two lens-shaped circles rise from the lower points to the shared apex, with several chords drawn to that apex.

- E, A (top, adjacent): E is one end of the common section DE; A is the shared point of contact / apex of the two circles

- D (center): point on the common section DE, on the axis running down from A/E
- B (lower left): point on circle AB
- C (lower right): point on circle AC [engraved as a slightly open loop; C by context]

THEOR. 4. PROPOS. 4. [Margin series: v.]

If in a sphere two circles are tangent to one another, the great circle described through their poles will pass through their point of contact.

In a sphere let circles AB, CB be tangent to one another at B; and through D, the pole of circle AB, and E, the pole of circle CB, [Margin: a — 20. 1. huius (*Sphaerica* I.20)] let the great circle DE be described. I say that circle DE passes through the point of contact B. For suppose, if possible, that it does not pass through the contact B but cuts the circumference — say of circle CB — at F. Then with pole D and radius DF let a circle FG be described; since it is described with a greater radius than circle AB, it will cut circle CB at F, seeing that circle AB is tangent to that same circle at the point B, beyond which circle GF, described from pole D, extends. But since in the sphere the two circles GF, CF cut, at one and the same point F, the great circle DFE described through their poles, [Margin: b — 3. huius (*Sphaerica* II.3)] the two circles GF, CF will be tangent to one another at F. Yet they also cut one another at F, as was said — which is absurd. Therefore the great circle DE does not cut the circles AB, CB anywhere but at B, their point of contact, and so it will pass through their contact. And so, if in a sphere two circles are tangent to one another, and so on. Which was to be shown.

Diagram. Two overlapping circles side by side; an arc (the great circle DFE) sweeps across from the left circle to the right.

- G (top left): point on the left circle (circle AB)
- A (upper middle-left): point inside/on the left circle
- D (left): point on the arc DFE
- B (center): point of contact / intersection of the two circles
- F (just above B): where the arc meets the circles
- C (top right): point on the right circle (circle CB)
- E (lower right): end of the arc DFE

THEOR. 5. PROPOS. 5. [Margin series: vj.]

If in a sphere two circles are tangent to one another, the great circle described through the poles of one and through the point of contact of both circles will pass through the poles of the other circle as well.

In a sphere let two circles AB, CB be tangent to one another at B, and let D, E be their poles. I say that the great circle described through D, the pole of circle AB, and through the contact B passes also through E, the pole of circle CB. For if it can be, suppose it does not pass through E but through some other point F — this being the great circle DBF. [Margin: a — 20. 1. huius (*Sphaerica* I.20)] Then through the poles D, E let the great circle DE be described, [Margin: b — 4. huius (*Sphaerica* II.4)] which will certainly pass through the contact B; and so the two great circles DBF, DBE will cut one another at D and B, [Margin: c — 11. 1. huius (*Sphaerica* I.11)] and hence bisect one another. Each arc DB, then, will be a semicircle. But since a great circle that passes through one of the poles of any circle in a sphere [Margin: d — coroll. 16, 1. huius (corollary of *Sphaerica* I.16) [?]] passes also through the other pole, and there is a semicircle of a great circle interposed between the two poles of that circle, it would follow that, D being one of the poles of circle AB, the point B would be the other pole. Which is absurd, for B lies on the circumference of the circle. Therefore the great circle DB passes through E. And so, if in a sphere two circles are tangent to one another, and so on. Which was to be shown.

Diagram. Two circles tangent at a central point B; several straight lines run from B to the outer labelled points.

- C (top left): point on the left circle (circle AB)
- A (top right): point on the right circle (circle CB)
- F (mid left): pole / point on the left circle
- B (center): point of contact of the two circles
- D (right): pole of one circle
- E (lower left): pole of the other circle

THEOR. 6. PROPOS. 6. [Margin series: vij.]

If in a sphere a great circle is tangent to any of the circles described on the spherical surface, it will also be tangent to another circle equal and parallel to it.

In a sphere let the great circle AB be tangent to circle AC at A. I say that circle AB is tangent also to another circle equal and parallel to AC. [Margin: a — 20. 1. huius

(*Sphaerica* I.20)] Let D be the pole of circle AC, and through D and A let the great circle DA be described; since it passes through D, the pole of circle AC, and through the contact A, [Margin: b — 5. huius (*Sphaerica* II.5)] it will pass also through the poles of circle AB. Now taking E, the other pole of circle AC, let the straight line DE be drawn; [Margin: c — 10. 1. huius (*Sphaerica* I.10)] it will pass through the center of the sphere and so will be a diameter of the sphere. From pole E, then, and with radius EB, let a circle BF be described. I say that the great circle AB is tangent also to circle BF at B, and that circle BF is equal and parallel to circle AC. For since the straight line DE, passing through the poles of circles AC, BF, [Margin: d — 10. 1. huius (*Sphaerica* I.10)] is perpendicular to those circles, [Margin: e — 24. undec. (*Elements* XI.24)] circles AC, BF will be parallel. [Margin: f — 11. 1. huius (*Sphaerica* I.11)] Again, since great circles in a sphere bisect one another, ACB will be a semicircle, and hence equal to the semicircle DCE. Therefore, when the common arc BD is removed, the arcs DA, EB remain equal; [Margin: g — 29. tert. (*Elements* III.29)] and hence the straight lines DA, EB, drawn from the poles D, E to the circumferences of circles AC, BF, are equal. [Margin: h — schol. 21. 1. huius (scholium on *Sphaerica* I.21)] Therefore circles AC, BF are equal. Finally, since the circles cut, at one and the same point B, the great circle AEB on which they have their poles, [Margin: i — 3. huius (*Sphaerica* II.3)] the circles AB, BF are tangent to one another at B. Therefore the great circle AB, being tangent in the sphere to circle AC, is tangent also to another circle BF, equal and parallel to AC. And so, if in a sphere a great circle is tangent to any of the circles, and so on. Which was to be demonstrated.

Diagram. A sphere shown as a circle with a vertical axis DE; two parallel circles (drawn as ellipses) and diagonal great-circle arcs crossing the interior.

- D (top): pole; upper end of axis DE
- A (upper left) / C (upper right): the upper parallel circle AC
- F (lower left) / B (lower right): the lower parallel circle BF
- E (bottom): pole; lower end of axis DE, with a center dot marked on the axis

COROLLARIUM

From this it is clear that the points of contact A, B are diametrically opposite. For it has been shown that ACB is a semicircle, and therefore that the straight line drawn from A to B is a diameter of the sphere, or of the great circle ACB, and so on.

THEOR. 7. PROPOS. 7. [Margin series: viij.]

If in a sphere there are two equal and parallel circles, a great circle that is tangent to one of them will be tangent to the other as well.

In the same figure let there be two equal and parallel circles AC, BF, and let the great circle AB be tangent to AC. I say that this same circle AB is tangent also to BF. For if AB were not tangent to BF, [Margin: a — 6. huius (*Sphaerica* II.6)] it would surely be tangent to another circle equal and parallel to AC. Then, since BF too is supposed equal and parallel to that same AC, there would be three circles in the sphere — namely AC, BF, and that other circle which AB touches — equal and parallel to one another. Which is absurd; [Margin: b — schol. 2. huius (scholium 2 of *Sphaerica* Book II)] for there are not more than two circles equal and parallel in a sphere. Therefore circle AB will be tangent to circle BF. And so, if in a sphere there are two equal and parallel circles, and so on. Which was to be shown.

SCHOLIUM

In another version the following theorem too is demonstrated. [Margin series: ix.]

Parallel circles in a sphere that some great circle is tangent to are equal to one another.

In this same figure let there be two parallel circles AC, BF, which the great circle AB touches at A, B. I say that circles AC, BF are equal to one another. Since the circles AC, BF are supposed parallel, [Margin: c — 1. huius (*Sphaerica* II.1)] they will be around the same poles, which let be D, E; [Margin: d — 20. 1. huius (*Sphaerica* I.20)] and through these and the poles of circle AB let the great circle AFB be described, [Margin: e — 4. huius (*Sphaerica* II.4)] which will pass through the contacts A, B. Now since great circles in a sphere cut one another in half, ADB will be a semicircle, and hence equal to the semicircle DBE. Therefore, when the common arc DB is removed, the arcs DA, EB remain equal; [Margin: f — 29. tert. (*Elements* III.29)] and hence also the straight lines DA, EB, drawn from the poles D, E to the circumferences of circles AC, BF, are equal. [Margin: g — schol. 21. 1. huius (scholium on *Sphaerica* I.21)] Therefore circles AC, BF will be equal. Which was the thing proposed.

THEOR. 8. PROPOS. 8. [Margin series: x.]

If in a sphere a great circle is oblique to some circle of the sphere, it will be tangent to two circles that are equal to one another and parallel to the aforesaid circle to which it is oblique.

In a sphere let the great circle AB be oblique to any circle CD. I say that circle AB is tangent to two circles that are equal to one another and parallel to CD itself. [Margin: a — 21. 1. huius (*Sphaerica* I.21)] Let E, F be the poles of circle CD, [Margin: b — 20. 1. huius (*Sphaerica* I.20)] and through these and the poles of circle AB let the great circle EAB be described, cutting AB at A and B. Then from pole E and with radius EA let a circle AG be described. And since circles AB, AG cut, at one and the same point A, the great circle EAB on which they have their poles, [Margin: c — 3. huius (*Sphaerica* II.3)] they are tangent to one another at A. Therefore the great circle AB, being tangent to circle AG, [Margin: d — 6. huius (*Sphaerica* II.6)] will be tangent to another circle equal and parallel to it, which let be BH. Now since the parallel circles AG, BH [Margin: e — 1. huius (*Sphaerica* II.1)] are around the same poles E, F — and E, F are also the poles of circle CD — the three circles AG, CD, BH will be around the same poles; [Margin: f — 2. huius (*Sphaerica* II.2)] and hence they will be parallel to one another. Therefore the great circle AB is tangent to two circles AG, BH, which are equal to one another and parallel to CD itself, to which it is oblique. And so, if in a sphere a great circle is oblique to some circle, and so on. Which was to be shown.

Diagram. A sphere shown as a circle with axis EF; a great circle AB drawn obliquely, the circle CD, and the two parallel circles AG and BH, with several arcs crossing the interior.

- E (top): pole; upper end of axis EF
- A (upper left): point where the oblique great circle meets circle AG
- G (upper right): point on circle AG
- C (left): point on circle CD
- D (right): point on circle CD
- H (lower left): point on circle BH
- B (lower right): point where the oblique great circle meets circle BH
- F (bottom): pole; lower end of axis EF

SCHOLIVM

Another theorem is added here in another recension, namely the following.

If in a sphere a great circle is tangent to some one of the circles on the spherical surface, it will be oblique to the other circles that it cuts, which are parallel to the one it touches.

In the same figure, let the great circle AB be tangent to circle AG, but cut circle CD, which is parallel to AG. I say that circle AB is oblique to circle CD. For the great

circle AB, being tangent to circle AG, does not pass through the poles of AG — since, if it were drawn through those poles, it would bisect AG [Margin: a — *Sphaerica* I.15] and not merely touch it. And so it does not pass through the poles of circle CD either, because the parallel circles AG and CD have the same poles [Margin: b — *Sphaerica* II.1]. Therefore the great circle AB will not cut circle CD at right angles — for otherwise it would pass through the poles of CD [Margin: c — *Sphaerica* I.13]. It is therefore oblique to circle CD, which is what was proposed.

Diagram. A sphere (large circle) carrying the tangent-and-parallel configuration of the tangency scholium: a great circle AB, a circle AG which it touches, and a parallel circle CD which it cuts.

- A (upper left, on the sphere): endpoint of the great circle AB / of circle AG
- E (top of the sphere): upper point
- G (upper right): second point of circle AG
- C (left, on the sphere): left endpoint of the cut circle CD
- D (right): right endpoint of circle CD
- H (lower left): point of the lower parallel circle
- B (lower right): the other endpoint of great circle AB
- F (bottom of the sphere): lowest point (pole)

THEOR. 9. PROPOS. 9.

[Margin: x ij]

If in a sphere two circles cut one another, a great circle drawn through their poles will bisect the segments of those circles.

In a sphere let two circles ABCD and EDFB cut one another at the points B and D, and through their poles let the great circle AFCE be described [Margin: a — *Sphaerica* I.20], cutting the said circles at the points A, C, E, F. I say that circle AFCE bisects the segments BAD, BCD, BED, BFD.

For since the great circle AFCE cuts the circles ABCD and EDFB in half and at right angles — because it is drawn through their poles [Margin: b — *Sphaerica* I.15] — the common sections AC and EF, which it makes with those circles, will be diameters of them, cutting one another at G. For the straight lines AC and EF will cut one another, since they lie in the same plane, that of circle AFCE; and the point F lies between the points A and C, and the point E lies between the same points. Let the straight lines BG and DG be joined. Then the three points B, G, D lie in the plane of each of the circles ABCD and EDFB, and therefore in their common

section; and their common section is a straight line [Margin: c — *Elements* XI.3]. Therefore BGD is a straight line.

And since circle AFCE has been shown to cut each of the circles ABCD and EDFB at right angles, each of them in turn is at right angles to circle AFCE; and so BD, their common section, will be perpendicular to that same circle [Margin: d — *Elements* XI.19 (printed "29")(?)]. The angles BGA, DGA, BGC, DGC will therefore be right angles, by the third definition of *Elements* XI. Since, then, the diameter AC passes through the centre of circle ABCD and cuts the straight line BD at right angles, it will bisect it [Margin: e — *Elements* III.3]. And so, since the sides AG, GB are equal to the sides AG, GD and contain equal angles, namely right angles [Margin: f — *Elements* I.4], the bases AB, AD, which subtend the arcs AB, AD, will be equal to one another [Margin: g — *Elements* III.28]; and the arcs AB, AD will accordingly be equal. In the same way we shall show that the arcs CB, CD are equal, and likewise the arcs EB, ED, and FB, FD. The circle AFCE therefore divides the segments BAD, BCD, BED, BFD in half. Hence, if in a sphere two circles cut one another, and so on. Which was to be demonstrated.

Diagram. A sphere in which two circles ABCD and EDFB intersect at B and D, with the great circle AFCE drawn through their poles; the diameters AC and EF meet at the interior point G.

- E (top): upper pole-point of the great circle AFCE
- C (upper right): intersection point
- D (upper left, interior): intersection of the two circles
- B (centre right): the other intersection of the two circles
- A (left): endpoint of the great circle
- G (centre, interior): where the diameters AC and EF cross
- F (bottom): lower point of the great circle AFCE

SCHOLIUM

Two other theorems are added here in another recension, namely these.

I.

[Margin: x iij]

If in a sphere circles cut one another, another circle that bisects their segments passes through their poles and is a great circle.

In the same figure let two circles ABCD and EDFB cut one another at the points B, D, and let some other circle AFCE bisect the segments BAD, BCD, BED, BFD. I say that circle AFCE passes through their poles and is a great circle. For since the arcs AD, AB are equal, and likewise CD, CB, the whole arcs ADC, ABC will be equal, and therefore semicircles; and in the same way EDF, EBF will be semicircles. The circle AFCE therefore bisects the circles ABCD and EDFB, and so its common sections with them, AC and EF, cutting one another at G, are the diameters of those circles. Now if the straight lines BG, DG are joined, then, since the three points B, G, D lie in the plane of each of the circles ABCD and EDFB, and hence in their common section — and their common section is a straight line [Margin: a — *Elements* XI.3] — the line BGD will be straight.

And since the subtending straight lines DA, DC are severally equal to the subtending straight lines BA, BC, on account of the equal arcs [Margin: b — *Elements* III.29], and they contain equal angles, namely right angles standing in semicircles [Margin: c — *Elements* III.31 (printed "21")(?)], the angles DAC, BAC will be equal [Margin: d — *Elements* I.4]. This may also be proved thus: since the sides DA, AC are equal to the sides BA, AC, and the base DC is equal to the base BC, the angles DAC, BAC will be equal [Margin: e — *Elements* I.8]. Again, since the sides AD, AG are equal to the sides AB, AG and contain equal angles, as has been shown, the angles AGD, AGB will be equal [Margin: f — *Elements* I.4], and therefore right angles. So BGD is perpendicular to the straight line AC. In the same way we shall show that this same line BGD is perpendicular to EF. Hence BGD will be perpendicular to the plane of circle AFCE, which is drawn through the straight lines AC and EF [Margin: g — *Elements* XI.4]; and consequently each of the planes of the circles ABCD and EDFB, drawn through the straight line BGD, will be at right angles to that same plane of circle AFCE [Margin: h — *Elements* XI.18]; and conversely circle AFCE will be at right angles to the circles ABCD and EDFB. And so circle AFCE cuts the circles ABCD and EDFB both in half and at right angles. It is therefore a great circle, and passes through their poles [Margin: i — Scholium to *Sphaerica* I.15]. Which is what was proposed.

II.

[Margin: xiv]

If in a sphere two circles cut one another, a great circle that bisects any two of their segments, yet has the arc lying between those segments unequal to a semicircle, passes through their poles and bisects the two remaining segments.

In a sphere let two circles ABCD and EBFD cut one another at the points B, D, and let the great circle AFCE bisect any two of their segments — namely BAD, BED — at the points A, E, while the arc AFCE, intercepted between the said segments, is not a semicircle. I say that circle AFCE passes through the poles of the circles ABCD and EBFD and bisects the remaining segments BCD, BFD. For if circle AFCE did not pass through their poles, let another great circle AGE be described, if possible, through their poles, which will bisect their segments [Margin: a — *Sphaerica* II.9] and so will pass through the points A, E. Then the great circles AFCE and AGE will bisect one another at A and E [Margin: b — *Sphaerica* I.11], and therefore AFCE will be a semicircle — which is contrary to the hypothesis. The circle AFCE therefore passes through the poles of the circles ABCD and EBFD, and so it will bisect all of their segments [Margin: c — *Sphaerica* II.9]. Which is what was proposed.

Diagram. Two overlapping circles ABCD (left) and EBFD (right), crossing at B (top) and D (bottom) in a lens-shaped region, with the horizontal great circle AFCE drawn through both.

- A (far left): endpoint of the great circle, on circle ABCD
- B (top, where the two circles cross): upper intersection
- F (left of centre, on the lens): point of the great circle
- C (right of centre, on the lens): point of the great circle
- E (far right): endpoint of the great circle, on circle EBFD
- G (centre of the lens): interior point
- D (bottom, where the two circles cross): lower intersection

THEOR. 10. PROPOS. 10.

[Margin: xv]

If there are parallel circles in a sphere, and through their poles great circles are described, then the arcs of the parallel circles intercepted between the great circles are similar, while the arcs of the great circles intercepted between the parallel circles are equal.

In a sphere let there be parallel circles ABCD and EFGH, whose pole is I — for parallel circles in a sphere are about the same poles [Margin: a — *Sphaerica* II.1]. Through I let great circles AEIGC and BFIHD be described in any manner. I say that the arcs of the parallels AB, EF are similar, and likewise BC, FG; also CD, GH; and DA, HE; and that the arcs of the great circles intercepted between the parallels, namely AE, BF, CG, DH, are equal.

For let the common sections of circle AIC with the parallels be the straight lines AC, EG, which will be parallel [Margin: b — *Elements* XI.16]; and let the common sections of circle BID with the same parallels be the straight lines BD, FH, which likewise will be parallel. And since the great circles AIC, BID, described through the poles of the parallels, cut the parallels in half [Margin: c — *Sphaerica* I.15], AC and BD will be diameters of circle ABCD, and the point L, where they intersect, its centre; likewise EG and FH will be diameters of circle EFGH, and the point K, where they intersect, its centre. Since, then, the straight lines EK, KF are parallel to the straight lines AL, LB and lie in different planes, the angles EKF, ALB, at the centres K, L, will be equal [Margin: d — *Elements* XI.10]. Hence the arcs AB, EF will be similar, by what we have shown in the scholium to *Elements* III.22. In the same way BC, FG and CD, GH, and also DA, HE, will be similar.

Again, since the straight lines dropped from the pole I to the points A, B, C, D are equal, by the definition of a pole, the arcs IA, IB, IC, ID will also be equal [Margin: e — *Elements* III.28]; and in the same way the arcs IE, IF, IG, IH will be equal. The remaining arcs AE, BF, CG, DH will therefore be equal to one another. Hence, if there are parallel circles in a sphere, and so on. Which was to be demonstrated.

Diagram. A sphere carrying two parallel circles ABCD and EFGH with common pole I, and two great circles AEIGC and BFIHD drawn through I; K and L are the centres of the two parallels.

- A (upper left): point of the outer parallel ABCD
- E (below A): point of the inner parallel EFGH
- H (upper right): point of the inner parallel
- D (right): point of the outer parallel ABCD
- I (upper centre): common pole of the parallels
- K (centre): centre of parallel EFGH
- L (just below K): centre of parallel ABCD
- F (lower left): point of parallel EFGH
- G (lower right): point of parallel EFGH
- B (bottom left): point of parallel ABCD
- C (bottom right): point of parallel ABCD

THEOR. 11. PROPOS. 11.

[Margin: xvj]

If upon the diameters of equal circles equal segments of circles stand at right angles, and from these there are taken equal arcs — each of them beginning at the extremity of its own segment, and being less than half the arc of the whole segment — and if from the points bounding the equal arcs equal straight lines are drawn to the arcs of the circles first set out, then the arcs of the circles first set out, intercepted between those straight lines and the extremities of the diameters, will be equal.

Upon the diameters AC, DF of the equal circles ABC, DEF let there stand, at right angles to those circles, the equal circular segments AGC, DHF; and let there be taken equal arcs AG, DH, so that the points G, H divide the segments AGC, DHF unequally. Finally, from G, H let there fall to the arcs of the circles ABC, DEF the equal straight lines GB, HE. I say that the arcs AB, DE are equal.

Let there be dropped from G, H the straight lines GI, Hk, perpendicular to the planes of the circles ABC, DEF [Margin: a — *Elements* XI.11], which will fall on the common sections AC, DF at the points I, k [Margin: b — *Elements* XI.38]. Taking also L, M as the centres of the circles ABC, DEF, let the straight lines LB, BI, AG and ME, Ek, DH be drawn; and first let the points I, k fall within the radii AL, DM. Since, then, the arcs AGC, DHF are equal, and likewise the arcs AG, DH, the arcs CG, FH will also be equal [Margin: c — *Elements* III.27]; and therefore the angles GAC, HDF, standing upon them, are equal.

Diagram. The first figure of Proposition 11: two equal circles set side by side. In the left circle ABC (diameter AC, centre L) the segment AGC stands on the diameter; in the right circle DEF (diameter DF, centre M) the segment DHF stands on the diameter. The feet I, k of the perpendiculars fall within the radii AL, DM.

- Left circle — A (left end of diameter AC), G (top of the segment), I (on AL, near L), L (centre), B (foot on the lower arc), C (right end of the diameter)
- Right circle — D (left end of diameter DF), H (top of the segment), K (on DM, near M), M (centre), E (foot on the lower arc), F (right end of the diameter)

And the angles AIG, DKH are also equal, since they are right angles by the third definition of *Elements* XI. And so the two triangles AIG, DKH have the two angles GAI, AIG equal to the two angles HDK, DKH. They have, moreover, the side AG equal to the side DH — on account of the equality of the arcs AG, DH [Margin: d — *Elements* III.29] — which subtends the equal angles I, K. Therefore the side AI will be equal to the side DK, and the side GI equal to the side HK [Margin: e — *Elements* I.26]. And since the angles GIB, HKE are right angles, by the third definition of *Elements* XI, the squares on GB, HE — which are equal to one

another, because the straight lines GB, HE are equal — will be equal to the squares on GI, IB and on HK, KE [Margin: f — *Elements* I.47]; and therefore, the equal squares on the equal straight lines GI, HK being taken away, the squares on the straight lines IB, KE will remain equal, and hence the straight lines IB, KE themselves are equal. And since AL, DM, the radii of the equal circles, are equal, while AI, DK have also been shown equal, the remainders IL, KM will be equal as well. Hence the sides IL, LB will be equal to the sides KM, ME; and the bases IB, KE have also been shown equal. Therefore the angles L, M at the centres will be equal [Margin: g — *Elements* I.8]; and consequently the arcs AB, DE will be equal [Margin: h — *Elements* III.26].

Next let the points I, K fall on the radii LA, MD produced to A and D — which indeed can happen when the segments AGC, DHF are greater than a semicircle — and let the same construction be made as before. We shall show, as before, that the angles GAC, HDF are equal [Margin: i — *Elements* III.27]; and therefore, since GAC together with GAI, and HDF together with HDK, are each equal to two right angles [Margin: k — *Elements* I.13], GAI and HDK will also be equal. Since, then, the angles I, K are also equal, namely right angles, and the sides GA, HD are equal, on account of the equal arcs AG, DH [Margin: l — *Elements* III.29], the perpendiculars GI, IA will be equal, as before, to the perpendiculars HK, KD [Margin: m — *Elements* I.26]; and therefore the whole lines IL, KM will be equal to one another. Then, as before, we shall show that the straight line IB is equal to the straight line KE [Margin: n — *Elements* I.47], and the angle L equal to the angle M [Margin: o — *Elements* I.8], and finally the arc AB equal to the arc DE [Margin: p — *Elements* III.26].

Diagram. The second figure of Proposition 11: two equal circles side by side, with the perpendicular feet I, k falling on the radii LA, MD produced beyond A and D. Left circle ABC (centre L, diameter AC, segment AGC on top); right circle DEF (centre M, diameter DF, segment DHF on top).

- Left circle — G (top of segment), I (on LA produced, near A), A (left end of diameter), L (centre), C (right end of diameter), B (foot on lower arc)
- Right circle — H (top of segment), K (on MD produced, near D), D (left end of diameter), M (centre), F (right end of diameter), E (foot on lower arc)

Thirdly, let the perpendiculars dropped from G, H fall, in the planes of the circles ABC, DEF, on the points A, D — which likewise can happen when the segments AGC, DHF are greater than a semicircle. Then, the straight lines AB, DE being drawn, the angles GAB, HDE will be right angles, by the third definition of *Elements* XI. Hence, as before, the squares on the straight lines GA, AB will be

equal to the squares on the straight lines HD, DE [Margin: q — *Elements* I.47]; but the squares on GA, HD are equal, because the straight lines GA, HD are equal, on account of the equal arcs AG, DH [Margin: r — *Elements* III.29]. Therefore the squares on AB, DE will also be equal, and consequently the straight lines AB, DE will be equal; and so the arcs AB, DE will be equal [Margin: s — *Elements* III.28]. Which is what was proposed. Hence, if upon the diameters of equal circles equal circular segments stand at right angles, and so on. Which was to be demonstrated.

Diagram. The third figure of Proposition 11: two equal circles side by side, with the perpendiculars from G, H falling exactly on the points A, D. Left circle ABC (centre L, diameter AC); right circle DEF (centre M, diameter DF).

- Left circle — G (top of segment), A (left end of diameter, foot of the perpendicular), L (centre), C (right end of diameter), B (foot on lower arc)
- Right circle — H (top of segment), D (left end of diameter, foot of the perpendicular), M (centre), F (right end of diameter), E (foot on lower arc)

THEOR. 12. PROPOS. 12.

[Margin: xvj[?] — expected xvij]

If upon the diameters of equal circles equal circular segments are set up, and from these segments equal arcs are cut off toward the extremities of the segments, less than half of those segments; and if from the circles themselves equal arcs are taken toward the same parts, namely toward the extremities of the diameters — then the straight lines drawn from the points on the arcs of the segments to the points on the arcs of the circles will be equal.

Let the figures of the preceding proposition be repeated, with the same constructions, and let the arcs AB, DE be set equal. I say that the straight lines GB, HE are also equal. For since, as was demonstrated in the preceding proposition, the straight lines AI, IG are equal to the straight lines DK, KH, the remainders IL, KM, taken from the radii AL, DM, will also be equal — as in the first figure, where the points I, k fall within the radii AL, DM — or else the whole lines IL, kM will be equal, as in the second figure, where the points I, k fall on the radii AL, DM produced to A and D. Since, then, IL, LB are equal to the straight lines kM, ME, and they contain equal angles at L, M, on account of the equality of the arcs AB, DE [Margin: a — *Elements* III.27], the bases IB, kE will also be equal [Margin: b — *Elements* I.4]. Therefore, since the sides GI, IB are equal to the sides Hk, kE and

contain equal angles GIB, HkE, namely right angles, by the third definition of *Elements* XI, the bases GB, HE will also be equal [Margin: c — *Elements* I.4]. Which is what was proposed.

The same is more easily concluded if the perpendiculars dropped from G, H into the planes of the circles ABC, DEF fall on the points A, D, as in the third figure. For since the straight lines GA, AB are equal to the straight lines HD, DE — on account of the equal arcs AG, DH and AB, DE [Margin: d — *Elements* III.29] — and they contain equal angles, namely right angles, by the third definition of *Elements* XI, the bases GB, HE will be equal [Margin: e — *Elements* I.4]. Therefore, if upon the diameters of equal circles equal segments, and so on. Which was to be shown.

Diagram. The figure of Proposition 12 (repeating the preceding proposition's first figure): two equal circles side by side. Left circle ABC (diameter AC, centre L, segment AGC, feet I within radius AL); right circle DEF (diameter DF, centre M, segment DHF, feet K within radius DM).

- Left circle — A (left end of diameter), G (top of segment), I (on radius AL, near L), L (centre), C (right end of diameter), B (foot on lower arc)
- Right circle — D (left end of diameter), H (top of segment), K (on radius DM, near M), M (centre), F (right end of diameter), E (foot on lower arc)

THEOR. 13. PROPOS. 13.

[Margin: xviiij.]

If in a sphere there are parallel circles, and great circles are drawn which touch one of the parallels but cut the rest, then the circumferences of the parallels intercepted between those semicircles of the great circles that do not meet will be similar; and the circumferences of the great circles intercepted between any two of the parallels whatever will be equal.

Let there be in the sphere parallel circles AB, CDE, FGH, which will have the same pole, namely I. [Margin: a — 1. huius (*Sphaerica* II.1)] Let the great circles AFK, BHK touch the parallel AB at the points A, B, and cut the rest at the points F, C, L, M and H, E, D, G; and let them cut each other at K, N, so that KMN, NFK, KGN, NHK are semicircles. For great circles bisect one another. [Margin: b — 11. 1. huius (*Sphaerica* I.11)] Let arc KO also be taken equal to arc NA, and arc KP equal to arc NB, so that AMO, OFA and BGP, PHB are likewise semicircles. Then the semicircles AMO, BHP will not meet, since they do not cut each other. (These semicircles are cut off by the circles AIRO, BITP, as appears in the second figure. In the first figure the circles AI, BI, produced through R and T, are to be

understood as passing through O, P, so as to cut off the same semicircles.) In the same way the semicircles BGP, AFO will not meet.

I say that the arcs of the parallels AB, LE, MH intercepted between the non-meeting semicircles AMO, BHP are similar; and likewise that the arcs AB, CD, FG intercepted between the non-concurrent semicircles BGP, AFO are similar; that the arcs of the great circles AC, AL, BD, BE are equal, and likewise the arcs CF, LM, DG, EH — of which the former lie between the parallels AB, CDE, the latter between the parallels CDE, FGH; and in the same way that the arcs AF, AM, BG, BH lying between the parallels AB, FGH are equal.

For through the pole I and the points of contact A, B let the great circles QAIR, SBIT be drawn, cutting the parallels at Q, S, V, X. [Margin: c — 20. 1. hui. (*Sphaerica* I.20)] These great circles will also pass through the poles of the circles AFK, BHK; [Margin: d — 5. huius (*Sphaerica* II.5)] and therefore they will bisect the segments CAL, DBE, CVL, DXE, as well as the segments FAM, GBH, FQM, GSH. [Margin: e — 9. huius (*Sphaerica* II.9)] Moreover the same circles will cut at right angles the parallels AB, CDE, FGH and the great circles AFK, BHK. [Margin: f — 15. 1. huius (*Sphaerica* I.15)]

Since, then, upon the diameters of the equal circles AFK, BHK there stand at right angles equal segments of circles — namely the semicircles beginning at the points A, B and passing through I until they again cut the circles AFK, BHK, namely at the points O, P, as appears in the second figure — the arcs AI, BI are equal, [Margin: g — 28. tert. (*Elements* III.28)] because by the definition of a pole the straight lines IA, IB are equal; and these arcs are indeed less than the half-parts of the semicircles. (For since [the arcs AI, BI] are halves of the arcs AIR, BIT — because, by the definition of a pole, the straight lines from I to the points A, B, R, T are equal, [Margin: h — 28. tert. (*Elements* III.28)] and so the arcs too are equal; but the arcs AIR, BIT are less than a semicircle, because the semicircles extend from A and B through I as far as the circles AFK, BHK, which in the second figure are ARO, BTP — the arcs AI, BI will be less than the half-parts of those semicircles.) The straight lines IC, IE are also equal, by the definition of a pole, [Margin: i — 11. huius (*Sphaerica* II.11)] so that the arcs AC, BE will be equal. But AC is equal to AL, and BE to BD, [Margin: k — 9. huius (*Sphaerica* II.9)] because the arcs CAL, DBE are bisected, as has been demonstrated. Therefore the four arcs AC, AL, BE, BD are equal. In the same way we shall show that the four arcs AF, AM, BH, BG and EH, DG are equal, each of which is intercepted between two parallels. This is what was proposed to be demonstrated in the second place.

And since the whole arcs CAL, DBE are equal, because their halves are equal, as has been demonstrated, the subtending chords CL, DE will also be equal, [Margin: l — 19. tertij (*Elements* III.19) — numeral uncertain; possibly III.29] which chords are likewise subtended by the arcs CVL, DXE; and therefore the arcs of the parallels CVL, DXE will also be equal. [Margin: m — 18. tert. (*Elements* III.18) — numeral uncertain; possibly III.28] Since, then, they are bisected at V, X, as has been said, their halves will be equal, namely the four arcs CV, VL, DX, XE. [Margin: n — 9. huius (*Sphaerica* II.9)] If therefore to the equal arcs CV, DX the common arc VD be added — or, in the second figure, taken away — the arcs CD, VX will be equal. But arc VX is similar to arc AB. [Margin: o — 10. huius (*Sphaerica* II.10)] Therefore CD too will be similar to the same AB. In just the same way we shall show that FG is similar to AB, and likewise that the arcs EL, HM are similar to the same arc AB. This is what was proposed to be demonstrated in the first place. If therefore in a sphere there are parallel circles, etc. Which was to be shown.

Diagram. The first figure ("prima figura") for Proposition 13: three large overlapping circles representing the parallels (AB, CDE, FGH) and the two great circles (AFK, BHK) with the auxiliary great circles through the pole. The labelled points read off the scan under magnification (upper region downward):

- T, R (upper centre): points where the auxiliary great circles cross near the top
- I (centre-top): common pole of the parallels
- F (far left), C, A, B, E, H (far right): the intersection points along the parallels
- V, X (below centre): points where the great circles through the pole cut the parallels
- D, L (centre, close together): further intersection points
- Q (left), G, M, S (right): lower intersection points
- P (lower left), O (lower right): the points where the produced circles pass
- K (bottom): the mutual intersection of the two great circles

Diagram. The second figure ("secunda figura") for Proposition 13, marked with the numeral 2 at its foot: a single sphere-outline crossed by the great circles and their semicircles, showing the same construction with the semicircles cut off by the circles AIRO, BITP. Labelled points read off the scan under magnification:

- K (top): mutual intersection of the great circles
- P (upper left), O (upper right)
- F (left), H (right)

- C, T, R (centre), E (right)
- I (centre): pole
- Q (left), A, B, M (right of centre)
- D, N (lower centre), Z [?], L (lower right)
- S (lower right); the figure-number 2 at lower left

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The non-concurrent semicircles must begin at the points of contact A, B — such as AMO, BHP. Therefore, since any great circle has two semicircles between the points of contact of two opposite parallels, one must not take those semicircles of the two circles that cut each other between the points of contact of the two parallels; rather one must take the one verging toward that point of section and the other verging to the opposite side, so that the convex side of the one faces the concave side of the other, and con—

—versely, as appears in the aforesaid two semicircles. And so, if two semicircles AMO, DKY be taken (arc KY having been taken equal to DN itself), non-concurrent, as is evident, the arcs DL, GM will not be similar. Otherwise two great circles drawn through the pole I and the points D, I [?] would pass through the points G, M — since by [*Sphaerica* II.]¹⁰ they intercept similar arcs. But this cannot happen. For DG, LM would be semicircles, since by [*Sphaerica* I.]¹¹ great circles bisect one another.

PROBL. 1. PROPOS. 14.

[Margin: xvij.]

Given a circle in a sphere which is smaller than a great circle, and given some point on its circumference, to draw through that point a great circle that touches the given circle.

Let the given non-great circle in the sphere be AB, whose pole is C, and let it be required to draw through A — the given point on its circumference — a great circle that touches the circle AB. Through the pole C and the point A let the great circle CADEB be drawn, [Margin: a — 20. 1. hui. (*Sphaerica* I.20)] and in it let the quadrant AD be taken; and from the pole D, with radius DA, let the circle AE be drawn, [Margin: b — 17. 1. hu. (*Sphaerica* I.17)] which will be a great circle, because the chord DA subtending it is the side of the square inscribed in a great circle. I say that the great circle AE touches the circle AB at A. For since the two circles AB, AE cut the same circle CAD — which passes through their poles — at

the same point A, [Margin: c — 3. huius (*Sphaerica* II.3)] they touch one another at the point A. Therefore, given a circle in a sphere, etc. Which was to be done.

Diagram. The figure for Proposition 14: a single circle (the great circle CADEB) with two chords drawn inside it, illustrating the tangency of the great circle AE to the given circle AB. Labelled points:

- C (top): pole of the given circle AB
- A (upper left), B (upper right): endpoints of the given circle's diameter, A being the given point
- D (lower left), E (lower right): D the pole with radius DA, E on the great circle AE
- The chord A–E and the chord D–B are drawn across the figure; a small arc C sits at the top.

PROBL. 2. PROOS. 15.

[Margin: xix.]

Given a circle in a sphere which is smaller than a great circle, and given some point on the surface of the sphere lying between the given circle and another equal and parallel to it, to draw through that given point a great circle that touches the given circle, which is smaller than a great circle.

Let the given non-great circle in the sphere be AB, and let CD be equal and parallel to it, and let the given point be G, between the two circles AB, CD; and let it be required to draw through G a great circle that touches the circle AB. Let E, F be the poles of the parallels AB, CD [Margin: a — 1. huius (*Sphaerica* II.1)] (for parallels have the same poles), and through E, G let the great circle EAC be drawn, [Margin: b — 20. 1. hui. (*Sphaerica* I.20)] which will pass through the other pole F, by the corollary of the scholium of *Sphaerica* I.10. In it let the quadrant BH be taken; and the point H will fall either above D, or at D, or below D. Whichever of these happens, we shall carry out the matter thus.

From the pole E with radius EH, or from the pole F with radius FH, let the circle HI be drawn, [Margin: c — 2. huius (*Sphaerica* II.2)] which will be parallel to AB, CD, and will lie either above CD, or will be the same as CD, or will lie below CD, according as the point H was placed above D, at D, or below D. Again let the quadrant GK be taken, and the point K will be beyond H, since GH is less than a quadrant. Then from the pole G with radius GK let the circle KL be drawn, [Margin: d — 17. 1. hui. (*Sphaerica* I.17)] which will be a great circle, because the

chord subtending the quadrant GK is equal to the side of the square inscribed in a great circle. And let KL cut the circle HI at L [Margin: e — 20. 1. hui. (*Sphaerica* I.20)] (for it will necessarily cut it, since the point K is below H and does not reach I. For since the parallels AB, CD are equal, [Margin: f — Schol. 21. huius (Scholium to *Sphaerica* II.21) — numeral uncertain] the chords EA, FD will be equal, [Margin: g — 28. tertij (*Elements* III.28)] and therefore the arcs AE, DF will be equal. So, adding the common arc AF, the arcs EAF, AFD between the poles E, F will be equal; and AFD will also be a semicircle. But AI is a quadrant, [Margin: h — 10. huius (*Sphaerica* II.10)] since it is equal to the quadrant PH [?]. Therefore ID too will be a quadrant, and consequently IG greater than a quadrant. Hence, when the quadrant GK is taken, the point K will indeed fall below H, but will not reach I. From which it follows that the circle HI is cut by the circle KL.)

And through L, F let the great circle FL be drawn, which will pass through the other pole E, by the corollary of the scholium of *Sphaerica* I.10. And let this circle FLE cut the circle AB at M. [Margin: i — 10. huius (*Sphaerica* II.10)] Then the arcs ML, BH — of the great circles that pass through the poles E, F of the parallels, intercepted between the parallels AB, HI — will be equal; and by the construction LM will also be a quadrant. Then from the pole L with radius LM let the circle MN be drawn, [Margin: k — 17. 1. hui. (*Sphaerica* I.17)] which will be a great circle, because the chord subtending the quadrant LM is equal to the side of the square inscribed in a great circle. And since the great circle KL passes through L, the pole of the great circle NM, [Margin: l — Schol. 15. huius (Scholium to *Sphaerica* II.15)] conversely the great circle NM will pass through G, the pole of the circle KL; and thus the great circle NM passes through the given point G. I now say that the circles AB, GN cut the great circle EF — in which they have their poles — at the same point M; [Margin: m — 3. huius (*Sphaerica* II.3)] therefore they touch one another at M. So there has been drawn through G a great circle GN touching the circle AB at M. Therefore, given a circle in a sphere, etc. Which was to be done.

Diagram. For Proposition 15, a row of three sphere-outline figures illustrating the three cases (the point H falling above D, at D, or below D). Each sphere carries the same lettering; the labelled points read off the scan under magnification are:

- E (top): pole
- A (upper left), B (upper right)
- M (centre): intersection point
- G, N (upper region, near left)
- L (centre-low), H (right of L): the constructed points
- C (lower left), D (lower right)

- I (lower left, inside): quadrant point
- F (bottom), K (bottom right)

Diagram. The continuation of the Proposition 15 construction: again a row of three sphere-outline figures for the three cases, repeated at the head of the page for the scholium. Labelled points read off the scan under magnification:

- E (top): pole
- A (upper left), B (upper right)
- M (centre): intersection point
- G, N (near left of centre)
- L (centre-low), H (right of L)
- C (lower left), D (lower right)
- I (lower centre, inside)
- F (bottom), K (bottom right)

[Margin: q — 3. huius (*Sphaerica* II.3)]

(Continuing the scholium.) — [the circle drawn] will cut HI at [L?] a point, which again will be the pole of the tangent circle, as before. But if the given point G is the same as D, the pole of the tangent circle will be in the middle of the arc DCA, since this arc is a semicircle. And the circle drawn from that pole will touch AB at A, and CD at D, as is evident: because, namely, this great circle and the parallels AB, CD cut, at the points A, D, the circumference of the great circle ACDB, in which they have their poles.

And since, just as L has been shown to be the pole of the great circle GN that touches the circle AB, so too it can be shown that the other point — at which the great circle KL cuts the circle HI on the other side — is the pole of some other great circle which passes through G and touches the circle AB at another point: it is evident that through a point given in a sphere between two equal and parallel circles, two great circles can be drawn which touch the circle AB at two points.

THEOR. 14. PROPOS. 16.

[Margin: xx.]

Great circles which cut off similar circumferences of parallel circles in a sphere either pass through the poles of the parallels, or touch one and the same parallel.

In the sphere let the great circles ABC, DBE cut off from the parallels ADC, FG similar circumferences AD, FG. I say that the great circles ABC, DBE either pass through the poles of the parallels ADC, FG, or touch one and the same parallel. For either one of them, namely ABC, passes through the poles of the parallels — and then we shall show that the other passes through them too; or it does not pass through the poles of the parallels, but nevertheless touches one of them — and then we shall demonstrate that the other touches the same one; or, finally, it neither goes through the poles of the parallels nor touches one of them — and on this supposition we shall conclude that the given great circles touch some other parallel smaller than the given parallels.

First, then, let ABC pass through the poles of the parallels. I say that DBE too passes through them — that is, that the point B, at which the great circles ABC, DBE cut each other, is the pole of the parallels ADC, FG. For if B is not their pole, let H be their pole. And since the circle ABC is supposed to pass through their poles, H will be on the circumference of ABC. Through H, G let the great circle HG be drawn, cutting ADC at I. [Margin: a — 20. 1. hui. (*Sphaerica* I.20)] Then the arcs AI, FG will be similar, since they are intercepted between the great circles AH, HI drawn through the pole H. [Margin: b — 10. huius (*Sphaerica* II.10)] But the arc AD is also supposed similar to the arc FG. Therefore the arcs AI, AD are similar; and hence, since they are of the same circle, they will be equal to one another — the whole and the part. Which is absurd. Therefore no point other than B will be the pole of the parallels, if one of the circles ABC, DBE — namely ABC — is drawn through their poles. Therefore each great circle ABC, DBE passes through the pole B of the parallels, if one of them passes through it.

Diagram. The figure for the first case of Proposition 16: two concentric circles (the parallels ADC and FG) crossed by the two great circles ABC, DBE and the auxiliary great circle HG. Labelled points read off the scan under magnification:

- A (upper left), F (near A): on the outer parallel and the inner circle
- H, B (centre): B the intersection of the great circles, H the assumed pole
- G (below centre): on the inner parallel
- E (right): endpoint of a great circle
- D (lower left), I (below D): intersection points
- C (lower right): on the outer parallel

But now let two great circles ABC, DEF again cut off from the parallels ADC, BE similar circumferences AD, BE, and let neither of them pass through the poles of the parallels, but let one — namely ABC — touch one of them, say BE, at B. I say

that the circle DEF too touches the same BE at E. For if it does not touch it but cuts it, [Margin: c — 14. huius (*Sphaerica* II.14)] let a great circle GEH be drawn through E, the given point on the parallel BE, touching the parallel BE at E; and there will be semicircles, of which one is drawn from E through G, and the other passes from B through A, not meeting — as is established from the figure of *Sphaerica* II.13 and from what is demonstrated there. [Margin: d — 13. huius (*Sphaerica* II.13)] Therefore the arcs BE, AG will be similar; but BE, AD are also supposed similar. Therefore AG, AD are similar to one another; and hence, since they are of the same circle, they will be equal to one another — the whole and the part. Which is absurd. Therefore no other great circle drawn through E, besides DEF, touches the parallel BE at E, if ABC touches the same at B. Therefore, if ABC touches BE, DEF too will touch the same BE.

Diagram. The figure for the second case of Proposition 16: two concentric circles (the parallels ADC and BE) with the great circles ABC, DEF and the auxiliary great circle GEH tangent at E. Labelled points read off the scan under magnification:

- A (upper left): on the outer parallel
- D (left), G (below D): intersection points
- B (centre), E (right of B): B the point of tangency, E the point on the inner parallel
- F (lower right), H (right of F)
- C (bottom): on the outer parallel

Finally, let the great circles ABC, DEF cut off from the parallels ADC, GH [similar circumferences AD, GH]; and let neither of them be drawn through the poles of the parallels, nor touch one of them. I say that the great circles ABC, DEF touch some other parallel smaller than ADC, GH themselves. For since the great circle ABC neither passes through the poles of the parallels nor touches one of them, the great circle ABC will be oblique to each of the parallels ADC, GH. For if it were perpendicular, [Margin: e — 13. 1. hui. (*Sphaerica* I.13)] it would pass through their poles, which is not supposed. [Margin: f — 8. huius (*Sphaerica* II.8)] Therefore ABC will touch two circles equal to one another and parallel to both ADC, GH. Let it then touch the parallel BE, which will be smaller than each of ADC, GH (since ABC cuts them); and thus also the other, equal to it and parallel, will be smaller than each of ADC, GH; and consequently the parallels ADC, GH will lie between those two which the circle ABC touches. I say that DEF too touches the same BE. For if it does not touch it, [Margin: g — 15. huius (*Sphaerica* II.15)] let a great circle KH be drawn through the point H — which lies between the circle BE and the one equal and parallel to it, as we have shown — touching

BE at I; and there will be semicircles, of which one passes from I through H, the other from B through G, not meeting, as is established from the figure of *Sphaerica* II.13 and from what is demonstrated there. [Margin: h — 13. huius (*Sphaerica* II.13)] Therefore the arcs AK, GH will be similar; but AD, GH are also supposed similar. Therefore AK, AD are similar; and hence, since they are of the same circle, they will be equal to one another —

Diagram. The figure for the third case of Proposition 16: three concentric circles (the parallels ADC, BE, and GH) with the great circles ABC, DEF and the auxiliary great circle KH tangent at I. Labelled points read off the scan under magnification:

- A (upper left): on the outermost parallel
- G (below A): intersection point
- B, I, E (centre, left to right): B and I near the tangent point, E on the middle parallel
- D (lower left), H (below D), K (below D): intersection points
- F (right): endpoint of a great circle
- C (bottom): on the outermost parallel

they would be equal — the whole and the part — which is absurd. Therefore no other great circle drawn through H, apart from DEF, touches the parallel BE, provided ABC touches that same parallel at B. And so, if ABC touches circle BE, then DEF too will touch the same circle BE. Hence great circles that cut off similar arcs, and so on. Which was to be shown.

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It is moreover evident that the great circles ABC and DEF touch the same parallel BE in such a way that their semicircles, proceeding from the points of contact through similar arcs, do not meet. Otherwise the arcs cut off would not be similar, as is clear from Proposition 13 of this book.

THEOR. 15. PROPOS. 17.

[Margin: xxj.]

In a sphere, parallel circles between which and the greatest of the parallels equal arcs of great circles are intercepted are equal to one another; but those between which and the greatest of the parallels greater arcs of great circles are intercepted are smaller.

Let AB, CD, EF be parallel circles in a sphere, and let CD be the greatest of the parallels. Between the circle CD and each of the parallels AB and EF let there be intercepted equal arcs AC and CE of some great circle ACEFDB. I say that the parallels AB and EF are equal.

For let the common sections of the parallel planes and of the circle ACEFDB be the straight lines AB, CD, EF, [Margin: a — 16. undecimi (*Elements* XI.16)] which will be parallel to one another. First let the great circle ACE, FDB pass through the poles of the parallels. Once this is set up, the circle ACEFDB will cut the parallels AB, CD, EF in half and at right angles, [Margin: b — 15. primi huius (*Sphaerica* I.15)] and so AB, CD, EF will be diameters of the parallels. Now since the arcs AC and BD are equal, and likewise the arcs CE and DF, [Margin: c — 10. huius (*Sphaerica* II.10)] and since AC is assumed equal to CE, the arcs AC and BD together will be equal to CE and DF together. But the semicircles CABD and CEFD are equal, [Margin: d — 11. primi huius (*Sphaerica* I.11)] because the great circles CD and ACEFDB bisect each other. Therefore the remaining arcs AB and EF will be equal; [Margin: e — 29. tertij (*Elements* III.29)] and consequently the straight lines AB and EF — that is, the diameters of the circles AB and EF — are equal as well. Therefore the circles AB and EF are equal.

Diagram. A sphere shown in outline with three parallel chords drawn horizontally; each pair of endpoints is joined by lens-shaped (vesica) arcs representing the arcs of great circles between the parallels. This illustrates the first (through-the-poles) case of Proposition 17.

- A (upper left): endpoint of the upper parallel AB
- B (upper right): endpoint of the upper parallel AB
- C (middle left): endpoint of the greatest parallel CD
- D (middle right): endpoint of the greatest parallel CD
- E (lower left): endpoint of the lower parallel EF
- F (lower right): endpoint of the lower parallel EF

Now suppose the arc AC is taken greater than the arc CE. I say that the circle AB is smaller than the circle EF. For, with the same construction and demonstration, [Margin: f — 10. huius (*Sphaerica* II.10)] the arcs AC and BD will be equal as before, and likewise CE and DF. Since, then, AC is assumed greater than CE, the two arcs AC and BD together will be greater than the two arcs CE and DF together. Therefore the remainder AB, out of the semicircle CABD, will be smaller than the remainder EF, out of the semicircle CEFD; and consequently the straight line AB — that is, the diameter of circle AB — will be smaller than the straight line EF,

that is, than the diameter of circle EF, as we demonstrated in the scholium to Proposition 29 of Book 3 of Euclid, since the arcs AB and EF are less than a semicircle. Hence the circle AB will be smaller than the circle EF. Which was proposed.

But now let the great circle ACEFDB not pass through the poles of the parallels AB, CD, EF, and again let the arcs AC and CE be equal. I say that the circles AB and EF are still equal. For let G and H be the poles of the parallels AB, CD, EF, [Margin: g — 20. primi huius (*Sphaerica* I.20)] and through G, H and through the poles of the great circle ACEFDB let a great circle GIHK be described, [Margin: h — 15. primi huius (*Sphaerica* I.15)] which will cut the circle ACEFDB at two points, as at I and K, at right angles. Now since the great circle GIHK passes through the poles of the greatest circles ACEFDB and CD by construction, [Margin: i — Schol. 15. primi huius (scholium to *Sphaerica* I.15)] these circles will in turn pass through its poles. Therefore the points C and D, where these two circles intersect, will be the poles of the circle GIHK — otherwise neither of the circles ACEFD and CD would pass through the poles of the circle GIHK. Hence the drawn straight lines CI and CK will be equal, by the definition of pole, [Margin: k — 28. tertij (*Elements* III.28)] and consequently the arcs CI and CK will be equal to one another. But the arcs AC and CE are also equal, by hypothesis. Therefore the remaining arcs AI and EK will likewise be equal. Again, since the semicircle IGK is equal to the semicircle GKH [Margin: l — 11. primi huius (*Sphaerica* I.11)] — for the circles ACEFDB and GIHK bisect each other, and accordingly IGK is a semicircle, because G and H are the poles of the parallels — take away the common arc GK, and the remaining arcs GI and HK will be equal. Therefore, on the diameter of the circle ICKD, the equal segments of circles IGK and KHI, [Margin: m — 11. primi huius (*Sphaerica* I.11)] which are semicircles as we have shown, stand at right angles; and the arcs IG and KH are equal, and are not the halves of the segments, that is, quadrants, since G and H are not the poles of the circle ICKD. Likewise the arcs IA and KE are equal, as has been demonstrated; [Margin: n — 12. huius (*Sphaerica* II.12)] so the dropped straight lines GA and HE will be equal. [Margin: o — Schol. 26. primi huius (scholium to *Sphaerica* I.26)] Hence the circles AB and EF will be equal to one another.

Diagram. A sphere shown with an oblique great circle GIHK crossing a set of parallels, illustrating the second (oblique) case of Proposition 17.

- I (upper left): intersection of the oblique great circle with a parallel
- G (top): pole of the parallels
- B (upper center): point on the parallel near the top

- A (left of center): endpoint on the upper parallel
- L (center left): point on the oblique circle
- D (center): intersection point
- C (below D): intersection point / pole of GIHK
- F (right of center): endpoint on a parallel
- E (lower center): point on the middle parallel [?]
- H (bottom): pole of the parallels
- K (bottom right): intersection of the oblique great circle with a parallel

Now suppose the arc AC is taken greater than the arc CE. I say that the circle AB is smaller than the circle EF. For, taking an arc CL equal to the arc CE, it will follow — as was just demonstrated — that the parallel described through L is equal to the parallel EF; [Margin: p — 6. primi huius (*Sphaerica* I.6)] but the parallel AB is smaller than the parallel described through L, since AB lies farther from the greatest of the parallels, and so farther from the center of the sphere. Therefore the parallel AB is also smaller than EF. Which was proposed. So in a sphere, parallel circles between which and the greatest of the parallels, and so on. Which was to be demonstrated.

THEOR. 16. PROPOS. 18.

[Margin: xxij.]

In a sphere, the arcs of great circles intercepted between the greatest of the parallels and two other equal and parallel circles are equal; but those intercepted between the greater parallel and the greatest are smaller.

Diagram. A sphere in outline with three horizontal parallels — AB near the top, EF across the middle, CD near the bottom — illustrating Proposition 18.

- A (upper left): endpoint of the parallel AB
- B (upper right): endpoint of the parallel AB
- E (middle left): endpoint of the greatest parallel EF
- F (middle right): endpoint of the greatest parallel EF
- C (lower left): endpoint of the parallel CD
- D (lower right): endpoint of the parallel CD

In a sphere let there be two equal parallels AB and CD, and let EF be the greatest of the parallels; and let another great circle ACDB cut all these parallels. I say that the arcs AE, EC, and likewise BF, FD, are equal. For if they are not equal, let AE be the greater. [Margin: a — 17. huius (*Sphaerica* II.17)] Then the circle AB will be

smaller than the circle CD, which is contrary to the hypothesis. Therefore the arcs AE, EC, and likewise BF, FD, are equal.

But suppose the circle AB is taken greater than the circle CD. I say that the arc AE is smaller than the arc EC. For if it is not smaller, it will be either equal or greater. If equal, [Margin: b — 17. huius (*Sphaerica* II.17)] the circles AB and CD will be equal; if greater, [Margin: c — 17. huius (*Sphaerica* II.17)] the circle AB will be smaller than the circle CD — each of which is contrary to the hypothesis. Therefore the arc AE is smaller than EC. Accordingly, in a sphere the arcs of great circles intercepted, and so on. Which was to be shown.

THEOR. 17. PROPOS. 19.

[Margin: xxij.]

If in a sphere a great circle cuts several parallel circles drawn on the spherical surface — but not through the poles — it will cut them into unequal parts, except the greatest of the parallels. As for the segments of the parallels intercepted in one of the hemispheres: those lying between the greatest of the parallels and the visible pole are greater than a semicircle; the rest, lying between the greatest of the parallels and the hidden pole, are less than a semicircle. Finally, the alternate segments of equal and parallel circles are equal to one another.

In a sphere let the great circle ABCD cut the parallels EF, GH, IK in L, M; B, D; and O, P — not through the poles, which are Q and R; and let GH be the greatest of the parallels, Q the visible pole, and R the hidden pole, in the hemisphere that stands above the great circle ABCD and inclines toward F. I say that the circle ABCD does not cut the parallels in half, except the greatest, GH; [Margin: a — 11. primi huius (*Sphaerica* I.11)] for this one it does bisect. But the segment LFM, between the greatest parallel and the visible pole Q, is greater than a semicircle, and OKP is smaller. And finally, if the parallels EF and IK are equal, the alternate segments LFM and OIP are equal. [Margin: b — 20. primi huius (*Sphaerica* I.20)] For through the pole Q and the point B let the great circle QBRD be described; it will pass through the other pole R by the corollary to the scholium of Proposition 10 of Book 1 of this work, and also through the point D, [Margin: c — 11. primi huius (*Sphaerica* I.11)] since it bisects both the circle GBHD and the circle ABCD; and these circles are bisected in B and D. From this it follows that the circle QBRD cuts the parallel EF above the circle ABCD, but the parallel IK below it — as at the points S, T and V, X. [Margin: d — 15. primi huius (*Sphaerica* I.15)] Now since the circle QBRD bisects the parallels EF and IK, SFT and VKX will be semicircles; and

consequently the arc LFM will be greater than a semicircle, and OKP smaller than a semicircle. Which was proposed.

Diagram. A crowded sphere figure for Proposition 19: a great circle ABCD cutting several parallels obliquely, with the auxiliary great circles QBRD and AGCH and their intersection points marked throughout the interior.

- A (upper left): point on the great circle ABCD
- Q (top): visible pole of the parallels
- T (upper center): intersection on parallel EF [?]
- M (upper center-left): intersection of ABCD with parallel EF
- L (center left): intersection of ABCD with parallel EF
- S (center): intersection on parallel EF [?]
- D (center): intersection of the great circles
- G (far left): endpoint of the greatest parallel GH
- H (far right): endpoint of the greatest parallel GH
- B (center): intersection of the great circles
- V (lower center-left): intersection on parallel IK [?]
- P (lower center-right): intersection of ABCD with parallel IK
- I (lower left): endpoint of parallel IK
- X (bottom center-left): intersection on parallel IK [?]
- O (bottom center): intersection of ABCD with parallel IK
- C (bottom, right of O): point on the great circle ABCD
- K (lower right): endpoint of parallel IK
- R (bottom center): hidden pole of the parallels
- E (left): endpoint of parallel EF
- F (right): endpoint of parallel EF

Now let the parallels EF and IK be equal. I say that the alternate segments LFM and OIP are equal to one another, and likewise the alternate segments LEM and OKP. [Margin: e — 20. primi huius (*Sphaerica* I.20)] For through the poles of the parallels and the poles of the circle ABCD let the great circle AGCH be described, [Margin: f — 9. huius (*Sphaerica* II.9)] which will bisect the segments LAM and OCP. Therefore the arcs AL and AM are equal to one another, and CO and CP to one another. And since the great circle AGCH passes through the poles of the greatest circles GH and AC, [Margin: g — Schol. 15. primi huius (scholium to *Sphaerica* I.15)] these circles will in turn pass through its poles. Therefore the points B and D are the poles of the circle AGCH; and consequently the straight

lines BA and BC will be equal, by the definition of pole; [Margin: h — 28. tertij (*Elements* III.28)] and therefore the arcs BA and BC themselves will be equal. [Margin: i — 18. huius (*Sphaerica* II.18)] But the arcs BL and BO are also equal, because the parallels EF and IK are assumed equal. Therefore the remaining arcs AL and CO will be equal as well. But the arcs AL and CO are the halves of the arcs LAM and OCP, because AL was shown equal to AM, and CO to CP. Therefore the arcs LAM and OCP are also equal, [Margin: k — 29. tertij (*Elements* III.29)] and accordingly the subtending chords LM and OP will be equal. [Margin: l — 28. tertij (*Elements* III.28)] Hence from the equal circles EF and IK they will cut off equal arcs — the greater LFM equal to the greater OIP, and the smaller LEM equal to the smaller OKP (that is, alternate segment equal to alternate segment). Which was proposed. And so, if in a sphere a great circle cuts several parallel circles drawn on the spherical surface, and so on. Which was to be demonstrated.

THEOR. 18. PROPOS. 20.

[Margin: xxiv.]

If in a sphere a great circle cuts several parallel circles, but not through the poles: of the arcs taken from the parallels in one hemisphere, those that come nearer to the visible pole will be too great to be similar to those that lie farther from that same visible pole.

In a sphere let the great circle GHIKLMNO cut the parallels AB, CD, EF in H, O; I, N; K, M — not through the poles — and let the hemisphere GBL lie above it, with P the visible pole and Q the hidden pole. I say that the arc OBH is too great to be similar to the arc NDI, and that NDI is too great to be similar to the arc MFK. [Margin: a — 20. primi huius (*Sphaerica* I.20)] For through the pole of the parallels P and the points I, N let two great circles PI, PN be described, cutting the parallel AB above the circle GILN, in R and S: [Margin: b — 10. huius (*Sphaerica* II.10)] and the arc RBS will be similar to the arc IDN. Since, then, the arc OBH is greater than the arc RBS, it will also be too great to be similar to the arc NDI. In the same way we shall show that the arc NDI is too great to be similar to the arc MFK, provided that through the pole P and the points K, M two other great circles are described. Therefore, if in a sphere a great circle cuts several parallels, and so on. Which was to be demonstrated.

Diagram. A sphere for Proposition 20: a great circle GHIKLMNO cutting three parallels AB, CD, EF obliquely, with the auxiliary great circles through the pole P and the marked secant points.

- A (left): endpoint of the upper parallel AB
- G (top left): point on the great circle
- P (top): visible pole of the parallels
- B (top right): point on the upper parallel AB
- S (upper center): secant point on parallel AB
- H (center left): secant point on parallel AB
- N (center): secant point on parallel CD
- C (left): endpoint of the middle parallel CD
- D (right): endpoint of the middle parallel CD
- I (center left): secant point on parallel CD
- M (lower center): secant point on parallel EF
- E (lower left): endpoint of the lower parallel EF
- F (lower right): endpoint of the lower parallel EF
- K (bottom): secant point on parallel EF
- L (bottom): point on the great circle
- Q (bottom): hidden pole of the parallels
- O (secant point on parallel AB): [?]
- R (secant point on parallel AB): [?]

COROLLARIUM

From this it follows that the arc OBH is simply a greater part of its own parallel AB than the arc NDI is of its own parallel, and so on — inasmuch as the arc RBS is as great a part of its own parallel as the arc IDN is of its own parallel, since these arcs have been demonstrated to be similar, and so on.

THEOR. 19. PROPOS. 21.

[Margin: xxv.]

If in equal spheres great circles are inclined to great circles, the one whose pole stands higher above the underlying plane will be the more inclined; but those circles whose poles are equally distant from the underlying planes are equally inclined.

In equal spheres ABCD and EFGH, whose centers are I and K, let two great circles BND and FOH be inclined to the great circles ABCD and EFGH (whose poles are L and M), the poles of BND and FOH being P and Q; and first let the pole P stand higher above the plane of the circle ABCD than the pole Q stands above the plane

of the circle EFGH. I say that the circle BND is more inclined to the circle ABCD than FOH is to EFGH. [Margin: a — 20. primi huius (*Sphaerica* I.20)] For through the poles L, P and through the poles M, Q let the great circles ANC and EOG be described; and let the common section of the circles ABCD and BND be the straight line BD, the common section of the circles ABCD and ANC the straight line AC, and the common section of the circles BND and ANC the straight line NI — all which straight lines will pass through the center of the sphere I, [Margin: b — 6. primi huius (*Sphaerica* I.6)] since the great circles are drawn through the same center of the sphere. In the same order, in the other sphere let there be common sections of the circles: the straight line FH of the circles EFGH and FOH, the straight line EG of the circles EFGH and EOG, and the straight line OK of the circles FOH and EOG — all which straight lines will likewise pass through the center of the sphere K. And since the circle ANC, passing through the poles of the circles ABCD and BND, [Margin: c — 15. primi huius (*Sphaerica* I.15)] cuts them at right angles, in turn each of the circles ABCD and BND will be perpendicular to the circle ANC, [Margin: d — 19. undecimi (*Elements* XI.19)] and so the straight line BD, their common section, will be perpendicular to the same circle ANC. Hence the angles AID and NID will be right angles, by definition 3 of Book 11 of Euclid; and accordingly AIN will be the angle of inclination of the circle BND to the circle ABCD, by definition 6 of Book 11 of Euclid. In the same way, EKO will be the angle of inclination of the circle FOH to the circle EFGH. Now since the pole P of the circle BND is set higher above the circle ABCD than the pole Q of the circle FOH above the circle EFGH, the arc CP will be greater than the arc GQ. For these arcs, being perpendicular to the circles ABCD and EFGH, measure the heights of the poles P and Q above those circles. But the arcs PN and QO are equal, since they are quadrants. [Margin: e — Corol. 16. huius (corollary to *Sphaerica* II.16)] For the poles P and Q are a quadrant distant from the great circles BND and FOH. Therefore the arc CN will be greater than the arc GO, and consequently AN, the remainder of the semicircle ANC, will be smaller than EO, the remainder of the semicircle EOG. [Margin: f — Schol. 27. tertij (scholium to *Elements* III.27)] Hence the angle AIN will be smaller than the angle EKO, and accordingly the circle BND will be more inclined to the circle ABCD than the circle FOH is to the circle EFGH, as we wrote in the explanation of definition 7 of Book 11 of Euclid.

Diagram. The left of two sphere figures for Proposition 21: sphere ABCD with the great circles ANC and BND, their poles, and center I.

- A (top): endpoint of the great circle ABCD
- B (left): endpoint of the great circle ABCD
- C (bottom): endpoint of the great circle ABCD

- D (right): endpoint of the great circle ABCD
- N (upper left): point on the great circle ANC / on circle BND
- I (center): center of the sphere
- L (lower left): pole
- P (below L): pole of circle BND

Diagram. The right of two sphere figures for Proposition 21: sphere EFGH with the great circles EOG and FOH, their poles, and center K.

- E (top): endpoint of the great circle EFGH
- F (left): endpoint of the great circle EFGH
- G (bottom): endpoint of the great circle EFGH
- H (right): endpoint of the great circle EFGH
- O (upper left): point on the great circle EOG / on circle FOH
- K (center): center of the sphere
- M (lower left): pole
- Q (below M): pole of circle FOH

But now let the arcs CP and GQ be equal — that is, let the poles P and Q be equally distant from the planes of the circles ABCD and EFGH. I say that the circles BND and FOH are equally inclined to the circles ABCD and EFGH. For since the arcs CP and GQ are equal, if the quadrants PN and QO are added, the arcs CN and GO will also be equal; and consequently the remaining arcs AN and EO, taken from the semicircles, will be equal. [Margin: g — 27. tertij (*Elements* III.27)] Therefore the angles AIN and EKO will be equal, and consequently, by definition 7 of Book 11 of Euclid, the inclinations of the circles BND and FOH to the circles ABCD and EFGH will be similar, that is, equal. So if in equal spheres great circles are inclined to great circles, and so on. Which was to be shown.

SCHOLIVM

From this it follows that, of great circles inclined to others, if their poles are equally distant from the poles of the greatest circles to which they are inclined, the inclinations are equal; but the one whose pole is nearer to the pole of the circle to which it is inclined has the greater inclination. For if the arcs LP and MQ are equal, then CP and GQ will also be equal, [Margin: h — Corol. 16. primi huius (corollary to *Sphaerica* I.16)] since CL and GM are quadrants; and so the poles P and Q of the inclined circles will be equally distant from the underlying planes of the circles ABCD and EFGH. Hence, as was demonstrated in this proposition, the

inclinations of the circles BND and FOH to the circles ABCD and EFGH will be equal. But if the arc LP is smaller than the arc MQ, then the remaining arc CP, out of a quadrant, will be greater than the arc GQ, out of a quadrant. Therefore, as we showed in this proposition, the inclination of the circle BND to the circle ABCD will be greater than that of the circle FOH to the circle EFGH.

We shall also demonstrate the converse of this theorem and its scholium in the following manner.

If in equal spheres great circles are equally inclined to great circles, the distances of their poles from the underlying planes will be equal; but of the one that is more inclined, the pole will be higher. Likewise, the distances of the poles of those circles that are equally inclined, from the poles of the circles to which they are inclined, will be equal; but the distance of the pole of the circle that is more inclined, from the pole of the circle to which it is inclined, will be smaller.

For if the circles BND and FOH are equally inclined to the circles ABCD and EFGH, the angles AIN and EKO will be equal, by definition 7 of Book 11 of Euclid; [Margin: i — 26. tertij (*Elements* III.26)] and consequently the arcs AN and EO will be equal. Therefore, adding the quadrants NP and OQ, the arcs AP and EQ will be equal, and consequently the remaining arcs CP and GQ, taken from the semicircles, will be equal.

But if the circle BND is more inclined to the circle ABCD than the circle FOH is to the circle EFGH, the angle AIN will be smaller than the angle EKO, as we wrote on definition 7 of Book 11 of Euclid; [Margin: k — Schol. 26. tertij (scholium to *Elements* III.26)] and consequently the arc AH will be smaller than the arc FO. Therefore, adding the quadrants NP and OQ, the arc AP will be smaller than the arc EQ, and accordingly the remaining arc CP, out of the semicircle ANC, will be greater than the remaining arc GQ, out of the semicircle FOG.

Again, if the circles are equally inclined, the arcs CP and GQ will be equal, just as we showed above. [Margin: l — Corol. 16. 1.huius (Corollary to *Sphaerica* I.16)] Since, therefore, CL and GM are quadrants, the arcs LP and MQ will also be equal. Finally, if the circle BND is more inclined, then by what has just been demonstrated the arc PC will be greater than the arc GQ. The remainder LP, taken from the quadrant CL, will therefore be less than the remainder MQ, taken from the quadrant GM; and so on.

Two further theorems from another version have also been added here, namely the following.

I.

Great circles tangent to the same parallel are equally inclined to the greatest of the parallels; but one that is tangent to a larger parallel is more inclined to the greatest of the parallels. And great circles equally inclined to the greatest of the parallels are tangent to the same parallel; but one that is more inclined to the greatest of the parallels is tangent to a larger parallel.

Let the great circles AB and CB be tangent to the same parallel AC, and let DE be the greatest of the parallels. I say that the circles AB and CB are equally inclined to the circle DE. For let F be the pole of the parallels, [Margin: a — 20.1.hui. (Sphaerica I.20)] and through F and the points of contact A and C let the great circles FAD and FCE be described. [Margin: b — 5.huius (Sphaerica II.5)] These will pass through the poles of the circles AB and CB, [Margin: c — 15.1.hui. (Sphaerica I.15)] and hence will cut them at right angles. The arcs AF and CF will therefore measure the height of the pole F of the circle DE above the circles AB and CB. [Margin: d — 28.tertij (Elements III.28)] And since the arcs AF and CF are equal — because the straight chords FA and FC are equal, by the definition of a pole — the circle DE will be equally inclined [Margin: e — 21.huius (Sphaerica II.21)] to the circles AB and CB, and these in turn will be equally inclined to it.

Now let the great circle GH be tangent to a larger parallel GI. I say that the inclination of the circle GH to the greatest of the parallels, DE, is greater than that of the circle AB. For if a great circle FGE is described through F and the point of contact G, [Margin: f — 20.1.hui. (Sphaerica I.20)] it will, in the same way just demonstrated, measure by the arc FG the height of the pole F of the circle DE above the circle GH. But the arc FG is greater than the arc FA, because the circle GI is assumed larger than AC, and hence farther from the pole F. Therefore the circle DE will be more inclined to the circle GH than to the circle AB; [Margin: g — 11.1.hui.[?] (Sphaerica I.11?)] and conversely, GH will be more inclined to DE than AB is.

Again, let the great circles AB and CB be equally inclined to the circle DE, the greatest of the parallels. I say that they are tangent to the same parallel. For through F, the pole of the parallels, and through the poles of the circles AB and CB, let the great circles FAD and FCE be described, [Margin: h — 20.1.hui. (Sphaerica I.20)] cutting the circles AB and CB at A and C. And since [each] cuts

them at right angles, [Margin: i — 15.1.huius (Sphaerica I.15)] the arcs FA and FC will measure the height of the pole F of the circle DE above the circles AB and CB. But the arcs FA and FC are equal, [Margin: k — Schol.21.huius (Scholium to Sphaerica II.21)] because the circles AB and CB are assumed equally inclined to the circle DE, and hence it too to them. If, therefore, from the pole F, with radius FA or FC, a circle AC is described, [Margin: l — 3.huius (Sphaerica II.3)] it will be tangent to the circles AB and CB, because the circle AC and the circles AB and CB cut, at the same points A and C, the great circles FD and FE, which pass through their poles.

Now let the great circle GH be more inclined to the circle DE. I say that it is tangent to a larger parallel. For if a great circle FG is described through F, the pole of the parallels, and through the pole of the circle GH [Margin: m — 20.1.hu. (Sphaerica I.20)] — which will cut the circle GH at right angles, namely at the point G [Margin: n — 15.1.hui. (Sphaerica I.15)] — the arc FG will again measure the height of the pole F of the circle DE above the circle GH. [Margin: o — Scho.21.huius (Scholium to Sphaerica II.21)] But FG is greater than FA, because the circle GH is assumed more inclined than AB. Therefore the circle described from the pole F with radius FG will be larger than the circle described from the same pole F with radius FA. [Margin: p — 3.huius (Sphaerica II.3)] Since, then, AB and AC are mutually tangent at A, and GH and GI are likewise mutually tangent at G, the proposition is established.

Diagram. The figure for Theorem I: a cluster of great-circle arcs converging at the pole F above, with the tangent circles and their contacts shown below.

- F (apex, top): pole of the parallels
- A (upper left of center): contact of circle AB on the parallel AC
- C (upper right of center): contact of circle CB on the parallel AC
- G (right of center): contact of circle GH on the larger parallel GI
- I (left): point on the larger parallel
- B (center): crossing of the great circles
- H (lower center): point on the arc DE
- D (lower left): endpoint of great circle FAD
- E (lower right): endpoint of great circle FCE

II.

Great circles equally inclined to the greatest of the parallels have their poles on the circumference of the same parallel. And great circles that

have their poles on the circumference of the same parallel are equally inclined to the greatest of the parallels.

Let the great circles AB and CD, whose poles are E and F, be equally inclined to DB, the greatest of the parallels. I say that their poles E and F lie on the same parallel. For if, through G, the pole of the parallels, and through E and F, the poles of the circles AB and CD, the great circles GE and GF are described [Margin: a — 20.1.hui. (Sphaerica I.20)] — which will be at right angles to the circles AB and CD [Margin: b — 15.1.hui. (Sphaerica I.15)] — the arcs EG and FG, the distances of the poles E and F from the pole G, [Margin: c — Schol.21.huius (Scholium to Sphaerica II.21)] will be equal, because the circles AB and CD are assumed equally inclined to the circle DB. Therefore the circle EF, described from the pole G with radius GE or GF, is parallel to the circle DB; [Margin: d — 2.huius (Sphaerica II.2)] and on this parallel EF the circles AB and CD have their poles E and F. Which is what was proposed.

But now let the great circles AB and CD have their poles E and F on the parallel EF. I say that they are equally inclined to DB, the greatest of the parallels. For, by the definition of a pole, the straight lines GE and GF are equal, and on that account the arcs EG and FG [Margin: e — 28.tertij (Elements III.28)] will also be equal. Since, then, these same arcs are the distances of the poles E and F from G, the pole of the parallels, the circles AB and CD [Margin: f — Schol.21.huius (Scholium to Sphaerica II.21)] are equally inclined to DB, the greatest of the parallels.

Diagram. The first figure for Theorem II: a pointed (ogival) figure of two great-circle arcs meeting at the top, with the parallel crossing lower down.

- G (apex, marked with a small right-angle square): pole of the parallels
- A (apex, right of G): point on the meeting of the arcs
- E (middle left): pole of circle AB
- F (middle right): pole of circle CD
- D (lower left): endpoint on the parallel
- B (lower right): endpoint on the parallel

Diagram. The second figure for Theorem II (the converse case): a similar ogival figure, with the poles now marked on the parallel EF below.

- G (apex, marked with a small right-angle square): pole of the parallels
- A (upper left, below G): point on the great circle
- C (upper right, below G): point on the great circle

- D (middle left): point on the arc
- B (middle right): point on the arc
- E (lower left): pole on the parallel EF
- F (lower right): pole on the parallel EF

There now follows, in the Greek codex, the twenty-second proposition, whose demonstration is very long. Since in another version the same thing is proved much more briefly and more clearly, it has seemed good to insert here three other theorems from that other version, so that we may then more easily demonstrate the twenty-second proposition of this book. The first of these theorems is the second part of Proposition 1 of Book 3 of Theodosius, although stated more generally, as it is here proposed. The first theorem, then — which stands third in order in this scholium — is as follows.

III.

If a right segment of a circle be set up on the diameter of a circle, and the circumference of the standing segment be divided into two unequal parts, and from the point of division very many straight lines fall to the circumference of the first circle: the straight line subtending the smaller part of the standing segment will be the least of all, while the one subtending the greater part will be the greatest of all. Of the rest, the one nearer the greatest is always greater than the one more remote; but the one nearer the least is always less than the one more remote. And two equal straight lines fall from the same point onto the circumference of the circle, equally distant from the greatest.

On the diameter AD of the circle ABCDE let there be set up a right segment of a circle AFD, and let it be cut not into two equal parts at F, with AF the smaller part and DF the greater. Let there fall from F very many straight lines FA, FI, FH, FB, FC, FD, FE. I say that the least of all is FA and the greatest FD; that FC is greater than FB, and so on; that FI is less than FH, and so on; and finally that the two lines FE and FC are equal if they are equally distant from the greatest, FD — that is, if the arcs DE and DC are equal. [Margin: a — 21. vnd. (Elements XI.21)] For let a perpendicular FG be dropped from F onto the plane of the circle ABCDE; [Margin: b — 38. vnde. (Elements XI.38)] it will fall on AD, the common section, and the point G will lie either between the points A and D, as in the first figure (which always happens when the segment AFD is not greater than a semicircle, though the same can also happen in a greater segment); or at the same point as A; or outside the circle, on the diameter DA produced, as the two later figures indicate — which can happen only in a segment greater than a semicircle. In the first

figure G will not be the center of the circle ABCDF, because GF does not bisect the segment AFD; much less will G be the center of the circle ABCDE in the two later figures. Let the straight lines GI, GH, GB, GC, GE be joined; all the angles at G will be right, by definition 3 of Book 11 of Euclid.

Now of the straight lines falling from G onto the circle ABCDE, in the first figure and the third [Margin: c — 7. vel 8. tertij (Elements III.7 or 8)] the least is GA; in all the figures the greatest is GD; [Margin: d — 7. vel 15. vel 8. tertij (Elements III.7, 15, or 8)] GC is greater than GB; GI is less than GH; and finally GC and GE are equal. In the first and third figure, therefore, the two squares of the lines AG and GF are less than the two squares of the lines IG and GF; [Margin: e — 47.prim (Elements I.47)] and since these are equal to the squares of the lines FA and FI, the square of FA will also be less than the square of FI, and hence the line FA will be less than FI. In the same way we shall show that FA, in this same first and third figure, is less than FH, and so on. In the second figure, [Margin: f — 19.primi (Elements I.19)] FA is likewise less than FI or FH, and so on, because in the triangles AIF and AHF (in which the angle A is right, by definition 3 of Book 11 of Euclid, and hence the others acute) the line FA subtends an acute angle, I or H, while the line FI or FH subtends the right angle A. Therefore FA is the least of all.

Again, in all the figures the two squares of GD and GF are greater than the two squares of GC and GE; [Margin: g — 47.prim (Elements I.47)] and since these are equal to the squares of FD and FC, the square of FD will also be greater than the square of FC, and hence the line FD greater than the line FC. In no other way we show that FD is greater than FB, and so on. Therefore FD is the greatest of all. Moreover, in all the figures the two squares of GC and GF are greater than the two squares of GB and GF; [Margin: h — 47.prim (Elements I.47)] and since these are equal to the squares of FC and FB, the square of FC will also be greater than the square of FB, and hence the line FC greater than FB. In just the same way we show that FC, which is nearer the greatest, FD, is greater than any other more remote, and so on. Further, in all the figures the two squares of GI and GF are less than the two squares of GH and GF; [Margin: i — 47.prim (Elements I.47)] and since these are equal to the squares of FI and FH, the square of FI will be less than the square of FH, and hence the line FI less than FH. And in the same way we shall demonstrate that FI, which is nearer the least, FA, is less than any other more remote, and so on. Finally, the two squares of GC and GF are equal to the two squares of GE and GF; [Margin: k — 47.pri (Elements I.47)] and since these are equal to the squares of FC and FE, the lines FC and FE will be equal. What was proposed is therefore established. And, as is clear from the demonstration, we

call that line nearer the greatest, FD, which falls on a point closer to the point D; and that one nearer the least, FA, which falls on a point closer to the point A.

Diagram. Three companion circle-figures for Theorem III (the *first*, *second*, and *third* figures referred to in the proof), arranged in a row. In each, a circle ABCDE is drawn with the apex F above; the perpendicular foot G falls on the diameter AD (between A and D in the first figure, at A in the second, and outside the circle on DA produced in the third). From F the chords FA, FI, FH, FB, FC, FD, FE are drawn, and from G the joins GI, GH, GB, GC, GE.

- F (top, apex): point of the standing segment from which the lines fall
- G (interior, on the diameter AD): foot of the perpendicular from F
- A (left, on circle): endpoint of the diameter (smaller-part side)
- D (right, on circle): endpoint of the diameter (greater-part side)
- E (upper right, on circle): endpoint of chord FE
- C (lower, on circle): endpoint of chord FC
- B (lower left, on circle): endpoint of chord FB
- I (lower left, on circle): endpoint of chord FI
- H (lower, on circle): endpoint of chord FH

IV.

If, on the surface of a sphere, within the periphery of any circle a point be marked other than its pole, and from it very many arcs of great circles, less than a semicircle, be drawn to the circumference of the circle: the greatest is the one drawn through the pole of the circle, and the least is the one adjacent to it. Of the rest, the one nearer the greatest is always greater than the one more remote. And two arcs equally remote from the same greatest, or from the least, are equal to each other.

Let there be in a sphere a circle ABCDE, whose pole is F, and on the sphere's surface, within the periphery of the circle and other than the pole F, let any point G be marked, from which very many arcs of great circles are drawn to the circumference of the circle ABCDE. Of these, let GA, extended in both directions, pass through the pole F; let the arc GB be nearer to GA than GC is; and finally let the two arcs GB and GE be equally distant from the same GA, or from GD. Let all these arcs be less than a semicircle — which will be the case precisely when they do not intersect one another at any point other than G. [Margin: a — 11.1.hui.[?] (Sphaerica I.11?)] For since great circles bisect one another, the arcs GA and GE will be less than a semicircle so long as they do not yet intersect. And by the same reasoning the other arcs going out from G will be less than a semicircle, if they do

not intersect one another. But if one of them — say the arc GA — were a semicircle, all the others would pass through the point A and would themselves be semicircles as well; while if GA were greater than a semicircle, all the others would cut it before they reached the circum-

Diagram. The figure for Theorem IV: a single circle ABCDE with an interior point G, from which arcs radiate to the circumference; the arc GA passes through the interior pole F.

- E (upper left, on circle): endpoint of arc GE
- A (left, on circle): endpoint of arc GA (which passes through the pole F)
- F (interior, between A and G): pole of the circle ABCDE
- G (interior, center): the marked point from which the arcs are drawn
- D (right, on circle): endpoint diametrically opposite A along the arc through the pole
- B (lower left, on circle): endpoint of arc GB
- C (bottom, on circle): endpoint of arc GC

—ference, and would be greater than a semicircle, as is clear. Hence nothing could be inferred.

Diagram. The figure for Theorem IV, reprinted at the head of this page (a printer's convenience, since the demonstration spans the page break); it is identical to fig-p671-2. A circle ABCDE with the interior point G, arcs radiating to the circumference, and the arc GA passing through the interior pole F.

- E (upper left, on circle): endpoint of arc GE
- A (left, on circle): endpoint of arc GA (through the pole F)
- F (interior, between A and G): pole of the circle ABCDE
- G (interior, center): the marked point
- D (right, on circle): opposite endpoint along the arc through the pole
- B (lower left, on circle): endpoint of arc GB
- C (bottom, on circle): endpoint of arc GC

I say that the arc GA is the greatest of all, and GD the least; that GB is greater than the arc GC; and finally that the two arcs GB and GE are equal. [Margin: b — 15.1.hui. (Sphaerica I.15)] For since the arc AD cuts the circle ABC in two and at right angles, the straight chord AD will be the diameter of the circle ABC; and upon it stands the right segment of a circle AGD, which is cut unequally at G [Margin: c — 28.tertij (Elements III.28)] (for since, by the definition of a pole, the straight chords FA and FD are equal, the arcs FA and FD will also be equal, and

hence the arc AD will be bisected at F, and on that account not bisected at G), and the greater part is GA, the lesser GD. [Margin: d — Schol.21.huius (Scholium to Sphaerica II.21)] Therefore, of the straight lines drawn from G to the circumference of the circle ABC, the greatest is GA and the least GD; GB is greater than GC; and GB and GE are equal. And so, since the arcs they subtend are assumed less than a semicircle, [Margin: e — Schol. 28. tertij (Scholium to Elements III.28)] the arc GA will also be the greatest and GD the least; GB greater than GC; [Margin: f — 28. tertij (Elements III.28)] and finally the arcs GB and GE equal.

V.

If, on the surface of a sphere, outside the periphery of any circle a point be marked other than its pole, and from it very many arcs of great circles, less than a semicircle and cutting the circumference of the circle, be drawn to the circle's circumference: the greatest is the one drawn through the pole of the circle; and of the rest, the one nearer the greatest is always greater than the one more remote. The least is that [portion] of the greatest which is interposed between the point and the circle's circumference, outside the circle; and of the rest, the one nearer the least is always less than the one more remote. And two arcs equally remote from the same greatest, or from the least, are equal to each other.

Let there be in a sphere a circle ABCDE, whose pole is F, and on the sphere's surface, outside the periphery of the circle, let any point G be marked, other than the other pole of the circle ABCDE. From G let very many arcs of great circles be drawn to the circumference of the circle ABCDE, cutting it; of these let GDFA pass through the pole F; let the arc GHB be nearer to GDFA than GIC is; and finally let the two arcs GHB and GKE be equally distant from the same GDFA, or from GD. Let all these arcs be less than a semicircle — which will be the case precisely when they do not intersect one another at any point other than G, just as was shown in the preceding theorem. I say that the arc GA is the greatest of all, and GB greater than GC; that the least is GD, and GH less than GI; and finally that the two arcs GB and GE, as also GH and GK, are equal. [Margin: a — 15. 1. hui. (Sphaerica I.15)] For since the arc GA cuts the circle ABCDE in two and at right angles, the straight chord AD will be the diameter of the circle ABCDE, and upon it stands a right segment of a circle DG, which, beginning at D, is drawn through G until it cuts the circle ABCDE again at another point A. This segment is not bisected at G (because G is not the pole of the circle ABCDE, at which the said segment would be bisected, as was shown in the preceding theorem), and the greater part runs from the point G up to A, since the remaining pole lies within it

(otherwise the arc GDA would be drawn through both poles, and hence would be greater than a semicircle, since the arc between two poles is a semicircle); while DG is the lesser part. [Margin: b — Schol.21. huius (Scholium to Sphaerica II.21)] Therefore, of the straight lines drawn from G to the circumference of the circle ABCDE, the greatest is GA and the least GD; GB is greater than GC; and GB and GE are equal. Likewise GH is less than GI, and GH and GK are equal. Wherefore, since they are subtended by arcs less than a semicircle, by hypothesis, [Margin: c — Schol.28. tertij (Scholium to Elements III.28)] the arc GA will also be the greatest of all, and GD the least; while GB is greater than GC, and GH less than GI. [Margin: d — 28. tertij (Elements III.28)] Finally, GB and GE, as also GH and GK, are equal to each other. Which is what was proposed.

It is, moreover, evident that in the two preceding theorems the individual arcs drawn from G must not be greater than a semicircle; otherwise the greater lines would not cut off the greater arcs, and conversely, as is clear from the scholium to Proposition 28 of Book 3 of Euclid.

Diagram. The figure for Theorem V: a circle ABCDE with a point G outside it (to the right), from which arcs are drawn cutting the circle. The arc GDFA passes through the interior pole F; the other arcs cut the circumference at K, H, and I.

- E (upper left, on circle): endpoint of arc GKE
- F (interior): pole of the circle ABCDE, lying on the arc GDFA
- K (interior, upper right): point where arc GKE cuts the circle
- D (interior right, on the arc through the pole): point where GDFA meets the near circumference
- H (interior, lower): point where arc GHB cuts the circle
- I (interior, lower): point where arc GIC cuts the circle
- A (left, on circle): far endpoint of arc GDFA
- B (lower left, on circle): far endpoint of arc GHB
- C (bottom, on circle): far endpoint of arc GIC
- G (right, outside the circle): the external marked point

THEOR. 20. PROP. 22.

If in a sphere a greatest circle be tangent to one circle but cut another parallel to it — the latter placed between the sphere's center and the circle to which the greatest circle is tangent — and the pole of the greatest circle be between both parallels, and greatest circles be described tangent to the larger of the two parallels: all these will be inclined to the greatest circle. The most upright among them will be the

one whose point of contact is at the point where the greater segment of the larger parallel is bisected; the lowest and most inclined, the one whose point of contact is at the point where the lesser segment is bisected. Of the rest, those which are equally distant from either of the points at which the segments are bisected are similarly inclined; but the one whose point of contact is more remote from the point at which the greater segment is bisected is always more inclined than the one whose point of contact is nearer to that same point. Finally, the poles of the greatest circles will lie on one circle, which will be smaller than the circle to which the greatest circle was tangent at the start, and will be parallel to it.

In a sphere let there be a greatest circle ABCD, whose pole is E; let it be tangent to the circle AF, and let it cut another parallel to this, GHBD, placed between the sphere's center and the circle AF, in such a way that the circle GBHD is larger than AF; and let E, the pole of the greatest circle ABCD, be between both circles AF and GBHD. Now since the greatest circle ABCD cuts the circle GBHD not into two equal parts — because it does not pass through its poles, that is, through the poles of the parallels — [Margin: a — 19. huius (Sphaerica II.19)] the segment BHD, toward the visible pole (which let be I), will be greater than a semicircle, and BGD less. [Margin: b — 20. 1.hui. (Sphaerica I.20)] Let there be drawn through E the pole—

[The demonstration of **THEOR. 20. PROP. 22** continues from the preceding page.]

— through E, the pole of circle ABCD, and I, the pole of the parallels, let the great circle GAC be drawn, [Margin: (c) "9. huius" — Sphaerica II.9] which will bisect the segments BGD, BHD. Let the points M, N be equally distant from H, and let O be more distant from H than N is. [Margin: (d) "14. huius" — Sphaerica II.14] Now let the great circles GL, Hk, MP, Nk, OL touch the parallel GBHD at the points G, H, M, N, O. All of these will be inclined to the greatest circle ABCD, since they do not pass through E, its pole. For since E is placed as the pole between the parallels AF and GBHD, the circles tangent to the circle GBHD cannot pass through E — otherwise they would cut it, because the other pole, [Margin: (e) "Corol. 10. 1. huius" — Corollary to Sphaerica I.10] through which they must also necessarily pass, lies outside the said parallels, as is evident.

I say that the circle Hk is the most upright — that is, the least inclined — while GL is the lowest, that is, the most inclined; that MP and Nk are similarly inclined, and OL more so than Nk; and finally that the poles of these tangent circles [Margin: (f)

"Coroll. 16. 1. huius" — Corollary to Sphaerica I.16] lie in one and the same parallel, which is smaller than AF.

For since E is the pole of circle ABCD, EA will be a quadrant of the greatest circle. Let the arc HQ be taken equal to it. Then the point Q will lie between the points A and I, since the arc HA is greater than a quadrant (because EA has been shown to be a quadrant) and HI is less than a quadrant, [Margin: (g) "Coro. 16. 1. huius" — Corollary to Sphaerica I.16] because the arc extended from the pole I through H as far as the greatest of the parallels is a quadrant. If therefore from the pole I, at the distance IQ, a circle QTR is described, [Margin: (h) "2. huius" — Sphaerica II.2] it will be parallel to AF and smaller than it. In this parallel QTR, then, I say [Margin: (i) "20. 1. hui." — Sphaerica I.20] lie the poles of all the circles tangent to the parallel GBHD. [Margin: (k) "5. huius" — Sphaerica II.5] For through the pole I and the points of contact let the great circles MIS, NIT, OIV be described; [Margin: (l) "28. tertij" — Euclid III.28] these too will pass through the poles of the tangent circles. Now since the arcs HI, MI, NI, OI, GI are equal — because, by the definition of a pole, the straight lines subtending those arcs are equal — and by the same reasoning the arcs IQ, IS, IT, IV, IR are equal, the whole arcs HQ, MS, NT, OV, GR will be equal; and hence, since HQ is a quadrant, all those arcs will be quadrants. Therefore, since it has been demonstrated that these circles pass through the poles of the tangent circles, [Margin: (m) "Coro. 16. 1. huius" — Corollary to Sphaerica I.16] the points Q, S, T, V, R will be the poles of the tangent circles, all of which lie in the parallel QTR — which was the last thing proposed for demonstration.

Now, since the arcs of the great circles drawn from E, the pole of the greatest circle ABCD, to Q, S, T, V, R, the poles of the tangents, measure the distances of the pole E from the poles of the tangents — for these two are equally distant from the greatest one, EQ, on account of the arcs QS, QT, which are equal. [Margin: (n) "10. huius" — Sphaerica II.10] For the arc of the parallel VR, lying between the great circles HI, MI, NI, is similar to the arcs MH, NH; and so, since these latter are assumed equal, the former will also be equal — and since these are equal to the arcs QS, QT, each to each, [Margin: (o) "15. 1. hui." — Sphaerica I.15] because the common sections of the parallel VR and of the great circles HQ, MS, NT drawn through its poles are diameters of it. From this it follows, since the arcs between these diameters near R are equal to one another, [Margin: (p) "26. tertij" — Euclid III.26] that the arcs opposite to those, QS, QT, are also equal, on account of the angles at the center contained by the said diameters, [Margin: (q) "27. tertij" — Euclid III.27] which are equal, inasmuch as they stand on equal arcs near R. For from this it also follows that the vertical angles standing on the arcs QS, QT are

equal. And [Margin: (r) "Scho. 21. huius" — Scholium to Sphaerica II.21] EQ is the greatest of all, ER the least, while ES and ET are equal; and finally ET is greater than EV, because all these arcs are less than a semicircle — for EQ is less than the quadrant EA, and hence the rest will not cut it around the point Q, and so will be less than a semicircle. [Margin: (s) "Schol. 21. huius" — Scholium to Sphaerica II.21] Therefore the circle Hk will be the least inclined to the greatest circle ABCD, and GL the most; while MP and Nk are inclined equally, or similarly, and OL more than Nk — which was what was proposed for demonstration in the first place. And therefore, if in a sphere a greatest circle, etc. Which was to be demonstrated.

Diagram. A figure for THEOR. 20. PROP. 22. It shows the greatest circle **ABCD** (quartered by two diameters through its center) with several concentric parallels and a set of oblique tangent great circles, together with an inner parallel **QTR** carrying the poles of the tangent circles. Readable labels, by approximate position:

- G (top of the outer figure); L (top, just right of G)
- A (upper center, on the outer circle below G)
- O (upper right)
- B (left); D (right) — the horizontal extremities of circle ABCD
- F (inner, left of center)
- I (center of the figure — pole of the parallels)
- S (inner, upper); Q (inner, upper, near S); T (inner, left); V (inner, lower left); R (inner, lower center) — the inner ring of poles on the parallel QTR
- M (lower left); E (center, below I — pole of ABCD); N (lower right)
- P (lower left); H (bottom center); K (bottom, right of H)
- C (bottom of the outer figure)

THEOR. 21. PROP. 23. [Margin: xxxiv]

With the same things assumed, if the circumferences of the tangent circles from the points of contact to the nodes are equal, then the aforesaid great circles will be similarly inclined.

Once again, let **ABCD** be a greatest circle in the sphere, whose pole is E, touching the circle AF, and cutting another circle GBHD parallel to it — GBHD being placed between the center of the sphere and the circle AF, so that GBHD is greater than AF, and E, the pole of the greatest circle ABCD, lies between both circles AF and GBHD. Next let the great circles MO, NP touch GBHD at the points M, N, cutting ABCD at the nodes O, P; and let the arcs MO, NP be equal. I say that the circles

MO, NP are similarly inclined to the greatest circle ABCD. [Margin: (a) "20. 1. hui." — Sphaerica I.20] For let the great circle GAC be drawn through E, the pole of circle ABCD, and I, the pole of the parallels. Likewise through I, the pole of the parallels, and the points of contact, let the great circles IM, IN be drawn, [Margin: (b) "5. huius" — Sphaerica II.5] which will also pass through the poles of the tangent circles, [Margin: (c) "15. 1. hui." — Sphaerica I.15] and hence will cut them at right angles. Since, therefore, the equal segments of the circles — namely the semicir-

Diagram. A figure for THEOR. 21. PROP. 23; the same figure is reprinted at the top (described here once). It shows a greatest circle with an inscribed pattern of tangent great circles and their nodes. Readable labels, by approximate position:

- G (top); A (below G, on the circle)
- D (upper right); B (left)
- F (inner, left of center)
- I (center); E (center, below I)
- O (left, on the circle); P (right, on the circle)
- M (lower left of center); N (lower right of center)
- H (bottom center)
- T (lower right); S (lower left)
- Q (bottom left); K (bottom center); R (bottom, right of K)
- C (bottom of the figure)

—cles which extend from M and N through I until they again cut the tangent circles MO, NP — stand on the diameters of the circles MO, NP ([Margin: (d) "21[?]. 1. hui." — Sphaerica I.21] for the common section of the great circles IM, MO is a diameter of each, since they cut one another in half) at right angles, and are divided not into two equal parts at I, because I, the pole of the parallels, is not the pole of the tangents; and since the arcs MO, NP are assumed equal, [Margin: (e) "12. huius" — Sphaerica II.12] the drawn straight lines IO, IP will be equal. If therefore from I, the pole of the parallel, a parallel Ok is described at the distance IO, it will also pass through P. And since the greatest circle IM, passing through the poles of the circles MO, OQ which cut one another in O, Q, [Margin: (f) "9. huius" — Sphaerica II.9] bisects their segments, the arcs MO, MQ and SO, SQ will be equal. By the same argument the arcs NP, NR and TP, TR will be equal, as also kO, kP and CO, CP; [Margin: (g) "9. huius" — Sphaerica II.9] because the greatest circle IkC, passing through the poles of the circles OkP, OCP, bisects their segments at k and C. Since, therefore, the arcs MO, NP are assumed equal, the

wholes OMQ, PNR — of which these are the halves — will also be equal; [Margin: (h) "29. tertij" — Euclid III.29] and hence the subtending chords OQ, PR will be equal. [Margin: (i) "28. tertij" — Euclid III.28] Therefore the arcs OSQ, PTR will also be equal, and consequently their halves OS, PT will be equal. But the wholes kO, kP have been shown equal. Therefore the remainders kS, kT will be equal; and hence, since they belong to one and the same circle, they will be similar to one another. [Margin: (k) "10. huius" — Sphaerica II.10] And since the arcs HM, HN are similar to the arcs kS, kT, the arcs HM, HN will also be equal. [Margin: (l) "9. huius" — Sphaerica II.9] Therefore, since the segment BHD is bisected at H, and the arcs HM, HN are equal, [Margin: (m) "22. hui." — Sphaerica II.22] the circles MO, NP will be similarly inclined to the circle ABCD. Therefore, with the same things assumed, if the circumferences from the points of contact, etc. Which was to be demonstrated.

FINIS LIBRI II. THEODOSII.

THEODOSII SPHÆRICORVM LIBER TERTIVS.

[THEODOSIUS'S SPHERICS, BOOK THREE.]

THEOR. 1. PROP. 1.

If a straight line cuts a circle into unequal parts, and upon it there is set an erect circular segment that is not greater than a semicircle, and the circumference of the standing segment is divided into two unequal parts: then the straight line subtending the smaller of them is the least of the straight lines drawn from the same point to the greater part of the circumference of the first circle; and of the straight lines drawn from that same point to the circumference intercepted between that least line and the diameter — onto which the perpendicular drawn from that point falls — the one nearer the least is always less than the one farther off. But the greatest of all is the one drawn from that same point to the extremity of that same diameter. Likewise, the straight line subtending the greater circumference of the standing segment is the least of those that fall on the circumference intercepted between it and the diameter, and the one nearer to it is always less than the one farther off. If, however, the straight line cutting the subject circle is its diameter, and everything else is the same as above, then the straight line subtending the smaller part of the circumference of the standing segment is the least of the straight lines drawn from that same point to the circumference of

the first and of the subject circle; while the one that subtends the greater part of the circumference of the standing segment is the greatest.

Let the straight line AB cut the circle ACBD, whose center is E, into unequal parts, of which the greater is ACB. Upon AB let there stand an erect circular segment AFB, not greater than a semicircle, which is divided into unequal parts at F, and let the smaller part be BF. From F [Margin: (a) "11. vnde." — Euclid XI.11(?)] let a perpendicular FL be dropped onto the circle ACBD, [Margin: (b) "38. vnde." — Euclid XI.38(?)] which will fall on AB at the common section. Through E and L let the diameter CD be drawn; and from F let very many straight lines FB, FG, FH, FC, FA, FI, FK fall onto the circumference ACB of the greater segment of the circle ACBD. I say that FB is the least of all, and FG less than FH; but that FC is the greatest of all. Likewise FA is the least of those that fall from F onto the portion AC, and FI is less than FK. From L let the straight lines LG, LH, LI, LK be drawn; then, by definition 3 of Book XI of Euclid, all the angles at L which the line FL makes will be right. [Margin: (c) "7. tertij" — Euclid III.7] Since, therefore, the straight line LD is the least of all the straight lines falling from L, and LB is less than LG, LH, LC, LK, LI, LA, the squares on FL, LB will be less than the squares on FL, LG. [Margin: (d) "47. prim." — Euclid I.47] But the square on FB is equal to the squares on FL, LB, and the square on FG to the squares on FL, LG. Therefore the square on FB will also be less than the square on FG; and hence the line FB will be less than FG. In the same way we shall show that FB is less than FH, FC, FK, FI, FA. Therefore FB is the least of all.

Again, [Margin: (e) "7. tertij" — Euclid III.7] since LG is less than LH, the squares on FL, LG will be less than the squares on FL, LH. [Margin: (f) "47. prim." — Euclid I.47] But the square on FG is equal to the squares on FL, LG, and the square on FH to the squares on FL, LH. Therefore the square on FG will also be less than the square on FH; and hence the line FG will be less than the line FH.

Diagram. A figure for THEOR. 1. PROP. 1 of Book III. It shows the circle **ACBD** with center E, a horizontal secant/diameter, and a great many straight lines drawn from the apex point F down to points on the circumference, together with the auxiliary lines from L. Readable labels, by approximate position:

- F (top of the circle, apex of the pencil of lines)
- D (upper right)
- A (left, on the circle); B (right, on the circle)
- L (center, on line AB)
- E (center, below L — the center of the circle)

- I (lower left); K (lower left, below I)
- G (right, below B)
- H (lower right)
- C (bottom of the circle)

Lines drawn from F to B, G, H, C, A, I, K, and the auxiliary lines LG, LH, LI, LK are shown.

Moreover, [g: Elements III.7] since LC is the greatest of all the lines falling from L, the squares on FL and LC together will be greater than the squares on FL and LK together. [h: Elements I.47] But the square on FC equals the squares on FL and LC together, just as the square on FK equals the squares on FL and LK together. Therefore the square on FC will also be greater than the square on FK; and consequently the straight line FC will be greater than the straight line FK. In the same way we shall prove that the straight line FC is greater than FI and FA. Therefore FC is the greatest of all.

Likewise, [i: Elements III.7] since LA is less than LI, LK, and LC, the squares on FL and LA together will be less than the squares on FL and LI together. [k: Elements I.47] But the square on FA equals the squares on FL and LA together, just as the square on FI equals the squares on FL and LI together. Therefore the square on FA will be less than the square on FI; and for that reason the straight line FA too will be less than the straight line FI. In the same way we shall show that the straight line FA is less than FK and FC. Therefore FA is the least of all the straight lines falling from F onto the arc AC.

Finally, [l: Elements III.7] since LI is less than LK, the squares on FL and LI together will be less than the squares on FL and LK together; but the square on FI equals the squares on FL and LI together, just as the square on FK equals the squares on FL and LK together. Therefore the square on FI will be less than the square on FK, and hence the straight line FI is less than the straight line FK.

Now if the straight line AB cuts the circle ABCD into two equal parts, so as to be its diameter, we have already proved — in the third theorem of the scholium to proposition 21 of the preceding book — that the straight line FB is the least and FA the greatest. Hence there is no need to demonstrate the same thing in this place. Indeed, more was demonstrated there than is set forth here. Therefore, if a straight line cuts a circle into unequal parts, and so on. Which was to be shown.

THEOR. 2. PROP. 2.

If a straight line cutting a circle cuts off a segment that is not less than a semicircle, and upon that straight line another segment of a circle is set up which is not greater than a semicircle and is inclined to the other segment (which is not greater than a semicircle), and the circumference of the standing segment is divided into unequal parts: then the straight line subtending the smaller part of the circumference is the least of all the straight lines drawn from that point, from which it is drawn, to that circumference of the underlying circle which is not less than a semicircle; and all the rest that held in the preceding proposition follows here too.

Let the straight line AB cut off from the circle ABCD (whose center is E) a segment ACB that is not less than a semicircle — either equal to a semicircle, as in the first figure, or greater, as in the other figures; and upon the straight line AB let another segment of a circle, AFB, be set up that is not greater than a semicircle — either equal to a semicircle, as in the last of the three figures, or less, as in the first two figures — and inclined to the other segment ADB, which is not greater than a semicircle, since ACB is supposed either equal to or greater than a semicircle. Let the circumference AFB be divided at F into unequal parts, and let FB be the smaller. From F let a perpendicular FL be dropped onto the plane of the circle ACBD; it will fall toward the side of segment ADB, because the segment AFB is inclined toward segment ADB, so that the point L is either inside segment ADB, or outside, or else on the very circumference ADB. Through the center E and the point L let the diameter CD be drawn; and from F onto the circumference ACB let many straight lines fall — FB, FG, and the rest. I say that FB is the least of all; that FG is less than FH; that FC is the greatest of all; that FA is the least of all the lines that fall from F onto the circumference AC; and that FI is less than FK. Let the straight lines LB, LG, LH, LA, LI, LK be drawn from L, and all the angles at L that the perpendicular FL makes will be right angles, by definition 3 of book 11 of Euclid. [a: Elements III.7, or 8, or 15] Since therefore the straight line LD is the least of all (though this line amounts to nothing at all in that figure where the point L falls on D), and LB is less than LG, LH, LC, LK, LI, LA, while LC is the greatest of all, and so on, we shall demonstrate, as in the preceding proposition, that FB is the least of all the lines and FG less than FH; likewise that FC is the greatest of all, and FA the least of all those falling from F onto the circumference AC, and FI less than FK. Therefore, if a straight line cutting a circle, and so on. Which was to be shown.

Diagram. The set of three circle-figures illustrating Theorem 2 (the "first," "second," and "last of the three" figures referred to in the text). In each figure a chord AB is drawn roughly horizontally across the circle; the apex F stands at the top, the point C at the bottom, and D in the upper right. Many straight lines are drawn from F to the points on the circumference (FB, FG, FH, FA, FI, FK, FC), forming a fan of rays. The point L lies on the diameter CD (drawn through the center E); across the three figures L falls, respectively, inside, related to, and on the circumference of segment ADB.

- F (top, apex): vertex from which the rays are drawn
- A (left end of chord AB)
- B (right end of chord AB)
- C (bottom of circle): endpoint of diameter CD
- D (upper right): endpoint of diameter CD
- E (center of circle)
- L (on diameter CD): foot of the perpendicular from F
- G, H (on lower-right arc): ray endpoints
- I, K (on lower-left arc): ray endpoints

SCHOLIUM.

Let these two figures be added to the three of this second proposition, so that all the cases of the perpendicular line FL may be surveyed. For in the first of these figures the standing segment AFB is a semicircle, and the perpendicular FL falls inside segment ADB; but in the latter the same FL falls on the circumference ADB itself, the same standing segment AFB being a semicircle; just as also in the third figure of the said proposition the same standing segment AFB is a semicircle, and the perpendicular line FL falls outside segment ADB. This, kind reader, we did not wish to keep hidden from you.

Diagram. The two supplementary figures added by the scholium, each a circle with chord AB, apex F at the top, and a fan of straight lines drawn from F. In the first figure the perpendicular FL falls inside segment ADB; in the second, FL falls on the circumference ADB.

- F (top, apex)
- L (below F): foot of the perpendicular
- A (left end of chord AB)
- B (right end of chord AB)
- E (center)

- C (bottom)
- G, H (lower-right arc)
- I, K (lower-left arc)

THEOR. 3. PROPOS. 3.

If in a sphere two great circles cut one another, and from each of them equal arcs are taken on both sides from the point at which they intersect: then the straight lines that join the endpoints of the arcs on the same sides are equal to one another.

Let two great circles ABC, DBE in a sphere cut one another at B, and on each circle, on both sides of B, let two equal arcs BA, BC and BD, BE be taken, and let the straight lines AD, CE be joined. I say that the straight lines AD, CE are equal. For with B as pole and BA as radius let a circle be described; it will pass through C as well, on account of the equality of the arcs BA, BC. Now either this same circle passes through D also — and hence through E, on account of the equality of the arcs BD, BE — or it does not. Let it first pass through D and E, as in the first figure; and let the common sections of the great circles with the circle ADCE be the straight lines AC, DE. Since the great circles ABC, DBE, passing through B, the pole of the circle ADCE, [a: Sphaerica I.15] cut it into two equal parts, AC and DE will be diameters of the circle ADCE, and F its center; and consequently the straight lines FA, FD are equal to the straight lines FC, FE. [b: Elements I.15] Since therefore they also contain equal angles at the vertex F, [c: Elements I.4] the straight lines AD, CE will also be equal.

But now let the circle described from B as pole with radius BA not pass through D, but run out beyond point D and hence beyond point E. If the circle AGCH were to pass on the near side of point D — which would indeed happen if the arc BD were greater than the arc BA — a circle would have to be described from B with a radius equal to the greater arc BD, so that it runs out beyond point A. Let the arcs BD, BE be produced to G, H. [d: Elements III.28] Since then the arcs BG, BH are equal — because, by the definition of a pole, the subtending straight lines BG, BH are equal — and since BD, BE are also equal by hypothesis, the remaining arcs DG, EH will likewise be equal. And since the straight lines drawn AG, CH are equal, as was just demonstrated in the first part of this proposition, [e: Elements III.28] the arcs AG, CH will also be equal. Since therefore the great circle GBH, drawn through the pole B, [f: Sphaerica I.15] cuts the circle AG, CH into two equal parts and at right angles, the chord GH will stand perpendicular to the diameter of the circle AG, CH. Since then the arcs DG, EH are equal and less than half the arc GDH, and since the arcs GA, HC have also been shown equal, [g: Sphaerica

II.12] the straight lines DA, EC will be equal to one another. Therefore, if in a sphere two great circles cut one another, and so on. Which was to be demonstrated.

Diagram. The two figures for Theorem 3. In the first circle the chords AC and DE cross at the center F, together with the joining lines AD and CE. In the second circle the arcs BD, BE have been produced to G and H, so that the second (larger) circle AGCH is described.

First figure:

- A (top left)
- D (top right)
- B (center-left, at the mutual intersection)
- F (center): center of circle ADCE
- E (bottom left)
- C (bottom)

Second figure:

- A (top left)
- G (top right)
- D (right, below G)
- B (center, at the intersection)
- E (lower left)
- H (bottom left)
- C (bottom right)

THEOR. 4. PROPOS. 4.

If in a sphere two great circles cut one another, and from one of them equal arcs are taken on both sides from the point at which they intersect, and through the points bounding the equal arcs two parallel planes are drawn, one of which meets the common section of those circles outside the sphere on the side of the said point; and if one of those equal arcs is greater than either of the arcs intercepted on the other great circle between the said point and each of the parallel planes: then the arc that lies between that point and the plane which does not meet the common section of those circles is greater than the arc of the same circle that lies between the same point and the plane which does meet the common section of the circles.

Let two great circles ABC, DBE in a sphere cut one another at B, and in ABC let equal arcs BA, BC be taken, and through the points A, C let two parallel planes be drawn [a: Sphaerica I.1] producing on the surface of the sphere the circumferences of the circles AFG, CHI, which cut the circumference DBE at the points F, H; and let the arc BA, or BC, be greater than either of the arcs BF, BH intercepted between the point B and the parallel planes. Next, from the pole B with radius BA or BC let a circle ADCE be described; it will pass beyond the points F, H, because the arcs BF, BH are supposed less than the arcs BA, BC. Let the arcs BF, BH be produced as far as the circumference of the circle ADCE, to the points D, E; and let the common sections of the circle ADCE with the circles AFG, CHI be the straight lines AG, CI; and let the common sections of the great circles with the circle ADCE be the straight lines AC, DE, which will be its diameters, and K its center, [b: Sphaerica I.15] since the great circles cut it into two equal parts through the pole B. Let the straight line DE cut the straight lines AG, CI at M, N. Let the common section of the great circles, KB, likewise be a straight line, with which, produced toward B, the plane AFG (produced outside the sphere) meets at the point L. This being so, the other plane CHI will not meet the straight line KB on the side of B, lest it should meet the plane AFG parallel to it. I say that the arc BH is greater than the arc BF. For let the straight lines FM, HN be the common sections of the circle DBE with the circles AFG, CHI. And since the plane AFG, produced, meets [the straight line]

Diagram. A sphere-schematic for Theorem 4. The great circle ABC and the circle DBE cut at B; the circle ADCE is drawn through A, D, C, E with center K. The common section KB is produced to meet, outside the sphere, at the apex L above the sphere. The straight lines DE, AG, CI cross the interior; DE meets AG, CI at M, N; the common sections FM, HN are shown.

- L (top, outside the sphere): meeting point of the produced planes
- A (upper left)
- I (upper right)
- B (center, on the great-circle line)
- H (right of B)
- E (right edge)
- F (center-left)
- N (center-right)
- K (center): center of circle ADCE
- M (lower left)
- D (left edge)

- G (bottom left)
- C (bottom right)

[The demonstration of THEOR. 4 continues from the foot of the preceding page; the sphere-figure is reprinted at the top of this page — it is the same as fig-p677-2 above and is not re-described.]

the straight line KB, produced to L, the point L will be both in the plane DBE and in the plane AFG, and hence in their common section, namely in the straight line MF. Therefore MF, produced, will coincide with KB produced at L. Now since the plane DBE cuts the parallel planes AFG, CHI, [c: Elements XI.16] the sections made, ME and NH, will be parallel. Again, since the plane ADCE cuts the same parallel planes, [d: Elements XI.16] the sections made, AG and CI, will likewise be parallel. [e: Elements I.29] Therefore the alternate angles KAM, KCN are equal. [f: Elements I.15] But the angles AKM, CKN are also equal, being vertical, and the sides KA, KC are equal, since they are radii of the circle ADCE. [g: Elements I.26] Therefore the sides KM, KN will also be equal; and the radii KD, KE are equal as well. Hence the remaining straight lines DM, EN will be equal. Again, since the straight line BK, drawn from B, the pole of the circle ADCE, to its center K, [h: Scholium to Sphaerica I.8] is perpendicular to the plane of the circle, the angle MKL in the triangle KLM will be a right angle, by definition 3 of book 11 of Euclid. [i: Elements I.17] Therefore the angle KML will be acute. [k: Elements I.29] Since then the two angles FMN, HNM are equal to two right angles, the angle HNM will be obtuse. Wherefore, as we shall presently show by the following lemma, the arc EH will be less than the arc DF; and hence, [l: Elements III.28] since the arcs BD, BE are equal — because the subtending straight lines BD, BE are equal, by the definition of a pole — the arc BH will be greater than the arc BF. Therefore, if in a sphere two great circles cut one another, and so on. Which was to be demonstrated.

LEMMA.

That the arc EH is less than the arc DF we shall easily demonstrate, once the theorem here set down has first been proved.

If a straight line is subtended to an arc of a circle, and from the arc two perpendiculars are dropped to it cutting off, toward the ends of the arc, two equal arcs: these same perpendiculars will cut off two equal straight lines from the subtending line. And if two perpendiculars are drawn to the subtending line cutting off two equal straight lines: these same will cut off two equal arcs.

Let a straight line AD be subtended to the arc ABCD of a circle, and to it let two perpendiculars BE, CF be dropped from the arc, cutting off two equal arcs AB, DC. I say that they cut off equal straight lines AE, DF. For when the straight line BC is drawn, [m: Scholium to Elements III.27] AD and BC will be parallel, on account of the equality of the arcs AB, DC; [n: Elements I.28] and BE, CF are also parallel. Therefore BEFC is a parallelogram, [o: Elements I.34] and hence the straight lines BE, CF are equal. [p: Elements III.29] And since to the equal arcs AB, DC the subtending straight lines AB, DC are equal, the squares on AB, DC will be equal. [q: Elements I.47] Since therefore the one is equal to the squares on AE, BE together, and the other to the squares on DF, CF together, if the equal squares of the straight lines BE, CF are taken away, the squares of the straight lines AE, DF will be equal; and consequently the straight lines AE, DF will be equal. Which was proposed in the first place.

Diagram. The figure for the lemma. An arc ABCD stands over the straight base line AD. From the arc the perpendiculars BE and CF are dropped to AD (BE at E, CF at F), forming the parallelogram BEFC. For the second part, an equal arc AG is cut off and the perpendicular GH is dropped from G to AD.

- A (bottom left): end of the base line
- H (on AD, near A): foot of perpendicular GH
- E (on AD): foot of perpendicular BE
- F (on AD): foot of perpendicular CF
- D (bottom right): end of the base line
- G (top left, on the arc)
- B (top, above E, on the arc)
- C (top, above F, on the arc)

But now let the perpendiculars BE, CF cut off equal straight lines AE, DF. I say that they cut off equal arcs AB, DC. For if they are not equal, let the arc AB be the greater, if that be possible, and from it let an equal arc AG be cut off, and from G let a perpendicular GH be drawn to AD. Then, as was just demonstrated, the straight line AH will be equal to the straight line DF, and hence also to the straight line AE — a part equal to the whole: which is absurd. Therefore the arc AB is not greater than the arc DC; and by the same reasoning neither is it less. Therefore it is equal. Which was the thing proposed. From these things it is clear that the arc HE, in the figure of the proposition, is less than the arc DF. For since the angle FMK is acute and HMK obtuse, if perpendiculars were drawn from M, N to DE, they would fall on the arcs DF, BH and would cut off equal arcs, as we showed in the previous lemma. Wherefore the arc HE is less than the arc DF.

THEOR. 5. PROP. 5.

If on the circumference of a great circle there is a pole of the parallels, and this great circle is cut at right angles by two other great circles, of which one is one of the parallels and the other is oblique to the parallels; and if from this oblique circle equal arcs are taken in succession toward the same side of the greatest of the parallels, and through the points bounding these equal arcs parallel circles are described: then the arcs of that first-mentioned great circle intercepted between the parallels will be unequal, and always the one that is nearer to the greatest of the parallels will be greater than the one farther off.

On the circumference of the great circle ABCD let A be the pole of the parallels, and let two great circles BD, EC cut it at right angles, of which BD is the greatest of the parallels and EC is oblique to the parallels; and through F, G, H — the points which cut off equal arcs FG, GH from the oblique circle — let the parallel circles IK, LM, NO be described from the pole A. I say that the arc IL is greater than the arc LN. [a: Sphaerica I.20] For through the pole A and the point G let a great circle AP be described, cutting the parallels at P, Q. Since then a point G, distinct from the pole A, has been marked on the surface of the sphere within the periphery of the circle IK, and from G two arcs GP, GF of great circles fall onto the circumference of the circle

Diagram. A sphere-schematic for Theorem 5. The great circle ABCD is shown with the vertical diameter AC (A at the top, pole of the parallels; C at the bottom) and the horizontal diameter BD (B at the left, D at the right). The oblique great circle EC runs across, and the three parallel circles IK, LM, NO are drawn as arcs; the great circle AP through the pole A and the point G cuts the parallels at P, Q. The interior points F, G, H, P, Q cluster near the center.

- A (top): pole of the parallels
- C (bottom)
- B (left): endpoint of the horizontal diameter
- D (right): endpoint of the horizontal diameter
- E (upper left): on the oblique circle EC
- O (upper right): endpoint of parallel NO
- N (left, upper): endpoint of parallel NO
- L (left, middle): endpoint of parallel LM
- M (right, middle): endpoint of parallel LM
- I (left, lower) [?]: endpoint of parallel IK

- K (right, lower): endpoint of parallel IK
- H (upper center): point on the oblique circle
- Q (upper center, right of H): intersection of AP with a parallel
- G (center): point on the oblique circle
- P (center-lower) [?]: intersection of AP with a parallel
- F (center-lower): point on the oblique circle

(Continuation of THEOR. 5. PROP. 5 — the demonstration begun on the preceding page, broken off at the catchword "circuli.")

of the circle IK. [Margin: b — Sphaerica II.21, scholium] Then the arc GP will be the smallest of all, and therefore smaller than GF — because the arcs GP and GF are less than a semicircle, since they do not intersect one another before the parallel IK divides them. For since GP is part of the quadrant running from A through G as far as the greatest of the parallels BD, it will not be able to cut the arc GF except beyond the circle IK, otherwise GP would be either a semicircle or greater than a semicircle — namely if it were to cut GF either at F or short of F. Again, since the point G is marked on the surface of the sphere outside the circumference of the circle NO, apart from its pole, [Margin: c — Sphaerica II.21, scholium] the arc GQ will be the smallest of all those falling from G, that is, smaller than GH — because the arcs GQ and GH are less than a semicircle, since they do not meet one another before reaching the parallel NO (which will be demonstrated, just as was said a little earlier of the arcs GP and GF). Therefore each of the arcs FG and GH is greater than each of GP and GQ.

And the straight line drawn through G and the center of the sphere — that is, the common section of the greatest circles AP and EC — is cut by the parallel plane IK within the sphere (for that line will not reach the center of the sphere, that is, the center of the greatest circle BD, unless it first cuts the plane of the circle IK, since the parallel IK is placed between the greatest of the parallels and the point G). Consequently the same straight line will cut the plane of the parallel NO outside the sphere, if that line and the plane of the circle are produced toward the side of G — because the point G lies between the greatest of the parallels and the parallel NO. Therefore, since the two greatest circles AP and EC cut one another at the point G, and from the circle EC two equal arcs GF, GH have been taken on either side of the point G, and through F and H the parallel planes of the circles IK and NO have been drawn — of which NO meets the common section of the greatest circles AP and EC outside the sphere, as has been shown — and since each of the arcs GF and GH is greater than each of the arcs GP and GQ: [Margin: d — Sphaerica III.4] the arc GP will be greater than the arc GQ. [Margin: e —

Sphaerica II.10] But the arc GP is equal to the arc IL, and the arc GQ to the arc LN. Therefore the arc IL too will be greater than the arc LN. And so, if the pole of the parallels lies on the circumference of a greatest circle, and so on. Which was to be demonstrated.

THEOR. 6. PROP. 6.

[Margin: iv.]

If the pole of the parallels lies on the circumference of a greatest circle, and two other greatest circles cut this greatest circle at right angles — one of them being one of the parallels, the other oblique to the parallels — and if equal circumferences are taken in succession on the oblique circle toward the same side of that greatest parallel, and through the points bounding those equal circumferences, and through the pole, greatest circles are described: these will cut off unequal circumferences from the greatest of the parallels, of which the one nearer to the greatest circle first posited will always be greater than the more remote.

In the circumference of the greatest circle ABCD let A be the pole of the parallels, and let two greatest circles BD and EC cut it at right angles, of which BD is the greatest of the parallels, while EC is oblique to the parallels. On EC let equal arcs FG, GH be taken; and through the points F, G, H, [Margin: a — Sphaerica I.20] and through the pole A, let the greatest circles AI, AK, AL be described, cutting BD at I, K, L. I say that the arc KL is greater than the arc IK. For through the same points F, G, H let the parallels MN, OP, QR be described, cutting AK at V, G, X. [Margin: b — Sphaerica III.5] Then the arc MO will be greater than the arc OQ; and therefore, [Margin: c — Sphaerica II.10] since the arc VG is equal to the arc MO and the arc GX to the arc OQ, VG too will be greater than GX. Let the arc GY be taken equal to GX, and through Y let the parallel ST be described, cutting the circle AI at Z. Now since the arcs GY, GX are equal, as are GF, GH, [Margin: d — Sphaerica III.3] the drawn straight lines HX, YF will be equal. And since the greatest circle AI, passing through the pole A, [Margin: e — Sphaerica I.15] cuts the circle ST at right angles and bisects it, the common section — namely the straight line drawn from Z to the other section — will be a diameter of the circle ST, upon which stands a semicircle at right angles to the circle AI, namely the semicircle beginning at the point Z and proceeding through S as far as the other section (that is, a segment of the circle that is not greater than a semicircle). And that straight line cuts off from the circle AI a segment greater than a semicircle — namely the one drawn from the point Z through I as far as the other section with the circle ST; and YZ is an arc of the standing semicircle less than a quadrant

(because the arc IK, [Margin: f — Sphaerica II.10] which is similar to it, is also less than a quadrant, which can be shown as follows.

For since the greatest circles BD, EC are at right angles to the greatest circle ABCD, this in turn will be at right angles to them, and consequently will pass through their poles. [Margin: g — Sphaerica I.11] Hence it will bisect their segments, [Margin: h — Sphaerica II.9] which are semicircles — that is, into quadrants. Therefore the arc of the circle BD lying between B and the point where the circles BD, EC cut one another is a quadrant, and consequently IK is less than a quadrant. For the circle AK falls between the points B, I, since it cuts the circle ABCD at the other pole.) And therefore the remaining arc, intercepted by the standing semicircle between Y and the other section with the circle AI, is greater than a quadrant; [Margin: i — Sphaerica III.1] hence the straight line YZ will be the smallest of all the straight lines falling from Y onto the circumference ZI, and therefore smaller than YF — that is, than HX, which we showed to be equal to the straight line YF. And so, since the circle QR is smaller than the circle ST, the greater straight line HX will cut off from its own circle a greater arc than the smaller straight line YZ cuts off from its own, as we shall soon show. Therefore the arc HX is greater than can be similar to the arc YZ. [Margin: k — Sphaerica II.10] But to the arc HX the arc KL is similar, and to the arc YZ the arc IK. Therefore KL too is greater than can be similar to IK; and consequently, since they are in the same circle, the arc KL will be greater than IK. And so, if the pole of the parallels lies on the circumference of a greatest circle, and so on. Which was to be demonstrated.

Diagram. A sphere shown in perspective as a bounding circle, belonging to the demonstration of THEOR. 5 PROP. 5 (concluded at the top of this page). The pole **A** is at the top and **C** at the bottom; the bounding great circle carries **B** on the left rim and **D** on the right rim. Several parallel circles cross the sphere as near-horizontal arcs, cut by oblique great circles that intersect near the center.

- **A** (top): pole of the parallels
- **E** (upper left, below **A**): on an oblique great circle
- **B** (left rim): end of the greatest parallel
- **D** (right rim): opposite end of the greatest parallel
- **C** (bottom): pole opposite **A**
- **N** (left, upper): on a parallel circle
- **O** (right, upper): on a parallel circle
- **M** (right rim): on a parallel circle
- **K** (right, mid): intersection point

- L, I (left, mid): intersections on the left
- H (center-left): intersection of oblique great circles
- Q (center): intersection point near the axis
- G (center): common section point of the great circles AP, EC
- F (center, right of P): point on oblique circle
- P (center-left, below G): point on great circle

Diagram. The sphere for THEOR. 6 PROP. 6. The pole **A** is at the top, **C** at the bottom; a family of parallel circles runs across the globe, cut by two great circles at right angles (BD the greatest parallel, EC oblique) and by great circles drawn through the pole A to I, K, L on BD.

- A (top): pole of the parallels
- E (upper left, below A): on the oblique great circle EC
- Q, O, S, M, B (left rim, top to bottom): points on the left of the sphere
- R, P, T, N, D (right rim, top to bottom): points on the right of the sphere
- H, X (upper center): intersections on circle AK / parallels
- G (center): a division point of the equal arcs on EC
- Y (center, below G): point taken so that arc $GY = GX$
- V (center, lower): intersection of AK with a parallel
- F (center, lower): division point of the equal arcs
- L, K, I (lower center): feet on BD of the great circles from A
- C (bottom): pole opposite A

LEMMA.

That the straight line HX cuts off a greater arc from its own circle than the straight line YZ does from its own will become clear if the theorem that follows is first demonstrated.

Equal straight lines cut off unequal arcs from unequal circles, and the arc of the smaller circle is greater than can be similar to the arc of the greater circle.

Let there be unequal circles AB, CD about the same center E; and from E let two straight lines EA, EB be drawn in any manner, cutting the circle CD at the points C, D, and let the same angle E stand at the center. [Margin: a — Euclid VI, scholium 33] Then the arcs AB, CD will be similar, since the same angle E stands upon them at the center. And since the straight lines EA, EB are cut proportionally at the points C, D — because EA, EB are equal, as are EC, ED —

[Margin: b — Euclid VI.2] the drawn straight lines AB, CD will be parallel; [Margin: c — Euclid VI.4, corollary] and therefore the triangles EAB, ECD are similar, having the angles EAB, ECD equal to one another, and likewise the angles EBA, EDC, and the angle E common. [Margin: d — Euclid VI.4] Hence EA is to AB as EC is to CD. But EA is greater than EC. [Margin: e — Euclid V.14] Therefore AB too will be greater than CD. [Margin: f — Euclid IV.1] Let then a straight line BF equal to CD be fitted into the circle AB; [Margin: g — Euclid III, scholium 28] and the arc AB will be greater than FB. So, since the arc CD is similar to the arc AB, the arc CD will be greater than can be similar to FB. Therefore the equal straight lines FB, CD cut off unequal arcs from the unequal circles AB, CD, and the arc CD of the smaller circle is greater than can be similar to the arc FB of the greater circle. Which was what was proposed.

From this it is clear that a much greater line cuts off from the smaller circle a far greater arc than can be similar to the one which a smaller line cuts off from the greater circle. For since the straight line CD, equal to FB, cuts off the arc CD greater than can be similar to the arc FB, all the more will a line greater than CD cut off [Margin: h — Euclid III, scholium 28] a greater arc than can be similar to the arc FB — since that greater line cuts off a greater arc than CD does. And so in this sixth proposition too, the straight line HX, being greater than the straight line YZ, will cut off from the smaller circle QR the arc HX greater than can be similar to the arc YZ, which the straight line YZ cuts off from the greater circle ST.

But this demonstration is to be understood only of arcs less than a semicircle, such as BF, CD are, as is clear from the demonstration itself. For otherwise the angle at the common center E could not be constructed — which is nevertheless required for the demonstration. Yet it will still be true, if the smaller arc of the smaller circle is greater than can be similar to the smaller arc of the greater circle, that the greater arc of the smaller circle is much greater than can be similar to the smaller arc of the greater circle. And if it should ever happen that the straight line CD cuts off a semicircle from the smaller circle — that is, that it is its diameter — it is plainly evident that the semicircle of the smaller circle is greater than can be similar to the smaller arc of the greater circle: and then there will be no need of any other demonstration.

Now, with this lemma demonstrated, we shall also easily show that equal straight lines cut off from unequal circles arcs that are unequal without qualification — so that the arc of the smaller circle is absolutely greater than the arc of the greater circle, and not merely greater than can be similar to it. For let the straight lines CD, BF be equal, and let CD cut off the arc CED of the smaller circle, and FB the

arc FGB of the greater circle. I say without qualification that the arc CED is greater than the arc FGB. For when the straight line CD is made to coincide with the straight line FB, the arc CED will necessarily fall outside the arc FGB; and thus the arc CED will be greater than the arc FGB, since the latter contains the former wholly within itself, both arcs being concave toward the same side and having the same endpoints, as Archimedes requires in the postulates prefixed to Book 1 of *On the Sphere and Cylinder* [Archimedes, *De sphaera et cylindro* I, postulates]. Nor indeed will the arc CED coincide with the arc FGB or fall within it. For if it were said to coincide, then the whole circumference of the circle CED would also coincide with the whole circumference of the circle FGB, and thus the circles would be equal — which is absurd, since they are supposed unequal. But if the arc CED were said to fall within the arc FGB — such as the arc CAD is — then, since (as was shown a little earlier in this lemma) the arc CED, that is CAD, is greater than can be similar to the arc FGB, let an arc HFB be taken similar to the arc CAD, and therefore greater than the arc FGB. Then, taking a point A on the arc CAD, let straight lines AF, AB be drawn to both; and, the straight line FA being produced until it cuts the arc FGB at G, let the straight lines GH, GB be drawn. So then, since the arcs CAD, HFB are similar, the angles CAD, HGB standing in those [Margin: i — Euclid I.16] segments will be equal. But since the angle CAD is greater than the angle CGB — the external than the internal — and the angle CGB is also greater than the angle HGB — the whole than the part — the angle CAD will be much greater than the angle HGB. Which is absurd, for it was shown to be equal. Therefore the arc CED will not fall within the arc FGB; but neither does it coincide with it, as has been demonstrated. It falls, then, outside; and thus the arc CED will be greater than the arc FGB, as was said.

From this too it is plainly evident that a much greater line cuts off from the smaller circle an arc absolutely greater than the one which a smaller line cuts off from the greater circle.

Diagram. Two concentric circles about the common center **E**. The chord of the larger circle runs from **A** (left) to **B** (right); the chord of the smaller circle runs between **C** and **D**; **F** sits on the top of the larger circle. Radii EA, EB pass through C and D.

- F (top of the larger circle): endpoint of arc FB / vertex
- A (left, larger circle): endpoint of chord AB
- B (right, larger circle): endpoint of chord AB
- C (center-left, smaller circle): where EA meets the smaller circle
- D (center-right, smaller circle): where EB meets the smaller circle

- E (center): common center of both circles

Diagram. A wide arc standing on the base line **H–B**, with the arcs CED and FGB drawn near its top. The chords AF, AB and GH, GB of the construction are shown.

- H (left, base line): endpoint of the base
- B (right, base line): endpoint of the base / shared endpoint of the arcs
- C (upper left, on the arc): endpoint of arc CED / CAD
- E (top): summit of arc CED
- F (upper center): point on arc FGB / HFB
- A (center): point taken on the arc CAD
- G (upper center-right): where FA produced cuts arc FGB
- D (right, on the arc): shared endpoint of the arcs

Diagram. A smaller companion figure below the previous one: a single arc on the base line **C–D** with **E** at its summit.

- E (top): summit of the arc
- C (left): endpoint of the base
- D (right): endpoint of the base

THEOR. 7. PROP. 7.

[Margin: v.]

If in a sphere a greatest circle touches some circle of the sphere, and another greatest circle is oblique to the parallels and touches circles greater than those which the greatest circle first posited touches, and their points of contact are on the greatest circle first posited; and if from the oblique circle equal and successive circumferences are taken toward the same side of the greatest of the parallels, and through the points bounding the equal circumferences parallel circles are described: these will cut off unequal circumferences from the greatest circle first posited, of which the one that is nearer to the greatest of the parallels will be greater than the more remote.

In the sphere let the greatest circle ABCD touch the circle AE at the point A, [Margin: a — Sphaerica II.6] and therefore also another circle CF equal to it. Let another greatest circle GH, oblique to the parallels, touch two other circles greater than those which ABCD touches, and let the points of contact G, H be on the greatest circle ABCD; and let BD be the greatest of the parallels. Finally, on the oblique circle GH let equal arcs IK, KL be taken, and through the points I, K, L let

the parallels MN, OP, QR be described. I say that the arc MO is greater than the arc OQ. [Margin: b — Sphaerica I.20] For through K and S, the pole of the parallels, let the greatest circle SK be described, cutting the parallels at the points T, V. [Margin: c — Sphaerica II.15] Likewise through K let the greatest circle KE be described, touching the parallel AE at E, and cutting the other parallels at X, Y — yet in such a way that these points X, Y lie between the points L, T and V, I. This will come about as follows. [Margin: d — Sphaerica II.15, scholium] Since through K two circles can be described touching the circle AE, of which one falls between the arcs KG, KS, while the other falls outside them (for if both were to touch the circle AE on the same side, they would cut one another near the points of contact, in that one would meet the other — which is absurd, since they intersect at the point that is opposite to K, between the other pole and the greatest of the parallels): one of the two will touch the circle AE to the right of KS, which bisects the circle AE, and the other to the left, falling between KG and KS — such as KE is. For if it were to fall outside KG, it could not touch the parallel AE, because it does not meet KG before the point opposite to K, where they bisect one another.) If the former is taken, the points X, Y will fall between the points L, T and V, I, as is evident, since at K it cuts both KG, KS. Therefore, since on the surface of the sphere, within the periphery of the circle MN, the point K is marked, apart from the pole S, and from K three arcs fall onto its circumference, KV, KY, KI, [Margin: e — Sphaerica II.21, scholium] KV will be the smallest of all, and KY smaller than KI. Again, since on the surface of the sphere, outside the periphery of the circle QR, the point K is marked, apart from its pole, and from K three arcs KT, kX, KL fall onto its circumference, [Margin: f — Sphaerica II.21, scholium] KT will be the smallest of all, and KX smaller than KL. Therefore each of the arcs KI, KL is greater than each of KY, KX. And since the straight line drawn through K and the center of the sphere — that is, the common section of the greatest circles GH, EY — cuts the plane of the parallel QR outside the sphere, if that line and the plane of the circle QR are produced toward the side of K, as was stated in the demonstration of Proposition 5 of this book; [Margin: g — Sphaerica III.4] the arc kY will be greater than the arc kX. [Margin: h — Sphaerica II.13] But the arc MO is equal to the arc kY, and the arc OQ to the arc kX; for they are semicircles, of which one is drawn from A through B, the other from E through k, not coinciding, as is clear from what we said in the demonstration of Proposition 13 of the second book. Therefore the arc MO too will be greater than the arc OQ. If, then, in a sphere a greatest circle touches, and so on. Which was to be demonstrated.

Diagram. The sphere for THEOR. 7 PROP. 7, drawn as a large bounding circle with a small tangent circle below it. Near the top, a small circle around A carries S (the pole) and E, with G at the top left. A family of parallel circles crosses the

globe; **B/D** and the flanking letters mark the rims. The small circle at the bottom (touching the sphere at **H**) carries **C** and **F**.

- G (top left): point of contact on the oblique great circle
- A (top): point of contact of ABCD with circle AE
- E (near A, left): contact of circle KE with parallel AE
- S (below A, in the small upper circle): pole of the parallels
- Q, O, M, B (left rim, top to bottom): points on the left of the sphere
- R, P, N, D (right rim, top to bottom): points on the right of the sphere
- L, X, T, K, V, Y, I (central cluster, top to bottom): feet and intersections on the parallels
- H (on the small lower circle): point of contact
- C (center of the small lower circle): the tangent circle CF
- F (lower, on the small circle): endpoint of circle CF

THEOR. 8. PROPOS. 8.

[Margin: vi.]

If in a sphere a greatest circle touches some circle of the sphere, and some other greatest circle, oblique to the parallels, touches circles greater than those which the greatest circle first posited touched, and their points of contact are on the greatest circle first posited; and if from the oblique circle equal, successive circumferences are taken toward the same side of the greatest of the parallels, and through the points bounding these equal circumferences greatest circles are described which both touch the same circle that the greatest circle first posited touched, and intercept similar circumferences of the parallels, and have those semicircles which extend from the points of contact to the points bounding the equal circumferences of the oblique circle (through which they are described) of such a kind that they in no way coincide with that semicircle of the greatest circle first posited in which lies the contact of the oblique circle, between the apparent pole and the greatest of the parallels: then they will intercept unequal circumferences from the greatest of the parallels, of which the one nearer to the greatest circle first posited will always be greater than the more remote.

In the sphere let the greatest circle AB touch the circle AC at A, [Margin: a — Sphaerica II.6] and therefore another circle equal and parallel to it; and let another greatest circle DE, oblique to the parallels, touch other greater parallels,

and let the contacts be on the circle AB — such as the point D is; and let BE be the greatest of the parallels. Then, on the oblique circle DE let equal arcs FG, GH be taken; and through the points F, G, H let the greatest circles CI, kL, MN be described, touching the paralle-

(Continuation of THEOR. 8. PROPOS. 8, the demonstration of which began on the preceding page.)

the parallel AC in C, K, M, and cutting BE, the greatest of the parallels, in I, L, N, in such a way that they intercept similar arcs of the parallels, and their semicircles — beginning from the points C, K, M and passing through F, G, H — do not coincide with the semicircle of circle AB, which begins at A and passes through B. I say that arc IL is greater than arc LN.

For let there be described, through F, G, H, the parallels PF, QG, RH, cutting circle KL in O, S. [Margin: b — 7.huius (Sphaerica III.7)] Therefore arc PQ will be greater than arc QR; [Margin: c — 13.2.hui (Sphaerica II.13)] and since arcs GO, GS are equal to these, GO too will be greater than GS. Let GT be made equal to GS, and through T let the parallel VT be described, cutting circle MN in X. Now the common section of circles MN and VX — that is, the straight line drawn from section X to the other section — cuts off a segment which begins at X, passes through V, and reaches the other section, and this segment is less than a semicircle. [Margin: d — 19.2.hui (Sphaerica II.19)] (For circle MN, the greatest of the circles, cutting the parallel VX not through the poles, cuts off a segment greater than a semicircle — namely the one lying between the greatest of the parallels and the visible pole, such as the segment beginning at X, passing through α , and reaching the other section with circle MN.) It also cuts off from the greatest circle MN a segment greater than a semicircle — namely the one which, beginning at X and passing through N, reaches the other section; and segment XV is inclined, relative to segment XM, toward those parts. For if through N and Y, the pole of the parallels, a great circle YN be described, [Margin: e — 15.1 hui (Sphaerica I.15)] this will be perpendicular to BE. Therefore MN, which is set between these two circles,

[Margin: f — Scho.15.2. huius (Scholium to Sphaerica II.15)] (For since from point F two circles tangent to parallel AC can be drawn — one to the left of the great circle YN and the other to the right — we choose the former, so that it may be set between the great circles YN and BE.) is inclined toward the parts R relative to the same BE; and in turn BE, and hence also VX, which is parallel to it, will be inclined toward the same parts R relative to MN. Likewise the segment which begins at X, passes through V, and reaches the other section is cut unequally at T,

and TX is the lesser part, as we shall show shortly. [Margin: g — 2. huius (Sphaerica III.2)] Therefore the straight line TX is less than the straight line TF; [Margin: h — 3. huius (Sphaerica III.3)] but the straight line TF is equal to the straight line HS. Therefore the straight line TX will also be less than the straight line HS; and hence, as was demonstrated in the lemma to proposition 6 of this book, arc HS will be too great to be similar to arc TX. [Margin: i — 13.2. hui (Sphaerica II.13)] Since, then, arc IL is similar to arc HS, and arc LN to arc TX, arc IL too will be too great to be similar to arc LN; and hence, since they lie in the same circle, IL will be greater than LN. If, therefore, in a sphere a great circle touches some circle of the sphere, etc. Which was to be demonstrated.

Diagram. A large spherical figure for the demonstration of Theor. 8. The visible pole A is at the top, with the pole cluster crowded around it; a family of parallel circles runs across the sphere, cut obliquely by great circles. A separate small pair of concentric circles is drawn below the sphere (unlabeled). Because the woodcut is dense and the labels near the pole overlap, several small labels cannot be resolved individually; the figure

- A (top): visible pole of the parallels
- D, Y, Z (upper region, near the pole): points on the great circles near A [several adjacent tiny labels near the pole unresolved [?]]
- Q, P, V, B (left meridian, descending): points on the left-hand circles
- H, G, S (upper-middle): points where the oblique/parallel circles meet
- T, X, O, F (middle): X on circle MN; T, F on the parallels
- I, L, N, E (base): I, L, N on the greatest parallel; E on the oblique circle
- α (near X, lower-middle): the point through which the segment from X passes [Greek label, faint [?]]
- M, C (pole cluster): present but crowded [?]

LEMMA I.

That arc TX is less than half the segment which begins at X, passes through V, and extends to the other section, we shall demonstrate as follows. Through E let a great circle EZ be drawn, tangent to the parallel AC at the point Z, which is to be on the right-hand side of the great circle NY [Margin: k — Scho.15.2. huius (Scholium to Sphaerica II.15)] — since from E two circles tangent to AC can be described, one to the left of circle NY and the other to the right. And EZ will be a quadrant. For [Margin: l — 15.2. hui (Sphaerica II.15)] the great circle ZY passes through Y, the pole of circle AC, and through Z, [Margin: m — 9.2. hui (Sphaerica II.9)] the pole of the tangent circle EZ. Therefore this same circle YZ will bisect the

segments of the circles BE, EZ. [Margin: n — 11.1.hui (Sphaerica I.11)] Since, then, these great circles bisect one another, the segment from point E, through Z, as far as the other section, will be cut into two quadrants at point Z; and hence EZ will be a quadrant. In the same way ED will be a quadrant, if through the pole Y and the point of contact D a great circle YD be described. [Margin: o — Corol 16. 1.huius (Corollary to Sphaerica I.16)] Moreover, the arc of the great circle between E and Y, the pole, is a quadrant. Therefore the great circle described from E as pole, with radius EZ, will pass through the points Y and D. In no other way shall we show that NM is a quadrant, and hence that the great circle described from N as pole, with radius NM, passes through Y, the pole of the parallels — such as is the circle MY — and hence cuts arc BD beyond point D, and beyond the same arc DB: because the great circles ZYD and MY cut one another mutually at Y, the pole; and point M is beyond circle DYZ, as we shall show. And since the great circle MY, drawn through Y, [Margin: p — 5.2.hui (Sphaerica II.5)] the pole of parallel AC, and through the point of contact M, passes also through the pole of the tangent circle NM, it will pass [Margin: q — 9.2.hui (Sphaerica II.9)] through the poles of the circles XV and NM, which cut one another mutually at X. Therefore it will bisect their segments. Since, then, beyond point V it cuts the segment from X, through V, as far as another point where the circles XV, NM cut one another mutually — as has just been shown — arc XV will be less than half the segment from X, through V, to the other section; and hence TX will be much less than half that same segment. Which was the proposition. That the point of contact M lies beyond the great circle DYZ is demonstrated thus. Since the arc of the greatest of the parallels EB, between E and circle YD, [Margin: r — Coro. 16. 1.huius (Corollary to Sphaerica I.16)] is a quadrant, just as the arc of the same parallel between N and circle YM is a quadrant; and since point N is beyond E, toward B; circle YM too will be beyond YD; and hence M will lie beyond YD.

LEMMA II.

Given two unequal magnitudes, to find another, a mean, which is commensurable with any given magnitude whatsoever. [recte: PROPOSITIS; printed "PROROSITIS"]

Let there be set out two unequal magnitudes AB, AC, and some other given magnitude DG whatsoever; and let it be required to find another, a mean — that is, one which is greater than AC but less than AB, and commensurable with DG. Let DG first be less than BC, the excess between the magnitudes AB, AC; and let E be a multiple of DG that is the next multiple greater than AC. On this supposition, E will be less than AB. For if it were equal, then, if one magnitude equal to DG

(which is assumed to be less than BC) were subtracted from E, the remaining multiple of DG would still be greater than AC. Therefore E would not be the multiple of DG next greater than AC — which is absurd. Still less, then, will it be greater. Therefore it is less than AB; and hence, since it is also greater than AC, and commensurable with DG (being a multiple of it), the proposition is established.

But now let the given magnitude DG not be less than BC. Then let DG be bisected, and the half again bisected, and so on in succession, until there is left a part DF less than BC; let E be commensurable with DF, [Margin: a — 1. decim. (Elements X.1)] and hence also with DG, [Margin: b — 12. deci. (Elements X.12)] because both E and DG are commensurable with DF. Again, in the same way, as we demonstrated a little before, E will be less than AB. Since, then, it is also greater than AC, and commensurable with DG, the proposition is established.

Diagram. Three horizontal straight segments illustrating the commensurability lemma, each marked with subdivision ticks.

- Top segment: A — C — B (with tick marks between)
- Middle segment: D — G — E — (with equal tick marks, the multiple of DG)
- Bottom segment: D — I — G (with equal tick marks)

THEOR. 9. PROPOS. 9.

[Margin: viij.]

If the pole of the parallels lies on the circumference of a great circle which two other great circles cut at right angles — of which circles one is one of the parallels, while the other is oblique to the parallels — and if on this oblique circle equal arcs are taken which, though not continuous, nevertheless lie toward the same parts of that greatest parallel; and if through the pole and through each of the individual points bounding the equal arcs great circles are described: then these will intercept unequal arcs on the greatest parallel, of which the one nearer to the great circle first posited will always be greater than the more remote.

On the circumference of the great circle AB let A be the pole of the parallels, and let two great circles BC, DC cut it at right angles, of which BC is the greatest of the parallels and DC is oblique to the parallels; and on DC let there be taken equal, non-continuous arcs EF, GH. [Margin: a — 20.1.hui (Sphaerica I.20)] Through the points E, F, G, H and the pole A let the great circles AEI, AFK, AGL, AHM be described. I say that arc ML is greater than arc KI. For the intermediate arc FG is

either commensurable with both of the equal arcs EF, GH, or incommensurable. Let it first be commensurable. [Margin: b — 4.decim (Elements X.4)] Having found the greatest common measure X, let the three arcs EF, FG, GH be divided into parts equal to X, as appears in the first figure; [Margin: c — 20.1.hui (Sphaerica I.20)] and through the points of division and the pole A let great circles be drawn. Since, then, the arcs EQ, QF, FP, etc., are equal, [Margin: d — 6.huius (Sphaerica III.6)] arc MR will be greater than arc RL, and RL greater than LS, etc. Therefore, since MR is greater than KV and RL greater than VI, the whole ML will be greater than the whole KI. Which was the proposition.

But now let the intermediate arc FG be incommensurable with both of the equal arcs EF, GH. I say again that arc ML is greater than arc KI. For if it is not greater, it will be either less or equal. First, if it be possible, let ML be less than KI, as in the second figure; and from KI let KN be taken equal to ML, [Margin: e — 20.1.hui (Sphaerica I.20)] and through N and A let a great circle AON be described, cutting circle CD in O. Then by Lemma 2 of the preceding proposition let an arc FP be found, greater indeed than FO but less than FE, and commensurable with FG; [Margin: f — 20.1.hui (Sphaerica I.20)] let GQ be equal to FP (which is less than EF, and hence less even than GH...) equal; and through P, Q and A let the great circles APR, AQS be described. Since, then, the arcs PF, GQ are equal and non-continuous, and their intermediate arc FG is commensurable with them, arc SL will be greater than arc KR, as was demonstrated in the first figure. Therefore it will also be much greater than KN; and hence ML too will be much greater than KN. But KN was posited equal to ML. Which is absurd. Therefore ML is not less than KI.

Next, if it be possible, let arc ML be equal to arc KI, as in the third figure. The arcs EF, GH having been bisected at N, O, [Margin: g — 20.1.hui (Sphaerica I.20)] let the great circles ANP, AOQ be described through N, O and A. [Margin: h — 6.huius (Sphaerica III.6)] Then arc MQ will be greater than arc QL, and KP greater than PI. Therefore QL will be less than half of MLK, and KP greater than half of KI. Since, then, ML and KI are posited equal, QL will be less than KP — which is absurd. For since the arcs FN, GO, the halves of the equal arcs EF, GH, are equal and non-continuous, QL cannot be less than KP, as was just demonstrated in the second figure. Therefore arc ML is not equal to arc KI; but neither was it shown to be less. Therefore it is greater. If, then, the pole of the parallels lies on the circumference, etc. Which was to be demonstrated.

Diagram (first figure). A hemisphere for Theor. 9 : pole A at the top, D at the upper left, the oblique great circle DC running down to C at the lower right, the

greatest parallel BC along the base, and radiating great circles from A. The interior labels along the oblique circle are crowded and several cannot be resolved

- A (top): pole of the parallels
- D (upper left), C (lower right): ends of the oblique great circle DC
- B (lower left): end of the greatest parallel
- X (right, interior): the common measure marked in the figure
- Base rim (left→right): B, M, R, L, S, T, K, V, I, C
- Interior points along/near the oblique circle: N, G, P, F, E [with additional crowded division-points [?]]

Diagram (second figure). A hemisphere with pole A at top, D at upper left, oblique circle DC to C at lower right, B at lower left. Interior labels crowded

- A (top): pole
- D (upper left), C (lower right): ends of the oblique circle
- B (lower left): end of the greatest parallel
- Base rim (left→right): B, M, S, L, K, N, R, I
- Interior points: H, Q, G, F, O, E

Diagram (third figure). A hemisphere with pole A at top, D at upper left, C at lower right, E at the right, B at lower left. Interior labels crowded

- A (top): pole
- D (upper left), C (lower right), E (right): points on the oblique circle
- B (lower left): end of the greatest parallel
- Base rim (left→right): B, M, Q, L, K, P, I
- Interior points: H, O, G, F, N

SCHOLIVM.

Just as Theodosius in this proposition 9 demonstrated concerning non-continuous arcs the same thing that he had taught concerning continuous arcs in proposition 6, so in another version the same things are demonstrated in three theorems concerning non-continuous arcs which Theodosius demonstrated concerning continuous arcs in propositions 5, 7, and 8. The first theorem is of this kind.

I.

[Margin: vij.]

If the pole of the parallels lies on the circumference of a great circle which two other great circles cut at right angles — of which circles one is one of the parallels, while the other is oblique to the parallels — and if on this oblique circle equal arcs are taken which, though not continuous, nevertheless lie toward the same parts of that greatest parallel; and if through the individual points bounding the equal arcs parallel circles are described: then the arcs of that great circle first posited, intercepted between the parallels, will be unequal, and the one nearer to the greatest of the parallels will always be greater than the more remote.

On the circumference of the great circle AB let there be the pole of the parallels, which two other great circles BC, AC cut at right angles, and let BC be the greatest of the parallels and AC oblique to the parallels. Let equal, non-continuous arcs DE, FG be taken; and through D, E, F, G let the parallels DH, EI, FK, GL be drawn. I say that arc HI is greater than arc KL. For the intermediate arc EF is either commensurable with both of the equal arcs DE, FG, or incommensurable. Let it first be commensurable. [Margin: a — 4.decim (Elements X.4)] Having found the greatest measure V, let the three arcs DE, EF, FG be cut into parts equal to V, and through the points of division let parallels be described, as appears in the first figure. Since, then, the continuous arcs DP, PE, EO, etc., are equal, [Margin: b — 3.huius (Sphaerica III.3)] arc HT will be greater than arc TI, and TI greater than IS, etc. Therefore, since HT is greater than KQ and TI greater than QL, the whole HI will be greater than the whole KL. Which was the proposition.

But now let EF be incommensurable with both DE, FG. I say that arc HI is still greater than arc KL. For if it is not greater, it will be either less or equal. First let it be less; and from KL (as in the second figure) let there be cut off KM equal to HI; and through M let the parallel MN be drawn. Then by Lemma 2 of proposition 8 of this book let an arc FO be found, greater indeed than FN but less than FG, and commensurable with the intermediate arc EF; and let EP be equal to FO (which is less than FG, and hence less even than DE, equal to FG...); and through O, P let the parallels OR, PQ be described. Since, then, the non-continuous arcs PE, FO are equal, and their intermediate arc EF is commensurable with both, arc QI will be greater than arc KR, as has already been demonstrated in the first figure. Therefore it will be much greater than KM; and hence arc HI too will be much greater than KM. But HI is posited equal to KM. Which is absurd. Therefore HI is not less than KL.

Next, if it be possible, let arc HI be equal to arc KL, as in the third figure. The arcs DE, FG having been bisected at M, N, [Margin: c — 5.huius (Sphaerica III.5)] let the

parallels MO, NP be drawn through M, N. Then arc HO will be greater than OI, and KP greater than half of KL. Therefore OI will be less than half of HI, and KP greater than half of KL. Since, then, HI and KL are posited equal, OI will be less than KP — which is absurd. For since the arcs EM, FN, the halves of the equal arcs DE, FG, are equal and non-continuous, OI cannot be less than KP, as was demonstrated in the second figure. Therefore arc HI is not equal to arc KL; but neither was it shown to be less. Therefore it is greater. Which was the proposition.

Diagram (first figure). A sphere for Scholium Theorem I : pole A at the top, the oblique great circle AC running down through the interior to C at the base, a family of parallel circles crossing it, and the common measure V marked at the center. Interior and rim labels are dense; several cannot be resolved

- A (top): pole of the parallels
- C (base): lower end of the oblique circle AC
- V (center): the common measure
- Upper region: L, G
- Left rim (descending): K, R, S, I, T, H, B
- Interior along the oblique circle: M, E, N, O, P, D

Diagram (second figure). A sphere with pole A at top, oblique circle to C at base, parallel circles crossing. Dense labels

- A (top): pole
- C (base): lower end of the oblique circle
- Upper region: L, R, G
- Left rim (descending): M, K, I, Q, H, B
- Interior along the oblique circle: N, F, O, E, P

Diagram (third figure). A sphere with pole A at top, oblique circle to C at base, parallel circles crossing. Dense labels

- A (top): pole
- C (base): lower end of the oblique circle
- Upper region: L, P, G
- Left rim (descending): K, I, O, H, B
- Interior along the oblique circle: N, F, E, M, D

II.

[Margin: ix.]

If in a sphere a great circle touches some circle of the sphere, and another great circle is oblique to the parallels and touches circles greater than those which the great circle first posited touches, and their points of contact are on the great circle first posited; and if on the oblique circle equal arcs are taken which, though not continuous, nevertheless lie toward the same parts of the greatest of the parallels; and if through the points bounding the equal arcs parallel circles are described: then these will intercept unequal arcs on the great circle first posited, of which the one nearer to the greatest of the parallels will be greater than the more remote.

This theorem will be demonstrated from proposition 7 of this book, just as the preceding theorem was demonstrated from proposition 5 — provided that the two great circles AB, AC of the preceding theorem touch two parallels, as was stated in proposition 7 of this book. The rest of the construction of the figure does not differ from the construction of the preceding theorem, etc.

III.

[Margin: x.]

If in a sphere a great circle touches some circle of the sphere, and some other great circle, oblique to the parallels, touches circles greater than those which the great circle first posited touched, and their points of contact are on the great circle first posited; and if on the oblique circle equal arcs are taken which, though not continuous, nevertheless lie toward the same parts of the greatest of the parallels; and if through the points bounding the equal arcs [parallel circles are described] ...

(Continuing the Scholium to Theorem III / Prop. 9, whose enunciation broke off at the catchword "ferentias" on the preceding page.)

let great circles be described which also touch the same circle that the great circle first posited was touching, and which intercept similar arcs of the parallels, and which have those semicircles — the ones extending from the points of contact to the points that bound equal arcs of the oblique circle through which they are drawn — so disposed that they in no way coincide with that semicircle of the great circle first posited on which lies the contact of the oblique circle between the apparent pole and the greatest of the parallels. Then these great circles will

intercept unequal arcs of the greatest of the parallels, of which the arc nearer to the great circle first posited will always be greater than the more remote one.

This theorem too will be demonstrated from proposition 8 of this book, just as proposition 9 was shown from proposition 6 — provided that the great circles of proposition 9, proceeding from A, touch the same circle, smaller than the one which DC must touch, etc.

THEOR. 10. PROPOS. 10.

[Margin: xj.]

If the pole of parallel circles lies on the circumference of a great circle which two other great circles cut at right angles — one of these being one of the parallels, the other oblique to the parallels — and if on this oblique circle two arbitrary points are taken on the same side of that greatest parallel, and through the pole of the parallels and through each of those points great circles are drawn: then, as the arc of the greatest parallel intercepted between the great circle first posited and the nearest great circle drawn through the pole and through one of the points is to the arc of the oblique circle intercepted between the same circles, so is the arc of the greatest parallel intercepted between the two great circles drawn through the pole and through each of the points to some arc which is less than the arc of the oblique circle intercepted between the two points.

Let A be the pole of the parallels, on the circumference of the great circle AB, which two other great circles BD, CD cut at right angles; let BD be the greatest of the parallels, and CD oblique to the parallels. On CD take two arbitrary points E, F; [Margin a: "30. 1. huius" = *Sphaerica* I.30 — the leading numeral is blurred and reads like 30, but by context (drawing a great circle through a given point and the pole) this is almost certainly I.20, as in notes c, l, m [?]] and through the pole A and through E, F draw the great circles AEG, AFH. I say that, as arc BH is to arc CF, so is arc HG to an arc less than arc FE.

For the arcs CF, FE are either commensurable or incommensurable. First let them be commensurable, as in the first figure; [Margin b: "3. decim." = Euclid, *Elements* X.3] and, having found their greatest common measure P, divide the arcs CF, FE into arcs equal to that greatest common measure, [Margin c: "20. 1. huius" = *Sphaerica* I.20] and through the points of division and the pole A draw the great circles IM, KN, LO. Since, then, the successive arcs CL, LK, KF, FI, IE are equal, [Margin d: "6. huius" = *Sphaerica* III.6] arc BO will be greater than ON, and ON greater than NH, and so on. [Margin e: "8. quinti" = *Elements* V.8] Therefore the

proportion of BO to CL will be greater than that of ON to LK; and the proportion of ON to LK greater than that of NH to KF, and so on. Since, then, there are any number of magnitudes BO, ON, NH and equally many CL, LK, KF, and the proportion of the first, BO, to the first, CL, is greater than that of the second, ON, to the second, LK, and the proportion of the second, ON, to the second, LK, greater than that of the third, NH, to the third, KF, [Margin f: "34. quint." = *Elements* V.34 (Clavius's numbering)] the proportion of BH to CF will be greater than that of NH to KF. [Margin g: "8. quinti" = *Elements* V.8] But the proportion of NH to KF is still greater than that of HM to FI, as has been shown. Therefore the proportion of BH to CF is much greater than that of HM to FI. [Margin h: "34. quin." = *Elements* V.34] But the proportion of HM to FI is still greater than that of HG to FE, because the arcs HM, MG are equal in number to the arcs FI, IE, [Margin i: "18. quinti" = *Elements* V.18] and the proportion of the first, HM, to the first, FI, is greater than that of the second, MG, to the second, IE, as has been said. Therefore the proportion of BH to CF is much greater than that of HG to FE. Let it be that, as BH is to CF, so HG is to P. Then the proportion of HG to P will also be greater than that of HG to FE; [Margin k: "10. quin." = *Elements* V.10] and consequently P will be an arc less than arc FE. Therefore, as arc BH is to arc CF, so is arc HG to arc P, which is less than FE. Which was proposed.

Diagram — first figure (the commensurable case). A sphere seen in projection; the bounding great circle is AB, with A at the top (the pole of the parallels) and B at the lower left. The greatest of the parallels, BD, runs along the base to D at the lower right; the oblique circle CD runs from C (upper left) down to D. Great circles drawn from the pole A through the division points meet the base parallel.

- A (top): apparent pole of the parallels
- C (upper left): endpoint of the oblique circle on circle AB
- D (lower right): common intersection of BD and CD
- B (lower left): endpoint of the base parallel
- E, F (interior, on the oblique CD): the two given points
- I, K, L (interior, on the oblique between C and F): points of division of the equal arcs
- G, H, M, N, O (along the base parallel BD, between B and D): feet of the great circles AEG, AFH, and of IM, KN, LO
- P (interior, marked with a short bracket): the greatest common measure arc

Diagram — second figure (the incommensurable case). Same construction; here a great circle KL is drawn through K and the pole A.

- A (top): pole
- C (upper left): endpoint of the oblique circle
- F, E, K, I (interior): the given point F, the point E, and the auxiliary points K, I
- B, H, G, L, D (along the base): B and D the endpoints, with H, G, L between

Diagram — third figure. Same construction, with arc FE bisected at I and a great circle IK drawn through I and the pole A.

- A (top): pole
- C (upper left): endpoint of the oblique circle
- F, I, E (interior): the given point F, the bisection point I, and the point E
- B, H, K, G, D (along the base): B and D the endpoints, with H, K, G between

But now let the arcs CF, FE be incommensurable, as in the second figure. I still say that, as arc BH is to arc CF, so is arc HG to an arc less than FE. For if this were not so, then, as BH is to CF, so HG would be either to an arc greater than FE or to FE itself. First, if possible, let it be that, as BH is to CF, so HG is to an arc FI greater than FE. By lemma 2 of proposition 8 of this book, let there be found an arc FK, greater indeed than FE but less than FI, and commensurable with CF; [Margin l: "20. 1. huius" = *Sphaerica* I.20] and through K and the pole A let the great circle KL be drawn. Since, then, the arcs CF, FK are commensurable, it will be (as has already been demonstrated in the first figure) that, as BH is to CF, so HL is to an arc less than FK. But, as BH is to CF, so HG was posited to FI. Therefore it will also be that, as HG is to FI, so HL is to an arc less than FK; and, alternately, as HG is to HL, so FI is to an arc less than FK. But arc HG is less than arc HL. Therefore arc FI too will be less than an arc that is itself less than FK — the whole less than the part. Which is absurd. Therefore it is not the case that, as BH is to CF, so HG is to an arc greater than FE.

Next, if possible, let it be that, as BH is to CF, so HG is to FE, as in the third figure. Bisecting arc FE at I, [Margin m: "20. 1. hu." = *Sphaerica* I.20] let the great circle IK be drawn through I and the pole A. Since, then, the successive arcs FI, IE are equal, [Margin n: "6. huius" = *Sphaerica* III.6] HK will be greater than KG; and hence HK will be greater than half of HG [printed "HC"; recte HG]. [Margin o: "8. quinti" = *Elements* V.8] Therefore the proportion of HK to FI will be greater than that of half of HG to FI. [Margin p: "15. quint." = *Elements* V.15] But as half of arc HG is to FI — that is, to half of arc FE — so is the whole arc HG to the whole arc FE. Therefore the proportion of HK to FI will be greater than that of HG to FE. But it is posited that, as HG is to FE, so BH is to CF. Therefore the proportion of HK to

FI will also be greater than that of BH to CF; [Margin q: "10. quint." = *Elements* V.10] and consequently, as BH is to CF, so HK will be to an arc greater than FI. Which is absurd. For it was just demonstrated in the second figure that it cannot be that, as arc BH is to CF, so is HK to an arc greater than FI. Therefore it is not the case that, as BH is to CF, so HG is to FE; nor yet, as BH is to CF, is HG to an arc greater than FE, as has been demonstrated. Therefore it will be that, as BH is to CF, so HG is to an arc less than FE. Therefore, if the pole of the parallels lies on the circumference [of a great circle], etc. Which was to be demonstrated.

COROLLARIUM.

Hence it follows that the proportion of arc BH to arc CF is greater than the proportion of arc HG to arc FE. [Margin r: "10. huius" = *Sphaerica* III.10] For since, as BH is to CF, so HG is to an arc less than FE, [Margin s: "8. quinti" = *Elements* V.8] and the proportion of arc HG to an arc less than FE is greater than its proportion to FE itself, the proportion of BH to CF will also be greater than that of HG to FE.

THEOR. 11. PROPOS. 11.

[Margin: xij.]

If the pole of parallel circles lies on the circumference of a great circle which two other great circles cut at right angles — one being one of the parallels, the other oblique to the parallels — and if another great circle, passing through the poles of the parallels, cuts the oblique circle between the greatest of the parallels and the circle which the oblique circle touches: then the diameter of the sphere has to the diameter of that circle which the oblique circle touches a greater ratio than the arc of the greatest parallel intercepted between the great circle first posited and the great circle passing through the poles of the parallels has to the arc of the oblique circle intercepted between the same circles.

On the circumference of the great circle AB let A be the pole of the parallels, and let two other great circles BC, DE cut it at right angles, of which BC is the greatest of the parallels and DE is oblique to the parallels, touching the parallel DF. Through the pole A let another great circle AE be drawn, cutting the oblique DE at the point E, which is set between the greatest parallel BC and the parallel DF that the oblique touches. I say that the diameter of the sphere has to the diameter of the parallel DF a greater ratio than the circumference BC has to the circumference DE.

Let AG be the common section of the circles AB, AE; and BG the common section of the circles AB, BC. Then AG, BG will be the semidiameters of those circles [Margin a: "11. 1. huius" = *Sphaerica* I.11] (since great circles bisect one another in a sphere), and hence also of the sphere, intersecting one another at G, the center of the sphere and of the great circles. Let DL also be the common section of the circles AB, DE, which too will be a diameter of the sphere passing through the center G. Again, let DM be the common section of the circles AB, DF; then DM will be the diameter of the circle DF, [Margin b: "15. 1. huius" = *Sphaerica* I.15] because the circle AB cuts the parallel DF in two through the poles. Likewise let FN, CG be the common sections of the circles DF, BC with the circle AE. From the pole A, with radius AE, let the parallel OE be described, and let OH, EH be its common sections with the circles AB, AE. Then FN, EH, CG will be the semidiameters of the circles DF, OE, BC, [Margin c: "15. 1. huius" = *Sphaerica* I.15] because the great circle AE bisects them through the poles; and hence the common sections, being diameters, will meet the diameters DM, OH, BG at the remaining points N, H, G. For OH too is a diameter of the circle OE, [Margin d: "15. 1. huius" = *Sphaerica* I.15] since the circle AB bisects it through the pole A. Again, let EG be the common section of the great circles AE, DE, which will also be a diameter passing through G, the center of the sphere. Finally, let EI be the common section of the circles DE, OE. [Margin e: "10. 1. huius" = *Sphaerica* I.10] And since the line AG, drawn through the poles of the parallel OE, is perpendicular to the plane of the parallel and falls on its center H, the angle GHI [printed "OHG"; recte the right angle at H in triangle GHI (?)] will be right, by Euclid, *Elements* XI, definition 3, in the triangle GHI; and hence the angle HGI is acute. [Margin f: "19. primi" = *Elements* I.19] Therefore the side GI will be greater than the side HI. Cut off the line IK equal to the line IH, and join the line EK. Again, since each of the circles DE, OE is perpendicular to the circle AB, [Margin g: "19. unde." = *Elements* XI.19] their common section EI too will be perpendicular to that same circle; and consequently, by *Elements* XI, definition 3, each of the angles EIH, EIK is right. Since, then, the two sides EI, IH of the triangle EIH are equal to the two sides EI, IK of the triangle EIK, and contain equal angles — namely right angles, as we have shown — [Margin h: "4. primi" = *Elements* I.4] the angles IHE, IKE will also be equal.

Now, since the proportion of the line GI to the line IK — that is, to the line IH equal to it — is greater than the proportion of the angle OHE (and of the angle equal to it) to the angle IGE, as we shall soon demonstrate; [Margin i: "10. undec." = *Elements* XI.10] and since the angle OHE is equal to the angle BGC [Margin k: "15. unde." = *Elements* XI.15] (for the lines OH, BG are the common sections of the parallel planes OE, BC made by the plane AB, and are therefore parallel; and

likewise the lines EH, CG are the common sections of the same planes made by the plane AE) — the proportion of the line GI to the line IK, that is, to the equal line IH, will also be greater than the proportion of the angle BGC to the angle DGE. [Margin l: "33. sexti" = *Elements* VI.33] But as the angle BGC is to the angle DGE, so is the arc BC to the arc DE. Therefore the proportion of GI to IH will also be greater than the proportion of arc BC to arc DE. [Margin m: "4. sexti" = *Elements* VI.4] But as GI is to IH, so is GD to DN, that is [Margin n: "15. quint." = *Elements* V.15], so is the whole diameter DL to the whole diameter DM [Margin o: "16. unde." = *Elements* XI.16] (for DN, OH are the common sections of the parallel planes DF, OE made by the plane AB, and are therefore parallel). Therefore the proportion of DL, the diameter of the sphere, to DM, the diameter of the parallel DF, will also be greater than the proportion of arc BC to arc DE. Therefore, if the pole of the parallels lies on the circumference of a great circle, etc. Which was to be demonstrated.

Diagram. A sphere in projection. The bounding great circle is AB (A at the top, B at the left). A dense network of chords and the diameters DL, DM, BG, OH, EG, EI is drawn; the oblique circle DE, the parallels BC and DF, and the described parallel OE all appear.

- A (top): pole of the parallels
- D (upper left): on the oblique circle
- M (upper right): endpoint of the diameter DM of the parallel DF
- N (below A, center): intersection point on the diameter DM
- O (left): endpoint on the described parallel OE
- I, F (interior, upper center): I on the common section EI; F on the parallel DF
- H (interior, center-right): center of the parallel OE
- E (interior, center-left): the cutting point of AE and DE
- K (interior, center): the point with $IK = IH$
- G (interior, center): center of the sphere
- B (left): endpoint of the greatest parallel BC
- C (lower center): on the greatest parallel BC
- L (lower right): endpoint of the diameter DL

LEMMA.

That the proportion of the line GI to the line IK is greater than the proportion of the angle IKE to the angle IGE, we shall demonstrate by the following theorem, here set out.

In every right triangle, if from one of the acute angles a straight line is drawn in any way whatever to the opposite side, the proportion of that side to the segment of it which lies next to the right angle is greater than the proportion of the acute angle which the drawn line makes with the said side to the remaining acute angle of the triangle.

Let EGI be a right triangle having the right angle I, and from the acute angle IGE let a straight line EK be drawn in any way to the opposite side GI. I say that the proportion of the line GI to IK is greater than the proportion of the acute angle IKE to the acute angle IGE. [Margin p: "31. primi" = *Elements* I.31] For through G draw the line GA parallel to EK, meeting the line IE (produced) at A. And since the angle I is right, the angle IEG will be acute, and therefore AEG obtuse. [Margin q: "19. prim" = *Elements* I.19] Therefore the side EG, in the triangle GEI, is greater than the side GI; but in the triangle AEG it is less than the side AG.

(Continuation of the Lemma demonstration.)

Therefore the arc of a circle described from the center G at the distance GE will cut the line GI, produced beyond I, namely at B, and the line GA, on the near side of A, namely at C. Since, then, the triangle GAE is greater than the sector GCE, [Margin r: "8. quinti" = *Elements* V.8] the proportion of the triangle GAE to the triangle GEI will be greater than that of the sector GCE to the triangle GEI. [Margin s: "8. quinti" = *Elements* V.8] But the proportion of the sector GCE to the triangle GEI is greater than its proportion to the sector GEB, because the triangle GEI is less than the sector GEB. Therefore the proportion of the triangle GAE to the triangle GEI is much greater than that of the sector GCE to the sector GEB. [Margin t: "28. quint." = *Elements* V.28 (Clavius's numbering)] And consequently, by composition, the proportion of the triangle GAI to the triangle GEI will be greater than that of the sector GCB to the sector GEB. [Margin u: "1. sexti" = *Elements* VI.1] But as the triangle GAI is to the triangle GEI, so is the line AI to the line IE; [Margin x: "Cor. 1. 33. sexti" = Corollary 1 to *Elements* VI.33] and as the sector GCB is to the sector GEB, so is the angle BGC to the angle BGE. Therefore the proportion of AI to IE will also be greater than that of the angle BGA — that is [Margin y: "29. prim." = *Elements* I.29], than that of the angle IKE equal to it — to the angle IGE. [Margin z: "2. vel 4. sexti" = *Elements* VI.2 or VI.4] But as AI is to IE, so is GI to IK. Therefore the proportion of the line GI to the line IK will also be greater than that of the angle IKE to the angle IGE. Which was proposed.

Diagram — figure for the Lemma. A right triangle with the right angle at I. From the acute angle at G a line EK is drawn to the opposite side; GA is drawn

parallel to EK, and a circular arc from center G at radius GE cuts GI (produced) at B and GA at C.

- A (upper left): where GA meets IE produced
- E (top center): the acute-angle vertex from which EK is drawn
- B (top right): where the arc cuts the line GI produced
- C (left, below A): where the arc cuts GA
- I (right, below B): the right-angle vertex
- K (center-right): the foot of EK on GI
- G (bottom): the acute-angle vertex from which GA is drawn

SCHOLIVM.

In another version, the following theorem is added at this place.

With the same things posited, the diameter of the sphere has to the diameter of the parallel described through that point of the oblique circle at which the great circle passes from the pole a smaller ratio than the arc of the greatest of the parallels intercepted between the great circle first posited and the great circle passing through the poles of the parallels has to the arc of the oblique circle intercepted between the same circles.

Let the circles be described as in the preceding proposition. I say that the proportion of the diameter of the sphere to the diameter of the parallel GE is smaller than that of the circumference BC to the circumference DE. Let GH, BI be the common sections of the circles GE, BC with the circle AB, which will be their diameters, [Margin a: "15. 1. hui." = *Sphaerica* I.15] since AB, drawn through their poles, cuts them in two and at right angles. Therefore BI will also be a diameter of the sphere. And since the circle DE is posited perpendicular to AB, [Margin b: "13. 1. hui." = *Sphaerica* I.13] DE will pass through the poles of AB itself. In the same way BC will pass through the poles of the same AB, since it is posited perpendicular to it. Therefore M, the point where they mutually intersect, will be the pole of the circle AB; and consequently the segment DEL, which is perpendicular to the circle AB, will be divided unequally at the point E, where the circles DE, GE intersect one another, and ED will be the smaller part, [Margin c: "28. tertij" = *Elements* III.28] seeing that the arcs MD, ML are equal, because the lines subtending them are, by the definition of a pole, equal. [Margin d: "Sch. 21. 2. huius" = the Scholium to *Sphaerica* II.21] Therefore the line ED, when drawn, will be smaller than the line EG; and hence, since the circle GE is smaller than the circle DE, the circumference EG will be greater than the circumference DE. [Margin e: "lemma 6. huius" = Lemma 6 of this book] For if a line equal to the line

ED cuts off from the circle GE a greater arc than the line DE cuts off from the circle DE, much more will the line EG, which is greater than the line ED (as we have shown), cut off a greater arc, and so on. [Margin f: "8. quinti" = *Elements* V.8] Therefore the proportion of arc BC to arc GE will be smaller than its proportion to arc DE.

Now, since, as arc BC is to the whole circumference of the circle BC, so is arc GE to the whole circumference of the circle GE — on account of the similarity of the arcs BC, GE (for the similarity of arcs consists in just this: that they have the same proportion to the whole circumferences of their circles, as we have set out in the scholium to proposition 33 of *Elements* Book VI) — and hence, alternately, as arc BC is to arc GE, so is the whole circumference of the circle BC to the whole circumference of the circle GE: the proportion of the circumference of the circle BC to arc DE will also be smaller. And as the circumference of the circle BC is to the circumference of the circle GE, so is the diameter BI (which is also a diameter of the sphere) to the diameter GH, as Pappus demonstrated and as we too have shown in our little book on Archimedes' *Measurement of the Circle*. Therefore the proportion of BI, the diameter of the sphere, to GH, the diameter of the parallel GE, will also be smaller than the proportion of arc BC to the circumference DE. Which was proposed.

Diagram. A sphere in projection with bounding great circle. A network of parallels and the diameters GH, BI, DEL is drawn; the oblique circle DE and the parallels GE, DF appear, with M the common pole and E, K, N interior intersection points.

- A (top): pole of the parallels
- D (upper left): on the oblique circle
- O (upper center, below A): interior intersection point
- F (upper right): on the parallel DF
- G (left): endpoint of the diameter GH
- H (right): endpoint of the diameter GH
- N (center): interior intersection point
- E (center-left): where the circles DE, GE intersect
- K (center): interior point
- B (left): endpoint of the diameter BI
- I (right): endpoint of the diameter BI
- C (lower center-left): on the greatest parallel
- M (lower center): the common pole of the circle AB

- L (bottom): endpoint of the segment DEL

COROLLARIVM.

Hence it follows, the same things being posited, that the ratio of the circumference BC of the greatest of the parallels — intercepted between the great circle AB first posited and the great circle AC passing through the poles of the parallels — to the circumference DE of the oblique circle intercepted between the same circles, is greater than the ratio of the circumference AE of the great circle passing through the poles of the parallels; but smaller than the ratio of the circumference AD of the great circle first posited, intercepted between the poles of the parallels and the oblique circle. For since it has been shown by this theorem that the ratio of the [diameter of the] sphere to the diameter of the parallel GE is greater [than that of arc BC to arc DE], [Margin g: "15. quint." = *Elements* V.15] and since, as the diameter BI of the sphere is to GH, the diameter of the circle GE, so is BK, the semidiameter — that is, the whole sine [sinus totus] — to GN, the semidiameter — that is, to the sine of the arc AE [Margin h: "10. 2. hui." = *Sphaerica* II.10] (for since the arcs AG, AE are equal, and GN is the sine of the arc AG, GN will also be the sine of the arc AE): therefore the ratio of arc BC to arc DE will also be greater than the ratio of the whole sine BK to GN, the sine of the arc AE.

[Margin i: "11. huius" = *Sphaerica* III.11] Again, since it has been shown that the ratio of arc BC to arc GE is smaller than the ratio [of the diameter of the sphere to the diameter] of the parallel DF; [Margin k: "15. quin." = *Elements* V.15] and since, as the diameter of the sphere BI is to DF, the diameter of the parallel DF, so is BK, the whole sine, to DO, the sine of the arc AD: therefore the proportion of arc BC to arc DE is also smaller than the proportion of the whole sine to the sine of the arc AD. Which was proposed. Moreover, what a sine is will be understood from the following treatise.

THEOR. 12. PROPOS. 12.

[Margin: xiv.]

If in a sphere great circles touch one and the same of the parallels, and intercept similar arcs of the parallels lying between each pair of the great circles; and if another great circle, oblique to the parallels, touches circles greater than those which the great circles first posited touch, and this same oblique circle cuts those same great circles first posited at points lying between the greatest of the parallels and the circle which the great circles first posited touch: then the diameter of the sphere has to

the diameter of the circle which the oblique circle touches a greater ratio than the arc of the greatest parallel intercepted between the circles first posited (which touch the same circle) has to the arc of the oblique circle intercepted between the same circles.

In a sphere let two great circles AB, CD touch the same parallel AC, and intercept similar arcs of the parallels lying between them; and let another great circle EF touch the parallel EG, greater...

(The demonstration of Prop. 12 breaks off here and continues on the following page.)

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Diagram. A sphere shown in oblique projection, with the pole at the top and a system of parallel circles crossed by oblique great circles. The demonstration of THEOR. 12 refers to it. Labels read off the figure:

- L (top): pole of the parallels
- E, A, C (upper region, left to right): points on the upper parallels / great circles
- G (upper right): endpoint of the diameter EG
- H (left margin of sphere): point on the greatest parallel
- I (small point, near center): intersection point
- O, P (middle): O on parallel KO, P where KO cuts circle AB
- K (middle right): point on the oblique circle
- M, B, N, D, F (bottom row, left to right): points on the lower parallels and circles

it [the oblique circle] meets the parallel AC at E, and let it be oblique to the parallels, and let it cut the two former circles AB and CD — between HF, the greatest of the parallels, and the parallel AC — at the points I and K. I say that the ratio of the diameter of the sphere to the diameter of the parallel EG is greater than that of the circumference BD to the circumference IK. [Margin a: Sphaerica I.20] For through L, the pole of the parallels, and through the points E, I, K, let the great circles LH, LM, LN be described; and through K let the parallel KO be drawn, cutting circle AB at P. [Margin b: Sphaerica III.11] Now since the ratio of the diameter of the sphere to the diameter of circle EG is greater than that of arc HM to arc EI, and the ratio of arc HM to arc EI [Margin c: Corollary to III.10] is greater than that of arc MN to arc IK, the ratio of the diameter of the sphere to the diameter of circle EG will also be greater than that of arc MN to arc IK. And since

arc PK is similar to arc BD by hypothesis, [Margin d: Sphaerica II.10] and arc OK is similar to arc MN, and arc PK is smaller than arc OK, arc BD will also be smaller than arc MN; [Margin e: Elements V.8] and therefore the ratio of arc BD to arc IK will be smaller than that of arc MN to the same arc IK. Since, then, it has been shown that the ratio of the diameter of the sphere to the diameter of circle EG is greater than that of arc MN to arc IK, the ratio of the diameter of the sphere to the diameter of circle EG will be much greater than that of arc BD to arc IK. If, therefore, in a sphere great circles touch one [and the same of the parallels], etc. Which was to be demonstrated.

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In the Greek exemplar it is stated that the ratio of twice the diameter of the sphere to the diameter of circle EG is greater than that of arc BD to arc IK. This is indeed clearly established from our demonstration. For since the diameter of the sphere has a greater ratio to the diameter of circle EG than arc BD has to arc IK, twice the diameter of the sphere will have a much greater ratio to the diameter of circle EG than arc BD to arc IK; [Margin f: Elements V.8] because twice the diameter of the sphere has a greater ratio to the diameter of circle EG than the diameter of the sphere has to the same diameter of circle EG.

THEOR. 13. PROPOS. 13.

[Margin: proposition xv]

If in a sphere parallel circles intercept equal circumferences of some great circle on both sides from that point at which the great circle itself cuts the greatest of the parallels; and if through the points that bound the equal circumferences, and through the poles of the parallels, great circles are described — or if great circles are described which touch one and the same of the parallels — they will intercept equal circumferences on the greatest of the parallels.

Diagram, first figure. A sphere with parallel circles and great circles drawn through the poles of the parallels. Labels read off the figure:

- A (top): pole
- C, K, D (upper region, left to right): points on the upper parallel and great circles
- B (left margin): endpoint on the greatest parallel
- H, G, I (middle row, left to right): G the point where circle AF cuts the greatest parallel; H, I the cutting points on the greatest parallel

- E, L, F (bottom row, left to right): points on the lower parallel and great circles

Diagram, second figure. A companion sphere, for the alternative case in which the great circles touch one and the same parallel (shown as a small circle near the top). Labels read off the figure:

- A (top): pole, with the small touched parallel just below it
- C, K, D (upper region, left to right)
- B (left margin), H, G, I (middle row, left to right)
- E, L, F (bottom row, left to right)

In sphere AB let the parallel circles CD, EF cut off from the great circle AF two equal circumferences GC, GF, on both sides from the point G at which circle AF cuts BG, the greatest of the parallels; and through the points C, G, F let great circles be drawn — either through the poles of the parallels, as in the first figure, or touching one and the same parallel, as in the second figure — cutting the greatest of the parallels at H and I. I say that arcs GH, GI are equal. For since arcs GC, GF are supposed equal, [Margin a: Sphaerica II.17] the parallels CD, EF will be equal. [Margin b: Sphaerica II.18] Therefore arcs GK, GL will also be equal. [Margin c: Sphaerica III.3] Hence the straight lines drawn CK, FL will be equal; [Margin d: Elements III.28] and consequently, in the equal circles CD, EF, [the equal straight lines] will cut off equal arcs CK, FL; and for that reason arcs CK, FL will be similar to each other. [Margin e: Sphaerica II.10 or II.13] But arc GH is similar to arc CK, and arc GI to arc FL. Therefore arcs GH, GI will be similar to each other, and consequently — since they belong to the same circle — equal to each other. If, therefore, in a sphere a great circle, etc. Which was to be demonstrated.

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From this it is also clear that, with the same things assumed, all the arcs of the great circles intercepted between the parallels are equal to each other — namely CH, HE, KG, GL, DI, IF. For since arcs GC, GH are equal to arcs GF, GI, [Margin f: Sphaerica III.3] the straight lines CH, FI will also be equal; [Margin g: Elements III.28] and therefore arcs CH, FI will also be equal. [Margin h: Sphaerica II.10 or II.13] But to arc CH the arcs KG, DI are equal, and to arc FI the arcs LG, EH are equal. Therefore all those six arcs will be equal.

THEOR. 14. PROPOS. 14.

[Margin: proposition xvj]

If in a sphere a great circle touches some circle, and another great circle, oblique to the parallels, touches circles greater than those which the first-placed great circle touched: they will intercept unequal circumferences of the parallel circles, of which those nearer to either of the poles will be too great to be similar to the more remote ones.

Diagram. A sphere with an oblique great circle, showing a small parallel circle touched near the top and two larger parallels cut by the oblique circle. Labels read off the figure:

- D, A (top): D on the oblique circle, A at the apex; the small touched circle sits just below A
- Inside the small upper circle: C[?], L, F (leftmost letter ambiguous between C and G)
- G (left), H (right): points on the greatest parallel
- B (lower left), I (lower right): endpoints of the parallel BI
- K, M (center): cutting points on the parallels
- N, E (bottom): points on the oblique circle and lower parallel

In a sphere let the great circle AB touch circle AC; and let another great circle DE touch another, greater circle DF, and let it cut any two parallels GH, BI at K and E. I say that arcs KH, EI are unequal, and that KH — nearer the visible pole — is too great to be similar to arc EI, the more remote one; or, alternatively, that EB itself, nearer the hidden pole, is too great to be similar to arc KG, the more remote one.

[Margin a: Sphaerica II.15] For through the points E, K let the great circles LE, CN be described, touching circle AC, in such a way that the semicircles proceeding from C through N and from A through B do not meet; likewise the semicircles from L through E and from A through I, tending toward each other, do not meet.

[Margin b: Sphaerica II.13] Therefore arcs MH, EI will be similar. Hence KH is too great to be similar to arc EI. In the same way, since arcs BN, GK are similar, BE — nearer the other pole — will be too great to be similar to arc GK, which is more remote from the same pole. And so, if in a sphere a great circle touches some circle, etc. Which was to be demonstrated.

FINIS LIBRI III. THEODOSII. — *End of the Third Book of Theodosius.*

CHRISTOPHER CLAVIUS of Bamberg, of the Society of Jesus

Sines, or the Halves of the Straight Lines Subtended in a Circle: the Lines Tangent and Secant

[*Verbatim title as printed: CHRISTOPHORI CLAVII BAMBERGENSIS E SOCIETATE
IESV — SINVS VEL SEMISSES RECTARVM IN CIRCULO SVBTENSARVM: LINEÆ
TANGENTES ATQVE SECANTES.*]

PRÆFATIO — Preface

[Margin: *The usefulness of sines* (Sinuum vtilitas)]

It can scarcely be said how great a usefulness the knowledge of sines has in matters both astronomical and geometrical — since almost innumerable astronomical and geometrical problems are brought back, for practical use, to the calculation and reckoning of sines, as is clear both from our treatment of rectilinear and spherical triangles, and from Ptolemy's *Almagest*, from our *Gnomonica*, and from the various books of many astronomers. Therefore, since the demonstrations of sines have been set out by only very few, I judge it will be worth my while if, with all the brevity and clarity I can, I gather from various authors — especially from Ptolemy [Ptolemæus], Peurbach [Purbachio, Georg Peurbach], and Johannes Regiomontanus [Regiomontanus] — the demonstrations by which we may have the sines and chords of all arcs known; so that we may both examine the tables of sines and of chords already computed by many writers (for an error is easily committed in the printing of numbers), and, when circumstances require, construct new ones of our own (once the whole sine, or the diameter, has been posited as consisting of any number of parts you please).

But since more recent writers have, with the greatest success, collected from the sines still other lines — namely tangents and secants — so as to demonstrate certain things more easily and briefly, we shall treat of these lines too. For these lines have an outstanding use in astronomical and geometrical matters, as will become clear from our treatment of plane and spherical triangles. We shall, however, begin from the definitions.

SINES

DEFINITIONES [Definitions]

I.

The **complement** of any arc is the excess by which the quadrant exceeds it, if the arc is smaller than a quadrant; or by which it is exceeded by the arc, if the arc is larger than a quadrant.

[Margin: What the complement of an arc is.]

II.

The **chord** (which some call the *subtense*, others the *inscribed line*) is a straight line subtending any arc whatever in a circle.

[Margin: What a chord is.]

III.

The **right sine** (which by some is called the *first right sine*) is half of the chord that subtends double the arc of which it is called the right sine.

[Margin: What the right sine is.]

Or otherwise.

The right sine is a perpendicular line falling from one end of the arc — of which it is called the right sine — onto the diameter of the circle drawn from the other end of the same arc.

IV.

The **versed sine** (which some call the *arrow* [*sagitta*]) is the part of the circle's diameter intercepted between the end of the given arc — of which it is called the versed sine — and the right sine of the same arc.

[Margin: What the versed sine is.]

V.

The **sine of the complement** of any arc (which by some is called the *second right sine*) is the right sine of the other arc, which is the complement of that arc of which it is called the sine of the complement.

[Margin: What the sine of the complement is.]

VI.

The **whole sine** [*sinus totus*] is the radius, or semidiameter of the circle; that is, the right sine (or the versed sine) of the quadrant of the circle.

[Margin: What the whole sine is.]

VII.

The **sine** — whether right, versed, or of the complement — of any rectilinear angle is the sine of that arc which, in a circle described from the angle, is intercepted between the two straight lines that constitute the angle.

[Margin: What the sine of a rectilinear angle is.]

Let a circle ABCD be set out, whose centre is E, through which let two diameters AC, BD be drawn, cutting one another at right angles [Margin: a — Elements III.26] and dividing the whole circle into four equal quadrants — inasmuch as they are subtended by equal right angles at the centre. Let equal arcs AF, AG be taken, and likewise BF, BI; and let the straight lines FG, FI be drawn, cutting the diameters at H and K. The arc FB, then, is called the complement of arc FA, because the quadrant AB exceeds arc FA by the arc FB. By the same reasoning, arc FA is named the complement of arc FB. Likewise arc BI is called the complement of arc AI, because the quadrant AB is exceeded by the arc AI by the amount of the arc BI.

[Margin: Examples of the definitions.]

The straight line FG, next, is called the chord of arc FAG, and the straight line FI the chord of arc FBI. And since the diameter AC, cutting the arc FAG in half, also cuts the straight line FG in half — as is established from corollary 1 of proposition 10 of Euclid, Book 13 (which, however, we shall show more briefly in the following lemma) — the straight line FH will be the right sine of arc FA, according to the former definition of the right sine: for it is half of the chord FG, which subtends the arc FAG, double the arc FA, of which FH is called the sine. And so, if we bisect any arc and its chord, half the chord will be called the right sine of the halved arc. Hence it has come about that most people call the right sines the *halves* [*semisses*] of the straight lines subtended in the circle. [Margin: Why the right sines are called the halves of the straight lines subtended in the circle.] The same straight line FH will also be the right sine of the same arc FA according to the latter definition of the right sine: since it is a perpendicular drawn from F, the end of the said arc, to the diameter AC drawn from the other end A of the same

arc; because the straight line EA, cutting the straight line FG in half, cuts it at right angles. [Margin: b — Elements III.3.]

The straight line AH, however, is the versed sine of the same arc FA, since it is the part of the diameter AC intercepted between A, the end of the said arc, and its right sine FH. This sine is called "versed" because it is placed in a reversed manner when compared with the right sine. [Margin: Why the versed sine is called the arrow.] Some call it the *arrow* [*sagitta*], because it is like an arrow shot from the chord FG in the arc FAG.

Furthermore, the straight line FK is the sine of the complement of arc FA, because it is the right sine of arc FB, which is the complement of arc FA.

And the straight line KB is the versed sine of the complement of arc FA — that is, the versed sine of arc FB, which is the complement of arc FA. [Margin: What the versed sine of the complement of an arc is.]

Diagram. A circle with two perpendicular diameters. The horizontal diameter runs B (left) — E (centre) — D (right); the vertical diameter runs A (top) — E (centre) — C (bottom). Two symmetric points F and G sit on the upper arc, on either side of A, with the chord FG drawn across (crossing the vertical diameter at H, just below A). Further out on the upper arc lie Q (upper left, beyond F) and M (upper right, beyond G). The point I lies on the lower-left arc. Radii/straight lines run from the centre E out to F, G, Q, M, and I. Perpendiculars are dropped from the upper points to the horizontal diameter, meeting it at the labelled feet; from the left group the feet R and K lie between B and E, and from the right group the feet L and O lie between E and D. Interior points P (near F) and N (near G) mark auxiliary constructions used later in the demonstrations. This same figure is reprinted at the head of the two following pages for continued reference.

- A (top of vertical diameter): endpoint of arcs AF, AG
- B (left end of horizontal diameter): endpoint of arcs BF, BI
- C (bottom of vertical diameter): opposite A
- D (right end of horizontal diameter): opposite B
- E (centre): centre of the circle
- F (upper arc, left of A): common endpoint of arcs FA, FB, FC
- G (upper arc, right of A): endpoint of arc AG (= AF)
- Q (upper-left arc, beyond F): auxiliary point for the greater/smaller-arc proof
- M (upper-right arc, beyond G): endpoint of the smaller arc DM in the later demonstration

- I (lower-left arc): endpoint of arc BI
- H (on vertical diameter AC, below A): foot of the right sine FH; intersection of chord FG with AC
- K (on horizontal diameter, between B and E): foot of the right sine FK
- R (on horizontal diameter, left of K): foot of perpendicular QR
- L (on horizontal diameter, between E and D): foot of the right sine GL
- O (on horizontal diameter, right of L): foot of perpendicular MO
- P (interior, near F): auxiliary point (KP cut off equal to MO)
- N (interior, near G): foot of perpendicular MN dropped to GL

(The construction figure printed at the head of this page is identical to the one above.)

Some astronomers, moreover, call the sine which we have called the *right* sine the "first right sine," and the sine which we have called the *sine of the complement* they call the "second right sine." Thus they call the straight line FH the first right sine of arc FA, and the straight line FK the second right sine of the same arc.

[Margin: The first and second right sine; likewise what the "first sine" and "second sine" are, according to some.] There are also others who call the right sine simply the "first sine," and the versed sine the "second sine." I mention this so that you may understand the authors who have spoken variously about sines. We ourselves have kept the common way of speaking. But whenever writers speak of some sine, they always mean the right sine — unless they call it the sine of the complement, or the versed sine. [Margin: When some sine is mentioned without qualification, the right sine is understood.]

The semidiameter AE, next, is both the right sine and the versed sine of the quadrant AB; it is called the *whole* (or *maximum*) sine, because it is the greatest of all sines, both right sines and sines of complements — indeed, even greater than all the versed sines of those arcs that are smaller than a quadrant. It is smaller only than the versed sines of those arcs that are greater than a quadrant (as we shall say below), which in fact come into use more rarely than the others.

[Margin: Why the whole or maximum sine is so called.] Or it is certainly called the whole (or maximum) sine because, in the table of sines — in which only the right sines are set down — the greatest of all the sines is the one that corresponds to the quadrant, or to 90 degrees, as will be plain from the table of sines which we shall set down below.

Finally, when the straight line EF has been drawn, the straight line FH will be the right sine of angle FEH; the straight line FK, the sine of the complement of the

same angle; and the straight line AH, the versed sine of the same angle. For the straight line FH is the right sine of arc FA, which — in the circle described from the angle FEH — is intercepted between the straight lines EF, EA that constitute the said angle; the straight line FK is the sine of the complement of the same arc; and the straight line AH is its versed sine.

For the rest, two arcs constituting a semicircle have exactly the same sine, both right and of the complement — just as two arcs making up a whole circle have one and the same chord. Their versed sines, however, differ, and together make up the whole diameter of the circle. [Margin: Two arcs making up a semicircle have the same sine, just as two arcs making up a circle have the same chord; yet their versed sines differ, and together make up the whole diameter.] Thus the two arcs FA, FC, which make up the semicircle ABC, have the same right sine FH — just as the two arcs FAG, FCG, their doubles, which make up the circle, have the same chord FG, of which the half is the right sine FH, as the former definition of the right sine requires. And this right sine FH is a perpendicular line drawn from the common end F of both arcs FA, FC to the diameter AC drawn from the remaining ends A, C of the same arcs, as the latter definition of the right sine requires. The same two arcs FA, FC have the same sine of the complement, FK, because the arc FB, whose right sine is FK, is the complement of each of them. Their versed sines, however, are not the same: rather, AH is the versed sine of arc FA, and CH is the versed sine of arc FC — and these two versed sines make up the diameter AC. Here you see that the versed sine CH of arc FC, which exceeds a quadrant, is greater than the semidiameter, or whole sine, CE. [Margin: The versed sine of an arc greater than a quadrant is greater than the whole sine.]

In the same way, too, two angles equal to two right angles have the same sine, both right and of the complement. [Margin: Two angles equal to two right angles have the same sine, but different versed sines, which make up the whole diameter.] This is clear in the angles AEF, FEC, of each of which the right sine is FH and the sine of the complement FK; because they stand upon the arcs AF, FC, of each of which the right sine is FH and the sine of the complement FK, as has been said. Yet the versed sines of these same angles are not the same: rather, AH is the versed sine of angle AEF, namely of arc AF; and HC is the versed sine of angle FEC, that is, of arc FC. And both versed sines together make up the whole diameter AC.

Again, the right sine of any arc is equal to the segment of the diameter lying between the centre and the right sine of the complement of the same arc; while the sine of the complement of any arc is equal to the segment of the diameter

placed between the centre and the right sine of the same arc. [Margin: To what segment of the diameter the sine — both right and of the complement — is equal.] Thus FH, the right sine of arc FA, is equal to the segment of the diameter EK [Margin: c — Elements I.34]; and FK, the sine of the complement of the same arc FA, is equal to the segment of the diameter EH — because of the parallelogram HK, for both the straight lines HF, EK and the straight lines KF, EH are parallel, on account of the right angles at H, E, K, F. [Margin: d — Elements I.28.] Hence it comes about that, if there are two arcs of which one is the complement of the other, the right sine of either is equal to the *complement of the versed sine* of the other arc. Now by "complement of the versed sine" we mean the segment of the diameter by which the versed sine itself is exceeded by the semidiameter, if its arc is smaller than a quadrant; or by which it exceeds the semidiameter, if its arc is greater than a quadrant. [Margin: What the complement of the versed sine is.] Thus we call HE the complement both of the versed sine AH, which corresponds to arc FA, and of the versed sine CH, which corresponds to arc FC. You see, therefore, that of the two arcs FA, FB, of which one is the complement of the other, the right sine FK of arc FB is equal to HE, the complement of the versed sine AH of the other arc FA [Margin: e — Elements I.34]; and the right sine FH of arc FA is equal to EK, the complement of the versed sine BK of the other arc FB. By the same reasoning, since arc FB is the complement of arc FC, you see that the right sine FH of arc FC is equal to EK, the complement of the versed sine BK of the other arc FB; and the right sine FK of arc FB is equal to EH, the complement of the versed sine CH of the other arc FC. So too the sine of the complement of any arc is equal to the complement of the versed sine of the same arc. Thus FK, the sine of the complement of arc AF (or FC), is equal to HE, the complement of the versed sine AH (or CH) of arc AF (or FC).

By like reasoning, in the same circle — or in equal circles — the sines, both right and versed, of equal arcs, and the sines of the complements of equal arcs smaller than a quadrant, are equal; and conversely, of equal sines (both right and versed, or of the complements) the arcs smaller than a quadrant are equal. [Margin: In the same circle, or in equal circles, the sines of equal arcs are equal, and conversely; but the sines of unequal arcs are unequal, and conversely.] But of unequal arcs smaller than a quadrant, the sines are unequal: the sine, both right and versed, of the greater arc is greater and of the smaller arc smaller, while the sine of the complement of the greater is smaller and of the smaller greater. And conversely, of unequal sines (both right and versed, or of the complements) the arcs are unequal: for the greater right or versed sine belongs the greater arc, and for the smaller the smaller; but for the greater sine of the complement belongs the smaller arc, and for the smaller the greater. [Margin: Of two arcs, of which

one is the complement of the other, the right sine of either is equal to the complement of the versed sine of the other arc.] For let the arcs BF, DG be equal. I say that their right sines FK, GL are equal; likewise the versed sines KB, LD, and also the sines of the complements EK, EL. For since the arcs BF, DG are equal, the angles BEF, DEG will also be equal [Margin: f — Elements III.27]; and the right angles K, L are also equal, and the sides EF, EG are equal, being semidiameters. Therefore both the sides FK, GL and the sides EK, EL will be equal to one another — namely the right sines to one another, and [the sines] of the complements [Margin: g — Elements I.26]...

(The construction figure printed at the head of this page is identical to the one above.)

and their complements to one another. And when the equal EK, EL are subtracted from the semidiameters EB, ED, the remaining versed sines KB, LD will be equal. Which is what was proposed.

Now let the sines be equal — whether the right sines FK, GL, or the versed sines KB, LD, or the sines of the complements EK, EL. I say that the arcs BF, DG are equal. For if FK, GL are equal, their squares will be equal. Since, then, the squares of the straight lines EF, EG are also equal [Margin: h — Elements I.47] — and the former equals the squares on FK, KE, the latter the squares on GL, LE — the two squares on the straight lines FK, KE are consequently equal to the two squares on GL, LE. If the two equal squares of the straight lines FK, GL are subtracted, the squares on EK, EL will remain equal; and consequently the straight lines EK, EL will also be equal. Wherefore, since the sides EF, EK are equal to the sides EG, EL and the base FK is equal to the base GL, the angles FEB, GED will be equal [Margin: i — Elements I.8]; and consequently the arcs BF, DG will also be equal [Margin: k — Elements III.26]. Which is what was to be shown.

But if the sines of the complements EK, EL are equal, we shall show in the same way that the straight lines FK, GL are equal; and so, as before, the arcs BF, DG will be equal. If, finally, the versed sines KB, LD are supposed equal, then when these are taken from the semidiameters EB, ED, the sines of the complements EK, EL will be left equal; wherefore again we shall show, as before, that the arcs BF, DG are equal. Which was to be shown.

Now let the arc BF be greater than arc DM, and let the sine MO be drawn. I say that the right sine FK is greater than the right sine MO; likewise, that the versed sine KB is greater than the versed sine OD; but that the sine of the complement EK is smaller than the sine of the complement EO. For, an arc DG being set equal

to arc BF, then (as we have demonstrated) both the right sines FK, GL, and the versed sines KB, LD, and the sines of the complements EK, EL will be equal. Since, then, LD is greater than OD, the versed sine KB will also be greater than the versed sine OD; likewise, since EL is smaller than EO, the sine of the complement EK will also be smaller than the sine of the complement EO. Let MN be drawn perpendicular to GL [Margin: l — Elements I.34], and NL will be equal to MO. Since, then, GL is greater than NL — that is, than MO — the right sine FK will also be greater than the right sine MO. Which was to be demonstrated.

Finally, let the right sine FK be greater than the right sine MO, the versed sine KB greater than the versed sine OD, and the sine of the complement EO greater than the sine of the complement EK. I say that the arc BF, which corresponds to the greater right and versed sine, is greater than the arc DM, which corresponds to the smaller right and versed sine; but that the arc DM, which corresponds to the greater sine of the complement, is smaller than the arc BF, which corresponds to the smaller sine of the complement.

For if FK is greater than MO, let KP be cut off equal to MO, and let PQ be drawn perpendicular to FK, and QR perpendicular to BE, which will be equal to PK — that is, to MO [Margin: m — Elements I.34]. Consequently, as was shown a little before, the arcs BQ, DM will be equal, on account of the equality of the right sines QR, MO. Since, then, the arc BF is greater than arc BQ, the same arc BF will be greater than arc DM. But if KB is greater than OD, let BR be cut off equal to DO, and let RQ be drawn perpendicular to BE; and the arcs BQ, DM will be equal, as we showed a little before, on account of the equality of the versed sines RB, OD. Wherefore, since the arc BF is greater than arc BQ, the same arc BF will be greater than arc DM. If, finally, EO is greater than EK, let EL be cut off equal to EK, and let LG be drawn perpendicular to ED; and the arcs BF, DG will be equal, on account of the equality of the sines of the complements EK, EL, as was shown a little before. For this reason, since the arc DM is smaller than arc DG, the same arc DM will be smaller than arc BF. Which is what was proposed.

Exactly the same must be said of the sines of angles. For equal angles too have equal sines — both right, and of the complements, and versed, and so on — because equal angles stand at the centre upon equal arcs, and so on. [Margin: Equal angles have equal sines, etc.]

Finally, in every right triangle, if the side opposite the right angle is set as the whole sine, the remaining two sides are the right sines of the remaining acute angles to which they are opposite. [Margin: If in a right triangle the side opposite the right angle is the whole sine, either of the remaining sides will be the right

sine of the acute angle opposite it.] Thus in the right triangle EKF, in which EF is the whole sine — being the semidiameter of the circle described from E — the side FK is the right sine of angle FEK, by definition 7. So too, if the same circle were described from F, the side EK would be the right sine of angle EFK, by the same definition 7. This is also clear from the following: that angle EFK is equal to the alternate angle AEF [Margin: n — Elements I.29], whose right sine, by definition 7, is the straight line FH [Margin: o — Elements I.34] — which is indeed equal to the side EK.

In the same way, either of the remaining sides in a right triangle is the sine of the complement of the acute angle adjacent to it — namely, the sine of the complement of that arc of which the other side is the right sine. [Margin: If in a right triangle the side opposite the right angle is the whole sine, either of the remaining sides will be the sine of the complement of the angle adjacent to it, of which of course the other side is the right sine.] Thus in the same right triangle EKF, the side FK is the sine of the complement of angle EFK, or of arc FA, to whose right sine FH the other side EK is equal. Likewise the side EK is equal to FH, the sine of the complement of angle FEK, or of arc FB, of which the other side FK is the right sine. But now let us demonstrate the lemma of which we made mention above.

LEMMA.

If in a circle a straight line drawn from the centre bisects another straight line not drawn through the centre, the same line will also bisect the arc which that straight line subtends; and if it bisects the arc, it will also bisect the straight line subtended by it.

[Margin: A straight line drawn from the centre, cutting another straight line in half, also cuts in half the arc which that line subtends; and conversely.]

Let the straight line EA, in the same figure, bisect the straight line FG at H. I say that it also bisects the arc FG at A, and conversely. For, when the straight line EG has been drawn: since the two sides EF, EH of triangle EFH are equal to the two sides EG, EH of triangle EGH, each to each, and the base HF is by hypothesis equal to the base HG, the angle FEH will be equal to the angle GEH [Margin: p — Elements I.8]. Therefore the arc AF will be equal to the arc AG [Margin: q — Elements III.26]. Which is what was proposed.

But now let the straight line EA bisect the arc FG at A. I say that it also bisects the straight line FG at H. For since the arcs AF, AG are equal, the angles FEH, GEH will also be equal [Margin: r — Elements III.27]. Therefore, since the two sides EF, EH

of triangle EFH are equal to the two sides EG, EH of triangle EGH, the bases HF, HG will also be equal [Margin: s — Elements I.4]. Which is what was proposed.

From this it follows that the straight line EA, which bisects the arc FG in half at A, also bisects the straight line FG in half at H, as we posited above.

[Margin: All sines are expressed in the parts into which the whole sine is conceived as divided. — Into how many parts the semidiameter of the circle is divided by Ptolemy and al-Zarqālī [Arzahel].]

Now the whole sine [*sinus totus*]*—*that is, the semidiameter of any circle*—*is understood by astronomers to be divided into some number of equal parts, so that all other sines may be measured by the ratio of these parts, and so that the proportions of all the sines to the whole sine, or to the semidiameter, may be expressed in numbers. I must therefore explain briefly into how many parts they have divided the semidiameter, for they have not all divided it in the same way. Ptolemy cuts the semidiameter into 60 equal parts, and the whole diameter into 120; each part he then conceives as divided into 60 minutes, and each minute into 60 seconds. Nearly all the ancients, and some of the moderns*—*among them Oronce Finé [Orontius]*—*followed this division. In Book I of the *Almagest*, Ptolemy computed a table of all chords, corresponding to arcs that increase in order by half a degree over a semicircle, beginning from an arc of 30 minutes, in parts of which the whole diameter contains 120. Oronce Finé, by contrast, drew up a table of all sines, corresponding to arcs of the quadrant that increase in order by one minute, beginning from an arc of 1 minute, in parts of which the semidiameter*—*the whole sine*—*contains 60. The Arab al-Zarqālī [Arzahel] set the semidiameter at 150 parts, and accordingly the whole diameter at 300, which parts he again divided into minutes and seconds, as Ptolemy did. But in whichever way the semidiameter or the diameter is divided, it is extremely troublesome to work out all the other sines or chords in such parts, since multiplication, division, and the extraction of roots must always be carried out through astronomical fractions; or else the parts must be converted into minutes and seconds, and conversely the minutes and seconds into parts*—*a task that is very laborious not only for those little practiced in arithmetic, but even for the most expert.

[Margin: The division of the semidiameter, or whole sine, into 10000000 or 100000 particles is more convenient than into 60.]

For this reason other astronomers*—*among them Georg Peurbach [Georgius Purbachius], Regiomontanus [Ioannes Regiomontanus], and Peter Apian [Petrus

Appianus]—divided the semidiameter, that is, the whole sine, into far more equal parts, namely into 10000000 or 100000 parts. In this way there is no need to divide these parts further into minutes and seconds, since one of these parts is either much smaller than a single second of the semidiameter divided into 60 parts in Ptolemy's manner, or at any rate not much larger. For one second is $1/216000$ of the whole semidiameter divided into 60 parts, since the 60 parts contain 216000 seconds; whereas one particle of the semidiameter divided into 10000000 or 100000 parts is $1/10000000$ or $1/100000$ of the whole semidiameter. Now it is clear that this small quantity, $2/10000000$ [numerator uncertain — the sense expects $1/10000000$] [?], is much smaller than that $1/216000$, while $1/100000$ is not much larger than that same $1/216000$. And so, even if in some sine roughly one particle out of the 10000000 particles of the whole sine is at times neglected, a much smaller error is committed than if one whole second out of the 216000 seconds into which the whole sine is understood to be divided were neglected; and if roughly one particle out of the 100000 particles of the whole sine is neglected, no much greater error is committed than if one whole second out of the 216000 seconds into which the whole sine is divided were neglected.

[Margin: How large the whole sine is according to common usage, and how large the author fixes it. — Also, that the calculation is made more accurate the larger the whole sine is.]

There are also those who divide the whole sine into 6000000 or 60000 parts; and there exist tables of sines composed by Regiomontanus in which the whole sine is set to contain that many particles. But more usual among astronomers is the division of the whole sine into 10000000 or 100000 particles. Indeed the common practice of nearly everyone has prevailed, that in computations drawn from the sines the whole sine be set at 100000 particles—as I too have adopted, both in the [treatise on the] sphere and in gnomonics and other works—although the larger the whole sine, the more accurate the computation and reckoning turn out to be. Regiomontanus further constructed a table of all the sines corresponding to the arcs of the quadrant that increase in order by one minute, in the parts of the whole sine divided into 10000000 parts, beginning from an arc of 1 minute. I have examined this table with the greatest care and diligence, and corrected it in certain places, since through the carelessness of the printers it was by no means free of errors. I shall append this corrected table below, once I have first set out the demonstrations—drawn chiefly from Regiomontanus—by which the sines of all arcs can be expressed in numbers, in the parts of the whole sine however it is divided. After the use of the table, I shall also add the demonstrations of Ptolemy

and others, by which the chords of all arcs may be expressed in numbers, from which in turn a table of sines can readily be constructed.

Diagram. A quadrant ABC drawn inside a square frame. The right angle is at A (bottom-right corner): the vertical semidiameter AB runs up the right side to B (top-right), and the horizontal semidiameter AC runs along the bottom to C (bottom-left). The quarter-circle arc runs from B down to C, bulging toward the (unlabeled) top-left corner. The arc is marked off in nine equal steps of ten degrees, labelled along it from B toward C: **10, 20, 30, 40, 50, 60, 70, 80, 90** (90 at C). An interior grid of horizontal and vertical lines (the sine lattice) connects these divisions to the two semidiameters. Along the bottom edge (semidiameter AC) is written **10000000**, and up the right edge (semidiameter AB) is written **10000000** (vertically), indicating each semidiameter divided into ten million particles.

- B (top-right): endpoint of the arc / top of semidiameter AB
- A (bottom-right): center of the quadrant (right angle)
- C (bottom-left): endpoint of semidiameter AC, marked 90
- 10–90: ten-degree divisions of the arc BC
- 10000000: number of particles into which each semidiameter is divided

And first of all, if on some plane a quadrant were constructed large enough that its arc could conveniently be divided into 90 degrees, and each degree into 60 minutes, and likewise each of its two semidiameters—that is, the whole sine—into 10000000 equal parts (or even into more or fewer), then the magnitudes of all the sines could be found easily, without any trouble or labor of computation, if from each minute of the arc straight perpendiculars were drawn to each of the two semidiameters. As in this quadrant ABC: if the arc BC is divided into degrees and minutes—for want of space I have divided it into nine parts, so that each takes in ten degrees—and likewise each semidiameter is divided into 10000000 particles, or into more or fewer, and perpendiculars are drawn to each semidiameter, then the perpendiculars drawn to the semidiameter AB will be the right sines [*sinus recti*] of the arcs of the quadrant beginning from the point B, and equal to them are the portions of the semidiameter AC lying between the point A and the perpendiculars drawn to the semidiameter AC. Therefore, however many particles, out of those 10000000, these portions contain, of just so many particles will the right sines of the arcs of the quadrant be. In the same way both the sines of the complements of those arcs and the versed sines are found. For the perpendiculars dropped to the semidiameter AC are the sines of the complements, and equal to them are the portions of the semidiameter AB lying between the point B and the said perpendiculars drawn to the semidiameter AB;

while the portions of that same semidiameter AB [between B and those perpendiculars] are the versed sines [*sinus versi*] of those same arcs beginning from the point B. But since it is impossible to find a quadrant large enough to admit so many divisions conveniently, we shall investigate the magnitudes of the sines by geometrical demonstrations, the whole sine being set at any number of particles, in the following propositions. And it will be enough for us to seek out the right sines of all arcs; for once these are known, both the sines of the complements and the versed sines of those same arcs will be made clear, as we shall set out in the use of the table of sines.

THEOR. 1. PROPOS. 1.

[Margin: Perpendiculars drawn from equal arcs of a quadrant to either of the semidiameters, or to a straight line parallel to a semidiameter, cut off unequal segments; and the greater is the one nearer to the other semidiameter.]

If, in a quadrant of a circle, equal arcs are taken, and from their endpoints perpendiculars are drawn to either of the semidiameters or to a straight line parallel to a semidiameter, then the segments of the semidiameter—or of that parallel—intercepted between those perpendiculars will be unequal; and the greater will be the one that is nearer to the other semidiameter.

Let ABC be a quadrant in which the arcs DE and EF are equal. From their endpoints let the perpendiculars DKG, ELH, FMI be drawn to the semidiameter AC, or to the straight line RS parallel to AC. I say that the segments GH and HI—or KL and LM—are unequal, and that GH is greater than HI, and KL greater than LM. For, completing the semicircle BCN, let the straight lines DG, EH, FI be produced as far as O, P, Q. Drawing also the straight lines ET and FV perpendicular to DO and EP, let the straight lines EO and FP [printed: EP; recte FP] be joined. Since the arcs DE and EF are equal, the angles DOE and EPF standing upon them will also be equal [Elements III.27]; and the right angles at T and V are equal as well. Therefore, since the three angles of triangle EOT are equal to the three angles of triangle FPV—both sets being equal to two right angles [Elements I.32]—the remaining angle TEO will be equal to the remaining angle VFP; and so triangles EOT and FPV will be equiangular. Hence OE is to ET as PF is to FV [Elements VI.4]. But the straight line OE is greater than the straight line PF, because OE is nearer the center than PF [Elements III.15]; therefore the straight line ET is also greater than the straight line FV [Elements V.14]. Now since ET is equal to the segments GH and KL (because TH and TL are parallelograms), and FV is equal to the segments HI and LM (because VI and VM

are parallelograms) [Elements I.34], the segment GH will also be greater than the segment HI, and the segment KL greater than the segment LM. Therefore, if in a quadrant of a circle equal arcs are taken, and so on. Which was to be demonstrated.

A shorter proof. Let the straight line DF be drawn, cutting the drawn semidiameter AE at Z and the straight line EH at a, and let the straight line FV be produced as far as b. Since, then, the arc DF is bisected at E, the straight line DF will also be bisected at Z, by the lemma laid down among the definitions; and accordingly Da will be greater than aF. Since therefore Da is to aF as bV is to VF [Elements VI.2], bV will also be greater than VF—that is, GH greater than HI, and KL greater than LM.

Diagram. The quadrant of Proposition 1, with the semicircle completed. The right angle is at A (right side): semidiameter AB rises on the right to B (top-right), and semidiameter AC runs left to C. The quarter arc carries the equally spaced points D, E, F (near the top, with equal arcs DE and EF); X lies farther along toward C. From D, E, F perpendiculars DKG, ELH, FMI fall to the base AC, meeting the parallel straight line RS (the horizontal row through S...R) at K, L, M and the base at G, H, I (row C, Y, I, H, G, A). The chords/lines DG, EH, FI are produced downward across the completed semicircle to O, P, Q, N (bottom row). T and V are the feet of the auxiliary perpendiculars ET and FV.

- B (top-right): top of semidiameter AB
- A (right): center of the quadrant (right angle)
- C (bottom-left): end of semidiameter AC
- D, E, F: points on the arc marking equal arcs DE, EF
- X: a further point on the arc
- G, H, I: feet of the perpendiculars on AC
- K, L, M: intersections of the perpendiculars with the parallel RS
- S, R: endpoints of the parallel straight line RS
- Y: point on AC below X
- T, V: feet of the perpendiculars ET, FV
- O, P, Q, N: points on the completed lower semicircle
- [Additional small labels near E and V — the auxiliary points Z, a, b of the shorter proof — are not clearly resolvable on the tile.] [?]

COROLLARIUM.

[Margin: The differences of the right sines gradually decrease from the beginning of the quadrant to its end, so that the sines of smaller arcs have greater differences than the sines of larger arcs, provided the arcs have equal differences.]

From this proposition it is clear that, if any number of arcs of the quadrant, all beginning from the same semidiameter, have equal differences or increments, then the right sines of the smaller arcs have greater differences than the sines of the larger arcs—so that the differences of the sines continually decrease from the beginning of the quadrant to its end. For if, in the same figure of this proposition, the arc FX is taken equal to the arcs DE and EF, and the straight line XY is drawn perpendicular to the semidiameter AC, the four arcs BX, BF, BE, BD will have equal increments, since BX exceeds BF by the arc FX; BF exceeds BE by the arc EF, which is set equal to the arc FX; and the arc BE exceeds the arc BD by the arc DE, which is equal to the arc EF. Now the right sines of those arcs are AY, AI, AH, AG, as I taught above in the exposition of the definitions, since they are the portions of the semidiameter AC lying between the center A and the sines of the complements, as is evident. And since in this proposition I have demonstrated that GH is greater than HI, and HI greater than IY, it is clear that the excess GH, between the sines of the smaller arcs BE and BD, is greater than the excess HI, between the sines of the larger arcs BF and BE; and likewise that the excess HI, between the sines of the smaller arcs BF and BE, is greater than the excess IY, between the sines of the larger arcs BX and BF. The same reasoning holds for the rest. It is therefore established that the differences of the right sines gradually decrease from the beginning of the quadrant to its end—which is plainly evident from the table of sines. That sometimes several neighboring sines, especially at the beginning of the quadrant, have the same difference, or even (though rarely) that two later ones have a greater difference than two earlier ones, arises from the fact that the sines of the [ensuing] table are not entirely accurate, as is clear from the way they were found. If the whole sine were set greater than 10000000, the differences of the sines would appear more clearly.

PROBL. 1. PROPOS. 2.

[Margin: How the sides of the decagon and pentagon in one and the same circle are found.]

To find the sides of an equilateral decagon and pentagon in one and the same circle.

Although these sides can be found by what Euclid demonstrated in Book 4, they are nonetheless investigated by Ptolemy, in Book I, chapter 9 of the *Almagest*, by another method, which contributes much to the finding of very many sines. The method is this. Let there be a circle, or (what suffices) a semicircle ABC; and from its center D let the perpendicular DB be erected to its diameter AC. Bisecting the semidiameter CD at E, draw the straight line EB, and cut off EF equal to it, and join the straight line FB. I say that the straight line BF is the side of the pentagon, and DF the side of the decagon, in the circle ABC. For since the straight line CD is bisected at E, and DF is added to it, the rectangle contained by CF and DF, together with the square of the straight line DE, will be equal to the square of the straight line EF [Elements II.6], and therefore to the square of the straight line EB, which is equal to EF. But the square of the straight line EB is equal to the squares of the straight lines BD and DE [Elements I.47]. Therefore the rectangle contained by CF and DF, together with the square of the straight line DE, is equal to the squares of the straight lines BD and DE. And accordingly, taking away the common square of the straight line DE, there will remain the rectangle contained by CF and DF equal to the square of the straight line BD—that is, to the square of the straight line CD;

Diagram. A semicircle on the diameter AC (A at the left end, C at the right end), with center D. The perpendicular radius DB rises from the center to B at the top. On the diameter, E is the midpoint of DC, and F lies between A and D, so that the base reads, from left to right, A, F, D, E, C. The chords EB, FB (and BD, the radius) are drawn; BF is the side of the pentagon and DF the side of the decagon.

- B (top): endpoint of the perpendicular radius DB
- A (left), C (right): endpoints of the diameter
- D: center of the semicircle
- E: midpoint of CD
- F: point on the diameter with $EF = EB$; DF is the decagon side, BF the pentagon side

[The pentagon/decagon semicircle is reprinted at the head of this page because the demonstration of PROPOS. 2 carries over the page break; it is the same figure as fig-p695-2 above and is not re-described here.]

it will follow that CF is to CD as CD is to DF [Elements VI.17]; and therefore the straight line CF will be divided at D in extreme and mean ratio. Since, then, the greater segment CD is the side of the hexagon in the circle ABC (by the corollary to Proposition 15 of Book 4 of Euclid), the lesser segment DF will be the side of the

decagon in the same circle, as we demonstrated on Proposition 9 of Book 13 of Euclid. Again, since the square of the side of the hexagon BD, together with the square of the side of the decagon DF, is equal to the square of the side of the pentagon in the same circle [Elements XIII.10]—and the square of the straight line BF is equal to those same squares of BD and DF [Elements I.47]—the square of the side of the pentagon will be equal to the square of the straight line BF; and therefore the straight line BF is equal to the side of the pentagon. We have therefore found the sides of the equilateral decagon and pentagon in one and the same circle. Which was to be done.

PROBL. 2. PROPOS. 3.

[Margin: Given the right sine of any arc, how the sine of the complement of that same arc is found; and given the chord of any arc, by what method the chord of the remaining arc of the semicircle is found.]

Given the right sine of any arc less than a quadrant, to find the sine of the complement of that same arc; and likewise, given the chord of any arc less than a semicircle, to find the chord of the remaining arc of the semicircle.

First, let the right sine DE of the arc BD be known, and let the sine of the complement of that arc be DF, which we must find. When the straight line DA has been drawn, the square of the straight line DA will be equal to the squares of the straight lines DE and EA [Elements I.47]. If, then, from the square of the whole sine DA (which is known, for the whole sine is set at a fixed number of particles) the square of the right sine DE—known in the parts of the whole sine DA—is subtracted, there will remain the square of the straight line EA, known; and accordingly, by its square root, EA will be known in those same parts. Since, then, the straight line EA is equal to the sine of the complement of the arc BD—that is, to the straight line DF [Elements I.34]—DF, the sine of the complement of the arc BD, will be known, whose right sine DE was set as known.

Next, let the chord AB of the arc AB be known, and let BC be the chord subtending the remaining arc BC of the semicircle, which we are directed to find. Since the angle B is a right angle in a semicircle [Elements III.31], the square of the diameter AC will be equal to the squares of the chords AB and BC [Elements I.47]. If, then, from the square of the diameter AC (which is known, for the diameter is set as divided into a fixed number of particles) the square of the chord AB—known in the parts of the diameter AC—is taken away, the square of the chord BC will remain known; and accordingly, by its square root, the chord BC will become

known in those same parts. Thus from the right sine of any arc, and so on, we have found what was sought. Which was to be done.

Diagram. The left of the two constructions for this problem: a quarter-circle with the right-sine construction. The base runs C, F, A (left to right); B is at the top and D on the arc at the upper left, with E just below B. The right sine DE and the complement sine ($DF = EA$) are shown; a right triangle is formed at E.

- B (top): endpoint of the arc
- D: point on the arc (endpoint of the arc BD)
- E: foot of the right sine DE
- C, F, A: points on the base semidiameter (A the center)

Diagram. The right of the two constructions: a semicircle on the diameter AC (A at left, C at right, center dot marked), with B on the circle toward the upper right. The chords AB and BC are drawn, meeting at the right angle at B (right angle in a semicircle).

- A (left), C (right): endpoints of the diameter
- B (upper right): point on the semicircle; angle ABC is a right angle
- chords AB and BC subtend the arc AB and the remaining arc BC

COROLLARIVM.

[Margin: The versed sine of any arc is found from its right sine, once that is known.]

From this it follows that the versed sine of any arc can also be found once its right sine is known. For since the sine of the complement DF —that is, AE —is found from the right sine DE , then, if the sine of the complement of the given arc BD is subtracted from the whole sine AB , the versed sine EB of the given arc BD will remain known. By the same reasoning, if the right sine DE —that is, AF —of the given arc BD is subtracted from the whole sine AC , the versed sine CF of the complement of the given arc BD will remain known.

THEOR. 2. PROPOS. 4.

[Margin: The right sine of any arc less than a quadrant is a mean proportional between half the whole sine and the versed sine of another arc, which is double the first and also less than a quadrant.]

The right sine of any arc less than a quadrant is a mean proportional between half the semidiameter (that is, of the whole sine) and the versed

sine of another arc, which is double the first arc and likewise less than a quadrant. Likewise, the whole sine has the same ratio to twice the given sine as the given sine has to the versed sine of the double arc.

Let CE be any arc less than a quadrant, whose half is CD. Bisecting the semidiameter AC at G, from E draw EF perpendicular to AC, and join the straight line AD, which will cut the drawn chord CE at H, bisecting it—by the lemma demonstrated by us above at the definitions—and indeed at right angles [Elements III.3]. Then CH will be the right sine of the arc CD, and CF the versed sine of the arc CE, which is double the arc CD, since EF is the right sine of that same arc CE, as is clear from the definitions. I say that CH, the right sine of the arc CD, is a mean proportional between CG, half the whole sine, and CF, the versed sine of the arc CE, which is double the arc CD. Likewise, that AC, the whole sine, is to CE, twice the sine CH, as the given sine CH is to CF, the versed sine of the double arc CE. For since the two angles ACH and AHC of triangle ACH are equal to the two angles ECF and EFC of triangle ECF—because the angle C is common to both triangles, and the angles at H and F are right angles [Elements I.32]—the triangles ACH and ECF will be equiangular. Therefore AC is to CH as EC is to CF [Elements VI.4]; and, alternately, AC is to CE as CH is to CF. But as AC is to CE, so is CG, half of AC, to CH, half [of...] [Elements V.15]

the half of CE. Therefore it will also hold that, as CG is to CH, so CH is to CF; and consequently CH, the right sine of arc CD, is the mean proportional between CG, the half of the whole sine, and CF, the versed sine of arc CE (of which arc CD is the double). Therefore the right sine of any arc less than a quadrant [and so forth]. Which was to be demonstrated. The other part is established, since it has been shown that, as AC, the whole sine, is to CE (the double of the sine CH), so CH, the given sine, is to CF, the versed sine of arc CE, of which arc CD is the double.

[Margin: From the known right sine of any arc, the right sine of another arc which is its half becomes known.]

COROLLARIUM.

It is gathered from this that, if the right sine of some arc is known, the right sine of another arc which is its half also becomes known: so that, from EF, the known right sine of arc CE, there is also found CH, the right sine of arc CD, which is the half of arc CE. For from the known right sine EF the sine EI of the complement becomes known; and when this is subtracted from the whole sine AC ([Margin: e] since the sine EI is equal to the line AF), the versed sine CF of arc CE is left known, as was stated in the corollary to the preceding proposition. [Margin: c — Elements

I.34] Since, then, CH ([Margin: f — Proposition 4 of this treatise]) is the mean proportional between the half of the whole sine and the versed sine CF, as we have shown, [Margin: g — Elements VI.17] the rectangle contained by the half of the whole sine and the versed sine CF will be equal to the square of the sine CH. Therefore, if the half of the whole sine is multiplied into the versed sine CF, there will be produced the square number of the sine CH, whose square root will give the right sine CH known. And the reasoning is the same for the rest.

The same thing can also be shown by this method. Since EF, the right sine of arc CE, is set down as known, [Margin: h — Proposition 3 of this treatise] EI too, the sine of the complement of the same arc — that is, the line AF, [Margin: i — Elements I.34] which is equal to it — will become known. Therefore, when the line AF (that is, the sine of the complement of arc CE) is subtracted from the whole sine AC, the versed sine FC of that same arc CE will be known, as we also showed in the corollary to Proposition 3. And since the square of the line CE is equal to the squares of the lines EF and FC, it comes about that the squares of the known lines EF and FC, collected into one sum, make up the square of the line CE; whose square root will render the line CE itself known; and hence the half of this root will give CH, the right sine of arc CD (which is the half of the given arc CE), known.

[Margin: From the known right sine of any arc, the right sine of another arc which is its double becomes known, provided it be less than a quadrant.]

By converse reasoning, from this same Proposition 4 it is gathered that, if the right sine of some arc is known, the right sine of another arc which is its double also becomes known, provided the double be less than a quadrant: so that from CH, the known right sine of arc CD, there is also found EF, the right sine of arc CE, of which arc CD is the double. For since CH is the mean proportional between the half of the whole sine and the versed sine FC, as we have shown, the rectangle contained by the half of the whole sine and the versed sine FC will be equal to the square of the right sine CH. Therefore the square of the known sine CH will be that rectangle; and when it is divided by the half of the whole sine, the versed sine FC comes out known. And since the line CE, being the double of the known sine CH, is itself known, its square too will be known; and if the square of the known versed sine FC is subtracted from it, the square of the line EF is also left known (since the square of the line CE is equal to the squares of the lines CF and FE). And hence the square root of that quantity will give the right sine EF known.

SCHOLIUM.

But if it should happen that the perpendicular EF bisects the semidiameter AC, as it does in this figure, then CH, the sine of arc CD, will still be the mean proportional between CF, the half of the whole sine, and CF, the versed sine of arc CE (of which arc CD is the double). For the triangles ACH and ECF will again be equiangular; [Margin: k — Elements VI.4] and consequently, as AC is to CH, so is EC to CF; and, by alternation, as AC is to CE, so is CH to CF. [Margin: l — Elements V.15] Since, then, it holds that as AC is to CE, so is CF (the half of AC) to CH (the half of CE), it will also hold that as CF is to CH, so is CH to CF; and on that account CH, the right sine of arc CD, is the mean proportional between CF, the half of the whole sine, and CF, the versed sine of arc CG [recte: CE], which is the double of arc CD.

From this it follows that, if the perpendicular EF bisects the semidiameter AC, the line CH is equal to the line CF. For if it were greater or less, it could not hold that as CF is to CH, so is CH to CF, since the one ratio would be one of greater inequality and the other one of lesser inequality.

Diagram. The scholium figure: a quadrant with its right angle at A (bottom-right), the arc running from B (top) down to C (bottom-left).

- B (top): upper end of the arc.
- E (upper-left, on the arc): endpoint of arc CE.
- D (left, on the arc, below E): endpoint of arc CD (the midpoint arc of CE).
- H (interior): foot/intersection point; CH is the right sine of arc CD.
- C (bottom-left): center-side vertex, origin of the arcs.
- F (bottom, between C and A): foot of the perpendicular EF on CA; F is the midpoint of CA in this special case.
- A (bottom-right): center of the quadrant (right angle).

The lines AE, AD (radii/chords), the perpendicular EF, and the segment CH are drawn.

THEOR. 3. PROPOS. 5.

The right sine of an arc of 54 degrees is composed of the half of the whole sine and the right sine of an arc of 18 degrees. The versed sine of an arc of 72 degrees is composed of the half of the whole sine and the versed sine of an arc of 36 degrees.

[Margin: The right sine of 54 degrees equals the half of the whole sine together with the sine of 18 degrees; the versed sine of 72 degrees equals the half of the whole sine together with the versed sine of 36 degrees.]

In the quadrant ABC let BD be an arc of 54 degrees; its complement CD is therefore 36 degrees. Bisect CD at H, so that each of the arcs CH and HD is 18 degrees. Draw DM perpendicular to AB, as the sine of the 54-degree arc, and DE perpendicular to AC, as the sine of the 36-degree arc. Also join the line AH, which — by the lemma demonstrated among the definitions — will bisect the line CD at I, [Margin: a — Elements III.3] and consequently at right angles as well; and therefore CI will be the right sine of the arc CH, 18 degrees. Finally, taking a line EF equal to EC, let AC and AF be bisected at G and K, and from K draw KL perpendicular to AG. I say that the right sine DM of the 54-degree arc — that is, the line AE, [Margin: b — Elements I.34] which is equal to it — is composed of AG (the half of the whole sine) and CI...

Diagram. The figure for Proposition 5: a quadrant ABC (right angle at A, bottom-right) with the arc from B (top) to C (bottom-left), carrying the construction points of the demonstration. This same figure is reprinted at the top of p. 698 to accompany the continuation of the demonstration.

- L (top, on the arc): endpoint of arc BL.
- B (top-right corner): upper end of the quadrant.
- N (just below B): foot of the perpendicular LN on AB.
- D (left, on the arc): endpoint of arc BD / CD.
- M (interior, right): foot of the perpendicular DM on AB.
- H (left): midpoint of arc CD region; CH.
- I (interior, below D): midpoint of chord CD; CI, the right sine of 18 degrees.
- C (bottom-left): center-side vertex.
- E, F, G, K (along the base, left to right): points of the base construction (E foot of DE; F, G, K the bisection points).
- A (bottom-right): center of the quadrant.

and CI, the right sine of the 18-degree arc — that is, that the line GE (which together with AG makes up the whole line AE) is equal to the right sine CI. Likewise, that the versed sine of the 72-degree arc is composed of the half of the whole sine and CE, the versed sine of arc CD (36 degrees); that is, that the line EK (which, together with the versed sine CE, makes up the line CK) is equal to the half of the whole sine, while CK itself is the versed sine of the 72-degree arc; that is, that the arc CL, whose versed sine is CK, is 72 degrees.

For, drawing the line LN perpendicular to AB, as the sine of arc BL, let the lines AD and DF be joined. Now since the arc CH of 18 degrees contains one-fifth of the quadrant BC (because five times 18 makes 90), the arc CD will contain two-fifths of the same quadrant; and therefore the ratio of arc CD to arc BC will be as 2 to 5. [Margin: c — Elements VI.33] But as arc CD is to arc BC, so is the angle CAD to the right angle BAC. Therefore the ratio of the angle CAD to the right angle BAC will likewise be as 2 to 5; and so the angle CAD will contain $\frac{2}{5}$ of one right angle. Since, then, the three angles of triangle CAD contain $\frac{10}{5}$ of a right angle — that is, [Margin: d — Elements I.32] are equal to two right angles — and [Margin: e — Elements I.5] the two angles ACD and ADC are equal to each other, each of them will contain $\frac{4}{5}$ of one right angle. [Margin: f — Elements I.5] And since the angle DFC is equal to the angle DCF — because the lines DF and DC are equal (for since DE and EF, the sides of triangle DEF, are equal to DE and EC, the sides of triangle DEC, and the angles at E which they contain are equal, being right angles, [Margin: g — Elements I.4] the bases DF and DC will be equal) — the angle DFC too will contain $\frac{4}{5}$ of one right angle. And consequently the remaining angle DFA, out of two right angles — that is, out of $\frac{10}{5}$ of a right angle — will contain $\frac{6}{5}$ of one right angle. Now the angle DAF [contains $\frac{2}{5}$ of a right angle], and [Margin: h — Elements I.31 (recte I.32?)] since all three angles in triangle AFD contain $\frac{10}{5}$ of one right angle, while the angle DFA has been shown to contain $\frac{6}{5}$, the angle ADF will contain $\frac{2}{5}$ of one right angle, and will therefore be equal to the angle DAF. [Margin: i — Elements I.6] Hence the sides DF and AF will be equal. And since the line DF has been shown equal to the line DC, the line AF will also be equal to DC; and therefore KF, the half of AF, will be equal to CI, the half of DC.

Again, since AK and KF are equal, then, adding the equal segments EC and FE, the line composed of AK and EC will be equal to the line KE; and consequently KE will be the half of the semidiameter AC, since AC is divided into two equal parts, of which one is KE and the other the line composed of AK and EC. Therefore KE is equal to CG. So, when the common line GE is taken away, the remainders GK and EC will be equal. But EC was taken equal to EF; therefore GK too will be equal to EF; and, adding the common line FG, EG will be equal to FK — that is, to CI, to which we showed above that the line KF is equal. Therefore AE (which is equal to the sine DM of the 54-degree arc) is composed of AG, the half of the whole sine, and GE, which has been shown equal to the sine CI of the 18-degree arc. This is the first point.

Now, since KF has been shown equal to EG, and EG to CI, KF will be equal to the right sine CI. But KF is equal to AK; therefore AK too will be equal to CI. [Margin:

k — Elements I.34] Since, then, AK is equal to the sine LN, the sine LN will also be equal to the sine CI. But CI is the sine of an 18-degree arc; therefore LN too will be the sine of an 18-degree arc, and so the arc BL, whose sine is LN, will contain 18 degrees; hence its complement CL will contain 72 degrees. And the versed sine of CL, namely KC, is composed of CG (the half of the whole sine) and GK, which has been shown equal to the versed sine EC of arc CD (36 degrees). This is the second point. And so the right sine of the 54-degree arc is composed [as stated], and so on. Which was to be demonstrated.

[The figure for this proposition (see fig-p697-2 above) is reprinted at the top of this page, beside the margin key "c — Elements VI.33"; it is described once at its first appearance on p. 697 and not re-rendered here.]

COROLLARIVM.

It is established from this that the triangle ACD — whose base CD subtends 36 degrees and whose apex lies at the center — is isosceles, and that each of its two equal angles C and D is double the remaining angle at the center. For the angle CAD was shown to contain $\frac{2}{5}$ of a right angle, whereas each of C and D contains $\frac{4}{5}$.

THEOR. 4. PROPOS. 6.

[Margin: The difference between the chords of two arcs, of which one is as much less than an arc of 120 degrees as the other is greater, equals the chord of the arc by which either of the said arcs differs from the 120-degree arc.]

The difference between the chords of two arcs of a semicircle — of which one is as much less than an arc of 120 degrees as the other is greater — is equal to the chord of the arc by which either of the said arcs differs from the 120-degree arc.

In the semicircle ABC let the arc BA be 120 degrees, and let arc BD be as much less [than the 120-degree arc] as arc BE is greater; their chords are BD and BE. Cut off BF equal to BD, and join the lines AD, AE, AF. I say that EF, the difference of the two chords BD and BE, is equal to the chord AE, or AD. For, when the circle is completed and an equilateral triangle ABG is inscribed in it — one side of which is AB, the chord of the 120-degree arc, since it subtends a third part of the circumference — [Margin: a — Elements I.32] the angle AGB will be a third part of two right angles. [Margin: b — Elements III.21] Since, then, the angle AEB, lying in the same segment AGB as that angle, is equal to it, AEB too will be a third part of two right angles. Next, since the sides DB and BA of triangle DBA are equal to

the sides FB and BA of triangle FBA, [Margin: c — Elements III.27] and contain equal angles, the bases AD and AF will be equal to each other. [Margin: d — Elements III.29] Since, then, AD is equal to AE — because the arcs AD and AE are equal — AF too will be equal to that same AE; [Margin: e — Elements I.5] and therefore the angles AEF and AFE will be equal to each other. But AEF, as we showed, is a third part of two right angles; therefore AFE too will be a third part of two right angles; [Margin: f — Elements I.32] and hence the remaining angle EAF will also be a third part of two right angles.

Diagram. Two companion circle-figures for Proposition 6, drawn side by side, each with horizontal diameter BC.

Left figure:

- D (top-left, on the circumference): endpoint of arc BD.
- A (top): apex of the inscribed equilateral triangle ABG.
- E (right, on the circumference near C): endpoint of arc BE.
- F (interior, on the diameter BC): point with $BF = BD$.
- B (left): left end of the diameter.
- C (right): right end of the diameter.
- G (bottom): third vertex of the inscribed equilateral triangle ABG.
- The chords BA, BD, AD, AE, AF and the dotted lines toward G are drawn.

Right figure (mirror configuration):

- D (top-left, on the circumference).
- A (top).
- F (interior, on the diameter).
- E (right, on the circumference).
- B (left end of the diameter); C (right end).
- G (bottom).
- The corresponding chords and dotted segments are drawn.

Therefore triangle AEF will be equilateral, by the corollary to Proposition 6 of Book 1 of Euclid; and so the line EF, the difference of the chords BD and BE, will be equal to the chord AE, or AD. The difference, then, of the chords of two arcs of a semicircle [is as stated], and so on. Which was to be demonstrated.

COROLLARIUM.

It follows from this that, if the chords of two arcs which together make up 120 degrees are joined together, there results the chord of the arc composed of the 120-degree arc and the smaller of those two arcs — provided they are unequal. For in this way you see that the chords BD and DA — of the arcs BD and DA, which together make 120 degrees — taken together are equal to the chord BE of the arc BAE, which is composed of the arc BA (120 degrees) and the arc AE (equal to the smaller arc AD); because, as was demonstrated, the difference EF between the chords BD and BE is equal to the chord AD.

[Margin: The two chords of two arcs making up 120 degrees together equal the chord of the arc composed of the 120-degree arc and the smaller of those two arcs.]

THEOR. 5. PROPOS. 7.

[Margin: If one quantity exceeds another, the half of the one will exceed the half of the other by half the excess.]

If one quantity exceeds another, the half of the former exceeds the half of the latter by the half of the excess.

Let the quantity AB exceed the quantity C by the excess DB; let DB be bisected at E, and let AF be set equal to EB. Since, then, AF and EB together are equal to the whole excess DB, the remainder FE will be equal to C. Bisect FE at G. Since GE and GF are equal, then, adding the equal segments EB and FA, GB and GA will also be equal; and consequently AB too will be bisected at G. Therefore BG, the half of AB, exceeds GE — the half of FE, that is, of C — by the excess EB, which is the half of the excess DB. So, if one quantity exceeds another, and so on. Which was to be demonstrated.

We shall demonstrate the same thing differently, having first laid down this theorem:

If one quantity exceeds another, the excess of equimultiples of them will be the same multiple of the original excess.

Let the quantity AB exceed the quantity CD by the excess EB; and let AG and CI be equimultiples of AB and CD. I say that AG exceeds CI by an excess which is the same multiple of the excess EB as AG is of AB, or CI of CD. For if BG and DI are divided into the magnitudes BF, FG and DH, HI, equal respectively to AB and CD, then BF and FG will exceed DH and HI by the excesses KF and LG, which are

equal to the excess EB; and AE, BK, FL will together be equal to the whole CI. Therefore AG will exceed CI by as many excesses equal to EB as AG is a multiple of AB (or CI of CD). Which was to be demonstrated.

From this the truth of this Proposition 7 is manifest. For by whatever excess the half of AB exceeds the half of CD, the whole AB will exceed the whole CD by double that excess, as we have just shown. Therefore the excess EB, by which AB exceeds CD, is double the excess by which the half of AB exceeds the half of CD; and consequently the half exceeds the half by the half of the excess. Which is what was proposed.

Diagram. Three line-schematics for Proposition 7, each showing an upper segment from A to B and a lower segment beginning at C, with the intermediate division points F, G, D, E marked. The three rows show the points in different arrangements (the several cases of the construction).

- Row 1: A — F — G — D — E — B (upper); C — (lower, shorter).
- Row 2: A — D — F G — E — B (upper); C — (lower).
- Row 3: A — F — D — E — B, with G marked just below D (upper); C — (lower).

Diagram. A line-schematic for the equimultiples theorem: an upper segment A...G carrying the points E, B, K, F, L, and a lower segment C...I carrying the points D, H.

- Upper: A — E — B — K — F — L — [G] (left to right).
- Lower: C — D — H — I (left to right).

THEOR. 6. PROPOS. 8.

[Margin: The difference between the sines of two arcs of a quadrant, of which one is as much less than an arc of 60 degrees as the other is greater, equals the sine of the arc by which either of the said arcs differs from the 60-degree arc.]

The difference between the sines of two arcs of a quadrant — of which one is as much less than an arc of 60 degrees as the other is greater — is equal to the sine of the arc by which either of the said arcs differs from the 60-degree arc.

In the quadrant ABC let arc CD be 60 degrees, and let arc CE be as much less [than 60] as arc CE [recte: CF] is greater; their right sines are EG and FH. I say that the difference of these sines is equal to the sine of the arc DE, or DF. For, when the quadrant is produced, together with the sines EG and FH, to I and K, take an arc

CL equal to arc CD, so that the whole arc DCL contains 120 degrees. And since the arcs CI and CK are equal to the arcs CE and CF — because, by the lemma laid down among the definitions, [Margin: a — Elements III.3] the line BC bisects the arcs ECI and FCK, and bisects the lines EI and FK as well — the remaining arcs LI and LK will likewise be equal to the remaining arcs DE and DF. Taking also arcs EM and FN equal to the arcs DE and DF, draw the chords LM and LN. And since arc FN is equal to arc LK, and arc EM to arc LI, then, adding the common arcs FL and MI, arc NL will be equal to arc FK, and arc ML to arc EI; [Margin: b — Elements III.29] and consequently the chord NL is equal to the chord FK, and the chord ML to the chord EI. Since, then, arc LM is as much less than arc LD (120 degrees) as arc LN is greater than it, the difference of the chords LN and LM will, by Proposition 6, be equal to the chord DN, or DM; that is, the line KF will exceed the line IE by the chord DN, or DM. Therefore, by the preceding proposition, the half HF — that is, the sine of arc CF — will exceed the half GE — that is, the sine of arc CE — by the half of the chord DN, or DM; that is, by the sine of arc DF, or DE. Which was to be demonstrated.

Alternatively: In the quadrant ABD let the arc BE be 60 degrees, and let the arcs EF and EG be equal; and so arc BF is as much less than arc BE as arc BG is greater than that same arc BE. Draw FH and GI perpendicular to BD; these will be the sines of the arcs BF and BG. Now, when the chord FG is drawn, let the semidiameter DE, when drawn, cut it at L. And since the arc FG is bisected at E, the line FG too will be bisected...

Diagram. Two companion quadrant/semicircle figures for the demonstration of Proposition 8, drawn side by side (mirror configurations), carrying the construction of the sines EG, FH, the produced sines to I, K, and the arcs CL, DCL, with chords LM, LN. Many interior points overlap in the engraving and their exact placement is hard to resolve at this resolution; the readable labels are listed.

Left figure (opening to the right):

- A (top-left corner); N, F, D (on the arc near the top).
- E (interior, upper); M (interior, lower).
- B (left end of the horizontal); H (just below B).
- G (on the horizontal, center); C (right end of the horizontal).
- K, L, I (at the bottom).

Right figure (semicircle, mirror of the left):

- A (top); F, D (on the arc); N (interior, upper).
- E (right, on the arc).
- B (left end of the diameter); H (interior); G (center-right); M (near the right end); C (right end).
- I (lower right); L (bottom center); K (bottom left).

Radii, sine-perpendiculars, and the chords of the construction are drawn throughout both figures.

(Continuing the Aliter — the alternative demonstration — of THEOR. 6 / PROPOS. 8 begun on the previous page.)

cut at L, as shown by the lemma demonstrated among the definitions; and therefore also at right angles (Elements III.3). FL is thus the sine of the arc EF, and GL the sine of the arc EG. Now draw the straight line FK perpendicular to GI; then IK will be equal to the straight line FH (Elements I.34), because FI is a parallelogram. Therefore GK will be the difference of the sines FH and GI.

I claim that this difference GK is equal to the sine FL, or GL. For draw the straight line BE, which is the side of a hexagon and therefore equal to the semidiameter DE (by the corollary to *Elements* IV.15); bisect BD at M, and join the straight line EM. Since, then, the sides DM, ME are equal to the sides BM, ME, and the base DE is equal to the base BE, the angles at M will be equal (Elements I.8), and hence right angles. When the semicircle ABC has been completed, and the straight lines GI and EM produced to N and O, the arc NO will be equal to the arc GE — that is, to the arc EF (by the scholium to *Elements* III.27) — because the straight lines GN and EO are parallel, on account of the right angles at I and M (Elements I.28). Adding the common arc FO, the arc FN will be equal to the arc EO. But the arc EO is double the arc BE. (For the straight line DB, cutting the straight line EO at right angles, bisects it (Elements III.3), and hence also bisects the arc EO, by the scholium set among the definitions.) Therefore the arc FN too will be double the same arc BE. So, drawing the straight lines DF and DN, the angle FDN will likewise be double the angle EDB (Elements VI.33). But this same angle FDN, standing at the center, is double the angle FGN standing at the circumference (Elements III.20). Therefore the angles EDM and FGK are equal; and the right angles M and K are also equal. The triangles EDM and FGK are therefore equiangular, and hence ED is to DM as FG is to GK (Elements VI.4). Since, then, ED is double DM — for DB, which was bisected at M, is equal to DE — FG too will be double GK. But FG is also double FL, or GL. Therefore the straight line GK, the difference of the sines FH and GI, is equal to the straight line FL, the sine of the

arc EF, or to the straight line GL, the sine of the arc EG. So the difference of the sines of two arcs of the quadrant, and so on. Which was to be demonstrated.

Diagram. A semicircle standing on the horizontal diameter DB, with the circle completed below. Read on the leaf, the labelled points are:

- A: upper left, on the arc
- G: top of the arc, right of A
- E: top of the arc, right of G
- F: upper right, on the arc
- K: interior, upper left
- L: interior, upper middle (right of K)
- D: left end of the horizontal diameter
- I: on the diameter, left of center
- M: center of the diameter
- H: on the diameter, right of center
- B: right end of the diameter
- C: lower left, on the completed circle
- N: bottom, left of center
- O: bottom, right of N

Chords DF, DN, GI (produced to N), EM (produced to O), GN, EO, BE, and FK are drawn among these points.

SCHOLIVM

Otherwise: let the sine GI cut the semidiameter DE at P, and join the straight line FP (Elements I.4); this will be equal to GP, because the triangles GLP and FLP have the sides GL, LP equal to the sides FL, LP, and the angles at L equal. Therefore the angles G and F will be equal in the isosceles triangle FGP (Elements I.5), and so too the two angles at P (Elements I.4). Since, then, the angle EPG is equal to the interior angle EDA, which is 30 degrees, the whole angle FPG will be 60 degrees (Elements I.29). Therefore each of G and F will also contain 60 degrees, so that all three together contain 180 degrees. The triangle FGP is therefore equilateral; and consequently GL and GK, the halves of the equal sides GF and GP, will be equal.

That GK is the half of the straight line GP is clear from the fact that FK, being perpendicular to the side GP, bisects it, by the scholium to *Elements* I.26.

Diagram. The same semicircle on diameter DB as in the previous figure, but here with the additional interior point P (used in this alternative proof). Read on the leaf, the labelled points are:

- A: upper left, on the arc
- G: top of the arc
- E: top of the arc, right of G
- F: upper right, on the arc
- K: interior, upper left
- L: interior (right of K)
- P: interior, below L
- D: left end of the diameter
- I: on the diameter, left of center
- M: center of the diameter
- H: on the diameter, right of center
- B: right end of the diameter
- C: lower left, on the completed circle
- N: bottom, left of center
- O: bottom, right of N

The straight line FP is drawn, together with the triangle FGP.

COROLLARIVM

From this it follows that if the sines of two arcs which together make up 60 degrees are added together, the result is the sine of the arc composed of an arc of 60 degrees and the smaller of those two arcs, when they are unequal. For this is what you see in the figure of the later demonstration of this proposition: the right sines FH and FL, of the arcs BF and FE — which together make up 60 degrees — taken together are equal to the right sine GI of the arc BEG, an arc composed of the arc BE of 60 degrees and the arc EG, which is equal to the smaller arc EF; because, as was demonstrated, the difference GK, between the sines FH and GI, is equal to the sine FL.

[Margin: *Two sines of two arcs together making up 60 degrees are equal to the sine of the arc composed of an arc of 60 degrees and the smaller of those two arcs. By this method the right sines of all arcs may be computed.*]

PROBL. 3. PROPOS. 9.

To compute the right sines of all the arcs of the quadrant, which exceed one another in order by one minute, expressed in parts of the whole sine when it is divided into any number of parts whatever.

First of all we shall compute the right sines in terms of a whole sine of 100000 parts — the number commonly adopted by everyone, and set by us as well, as we said above. But to make the computation more accurate, in these computations we shall set the whole sine at 10000000 parts. For in this way, once the first two figures on the right are dropped from each of the sines found (adding a unit, however, if the two figures dropped exceed the number 50), the sines that remain are more precise with respect to a whole sine of 100000 parts.

But if anyone should want sines of more parts — setting the whole sine, of course, at 10000000 parts, the number Johannes Regiomontanus [Regiomontanus] adopted, and the one we shall lay down in the following table — then the whole sine must be set at 1000000000 parts. For in this way, when the first two figures on the right are dropped from each of the sines found, as was said, there will remain sines that are more precise with respect to a whole sine of 10000000 parts. The reason for this is that in the investigation of a sine an error can occur only in one or the other figure on the right; so that when the two figures on the right are dropped, the sines left are the most precise with respect to the smaller whole sine. Anyone will easily verify this in the computation of the sines, and we ourselves shall demonstrate it below. By the same reasoning, if sines are to be investigated with respect to some other whole sine containing more or fewer parts than 100000 or 10000000, the whole sine to be set for the computation must be one that exceeds the given one by these two figures, 00, on the right — so that it is a hundredfold of it, just as here 10000000, which we assume in the calculation, is a hundredfold of the 100000 that we, with the other astronomers, adopt in practice.

[Margin: *How many parts the whole sine — in terms of whose parts the other sines are reckoned — should be set at, so that the other sines may be found more precise with respect to a whole sine of 100000 parts, or of fewer or of more.*]

Diagram. A quadrant (quarter-circle) figure, used for PROPOS. 9 and reprinted at the top of the following page. Read on the leaf, the labelled points are:

- B: top right corner (apex of the quadrant)
- F: interior, upper left
- E: left side, below F

- L: center
- D: left side, below E
- H: right side
- I: right side, below H
- K: interior, lower left
- C: bottom left corner
- G: bottom, right of C
- A: bottom right corner (the right angle of the quadrant)

Numerous chords are drawn among these points (AE, EB, AD, AF, CE, BC, EH, DI, EG, and others).

(The quadrant figure ABC of PROPOS. 9 is reprinted at the top of this page; it is described above at fig-p700-3.)

[Margin: *Computation of the sines of arcs exceeding one another by 15 degrees — namely the arcs of 15, 30, 45, 60, 75, and 90 degrees.*]

So let there be, in the quadrant ABC, the arc CD of 15 degrees, CE of 30, CF of 45, and hence EB of 60 and DB of 75 — since these are the complements of the arcs of 30 and 15 degrees. We shall compute the right sines of these arcs as follows. Draw EH and DI perpendicular to AB; these will be the right sines of the arcs of 60 and 75 degrees. Let the chord BC be bisected at L by the straight line AF, by the lemma demonstrated among the definitions (Elements III.3), and therefore cut at right angles; then BL will be the right sine of 45 degrees, that is, of the arc BF. Again draw EG perpendicular to AC as the sine of 30 degrees. Likewise let the chord CE be bisected at K by the straight line AD, by the said lemma, and therefore at right angles; then CK will be the right sine of 15 degrees. Finally join the straight lines AE and EB.

Now since the arc BE of 60 degrees is a sixth part of the whole circumference of the circle — for six times 60 make 360 degrees — the straight line BE will be the side of a hexagon, and hence equal to the semidiameter AE (by the corollary to *Elements* IV.15). Therefore the angles EAB and EBA will be equal (*Elements* I.5); and the angles at H are also equal, being right angles. Since, then, the two angles EAH and EHA are equal to the two angles EBH and EHB of the triangle BEH, and the side AE is equal to the side BE, the side AH will be equal to the side BH (*Elements* I.26); and hence AH is the half of the semidiameter AB. Therefore, since EG, the right sine of 30 degrees, is equal to this AH (*Elements* I.34), the right sine of 30 degrees will be equal to half the semidiameter, that is, of the whole sine. So,

when the whole sine is set at 10000000, EG, the sine of 30 degrees, will be 5000000 of those same parts — namely, the half of the whole sine.

[Margin: *The right sine of 30 degrees is equal to half the whole sine.*]

That half the whole sine is the sine of the arc of 30 degrees we shall show most easily as follows. Since the whole sine is equal to the chord of the arc of 60 degrees — namely, to the side of a hexagon (by the corollary to *Elements* IV.15) — it follows, from the definition of the right sine, that half the whole sine, or of the chord of the arc of 60 degrees, is the sine of the arc of 30 degrees, which is the half of the arc of 60 degrees.

From this sine, by PROPOS. 3, the sine of the complementary arc will be known — namely the sine EH of 60 degrees — provided that the square of the sine, 5000000, is subtracted from the square of the whole sine, 10000000, and the square root of the remaining number is taken, which is about 8660254.

Next, since the straight line AF, bisecting the arc BC, likewise bisects the straight line BC by the lemma of the definitions, and therefore also at right angles (*Elements* III.3), CL will be the sine of the arc CF of 45 degrees, which we shall find thus. Since in the triangle CAL the angle L is a right angle, and the angle CAL is half a right angle, the angle ACL will also be half a right angle (*Elements* I.32), and hence equal to the angle CAL. Therefore the straight lines AL and CL will be equal (*Elements* I.6). Since, then, the square of the straight line AC is equal to the squares of the straight lines AL and CL together (*Elements* I.47), the square of the whole sine AC will be double the square of the sine CL of 45 degrees. Half the square of the whole sine will therefore be the square of the straight line CL, whose square root will give the sine CL, about 7071068, for the arc of 45 degrees. This will also be found in the following way. Since the square of the straight line BC is equal to the squares of the straight lines AB and AC (*Elements* I.47), and hence to double the square of the whole sine AC, then if the square of the whole sine is doubled, we obtain the square of the straight line BC, whose square root will give the straight line BC, of 14142136 parts, about; and its half, 7071068, will give the sine CL of 45 degrees.

Again, since the straight line AD, bisecting the arc CE, likewise bisects the straight line CE at K by the lemma of the definitions, and therefore at right angles (*Elements* III.3), CK will be the sine of the arc CD of 15 degrees, which we shall find as follows. Since, by PROPOS. 4, the sine CK is the mean proportional between half the whole sine and the versed sine CG (which is indeed obtained if EH, the sine of 60 degrees, is subtracted from the whole sine AC, as we said in the

corollary to PROPOS. 3), CG being the half of the arc CE — for if half the whole sine is multiplied into the known versed sine CG, the square of the straight line CK is produced, since the rectangle contained by half the whole sine and the versed sine CG is equal to the square of the mean proportional CK (Elements VI.17). If, then, the root of the square of the straight line CK is extracted, the sine CK will be obtained, of 2588190 parts, as we taught in the said corollary to PROPOS. 4. This sine we shall also investigate in this other way. Since the square of the straight line EC is equal to the squares of the known straight lines EG and GC (Elements I.47), the square of the straight line EC will become known; and the square root of its square will give the straight line EC as known, whose half will be the known sine CK.

From the sine of 15 degrees, once known, there will also be known, by PROPOS. 3, the sine DI of the complement of the arc CD — that is, the sine of the arc BD of 75 degrees — which will be found to be 9659258 parts. And so the right sines of the arcs containing 15, 30, 45, 60, 75, and 90 degrees have now been found, as appears in this table.

Arcus (G.)	Sinus
15	2588190
30	5000000
45	7071068
60	8660254
75	9659258
90	10000000

Now with the help of these sines we shall proceed to the investigation of others as follows. Since, by the corollary to PROPOS. 4, from the known right sine of any arc the right sine of half that arc is also known, from the sine of the arc of 15 degrees we shall know the sine of the arc of 7 degrees 30 minutes; and from this, the sine of the arc of 3 degrees 45 minutes — an arc which cannot be further bisected without seconds (for its half contains 1 degree 52 minutes 30 seconds), which are neglected in the treatment of sines. Then, since by PROPOS. 3 from the known right sine of any arc the sine of that arc's complement is also made known, from the sine of the arc of 7 degrees 30 minutes we shall know the sine of the arc of 82 degrees 30 minutes; and from this, by the corollary to PROPOS. 4, the sine of the arc of 41 degrees 15 minutes; and from this, the sine of the arc of 48 degrees 45 minutes. Likewise, from the sine of the arc of 3 degrees 45 minutes we shall know, by PROPOS. 3, the sine of the arc of 86 degrees 15 minutes. And if the

other arcs whose sines have been found are likewise bisected, and their halves again bisected, and so on in succession, until arcs are reached whose minutes are odd in number and so cannot be further divided in two without seconds; and if the complements of these halves are taken, which again are to be continually bisected in the same way until arcs of odd minutes are reached; and the complements of the halves are taken, and so on — then, by the corollary to PROPOS. 4 and by PROPOS. 3, we shall know the sines of all these halves and complements. These sines, together with the six first found, will constitute the sines of 24 arcs exceeding one another in order by 3 degrees 45 minutes, as you see in this table.

[Margin: *Computation of the sines of arcs exceeding one another by 3 degrees 45 minutes.*]

Arcus (G M)	Sinus	Arcus (G M)	Sinus	Arcus (G M)	Sinus	Arcus (G M)	Sinus
3 45	65403 1	26 15	44228 87	48 45	75183 98	71 15	946930 1
7 30	13052 62	30 0	50000 00	52 30	79335 33	75 0	965925 8
11 15	19509 03	33 45	55557 02	56 15	83146 96	78 45	980785 3
15 0	25881 90	37 30	60876 14	60 0	86602 54	82 30	991444 9
18 45	32143 95	41 15	65934 58	63 45	89687 27	86 15	997858 9
22 30	38268 34	45 0	70710 68	67 30	92387 95	90 0	100000 00

[Margin: *Computation of the sine of the arc of 36 degrees.*]

After this, by means of what we demonstrated in PROPOS. 2 concerning the finding of the side of a decagon and of an equilateral pentagon in the same circle, we shall seek the sine of the arc of 36 degrees in this way. Let the figure of Proposition 2 be repeated, where it was demonstrated that DF is the side of a decagon and BF the side of an equilateral pentagon. And since the squares of the known straight lines BD and DE (for BD is the whole sine and DE its half) are equal to the square of the straight line EB (Elements I.47), the square of the straight line EB will be known, and therefore the straight line EB itself — that is,

the straight line EF, which is equal to it — will be known, namely about 11180339 parts. If from this the straight line ED, half the whole sine of 5000000 parts, is subtracted, the straight line DF will become known, of 6180339 parts; and if the square of DF is added to the square of the whole sine BD, the sum of the squares described on the straight lines DF and BD will become known — and hence also the square of the straight line BF, which is equal to those two (Elements I.47), will be known; whose square root will give the straight line BF as known. And since BF subtends 72 degrees in the circle ABC, being a fifth part of the circumference — because it is the side of a pentagon — it will be the known chord of the arc of 72 degrees; and its half, of 5877852 parts, will give the right sine of the arc of 36 degrees, which is half of it.

Diagram. A semicircle standing on the horizontal diameter AC, the figure repeated from PROPOS. 2. Read on the leaf, the labelled points are:

- B: apex, at the top of the arc
- A: left end of the diameter
- F: on the diameter, right of A
- D: on the diameter, near the center
- E: on the diameter, right of D
- C: right end of the diameter

Straight lines are drawn from B down to A, F, D, E, and C.

[Margin: *Computation of the sines of arcs exceeding one another by 2 degrees 15 minutes, and of certain others.*]

Furthermore, from this sine of the arc of 36 degrees we shall find, by PROPOS. 3, the sine of its complement — that is, the sine of the arc of 54 degrees. And if we bisect these two arcs, and again bisect their halves, and so on in succession, until we come to arcs whose minutes are odd in number and can no longer be divided, we shall likewise find, by the corollary to PROPOS. 4, the sines of these halves, and also, by PROPOS. 3, the sines of their complements. And if we again bisect these complements as far as we can, and take the complements of these halves, which again we bisect, and so on, we shall find in the same way the sines of all the arcs contained in this table.

Arcus (G M)	Sinus	Arcus (G M)	Sinus	Arcus (G M)	Sinus	Arcus (G M)	Sinus
36 0	58778 52	2 15	39259 8	40 30	64944 80	15 45	27144 05

Arcus (G M)	Sinus	Arcus (G M)	Sinus	Arcus (G M)	Sinus	Arcus (G M)	Sinus
54 0	80901 70	87 45	99922 90	49 30	76040 60	74 15	96245 52
18 0	30901 70	27 0	45399 05	20 15	34611 71	38 15	61909 40
72 0	95105 65	63 0	89100 65	69 45	93819 13	51 45	78531 69
9 0	15643 45	13 30	23344 54	42 45	67880 07	24 45	41865 97
81 0	98768 83	76 30	97236 99	47 15	73432 25	65 15	90814 32
4 30	78459 1	6 45	11753 74	31 30	52249 86	29 15	48862 12
85 30	99691 73	83 15	99306 85	58 30	85264 02	60 45	87249 60

And if we now arrange in order these sines together with those found up to this point, we shall have the right sines of all the arcs exceeding one another in order by 2 degrees 15 minutes, beginning from the arc of 2 degrees 15 minutes up to the arc of 87 degrees 45 minutes inclusive; and in addition the sines of 17 other arcs which do not keep to this increment. As appears in this table.

Arcus (G M)	Sinus	Arcus (G M)	Sinus	Arcus (G M)	Sinus	Arcus (G M)	Sinus
2 15	39259 8	33 45	55557 02	65 15	90814 32	18 45	321439 5
4 30	78459 1	36 0	58778 52	67 30	92387 95	26 15	442288 7
6 45	11753 74	38 15	61909 40	69 45	93819 13	30 0	500000 0
9 0	15643 45	40 30	64944 80	72 0	95105 65	37 30	608761 4
11 15	19509 03	42 45	67880 07	74 15	96245 52	41 15	659345 8
13 30	23344 54	45 0	70710 68	76 30	97236 99	48 45	751839 8

Arcus (G M)	Sinus	Arcus (G M)	Sinus	Arcus (G M)	Sinus	Arcus (G M)	Sinus
15 45	27144 05	47 15	73432 25	78 45	98078 53	52 30	793353 3
18 0	30901 70	49 30	76040 60	81 0	98768 83	60 0	866025 4
20 15	34611 71	51 45	78531 69	83 15	99306 85	63 45	896872 7
22 30	38268 34	54 0	80901 70	85 30	99691 73	71 15	946930 1
24 45	41865 97	56 15	83146 96	87 45	99922 90	75 0	965925 0
27 0	45399 05	58 30	85264 02	3 45	65403 1	82 30	991444 9
29 15	48862 12	60 45	87249 60	7 30	13052 62	86 15	997858 9
31 30	52249 86	63 0	89100 65	15 0	25881 90	90 0	100000 00

From this it is manifest that the sine of 54 degrees — that is, 8090170 — is composed exactly of the sines of 18 and 30 degrees — that is, of 3090170 and 5000000. And since the sine of 30 degrees is equal to half the whole sine, as was shown above, it is clear that the sine of 54 degrees is composed precisely of half the whole sine and the right sine of 18 degrees. This is what we demonstrated in PROPOS. 6.

Now we shall investigate the sine of the arc of 12°, using the sines of 30° and 54°, which have by now been determined. [Margin: Computation of the sine of the arc of 12°.] In the quadrant ABC let the arc BD be 30° and BE 54°, so that the arc DE, their difference, is 24°. Draw DF and EG perpendicular to AB; these will be the right sines of the arcs BD and BE. Likewise draw the lines DH and EI perpendicular to AC; these will be the right sines of the arcs CD and CE — that is, 60° and 36° — which are the complements of the arcs BD and BE. Let the line DH cut the line EG at K. Now since the sine DF (that is, KG, which equals it) and EG are known, their difference EK is also known. [Margin: Elements I.34] Similarly, since the sine EI (that is, KH, which equals it) and DH are known — by Prop. 3, being the sines of the complements of the arcs BE and BD — [Margin: Elements I.34] their difference DK is likewise known. Hence the two squares of the known

lines EK and DK are known; [Margin: Elements I.47] and since together they equal the square of the line DE, the square of DE is known too, and therefore the line DE itself is known — namely the chord of the arc of 24° . From this its half is known as well, that is, the right sine of the arc of 12° , which contains 2079117 particles.

Diagram. A quadrant ABC with center A at the lower right, the vertical radius AB rising on the right, and the arc BC sweeping from B at the top to C at the lower left. Two radii to points on the arc, DF and EG, are dropped perpendicular to AB, and DH, EI perpendicular to the horizontal radius AC (the bottom edge, C–I–H–A). The perpendiculars DH and EG cross at K.

- B (top right): endpoint of the vertical radius, top of the arc
- A (bottom right): center of the quadrant
- C (bottom left): endpoint of the horizontal radius / arc
- D (on the arc, upper middle): endpoint of arc BD (30°)
- E (on the arc, left of D): endpoint of arc BE (54°)
- F (on radius AB, below B): foot of perpendicular from D
- G (on radius AB, below F): foot of perpendicular from E
- H (on bottom radius): foot of perpendicular from D
- I (on bottom radius): foot of perpendicular from E
- K (interior): intersection of DH and EG

By this same method, if the sines of any two arcs whatsoever are known, the sine of half their difference will also be found. For example, if the sines of the arcs BD and BE — whatever they may be, even if they do not amount to 30° and 54° — are known, then the sines of their complements CD and CE are also known by Prop. 3; and so, as we have just shown, the chord DE will become known, and consequently its half as well, namely the sine of half the arc DE. [Margin: From the known sines of two arcs, the sine of half their difference likewise becomes known.]

Moreover, starting from the sine of the arc of 12° — if we bisect this arc repeatedly, as far as we can, and take the complements of the halves, together with the sine of the complement of the arc of 12° (which we again bisect repeatedly, and so on) — we shall find, by the corollary of Prop. 4 and by Prop. 3, the sines of all these arcs, which are contained in the following table. If we arrange these sines together with the sines of the immediately preceding table in order, we shall have the sines of 120 arcs. [Margin: Computation of the sines of arcs increasing successively by 45 minutes.]

Arc (G°M')	Sine	Arc (G°M')	Sine	Arc (G°M')	Sine	Arc (G°M')	Sine
12°0'	2079 117	42°0'	6691 306	12°45'	2206 974	33°0'	5446 390
78°0'	9781 476	48°0'	7431 448	77°15'	9753 423	57°0'	8386 706
6°0'	1045 285	21°0'	3583 679	35°15'	5771 452	16°30'	2840 153
84°0'	9945 219	69°0'	9335 804	54°45'	8166 416	73°30'	9588 197
3°0'	5233 60	10°30'	1822 355	24°0'	4067 366	8°15'	1434 926
87°0'	9986 295	79°30'	9832 549	66°0'	9135 455	81°45'	9896 514
1°30'	2617 69	5°15'	9150 16	34°30'	5664 062	27°45'	4656 145
88°30'	9996 573	84°45'	9958 049	55°30'	8241 262	62°15'	8849 876
0°45'	1308 96	43°30'	6883 546	17°15'	2965 416	28°30'	4771 588
89°15'	9999 143	46°30'	7253 744	72°45'	9550 199	61°30'	8788 171
39°0'	6293 204	21°45'	3705 574	39°45'	6394 390	14°15'	2461 533
51°0'	7771 460	68°15'	9288 096	50°15'	7688 418	75°45'	9692 309
19°30'	3338 069	44°15'	6977 905	23°15'	3947 439	36°45'	5983 246
70°30'	9426 415	45°45'	7163 019	66°45'	9187 912	53°15'	8012 538
9°45'	1693 495	25°30'	4305 111	32°15'	5336 145	30°45'	5112 931
80°15'	9855 561	64°30'	9025 853	57°45'	8457 278	59°15'	8594 064

which follow one another in continuous order by 45 minutes, the first being the arc of 0°45' and the last the whole quadrant of 90°, such as are the arcs of the

following table. For you will find that the sines of all these arcs have been obtained in the two immediately preceding tables, although they are not arranged there in this order.

Arc (G M)	Arc (G M)	Arc (G M)	Arc (G M)	Arc (G M)	Arc (G M)	Arc (G M)	Arc (G M)	Arc (G M)	Arc (M G)
0 45	9 45	18 45	27 45	36 45	45 45	54 45	63 45	72 45	81 45
1 30	10 30	19 30	28 30	37 30	46 30	55 30	64 30	73 30	82 30
2 15	11 15	20 15	29 15	38 15	47 15	56 15	65 15	74 15	83 15
3 0	12 0	21 0	30 0	39 0	48 0	57 0	66 0	75 0	84 0
3 45	12 45	21 45	30 45	39 45	48 45	57 45	66 45	75 45	84 45
4 30	13 30	22 30	31 30	40 30	49 0	58 30	67 30	76 30	85 30
[?]									
5 15	14 15	23 15	32 15	41 15	50 15	59 15	68 15	77 15	86 15
6 0	15 0	24 0	33 0	42 0	51 0	60 0	69 0	78 0	87 0
6 45	15 45	24 45	33 45	42 45	51 45	60 45	69 45	78 45	87 45
7 30	16 30	25 30	34 30	43 30	52 30	61 30	70 30	79 30	88 30
8 15	17 15	26 15	35 15	44 15	53 15	62 15	71 15	80 15	89 15
9 0	18 0	27 0	36 0	45 0	54 0	63 0	72 0	81 0	90 0

Finally, we shall find the sines of the other arcs as follows. In the quadrant ABC let the arc BD be 1° , the arc BE $1^\circ 30'$, and the arc BH $0^\circ 45'$. Draw the line EK perpendicular to AB; divide the arc BH into three equal parts BF, FG, GH, and the arc DE into two parts DI, IE, so that each small arc contains $15'$. From F, G, H, D, I drop the perpendiculars FL, GM, HN, DO, IP to EK. Then EK will be the right sine of the arc BE, and OK the right sine of

the arc BD, since OK equals the line drawn perpendicular from D to AB, which is indeed the sine of the arc BD. [Margin: Elements I.34] And by the same reasoning NK will be the right sine of the arc BH. Let it then be proposed to find the sine OK of the arc BD, 1° . Since NK, the sine of $0^\circ 45'$, has been found to be 130896, its third part — namely 43632 — will be greater than MN; for by Prop. 1, KL is greater than LM, and LM greater than MN, so that MN is less than a third part of NK, that is, less than 43632. Therefore the same third part of the line NK, namely 43632, will be much greater than NO, since by the same Prop. 1, MN is also greater than NO. Hence, if we add 43632 to 130896 — that is, to NK, the sine of $45'$ — we shall produce the number 174528, which will be greater than OK, the right sine of 1° ,

since to NK we are adding more than the line NO, as has been said. [Margin: Computation of the sine of 1° .]

Diagram. A quadrant ABC with center A at the lower right and vertical radius AB on the right. From B down the arc are marked, in succession, the small division-points F, G, H (dividing arc BH into three equal parts), then D and I (dividing arc DE), down to E. The horizontal line EK runs from E to K near A, and onto it the perpendiculars from the arc-points fall at L, M, N, O, P.

- B (top right): top of the arc
- A (bottom right): center of the quadrant
- C (bottom left): far end of the arc
- F, G, H (on the arc, descending from B): the points trisecting arc BH
- D (on the arc, below H): endpoint of arc BD
- I (on the arc, below D): midpoint of arc DE
- E (on the arc, lowest of the labelled points): endpoint of arc BE
- L, M, N, O, P (on line EK, from K toward E): feet of the perpendiculars from F, G, H, D, I respectively
- K (right end of line EK, near A): foot of the perpendicular from B

Again, since EK, the sine of $1^\circ 30'$, has been found to be 261769, if we subtract from it the sine NK of $45'$, namely 130896, there remains the line EN of 130873 particles, whose third part — namely 43624 approximately — will be less than NO; for by Prop. 1, NO is greater than OP, and OP greater than PE, so that NO is greater than a third part of EN, that is, greater than 43624. Hence, if we add 43624 to 130896 — that is, to NK, the sine of $45'$ — we shall produce the number 174520, which will be less than OK, the right sine of 1° , since to NK we are adding less than the line NO. It is therefore established that the right sine of 1° lies between these two numbers, 174528 and 174520, the former being greater and the latter less. Let us accordingly set it as 174524, exactly midway between those numbers; for this number will not differ appreciably from the true sine of 1° .

From this sine of 1° we shall find, by Prop. 3, the sine of its complement — that is, the sine of the arc of 89° — to be 9998477. We may find it in this way as well. Since the number 174528 is greater than the sine of 1° , as we said, the complement's sine derived from it, by the doctrine of Prop. 3 — the sine of 89° — comes out as 9998476 and a little more, which will necessarily be less than the true sine of the arc of 89° , since for the sine of 1° we took a number greater than the true sine. Again, since the number 174520 is less than the sine of 1° , as we said, the complement's sine derived from it, by the doctrine of Prop. 3 — the sine

of 89° — comes out as 9998477 and a little more, which will necessarily be greater than the true sine of 89° , since for the sine of 1° we took a number less than the true sine. The sine of 89° is therefore established between these two numbers — 9998476 and a little more, and 9998477 and a little more — and so, without appreciable error, we shall set it as 9998477, which is precisely what we found for it from the sine of 1° . From this it is clear that the sine of 1° has been correctly established as 174524 particles, since from it, by Prop. 3, the sine of its complement has been found to contain exactly as many particles as it truly and really ought to contain. [Margin: Computation of the sine of 89° .]

Furthermore, from the sine of the arc of 1° we shall find, by the corollary of Prop. 4, the right sine of the arc of $30'$; and from this, by Prop. 3, the sine of its complement, namely the sine of $89^\circ 30'$. Likewise, from the sine of the arc of $30'$ we shall find, by the corollary of Prop. 4, the sine of the arc of $15'$; and from this the sine of its complement, that is, the sine of $89^\circ 45'$. [Margin: Computation of the sine of $30'$ and the sine of $89^\circ 30'$; likewise the sine of $15'$ and the sine of $89^\circ 45'$.]

Now, just as above we found from the right sines of 30° and 54° the right sine of 12° — which is half the chord of the arc of 24° , by which the said arcs of 30° and 54° differ from each other — so too from the right sines of the arc of $30'$ and the arc of $52^\circ 30'$, once known, we shall find the right sine of 26° , which is half the chord of the arc of 52° by which the said arcs of $30'$ and $52^\circ 30'$ differ from each other, as we showed above. And from the sine of 26° there will also be made known, by Prop. 3, the sine of its complement, namely the sine of 64° . [Margin: Computation of the sine of 26° and of 64° .]

If, then, we bisect repeatedly, as far as we can, the arcs of 26° , 64° , and $89^\circ 30'$, whose sines have been found, and take the complements of the halves — which in turn we divide repeatedly in two, and so on — applying that doctrine by which from the known sines of two arcs the sine of half their difference is found, we shall obtain the sines of 240 arcs. When these are set in order together with the sines of the 120 arcs previously found, they will make up the sines of 360 arcs, following one another in order by 15 minutes. [Margin: Computation of the sines of arcs increasing successively by 15 minutes.]

But since it is laborious and tedious to hunt out so many sines in this way, it will be enough to have found, with such difficulty, those sines of the 120 higher arcs that follow one another in order by 45 minutes. For from these we shall easily find, by the rule of proportions, the sines of the arcs that follow one another in order by 15 minutes; then from these the sines of the arcs that follow one another by 5 minutes; and finally, from these, the sines of the arcs extended through

single minutes — and never in this computation will an appreciable error arise. For when we have set the whole sine a hundred times greater than 100000, it follows that, when the first two figures on the right are discarded (as was said above), sines of the greatest exactness are left with respect to the whole sine of 100000, since the whole error that can occur in this computation lies in the first figure on the right, or at most in the first two. Therefore, when the two first figures are discarded, there will remain the very same sines that would have been found by that earlier geometrical method, with respect to the same whole sine of 100000, if in the computation the whole sine were set a hundred times greater still — namely of 10000000 particles — and two figures were rejected from the sines so found. Experience will teach this. We shall carry out the matter as follows: let those arcs be set in order together with their sines, and to the right of each sine let there be written the difference by which it differs from the preceding sine, as you see done here in five arcs. [Margin: Another computation of the sines of arcs increasing successively by 45 [?] minutes.]

Arc (G°M')	Sine	Difference
0°0'	000000	—
0°45'	130896	130896
1°30'	261769	130873
2°15'	392598	130829
3°0'	523360	130762
3°45'	654031	130671

Then say: if 45 minutes require the difference 130896, to be added to the sine of 0 minutes so as to make the sine of 45 minutes, what difference do 15 minutes demand, to be added to the same sine of 0 minutes so as to make the sine of 15 minutes? You will find that the difference required is 43632, which added to the sine of 0 minutes will constitute the sine of 15 minutes, of 43632 particles. Again say: with the same things posited, what difference do 30 minutes require, to be added to the same sine of 0 minutes so as to make the sine of 30 minutes? You will find the difference 87264, which added to the sine of 0 minutes will make 87264, the sine of 30 minutes.

Likewise say: if 45 minutes — by which the arc of 45 minutes differs from the following arc of 1°30' — require the difference 130873, to be added to the sine of 45 minutes so as to make the sine of 1°30', what difference do 15 minutes demand, to be added to the same sine of 45 minutes so as to make the sine of 60 minutes, that is, the sine of 1°? You will find the difference 43624, which added to

130896, the sine of 45 minutes, makes 174520, the sine of 1° . This, although somewhat less than the one we found by the other method, yet, when the two first figures are discarded — 20 — leaves the sine 1745, exceedingly exact with respect to the whole sine of 100000.

Say further: with the same posited, what difference do 30 minutes demand, to be added to the same sine of 45 minutes so as to make the sine of 75 minutes, that is, the sine of $1^\circ 15'$? You will find the difference 87249, which added to 130896, the sine of 45 minutes, will make 218145, the sine of $1^\circ 15'$. This, although somewhat less than the one the earlier method yields, yet, when the two first figures are discarded — 45 — will leave the sine 2181, exceedingly exact with respect to the whole sine of 100000.

Likewise say: if 45 minutes — by which the arc of $1^\circ 30'$ differs from the following arc of 2°

15 minutes — require the difference 130829, to be added to the sine of $1^\circ 30'$ so as to make the sine of $2^\circ 15'$, what difference do 15 minutes demand, to be added to the same sine of $1^\circ 30'$ so as to make the sine of $1^\circ 45'$? And further, what difference do 30 minutes require, to be added to the same sine of $1^\circ 30'$ so as to make the sine of 2° ? You will find the former difference to be 43610, which added to 261769, the sine of $1^\circ 30'$, will yield 305379, the sine of $1^\circ 45'$. The latter difference you will find to be 87219, which added to 261769, the same sine of $1^\circ 30'$, will compose 348988, the sine of 2° . These two sines, although less than those the earlier method yields, yet, when the two first figures are rejected from each, leave the sines 3053 and 3489, exceedingly exact with respect to the whole sine of 100000.

By this same method one must proceed in the rest, until the sines of all the arcs extended by 15 minutes, up to the arc of 45° , have been found. Beyond this it is not expedient to proceed by this route, since the sines of the complements of those arcs can be investigated more exactly, by Prop. 3, with respect to the whole sine of 10000000, than by the rule of proportions. The reason is that, as we showed above in the corollary of Prop. 1, the differences of the sines toward the end of the quadrant are smaller than near its beginning; and so the differences of the sines change more often near the end of the quadrant than near the beginning. From this it comes about that sines are more correctly found by the rule of proportions near the beginning of the quadrant than near the end, since there one and the same difference serves several sines, whereas here it serves fewer, as has been said. Experience too demonstrates this most plainly. [Margin:

Why sines are found more exactly by the rule of proportions near the beginning of the quadrant than near the end.]

If, then, all the sines of the arcs proceeding by 15 minutes, beginning from the arc of 0 minutes, are set in order together with their differences, we shall find by the same device the sines of all the arcs extended by 5 minutes; and from these, in turn, by the same method, we shall explore the sines of all the arcs proceeding by single minutes. And so we shall complete the table of sines up to the arc of 45° . From the sines of these arcs we shall then elicit, by Prop. 3, the sines of the complements of the same arcs. Thus the whole table of sines will be complete; and accordingly we shall have computed the right sines of all the arcs of the quadrant, following one another in order by one minute, in the particles of the whole sine divided into however many parts. Which was to be done. [Margin: Computation of the sines of arcs by 5 minutes and by single minutes.]

SCHOLIUM

Moreover, that you may see the rationale of the computation of sines by the rule of proportions: in the quadrant ABC let the sines DF, EG of the arcs BD, BE be known, from which it is proposed to elicit the sine HI of the arc BH, which lies between those two arcs. [Margin: Demonstration of the computation of sines by the rule of proportions.] Draw the line DK perpendicular to EG; then the lines LI, KG will be equal to the sine DF, and so HL, EK will be the differences between the sines HI, EG and the sine DF. Now since the arc DE — such as we posit it, namely of 45, or 15, or 5 minutes (for it is only from these intermediate arcs that we seek the sines) — does not differ from a straight line as far as sense can tell, and much less does the arc DH, the triangles DEK, DHL will be in a manner rectilinear and equiangular to each other. [Margin: Elements VI.4] Therefore it will hold that: as DE, the difference between the arc BD and the arc BE, is to EK, the difference between the sine DF and the sine EG, so is DH, the difference between the arc BD and the arc BH, to HL, the difference between the sine DF and the sine HI. Since, then, the first three magnitudes are known — namely the arc DE, by which the arc BE exceeds the arc BD; the line EK, by which the sine EG exceeds the sine DF; and also the arc DH, by which the arc BH, whose sine is sought, exceeds the arc BD — the fourth magnitude will also be known, that is, the line HL, which must be added to the sine DF to make the sine HI. It is therefore established that the sines found by the rule of proportions do not differ appreciably from the true sines, especially when the arc DE, by which the arcs of the known sines differ from one another, is very small, so that it scarcely differs from a straight line.

Diagram. A quadrant ABC with center A at the bottom and vertical radius AB on the right. On the arc, from the top B downward, are D, H, E; from these the sines DF, EG, HI fall on the vertical radius at F, G, I. The line DK is drawn perpendicular to EG (with a small right-angle mark near L), producing the two small nearly-rectilinear triangles used in the proof.

- B (top right): top of the arc
- A (bottom): center of the quadrant
- D (on the arc, upper): endpoint of arc BD
- H (on the arc, below D): endpoint of arc BH (its sine HI is sought)
- E (on the arc, below H): endpoint of arc BE
- F (on radius AB): foot of the sine DF
- I (on radius AB, below F): foot of the sine HI
- G (on radius AB, below I): foot of the sine EG
- K (on line EG): foot of the perpendicular DK
- L (near H, at the right-angle mark): intersection on HI with the perpendicular from D

A great abridgement in this task will also be afforded us by Prop. 8. For from it we shall obtain very many sines from others already found, by mere addition or subtraction. [Margin: A wonderful shortcut for finding very many sines.] For if we add the sine of any arc that is not greater than 30° to the sine of the arc that falls short of 60° by that same arc taken, we shall compose the sine of the arc that exceeds 60° by that same arc taken; because the difference between the sines of these two greater arcs is equal to the sine of that taken arc, which is supposed not greater than 30° , as we demonstrated there. For example, if we add 3461171, the sine of the arc of $20^\circ 15'$, to 6394390, the sine of the arc of $39^\circ 45'$ — which falls short of the arc of 60° by the said arc of $20^\circ 15'$ — we shall produce 9855561, the sine of the arc of $80^\circ 15'$, which exceeds the arc of 60° by the same arc of $20^\circ 15'$. Likewise, if we add 5000000, the sine of the arc of 30° , to the sine 5000000 of the arc of 30° by which the arc of 60° exceeds the said arc of 30° , we shall compose 10000000, the sine of the arc of 90° , which exceeds the arc of 60° by the same arc of 30° taken.

Likewise, if we subtract the sine of any arc that is not greater than 30° from the sine of the arc that exceeds 60° by that same arc taken, there will be left the sine of the arc that falls short of 60° by that same arc taken. For example, if we subtract 3502075, the sine of the arc of $20^\circ 30'$, from 9862856, the sine of the arc of $80^\circ 30'$ — which exceeds the arc of 60° by the said arc of $20^\circ 30'$ — the remainder

will be 6360781, the sine of the arc of $39^{\circ}30'$, which the arc of 60° exceeds by the same arc of $20^{\circ}30'$.

Again, if from the sine of any arc greater than 60° we subtract the sine of the arc that is as much less than 60° as the former is greater, there will be left the sine of the arc by which each of them differs from the arc of 60° . For example, if from 9781476, the sine of the arc of 78° , we take away 6691306, the sine of the arc of 42° , the remainder will be 3090170, the sine of the arc of 18° , by which each of them differs from the arc of 60° . All of which are gathered from the aforesaid Prop. 8.

And so it is enough to find, by the rule of proportions, the sines of all the arcs from the beginning of the quadrant up to the arc of 30° . For if from these the sines of the complements are elicited, and from these found sines the former sines are subtracted, there will remain the sines of all the arcs between the arc of 30° and the arc of 60° . Likewise, if the sines of all the arcs from the arc of 30° up to the end of the quadrant were known, and the sines of all the arcs smaller than the arc of 60° were subtracted from the sines of all the arcs greater than 60° , there would remain the sines of all the arcs from the beginning of the quadrant up to the arc of 30° . Finally, if the sines of all the arcs from the beginning of the quadrant up to the arc of 60° were found, and the sines of all the arcs smaller than 30° were added to the sines of all the arcs greater than 30° , the sines of all the arcs greater than 60° would be composed.

But let us now append the table of sines of all the arcs of the quadrant, extended through single minutes, computed by Johannes Regiomontanus — a table which, however, we have purged in very many places of the errors that had crept in through the carelessness of the printers. [Margin: What is to be done so that the following table of sines...] In this table the whole sine contains 10000000 particles, on which basis all the other sines have been found. If from each the first two figures on the right are rejected (adding, however, a unit if the two figures discarded make up a number greater than 50), there will be left the sine with respect to the whole sine of 100000. If, however, anyone should wish to construct the same table by his own effort, following the precepts already handed down, then, with the whole sine set at 10000000000 [?] particles, he must set the whole sine in the computation at 10000000000 [?] particles. For if from each of the sines found

with the two first figures on the right dropped, as we said, the remaining ones will be sines with respect to the whole sine 10000000 — the very same ones that are set out in the table of Johann Regiomontanus [Regiomontanus]. I say this so that you will not be surprised that not all the sines found by the rule of proportions from the sines of arcs increasing successively by 45 minutes — the whole sine being put at 10000000 — answer exactly, to the last digit, to the sines of this table. For in order that a sine be found exactly with respect to some whole sine, the whole sine must be made a hundred times greater in the calculation, as was said above.

[Margin: Why, when the two first figures are dropped from any sine taken with respect to the whole sine 10000000, the same sine remains with respect to the whole sine 100000 — even though the former was not found exactly.]

It is evident, moreover, that when the two first figures on the right are dropped from each individual sine, exact sines remain with respect to the whole sine 100000, even though those earlier ones were not found exactly. For since the whole sine bears some determinate ratio to any given right sine, it follows that all the parts of that whole sine — however many they be — bear the same ratio to the parts found for this [smaller sine] with respect to those parts of the whole sine, as all the parts of the same whole sine (fewer than the former) bear to the parts of the same whole sine taken in fewer parts. Otherwise the whole sine would not always bear the same ratio to the same sine, but would sometimes be of greater magnitude with respect to it and sometimes of less — which is absurd. Therefore, if the sine of any arc — for example, of the one containing 28 degrees — has been found with respect to the whole sine divided into any number of particles whatever, we shall easily find, by the rule of proportions, the same sine with respect to the same whole sine divided into fewer parts. Let us put it this way. If the whole sine of 10000000 parts gives the sine of the arc of 28 degrees as 4694716 parts, how many parts will the same whole sine of 100000 parts give for the sine of the same arc of 28 degrees? We shall in fact find the sine to be 46947 parts, and a further $\frac{1600000}{10000000}$. But if this fraction is omitted (since it is less than unity), the same sine will contain just 46947 parts — exactly as it is set down in the tables of sines, in which the whole sine contains 100000 parts. Here you see that this is the sine that is left when the two first figures on the right are dropped from it.

[Margin: If two figures are added on the right to any sine taken with respect to the whole sine 100000, one obtains the same sine with respect to the whole sine 10000000.]

The reason for this dropping is as follows. Since that sine of 28 degrees, multiplied by the whole sine 100000 — which multiplication is done by appending five ciphers, thus: 469471600000, as we taught in chapter 4 of our *Arithmetic* — is then divided by the whole sine 10000000 (as the rule of proportions prescribes; it is enough, if from the product-number 469471600000 there be removed as many figures as there are ciphers in the divisor 10000000, that is, seven figures, namely, the first figure, 4694716; and moreover $\frac{1600000}{10000000}$, as is plain from what has been said), from which precisely there is obtained that sine 4694700, such as we found it. But the prior sine [4694716], from which the two figures are dropped, is not omitted altogether with respect to the whole sine 100000; it only [becomes] this 4694700. And it ought indeed to be 46947, and over and above that $\frac{1600000}{10000000}$. Yet although these 16 units are neglected, and there be taken instead a sine of these units with respect to the whole sine $\frac{16}{10000000}$, which with respect to the whole sine 60 may deservedly be neglected — for this small quantity, although it be somewhat greater than $\frac{22}{500}$ [?], is nonetheless much less than $\frac{1}{3850}$ [?], that is, than one Minute with respect to the whole sine 60. (What we said of the whole sines of 10000000 and 100000 parts is to be understood likewise of all others.)

[Margin: In what way larger sines are made smaller, and conversely, whatever number of particles the whole sine be set at.]

For always, from sines taken with respect to a larger whole sine, in order to obtain the sines with respect to a smaller whole sine, as many figures are to be dropped as the number of ciphers by which the larger whole sine exceeds the smaller. And it is demonstrated in the same way: if, when the three first figures on the right are dropped from each sine, the remaining ones will be sines with respect to the whole sine 10000000 — that is, when sines are reckoned with respect to the whole sine 10000000, and to the [sines] found with respect to 10000. three ciphers are added — they become sines with respect to the whole sine 100000000, and so of the rest.

From these things the reason of that operation is plain, which I have often used, both in my *Gnomonics* and in other places: namely, when, of two straight lines, one is known with respect to some measure as though it were the whole sine, and then one of them is again known with respect to another straight line, to find in what way the other is known with respect to this other measure. This is a thing very familiar to astronomers.

[Margin: Given two straight lines with respect to some measure, and then one of them known with respect to another measure, in what way the other may be

known with respect to this other measure — which is most familiar to astronomers.]

Diagram. Near the foot of the page stand six horizontal straight line-segments of differing lengths, each drawn with terminal ticks and labelled at its left end. From top to bottom they are lettered A through F; A is the longest, the others shorter in varying degree, illustrating the proportion of measured lines discussed in the text.

- A: longest segment (given below as of 91354 parts, and again of 80901 parts)
- B: a shorter segment (40673 parts, and again 36019 parts)
- C: a long segment (serving as one "whole sine" measure of 100000 parts)
- D: a long segment (serving as another "whole sine" measure of 100000 parts)
- E: a short segment (equal to one palm)
- F: a segment of several palms (namely five)

For example: two straight lines A and B being known with respect to the straight line C — as though C were the whole sine, containing 100000 particles — line A being of 91354 parts and line B of 40673 parts; then A being found again, with respect to another, larger line D (as though D were the whole sine, likewise embracing 100000 particles), to be of 80901 parts, or of 4 palms with respect to the measure E, which is equal to one palm, or with respect to that said measure E, or F, which contains several palms, namely five — I ask, how many parts, or palms, will the same line B give? For line B will be found to be of 36019 parts, and a little more, or of $1\frac{3566}{4567}$ palms [?]. The reason of this operation does not differ from that above, since the straight line A bears always one and the same, determinate ratio to the straight line B.

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SEQVITVR — There follows:

TABLE OF RIGHT SINES, EXTENDED THROUGH THE INDIVIDUAL MINUTES OF THE QUADRANT, and once computed by Johann Regiomontanus [Regiomontanus], but now examined by me, and in very many places amended and corrected.

[Latin display title, verbatim: *TABVLA SINNVVM RECTORVM PER SINGVLA QVADRANTIS MINVTA EXTENSA, ET à Ioan. Regiomontano quondam supputata, nunc autem per me examinata, & plerisque in locis castigata, atque correcta.*]

(The remainder of the page is blank.)

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TABLE. Degrees of the Quadrant for the right sines of the arcs of the same Quadrant. [Verbatim banner (top): *T A B V L A. Gradus Quadrantis pro sinibus rectis arcuum eiusdem Quadrantis.*]

Two-way sine table. Read top-down with the left-hand minute column (0 → 60) under the degree-headings **0, 1, 2, 3, 4**: this gives the right sine of the arc (whole sine = 10000000, here truncated to the six/seven leading figures — degree 0 shows only the significant leading figures). Read bottom-up with the right-hand minute column (0 → 60, i.e. the labels 60 → 0 as printed top-to-bottom) under the foot-banner degrees **89, 88, 87, 86, 85**: this gives the right sines of the complements. [Left vertical label: *Minuta Graduum Quadrantis pro sinibus rectis arcuum eiusdem Quadrantis.* — Right vertical label: *Minuta Graduum Quadrantis pro sinibus rectis complementorum arcuum eiusdem Quadrantis.*]

Min ↓	0°	1°	2°	3°	4°	Min ↑
0	0000	174524	348995	523360	697565	60
1	2909	177433	351902	526265	700467	59
2	5818	180341	354809	529170	703369	58
3	8727	183250	357716	532075	706270	57
4	11636	186158	360623	534980	709172	56
5	14544	189066	363530	537884	712073	55
6	17453	191975	366437	540789	714975	54
7	20362	194883	369344	543694	717876	53
8	23271	197792	372251	546598	720777	52
9	26180	200700	375158	549503	723678	51
10	29088	203608	378064	552407	726579	50
11	31997	206517	380971	555312	729480	49
12	34906	209425	383878	558216	732381	48
13	37815	212333	386785	561120	735282	47
14	40724	215241	389692	564024	738183	46
15	43632	218149	392598	566928	741084	45
16	46541	221057	395505	569832	743985	44
17	49450	223965	398412	572736	746886	43

Min ↓	0°	1°	2°	3°	4°	Min ↑
18	52359	226873	401318	575640	749787	42
19	55268	229781	404225	578544	752688	41
20	58177	232689	407131	581448	755588	40
21	61086	235597	410038	584352	758489	39
22	63995	238505	412944	587256	761389	38
23	66904	241413	415851	590160	764290	37
24	69813	244321	418757	593064	767190	36
25	72721	247229	421663	595967	770090	35
26	75630	250137	424570	598871	772991	34
27	78539	253045	427476	601775	775891	33
28	81448	255953	430382	604678	778791	32
29	84357	258861	433288	607582	781691	31
30	87265	261769	436194	610485	784591	30
31	90174	264677	439100	613389	787491	29
32	93083	267585	442006	616292	790391	28
33	95992	270493	444912	619196	793291	27
34	98901	273401	447818	622099	796191	26
35	101809	276308	450724	625002	799090	25
36	104718	279216	453630	627905	801990	24
37	107627	282124	456536	630808	804889	23
38	110536	285032	459442	633711	807789	22
39	113445	287940	462348	636614	810688	21
40	116353	290847	465253	639517	813587	20
41	119262	293755	468159	642420	816486	19
42	122171	296663	471065	645323	819385	18
43	125079	299570	473970	648226	822284	17
44	127988	302478	476876	651129	825183	16
45	130896	305385	479781	654031	828082	15
46	133805	308293	482687	656934	830981	14
47	136714	311200	485592	659837	833880	13
48	139622	314108	488498	662739	836778	12

Min ↓	0°	1°	2°	3°	4°	Min ↑
49	142531	317015	491403	665642	839677	11
50	145439	319922	494308	668544	842576	10
51	148348	322830	497214	671447	845474	9
52	151257	325737	500119	674349	848372	8
53	154165	328645	503024	677251	851271	7
54	157074	331552	505929	680153	854169	6
55	159982	334459	508834	683055	857067	5
56	162891	337367	511740	685957	859965	4
57	165799	340274	514645	688859	862863	3
58	168708	343181	517550	691761	865761	2
59	171616	346088	520455	694663	868659	1
60	174524	348995	523360	697565	871557	0

Foot-banner (bottom): *Gradus Quadrantis pro sinibus rectis complementorum arcuum eiusdem Quadrantis.* — complementary degree-labels beneath the columns, left→right: **89** | **88** | **87** | **86** | **85**.

Canon of Sines — Degrees 5°–19° (right sines of arcs of the quadrant)

Three full-page numeric tables from the *Canon sinuum* [Canon of sines]. Each page carries five degree columns. The **top** banner gives the degree (*Gradus*) whose right sine [sinus rectus] is tabulated, read with the left-hand minute column running 0'→60' downward. The **bottom** banner gives the complement degree [complementum], read with the right-hand minute column running 60'→0' downward (heading: *pro sinibus rectis complementorum arcuum eiusdem Quadrantis* — "for the right sines of the complements of the arcs of the same quadrant"). Sine values are given to a radius of 10,000,000 (leading zero of the fraction omitted in print, as is usual). The narrow *Differentia* value printed beside each column is the difference per second [per secundum] of arc; it is listed after each table with the minute at which it changes.

Left margin (vertical), all pages: *Minuta Graduum Quadrantis pro sinibus rectis [arcuum eiusdem Quadrantis]* — "Minutes of the degrees of the quadrant, for the right sines." Right margin (vertical), all pages: *...bus rectis complementorum*

arcuum eiusdem Quadrantis — "...right [sines] of the complements of the arcs of the same quadrant."

Source p. 709 (printed p. 69) — Degrees 5°–9° (complements 84°–80°)

Top banner: 5 | 6 | 7 | 8 | 9 Bottom banner: 84 | 83 | 82 | 81 | 80

'	5°	6°	7°	8°	9°	' (compl.)
0	8715 57	1045285	1218 693	1391 731	1564345	60
1	8744 55	1048178	1221 580	1394 612	1567218	59
2	8773 53	1051071	1224 467	1397 492	1570091	58
3	8802 50	1053964	1227 354	1400 373	1572964	57
4	8831 48	1056857	1230 241	1403 253	1575837	56
5	8860 45	1059749	1233 128	1406 133	1578709	55
6	8889 43	1062642	1236 015	1409 013	1581581	54
7	8918 40	1065534	1238 901	1411 893	1584453	53
8	8947 37	1068426	1241 788	1414 772	1587325	52
9	8976 34	1071318	1244 674	1417 652	1590197	51
1 0	9005 31	1074210	1247 560	1420 531	1593069	50
1 1	9034 28	1077102	1250 446	1423 410	1595941	49
1 2	9063 25	1079994	1253 332	1426 289	1598812	48

'	5°	6°	7°	8°	9°	' (compl.)
1	9092	1082886	1256	1429	1601684	47
3	22		218	168		
1	9121	1085778	1259	1432	1604555	46
4	19		104	047		
1	9150	1088669	1261	1434	1607426	45
5	16		990	926		
1	9179	1091561	1264	1437	1610297	44
6	13		876	805		
1	9208	1094452	1267	1440	1613168	43
7	09		761	684		
1	9237	1097344	1270	1443	1616038	42
8	06		647	562		
1	9266	1100235	1273	1446	1618909	41
9	02		532	441		
2	9294	1103126	1276	1449	1921779 [recte: 1621779?]	40
0	98		417	319		
2	9323	1106017	1279	1452	1624649	39
1	95		302	197		
2	9352	1108908	1282	1455	1627519	38
2	91		187	075		
2	9381	1111799	1285	1457	1630389	37
3	87		072	953		
2	9410	1114690	1287	1460	1633259	36
4	83		957	831		
2	9439	1117580	1290	1463	1636129	35
5	79		841	708		
2	9468	1120471	1293	1466	1638999	34
6	75		726	586		
2	9497	1123361	1296	1469	1641868	33
7	71		610	463		
2	9526	1126252	1299	1472	1644738	32
8	67		495	340		
2	9555	1129142	1302	1475	1647607	31
9	63		378	217		

'	5°	6°	7°	8°	9°	' (compl.)
3	9584	1132032	1305	1478	1650476	30
0	58		262	094		
3	9613	1134922	1308	1480	1653345	29
1	54		146	971		
3	9642	1137812	1311	1483	1656214	28
2	49		030	848		
3	9671	1140702	1313	1486	1659082	27
3	44		914	724		
3	9700	1143592	1316	1489	1661951	26
4	39		798	601		
3	9729	1146482	1319	1492	1664819	25
5	34		681	477		
3	9758	1149372	1322	1495	1667687	24
6	29		564	353		
3	9787	1152261	1325	1498	1670555	23
7	24		447	229		
3	9816	1155151	1328	1501	1673423	22
8	19		330	105		
3	9845	1158040	1331	1503	1676291	21
9	14		213	981		
4	9874	1160929	1334	1506	1679159	20
0	08		096	857		
4	9903	1163818	1336	1509	1682027	19
1	03		979	733		
4	9931	1166707	1339	1512	1684894	18
2	98		862	608		
4	9960	1169596	1342	1515	1687761	17
3	92		744	484		
4	9989	1172485	1345	1518	1690628	16
4	87		627	359		
4	1001	1175374	1348	1521	1693495	15
5	881		509	234		
4	1004	1178263	1351	1524	1696362	14
6	775		392	109		

'	5°	6°	7°	8°	9°	' (compl.)
4	1007	1181151	1354	1526	1699229	13
7	669		274	984		
4	1010	1184040	1357	1529	1702095	12
8	563		156	859		
4	1013	1186928	1360	1532	1704962	11
9	457		038	734		
5	1016	1189816	1362	1535	1707828	10
0	351		920	608		
5	1019	1192704	1365	1538	1710694	9
1	245		802	482		
5	1022	1195592	1368	1541	1713560	8
2	139		683	356		
5	1025	1198480	1371	1544	1716426	7
3	032		564	230		
5	1027	1201368	1374	1547	1719292	6
4	926		446	104		
5	1030	1204255	1377	1549	1722157	5
5	819		327	978		
5	1033	1207143	1380	1552	1725022	4
6	713		208	852		
5	1036	1210031	1383	1555	1727887	3
7	606		089	725		
5	1039	1212198 [recte:	1385	1558	1730752	2
8	499	1212919?]	970	599		
5	1042	1215806	1388	1561	1730617 [recte:	1
9	392		851	472	1733617?]	
6	1045	1218693	1391	1564	1736482	0
0	285		731	345		

Differentiae (per second), as printed beside each column, top→bottom:

- 5°: 48.3 → 48.2 (near foot, printed at 60')
- 6°: 48.2 → 48.1 (printed at 60')
- 7°: 48.1 → 48.0 (printed at 60')
- 8°: 48.0 → 47.9 (printed at 60')

- 9°: 47.9 → 47.8 (mid-column, ≈13') → 47.7 (printed at 60')

Source p. 710 (printed p. 70) — Degrees 10°–14° (complements 79°–75°)

Top banner: **TABVLA** / 10 | 11 | 12 | 13 | 14 Bottom banner: 79 | 78 | 77 | 76
| 75

'	10°	11°	12°	13°	14°	' (compl.)
0	17364 82	19080 90	20791 17	2249511	2419219	60
1	17393 47	19109 45	20819 62	2252345	2422041	59
2	17422 11	19138 00	20848 07	2255179	2424863	58
3	17450 75	19166 55	20876 52	2258013	2427685	57
4	17479 39	19195 10	20904 97	2260847	2430507	56
5	17508 03	19223 65	20933 42	2263680	2433329	55
6	17536 67	19252 20	20961 86	2266513	2436150	54
7	17565 31	19280 74	20990 30	2269346	2438971	53
8	17593 94	19309 28	21018 74	2272179	2441792	52
9	17622 58	19337 82	21047 18	2275012	2444613	51
1 0	17651 21	19366 36	21075 62	2277844	2447434	50
1 1	17679 84	19394 90	21104 05	2280676	2450254	49
1 2	17708 47	19423 44	21132 48	2283508	2453074	48
1 3	17737 10	19451 97	21160 91	2286340	2455894	47

'	10°	11°	12°	13°	14°	' (compl.)
1	17765	19480	21189	2289172	2458714	46
4	63	50	34			
1	17794	19509	21217	2292004	2461533	45
5	35	03	77			
1	17822	19537	21246	2294835	2464352	44
6	98	56	20			
1	17851	19566	21274	2297666	2467171	43
7	60	09	62			
1	17880	19594	21303	2300497	2469990	42
8	22	62	04			
1	17908	19623	21331	2303328	2472809	41
9	84	14	46			
2	17937	19651	21359	2306159	2475628	40
0	46	66	88			
2	17966	19680	21388	2308985	2478446	39
1	08	18	30	[?]		
2	17994	19708	21416	2311819	2481264	38
2	69	70	71			
2	18023	19737	21445	2314649	2484082	37
3	31	22	12			
2	18051	19765	21473	2317479	2486900	36
4	92	74	53			
2	18080	19794	21501	2320309	2489717	35
5	53	25	94			
2	18109	19822	21530	2323138	2492534	34
6	14	76	35			
2	18137	19851	21558	2325967	2495351	33
7	74	27	76			
2	18166	19879	21587	2328796	2498168	32
8	34	78	16			
2	18194	19908	21615	2331625	2590984 [recte: 2500984?]	31
9	95	29	56			
3	18223	19936	21643	2334454	2503800	30
0	55	79	96			

'	10°	11°	12°	13°	14°	' (compl.)
3	18252	19965	21672	2337282	2506616	29
1	15	30	36			
3	18280	19993	21700	2340110	2509431	28
2	75	80	76			
3	18309	20022	21729	2342938	2512248	27
3	35	30	16			
3	18337	20050	21757	2345766	2515064	26
4	95	80	55			
3	18366	20079	21785	2348594	2517879	25
5	54	30	94			
3	18395	20107	21814	2351421	2520694	24
6	13	80	33			
3	18423	20136	21842	2354248	2523509	23
7	72	29	72			
3	18452	20164	21871	2357075	2526324	22
8	31	78	11			
3	18480	20193	21899	2359902	2529138	21
9	90	27	49			
4	18509	20221	21927	2362729	2531952	20
0	49	76	87			
4	18538	20250	21956	2365555	2534766	19
1	08	25	25			
4	18566	20278	21984	2368381	2537580	18
2	66	74	63			
4	18595	20307	22013	2371207	2540393	17
3	24	22	00			
4	18623	20335	22041	2374033	2543206	16
4	82	70	37			
4	18652	20364	22069	2376859	2546019	15
5	40	18	74			
4	18680	20392	22098	2379684	2548832	14
6	98	66	11			
4	18709	20421	22126	2382509	2551645	13
7	56	14	48			

'	10°	11°	12°	13°	14°	' (compl.)
4	18738	20449	22154	2385334	2554458	12
8	13	62	85			
4	18766	20478	22183	2388159	2557270	11
9	70	09	22			
5	18795	20506	22211	2390983	2560082	10
0	27	56	58			
5	18823	20535	22239	2393808	2562894	9
1	84	03	94			
5	18852	20563	22268	2396632	2565706	8
2	41	50	30			
5	18880	20591	22296	2399456	2568517	7
3	98	97	66			
5	18909	20620	22325	2402280	2571328	6
4	54	43	02			
5	18938	20648	22353	2405104	2574139	5
5	10	89	37			
5	18966	20677	22381	2407927	2576950	4
6	66	35	72			
5	18995	20705	22410	2410750	2579760	3
7	22	81	07			
5	19023	20734	22438	2413573	2582570	2
8	78	27	42			
5	19052	20762	22466	2416396	2585380	1
9	34	72	77			
6	19080	20791	22495	2419219	2588190	0
0	90	17	11			

Differentiae (per second), as printed beside each column, top→bottom:

- 10°: 47.7 → 47.6 ($\approx 34'$)
- 11°: 47.6 → 47.5 ($\approx 13'$) → 47.4 ($\approx 47'$)
- 12°: 47.4 → 47.3 (mid-column) → 47.2 (printed at 60')
- 13°: 47.2 → 47.1 ($\approx 25'$) → 47.0 (printed at 60')
- 14°: 47.0 → 46.9 ($\approx 24'$) → 46.8 (near foot)

Source p. 711 (printed p. 71) — Degrees 15°–19° (complements 74°–70°)

Top banner: **SINVVM.** / 15 | 16 | 17 | 18 | 19 Bottom banner: 74 | 73 | 72 | 71
| 70

	15 °	16°	17°	18°	19°	(comp l.)
0	25 88 19 0	2756373	2923717	3090170	3255682	60
1	25 91 00 0	2759169	2926499	3092936	3258532 [recte: 3258432?]	59
2	25 93 80 9	2761965	2929280	3095702	3261182	58
3	25 96 61 8	2764761	2932061	3098468	3263931	57
4	25 99 42 7	2767556	2934842	3101234	3266681	56
5	26 02 23 6	2770351	2937623	3103999	3269430	55
6	26 05 04 5	2773146	2940403	3106764	3272179	54
7	26	2775941	2943183	3109529	3274927	53

15°		16°	17°	18°	19°	(comp l.)
8	07					
	85					
	3					
26		2778735	2945963	3112294	3277675	52
9	10					
	66					
	1					
26		2781529	2948743	3115058	3280423	51
1	13					
	46					
	9					
26		2784323	2951523	3117822	3283171	50
0	16					
	27					
	7					
26		2787117	2954302	3120586	3285918	49
1	19					
	08					
	4					
26		2789911	2957081	3123349	3288665	48
2	21					
	89					
	1					
26		2792704	2959860	3126112	3291412	47
3	24					
	69					
	8					
26		2795497	2962638	3128875	3294159	46
4	27					
	50					
	5					
26		2798290	2965416	3131638	3296906	45
5	30					
	31					

15°		16°	17°	18°	19°	(comp l.)
2						
1	26	2801082	2968194	3134400	3299652	44
6	33					
11						
8						
1	26	2803874	2970972	3137162	3302398	43
7	35					
92						
4						
1	26	2806666	2973750	3139624	3305144	42
8	38			[recte: 3139924?]		
73						
0						
1	26	2809458	2976527	3142686	3307889	41
9	41					
53						
6						
2	26	2812250	2979304	3145448	3310634	40
0	44					
34						
2						
2	26	2815041	2982081	3148209	3313379	39
1	47					
14						
7						
2	26	2817832	2984857	3150970	3316123	38
2	49					
95						
2						
2	26	2820623	2987633	3153731	3318867	37
3	52					
75						
7						
2	26	2823414	2990409	3156491	3321611	36

15°		16°	17°	18°	19°	(comp l.)
4	55					
	56					
	2					
2	26	2826204	2993185	3159251	3324355	35
5	58					
	36					
	6					
2	26	2828994	2995960	3162011	3327098	34
6	61					
	17					
	0					
2	26	2831784	2098735	3164770	3329841	33
7	63		[recte: 2998735?]			
	97					
	4					
2	26	2834574	3001510	3167529	3332585	32
8	66					
	77					
	7					
2	26	2837264	3004284	3170288	3335327	31
9	69	[recte: 2837364?]				
	58					
	0					
3	26	2840153	3007058	3173047	3338069	30
0	72					
	38					
	3					
3	26	2842942	3009832	3175805	3340811	29
1	75					
	18					
	6					
3	26	2845731	3012606	3178563	3343553	28
2	77					
	98					

15°		16°	17°	18°	19°	(comp l.)
9						
3	26	2848520	3015380	3181321	3346294	27
3	80					
79						
2						
3	26	2851308	3018153	3184079	3349035	26
4	83					
59						
5						
3	26	2854096	3020926	3186837	3351776	25
5	86					
39						
7						
3	26	2856884	3023699	3189594	3354516	24
6	89					
19						
9						
3	26	2859672	3026472	3192351	3357256	23
7	92					
00						
1						
3	26	2862459	3029244	3195108	3359996	22
8	94					
80						
2						
3	26	2865246	3032016	3197864	3362736	21
9	97					
60						
3						
4	27	2868033	3034788	3200620	3365475	20
0	00					
40						
4						
4	27	2870819	3037559	3203375	3368214	19

15		16°	17°	18°	19°	(comp l.)
'	°					
1	03					
	20					
	5					
4	27	2873605	3040330	3206130	3370953	18
2	06					
	00					
	5					
4	27	2876391	3043101	3208885	3373691	17
3	08					
	80					
	5					
4	27	2879177	3045872	3211640	3376429	16
4	11					
	60					
	5					
4	27	2881963	3048643	3214395	3379167	15
5	14					
	40					
	5					
4	27	2884748	3051413	3217150	3381905	14
6	17					
	20					
	4					
4	27	2887533	3054183	3219904	3384642	13
7	20					
	00					
	3					
4	27	2890318	3056953	3212658	3387379	12
8	22			[recte: 3222658?]		
	80					
	2					
4	27	2893103	3059723	3225412	3390116	11
9	25					
	60					

15°		16°	17°	18°	19°	(comp l.)
1						
5	27	2895888	3062492	3228165	3392852	10
0	28					
	40					
	0					
5	27	2898672	3065261	3230918	3395588	9
1	31					
	19					
	8					
5	27	2901456	3068030	3233671	3398324	8
2	33					
	99					
	6					
5	27	2904240	3070798	3236423	3401060	7
3	36					
	79					
	4					
5	27	2907023	3073566	3239175	3403795	6
4	39					
	59					
	2					
5	27	2909806	3076334	3241927	3406530	5
5	42					
	38					
	9					
5	27	2912589	3079102	3244679	3409265	4
6	45					
	18					
	6					
5	27	2915371	3081869	3247430	3411999	3
7	47					
	98					
	3					
5	27	2918153	3084636	3250181	3414733	2

	15 °	16°	17°	18°	19°	(comp l.)
8 50						
78						
0						
5 27	2920935	3087403	3252932	3417467	1	
9 53						
57						
7						
6 27	2923717	3090170	3255682	3420201	0	
0 56						
37						
3						

Differentiae (per second), as printed beside each column, top→bottom:

- 15°: 46.8 → 46.7 ($\approx 20'$) → 46.6 ($\approx 45'$)
- 16°: 46.6 → 46.5 ($\approx 12'$) → 46.4 ($\approx 37'$)
- 17°: 46.4 → 46.3 ($\approx 2'$) → 46.2 ($\approx 25'$) → 46.1 ($\approx 49'$)
- 18°: 46.1 → 46.0 ($\approx 11'$) → 45.9 ($\approx 35'$) → 45.8 ($\approx 56'$)
- 19°: 45.8 → 45.7 ($\approx 17'$) → 45.6 ($\approx 40'$)

Canon of Sines (Tabula Sinuum) — degrees 20°–34°

Christoph Clavius [Christophorus Clavius], Opera Mathematica, Tomus I (Mainz, 1611). Full-page numeric sine tables. Radius (sinus totus) = 10,000,000; entries are seven-figure right sines. Each degree-column runs from 0' (top) to 60' (bottom); the left minute index counts 0→60 (direct sines, *sinus recti*), the right minute index counts 60→0 (complements, *sinus recti complementorum*). The 60' value of each column equals the 0' value of the next column (verified — see QA).

Running head across the opening: **TABVLA SINVVM** ("Table of Sines"). Verso pages carry TABVLA, the recto carries SINVVM. Left-margin label (verso, direct): *Minuta Graduum Quadrantis pro sinibus rectis arcuum eiusdem Quadrantis*. Right-margin label (complements): *Minuta Graduum Quadrantis pro sinibus rectis complementorum arcuum eiusdem Quadrantis*.

degrees 20–24

Top banner (verbatim): 20 | 21 | 22 | 23 | 24 Bottom banner (verbatim): 69 | 68 | 67 | 66 | 65 Sub-head: *T A B V L A / Gradus Quadrantis pro sinibus rectis arcuum eiusdem Quadrantis.*

' (L)	20	21	22	23	24	' (R)
0	34202 01	358367 9	3746066	39073 11	4067366	60
1	34229 34	358639 5	3748763	39099 89	4070023	59
2	34256 67	358911 0	3751460	39126 66	4072680	58
3	34284 00	359182 5	3754156	39153 43	4075337	57
4	34311 33	359454 0	3756852	39180 20	4077993	56
5	34338 65	359725 4	3759548	39206 96	4080649	55
6	34365 97	359996 8	3762243	39233 72	4083305	54
7	34393 29	360268 2	3764938	39260 48	4085960	53
8	34420 60	360539 5	3767633	39287 23	4088615	52
9	34447 91	360810 8	3770327	39313 98	4091269	51
10	34475 22	361082 1	3773021	39340 72	4093923	50
11	34502 53	361353 3	3775715	39367 46	4096577	49
12	34529 83	361624 5	3778408	39394 20	4099231	48
13	34557 13	361895 7	3781101	39420 93	4101884	47
14	34584	362166	3783794	39447	4104537	46

' (L)	20	21	22	23	24	' (R)
	42	9		66		
15	34611	362438	3786486	39474	4107189	45
	71	0		39		
16	34639	362709	3789178	39501	4109841	44
	00	1		12		
17	34666	362980	3791870	39527	4112493	43
	29	2		84		
18	34693	363251	3794562	39554	4115144	42
	57	2		56		
19	34720	363522	3797253	39581	4117795	41
	85	2		28		
20	34748	363793	3799944	39607	4110446 [recte:?]	40
	13	2		99		
21	34775	364064	3802635	39634	4123096	39
	40	2		70		
22	34802	364335	3805325	39661	4125746	38
	67	1		40		
23	34829	364606	3808015	39688	4128395	37
	94	0		10		
24	34857	364876	3810704	39714	4131044	36
	21	8		80		
25	34884	365147	3813393	39741	4133693	35
	47	6		49		
26	34911	365418	3816082	39768	4136341	34
	73	4		18		
27	34938	365689	3818771	39794	4138989	33
	99	2		87		
28	34966	365959	3821459	39821	4141637	32
	24	9		55		
29	34993	366230	3824147	39848	4144285	31
	49	6		23		
30	35020	366501	3826834	39874	4146932	30
	75	2		91		
31	35047	366771	3829521	39901	4149579	29

' (L)	20	21	22	23	24	' (R)
	99	8		59		
32	35075	367042	3832208	39928	4152226	28
	23	4		26		
33	35102	367313	3834895	39954	4154872	27
	47	0		93		
34	35129	367583	3837581	39981	4157518	26
	71	5		59		
35	35156	367854	3840267	40008	4160163	25
	94	1		25		
36	35184	368124	3842953	40034	4162808	24
	17	6		91		
37	35211	368395	3845638	40061	4165453	23
	40	1		56		
38	35238	[cell:?]	3848323	40088	4168097	22
	62			21		
39	35265	368935	3851008	40114	4170741	21
	84	9		86		
40	35293	369206	3853692	40141	4173385	20
	06	2		50		
41	35320	369476	3856376	40168	4176028	19
	27	5		14		
42	35347	369746	3859060	40194	4178671	18
	48	8		78		
43	35374	370017	3861743	40221	4181313	17
	69	0		41		
44	35401	370287	3864426	40248	4183955	16
	90	2		04		
45	35429	370557	3867109	40274	4186597	15
	10	2		67		
46	35456	370827	3869791	40301	4189239	14
	30	6		30		
47	35483	371097	3872473	40327	4191880	13
	50	7		92		
48	35510	371367	3875155	40354	4194521	12

' (L)	20	21	22	23	24	' (R)
	70	8		54		
49	35537 89	371637 9	3877837	40381 15	4197162	11
50	35565 08	371908 0	3880518	40407 76	4199802	10
51	35592 27	372178 0	3883199	40434 37	4020442 [recte:?]	9
52	35619 45	372448 0	3885880	40460 97	4205081	8
53	35646 63	372717 9	3888560	40487 57	4207720	7
54	35673 80	372987 8	3891240	40514 16	4210359	6
55	35700 97	373257 7	3893919	40540 75	4212997	5
56	35728 14	373527 5	3890598 [recte:?]	40567 34	4215635	4
57	35755 31	373797 3	3899277	40593 92	4218273	3
58	35782 47	374067 1	3901955	40620 50	4220910	2
59	35809 63	374336 9	3904633	40647 08	4223547	1
60	35836 79	374606 6	3907311	40673 66	4226183	0

Differentiae (per-second differences, printed at block boundaries, in column order 20→24): 20°: 45.5 / 45.4 / 45.3 / 45.3 · 21°: 45.3 / 45.2 / 45.1 / 45.0 · 22°: 44.9 / 44.8 / 44.7 / 44.6 · 23°: 44.6 / 44.5 / 44.4 / 44.3 · 24°: 44.3 / 44.2 / 44.0 / 43.9.

degrees 25–29 (printed top banner 29–25)

Top banner (verbatim, as printed): 29 | 28 | 27 | 26 | 25 Bottom banner (verbatim): 64 | 63 | 62 | 61 | 60 Sub-head: *S I N V V M. / Gradus Quadrantis pro sinibus rectis arcuum eiusdem Quadrantis.*

Note on this page's header: the printed top-banner degree labels run **29, 28, 27, 26, 25** left-to-right, but the columns' actual right-sine content is **sin 25°, 26°, 27°, 28°, 29°** left-to-right (0' heads 4226183 = sin 25° ... 4848096 = sin 29°; 60' of the last column = 5000000 = sin 30°). The bottom banner (64, 63, 62, 61, 60 = 89 – true degree) is internally consistent with the true degrees. The top-banner inversion is a feature/anomaly of the print — see QA. Column headers below are given as printed-banner (= true degree).

' (L)	29 (25°)	28 (26°)	27 (27°)	26 (28°)	25 (29°)	' (R)
0	4226183	438371 2	453990 5	4694716	4848096	60
1	4228819	438632 6	454249 7	4697284	4850640	59
2	4231455	438894 0	454508 8	4699852	4853184	58
3	4234090	439155 4	454767 9	4702419	4855727	57
4	4236725	439416 7	455027 0	4704986	4858270	56
5	4239360	439678 0	455286 0	4707553	4860812	55
6	4241994	439939 2	455545 0	4710119	4863354	54
7	4244628	440200 4	455803 9	4712685	4865895	53
8	4247362 [recte:?]	440461 6	456062 8	4715250	4868436	52
9	4249895	440722 7	456321 6	4717815	4870977	51
10	4252528	440983 8	456580 4	4720380	4873517	50
11	4255161	441244 9	456839 2	4722944	4876057	49
12	4257793	441505 9	457097 9	4725508	4878596	48
13	4260425	441766	457356	4728071	4881135	47

' (L)	29 (25°)	28 (26°)	27 (27°)	26 (28°)	25 (29°)	' (R)
		9	6			
14	4263056	442027	457615	4730634	4883674	46
		8	3			
15	4265687	442288	457873	4733197	4886212	45
		7	9			
16	4268318	442549	458132	4735759	4888750	44
		6	5			
17	4270949	442810	458391	4738321	4891287	43
		4	1			
18	4273579	443071	458649	4740882	4893824	42
		2	6			
19	4276209	443332	458908	4743443	4896361	41
		0	1			
20	4277838	443592	459166	4746004	4898897	40
	[recte:?]	7	5			
21	4281467	443853	459424	4748564	4901433	39
		4	9			
22	4284096	444114	459683	4751124	4903968	38
		0	3			
23	4286724	444374	459941	4753683	4906503	37
		6	6			
24	4289352	444635	460199	4756242	4909037	36
		2	9			
25	4291979	444895	460458	4758801	4911571	35
		7	1			
26	4294606	445156	460716	4761359	4914105	34
		2	3			
27	4297233	445416	460974	4763917	4916638	33
		7	4			
28	4299859	445677	461232	4766474	4919171	32
		1	5			
29	4302485	445937	461490	4769031	4921703	31
		5	6			
30	4305111	446197	461748	4771588	4924235	30

' (L)	29 (25°)	28 (26°)	27 (27°)	26 (28°)	25 (29°)	' (R)
		8	6			
31	4307736	446458	462006	4774144	4926767	29
		1	0			
32	4310361	446718	462264	4776700	4929298	28
		4	6			
33	4312986	446978	462522	4779255	4931829	27
		6	5			
34	4315610	447238	462780	4781810	4934359	26
		8	4			
35	4318234	447499	463038	4784365	4936889	25
		0	2			
36	4320858	447759	463296	4786919	4939418	24
		1	0			
37	4323481	448019	463553	4789473	4941947	23
		2	8			
38	4326104	448279	463811	4792026	4944476	22
		2	5			
39	4328726	448539	464069	4794579	4947004	21
		2	2			
40	4331348	448799	464326	4797132	4949532	20
		2	8			
41	4333970	449059	464584	4799684	4952059	19
		1	4			
42	4336591	449319	464842	4802236	4954686	18
		0	0		[recte:?]	
43	4339212	449578	465099	4804787	4957113	17
		8	5			
44	4341833	449838	465357	4807338	4959639	16
		6	0			
45	4344453	450098	465614	4809888	4962165	15
		4	5			
46	4347073	450358	465871	4812438	4964690	14
		2	9			
47	4349693	450617	466129	4814988	4967215	13

' (L)	29 (25°)	28 (26°)	27 (27°)	26 (28°)	25 (29°)	' (R)
		9	3			
48	4352312	450877	466386	4817537	4969740	12
		6	6			
49	4354931	451137	466643	4820086	4972264	11
		2	9			
50	4357549	451396	466901	4822635	4974788	10
		8	2			
51	4360167	451656	467158	4825183	4977311	9
		3	4			
52	4362785	451915	467415	4827731	4979834	8
		8	6			
53	4365402	452175	467672	4830278	4982356	7
		3	7			
54	4368019	452434	467929	4832825	4984878	6
		7	8			
55	4370635	452694	468186	4835371	4987399	5
		1	9			
56	4373251	452953	468443	4837917	4989920	4
		5	9			
57	4375867	453212	468700	4840462	4992441	3
		8	9			
58	4378482	453472	468957	4843007	4994961	2
		1	8			
59	4381097	453731	469214	4845552	4997481	1
		3	7			
60	4383712	453990	469471	4848096	5000000	0
		5	6	[?]		

Differentiae (per-second, in printed column order 29→25): sin 25°: 43.9 / 43.8 / 43.7 / 43.6 · sin 26°: 43.5 / 43.4 / 43.3 / 43.2 · sin 27°: 43.2 / 43.1 / 43.0 / 42.9 / 42.8 · sin 28°: 43.1 / 43.0 / 42.9 / 42.6 / 42.5 · sin 29°: 42.8 / 42.7 / 42.6 / 42.4 / 42.2 / 42.1 / 42.0.

Foot of page: gathering signature. Printed page number 73.

degrees 30–34

Top banner (verbatim): 30 | 31 | 32 | 33 | 34 Bottom banner (verbatim): 59 | 58 | 57 | 56 | 55 Sub-head: *T A B V L A / Gradus Quadrantis pro sinubus rectis arcuum eiusdem Quadrantis.*

' (L)	30	31	32	33	34	' (R)
0	5000000	51503 81	5299192	54463 90	55919 29	60
1	5002519	51528 74	5301659	54488 29	55943 40	59
2	5005038	51553 67	5304125	54512 68	55967 51	58
3	5007556	51578 59	5306591	54537 07	55991 61	57
4	5010074	51603 51	5309056	54561 45	56015 71	56
5	5012591	51628 43	5311521	54585 83	56039 81	55
6	5015108	51653 34	5313985	54610 20	56063 90	54
7	5017624	51678 25	5316449	54634 56	56087 98	53
8	5020140	51703 15	5318913	54658 92	56112 06	52
9	5022656	51728 05	5321376	54683 28	56136 14	51
10	5025171	51752 94	5323839	54707 63	56160 21	50
11	5027686	51777 83	5326301	54731 98	56184 27	49
12	5030200	51802 71	5328763	54756 32	56208 33	48
13	5032714	51827 59	5331224	54780 66	56232 39	47
14	5035227	51852	5333685	54804	56256	46

' (L)	30	31	32	33	34	' (R)
		46		99	44	
15	5037740	51877	5356145 [recte:?)	54829	56280	45
		33		32	49	
16	5040253	51902	5338605	54853	56304	44
		20		64	53	
17	5042765	51927	5341065	54877	56328	43
		06		96	57	
18	5045277	51951	5343524	54902	56352	42
		92		28	60	
19	5047788	51976	5345983	54926	56376	41
		77		59	63	
20	5050299	52001	5348441	54950	56400	40
		62		90	66	
21	5052809	52026	5350898	54975	56424	39
		46		20	68	
22	5055319	52051	5353355	54999	56448	38
		30		50	69	
23	5057829	52076	5355812	55023	56472	37
		14		79	70	
24	5060338	52100	5358268	55048	56496	36
		97		08	70	
25	5062847	52125	5360724	55072	56520	35
		80		36	70	
26	5065355	52150	5363179	55096	56544	34
		62		64	69	
27	5067863	52175	5365634	55120	56568	33
		44		91	68	
28	5070370	52200	5368088	55145	56592	32
		25		18	66	
29	5072877	52225	5370542	55169	56616	31
		06		44	64	
30	5075384	52249	5372996	55193	56640	30
		86		70	62	
31	5077890	52274	5375449	55217	56664	29

' (L)	30	31	32	33	34	' (R)
		66		95	59	
32	5080396	52299	5377902	55242	56688	28
		46		20	56	
33	5082901	52324	5380354	55266	56712	27
		25		45	52	
34	5085406	52349	5382806	55290	56736	26
		04		69	48	
35	5087911	52373	5385258	55314	56760	25
		82		93	43	
36	5090415	52398	5387709	55339	56784	24
		60		16	38	
37	5092619 [recte:?)	52423	5390159	55363	56808	23
		37		38	32	
38	5095422	52448	5392609	55387	56832	22
		14		60	26	
39	5097925	52472	5395058	55411	56856	21
		90		82	19	
40	5100427	52497	5397507	55436	56880	20
		66		03	12	
41	5102929	52522	5399955	55460	56904	19
		41		24	04	
42	5105430	52547	5402403	55484	56927	18
		16		44	96	
43	5107931	52571	5404851	55508	56951	17
		91		64	87	
44	5110431	52596	5407298	55532	56975	16
		65		83	78	
45	5112931	52621	5409745	55557	56999	15
		39		02	68	
46	5115431	52646	5412191	55581	57023	14
		12		20	58	
47	5117930	52670	5414637	55605	57047	13
		85		38	47	
48	5120429	52695	5417082	55629	57071	12

' (L)	30	31	32	33	34	' (R)
		57		56	36	
49	5122927	52720	5419527	55653	57095	11
		29		73	24	
50	5125425	52745	5421972	55677	57119	10
		01		90	12	
51	5127922	52769	5424416	55702	57142	9
		72		06	99	
52	5130419	52794	5426859	55726	57166	8
		43		22	86	
53	5132919	52819	5429302	55750	57190	7
		13		37	72	
54	5135412	52843	5431745	55774	57214	6
		83		52	58	
55	5137908	52868	5434187	55798	57238	5
		52		66	44	
56	5140403	52893	5436629	55822	57262	4
		21		80	29	
57	5142898	52917	5439070	55846	57286	3
		89		93	13	
58	5145393	52942	5441510	55871	57309	2
		57		06	97	
59	5147887	52967	5443950	55895	57333	1
		25		18	81	
60	5150381	52991	5446390	55919	57357	0
		92		29	64	

Differentiae (per-second, in column order 30→34): 30°: 42.0 / 41.9 / 41.8 / 41.7 / 41.6
 · 31°: 41.5 / 41.4 / 41.3 / 41.2 / 41.1 · 32°: 41.1 / 41.0 / 40.9 / 40.8 / 40.7 · 33°: 40.6 /
 40.5 / 40.4 / 40.3 / 40.2 · 34°: 40.2 / 40.1 / 40.0 / 39.9 / 39.8 / 39.7.

Printed page number 74. Column-34 60' entry 5735764 = sin 35° (0' head of the next column, continued in the following opening).

Canon of Sines (*Sinus recti*) — degrees 35°–49°

Opera Mathematica, Tomus I (Clavius, Mainz 1611), trigonometric tables. Global (printed folios 75–77).

Structure of each page (a standard Clavius/Rheticus double-entry sine canon): a top degree banner names five whole degrees (ascending left→right); the leftmost column gives the minute argument 0'→60' (top→bottom) for the **direct sine** reading; the rightmost column gives the minute argument 60'→0' for reading the same columns as **complement sines** against the bottom banner. Each seven-digit entry is the *sinus rectus* to radius 10,000,000. To the right of each *Sinus* column stands its **Differentia** column — the proportional part per second of arc (i.e. the per-minute first difference ÷ 60), printed sparsely and decreasing toward 90°. The degree banner is printed ascending and matches the true column identity (no mirrored/stale banner on these three pages).

Continuity: the batch continues the canon from the prior opening, whose final value was $\sin 35^\circ 0' = 5735764$ — reproduced here as the first cell (35°, 0'). All internal and cross-page column seams were checked (each column's 60' value = the next column's 0' value).

Folio 75 — degrees 35°–39°

Gradus Quadrantis pro sinibus rectis arcuum eiusdem Quadrantis. (*The degrees of the quadrant, for the right sines of the arcs of the same quadrant.*)

0'→	35	36	37	38	39	←60'
0	5735764	587785 2	601815 0	615661 5	629320 4	60
1	5738147	588020 5	602047 3	615890 7	629546 4	59
2	5740529	588255 8	602279 6	616119 8	629772 4	58
3	5742911	588491 0	602511 8	616348 9	629998 3	57
4	5745292	588726 2	602743 9	616578 0	630224 2	56
5	5747672	588961 3	602976 0	616807 0	630450 1	55

0'→	35	36	37	38	39	←60'
6	5750052	589196 4	603208 0	617035 9	630675 9	54
7	5752432	589431 4	603440 0	617264 8	630901 6	53
8	5754811	589666 4	603671 9	617493 6	631127 3	52
9	5757190	589901 3	603903 8	617722 4	631352 9	51
10	5759568	590136 1	604135 7	617951 2	631578 4	50
11	5761946	590370 9	604367 5	618179 9	631803 9	49
12	5764323	590605 6	604599 2	618408 5	632029 3	48
13	5766700	590840 3	604830 9	618637 1	632254 7	47
14	5769076	591075 0	605062 5	618865 6	632480 0	46
15	5771452	591309 6	605294 0	619094 0	632705 3	45
16	5773827	591544 2	605525 5	619322 4	632930 5	44
17	5776202	591778 7	605757 0	619550 8	633155 7	43
18	5778576	592013 2	605988 4	619779 1	633380 8	42
19	5780950	592247 6	606219 8	620007 4	633605 9	41
20	5783324	592482 0	606451 1	620235 6	633831 0	40
21	5785697	592716 3	606682 4	620463 8	634056 0	39
22	5788069	592950 5	606913 6	620691 9	634280 9	38

0'→	35	36	37	38	39	←60'
23	5790441	593184 7	607144 8	620919 9	634505 8	37
24	5792812	593418 9	607375 9	621147 9	634730 6	36
25	5795183	593653 0	607606 9	621375 8	634955 3	35
26	5797553	593887 1	607837 9	621603 7	635180 0	34
27	5799923	594121 1	608068 8	621831 5	635404 6	33
28	5802292	594355 1	608299 7	622059 3	635629 2	32
29	5804661	594589 0	608530 6	622287 0	635853 7	31
30	5807030	594822 8	608761 4	622514 6	636078 2	30
31	5809398	595056 6	608992 2	622742 2	636302 6	29
32	5811766	595290 4	609222 9	622969 8	636527 0	28
33	5814133	595524 1	609453 6	623197 3	636751 3	27
34	5816499	595757 8	609684 2	623424 8	636975 6	26
35	5818865	595991 4	609914 7	623652 2	637199 9	25
36	5821230	596225 0	610145 2	623879 6	637424 1	24
37	5823595	596458 5	610375 6	624106 9	637648 2	23
38	5825959	596691 9	610606 0	624334 2	637872 2	22
39	5828323	596925 3	610836 4	624561 4	638096 2	21

0'→	35	36	37	38	39	←60'
40	5830687	597158 6	611066 7	624788 5	638320 1	20
41	5833050	597391 9	611297 0	625015 6	638544 0	19
42	5835412	597625 1	611527 2	625242 6	638767 8	18
43	5837774	597858 3	611757 3	625469 6	638991 6	17
44	5840136	598091 5	611987 3	625696 6	639215 3	16
45	5842497	598324 6	612217 3	625923 5	639439 0	15
46	5844858	598557 7	612447 3	626159 3	639662 6	14
47	5847218	598790 7	612677 2	626377 1	639886 2	13
48	5849578	599023 7	612907 1	626603 8	640109 7	12
49	5851937	599256 6	613136 9	626830 5	640333 2	11
50	5844195 [recte:?)	599489 4	613366 7	627057 2	640556 6	10
51	5856653	599722 2	613596 4	627283 8	640779 9	9
52	5859010	599954 9	613826 1	627510 3	641003 2	8
53	5861367	600187 6	614055 7	627736 8	641226 4	7
54	5863724	600420 2	614285 3	627963 2	641449 6	6
55	5866080	600652 8	614514 8	628189 5	641672 8	5
56	5868436	600885 3	614744 2	628415 8	641895 9	4

0'→	35	36	37	38	39	←60'
57	5870791	601117 8	614973 6	628642 0	642118 9	3
58	5873145	601350 2	615203 0	628868 2	642341 9	2
59	5875499	601582 6	615432 3	629094 3	642564 8	1
60	5877852	601815 0	615661 5	629320 4	642787 6	0

[bottom banner] 54 | 53 | 52 | 51 | 50 Gradus Quadrantis pro sinubus rectis complementorum arcuum eiusdē Quadrantis. (*The degrees of the quadrant, for the right sines of the complements of the arcs of the same quadrant.*)

Differentiae (proportional parts, per second; printed sparsely, top→bottom, positions approximate):

- 35°: 39.7 (0'), 39.6 (≈7'), 39.5 (≈20'), 39.4 (33'), 39.3 (≈45'), 39.2 (≈57')
- 36°: 39.2 (0'), 39.1 (≈8'), 39.0 (≈21'), 38.9 (≈33'), 38.8 (≈43'), 38.7 (≈56')
- 37°: 38.7 (0'), 38.6 (≈6'), 38.5 (≈20'), 38.4 (≈31'), 38.3 (≈43'), 38.2 (≈56')
- 38°: 38.2 (0'), 38.1 (≈6'), 38.0 (≈18'), 37.9 (29'), 37.8 (≈40'), 37.7 (≈51')
- 39°: 37.7 (0'), 37.6 (≈3'), 37.5 (≈14'), 37.4 (≈25'), 37.3 (≈38'), 37.2 (≈49'), 37.1 (≈59')

Folio 76 — degrees 40°–44°

Gradus Quadrantis pro sinubus rectis arcuum eiusdem Quadrantis.

0'→	40	41	42	43	44	←60'
0	6427876	6560 590	6691 306	6819984	6946584	60
1	6430104	6562 785	6693 468	6822111	6948676	59
2	6432331	6564 979	6695 629	6824237	6950767	58
3	6434558	6567 173	6697 789	6826363	6952858	57
4	6436785	6569 367	6699 949	6828489	6954949	56

0'→	40	41	42	43	44	←60'
5	6439011	6571 560	6702 108	6830614	6957039	55
6	6441236	6573 753	6704 267	6832738	6959128	54
7	6443461	6575 945	6706 425	6834861	6961216	53
8	6445685	6578 136	6708 582	6836984	6963304	52
9	6447909	6580 326	6710 739	6839107	6965392	51
10	6450132	6582 516	6712 895	6841229	6967479	50
11	6452355	6584 705	6715 051	6843350	6969565	49
12	6454577	6586 894	6717 206	6845471	6971651	48
13	6456799	6589 082	6719 361	6847591	6973736	47
14	6459020	6591 270	6721 515	6849711	6975821	46
15	6461240	6593 458	6723 668	6851830	6977905	45
16	6463460	6595 645	6725 821	6853949	6979988	44
17	6465679	6597 831	6727 973	6856067	6982071	43
18	6467898	6600 016	6730 125	6858184	6984153	42
19	6470116	6602 201	6732 276	6860301	6986235	41
20	6472333	6604 386	6734 427	6862417	6988316	40
21	6474550	6606 570	6736 577	6864533	6990396	39

0'→	40	41	42	43	44	←60'
22	6476766	6608 753	6738 726	6866648	6992476	38
23	6478982	6610 936	6740 875	6868762	6994555	37
24	6481198	6613 118	6743 024	6870876	6996634	36
25	6483413	6615 300	6745 172	6872989	6998712	35
26	6485628	6617 481	6747 319	6875102	7000789	34
27	6487842	6619 661	6749 465	6877214	7002866	33
28	6490055	6621 841	6751 611	6879325	7004942	32
29	6492268	6624 021	6753 757	6881436	7007018	31
30	6494480	6626 200	6755 902	6883546	7009093	30
31	6496692	6628 379	6758 047	6885656	7011167	29
32	6498903	6630 557	6760 191	6887765	7013241	28
33	6501114	6632 734	6762 334	6889874	7015314	27
34	6503324	6634 911	6764 477	6891982	7017387	26
35	6505533	6637 087	6766 619	6894089	7019459	25
36	6507742	6639 263	6768 760	6896196	7021530	24
37	6596950 [recte:?]	6641 438	6770 901	6898302	7023601	23
38	6512158	6643 612	6773 041	6900408	7025671	22

0'→	40	41	42	43	44	←60'
39	6514365	6645 786	6775 181	6902513	7027741	21
40	6516572	6647 959	6777 320	6904617	7029810	20
41	6518778	6650 132	6779 459	6906721	7031879	19
42	6520984	6652 304	6781 597	6908824	7033947	18
43	6523189	6654 476	6783 734	6910927	7036014	17
44	6525394	6656 647	6785 871	6913029	7038081	16
45	6527598	6658 817	6788 007	6915131	7040147	15
46	6529801	6660 987	6790 143	6917232	7042213	14
47	6532004	6663 156	6792 258	6919332	7044278	13
48	6534206	6665 325	6794 413	6921432	7046342	12
49	6536408	6667 493	6796 547	6923531	7048406	11
50	6538609	6669 661	6798 681	6925630	7050469	10
51	6540809	6671 828	6800 814	6927728	7052432 [recte:?]	9
52	6543009	6673 994	6802 946	6929725 [recte:?]	7054594	8
53	6545208	6676 160	6805 078	6931922	7056655	7
54	6547407	6678 326	6807 209	6934018	7058716	6
55	6549606	6680 491	6809 340	6936114	7060776	5

0'→	40	41	42	43	44	←60'
56	6551804	6682 655	6811 470	6938209	7062836	4
57	6554001	6684 818	6813 599	6940303	7064895	3
58	6556198	6686 981	6815 728	6942397	7066953	2
59	6558394	6689 144	6817 856	6944491	7069011	1
60	6560590	6691 306	6819 984	6946584	7071068	0

[bottom banner] 49 | 48 | 47 | 46 | 45 Gradus Quadrantis pro sinubus rectis complementorum arcuum eiusdē Quadrantis.

Differentiae (proportional parts, per second; top→bottom, positions approximate):

- 40°: 37.1 (0'), 37.0 (≈10'), 36.9 (≈20'), 36.8 (≈31'), 36.7 (≈43'), 36.6 (≈53')
- 41°: 36.6 (0'), 36.5 (≈5'), 36.4 (≈15'), 36.3 (≈25'), 36.2 (≈36'), 36.1 (≈47'), 36.0 (≈57')
- 42°: 36.0 (0'), 35.9 (≈8'), 35.8 (≈19'), 35.7 (≈30'), 35.6 (≈40'), 35.5 (≈51')
- 43°: 35.4 (≈1'), 35.3 (≈11'), 35.2 (≈21'), 35.1 (≈32'), 35.0 (≈42'), 34.9 (≈52')
- 44°: 34.8 (0'), 34.8 (≈2'), 34.7 (≈14'), 34.6 (≈23'), 34.5 (≈34'), 34.4 (≈44'), 34.3 (≈54')

Folio 77 — degrees 45°–49°

Gradus Quadrantis pro sinubus rectis arcuum eiusdem Quadrantis.

0'→	45	46	47	48	49	←60'
0	707106 8	719339 8	731353 7	7431448	754709 6	60
1	707312 5	719541 8	731552 1	7433394	754900 4	59
2	707518 1	719743 8	731750 4	7435339	755091 1	58
3	707723 6	719945 7	731948 6	7437284	755281 8	57

0'→	45	46	47	48	49	←60'
4	707929 1	720147 6	732146 8	7439229	755472 4	56
5	708134 5	720349 4	732344 9	7441173	755663 0	55
6	708339 9	720551 1	732542 9	7443116	755853 5	54
7	708545 2	720752 7	732740 9	7445058	756043 9	53
8	708750 4	720954 3	732938 8	7447000	756234 3	52
9	708955 6	721155 9	733136 7	7448941	756424 6	51
10	709160 7	721357 4	733334 5	7450882	756614 8	50
11	709365 8	721558 8	733532 2	7452822	756805 0	49
12	709570 8	721760 1	733729 8	7454761	756995 1	48
13	709775 7	721961 4	733927 4	7456699	757185 1	47
14	709980 6	722162 7	734125 0	7458637	757375 1	46
15	710185 4	722363 9	734322 5	7460574	757565 0	45
16	710390 2	722565 1	734519 9	7462511	757754 8	44
17	710594 9	722766 2	734717 3	7464447	757944 6	43
18	710799 5	722967 2	734914 6	7466382	758134 3	42
19	711004 1	723168 1	735111 8	7468317	758324 0	41
20	711208 6	723368 9	735309 0	7470251	758513 6	40

0'→	45	46	47	48	49	←60'
21	711413 1	723569 7	735506 1	7472184	758703 1	39
22	711617 5	723770 4	735703 1	7474117	758892 5	38
23	711821 8	723971 1	735900 1	7476249 [recte:?]	759081 9	37
24	712026 1	724171 8	736097 0	7477981	759271 3	36
25	712230 3	724372 4	736293 9	7479912	759460 6	35
26	712434 4	724572 9	736490 7	7481842	759649 8	34
27	712638 5	724773 3	736687 4	7483771	759838 9	33
28	712842 5	724973 7	736884 1	7485700	760028 0	32
29	713046 5	725174 1	737080 7	7487629	760217 0	31
30	713250 4	725374 4	737277 3	7489557	760406 0	30
31	713454 3	725574 6	737473 8	7491484	760594 9	29
32	713658 1	725774 7	737670 2	7493410	760783 7	28
33	713861 8	725974 8	737866 6	7495336	760972 5	27
34	714065 5	726174 9	738062 9	7497262	761161 2	26
35	714269 1	726374 9	738259 2	7499187	761349 8	25
36	714472 7	726574 8	738455 4	7501111	761538 4	24
37	714676 2	726774 6	738651 5	7503034	761726 9	23

0'→	45	46	47	48	49	←60'
38	714879 6	726974 4	738847 5	7504957	761915 3	22
39	715083 0	727174 1	739043 5	7506879	762103 7	21
40	715286 3	727373 7	739239 4	7508801	762292 0	20
41	715489 5	727573 3	739435 3	7510722	762480 2	19
42	715692 7	727772 8	739631 1	7512642	762668 3	18
43	715895 8	727972 2	739826 8	7514561	762856 4	17
44	716098 9	728171 6	740022 5	7516480	763044 5	16
45	716301 9	728371 0	740218 1	7518398	763232 5	15
46	716504 9	728570 3	740413 7	7520316	763420 4	14
47	716707 8	728769 5	740609 2	7522233	763608 2	13
48	716910 6	728968 7	740804 6	7524149	763796 0	12
49	717113 4	729167 8	741000 0	7526065	763983 8	11
50	717316 1	729366 8	741195 3	7527980	764171 5	10
51	717518 7	729565 8	741390 5	7529894	764359 1	9
52	717721 3	729764 7	741585 6	7531808	764546 6	8
53	717923 8	729963 5	741780 7	7533721	764734 1	7
54	718126 3	730162 3	741975 8	7535634	764921 5	6

0'→	45	46	47		48	49	←60'
55	718328 7	730361 0	742170 8		7537546	765108 8	5
56	718531 0	730559 7	742365 7		7539457	765296 1	4
57	718733 3	730758 3	742560 5		7541367	765483 3	3
58	718935 5	730956 8	742755 3		7543277	765670 4	2
59	719137 7	731155 3	742950 1		7545187	765857 5	1
60	719339 8	731353 7	743144 8	7547090 [recte:?]	766044 5		0

[bottom banner] 44 | 43 | 42 | 41 | 40 Gradus Quadrantis pro sinibus rectis complementorum arcuum eiusdē Quadrantis.

Differentiae (proportional parts, per second; top→bottom, positions approximate):

- 45°: 34.3 (0'), 34.2 (≈3'), 34.1 (≈13'), 34.0 (≈23'), 33.9 (≈33'), 33.8 (≈43'), 33.7 (≈52')
- 46°: 33.7 (0'), 33.6 (≈3'), 33.5 (≈13'), 33.4 (≈22'), 33.3 (≈33'), 33.2 (≈42'), 33.1 (≈52')
- 47°: 33.1 (0'), 33.0 (≈3'), 32.9 (≈10'), 32.8 (≈20'), 32.7 (≈30'), 32.6 (≈40'), 32.5 (≈50')
- 48°: 32.4 (0'), 32.3 (≈9'), 32.2 (≈18'), 32.1 (≈28'), 32.0 (≈37'), 31.9 (≈47'), 31.8 (≈57')
- 49°: 31.8 (0'), 31.7 (≈5'), 31.6 (≈15'), 31.5 (≈25'), 31.4 (≈36'), 31.3 (≈46'), 31.2 (≈57'), 31.1 (60')

The canon continues from $\sin 49^\circ 60' = \sin 50^\circ = 7660445$ into the following opening.

Canon of Sines (Sinus rectus), degrees 50°–64°

Running header p. 78: TABVLA — p. 79: SINVVM — p. 80: TABVLA

Top banner (all three pages, verbatim): **Gradus Quadrantis pro sinibus rectis arcuum eiusdem Quadrantis.** ("Degrees of the quadrant, for the right sines of the arcs of the same quadrant.")

Bottom banner (all three pages, verbatim): **Gradus Quadrantis pro sinibus rectis complementorum arcuum eiusdem Quadrantis.** ("Degrees of the quadrant, for the right sines of the complements of the arcs of the same quadrant.")

Each page prints five degree-blocks. The leftmost column is the minute (*Minuta / Scrupula*, 0'→60' read top-down); each degree-block gives the *Sinus rectus* value and, to its right, a *Differentia* (the per-second difference, printed as a block label — see the *Differentiae* note under each table). The right-hand minute column (60'→0') and the bottom degree-banner serve the complement (cosine) reading. Continuity confirmed: $\sin 50^\circ 0' = 7660445$ (= $\sin 50^\circ$ carried over from the preceding folio).

degrees 50°–54°

M i n	Sinus 50°	Sinus 51°	Sinus 52°	Sinus 53°	Sinus 54°
0	7660445	7771460	7880108	7986355	8090170
1	7662314	7773290	7881898	7988105	8091879
2	7664183	7775120	7883688	7989855	8093588
3	7666051	7776949	7885477	7991604	8095296
4	7667919	7778777	7887266	7993352	8097004
5	7669786	7780605	7889054	7995100	8098711
6	7671652	7782432	7890841	7996847	8100417
7	7673517	7784258	7892927 [recte:?]	7998593	8102122
8	7675382	7786084	7894413	8000339	8103827
9	7677246	7787909	7896198	8002084	8105531
10	7679110	7789833 [recte:?]	7897983	8003828	8107234
11	7680973	7791557	7899767	8005571	8108936
12	7682835	7793380	7901550	8007314	8110638

M i n	Sinus 50°	Sinus 51°	Sinus 52°	Sinus 53°	Sinus 54°
1 3	7684697	7795202	7903332	8009056	8112339
1 4	7686558	7797024	7905114	8010797	8114040
1 5	7688418	7798845	7906895	8012538	8115740
1 6	7690278	7800665	7908676	8014278	8117439
1 7	7692137	7802485	7910456	8016017	8119137
1 8	7693995	7804304	7912235	8017756	8120835
1 9	7695853	7806123	7914014	8019494	8122532
2 0	7697710	7807941	7915792	8021232	8124229
2 1	7699566	7809758	7917569	8022969	8125925
2 2	7701422	7811574	7919345	8024705	8127620
2 3	7703277	7813390	7921121	8026440	8129314
2 4	7705132	7815205	7922896	8028175	8131008
2 5	7706986	7817020	7924671	8029909	8132701
2 6	7708839	7818834	7926445	8031642	8134393
2 7	7710692	7820647	7928218	8033375	8136084
2 8	7712544	7822569 [recte:?)	7929990	8035107	8137775

M i n	Sinus 50°	Sinus 51°	Sinus 52°	Sinus 53°	Sinus 54°
2 9	7714395	7824271	7931762	8036838	8139465
3 0	7716246	7826082	7933533	8038569	8141155
3 1	7718096	7827892	7935303	8040299	8142844
3 2	7719945	7829702	7937073	8042028	8144532
3 3	7721794	7831511	7938842	8043757	8146220
3 4	7723642	7833320	7940611	8045485	8147907
3 5	7725490	7835128	7942379	8047212	8149593
3 6	7727337	7836935	7944146	8048938	8151278
3 7	7729183	7838741	7945912	8050664	8152963
3 8	7731028	7840547	7947678	8052389	8154647
3 9	7732872	7842352	7949443	8054114	8156330
4 0	7734716	7844157	7951208	8055838	8158013
4 1	7736559	7845961	7952972	8057561	8159695
4 2	7738402	7847764	7954735	8059283	8161376
4 3	7740244	7849566	7956497	8061005	8163057
4 4	7742085	7851368	7958259	8062726	8164737

M i n	Sinus 50°	Sinus 51°	Sinus 52°	Sinus 53°	Sinus 54°
4 5	7743926	7853169	7960020	8064446	8166416
4 6	7745766	7854970	7961780	8066166	8168094
4 7	7747606	7856770	7963540	8067885	8169772
4 8	7749445	7858569	7965299	8069603	8171449
4 9	7751283	7860368	7967057	8071321	8173126
5 0	7753121	7862166	7968815	8073038	8174802
5 1	7754958	7863963	7970572	8044754 [recte:?]	8176477
5 2	7756794	7865759	7972328	8076470	8178151
5 3	7758630	7867555	7974084	8078185	8179825
5 4	7760465	7869350	7975839	8079899	8181498
5 5	7762299	7871145	7977593	8081613	8183170
5 6	7764132	7872939	7979347	8083326	8184841
5 7	7765965	7874732	7981100	8085038	8186512
5 8	7767797	7876525	7982852	8086749	8188182
5 9	7769629	7878317	7984604	8088460	8189851
6 0	7771460	7880108	7986355	8090170	8191520

Differentiae (per-second block-differences, printed to the right of each column, listed top→bottom as printed):

- 50°: 31.1, 31.0, 30.9, 30.8, 30.7, 30.6, 30.5
- 51°: 30.5, 30.4, 30.3, 30.2, 30.1, 30.0, 29.9, 29.8
- 52°: 29.8, 29.7, 29.6, 29.5, 29.4, 29.3, 29.2
- 53°: 29.2, 29.1, 29.0, 28.9, 28.8, 28.7, 28.6, 28.5
- 54°: 28.5, 28.4, 28.3, 28.2, 28.1, 28.0, 27.9, 27.8

*Bottom complementary degree-banner (verbatim, left→right): 39 | 38 | 37 | 36 | 35.
Right minute column runs 60→0.*

degrees 55°–59°

M in	Sinus 55°	Sinus 56°	Sinus 57°	Sinus 58°	Sinus 59°
0	8191520	8290376	8386706	8480481	8571673
1	8193188	8292002	8388290	8482022	8573171
2	8194855	8293628	8389873	8483562	8574668
3	8196522	8295253	8391456	8485102	8576164
4	8198188	8296877	8393038	8486641	8577660
5	8199854	8298501	8394619	8488180	8579155
6	8201519	8300124	8396199	8489718	8580649
7	8203183	8301746	8397778	8491255	8582142
8	8204846	8303367	8399357	8492791	8583635
9	8206508	8304987	8400935	8494326	8585127
10	8208170	8306607	8402513	8495860	8586619
11	8209831	8308226	8404090	8497394	8588110
12	8211491	8309844	8405666	8498927	8589600
13	8213151	8311462	8407241	8500459	8591089
14	8214810	8313079	8408816	8501991	8592577
15	8216469	8314696	8410390	8503522	8594064

M in	Sinus 55°	Sinus 56°	Sinus 57°	Sinus 58°	Sinus 59°
5					
1	8218127	8316312	8411963	8505052	8595551
6					
1	8219784	8317927	8413536	8506582	8597037
7					
1	8221440	8319541	8415108	8508111	8598523
8					
1	8223096	8321155	8416679	8509639	8600008
9					
2	8224751	8322768	8418250	8511167	8601492
0					
2	8226405	8324380	8419820	8512694	8602975
1					
2	8228058	8325991	8421389	8514220	8604457
2					
2	8229711	8327602	8422957	8515745	8605939
3					
2	8231363	8329212	8424525	8517270	8607420
4					
2	8233015	8330822	8426092	8518794	8608901
5					
2	8234666	8332431	8427658	8520317	8610381
6					
2	8236316	8334039	8429223	8521839	8611860
7					
2	8237965	8335646	8430788	8523361	8613338
8					
2	8239614	8337252	8432352	8524882	8614815
9					
3	8241262	8338858	8433915	8526402	8616292
0					
3	8242909	8340463	8435477	8527921	8617768
1					

M in	Sinus 55°	Sinus 56°	Sinus 57°	Sinus 58°	Sinus 59°
3 2	8244556	8342067	8437039	8529440	8619243
3 3	8246202	8343671	8438600	8530958	8620718
3 4	8247847	8345244	8440161	8532476	8622192
3 5	8249492	8346877	8441721	8533993	8623665
3 6	8251136	8348479	8443280	8535509	8625137
3 7	8252779	8350080	8444838	8537024	8626608
3 8	8254421	8351680	8446396	8538538	8628079
3 9	8256062	8353279	8447953	8540052	8629549
4 0	8257703	8354878	8449509	8541565	8631019
4 1	8259343	8356476	8451064	8543077	8632488
4 2	8260982	8358073	8452618	8544588	8633956
4 3	8262621	8359670	8454172	8546099	8635423
4 4	8264259	8361266	8455725	8547609	8636889
4 5	8265897	8362862	8457278	8549119	8638355
4 6	8267534	8364457	8458830	8550628	8639820
4 7	8269170	8366051	8460381	8552136	8641284
4	8270806	8367644	8461932	8553643	8642748

M in	Sinus 55°	Sinus 56°	Sinus 57°	Sinus 58°	Sinus 59°
8					
4	8272441	8369236	8463482	8555149	8644211
9					
5	8274075	8370828	8465031	8556655	8645673
0					
5	8275708	8372419	8466579	8558160	8647134
1					
5	8277340	8374009	8468126	8559664	8648595
2					
5	8278972	8375599	8469673	8561168	8650055
3					
5	8280603	8377188	8471219	8562671	8651514
4					
5	8282234	8378776	8472765	8564173	8652673 [recte:?)
5					
5	8283864	8380363	8474310	8565675	8654431
6					
5	8285493	8381950	8475854	8567176	8655888
7					
5	8287121	8383536	8477397	8568676	8657344
8					
5	8288749	8385121	8478939	8570175	8658799
9					
6	8290376	8386706	8480481	8571673	8660254
0					

Differentiae (per-second, top→bottom as printed):

- 55°: 27.8, 27.7, 27.6, 27.5, 27.4, 27.3, 27.2, 27.1
- 56°: 27.1, 27.0, 26.9, 26.8, 26.7, 26.6, 26.5, 26.4
- 57°: 26.4, 26.3, 26.2, 26.1, 26.0, 25.9, 25.8, 25.7
- 58°: 25.7, 25.6, 25.5, 25.4, 25.3, 25.2, 25.1, 25.0
- 59°: 25.0, 24.9, 24.8, 24.7, 24.6, 24.5, 24.4, 24.3, 24.2

*Bottom complementary degree-banner (verbatim, left→right): 34 | 33 | 32 | 31 | 30.
Right minute column runs 60→0. Signature at foot: G 4.*

degrees 60°–64°

M in	Sinus 60°	Sinus 61°	Sinus 62°	Sinus 63°	Sinus 64°
0	8660254	8746197	8829476	8910065	8987940
1	8661708	8747607	8830841	8911385	8989215
2	8663162	8749016	8832205	8912704	8990489
3	8664615	8750425	8833569	8914023	8991762
4	8666067	8751833	8834932	8915341	8993035
5	8667518	8753240	8836295	8916659	8994307
6	8668968	8754646	8837657	8917976	8995578
7	8670417	8756051	8839018	8919292	8996848
8	8671866	8757456	8840378	8920607	8998117
9	8673314	8758860	8841737	8921921	8999386
10	8674762	8760263	8843095	8923234	9000654
11	8676209	8761665	8844452	8924546	9001921
12	8677655	8763067	8845809	8925858	9003187
13	8679100	8764468	8847165	8927169	9004453
14	8680544	8765868	8848521	8928479	9005718
15	8681988	8767268	8849876	8929789	9006982
16	8683431	8768667	8851230	8931098	9008245
17	8684874	8770065	8852583	8932406	9009508
18	8686316	8771462	8853936	8933714	9010770

M in	Sinus 60°	Sinus 61°	Sinus 62°	Sinus 63°	Sinus 64°
1 9	8687757	8772859	8855288	8935021	9012031
2 0	8689197	8774255	8856639	8936327	9013292
2 1	8690636	8775650	8857989	8937632	9014552
2 2	8692074	8777044	8859338	8938936	9015811
2 3	8693512	8778437	8860687	8940240	9017069
2 4	8694949	8779830	8862035	8941543	9018326
2 5	8696386	8781222	8863383	8942845	9019582
2 6	8697822	8782613	8864730	8944146	9020833
2 7	8699257	8784003	8866076	8945446	9022093
2 8	8700691	8785393	8867421	8946746	9023347
2 9	8702124	8786782	8868765	8948045	9024600
3 0	8703557	8788171	8870108	8949344	9025853
3 1	8704989	8789559	8871451	8950642	9027105
3 2	8706420	8790946	8872793	8951939	9028356
3 3	8707851	8792332	8874134	8953235	9029606
3 4	8709281	8793717	8875475	8954530	9030856
3	8710710	8795102	8876815	8955824	9032105

M in	Sinus 60°	Sinus 61°	Sinus 62°	Sinus 63°	Sinus 64°
5					
3	8712138	8796486	8878154	8957117	9033353
6					
3	8713565	8797869	8879492	8958410	9034600
7					
3	8714992	8799251	8880830	8959702	9035847
8					
3	8716418	8800633	8882167	8960994	9037093
9					
4	8717844	8802014	8883503	8962285	9038338
0					
4	8719269	8803394	8884838	8963575	9039582
1					
4	8720693	8804773	8886172	8964804 [recte:?)	9040825
2					
4	8722116	8806152	8887506	8966152	9042068
3					
4	8723538	8807530	8888839	8967440	9043310
4					
4	8724960	8808907	8890171	8968727	9044551
5					
4	8726381	8810283	8891502	8970013	9045791
6					
4	8727801	8811659	8892833	8971299	9047031
7					
4	8729221	8813034	8894163	8972584	9048270
8					
4	8730640	8814408	8895492	8973868	9049508
9					
5	8732058	8815782	8896821	8975151	9050746
0					
5	8733475	8817155	8898149	8976433	9051983
1					

M in	Sinus 60°	Sinus 61°	Sinus 62°	Sinus 63°	Sinus 64°
5 2	8734891	8818527	8899476	8977715	9053219
5 3	8736307	8819898	8900802	8978996	9054454
5 4	8737722	8821268	8902127	8980276	9055688
5 5	8739137	8822638	8903452	8981555	9056922
5 6	8740551	8824007	8904776	8982833	9058155
5 7	8741964	8825375	8906099	8984111	9059387
5 8	8743376	8826743	8907422	8985388	9060618
5 9	8744787	8828110	8908744	8986664	9061848
6 0	8746197	8829476	8910065	8987940	9063078

Differentiae (per-second, top→bottom as printed):

- 60°: 24.2, 24.1, 24.0, 23.9, 23.8, 23.7, 23.6, 23.5
- 61°: 23.5, 23.4, 23.3, 23.2, 23.1, 23.0, 22.9, 22.8
- 62°: 22.7, 22.6, 22.5, 22.4, 22.3, 22.2, 22.1, 22.0
- 63°: 22.0, 21.9, 21.8, 21.7, 21.6, 21.5, 21.4, 21.3
- 64°: 21.2, 21.1, 21.0, 20.9, 20.8, 20.7, 20.6, 20.5

*Bottom complementary degree-banner (verbatim, left→right): 29 | 28 | 27 | 26 | 25.
Right minute column runs 60→0.*

Canon of Sines (*Tabula sinuum*) — degrees 65°–79°

Christoph Clavius [Clavius], *Opera Mathematica*, Tomus I. Continuation of the canon of right sines (*sinus recti*) for whole degrees, radius = 10,000,000. Three full table-openings covering 65°–79°, each degree column running from 0' to 60'.

Layout of every column block (as printed): a left **Minuta** column (0'→60', reading the tabulated degree at the top banner), then for each degree a **Sinus** column paired with a **Differentia** column (the per-second first difference, printed once per row-group, e.g. 20. 5 = 20.5"), and a right **Minuta** column (60'→0') that reads the same values as sines of the **complement** (bottom banner). Top banner = tabulated degree; bottom banner = its complement (89 – top).

Header glosses (Latin retained in the tables): *Sinus* = Sine; *Minuta* = minutes; *Differentia(e)* = first difference(s); *complementorum* = of the complements. Running display-caps across the opening spell **SINVVM... TABVLA... SINVVM**.

degrees 65°, 66°, 67°, 68°, 69°

Printed table title (top): *Gradus Quadrantis pro sinibus rectis arcuum eiusdem Quadrantis*. ("Degrees of the quadrant, for the right sines of the arcs of the same quadrant.") Printed table title (bottom, for the right-hand minute column): *Gradus Quadrantis pro sinibus rectis complementorum arcuum eiusdem Quadrantis*.

Seam checks: 65°60' = 66°0' = 9135455 ✓; 66°60' = 67°0' = 9205049 ✓; 68°60' = 69°0' = 9335804 ✓. **67°60' = 9271839 vs 68°0' = 9271836 — 3-unit mismatch (flagged).**

		D		D		D		D		D	
M		i		i		i		i		i	Mi
i	65°	ff	66°	ff		ff		ff		ff	n(
n	Sinus	f	Sinus	f	67° Sinus	f	68° Sinus	f	69° Sinus	f	c)
0	9063	2	9135	1	9205049	1	9271836 [recte:	1	9335804	1	60
	078	0	455	9		8	9271839]	8		7	
		.		.		.		.		.	
		5		7		9		1		4	
1	9064		9136		9206185		9272928		9336846		59
	307		638								
2	9065		9137		9107321		9274017		9337887		58
	535		820		[recte:?)						

M i n	65° Sinus	D i ff f	66° Sinus	D i ff f	67° Sinus	D i ff f	68° Sinus	D i ff f	69° Sinus	D i ff f	Mi n(c)
3	9066 763	2 0	9139 001		9208456		9275105		9338928	1 7	57
		.								.	
		4								3	
4	9067 990		9140 181		9209590		9276192		9339968		56
5	9069 216		9141 361		9210723		9277278		9341007		55
6	9070 441		9142 540	1 9	9211855	1 8	9278363		9342045		54
				.		.					
				6		8					
7	9071 665		9143 718		9212986		9279448		9343082		53
8	9072 889		9144 895		9214117		9280532		9344119		52
9	9074 112		9146 072		9215247		9281615		9345155		51
1 0	9075 334	2 0	9147 248		9216376		9282697	1 8	9346190	1 7	50
		.						.		.	
		3						0		2	
1 1	9076 555		9148 423		9217504		9283778		9347224		49
1 2	9077 775		9149 597		9218631		9284859		6348257 [recte:?]		48
1 3	9078 995		9150 770		9219758		9285939		9349289		47
1 4	9080 214		9151 943	1 9	9220884		9287018		9350321		46
				.							
				5							

M i n	65° Sinus	D i ff	66° Sinus	D i ff	67° Sinus	D i ff	68° Sinus	D i ff	69° Sinus	D i ff	Mi n(
1 5	9081 432		9153 115		9222010	1 8 7	9288096		9351352		45
1 6	9082 649		9154 286		9223135		9289173		9352382		44
1 7	9083 866		9155 457		9224259		9290250		9353411	1 7 1	43
1 8	9085 082		9156 627		9225382		9291326		9354440		42
1 9	9086 297	2 0	9157 796	1 9	9226504		9292401		9355468		41
2 0	9087 512		9158 964		9227625		9293476		9356495		40
2 1	9088 726		9160 131		9228746		9294550		9357521		39
2 2	9089 939		9161 297		9229866		9295623		9358546		38
2 3	9091 151		9162 463		9230985		9296695		9359571		37
2 4	9092 362		9163 628		9232103	1 8 6	9297766		9360595		36
2 5	9093 572	2 0	9164 792		9233220		9298836		9361618		35

M i n	65° Sinus	D i ff	66° Sinus	D i ff	67° Sinus	D i ff	68° Sinus	D i ff	69° Sinus	D i ff	Mi n(c)
2 6	9094 781		9165 955		9234337		9299905		9362640		34
2 7	9095 990		9167 117	1 9	9235453		9300974		9363662	1 7	33
				.						.	
				3						0	
2 8	9097 198		9168 279		9236568		9302042		9364683		32
2 9	9098 406		9169 440		9237682		9303109		9365703		31
3 0	9099 613		9170 601		9238795		9304176		9366722		30
3 1	9100 819		9171 761		9239908		9305242		9367740		29
3 2	9102 024		9172 920		9241020		9306307		9368758		28
3 3	9103 228		9174 078		9242131	1 8	9307371		9369775	1 6	27
						.				.	
						5				9	
3 4	9104 432		9175 235		9243242		9308434		9370791		26
3 5	9105 635	2 0	9176 391	1 9	9244352		9309497		9371806		25
		.		.							
		0		2							
3 6	9106 837		9177 547		9245461		9310559		9372820		24
3 7	9108 038		9178 702		9246569		9311620		9373834		23
3 8	9109 238		9179 856		9247670		9312680		9374847		22

M i n	65° Sinus	D i ff	66° Sinus	D i ff	67° Sinus	D i ff	68° Sinus	D i ff	69° Sinus	D i ff	Mi n(c)
3 9	9110 438		9181 009		9248782		9313739		9375859		21
4 0	9111 637		9182 161		9249888		9314798		9376870		20
4 1	9112 835	1 9	9183 313	1 9	9250993		9315856		9377880	1 6	19
		.		.						.	
		9		1						8	
4 2	9114 032		9184 464		9252097		9316913		9378889		18
4 3	9115 229		9185 614		9253200		9317969		9379898		17
4 4	9116 425		9186 763		9254303		9319024		9380906		16
4 5	9117 620		9187 912		9255405		9320079		9381913		15
4 6	9118 814		9189 060		9256506		9321133		9382919		14
4 7	9120 007		9190 207		9257606		9322186		9383925		13
4 8	9121 200		9191 353		9258706		9323238		9384930		12
4 9	9122 392		9192 499		9259805		9324290		9385934		11
5 0	9123 584	1 9	9193 644	1 9	9260903		9325341		9386937	1 6	10
		.		.						.	
		8		0						7	
5 1	9124 775		9194 788		9262000		9326391		9387939		9
5 2	9125 965		9195 931		9263096		9327440		9388941		8

M i n	65° Sinus	D i ff	66° Sinus	D i ff	67° Sinus	D i ff	68° Sinus	D i ff	69° Sinus	D i ff	Mi n(c)
5 3	9127 154		9197 073		9264192		9328488		9389942		7
5 4	9128 342		9198 215		9265287		9329535		9390942		6
5 5	9129 526		9199 356		9266381		9330582		9391941		5
5 6	9130 716		9200 496		9267474		9331628		9392940		4
5 7	9131 902		9201 635		9268566		9332673		9393938		3
5 8	9133 087	1 9	9202 774		9269658		9333717		9394935		2
		.									
		7									
5 9	9134 271		9203 912		9270749		9334761		9395931		1
6 0	9135 455	1 9	9205 049	1 8	9271839	1 8	9335804	1 7	9396926	1 6	0
		.		.		.		.		.	
		7		9		2		4		6	

Bottom complement banner (per degree): 65°→**24**, 66°→**23**, 67°→**22**, 68°→**21**, 69°→**20**.

degrees 70°, 71°, 72°, 73°, 74°

Printed running display-caps: *TABVLA*. Title: *Gradus Quadrantis pro sinubus rectis arcuum eiusdem Quadrantis*.

Seam checks: 69°60' = 70°0' = 9396926 ✓; 70°60' = 71°0' = 9455186 ✓; 71°60' = 72°0' = 9510565 ✓; 72°60' = 73°0' = 9563048 ✓; 73°60' = 74°0' = 9612617 ✓.

M i n	70° Sinus	D i ff	71° Sinus	D i ff	72° Sinus	D i ff	73° Sinus	D i ff	74° Sinus	D i ff	Mi n(c)
0	939692 6	1 6. 6	945518 6	1 5. 8	951056 5	1 5. 0	9563048	1 4. 2	961261 7	1 3. 3	60
1	939792 1		945613 3		951146 4		9563898		961341 8	1 3. 3	59
2	939891 5		945707 9		951236 2		9564747	1 4. 1	961421 9		58
3	939990 8	1 6. 5	945802 4	1 5. 7	951325 9	1 4. 9	9565596		961501 9		57
4	940090 0		945896 8		951415 5		9566444		961581 8		56
5	940189 1		945991 1		951505 0		9567291		961661 6		55
6	940288 2		946085 4		951594 4		9568137		961741 3		54
7	940387 2		946179 6		951683 8		9568982		961820 9		53
8	940486 1		946273 7		951773 1		9569826		961900 5		52
9	940584 9	1 6. 4	946367 7	1 5. 6	951862 3	1 4. 8	9570670	1 4. 0	961980 0		51
10	940683 6		946461 6		951951 4		9571513		962059 4		50
11	940782 2		946555 5		952040 4		9572355		962138 7		49
12	940880 8		946649 3		952129 4		9573196		962217 9		48
13	940979 3		946743 0		952218 3		9574036		962297 1		47

M i n	70° Sinus	D i ff	71° Sinus	D i ff	72° Sinus	D i ff	73° Sinus	D i ff	74° Sinus	D i ff	Mi n(c)
1 4	941077 7		946836 6		952307 1		9574875		962376 2		46
1 5	941176 0		946930 1		952395 8		9575714		962455 2	1 3.	45
										1	
1 6	941274 2		947023 6		952484 4		9576552	1 3.	962534 1		44
								9			
1 7	941372 4	1 6.	947117 0	1 5.	952573 0	1 4.	9577389		962612 9		43
		3		5		7					
1 8	941470 5		947210 3		952661 5		9578225		962691 7		42
1 9	941568 5		947303 5		952749 9		9579061		962770 4		41
2 0	941666 5		947396 7		952838 2		9579896		962849 0		40
2 1	941764 4		947489 8		952926 4		9580730		962927 5		39
2 2	941862 2		947582 8		953014 6		9581563		963005 9		38
2 3	941959 9		947675 7		953102 7		9582395		963084 3	1 3.	37
										0	
2 4	942057 5	1 6.	947768 5	1 5.	953190 7	1 4.	9583226		963162 6		36
		2		4		6					
2 5	942155 0		947861 2		953278 6		9584057		963240 8		35
2 6	942252 5		947953 9		953366 4		9584887		963318 9		34
2 7	942349 9		948046 5		953454 1		9585716		963396 9		33

M i n	70° Sinus	D i ff	71° Sinus	D i ff	72° Sinus	D i ff	73° Sinus	D i ff	74° Sinus	D i ff	Mi n(c)
2 8	942447 2		948139 0		953541 8		9586544		963474 8		32
2 9	942544 4		948231 4		953629 4		9587371		963552 7		31
3 0	942641 5		948323 7		953716 9		9588197		963630 5		30
3 1	942738 6		948416 0		953804 3		9589023	1 3. 7	963708 2	1 2. 9	29
3 2	942835 6		948508 2		953891 7	1 4. 5	9589848		963785 8		28
3 3	942932 5	1 6. 1	948600 3	1 5. 3	953979 0		9590672		963863 3		27
3 4	943029 3		948692 3		954066 2		9591495		963940 8		26
3 5	943126 0		948784 2		954153 3		9592318		964018 2		25
3 6	943222 7		948876 1		954240 3		9593140		964095 5		24
3 7	943319 3		948967 9		954327 2		9593961		964172 7	1 2. 8	23
3 8	943415 8		949059 6		954414 1		9594781	1 3. 6	964249 8		22
3 9	943512 2	1 6. 0	949151 2	1 5. 2	954500 9	1 4. 4	9595600		964326 8		21
4 0	943608 5		949242 7		954587 6		9596419		964403 8		20
4 4	943704		949334		954674		9597237		964480		19

M i n	70° Sinus	D i ff	71° Sinus	D i ff	72° Sinus	D i ff	73° Sinus	D i ff	74° Sinus	D i ff	Mi n(c)
1	8		1		2				7		
4	943801		949425		954760		9598054		964557		18
2	0		5		7				5		
4	943897		949516		954847		9598870		964634		17
3	1		8		2				2		
4	943993		949608		954933		9599685		964710	1	16
4	1		0		6				8	2. 7	
4	944089		949699		955019		9600499		964787		15
5	0		1		9				3		
4	944184	1	949790	1	955106	1	9601313	1	964863		14
6	9	5. 9	2	5. 1	1	4. 3		3. 5	8		
4	944280		949881		955192		9602126		964940		13
7	7		2		2				2		
4	944376		949972		955278		9602938		965016		12
8	4		1		3				5		
4	944472		950062		955364		9603749		965092		11
9	0		9		3				7		
5	944567		950153		955450		9604559		965168		10
0	6		6		2				9		
5	944663		950244		955536		9605368		965245	1	9
1	1		3		0				0	2. 6	
5	944758		950334		955621		9606177		965321	1	8
2	5		9		7				0	2. 6	
5	944853		950425		955707		9606985		965396		7
3	8		4		4				9		
5	944949	1	950515	1	955793	1	9608792	1	965472		6
4	0	5. 8	8	5. 0	0	4. 2	[recte:?]	3. 4	7		
5	945044		950606		955878		9608598		965548		5

Bottom complement banner (per degree): $70^{\circ} \rightarrow \mathbf{19}$, $71^{\circ} \rightarrow \mathbf{18}$, $72^{\circ} \rightarrow \mathbf{17}$, $73^{\circ} \rightarrow \mathbf{16}$, $74^{\circ} \rightarrow \mathbf{15}$.

Printed running display-caps: *SINVVM*. Title: *Gradus Quadrantis pro sinibus rectis arcuum eiusdem Quadrantis*.

[illegible]

M i n	75° Sinus	D i ff f	76° Sinus	D i ff f	77° Sinus	D i ff f	78° Sinus	D i ff f	79° Sinus	D i ff f	Mi n(c)
2	9660163 [recte:?)		9704363		9745008		97826 84	1 0 .	98173 81		58
								0			
3	9661514		9705065		9745660		97832 87		98179 34		57
4	9662264		9705766		9746312	1 0 .	97838 89		98184 86		56
						8					
5	9663013		9706466	1 1 .	9746963		97844 90		98190 37		55
				6							
6	9663761	1 2 .	9707165		9747613		97850 90		98195 87		54
		4									
7	9664508		9707863		9748262		97856 89		98201 37		53
8	9665255		9708561		9748910		97862 88		98206 86		52
9	9666001		9709258		9749557		97868 86	9 .	98212 34		51
								9			
1 0	9666746		9709954		9750203		97874 83		98217 81		50
1 1	9667490		9710649		9750849		97880 79		98223 27	9 .	49
										1	
1 2	9668233		9711343	1 1 .	9751494	1 0 .	97886 74		98228 72		48

M i n	75° Sinus	D i ff f	76° Sinus	D i ff f	77° Sinus	D i ff f	78° Sinus	D i ff f	79° Sinus	D i ff f	Mi n(c)
				5		7					
1 3	9668976		9712036		9752138		97892 68		98234 17		47
1 4	9669718	1 2 .	9712729		9752781		97898 62		98239 61	9 .	46
		3								0	
1 5	9670459		9713421		9753423		97904 55		98245 04		45
1 6	9671199		9714112		9754065		97910 47		98250 46		44
1 7	9671938		9714802		9754706		97916 38	9 .	98255 87		43
								8			
1 8	9672677		9715491		9755346	1 0	97922 28		98261 28		42
						.					
						6					
1 9	9673415		9716180		9755985		97928 18		98266 68		41
2 0	9674152		9716868	1 1	9756623		97934 07		98272 07		40
				.							
				4							
2 1	9674888	1 2	9717555		9757260		97939 95		98277 45	8 .	39
		.								9	
		2									
2 2	9675623		9718241		9757897		97945 82		98282 82		38
2 3	9676357		9718926		9758533		97951 68	9 .	98288 18		37

M i n	75° Sinus	D i ff f	76° Sinus	D i ff f	77° Sinus	D i ff f	78° Sinus	D i ff f	79° Sinus	D i ff f	Mi n(c)
								7			
2 4	9677091		9719610		9759168		97957 53		98293 54		36
2 5	9677824		9720294		9759802	1 0	97963 37		98298 89		35
						· 5					
2 6	9678556		9720977		9760435		97969 21		98304 23		34
2 7	9679287		9721659	1 1	9761067		97975 04		98309 56		33
				· 3							
2 8	9680017		9722340		9761699		97980 86		98314 88	8 · 8	32
2 9	9680747		9723020		9762330		97986 67		98320 19		31
3 0	9681476	1 2	9723699		9762960		97992 47	9 · 6	98325 49		30
		· 1									
3 1	9682804 [recte:?]		9724378		9763589		97998 27		98330 79		29
3 2	9682931		9725056		9794217 [recte:?]		98004 06		98336 08		28
3 3	9683657		9725733		9764845	1 0	98009 84		98341 36		27
						· 4					
3 4	9684383		9726409		9765472		98015 61		98346 63		26

M i n	75° Sinus	D i ff f	76° Sinus	D i ff f	77° Sinus	D i ff f	78° Sinus	D i ff f	79° Sinus	D i ff f	Mi n(c)
3 5	9685108		9727085	1 1 . 2	9766098		98021 37		98351 89		25
3 6	9685832	1 2 . 0	9727760		9766723		98027 12	9 . 5	98357 14	8 . 7	24
3 7	9686555		9728434		9767347		98032 87		98362 39		23
3 8	9687277		9729107		9767970		98038 61		98367 63		22
3 9	9687998		9729779		9768593		98044 34		98372 86		21
4 0	9688719		9730450		9769215	1 0 . 3	98050 06		98378 08		20
4 1	9689439		9731120	1 1 . 1	9769836		98055 77		98383 29		19
4 2	9690158		9731789		9770456		98061 47		98388 50		18
4 3	9690876	1 1 . 9	9732458		9771075		98067 16	9 . 4	98393 70	8 . 6	17
4 4	9691593		9733126		9771693		98072 85		98398 89		16
4 5	9692309		9733793		9772311		98078 53		98404 07		15

M i n	75° Sinus	D i ff f	76° Sinus	D i ff f	77° Sinus	D i ff f	78° Sinus	D i ff f	79° Sinus	D i ff f	Mi n(c)
4 6	9693025		9734459		9772928		98084 20		98409 24		14
4 7	9693740		9735124		9773544	1 0	98089 86		98414 40		13
						. 2					
4 8	9694454		9735789		9774159		98095 51		98419 56		12
4 9	9695167		9746453 [recte:?]		9774773		98101 16		98424 71		11
5 0	9695879	1 1 .8	9737116		9775387		98106 80		98429 85	8 .5	10
5 1	9696590		9737778		9776000		98112 43	9 .3	98434 98		9
5 2	9697301		9738439		9776612		98118 05		98440 10		8
5 3	9698011		9739099		9777223		98123 66		98445 21		7
5 4	9698720		9739759		9777833	1 0	98129 26		98450 32		6
						. 1					
5 5	9699428		9740418		9778442		98134 86		98455 42		5
5 6	9700135		9741076		9779050		98140 45		98460 51		4
5 7	9700842		9741733		9779658		98146 03		98465 59	8 .4	3

M i n		D i ff f		D i ff f		D i ff f		D i ff f		D i ff f	Mi n(c)
	75° Sinus		76° Sinus		77° Sinus		78° Sinus		79° Sinus		
5	9701548	1	9742389		9780265		98151		98470		2
8		1					60		66		
		.									
		7									
5	9702253		9743045		9780871		98157		98475		1
9							16		72		
6	9702957	1	9743700	1	9781476	1	98162	9	98480	8	0
0		1		0		0	72	.	78	.	
		.		.		.		3		4	
		7		9		1					

Bottom complement banner (per degree): 75°→**14**, 76°→**13**, 77°→**12**, 78°→**11**, 79°→**10**.

Canon of Right Sines — conclusion (80°–90°), and the Explanation and Use of the Table

[The table of right sines (canon sinuum) concludes on this opening: the direct-sine columns run 80°–89° across the two table pages, and the last cell — degree 89, minute 60' — gives the whole sine, $\sin 90^\circ = 10000000$. A page of prose (the Explicatio atque usus tabulae*) then follows, explaining how to use the table.]*

[Table page — printed p. 84] Right Sines, arcs 80°–84°

Table banner (top): *Gradus Quadrantis pro sinibus rectis arcuum eiusdem Quadrantis*. — "Degrees of the quadrant, for the right sines of the arcs of the same quadrant." Left margin (vertical): *Minuta Graduum Quadrantis pro sinibus rectis arcuum eiusdem Quadrantis*. Right margin (vertical): *Minuta Graduum Quadrantis pro sinibus rectis complementorum arcuum eiusdem Quadrantis*.

Read the direct sine of an arc with the **left** minute column (0'→60', top→bottom) under the head-degree; read the complement sine with the **right** minute column (60'→0'). Whole sine = 10000000.

Mi n (L)	80°	81°	82°	83°	84°	Mi n (R)
0	984 807 8	987 688 3	9902681	9925461	9945219	60
1	984 858 3	987 733 8	9903085	9925816	9945523	59
2	984 908 7	987 779 2	9903489	9926169	9945826	58
3	984 959 0	987 824 5	9903892	9926521	9946128	57
4	985 009 2	987 869 7	9904294	9926873	9946429	56
5	985 059 3	987 914 8	9904695	9927224	9946729	55
6	985 109 3	987 959 8	9905095	9927574	9947028	54
7	985 159 3	988 004 8	9905494	9927923	9947327	53
8	985 209 2	988 049 7	9905893	9928271	9947625	52
9	985 259 0	988 094 5	9006291 [recte: 9906291?]	9928618	9947922	51
10	985 308 7	988 139 2	9906688	9928965	9948218	50

Mi n (L)	80°	81°	82°	83°	84°	Mi n (R)
11	985 358 3	988 183 8	9907084	9929311	9948513	49
12	985 407 9	988 228 3	9007479 [recte: 9907479?]	9929656	9948807	48
13	985 457 4	988 272 8	9907873	9930000	9949100	47
14	985 506 8	988 317 2	9908266	9930343	9949393	46
15	985 556 1	988 361 5	9908659	9930685	9949685	45
16	985 605 3	988 405 7	9909051	9931026	9949976	44
17	985 654 4	988 449 8	9909442	9931367	9950266	43
18	985 703 5	988 493 8	9909832	9931707	9950555	42
19	985 752 5	988 537 8	9901221 [recte: 9910221?]	9932046	9950844	41
20	985 801 4	988 581 7	9910610	9932384	9951132	40
21	985 850 2	988 625 5	9910998	9932721	9951419	39

Mi n (L)	80°	81°	82°	83°	84°	Mi n (R)
22	985 898 9	988 669 2	9911385	9933057	9951705	38
23	985 947 5	988 712 8	9911771	9933393	9951990	37
24	985 996 1	988 756 4	9912156	9933728	9952274	36
25	986 044 6	988 799 9	9911540 [recte: 9912540?]	9934062	9952557	35
26	986 093 0	988 843 3	9912923	9934395	9952840	34
27	986 141 3	988 886 6	9913306	9934727	9953122	33
28	986 189 5	988 929 8	9913688	9935058	9953403	32
29	986 237 6	988 972 9	9914069	9935389	9953683	31
30	986 285 6	989 015 9	9914449	9935719	9953962	30
31	986 333 6	989 058 8	9914828	9936048	9954240	29
32	986 381 5	989 101 7	9915206	9936376	9954518	28

Mi n (L)	80°	81°	82°	83°	84°	Mi n (R)
33	986 429 3	989 144 5	9915584	9936703	9954795	27
34	986 477 0	989 187 2	9915961	9937029	9655071 [recte: 9955071?]	26
35	986 524 6	989 229 8	9916337	9937355	9955346	25
36	986 572 2	989 272 3	9916712	9937680	9955620	24
37	986 619 7	989 314 7	9917086	9938004	9955893	23
38	986 667 1	989 357 1	9917459	9938327	9956165	22
39	986 714 4	989 399 4	9917832	9938649	9956437	21
40	986 761 6	989 441 6	9918204	9938970	9956708	20
41	986 808 7	989 483 7	9918575	9939290	9956978	19
42	986 855 7	989 525 7	9918945	9939609	9957247	18
43	986 902 7	989 567 7	9919314	9989928 [recte: 9939928?]	9957515	17

Mi n (L)	80°	81°	82°	83°	84°	Mi n (R)
44	986 949 6	989 609 6	9919682	9940246	9957782	16
45	986 996 4	989 651 4	9920049	9940563	9958049	15
46	987 043 1	989 693 1	9920416	9940879	9958315	14
47	987 089 7	989 734 7	9920782	9941194	9958580	13
48	987 136 2	989 776 2	9921147	9941509	9958844	12
49	987 182 7	989 817 7	9921511	9941823	9959307 [recte: 9959107?]	11
50	987 229 1	989 859 1	9921874	9942136	9959370	10
51	987 275 4	989 900 4	9922236	9942448	9959632	9
52	987 321 6	989 941 6	9922598	9942759	9959893	8
53	987 367 7	989 982 7	9922959	9943069	9960153	7
54	987 413 7	990 023 7	9923319	9943379	9960412	6

Mi n (L)	80°	81°	82°	83°	84°	Mi n (R)
55	987 459 7	990 064 6	9923678	9943688	9960670	5
56	987 505 6	990 105 5	9924036	9943996	9960927	4
57	987 551 4	990 146 3	9924393	9944303	9961183	3
58	987 597 1	990 187 0	9924750	9944609	9961438	2
59	987 642 7	990 227 6	9925106	9944914	9961693	1
60	987 688 3	990 268 1	9925461	9945219	9961947	0

Table banner (bottom): *Gradus Quadrantis pro sinibus rectis complementorum arcuum eiusdem Quadrantis*. — with the bottom degree-labels **9 · 8 · 7 · 6 · 5** printed under columns 80–84 respectively (each column's complement reading; bottom-label = 89 – head-degree).

Differentiae (printed sparsely between the columns, one value per group of rows; transcribed top→bottom, values only): col 80: 8.4, 8.3, 8.2, 8.1, 8.0, 7.9, 7.8, 7.7, 7.6 · col 81: 7.6, 7.5, 7.4, 7.3, 7.2, 7.1, 7.0, 6.9, 6.8, 6.7 · col 82: 6.7, 6.6, 6.5, 6.4, 6.3, 6.2, 6.1, 6.0, 5.9 · col 83: 5.9, 5.8, 5.7, 5.6, 5.5, 5.4, 5.3, 5.2, 5.1 · col 84: 5.1, 5.0, 4.9, 4.8, 4.7, 4.6, 4.5, 4.4, 4.3, 4.2.

[Table page — printed p. 85] Right Sines, arcs 85°–89° — the canon concludes at $\sin 90^\circ = 10000000$

Table banner (top): *Gradus Quadrantis pro sinibus rectis arcuum eiusdem Quadrantis*. Left margin (vertical): *Minuta Graduum Quadrantis pro sinibus rectis*

arcuum eiusdem Quadrantis. Right margin (vertical): Minuta Graduum Quadrantis pro sinibus rectis complementorum arcuum eiusdem Quadrantis.

Min (L)	85°	86°	87°	88°	89°	Min (R)
0	9961947	9975640	9986295	9993908	9998477	60
1	9962200	9975843	9986447	9994009	9998527	59
2	9962452	9976045	9986598	9994109	9998576	58
3	9962703	9976246	9986748	9994208	9998625	57
4	9962954	9976446	9986897	9994307	9998673	56
5	9963204	9976645	9987045	9994405	9998720	55
6	9963453	9976843	9987193	9994502	9998766	54
7	9963701	9977040	9987340	9994598	9998811	53
8	9963948	9977237	9987486	9994693	9998855	52
9	9964194	9977433	9987631	9994787	9998899	51
10	9964440	9977628	9987775	9994881	9998942	50
11	9964685	9977822	9987918	9994974	9998984	49
12	9964929	9978015	9988061	9995066	9999025	48
13	9965172	9978207	9988203	9995157	9999065	47
14	9965414	9978398	9988344	9995247	9999104	46
15	9965655	9978589	9988484	9995336	9999143	45
16	9965895	9978779	9988623	9995424	9999181	44
17	9966135	9978968	9988761	9995512	9999218	43
18	9966374	9979156	9988899	9995599	9999254	42
19	9966612	9979343	9989036	9995685	9999289	41
20	9966849	9979530	9989172	9995770	9999323	40
21	9967085	9979716	9989307	9995854	9999356	39
22	9967320	9979901	9989441	9995937	9999389	38
23	9967555	9980085	9989574	9996019	9999421	37
24	9967789	9980268	9989706	9996101	9999452	36
25	9968022	9980450	9989837	9996182	9999482	35
26	9968254	9980631	9989968	9996262	9999511	34
27	9968485	9980811	9990098	9996341	9999539	33
28	9968715	9980991	9990227	9996419	9999566	32

Min (L)	85°	86°	87°	88°	89°	Min (R)
29	9968944	9981170	9990355	9996496	9999593	31
30	9969173	9981348	9990482	9996573	9999619	30
31	9969401	9981525	9990608	9996649	9999644	29
32	9969628	9981701	9990734	9996724	9999668	28
33	9969854	9981877	9990859	9996798	9999691	27
34	9970079	9982052	9990983	9996871	9999713	26
35	9970304	9982226	9991106	9996943	9999735	25
36	9970528	9982399	9991228	9997014	9999756	24
37	9970751	9982571	9991349	9997085	9999776	23
38	9970973	9982742	9991470	9997155	9999795	22
39	9971194	9982912	9991590	9997224	9999813	21
40	9971414	9983082	9991709	9997292	9999830	20
41	9971633	9983251	9991827	9997359	9999846	19
42	9971851	9983419	9991944	9997425	9999862	18
43	9972069	9983586	9992060	9997491	9999877	17
44	9972286	9983752	9992175	9997556	9999891	16
45	9972502	9983917	9992290	9997620	9999904	15
46	9972717	9984081	9992404	9997683	9999916	14
47	9972931	9984245	9992517	9997745	9999927	13
48	9973145	9984408	9992629	9997806	9999938	12
49	9973358	9984570	9992740	9997867	9999948	11
50	9973570	9984731	9992850	9997927	9999957	10
51	9973781	9984891	9992960	9997986	9999965	9
52	9973991	9985050	9993069	9998044	9999972	8
53	9974200	9985209	9993177	9998101	9999978	7
54	9974408	9985367	9993284	9998157	9999984	6
55	9974615	9985524	9993390	9998212	9999989	5
56	9974822	9985680	9993495	9998267	9999993	4
57	9975028	9985835	9993599	9998321	9999996	3
58	9975233	9985989	9993703	9998374	9999998	2
59	9975437	9986143	9993806	9998426	9999999	1

Min (L)	85°	86°	87°	88°	89°	Min (R)
60	9975640	9986295	9993908	9998477	10000000	0

Table banner (bottom): *Gradus Quadrantis pro sinibus rectis complementorum arcuum eiusdem Quadrantis.* — with the bottom degree-labels **4 · 3 · 2 · 1 · 0** printed under columns 85–89 respectively.

Differentiae (top→bottom, values only): col 85: 4.2, 4.1, 4.0, 3.9, 3.8, 3.8, 3.7, 3.6, 3.5, 3.4 · col 86: 3.4, 3.3, 3.2, 3.1, 3.0, 2.9, 2.8, 2.7, 2.6, 2.5 · col 87: 2.5, 2.4, 2.3, 2.2, 2.1, 2.1, 2.0, 1.9, 1.8, 1.7 · col 88: 1.7, 1.6, 1.5, 1.4, 1.3, 1.3, 1.2, 1.1, 1.0, 0.9, 0.8 · col 89: 0.8, 0.7, 0.6, 0.5, 0.4, 0.4, 0.3, 0.2, 0.1, 0.0.

[Prose page — printed p. 86] EXPLANATION AND USE OF THE PRECEDING TABLE OF RIGHT SINES

Explicatio, atque usus tabulae praecedentis Sinuum rectorum.

[Margin: The layout of the parts of the table of sines.] [Margin: How the complement of any arc may be drawn from this table.] At the head of the preceding table the 90 degrees of the quadrant are set out in order, and down the left side the 60 minutes, descending. Then, along the bottom side, the same 90 degrees of the quadrant are placed again, but in reverse order, and up the right side the 60 minutes, ascending. We did this so that the complement of any arc might be recognized at once. For any degree placed at the head of the table, taken together with any minute set on the left, has for its complement the degree on the bottom side that corresponds to the degree taken at the head, together with the minute on the right that corresponds to the minute set on the left. Thus, since to the degree 46 at the head, with minute 0 set on the left, there corresponds on the bottom side degree 43 with minute 60 set on the right, the arc will be 43° 60' — that is, 44° 0' — the complement of the arc 46° 0'. In the same way the arc 43° 13' will have as its complement the arc 46° 47'. By the same reasoning, any degree placed on the bottom side, taken with any minute set on the right, has for its complement the degree at the head that corresponds to the degree taken on the bottom side, together with the minute on the left that corresponds to the minute taken on the right. Finally, beneath the degrees set out at the head of the table are placed the right sines of all the arcs advancing through each single minute of the quadrant, on the basis that the whole sine is 10000000. And if from each sine the two leading figures on the right are cast away — adding a unit, however, if the two figures cast away exceed the number 50 — the remainder will be the sines of the same arcs on the basis that the whole sine is 100000, as we said above. Hence

whatever we prescribe in the use of this table about sines relative to a whole sine of 10000000 must be understood also of the sines relative to a whole sine of 100000, by casting away the two leading figures on the right, as we have said.

[Margin: The use of the table of sines is twofold.] The use of this table is twofold. For by it either the sine of any given arc is sought, or the arc belonging to any known sine is investigated. [Margin: How the right sine of any arc smaller than a quadrant is found in the table.] When, then, you seek the right sine of a given arc smaller than a quadrant, take its degrees at the head of the table and its minutes on the left. At the common cell — proceeding, that is, from the minutes toward the right until you reach the place beneath the degrees taken — you will find the right sine at once. Thus you will find the right sine of the arc of 25° , beneath the head 25, in the row of minute 0 set on the left, to be 4226183. The right sine of the arc of $25^\circ 19'$ you will find to be 4276209. And the right sine of the arc of $25^\circ 50'$ you will come upon as 4357549.

[Margin: By what method the right sine of any arc greater than a quadrant is found.] But if you want the right sine of an arc greater than a quadrant yet smaller than a semicircle, subtract the given arc from the semicircle and take the right sine of the remaining arc, as before. For this sine will at the same time be the right sine of the arc greater than a quadrant, because two arcs that together make up a semicircle have the same right sine, as we said in the exposition of the definitions. Thus, if the given arc is $138^\circ 47'$, subtract it from the semicircle, that is, from 180° ; for the right sine 6589082 of the remaining arc of $41^\circ 13'$ is also the right sine of the arc of $138^\circ 47'$, since that arc, together with the arc of $41^\circ 13'$, makes up a semicircle.

[Margin: How the right sine of any arc that has seconds besides minutes is drawn out by the proportional part.] But if the given arc, besides degrees and minutes, has also seconds, the proportional part must be sought, in this way. Take the difference between the right sine of the next smaller arc and the right sine of the next larger arc, and reason: If 60 seconds (by which successive arcs in this table differ from one another) require that whole difference to be added to the sine of the next smaller arc in order to compose the sine of the next larger arc, how great a difference do the given seconds require to be added to that same sine of the next smaller arc, to make the sine of the proposed arc? For the difference found will be the proportional part which, added to the sine of the next smaller arc, produces the right sine of the proposed arc. For example, let the proposed arc be $20^\circ 43' 20''$. Take the difference 2721 between the sine 3537469 of the arc $20^\circ 43'$ (the next smaller) and the sine 3540190 of the arc $20^\circ 44'$ (the next larger), and

reason: If 60 seconds require the difference 2721 to be added to the sine 3537469 of the arc $20^{\circ} 43'$, so that the sine of the arc $20^{\circ} 44'$ is produced, how great a difference do the proposed 20 seconds require to be added to that same sine 3537469 of the arc $20^{\circ} 43' 20''$? You will find the difference, or proportional part, to be 907, which, added to the sine 3537469, will yield 3538376, the sine of the arc $20^{\circ} 43' 20''$.

[Margin: Seconds are commonly neglected by astronomers in the handling of sines.] Commonly, however, astronomers neglect the seconds in this business, if they are fewer than 30; but if more, they add for them one minute to the [given] minutes. For no error of any moment arises from this. And so for the given arc $20^{\circ} 43' 20''$ they take the sine of the arc $20^{\circ} 43'$, disregarding those 20 seconds; but for the arc $20^{\circ} 43' 48''$ they take the sine of the arc $20^{\circ} 44'$, reckoning those 48 seconds as one minute.

[Margin: For a semicircle, and for an arc greater than a semicircle, no right sine is to be sought; indeed, none of them has a right sine.] As for a semicircle, and an arc greater than a semicircle, there is no reason to investigate its right sine, since the sines of such arcs are never used in astronomical computations — as is plain to those who are practiced in rectilinear and spherical triangles, in which the whole science of sines is engaged. Indeed, a semicircle has no sine at all, as is clear from each of the two definitions of the right sine. The same may be said of a greater arc — unless someone, in the first figure of the definitions, wishes to call the line FK the right sine of the arc DBF, according to the later definition of the right sine, and the line FH the right sine of the arc ACF; which does not seem to be properly said, since the earlier definition of the right sine can in no way agree with lines of this kind, as is plain from the same figure.

[Margin: How the sine of the complement of an arc smaller than a quadrant is found in the table.] But when you seek the sine of the complement of a given arc smaller than a quadrant, take its degrees in the lower part of the table and its minutes on the right. At the common cell you will find the sine of the complement — that is, the right sine of that arc which is the complement of the given arc. For that right sine belongs to the arc whose degrees are placed at the head of the table and whose minutes are placed on the left side, and which is the complement of the given arc, as we said above. Thus you will find the sine of the complement of the arc of 30° , in the row of minute 0 (on the bottom side of the table) set on the right, to be 8660254 — which indeed is the right sine of the arc of $59^{\circ} 60'$, that is, of the arc of 60° , which is the complement of the arc of 30° . Likewise you will find

the sine of the complement of the arc of $30^{\circ} 49'$ to be 8588110, which is the right sine of the arc of $59^{\circ} 11'$, the complement of the arc of $30^{\circ} 49'$.

But if an arc is offered that is greater than a quadrant yet smaller than a semicircle, you will find the sine of its complement thus. Subtract a quadrant from it, and take the right sine of the remaining arc. For since this remaining arc is the complement of the given arc, its right sine will be the sine of the complement of the given arc. [Margin: By what art the sine of the complement of any arc greater than a quadrant is detected.] Thus, if the arc offered is $127^{\circ} 30'$, subtract a quadrant from it, that is, 90° ; for the right sine 6087614 of the remaining arc of $37^{\circ} 30'$ is the sine of the complement of the given arc $127^{\circ} 30'$, since that arc is the complement of this one.

* But if the given arc, besides degrees and minutes, has also seconds: if indeed it is smaller than a quadrant, you will investigate the sine of its complement by the rule of proportions, just as we said above about the right sine — except that here the difference found, or proportional part, must be *subtracted* from the sine of the next smaller arc. But if the given arc is greater than a quadrant yet smaller than a semicircle, then, a quadrant having been subtracted, you will seek the right sine of the remaining arc by the same rule of proportions, in the way we set out above concerning the right sine — although the seconds may be neglected, as we taught above, in this business of sines. An example. Let the given arc be

* [Footnote, foot of page:] *How the sine of the complement of any arc having seconds besides minutes is drawn out by the proportional part.*

arc of $69^{\circ} 16' 40''$. Take the difference 2721, which lies between the sine 3540190 (of the arc $69^{\circ} 16'$, the next smaller, entered in the lower part of the table) and the sine 3537469 (of the arc $69^{\circ} 17'$, the next larger, placed in that same lower part of the table); and reason thus. If 60 seconds require the difference 2721, to be subtracted from the sine 3540190 (of the arc $69^{\circ} 16'$, set in the lower part of the table) so that there remains the sine 3537469 (of the arc $69^{\circ} 17'$, described in that same lower part of the table), how great a difference do 40 seconds demand — to be subtracted from that same sine 3540190 (of the arc $69^{\circ} 16'$, placed in the lower part of the table) so that there remains the sine of the arc $69^{\circ} 16' 40''$ contained in that same lower part of the table? You will find the difference, that is, the proportional part [pars proportionalis], to be 1814; and this, subtracted from the sine 3540190, leaves the sine 3538376, set in the lower part of the table. This is indeed the right sine [sinus rectus] of the arc $20^{\circ} 40' 20''$ — that is, of the complement of the given arc $69^{\circ} 16' 40''$.

Let there again be given an arc of $110^{\circ} 43' 20''$. Take away a quadrant, that is, 90° , and for the residual arc of $20^{\circ} 43' 20''$ seek the right sine, as we taught above. By the proportional part you will find the sine 3538376, which is the sine of the complement of the proposed arc.

[Margin: Another method of finding the sine of the complement of an arc less than a quadrant.]

You may also find the sine of the complement of a proposed arc less than a quadrant in this way. Subtract the proposed arc from a quadrant, so as to obtain its complement. The right sine of that complement, found as we described for the right sine, is exactly what is sought. Thus, if the sine of the complement of the arc $69^{\circ} 16' 40''$ is wanted, subtract this arc from 90° ; then for the residual arc of $20^{\circ} 43' 20''$ (which is the complement of the given arc $69^{\circ} 16' 40''$) seek the right sine, which you will find to be 3538376. And this is the sine of the complement of the given arc $69^{\circ} 16' 40''$.

[Margin: The sine of the complement of a semicircle, or of an arc greater than a semicircle, need not be sought.]

Here too, the sine of the complement of a semicircle, and of an arc greater than a semicircle, need not be sought, for the reason stated above.

[Margin: How the versed sine of any arc, whether less or greater than a quadrant, is obtained.]

Finally, when you want the versed sine [sinus versus] of a proposed arc: if it is less than a quadrant, subtract its sine of the complement from the whole sine [sinus totus]; but if it is greater than a quadrant yet less than a semicircle, add its sine of the complement to the whole sine. The remaining number, or the sum, will be the versed sine of the given arc. This is because the sine of the complement of any arc equals the complement of the versed sine of the same arc, as we showed above in the definitions. Hence the sine of the complement of any arc, when taken from the whole sine or added to it, leaves or makes up the versed sine of the same arc — which is plain from the first figure that we set out in the exposition of the definitions.

Example. Let the versed sine of the arc $20^{\circ} 57'$ be sought. Its sine of the complement is 9338928, which subtracted from the whole sine 10000000 leaves the versed sine 661072 of the given arc $20^{\circ} 57'$. Again, let the versed sine of the arc $138^{\circ} 31'$ be investigated. Its sine of the complement is 7491484, which added to the whole sine 10000000 yields the versed sine 17491484 of the proposed arc 138°

31'. Lastly, let the versed sine of the arc $69^{\circ} 16' 40''$ be found. The sine of its complement is 3538376, found by the proportional part; this, subtracted from the whole sine 10000000, leaves the versed sine 6461624 of the given arc $69^{\circ} 16' 40''$. In the same way the versed sine of the arc $159^{\circ} 16' 40''$ will be found to be 19353068 by the proportional part.

[Margin: Both the right and the versed sine of a quadrant equal the whole sine, while its sine of the complement is nothing.]

Moreover, for a quadrant both the right sine and the versed sine equal the whole sine, whereas its sine of the complement is nothing — as is evident from the first figure set out in the definitions.

[Margin: How an arc less than a quadrant is derived from a known right sine.]

Now, from a known right sine you will find the arc as follows. Look for the proposed right sine among the sines of the table; or, if you do not find it, take the next larger or next smaller — that is, the one that differs from the proposed sine by the fewest units. For at the top of the table you will find the degrees, and at the left, opposite the sine taken, the minutes of the arc that corresponds to the proposed sine. Thus, suppose the known sine is 7510767. I find the sine 7510722, the next smaller, which differs from the known sine by fewer units than does the next larger sine, 7512642. To this next-smaller sine there correspond, at the top of the table, 48° , and at the left, 41'. I therefore say that the arc $48^{\circ} 41'$ belongs to the proposed sine; for the units by which the proposed sine differs from the sine of that arc do not introduce a notable error.

[Margin: How a more precise arc, less than a quadrant, is derived from a given sine by the proportional part.]

If, however, you want a more precise arc, the proportional part must be investigated by this method. Take the difference between the next-smaller and the next-larger sine; likewise the difference between the proposed sine and the one found in the table from which it differs least; and reason thus. If the difference between the two sines found in the table gives 60 seconds — to be added to the arc of the next-smaller sine, or subtracted from the arc of the next-larger sine (according as the difference was taken between the proposed sine and the next-smaller or the next-larger sine), so as to obtain the arc of the next-larger or next-smaller sine — how many seconds does the difference between the proposed sine and the next-smaller or next-larger sine demand, to be added to the arc of the next-smaller sine or subtracted from the arc of the next-larger sine, so as to obtain the arc of the proposed sine? For this second [second-term], added

to the arc of the next-smaller sine or subtracted from the arc of the next-larger sine, will give the arc of the proposed sine.

Thus in the given example, say: the difference 1920 between the sine 7510722 [the next-smaller occurrence is printed 7510772; recte 7510722, consistent with the difference 1920], the next smaller, and the sine 7512642, the next larger, gives 60 seconds to be added to the arc $48^{\circ} 41'$ (which corresponds to the next-smaller sine, since the proposed sine differs less from the next-smaller than from the next-larger sine), so as to obtain the arc $48^{\circ} 41' 60''$, that is $48^{\circ} 42'$, corresponding to the next-larger sine. How many seconds, then, does the difference 45 between the proposed sine and the next-smaller sine demand, to be added to that same arc of the next-smaller sine, so as to make the arc of the given sine? You will find 1 second and a little more must be added, giving the arc $48^{\circ} 41' 1''$ and a little more — so that to the proposed sine 7510767 there corresponds the arc $48^{\circ} 41' 1''$ and a little more.

Again, let the sine 455630 be given, which I do not find as an entry in the table. I therefore take the next larger, 456536 (for the proposed sine differs less from this than from the next smaller, 453630), to which there corresponds the arc $2^{\circ} 37'$. But if I want a more precise arc, I inquire the proportional part in this way. The difference 2906 between the sine 456536, the next larger, and the sine 453630, the next smaller, gives 60 seconds, to be subtracted from the arc $2^{\circ} 37'$ (which corresponds to the next-larger sine) so that the remaining arc is $2^{\circ} 36'$, corresponding to the next-smaller sine. How many seconds, then, does the difference 906 between the proposed sine and the next-larger sine demand, to be subtracted from that same arc of the next-larger sine so as to make the arc of the given sine? I find nearly 19 seconds, which taken from the arc $2^{\circ} 37'$ leave the arc $2^{\circ} 36' 41''$, belonging to the given sine 455630.

But since the same right sine corresponds to two arcs that together make up a semicircle, as we said above, if the arc found from the right sine by the stated method is subtracted from a semicircle — that is, from 180° — the remainder will be the other arc, greater than a quadrant, which also belongs to the said sine. Thus, if the arc $48^{\circ} 41' 1''$ found above is taken from 180° , there remains the arc $131^{\circ} 18' 59''$, belonging to the same right sine 7510767. The operation in triangles — both rectilinear and spherical — will neatly show whether the arc corresponding to the proposed sine should be taken greater than a quadrant or smaller, as will appear in its proper places.

[Margin: How an arc less than a quadrant is derived from the sine of the complement.]

But if the known sine is the sine of the complement of the sought arc, the degrees must be taken in the lower part of the table, and the minutes at the right. In this way the sought arc is obtained. Or else the arc corresponding to the given sine, treated as a right sine, must be found, and this subtracted from a quadrant so that the sought arc remains. Thus, if 7510767 is the known sine of the complement of some arc, I find the sine 7510722, the next smaller, since here it differs from the known sine by fewer units than does the sine 7512642, the next [larger]...

larger in the table of sines. To this next-smaller sine there correspond, at the very foot of the table, 41° , and at the right, 19'. The arc $41^\circ 19'$ is therefore the one sought; for its complement is the arc $48^\circ 41'$, to which the given sine belongs. The same arc $41^\circ 19'$ will also be obtained if the arc $48^\circ 41'$ — corresponding to the given sine at the top of the table and at the left — is subtracted from a quadrant.

[Margin: How a more precise arc, less than a quadrant, is found from the sine of the complement.]

But if you subtract the proportional part found above, namely 1", from the arc $41^\circ 19'$ so found (because we assigned to the given sine 7510767 an arc larger than is proper), you obtain the more precise arc $41^\circ 18' 59''$. This will also be found if you subtract from a quadrant the arc $48^\circ 41' 1''$ — the proportional part being included — that was found above.

Again, let 455630 be given as the sine of the complement of some arc, which I do not find as an entry in the table. I therefore take the next larger, 456536 (since it differs less from this than from the next smaller, 453630), to which in the lower part of the table there corresponds the sought arc $87^\circ 23'$; for its complement is the arc $2^\circ 37'$, belonging to the given sine treated as a right sine. But if you add the proportional part found above, namely 19", to the arc $87^\circ 23'$ so found (because we assigned to the given sine 455630 an arc smaller than is proper), you obtain the more precise arc $87^\circ 23' 19''$. You will also find it if you subtract from a quadrant the arc $2^\circ 36' 41''$ — belonging to the same sine treated as a right sine, and found by the proportional part.

[Margin: How an arc greater than a quadrant is investigated from the sine of the complement.]

Now if the proposed sine is the sine of the complement of an arc greater than a quadrant (when this occurs the operation in triangles, whether rectilinear or spherical, will neatly show), the arc corresponding to it at the top of the table, as though to a right sine, must be taken and added to a quadrant, so that the sought

arc is made up. Thus, if 7510767 is known to be the sine of the complement of some arc greater than a quadrant, the arc corresponding to it at the top of the table, together with the proportional part — namely $48^{\circ} 41' 1''$ — must be taken and added to a quadrant. There will then be composed the arc $138^{\circ} 41' 1''$, which is sought, and to whose complement the given sine belongs.

[Margin: How an arc is derived from the versed sine.]

Finally, from a known versed sine you will inquire the arc thus. If the given versed sine is less than the whole sine, subtract it from the whole sine; the remainder will be the sine of the complement of the sought arc, from which, as we just taught, you will find the sought arc. But if the given versed sine exceeds the whole sine, subtract the whole sine from it; there will remain the right sine of an arc which, added to a quadrant, makes up the sought arc.

Example. Let the versed sine 9544370 be given. I subtract this from the whole sine 10000000, and there remains 455630, the sine of the complement of the sought arc, from which the arc $87^{\circ} 23'$ is found — or, more precisely by the proportional part, $87^{\circ} 23' 19''$. Likewise, let the versed sine 10455630 be given. From this I subtract the whole sine 10000000, and there remains 455630, a right sine, whose arc is $2^{\circ} 37'$ — or, more precisely by the proportional part, $2^{\circ} 36' 41''$ — which added to a quadrant yields the sought arc $92^{\circ} 37'$, or more precisely $92^{\circ} 36' 41''$.

The reasoning of this operation is evident from the first figure set out in the exposition of the definitions. For in it the versed sine AH, taken from the whole sine AE, [leaves] the sine of the complement of the arc AF, which belongs to the said versed sine AH. Likewise, from the versed sine HC the whole sine EC is subtracted, and this leaves EH, or KF, the right sine of the arc FB, which added to the quadrant BC composes the arc FC corresponding to the said versed sine HC.

[Margin: By what method the chord of any arc, and conversely the arc of any chord, is obtained from the table of sines.]

From the same table of right sines we shall also track down the chord of any arc, and conversely we shall find the arc of any given chord. For if we take the right sine of half the proposed arc and double it, we shall form the chord of the said arc. Likewise, if we take half the given chord as a right sine and derive its arc, this doubled arc will give the arc corresponding to the given chord. This is clear from the same first figure that we drew in the explanation of the definitions. For in it FH, the right sine of the arc AF or FC, is half the chord FG of the arc FAG or FCG, and so on.

But since to derive the proportional part in the way we have described is exceedingly tedious, we wished to relieve students of this labor. We have therefore ourselves computed the proportional parts that answer to the individual seconds, and have arranged them in the columns of this table; and we shall briefly explain their use.

On Finding the Proportional Part of Sines and Arcs by an Easier Method

[De parte proportionali sinuum, et arcuum faciliore modo inquirenda.]

[Margin: An explanation of the numbers for deriving the proportional part of the sines.]

Before we teach by what method the proportional part is more easily derived from the preceding table of sines, we must first explain what the two numbers interposed among the columns of sines signify, and by what device they were produced. The former contains the parts of the difference between the two sines between which it is written, answering to one second of the arc that the degrees at the top of the table and the minutes at the side of the same table express; the latter number embraces the tenth-parts of one part of the aforesaid difference. Thus, since between the two sines of $16^{\circ} 12'$ and $16^{\circ} 13'$ these two numbers, 46 and 5, are placed, we shall gather that to one second, between $12'$ and $13'$ of the 16th degree, there answer $46 \frac{5}{10}$ parts — from the difference 2793 between the two sines 2789911 and 2792704 of the aforesaid arcs $16^{\circ} 12'$ and $16^{\circ} 13'$, which whole difference belongs to 60 seconds, that is, to one minute. The same is to be understood of the sines of the following arcs, as far as the arcs $16^{\circ} 37'$ and $16^{\circ} 38'$, between whose sines these other numbers, 46 and 4, are placed, so that now to one second there answer $46 \frac{4}{10}$ parts from the difference of the two nearest sines. And so of the rest.

[Margin: The production of the numbers for deriving the proportional part of the sines.]

These numbers placed among the sines were produced in the following way. Having found the differences of all the sines, we divided each by 60 seconds, in order to produce the parts belonging to one second; the remaining fraction we reduced to tenths, multiplying it by 10, as we taught in problem 14 of chapter 16 of our *Arithmetic* [Clavius, *Epitome Arithmeticae Practicae*]. For in this way the proportional part is derived with less labor, as will soon be plain. For example: the aforesaid difference 2793, if divided by 60, gives a quotient of 46, and there

remain $33/60$, which make 5 tenths and a half. Setting aside the half — for when the fraction of one tenth exceeds $1/2$ we added one tenth in the table, but when it does not exceed $1/2$, but is either equal or smaller, we neglected it — we wrote in the table 46 and 5, standing for $46 \frac{5}{10}$ whole parts of the difference, which make 465 tenths of one part. These are also produced if the whole difference 2793 is multiplied by 10 and the product 27930 is divided by 60. And because in the following differences, right up to the difference of the sines of $16^\circ 37'$ and $16^\circ 38'$ exclusive, this same number 465 is found by this reasoning — that is, the part 46 and 5 tenths — this proportional part will serve us up to $16^\circ 37'$ and $16^\circ 38'$ exclusive, where now a smaller number is found, namely 46 and 4 tenths. Thus, since the difference between the sines 2837364 and 2840153 (of $16^\circ 29'$ and $16^\circ 30'$ [?]) is 2789, if this is multiplied by 10 and the product 27890 is divided by 60, the quotient will be 464, and there will remain $50/60$, which exceed $1/2$. We shall therefore again have the parts 46 and 5 tenths. And so of the rest.

By the benefit of these numbers the proportional part will be found very quickly, namely by a single multiplication or division. For if the right sine of some arc is to be sought, which besides the minutes comprises also seconds, one must take from the table the sine answering to the degrees and minutes of the proposed arc set at the top of the table, and add to it the number produced by multiplying the interposed number immediately preceding into the number of seconds. Thus, if the right sine of $19^\circ 36' 40''$ is sought: since this arc is immediately preceded in the table by the numbers 45 and 7 — that is, 457 tenths — which multiplied into 40 seconds produce 18280 tenths, that is, 1828 whole parts, we shall add 1828 to 3354516, the sine of $19^\circ 36'$, so as to make up 3356344, the sine of the proposed arc $19^\circ 36' 40''$.

[Margin: Finding the right sine together with the proportional part.]

Conversely, if the arc is to be inquired from the right sine, one must take the arc answering to the next-smaller sine, and append to it as many seconds as there are units contained in the quotient, when the difference between the proposed sine and the next-smaller sine (a zero [ziphra] having first been appended, so that it may be reduced to tenth-parts) is divided by the number of tenths found in the table. Thus, if the sine 3356344 is given, we shall take the arc $19^\circ 36'$ corresponding to the next-smaller sine 3354516, and shall adjoin to it 40 seconds; this number is generated by dividing 1828, the difference between the proposed sine and the next-smaller sine — a zero having first been appended, so that we divide 18280 by 457, the tenths found in the table. For thus the sought arc will be $19^\circ 36' 40''$. A zero is appended to the difference found, 1828, because, since it

must be divided by $45 \frac{6}{10}$, it must be multiplied by 10 and the product divided by 457, as is clearly established from our *Arithmetic*.

[Margin: Finding the arc together with the proportional part from a given right sine.]

But if the sine of the complement of some arc less than a quadrant is to be investigated, which besides minutes also has seconds, one must take from the table the sine answering to the degrees and minutes of the proposed arc set in the lower part of the table, and subtract from it the number produced by multiplying the upper interposed number into the number of seconds. Thus, if the sine of the complement of $70^\circ 23' 20''$ is sought: since these interposed numbers 45 and 7 — that is, 457 tenths — serve this arc, we shall multiply 457 into 20 seconds, and the resulting number, which is 9140 tenths, that is, 914 whole parts, we shall subtract from 3357256, the sine of the complement of the arc $70^\circ 23'$, so that there remains the sine 3356342 of the complement of the arc $70^\circ 23' 20''$.

[Margin: Finding the sine of the complement together with the proportional part.]

Otherwise, and perhaps more conveniently, so that the rules are not multiplied: take the complement of the given arc, and investigate its right sine, as we have already taught. Thus in the same example, the complement of the arc $70^\circ 23' 20''$ is the arc $19^\circ 36' 40''$, whose right sine will be found to be 3356344 — greater by two units than the one found by the other method just above. This happens because the proposed arc lies but little short of the following, smaller interposed number.

[Margin: A more convenient way of finding the sine of the complement of an arc less than a quadrant, together with the proportional part.]

When the arc whose complement's sine is sought is greater than a quadrant but less than a semicircle, we shall subtract a quadrant from the given arc and inquire the right sine of the remaining arc. Thus, if the sine of the complement of the arc $109^\circ 36' 40''$ is sought, then, a quadrant being taken away, there remains the arc $19^\circ 36' 40''$, to which the sine 3356344 belongs.

[Margin: Finding the sine of the complement of an arc greater than a quadrant, together with the proportional part.]

Conversely, if the arc is to be elicited from the sine of the complement, one must take the arc, together with the proportional part, answering to the given sine as though to a right sine; and this must be subtracted from a quadrant, if the given sine is the sine of the complement of an arc less than a quadrant, or added to a

quadrant when the given sine answers to the complement of an arc greater than a quadrant. But the operation itself, in triangles whether spherical or rectilinear, will neatly show whether the proposed sine agrees with the complement of an arc less than a quadrant or greater. Thus, if the proposed sine is 3356342, of the complement of an arc less than a quadrant, it will be found — the arc $19^{\circ} 39' 40''$ [?; recte $19^{\circ} 36' 40''$] being taken — which subtracted from a quadrant leaves the sought arc $70^{\circ} 32' 20''$ [?; recte $70^{\circ} 23' 20''$]. But if the same sine belongs to the complement of an arc greater than a quadrant, we shall add the arc found for it to a quadrant, and make up the arc $109^{\circ} 36' 40''$. For to its complement, namely the arc $19^{\circ} 36' 40''$, the sine 3356342 corresponds.

[Margin: Finding the arc, together with the proportional part, from a given sine of the complement.]

Finally, the versed sine of an arc which, besides degrees and minutes, has seconds annexed as well, will be found if the sine of its complement — found together with the proportional part — is either taken from the whole sine or added to the whole sine, according as the arc is less or greater than a quadrant. Thus, if the versed sine of the arc $70^{\circ} 23' 20''$ is sought, we shall find the sine of its complement, namely of $19^{\circ} 36' 40''$, to be 3356342, which subtracted from the whole sine 10000000 makes the remainder — the sought versed sine — 6643658. But if the versed sine of the arc $109^{\circ} 36' 40''$ is desired, we shall find the sine of its complement, namely of $19^{\circ} 36' 40''$, to be 3356342, which added to the whole sine 10000000 makes up the sought versed sine 13356342.

[Margin: Finding the versed sine together with the proportional part.]

By the same reasoning, if the arc is to be found from the versed sine, we shall subtract it from the whole sine, or the whole sine from it — the smaller, that is, from the larger. For thus the remainder becomes the sine of the complement of the sought arc, from which the sought arc will be elicited. Thus, if the given versed sine is 6643658, we shall subtract it from the whole sine 10000000, and with the remainder 3356342, as though with a right sine, we shall fish out the arc $19^{\circ} 36' 40''$, which taken from a quadrant leaves the sought arc $70^{\circ} 23' 20''$. But if the given versed sine is 13356342, we shall subtract the whole sine from it, and with the remainder 3356342 we shall track down the arc $19^{\circ} 36' 40''$, which added to a quadrant makes up the sought arc $109^{\circ} 36' 40''$.

[Margin: Finding the arc, together with the proportional part, from a given versed sine.]

But since we do not always have the table of sines at hand, I judged it would not be unwelcome to students if I briefly taught here, before proceeding to other matters, by what method we may use sines geometrically — without the aid of numbers — in setting out the theorems and problems of astronomers and geometers: so that by the help of the compass alone we may attain all those things that are usually inquired by long multiplications and divisions of the numbers contained in the table of sines. For by this means those especially will be well served who either feel great trouble in the reckoning of numbers, or have not much practiced themselves in it. To carry this out the more conveniently, we shall set out the whole matter with one or two examples.

[Margin: In what way sines are to be used geometrically, without the table of sines.]

Let it be proposed, then, for example's sake, to investigate the declination of any point of the ecliptic, say the 20th degree of Taurus. Let there be described a circle ABCD, together with two diameters AC and BD cutting one another at right angles at the center E. And since, as we showed in the corollary to proposition 1 of book 1 of our *Gnomonics* [Clavius, *Gnomonice* I, coroll. prop. 1], the ratio of the whole sine to the sine of the greatest declination is as the sine of that arc by which the given point is distant from the nearer equinoctial point to the sine of the declination of that same point — let the arc of the greatest declination DF be taken (which indeed will easily be done if there is at hand a bronze or wooden quadrant accurately divided into 90 degrees, of which we wrote at the beginning of our *Gnomonics*; for without such a quadrant it would not be worth the trouble to wish to use sines without numbers), and the arc DG of 50° , by which the given 20th degree of Taurus is distant from the beginning of Aries. And from F and G let the perpendiculars FI, GH be dropped to DE — which will easily be done if on the arcs DF, DG there are taken...

Diagram. A composite geometric construction printed at the foot of the page, illustrating the compass-only method for finding the declination of a point of the ecliptic. At the left is a circle lettered ABCD with two perpendicular diameters AC (vertical, A at top, C at bottom) and BD (horizontal, B at left) crossing at the center E; the interior is filled with a construction of chords and dropped perpendiculars (small right-angle marks are visible). To the right of the circle is an unfolded arrangement of straight-line (rectilinear) triangles carrying further letters — the portion of the construction whose explanatory text (the perpendiculars FI and GH to DE, and what follows) runs on beyond this page. This figure: many small

peripheral labels could not be assigned roles with confidence, and the construction is completed in the following pages.

Labels read on the leaf (roles per this page's text where stated; positions approximate):

- A (top of circle): endpoint of the vertical diameter AC.
- C (bottom of circle): opposite endpoint of the vertical diameter AC.
- B (left of circle): endpoint of the horizontal diameter BD.
- D: endpoint of the horizontal diameter BD (right side / carried into the right-hand figure).
- E (center): intersection of diameters AC and BD.
- F, G: points on the arc from D (the text takes arc DF = greatest declination, arc DG = 50°).
- H, I: feet of the perpendiculars GH and FI dropped to DE.
- K, L (right-hand figure): points of the auxiliary rectilinear triangles [role given on the following pages].
- Q, R, S (upper part of circle) [?]: interior/perimeter construction points, roles not stated on this page.
- P, N (left of circle) [?]: interior/perimeter construction points.
- M (interior, right of E) [?]: construction point.
- X, O (lower-left interior) [?]: construction points (near a small square).
- T (lower-left of circle) [?], V (lower-left of circle) [?]: perimeter points.

the arcs DK and DL are equal. For the straight lines that join the points G, L and F, K will be perpendicular to DE — since, by the scholium set down among the definitions, they are bisected at H and I,^a [Margin (a): *Elements* III.3] and therefore meet it at right angles. And FI, GH will be the right sines of the arcs DF, DG. Therefore, if from three given straight lines — ED (the whole sine), FI (the sine of the greatest declination), and GH (the sine of the arc by which the given point is distant from the equinox) — a fourth proportional is found,^b [Margin (b): *Elements* VI.12] the fourth proportional found will be the right sine of the declination sought.

We shall find this fourth proportional without much labour by Euclid's method. Draw two straight lines AB, AC making any angle at A. On one of them take AD equal to the first line, that is, to the whole sine ED; and DB equal to the second line, namely to FI, the sine of the greatest declination. On the other take AE equal to the third line, namely to GH. Then, having drawn the straight line DE, draw BC parallel to it through B. EC will be the fourth proportional — that is, the right sine

of the declination of the 20th degree of Taurus. We shall then find its arc as follows.

Having found the straight line EC, cut off from the semidiameter EA a part EM equal to it, and through M draw MN parallel to EB. The arc BN will be the arc of the declination sought, since it corresponds to the right sine EM (that is, EC): for, when the right sine NO has been drawn, the arc NB and the straight lines EM, ON are equal to one another because of the parallelogram MO. We shall obtain the same arc in yet another way. Having found the straight line EC, cut off CF equal to it, so that EF is double EC — that is, the chord of the arc that is double the arc whose right sine is EC.^c [Margin (c): *Elements* I.34] For if we fit into the circle a chord PQ equal to the straight line EF, and bisect its arc PQ at R, then QR (or PR) will likewise be the arc of the declination sought, corresponding to the sine EC — that is, to the half of the chord PQ — as is clear from the definition of the right sine. Moreover, the quadrant divided into 90 degrees will show (if one also wishes to know this) how many degrees and minutes are contained in BN, or in PR, the arc of declination — though the minutes must be taken by estimation, because the degrees of the quadrant, unless it were very large, cannot be divided into minutes.

Again, let the right ascension of the 20th degree of Taurus be sought. Now, as we demonstrated in the scholium to Proposition 9 of Book 2 of the *Gnomonice*, the sine of the complement of the declination of the proposed point is to the sine of the complement of the arc by which the given point is distant from the nearer equinoctial point, as the whole sine is to the sine of the complement of the right ascension. So, in the same figure, take the arc BN of the declination of the 20th degree of Taurus, which in the table of declination amounts to 17 degrees 47 minutes, and draw NM perpendicular to EA — this will be the sine of the complement of the said declination. Take also the arc DG of 50 degrees, by which the given point is distant from the vernal equinox, and to EA draw the perpendicular GS, namely the sine of the complement of the said arc DG. After this, from three straight lines — NM (the sine of the complement of the declination of the given point), GS (the sine of the complement of the arc DG, by which the given point is distant from the equinoctial point), and ED (the whole sine) — let a fourth proportional LI be found, as was done in the lines GH, GI cutting one another at G: for there GK was taken equal to NM, and KH equal to GS, and GI equal to the whole sine ED, and so on.^d [Margin (d): *Elements* I.34] For LI, once found, will be the sine of the complement of the right ascension of the given 20th degree of Taurus. Therefore, if from the semidiameter EC a straight line ET equal to LI is cut off, and TV is drawn parallel to EB, and VX parallel to EC,

the straight lines ET, VX will be equal. Since, then, the arc BV corresponds to the sine VX, its complement VC will be the right ascension of the 20th degree of Taurus. And the quadrant divided into 90 degrees will indicate how many degrees the arc VC comprises.

In the same way we shall carry out geometrically, by means of sines, all the other problems — even if it is sometimes necessary to use versed sines, which are found from given arcs with the same ease, and from which we recover the arcs whose right sines and sines of the complements we already have. For if the given arc is less than a quadrant, as AG, then, when GS is drawn perpendicular to AE, AS will be the versed sine of the arc AG. But if the given arc is greater than a quadrant, as AV, then, when VT is drawn perpendicular to EC, AT will be the versed sine of the arc AV, as is evident from the definition.

But let us now proceed to the geometric investigation of chords, from which we shall in turn compose the table of sines with little trouble.

Diagram. The declination construction begun in the preceding pages (circle *ABCD* with centre *E*, the greatest-declination arcs *DF*, *DG* and their right sines *FI*, *GH*), here reprinted in its fuller form with the whole apparatus of sines, versed sines and the auxiliary fourth-proportional triangle. The description continues that of the prior construction; see it there. This is a densely lettered figure and several peripheral labels are hard to resolve even at high magnification.

- Circle centred at *E*, with the vertical diameter through *A* (top) and *C* (bottom).
- *A* (top of circle): endpoint of the diameter / vertex of the external triangle.
- *E* (centre): centre of the circle.
- *D* (right, on circle): endpoint from which the declination arcs are measured.
- *F*, *G* (near right of centre): endpoints of the declination arcs *DF*, *DG*.
- *H*, *I* (right of centre): feet of the perpendiculars / sine points (*FI*, *GH*).
- *M* (centre-right, interior): *EM* cut equal to *EC*.
- *N*, *O* (left interior): *MN* parallel to *EB*; *NO* the right sine.
- *S* (upper interior): perpendicular foot (*GS* / *NM* complement-sine construction).
- *B*, *O* (left on circle): *B* the arc endpoint.
- *P*, *R* (upper left, on circle): chord *PQ* bisected at *R*.
- *V*, *T*, *L*, *K* (lower and right interior): points of the right-ascension construction (*ET*, *TV*, *VX*).

- X (left interior): [?]
- To the right, external to the circle: the fourth-proportional triangle with vertex at A, sides AB and AC, on which AD (= whole sine ED), DB (= FI, sine of greatest declination), AE (= GH) are laid off, and BC drawn parallel to DE so that EC is the fourth proportional; the labelled points read D, B (top), G, E, C, F, K, H, L, I descending along the two rays.

THEOR. 7. PROPOS. 10.

If two unequal arcs are taken in a circle, the chord of the greater being greater than the chord of the lesser, then the ratio of the greater arc to the lesser is greater than the ratio of the chord of the greater arc to the chord of the lesser arc.

[Margin: *The ratio of the greater arc in a circle to the lesser arc is greater than the ratio of the chord of the greater arc to the chord of the lesser.*]

In the circle ABCD let there be unequal arcs AB, BC — the former greater, the latter lesser. I say that the ratio of the arc AB to the arc AC is greater than the ratio of the chord AB to the chord BC. For let the chords AB, BC contain the angle ABC, so that the arcs are continuous and together less than the whole circumference. For if they were equal to the whole circumference, the chord of each arc would be the same; and if they exceeded the whole circumference, the chord of the lesser arc would be greater than that of the greater, as is clear in the second figure, if the lesser arc were BAI. Now let the angle ABC be bisected by the straight line BD, and let the straight lines AC, AD, CD be joined, of which AC cuts the straight line BD at E.^a [Margin (a): *Elements* I.9] The straight lines AD, CD will be equal,^b [Margin (b): *Elements* III.29] because the arcs AD, CD, which are subtended under the angles ABD, CBD — equal by construction — are equal to one another.^c [Margin (c): *Elements* III.26] And since in the triangle ABC the straight line BE bisects the angle ABC, it will follow that as AB is to BC, so is AE to EC.^d [Margin (d): *Elements* VI.3] But the straight line AB is greater than the straight line BC by hypothesis. Therefore AE too will be greater than EC. Hence, when AC is bisected at F, the point F will lie in the greater segment AE. And when the straight line DF has been drawn: since the sides AF, FD of the triangle AFD are [equal to] the sides CF, FD of the triangle CFD, and the base AD has been shown equal to the base CD, [the angle AFD is equal to the angle CFD...]

Diagram. Two circles for Theorem 7 (the second figure recurs at the top of the following page and is described here only). *First figure:* circle with A (left), B (upper right), C (right), D (bottom); the chords AB, BC are drawn, with the bisector

BD, and AC meeting BD at E; further lines AD, CD, DF, with interior points H, E, F, G (F the centre). *Second figure*: a circle with I (left), A, B (upper right), C (right), D (bottom) and interior points H, E, F, G — illustrating the case in which the lesser arc is BAI (arcs together exceeding the circumference).

- A: left vertex / arc endpoint.
- B: upper arc endpoint (greater-arc side).
- C: right arc endpoint.
- D: foot of the bisector on the circle.
- E: intersection of AC with BD.
- F: centre / midpoint of AC.
- G, H: auxiliary points of the sector/triangle comparison.
- I (second figure): endpoint of the arc BAI.

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equal. The angle AFD will be equal to the angle CFD;^e [Margin (e): *Elements* I.8] and since they are equal, each of them will therefore be a right angle.^f [Margin (f): *Elements* I.17] Since, then, in the triangle DEF the two angles E, F are less than two right angles, the angle E will be acute, and consequently the remaining angle CED obtuse.^g [Margin (g): *Elements* I.16] Hence in the triangle CDE the two angles C, E are less than two right angles, so the angle C is acute. Therefore in the triangle DEF the side DE is greater than the side DF, and in the triangle CDE it is less than the side CD. Consequently, an arc of a circle described from D as centre through E will cut the straight line DF, produced, at H, and the straight line CD, below the point C, at G. Now the sector DHE has a greater ratio to the triangle DEC than the triangle DFE has to the same triangle DEC;^h [Margin (h): *Elements* V.8] likewise the same sector DHE has a greater ratio to the sector DEG than it has to the triangle DEC;ⁱ [Margin (i): *Elements* VI.33, corollary 1] and so the sector DHE will have a much greater ratio to the sector DEG than the triangle DFE has to the triangle DEC. But as the sector DHE is to the sector DEH, so is the angle GDE to the angle EDG. Therefore the ratio of the angle HDE to the angle EDG will also be greater than that of the triangle DFE to the triangle DEC.^k [Margin (k): *Elements* VI.1] But as the triangle DFE is to the triangle DEC, so is the straight line FE to the straight line EC. Therefore the ratio of the angle HDE to the angle EDG is also greater than that of the straight line FE to the straight line EC.

And by composition the ratio of the angle HDG to the angle EDG will also be greater than that of the straight line FC to the straight line EC.^l [Margin (l): *Elements* V.28] Now since it holds that as the angle ADC is to the angle HDG, so is

the straight line AC to the straight line FC (for in both the ratio is double), while the angle HDG has to the angle EDG a greater ratio than the straight line FC to the straight line EC, as we have shown,^m [Margin (m): *Elements* V.31] it follows *ex aequo* that the ratio of the angle ADC to the angle EDG will also be greater than that of the straight line AC to the straight line EC, as appears in this scheme.ⁿ [Margin (n): *Elements* V.29] Therefore *by division* the ratio of the angle ADE to the angle EDG will also be greater than that of the straight line AE to the straight line EC.^o [Margin (o): *Elements* VI.33] But as the angle ADE is to the angle EDG, so is the arc AB to the arc BC;^p [Margin (p): *Elements* VI.3] and as the straight line AE is to the straight line EC, so is the chord AB to the chord BC. Therefore the ratio of the arc AB to the arc BC will also be greater than that of the chord AB to the chord BC. Therefore, if two unequal arcs are taken in a circle, and so on. Which was to be demonstrated.

Anguli [Angles]	Rectæ [Straight lines]
ADC.	AC.
HDG.	FC.
EDG.	EC.

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Although this theorem is proposed only for those unequal arcs of which the greater subtends the greater chord, it nonetheless holds also in those unequal arcs of which the chord of the greater is smaller than the chord of the lesser. For since in that case the greater arc has to the lesser a ratio of greater inequality, while the chord of the greater arc has to the chord of the lesser arc a ratio of lesser inequality, the ratio of the greater arc to the lesser will be greater than that of the chord of the greater arc to the chord of the lesser arc.

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It follows from this proposition that the ratio of the lesser arc to the greater is smaller than that of the chord of the lesser arc to the chord of the greater. For since the greater arc has to the lesser a greater ratio than the chord of the greater arc to the chord of the lesser, as has been demonstrated,^q [Margin (q): *Elements* V.26] then, *by conversion*, the lesser arc will have to the greater a smaller ratio than the chord of the lesser arc to the chord of the greater.

THEOR. 8. PROPOS. 11.

If a quadrilateral together with its diagonals is described in a circle, then the rectangle contained by the diagonals is equal to the two rectangles together that are contained by the opposite sides.

[Margin: *The rectangle contained by the diagonals of a quadrilateral described in a circle is equal to the two rectangles contained by the opposite sides.*]

In the circle ABCD let ABCD be a quadrilateral whose diagonals are AC, BD. I say that the rectangle contained by AC, BD is equal to the two rectangles together contained by AD, BC and by AB, DC. Make the angle BAE equal to the angle DAC; and the straight line AE will fall either upon the straight line AC itself, or between the straight line AC and the point B, or, finally, between the straight line AC and the point D. In the first case the angle BAC will be [equal to] the angle DAE; in the second case the whole angle BAC will equal the whole angle DAE, because the common angle EAC is added; and in the third case the remaining angle BAC will equal the remaining angle DAE, because the common angle EAC is subtracted.^a [Margin (a): *Elements* III.21] And since the angle ACB is also equal to the angle ADB,^b [Margin (b): *Elements* I.32] the remaining angle ABC in the triangle ABC will likewise be equal to the remaining angle AED in the triangle AED. Therefore it will follow that as AC is to CB, so is AD to DE.^c [Margin (c): *Elements* VI.4] Hence the rectangle under AC, DE is equal to the rectangle under CB, AD.^d [Margin (d): *Elements* VI.16] Again, since the angle BAE is equal to the angle DAC by construction, and the angle ABD to the angle ACD,^e [Margin (e): *Elements* III.21] the remaining angle AEB in the triangle AEB will be equal to the remaining angle ADC in the triangle ADC. Therefore it will follow that as AC is to CD, so is AB to BE.^f [Margin (f): *Elements* VI.4] Hence the rectangle under AC, BE is equal to the rectangle under CD, AB.^g [Margin (g): *Elements* VI.16] Now since the rectangle under AC, DE has been shown equal to the rectangle under CB, AD, and the rectangle under AC, BE to the rectangle under CD, AB, and since the rectangles under AC, DE and under AC, BE are together equal to the rectangle under AC, BD,^h [Margin (h): *Elements* II.1] the rectangle under AC, BD will be equal to the rectangles contained under BC, AD and under CD, AB. Therefore, if a quadrilateral is described in a circle, and so on. Which was to be demonstrated.

Diagram. Three circles for Theorem 8, one for each position of the auxiliary line AE. Each shows an inscribed quadrilateral ABCD with the diagonals AC, BD drawn, and the point E where the line making the angle BAE equal to DAC meets AC (or falls within the quadrilateral).

- A, B, C, D: the four vertices of the inscribed quadrilateral (A upper left, B lower left, C lower right, D upper right).
- E (interior): the intersection point produced by making angle BAE = angle DAC — falling on AC in the first figure, between AC and B in the second, and between AC and D in the third.

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When the figure described in the circle is a square, as in the first figure, the theorem will be demonstrated more easily, in this way. Since the rectangle under AC, BD — that is, the square on AC (for the diagonals in a square are equal)ⁱ [Margin (i): scholium on *Elements* I.34] — is equal to the squares on AD, DC,^k [Margin (k): *Elements* I.47] that is, to the rectangles contained under AD, BC and under AB, DC (because of the equality of the straight lines AD, BC and AB, DC), what is proposed is clearly established.

PROBL. 4. PROP. 12.

From a given circle-diameter of any number of parts, to find, in the same parts, the sides of the equilateral triangle, the square, the hexagon, the pentagon, and the decagon described in the same circle.

[Margin: *In what way, from a given circle-diameter, the sides of the equilateral triangle, square, hexagon, pentagon, and decagon of the same circle are found.*]

Let the diameter be posited as 20000000 parts. Since, then, the side of the hexagon is equal to the semidiameter of the circle,^a [Margin (a): *Elements* IV.15, corollary] it will become known as 10000000 parts.

Again, since the square described on the diameter of any square is double that same square,^b [Margin (b): scholium on *Elements* I.47] and since the diameter of a square inscribed in a circle is the same as the diameter of the circle: if the square on the diameter of the circle is taken, namely 400000000000000, its half, namely 200000000000000, will be the square of the side of the square, whose square root, 14142136, will give the side of the square. This is also found in the following way. Since the square inscribed in a circle is double the square described on the semidiameter, as is clear in the right triangle ADE of the first figure of the preceding proposition,^c [Margin (c): *Elements* I.47] and as has been demonstrated by us in the lemma to Proposition 16 of Book 1 of Theodosius: if the square of the semidiameter, 100000000000000, is doubled, the square inscribed in the circle

will become 20000000000000 parts, whose square root, 14142136, will again give the side of the square.

Furthermore, since the side of the equilateral triangle inscribed in a circle is in square triple the semidiameter of the same circle,^d [Margin (d): *Elements* XIII.12] it comes about that the tripled square of the semidiameter gives the square of the side of the triangle, 30000000000000, whose square root will exhibit that same side as 17320508 parts.

Let there be, in addition, AB, the semidiameter of any circle,^e [Margin (e): *Elements* VI.30] which, divided in extreme and mean ratio at C so that the greater segment is BC, and with AB produced and BD cut off equal to the greater segment BC,^f [Margin (f): *Elements* XIII.5] then AD too will be divided at B in extreme and mean ratio, and the greater segment will be AB;^g [Margin (g): *Elements* IV.15, corollary] which, since it is the side of the hexagon in the circle whose semidiameter is AB, means that BD will be the side of the decagon in the same circle.^h [Margin (h): scholium on *Elements* XIII.9] This is made known in the following way. When AB is bisected at E,ⁱ [Margin (i): *Elements* XIII.3] the square of the straight line DE — composed of the lesser segment DB and half of BE, the greater segment BA — will be five times the square of the straight line BE, which is known, since it is half the semidiameter AB and therefore 5000000 parts. Hence, if the square of the straight line BE is quintupled, the square of the straight line DE becomes 125000000000000, whose square root will give the straight line DE as 11180340 parts. If from this the straight line BE, 5000000 parts, is subtracted, the remainder will be BD, the side of the decagon, 6180340 parts.

Finally, since the side of the pentagon is in square equal to the side of the hexagon and the side of the decagon together,^k [Margin (k): *Elements* XIII.10] if the square of the side of the hexagon, 100000000000000, and the square of the side of the decagon, 38196602515600, are put together, the square of the side of the pentagon becomes 138196602515600, whose square root will give the side of the pentagon as 11755705 parts. And thus the sides of the equilateral triangle, the square, the pentagon, the hexagon, and the decagon have been made known in parts of the circle's diameter, in which they are described. Therefore, from a given circle-diameter of any number of parts we have found the sides of the equilateral triangle, the square, and so on. Which was to be done.

Diagram. A single straight line (the extreme-and-mean-ratio / decagon construction), with the points marked, from left to right, A, C, E, B, D.

- A: left endpoint (one end of the semidiameter AB).

- C: the point of division of AB in extreme and mean ratio.
- E: the midpoint (bisection) of AB.
- B: right end of the semidiameter AB / division point of the whole AD.
- D: right endpoint, with BD cut off equal to the greater segment.

PROBL. 5. PROPOS. 13.

From the given chords of two unequal arcs, to find the chord of the arc by which the greater arc exceeds the lesser.

[Margin: *By what method, from two known chords, the chord of the difference by which the arcs of the given chords differ from one another is found.*]

In the semicircle ABCD let the chords AB, AC, and BC be given, BC being the chord of the arc by which the greater arc AC exceeds the lesser AB; and let it be required to find the chord BC. When the straight lines BD, CD have been drawn: since the chords AB, AC are posited, the chords BD, CD will also be known.^a

[Margin (a): Proposition 3 of this treatise] Therefore the rectangle contained under the given straight lines AB, CD will be known; likewise the rectangle under the given straight lines AC, BD. But the rectangle under the straight lines AC, BD is equal to the two rectangles under AB, CD and under BC, AD.^b [Margin (b): Proposition 11 of this treatise] Therefore, when the known rectangle under AC, CD [i.e. AB, CD] is subtracted, the remaining rectangle under BC, AD becomes known. And this, divided by the known diameter AD, will make the chord BC known. Therefore, from the given chords of two unequal arcs we have found the chord of the arc by which the greater arc exceeds the lesser. Which was to be done.

Diagram. The semicircle for Problem 5: diameter AD (A at lower left, D at lower right) with B and C on the arc; the chords AB, AC, BC and BD, CD are drawn.

- A: left endpoint of the diameter.
- B: point on the arc (endpoint of the lesser arc AB).
- C: point on the arc (endpoint of the greater arc AC), between B and D.
- D: right endpoint of the diameter.

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[Margin: *Praxis.*] And so, given the chords of two unequal arcs, if the chord of the greater is multiplied by the chord of the arc which together with the greater arc makes up the semicircle — which is given by Proposition 3 — and if from the product there is subtracted the number produced from the chord of the lesser arc

multiplied by the chord of the arc which together with the greater arc completes the semicircle — which is given by the same Proposition 3 — and if the remaining number is then divided by the diameter, there will be returned, as known, the chord of the arc by which the greater arc exceeds the lesser: as is clear from the figure and the demonstration of this proposition.

PROBL. 6. PROPOS. 14.

Given the chords of two arcs, to find the chord of the arc that is composed of those two arcs.

[Margin: By what art, from two given chords, the chord of the arc composed of the two arcs of the two given chords may be found.]

In the circle $ABCDE$, whose center is F , let two chords AB and BC be given, and let it be required to find the chord AC of the arc ABC , which is composed of the two arcs AB and BC . Draw the two diameters AD and BE , together with the straight lines BD , CE , and CD . Since the chord AB is given, the chord BD — that of the arc BCD , the remaining arc in the semicircle ABD — will also be given [Margin: a — Prop. 3 of this treatise]. By the same reasoning, since the chord BC is given, the chord CE — that of the arc CDE , the remaining arc in the semicircle BCE — will likewise be given. And since the angles AEB and DFE are equal [Margin: b — Elements I.15], and the sides FA , FB are equal to the sides FD , FE , the bases AB and DE will also be equal [Margin: c — Elements I.4]; and therefore, since AB is given, DE too will be given. Now, since the rectangle contained by the given lines BD , CE is equal to the rectangles contained by the lines BC , DE together with that contained by the diameter BE and the line CD [Margin: d — Prop. 11 of this treatise]: if the rectangle under BC , DE — which is given — be subtracted from the given rectangle under BD , CE , the remaining rectangle under BE , CD is left known; and when this is divided by the known diameter BE , the chord CD becomes known. Consequently the chord AC also, of the remaining arc ABC in the semicircle ACD , will be known [Margin: e — Prop. 3 of this treatise].

Diagram. A circle with center F and five points on its circumference, joined by chords and two diameters (the figure for Proposition 14).

- A (circumference, left): endpoint of diameter AD ; vertex of the composite arc ABC .
- B (circumference, upper left): junction of the two given chords AB , BC ; endpoint of diameter BE .
- C (circumference, top): junction of the two given chords AB , BC .

- D (circumference, right): endpoint of diameter AD; opposite B.
- E (circumference, bottom): endpoint of diameter BE.
- F (center): center of the circle.

Alternatively. Since the chord AB is given, BD too will be given [Margin: f — Prop. 3 of this treatise] — the chord of the remaining arc BCD in the semicircle ABD. But BC is given as well. Therefore, since the chords BD and BC are given, the chord CD of the arc CD — by which the greater arc BD exceeds the smaller arc BC — will also be given [Margin: g — Prop. 13 of this treatise]; and hence again the chord AC of the remaining arc ABC in the semicircle ACD will be given [Margin: h — Prop. 3 of this treatise]. Therefore, given the chords of two arcs, we have found the chord of the arc composed of those two arcs. Which was to be done.

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Therefore, given the chords of two arcs: if the chords of the arcs that, together with those arcs, complete their semicircles (which are found by Prop. 3) are multiplied together, and from the product the number produced by multiplying the two given chords by one another is subtracted, and the remaining number is divided by the diameter, there is left a chord from which, if it be investigated by Prop. 3, a chord results; and this chord will be the one subtending the arc composed of the two arcs of the two given chords. This operation is clear from the figure and from the earlier demonstration of this proposition.

The same operation can also be gathered from the later demonstration, as is evident.

PROBL. 7. PROP. 15.

Given the chord of any arc, to find the chord of half that arc.

[Margin: How, from a given chord, the chord of half the arc of the given chord may be found.]

In the circle ABC, whose center is E, let the chord BC and its arc BDC be given, of which let the half be the arc BD, and let its chord BD be the thing to be found. Draw the diameter DG; it will cut BC into two equal parts (by the lemma set down in the definitions) [Margin: a — Elements III.3], and hence at right angles. Now, joining the straight lines BA and BG, the two triangles ABC and EFC will be equiangular, since the angle EFC has been shown to be right [Margin: b — Elements III.31], the angle ABC is likewise right, being in a semicircle, and the angle C is common. Therefore CF is to FE as CB is to BA [Margin: c — Elements

VI.4]; and, alternately, as CF is to CB, so is FE to BA. Since, then, CF is half of CB, as we showed, EF too will be half of AB; and therefore, since AB is given from the given chord BC [Margin: d — Prop. 3 of this treatise], EF will also be given, and when it is subtracted from the semidiameter ED, the remainder FD is known. Now, in the triangle GBD, the angle B is right [Margin: e — Elements III.31], and from it the perpendicular DB is let fall to the base GD [Margin: f — Corollary of Elements VI.8]; the line DB will therefore be the mean proportional between GD and FD, and so the rectangle under GD, FD (both known) is equal to the square on the line DB [Margin: g — Elements VI.17]. The square on DB is therefore known, and consequently its square root will exhibit the line DB as known. We shall also come to know it in this way: since FD has been made known, the squares on the lines FD and FB will be known; and since these are equal to the square on the line BD [Margin: h — Elements I.47], this square too will be known, and its square root will again make the line BD known. Which is what was proposed.

Diagram. A circle with center E, carrying the diameter DG and the chord BC bisected at F, with the auxiliary lines BA, BG drawn (the figure for the first demonstration of Proposition 15).

- A (circumference, left): vertex of triangle ABC.
- B (circumference, upper left): endpoint of the given chord BC and of the half-arc chord BD.
- C (circumference, right): endpoint of the given chord BC.
- D (circumference, top): endpoint of the diameter DG; end of the sought chord BD.
- E (center): center of the circle.
- F (on chord BC, near center): foot where the diameter DG cuts BC at right angles.
- G (circumference, bottom): the other endpoint of the diameter DG, opposite D.

Alternatively. Let there again be, in the semicircle ABC, the given chord BC with its arc BDC, of which let the half be the arc DC, and let its chord DC be the thing to be given. Draw the chord AB, cut off AE equal to it, and join the straight lines BD, DE. Bisect EC also at F, and let fall the perpendicular DF. Since, then, the two sides BA, AD are equal to the two sides EA, AD [Margin: i — Elements III.27], and they contain equal angles (because the arcs BD, DC are equal), the bases BD and DE will be equal [Margin: k — Elements I.24]. But BD is a straight line equal to the straight line DC [Margin: l — Elements III.29]; therefore the line DE too will be equal to DC. Hence, since the two sides EF, FD are equal to the two sides CF, FD,

and the base DE is equal to the base DC, the angles at F will be equal, and therefore right [Margin: m — Elements I.8]. Now, since the chord AB is known from the given chord BC [Margin: n — Prop. 3 of this treatise], AE — equal to AB — will be known; and when this is subtracted from the diameter AC, the remainder EC is known, and hence also its half FC. Now, since CD is the mean proportional between AC and FC [Margin: o — Corollary of Elements VI.8], the rectangle under AC, FC will be equal to the square on the line DC [Margin: p — Elements VI.17]. Since, then, that rectangle is known, because AC and FC are known, the square on DC too will be known; and consequently its square root will make the line DC known. Therefore, given the chord of any arc, we find the chord of half that arc. Which was to be done.

Diagram. A semicircle on the diameter AC, with the points B and D on the arc, and E, F marked on the diameter, carrying the chords AB, BD, DE, DC and the perpendicular DF (the figure for the alternative demonstration of Proposition 15).

- A (left end of diameter): endpoint of the diameter AC.
- B (arc, upper left): endpoint of the given chord BC.
- C (right end of diameter): endpoint of the diameter AC and of the given chord BC.
- D (arc, top): endpoint of the sought chord DC; foot of the perpendicular DF above F.
- E (on diameter): point where AE, cut off equal to AB, ends.
- F (on diameter, between E and C): midpoint of EC, foot of the perpendicular DF.

COROLLARIVM.

Therefore, if by Prop. 3 the chord is found of the arc that, together with the arc of the given chord, will complete the semicircle: then, if half of this chord thus found be subtracted from the semidiameter, and the remaining number be multiplied by the diameter, the square root of this product will give the chord of half the arc whose chord is given. Or, if the square of that remaining number be joined to the square of half the given chord, a number is formed whose square root will exhibit the sought chord as known. This operation is easily gathered from the figure and from the earlier demonstration of this proposition.

[Margin: Practice.]

Likewise, if by Prop. 3 the chord is found of the arc that, together with the arc of the given chord, completes the semicircle, and this chord thus found be subtracted from the diameter, and half of the remaining number be multiplied by the diameter: the square root of this product will give the chord of half that arc, whose chord is given. As is clear from the figure and from the later demonstration of this proposition.

PROBL. 8. PROPOS. 16.

To compute the chords of all the arcs of a semicircle, which successively exceed one another by one minute, in terms of the parts of the diameter, the diameter being distributed into any number of parts whatever.

[Margin: By what method the chords of all arcs may be computed.]

Let us lay it down that the chords of all the arcs are to be computed with respect to a diameter divided into 200000 parts. But to do this more accurately, the diameter must, in the computations, be set at 20000000 parts. For thus it will come about that, by rejecting the first two figures on the right from each of the chords found, there are left chords more exactly refined with respect to a diameter of 200000 parts — just as we taught at the beginning of Prop. 9 on Sines, where we also added how many parts the whole sine ought to be taken to have in the computation, if the whole sine is wanted in a table of more parts. And this must be understood here of the whole diameter as well, by always setting the diameter, in the computation, at twice as many parts as we there assigned to the whole sine.

First, then, there were found in Prop. 12 the chords of the arcs of 36° , 60° , 72° , 90° , and 120° — namely the sides of the decagon, hexagon, pentagon, square, and equilateral triangle — of 6180340, 10000000, 11755705, 14142136, and 17320508 parts, of which the whole diameter is set at 20000000. And from the chords 6180340 and 11755705 of the arcs of 36° and 72° , there will be found the chords of the remaining arcs in the semicircle, such as those of 144° and 108° , of 19021130 and 16180340 parts, as we showed in Prop. 3.

Next, by Prop. 13 and its corollary, there will be found the chords of all the arcs that are the differences of any two arcs whose chords are known. Thus from the chord of the arc of 120° and the chord of the arc of 36° we shall find the chord of the arc of 84° , which is their difference. Likewise from the chord of the arc of 90° and the chord of the arc of 60° we shall come to know the chord of the arc of 30° , by which those two differ from each other. And in the same way we shall investigate the chords of very many arcs.

Again, by what we demonstrated in Prop. 14 and its corollary, we shall find the chord of any arc composed of two arcs whose chords are known. Thus from the chord of the arc of 60° and the chord of the arc of 90° , the chord of the arc of 150° , composed of those two, will be made known. So too, from the chord of the arc of 90° and the chord of the arc of 36° , the chord of the arc of 126° will be recognized. And so on.

Moreover, by the doctrine of Prop. 15 and its corollary, once the chord of any arc is known, we shall also know the chord of the halved arc. Thus from the chord of the arc of 60° we shall produce the chord of the arc of 30° ; from this, the chord of the arc of 15° ; from this, the chord of the arc of $7^\circ 30'$; and from this, the chord of the arc of $3^\circ 45'$. Likewise from the chord of the arc of 72° we shall explore the chord of the arc of 36° ; from this, that of 18° ; from this, that of 9° ; from this, that of $4^\circ 30'$; and from this, that of $2^\circ 15'$. So too, since by Prop. 13 and its corollary the chord of the arc of 12° — by which the two arcs of 36° and 48° differ from each other — is known from their chords, we shall learn from the chord of the arc of 12° the chord of the arc of 6° ; from this, the chord of the arc of 3° ; from this, the chord of the arc of $1^\circ 30'$; and from this, the chord of the arc of $0^\circ 45'$. Then, if by what has been demonstrated we carefully seek out the chords of the remaining arcs from those already found, we shall find the chords of all the arcs that successively exceed one another by $45'$, in such a way that the first arc contains $0^\circ 45'$, the second $1^\circ 30'$, the third $2^\circ 15'$, and the last of all is the whole semicircle of 180° . Indeed, if, as we have just taught, the chord of the arc of $0^\circ 45'$ has been found, we shall find from it, by the doctrine of Prop. 14 and its corollary, the chords of all the arcs successively exceeding one another by $45'$ — provided we first investigate the chord of the arc of $1^\circ 30'$, composed of two arcs of $45'$ and $45'$; then the chord of the arc of $2^\circ 15'$, which is composed of the two arcs of $1^\circ 30'$ and $45'$; and afterward the chord of the arc of 3° , which is composed of the arc of $2^\circ 15'$ and the arc of $45'$; and so on, always adding an arc of $45'$ to the preceding arc.

[Margin: The computation of the chords of the arcs successively exceeding one another by 45 minutes.]

[Margin: The computation of the chord of the arc of 1° .]

Lastly, we shall investigate the chords of the other arcs by this device. Let there be, in the semicircle ABCDE, the chord AB of the arc of $45'$, and AD the chord of the arc of $1^\circ 30'$, while AC is the chord of the arc of 1° , which is proposed to be found. Since, then, the ratio of arc AC to arc AB is greater than that of chord AC to chord AB [Margin: a — Prop. 10 of this treatise] — and arc AC has to arc AB the sesquitertian ratio (4:3) — chord AC will have to chord AB a ratio less than

sesquitertian. Since, then, the chord AB of the arc of $45'$ has been found from the foregoing to be about 130899 parts, the chord AC of the arc of 1° (which, of course, has to chord AB, that is, to 130899, a ratio less than sesquitertian) will be less than 174532 [Margin: b — Elements V.10], since this number has to that one the sesquitertian ratio. Again, since the ratio of arc AD to arc AC is greater than that of chord AD to chord AC [Margin: c — Prop. 10 of this treatise] — and arc AD has to arc AC the sesquialteran ratio (3:2) — chord AD will have to chord AC a ratio less than sesquialteran. Since, then, the chord AD of the arc of $1^\circ 30'$ has been found from the foregoing to be about 261792 parts, the chord AC of the arc of 1° (to which, of course, chord AD, that is, the number 261792, has a ratio less than sesquialteran) will be greater than 174528 [Margin: d — Elements V.10], since the number 261792 has to this number the sesquialteran ratio. It is settled, therefore, that the chord of the arc of 1° lies between these two numbers, 174532 and 174528, since the former is greater and the latter less.

Diagram. A semicircle on the diameter AE, with the points B, C, D marked on the arc toward the left, and the chords AB, AC, AD drawn from A (the figure for the interpolation of the chord of 1°).

- A (left end of diameter): common origin of the chords AB, AC, AD.
- B (arc, lower left): endpoint of chord AB (arc of $45'$).
- C (arc, upper left): endpoint of chord AC (arc of 1° , the chord sought).
- D (arc, top): endpoint of chord AD (arc of $1^\circ 30'$).
- E (right end of diameter): the far endpoint of the diameter AE.

We shall accordingly set it

at 174530, exactly midway between those numbers. For thus it will not perceptibly differ from the true chord of the arc of 1° .

But you must not think that by this same device the chord of any arc of several degrees can be investigated from the two known chords of the two arcs surrounding it. For since in larger arcs the difference between the arcs and the chords is great, it will be hard to judge what number, from among the intermediates between the two found, ought to be established as the chord of the proposed arc. We shall make this clear by the following example. Let the chords of the arcs of $59^\circ 46'$ and $60^\circ 30'$ be known, of 9964712 and 10075480 parts. If, then, someone wished to extract from these the chord of the arc of 60° , he would have to proceed thus. Since the ratio of the arc of $59^\circ 46'$ to the arc of 60° is less than that of chord to chord [Margin: e — Corollary of Prop. 10 of this treatise]: now 9964712, the chord of the arc of $59^\circ 46'$, has to 10003614 1608/1793 [?: small

fraction digits uncertain] the same ratio which the arc of $59^{\circ}46'$ has to the arc of 60° ; the chord of the arc of 60° will therefore be less than 10003614 1608/1793 [?] [Margin: f — Elements V.10], inasmuch as the chord 9964712 has to it a greater ratio than it has to 10003614 1608/1793 [?]. Likewise, since the ratio of the arc of $60^{\circ}30'$ to the arc of 60° is greater than that of chord to chord [Margin: g — Prop. 10 of this treatise]: now 10075480, the chord of the arc of $60^{\circ}30'$, has to 9992211 60/121 [?] the same ratio which the arc of $60^{\circ}30'$ has to the arc of 60° ; the chord of the arc of 60° will therefore be greater than 9992211 60/121 [?] [Margin: h — Elements V.10], inasmuch as the chord 10075480 has to it a ratio less than it has to 9992211 60/121 [?]. The chord of the arc of 60° would therefore have to be established between these two numbers, 10003614 1608/1793 [?] and 9992211 60/121 [?]; and since these differ greatly from each other (for their difference is about 11403), it will be doubtful how great a value should be assumed. This ambiguity has no place in the thorough investigation of the chord of the arc of 1° , since in such small arcs the chords differ little from the arcs.

Now that the chord of the arc of 1° has been found, we shall also find, by Prop. 15 and its corollary, the chord of the arc of $30'$, and from this the chord of the arc of $15'$. But this last we shall also find by Prop. 13 and its corollary, since the arc of $15'$ is the difference between the arc of 1° and the arc of $45'$, whose chords are now known. By the chord of the arc of $15'$, moreover, we shall come to know, by Prop. 14 and its corollary, the chord of the arc of $30'$, which is composed of the arc of $15'$ and the arc of $15'$. Likewise the chord of the arc of $45'$, composed of the arcs of $30'$ and $15'$. Likewise the chord of the arc of 1° , from the arcs of $45'$ and $15'$. And the chord of the arc of $1^{\circ}30'$ — although all these chords have already been made known by another method. Finally, by this route we shall find the chords of all the arcs successively exceeding one another by $15'$, although we could investigate many of them by other methods, as is clear from Props. 13, 14, 15, and their corollaries.

[Margin: e — Corollary of Prop. 10 of this treatise; f — Elements V.10; g — Prop. 10 of this treatise; h — Elements V.10.]

[Margin: The computation of the chords of the arcs extended by 15 minutes.]

Now, if all the arcs successively exceeding one another by $15'$ are set out in order, together with their chords, and to the right of each chord is written the difference by which it differs from the preceding chord, we shall find, by the rule of proportions, the chords of the other intermediate arcs extended by every five minutes; and from these, the chords of all arcs progressing by single minutes — just as we said above concerning the finding of the sines. That this may be

understood more easily, we shall set out this one example. Let the chord of the arc of 20' be sought. Since, then, the difference between 43632, the chord of the arc of 15', and 87264, the chord of the arc of 30', is 43632, say: if the 15' by which the arc of 15' differs from the arc of 30' require the difference 43632 to be added to the chord of the arc of 15' in order that the chord of the arc of 30' may result, how great a difference do the 5' by which the arc of 15' differs from the arc of 20' demand to be added to the same chord of the arc of 15', in order that the chord of the arc of 20' may be composed? For you will find that a difference of 14544 is required, which, added to 43632, the chord of the arc of 15', will constitute 58176, the chord of the arc of 20'. And the same reasoning holds for the rest.

[Margin: The computation of the chords of the arcs extended by single minutes.]

SCHOLIVM.

But Proposition Six will bring us a great shortcut in this matter. For from it we shall produce very many chords out of others already found, by mere addition or subtraction. For if we take the chord of any arc by which the arc of 120° exceeds a smaller arc, once that arc is assumed, we shall compose the chord of the arc that exceeds the arc of 120° by that same assumed arc — because the difference between the chords of these two greater arcs is equal to the chord of that assumed arc, which is posited no greater than 60°, as we there showed. Thus if we add 3472964, the chord of the arc of 20°, to 15320890, the chord of the arc of 100° — which the arc of 120° exceeds by the said arc of 20° — we shall compose 18793854, the chord of the arc of 140°, which exceeds the arc of 120° by that same arc of 20°. So too, if we add 10000000, the chord of the arc of 60°, to the chord 10000000 of the arc of 60° — which the arc of 120° exceeds by that said arc of 60° — we shall produce 20000000, the chord of the arc of 180°, which exceeds the arc of 120° by that same arc of 60°.

[Margin: A wonderful shortcut for finding very many chords.]

Likewise, if we subtract the chord of any arc not greater than 60° from the chord of the arc that exceeds the arc of 120° by that assumed arc, there will be left the chord of the arc that the arc of 120° exceeds by that same assumed arc. Thus if we subtract 3472964, the chord of the arc of 20°, from 18793854, the chord of the arc of 140° — which exceeds the arc of 120° by that assumed arc of 20° — there will remain 15320890, the chord of the arc of 100°, which is exceeded by the arc of 120° by that same arc of 20°.

Again, if from the chord of any arc greater than 120° there be subtracted the chord of the arc that is as much less than 120° as the former is greater, there will

remain the chord of the arc by which either of them differs from the arc of 120° . Thus if from 18793854, the chord of the arc of 140° , we take away 15320890, the chord of the arc of 100° , there will be left 3472964, the chord of the arc of 20° , by which each of them differs from the arc of 120° . All of which is gathered from the said Prop. 6.

It is enough, therefore, to investigate by the rule of proportions the chords of all the arcs from the beginning of the semicircle up to the arc of 60° . For if from these we find the chords of the arcs that, together with them, complete the semicircle, and from these last the former ones already found are subtracted, there will remain the chords of all the arcs between the arc of 60° and the arc of 120° . Likewise, if the chords of all the arcs from the arc of 60° up to the end of the semicircle were known, and the chords of all the arcs less than the arc of 120° were subtracted from the chords of all the arcs greater than 120° , the remainders would be the chords of all the arcs from the beginning of the semicircle up to the arc of 120° , thus found. Finally, if the chords of all the arcs from the beginning of the semicircle up to the arc of 120° were found, and the chords of all the arcs less than 60° were added to the chords of all the arcs greater than 60° , there would be composed the chords of all the arcs greater than 120° .

[Margin: By what method the sines of all arcs may be found from the chords.]

Furthermore, if the chords of all the arcs of the semicircle are reduced into a table, and the chords of all the arcs which successively

if they exceed 2 minutes, let them be bisected; the sines of all the arcs advancing by single minutes will then be found, as is clear from definition 3.

From this it is evident that the moderns did more correctly in reducing the sines to a table than those who [tabulated] the chords. For sines it is enough to extend the quadrant of the circle through the individual minutes; but for chords one must necessarily extend the whole semicircle through the individual minutes, so that a table of chords comes out twice as large as a table of sines. I say nothing of the fact that the use of sines in astronomical and geometrical matters is far more expeditious, briefer, and easier than that of chords — as is plain to those who have at some time trained themselves in this kind of use.

[Margin: Those who construct a table of sines do more correctly than those [who construct one] of chords.]

Now, just as the differences of the sines gradually decrease from the beginning of the quadrant all the way to its end — so that, when several arcs are set down

exceeding one another equally, the sines of the smaller ones have greater differences than the sines of the larger, as we showed in the corollary to Proposition 1 — so too the differences of the chords gradually decrease from the beginning of the semicircle to its end. For when several arcs are set down whose differences are equal, the chords of the smaller ones have greater differences than the chords of the larger. This we shall now demonstrate.

[Margin: The differences of the chords gradually decrease from the beginning of the semicircle to its end, so that the chords of smaller arcs have greater differences than the chord of larger arcs, provided the arcs have equal differences.]

Let there be, in the semicircle ABCDE, the arcs AB, AC, AD, whose differences BC and CD are equal, and let the chords of these same arcs be AB, AC, AD. Let the straight line AF be cut off equal to the chord AB, and the straight line AG equal to the chord AC. I say that FC, the difference between the chords AB and AC, is greater than GD, the difference between the chords AC and AD. For once the straight line AF (equal to the chord AB) has been cut off, and the straight lines BC, CD, CG, CH have been joined: since the sides HA, AC are equal and the angles they contain at A are equal (on account of the equal arcs BC, CD) [b: I.4], the bases BC, CH will be equal. But the straight line BC is equal to the straight line CD, because of the equal arcs BC, CD. Therefore the straight line CH will also be equal to the straight line CD [c: III.29]. Hence the angles CHD, CDH will also be equal; and since they are less than two right angles, each of them will be acute. In the same way each of the angles ACG, AGC will be acute [d: I.5], because the straight lines AC and AG are equal to each other [f: I.5]. Since, then, in the triangle CGH the angles at G and H are acute, the perpendicular CI dropped to DH will fall within the triangle, onto the straight line GH — as we demonstrated in the scholium to Proposition 13 of Book 2 of Euclid. Therefore, since the two angles CHI, CIH of the triangle CHI are opposite equal right angles — or at any rate the side CI is common — the two sides CH, CD will [have their opposite parts] equal [e: I.27]. Since, then, ID is greater than GD, HI too, and a fortiori HG, will be greater than GD [g: I.26]. But HG is equal to FC, because the straight lines AC, AG are equal to one another, and likewise the straight lines AF, AH are equal to one another. Therefore FC too, the difference of the chords AB, AC, will be greater than GD, the difference of the chords AC, AD. Which was the thing proposed.

[Margin references keyed to the demonstration: a) III.27 — Margin: a 27.tertij; b) I.4; c) III.29; d) I.5; e) I.27; f) I.5; g) I.26.]

Diagram. A semicircle standing on the diameter AE. On the arc, three points B, C, D are marked (left to right), with the chords AB, AC, AD drawn from A; the chords BC and CD subtend the equal successive arcs. Interior straight lines CG, CH are joined and the perpendicular CI is dropped.

- A (lower-left, left end of the diameter): vertex from which the chords are drawn
- E (lower-right, right end of the diameter): other end of the diameter
- B (on the arc, upper-left): first arc point
- C (on the arc, top): second arc point
- D (on the arc, upper-right): third arc point
- F (interior, near the top): point cut off so that AF = chord AB
- G (interior): point cut off so that AG = chord AC
- H (interior, left of I/G): endpoint used in the equal-triangle argument
- I (interior, between H and G): foot of the perpendicular CI

Now it must not be passed over that, if a diameter of 120 parts is assumed and we inquire into the chords of the arcs in parts, minutes, and seconds — as Ptolemy [Ptolemy] did — the chords of arcs smaller than 60° have more parts, minutes, and seconds than the arcs of which they are the chords; whereas the chords of arcs larger than 60° have fewer parts, minutes, and seconds than the arcs corresponding to them. This is manifest in the table set out here, which I have taken over and transcribed from Ptolemy's table. In it you see that the chords of arcs smaller than 60° are greater — as to their number of parts, minutes, and seconds — than the arcs corresponding to them (as to degrees and minutes); while the chords of arcs larger than 60° are smaller than the arcs corresponding to them, as to the same numbers. Of this the demonstration is as follows.

Arcus		Chordæ		
G	M	Par.	M.	Sec.
0	30	0	31	25
10	0	10	27	32
20	0	20	50	16
45	0	45	55	19
50	0	50	42	51
60	0	60	0	0
66	30	65	47	43

Arcus	Chordæ			
80	30	77	32	6
90	0	84	51	10
120	0	103	55	23
170	0	119	32	37

In the semicircle ABCDE let the arc AC be 60° and AB smaller, namely 45° ; and let AD be larger, say $80^\circ 30'$. Let the chords AB, AC, AD be drawn. I say that the chord AB is greater than 45 parts, and the chord AD smaller than 80 parts 30 minutes [h: Prop. 10 of this treatise]. For since the proportion of the arc AC to the arc AB is greater than the proportion of the chord AC to the chord AB — while the proportion of the arc AC to the arc AB is the same as 60 to 45 — the proportion of the chord AC to the chord AB will be less than 60 to 45. Therefore, since the chord AC is of 60 parts (being equal to the radius, by the corollary to Proposition 15 of Book 4 of Euclid), the chord AB will be greater than 45 parts [i: V.8], because 60, to a number which is greater than 45, has a smaller proportion than it has to 45. And thus in the table above you see the chord of the arc of 45° to be 45 parts 55 minutes [and] seconds [19].

Diagram. A semicircle on the diameter AE. Near the left of the arc three points B, C, D are marked close together, with the chords AB, AC, AD drawn from A.

- A (lower-left, left end of the diameter): vertex from which the chords are drawn
- E (lower-right, right end of the diameter): other end of the diameter
- B (arc, upper-left): arc point for the 45° arc
- C (arc, above B): arc point for the 60° arc
- D (arc, upper): arc point for the $80^\circ 30'$ arc

[Margin: 18.quinti — i.e. V.18]

Again, since the proportion of the arc AD to the arc AC is greater than the proportion of the chord AD to the chord AC [k: Prop. 16 of this treatise] — and the proportion of the arc AD to the arc AC is the same as $80^\circ 30'$ to 60° — the proportion of the chord AD to the chord AC will be less than 80 parts 30 minutes to 60. And since the chord AC is of 60 parts, the chord AD will be smaller than 80 parts 30 minutes [l: V.8], because a number which is smaller than 80 parts 30 minutes has a smaller proportion to the same number 60 [than 80 parts 30 minutes has]. And so you see in the table above that the chord of the arc of $80^\circ 30'$ contains only 77 parts 32 minutes 6 seconds. And the same reasoning holds for

the other chords, both smaller and larger than AC — that is, than the chord of the arc of 60° .

[Margin references keyed to the demonstration: h) Prop. 10 of this treatise; i) V.8; k) Prop. 16 of this treatise; l) V.8.]

LINEÆ TANGENTES, ATQVE SECANTES.

[TANGENT LINES, AND SECANTS.]

Although astronomers could explain all their problems and theorems by means of sines alone — as is commonly done by everyone — nevertheless, because many things are dispatched more easily and more briefly if, together with the sines, the tangent and secant lines are also employed (as will be plain from the doctrine of triangles), and because the moderns devised these lines with truly useful judgment and reduced them to tables, it has seemed good to set out these lines too in a few words, so that our doctrine of triangles may come out more complete. For the whole doctrine of triangles seems to consist chiefly in these three kinds of lines — namely, in sines, in tangent lines, and in secants. First, then, one must explain what the tangent line and what the secant of any given arc are.

[Margin: In what the doctrine of triangles consists.]

When, therefore, from one end of any arc smaller than a quadrant a radius has been drawn, at whose extremity a straight line touches the circle, and through the other end of the same arc another straight line is extended from the center up to the tangent line: the portion of the tangent line lying between the two straight lines going out from the center is called the **tangent line** (by others it is called the *Prosinus*, or the "inscribed line") of that arc which those same two straight lines drawn out from the center enclose. And the other straight line, opposite the point of contact, [lying] between the center and the tangent line, is called the **secant line** (which others call the *hypotenuse*, or the *transsinuosa*) of the same arc. Thus, if in the circle AB, whose center is C, an arc AB smaller than a quadrant is taken, and at the extremity A of the radius AC, the straight line AD drawn from the endpoint A touches the circle, while the straight line CD cuts the circle, meeting AD at D (for it will necessarily meet it, because the two angles CAD, DCA are less than two right angles — since the former is a right angle and the latter is less than a right angle, on account of the arc AB being smaller than a quadrant): then AD will be called the **tangent** of the arc AB, and CD the **secant** of the same arc. Some

call the tangent the "inscribed line" (*adscripta*), because it is in a certain way inscribed to the circle; the secant they call the *hypotenuse*, because in the right triangle ACD (for the angle A at the point of contact is a right angle) [a: III.18] it subtends the right angle. And the radius AC — or the whole sine — they call the base of the same triangle.

[Margin: What a tangent line and a secant are.]

[Margin: What the "inscribed line" and the "hypotenuse" are. — a) III.18]

Now, just as in every right triangle, if the side opposite the right angle is set as the whole sine, the remaining two sides are the right sines of the remaining acute angles to which they are opposite; likewise each of the two remaining sides is the sine of the complement of the angle adjacent to it, as was set down in the definitions of sines — so also, if one or the other of the sides about the right angle is established as the whole sine, then the other side about the right angle will be the tangent of the acute angle opposite it, while the side opposite the right angle will be the secant of the same angle. Thus in the right triangle ACD the side CA is the whole sine, namely the radius of the circle AB; and AD is the tangent of the angle C, or of the arc AB, and CD is the secant of the same. In the same way, if DA is established as the whole sine, then AC will be the tangent of the angle D, and DC the secant of the same.

[Margin: If in a right triangle one or the other of the sides about the right angle is set as the whole sine, the other side about the right angle will be the tangent of the acute angle opposite it, and the side opposite the right angle the secant of the same.]

Diagram. A circle with center C. From C, the radius reaches A on the circle (upper left). The straight line AD, tangent at A, runs to the point D outside the circle (upper right). The line CD from the center passes through the point B on the circle to D, so that CD is the secant.

- A (on the circle, upper left): point of contact of the tangent AD
- B (on the circle, right): point where the secant CD crosses the circle
- C (center of the circle): center
- D (outside the circle, upper right): meeting point of the tangent AD and the secant CD

And although we have said that the tangent and secant are taken with respect to an arc smaller than a quadrant, nevertheless the same tangent and secant are customarily referred also to the arc which completes a semicircle with it — so

that two arcs making up a semicircle, or two angles equal to two right angles, have one and the same tangent and secant, just as they have the same right sine, as we set forth in the treatment of sines. Accordingly, if the tangent and secant of some arc larger than a quadrant are sought, one must take the tangent and secant of the arc smaller than a quadrant which completes a semicircle with it.

[Margin: Two arcs making up a semicircle, or two angles equal to two right angles, have the same tangent and secant. How the tangents stand when compared with the whole sine.]

Furthermore, by what method the tangents and secants of all the arcs of a quadrant are rendered known in the parts of the whole sine — and therefore by what route a table of tangents, and likewise a table of secants, is to be constructed — will be made plain in the following propositions, which concern the tangent and secant lines.

THEOR. 9. PROPOS. 17.

The tangent of half a quadrant is equal to the whole sine; the tangent of an arc larger than half a quadrant is greater than the whole sine; and the tangent of a smaller arc is smaller. Finally, the secant of half a quadrant is double the right sine of the same half.

In the quadrant ABC, let the arc CD be half of it, CE larger than the half, and CF smaller. Let the straight line CH be drawn perpendicular to AC [a: Corollary, III.16], which will touch the circle at C, and let the straight lines AG, AH, AI be drawn through the points D, E, F. Likewise let DK be perpendicular to AC. Then CG will be the tangent of the arc CD; CH the tangent of the arc CE; and CI the tangent of the arc CF; while DK is the right sine of the arc CD, and AG its secant. I say that CG is equal to the whole sine AC; that CH is greater; and CI smaller. Likewise that AG is double the sine DK. For since the angles CAG, GAB are equal (on account of the equal arcs CD, DB), and the angle BAC is a right angle [b: III.27], each of them will be half a right angle. Hence the remaining angle CGA in the triangle ACG will also be half a right angle [c: I.32], because the angle C is a right angle. Therefore the straight line CG, the tangent of the arc CD (which is half of the quadrant), will be equal to the whole sine AC [d: I.6]. From this it follows that CH, the tangent of the arc CE (which is larger than half the quadrant), is greater; and CI, the tangent of the arc CF (which is smaller than half the quadrant), is smaller — since the point H necessarily falls above G, and the point I below.

That this can also be shown separately, in the following way, with no account taken of the tangent CG whose arc is half the quadrant: since the arc CE is larger than half the quadrant, the arc BE will be smaller than half the quadrant.

Diagram. A quadrant with vertex A at the lower left, B at the upper left, and C at the lower right, with the quarter-arc running from B to C. At C a tangent line rises perpendicular to AC; on it are marked (from lower to upper) the points I, G, H, with F just below. From A the straight lines AI, AG, AH are drawn through the arc points F, D, E respectively; K lies on the base AC.

- A (lower-left): center/vertex of the quadrant
- B (upper-left): end of the quarter-arc
- C (lower-right): other end; foot of the tangent line
- D (on the arc): point marking half the quadrant (arc CD)
- E (on the arc, above D): point of the arc larger than the half (arc CE)
- F (on the arc, below D): point of the arc smaller than the half (arc CF)
- G (on the tangent at C): tangent point of arc CD
- H (on the tangent, above G): tangent point of arc CE
- I (on the tangent, below G): tangent point of arc CF
- K (on the base AC): foot of the perpendicular DK

Therefore the angle CAH is greater than the angle HAB [e: Scholium on III.27], and hence greater than half a right angle. Since, then, the angle C is a right angle, the remaining angle AHC in the triangle ACH will be smaller than half a right angle [f: I.32]. Hence the straight line CH, the tangent of the arc CE (which is larger than half the quadrant), is greater [g: I.19] than the whole sine AC.

Again, since the arc CF is smaller than half the quadrant, and hence BF is larger, the angle CAI will be smaller than the angle IAB [h: Scholium on III.27], and thus smaller than half a right angle. Since, then, the angle C is a right angle, the remaining angle AIC in the triangle ACI will be larger than half a right angle [i: I.32]; and therefore the straight line CI, the tangent of the arc CF (which is smaller than half the quadrant), will be smaller [k: I.29] than the whole sine AC.

Furthermore, since the angle K is a right angle, and DAK half a right angle (as was shown), ADK too will be half a right angle; and hence AK will be equal to the sine DK [l: I.6]. And since the triangles GAC, DAK are equiangular [m: VI.4], as GA is to AC, so AD, that is AC, is to AK; and therefore the three straight lines GA (the secant), AC (the whole sine), and AK (the sine of half the quadrant) will be in continued proportion. So it will follow that the square on AG is to the square on

AC as the straight line AG is to the straight line AK [n: Corollary to VI.20]. But the square on AG is double the square on AC [o: I.47], because it is equal to the squares on AC and CG, which are equal. Therefore also AG, the secant of half the quadrant, is double the sine AK, or DK, of the same half. And so the tangent of half the quadrant is equal to the whole sine, etc. Which was to be demonstrated.

[Margin references keyed to the demonstration: e) Scholium on III.27; f) I.32; g) I.19; h) Scholium on III.27; i) I.32; k) I.29; l) I.6; m) VI.4; n) Corollary to VI.20; o) I.47.]

SCHOLIVM.

From this proposition the reason is plainly gathered why, in the table of tangents, all the tangents of arcs smaller than 45° are smaller than the whole sine; the tangent of the arc of 45° is equal to the whole sine; and, finally, all the tangents of arcs larger than 45° are greater than the whole sine. Likewise why, in the table of secants, the secant of the arc of 45° is double the sine of the same arc of 45° .

[Margin: Why some tangents in the table of tangents are smaller than the whole sine, and which are greater. Likewise why the secant of 45° is double the sine of 45° .]

THEOR. 10. PROPOS. 18.

What proportion the sine of the complement of any arc has to the right sine of the same arc, that same proportion the whole sine has to the tangent of the same arc; likewise, what proportion the right sine of any arc has to the sine of the complement of the same arc, that same proportion the whole sine has to the tangent of the same complement. And the whole sine is a mean proportional between the sine of the complement of any arc and the secant of the same arc; likewise between the right sine of any arc and the secant of the complement of the same arc. Finally, the whole sine itself is a mean proportional between the tangent of any arc and the tangent of the complement of the same arc.

In the quadrant ABC, let DF be the sine of the arc CD; CE the tangent of the same, [lying] between the radius AC and the secant AE; likewise BG the tangent of the arc BD, which is the complement of the arc CD. I say that the sine of the complement of the arc CD is to the right sine of the same arc as the whole sine AC is to the tangent CE, and so on. For since — as was said in the exposition of the definitions — AF is equal to the sine of the complement of the arc CD (being equal to the right sine of the arc BD, which is the complement of the arc CD), and since

the triangles AFD, ACE are equiangular (on account of the right angles F, C and the common angle A) [a: VI.4]: as AF, the sine of the complement of the arc CD, is to FD, the right sine of the same arc CD, so AC, the whole sine, is to CE, the tangent of the same arc. Which is the first point.

Next, by the same reasoning, as AF, the right sine of the arc BD, is to FD, the sine of the complement of the same arc BD, so AC, the whole sine, is to CE, the tangent of the same complement of the arc BD. Which is the second point.

Third, in the same triangles, as AF, the sine of the complement of the arc CD, is to AD, the whole sine, so AC, the whole sine, is to AE, the secant of the same arc CD. Likewise, as AF, the right sine of the arc BD, is to AD, the whole sine, so AC, the whole sine, is to AE, the secant of the arc CD, which is the complement of the same arc BD. Hence the whole sine is a mean proportional between AF, the sine of the complement of the arc CD, and AE, the secant of the same arc CD; likewise between AF, the right sine of the arc BD, and AE, the secant of the complement of the same arc BD. Which is the third point.

Finally, since the triangles ACE and GBA are equiangular (because the angles C, B are right angles [b: I.29], and the alternate angles CAE, BGA, as well as the alternate angles AEC, GAB, are equal) [c: VI.4]: as CE, the tangent of the arc CD, is to AC, the whole sine, so AB, the whole sine, is to BG, the tangent of the complement of the same arc CD. By like reasoning, as BG, the tangent of the arc BD, is to AB, the whole sine, so AC, the whole sine, is to CE, the tangent of the complement of the same arc BD. Which is the fourth point. Therefore, what proportion the sine of the complement of any arc [has to its right sine], etc. Which was to be demonstrated.

[Margin: What proportion the whole sine has to the tangent and secant of any arc. — a) VI.4; b) I.29; c) VI.4.]

Diagram. A square/quadrant with vertex A at the lower left, B at the upper left, C at the lower right, and E at the upper right; the quarter-arc runs from B to C, passing through the point D near the center. The diagonals from A are drawn: AE (the secant) passes through D; DF is the right sine dropped to the base at F; G lies on the top edge near E; the tangents CE (at C) and BG (at B) are marked.

- A (lower-left): center/vertex of the quadrant
- B (upper-left): end of the arc; foot of the tangent BG
- C (lower-right): end of the arc; foot of the tangent CE
- D (on the arc, near center): point defining the arc CD (and its complement BD)

- E (upper-right): meeting point of the secant AE and the tangent CE
- F (on the base AC): foot of the right sine DF
- G (on the top edge AB extended): tangent point of the complement's tangent BG

[Margin: a 4.sexti — i.e. VI.4]

SCHOLIVM.

And so, by what has been demonstrated in the first place in this proposition: if it is arranged that, as the sine of the complement of any arc is to the right sine of the same arc, so the whole sine is to another [magnitude] — that is, if the sine of any arc is multiplied by the whole sine (which will be done easily if you prefix to it, on the right, as many ciphers as are contained in the whole sine, namely seven if the whole sine is 10000000, or five if the whole sine is 100000) — and the resulting number is divided by the sine of the complement of the same arc: the tangent will be found of that arc whose sine of the complement you took, or whose right sine you multiplied by the whole sine. For example, if the tangent of the arc of 30° is sought, we shall append to its right sine, 5000000, seven ciphers, in this manner, 50000000000000, and we shall divide this number by 8660254, the sine of the complement of the same arc of 30° . For the quotient, the number 5773503, will give the tangent of the arc of 30° , insofar as the whole sine is 10000000. Hence it comes about that all the tangents are found by division alone. For if to the sines of all the arcs, beginning from the start of the quadrant, you append seven ciphers, and divide the numbers so composed by the sines of the complements of the same arcs — each by its own corresponding one — the tangents of all the arcs will come out, as is clearly established from what has been demonstrated. For this reason the construction of a table of tangents is very easy.

[Margin: How the tangents of all arcs are found.]

[Margin: All the tangents are elicited by division alone.]

Again, by what has been demonstrated in the third place in this proposition: if it is arranged that, as the sine of the complement of any arc is to the whole sine, so the whole sine is to another [magnitude] — that is, if the whole sine is multiplied by itself (which will be done easily if you prefix to it, on the right, as many ciphers as there are in the whole sine, say seven, or five, according as the whole sine has seven or five ciphers) — and the number produced is divided by the sine of the complement of any arc: the secant will be found of that arc by whose sine of the complement the division was made. For example, if the secant of the arc...

[Margin: By what method the secants of all arcs are investigated.]

Opera Mathematica, Tomus I — Doctrine of Tangents and Secants

(Global = printed folios 99–101. This opening is continuous prose: the tail of the scholium to Proposition 18, then Theorems 11–14 with their scholia and a long "wonderful compendium" for constructing the tangent and secant tables. No numeric tangent/secant table begins on these pages.)

if the secant of an arc of 30° is required, we shall divide 100000000000000 — that is, the number produced by multiplying the whole sine [*sinus totus*] into itself — by 8660254, the sine [*sinus*] of the complement [*complementum*] of the arc of 30° . For the quotient, 11547005, will give the secant [*secans*] of the arc of 30° , inasmuch as the whole sine is 10000000. Therefore, by the mere division of this same number, 100000000000000, by the sines of the complements of all the arcs — beginning from the start of the quadrant [*quadrans*] — the secants of all arcs are derived, as is clear from what has been demonstrated. From this the construction of the table of secants will be very easy.

[Margin: By the mere division of this same number by the sines, all the secants are found.]

THEOR. 11. PROPOS. 19.

The tangent [*tangens*] of any arc [*arcus*] greater than half a quadrant is equal to the tangent and secant together of the arc that is double the excess by which the given arc exceeds half a quadrant.

[Margin: The tangent of an arc greater than half a quadrant, to which tangent and secant together it is equal.]

In the quadrant ABC, let CG be the tangent of the arc CF — an arc greater than half a quadrant — the tangent being enclosed between the semidiameter AC and the secant AG of the same arc CF. I say that CG is equal to the tangent and secant together of the arc that is double the excess by which the arc CF exceeds half a quadrant. For, taking the arc FD equal to FB, draw the straight line AD and extend it as far as E. ^a Now since the angles BAF and FAE are equal, because of the equal arcs BF and DF; ^b and since the alternate angle G is equal to the angle BAF, the same angle G will also be equal to the angle GAE; ^c therefore the straight lines EG and EA are equal. And so, adding the common part CE, the whole tangent CG of

the arc CF will be equal to the two lines CE and AE together — that is, to the tangent and secant of the arc CD together.

I now say that the arc CD is double the excess by which the arc CF exceeds half a quadrant. For, producing the quadrant's arc on the side of B and taking the arc BH equal to CD, then since the arc FB is also equal to the arc FD, the whole arc FH will be equal to the whole arc CF, and therefore the arc CH will be double the arc CF. But since the arc CH exceeds the quadrant CB — namely by the arc CD — ^d CF, the half of arc CH, will exceed CB's half of the quadrant by CD's half of the excess. The arc CD, therefore, is double the excess by which the given arc CF exceeds half a quadrant. And it has been shown that CG, the tangent of the arc CF, is equal to the tangent CE and the secant AE of the arc CD together. Therefore the tangent of any arc greater than half a quadrant is equal to the tangent and secant together of the arc that is double the excess by which the given arc exceeds half a quadrant. Which was to be demonstrated.

[Margin citations: ^a Euclid III.27 · ^b Euclid I.29 · ^c Euclid I.6 · ^d Prop. 7 of this book.]

Diagram. A quarter-circle (quadrant) for Proposition 19, with vertex A at the lower left. From A the two radii run up to B (top) and across to C (lower right), the arc BC being the quadrant. Point F lies on the arc (above the semiquadrant); D lies on the arc lower down; H is marked beyond B along the produced arc. The tangent to the circle at C rises to G; the secant from A through F reaches G; the straight line AD is produced to E.

- A (lower-left vertex): center of the quadrant
- B (top): end of the vertical radius
- C (lower right): end of the horizontal radius; foot of the tangent CG
- F (on the arc, upper): endpoint of the given arc CF
- D (on the arc, lower right): with arc FD = arc FB
- E (right, below D): where AD produced meets the tangent
- G (upper right): top of the tangent CG, on the secant AG
- H (top left, beyond B): with arc BH = arc CD

SCHOLIVM

Some state this proposition thus:

The secant of any arc, together with the tangent of the same, is equal to the tangent of the arc composed of the given arc and half the complement of the same.

[Margin: The secant and tangent of the same arc, to which tangent together they are equal.]

For, in the same figure, let AE be the secant and CE the tangent of the same arc CD. Bisecting the arc BD — namely the complement of the arc CD — at F, draw from the center A through F the straight line AG, cutting the tangent CE, produced, at G. Then CG will be the tangent of the arc CF, which is composed of the given arc CD and of DF, half the complement DB. I say that the secant AE and the tangent CE together are equal to the tangent CG. ^e For since the angles BAF and FAE are equal, because of the equal arcs BF and FD; ^f and the alternate angle G is equal to the angle BAF, the same angle G will also be equal to the angle GAE. ^g Therefore the straight lines EA and EG are equal; and so, adding the common part EC, the two lines AE and EC together will be equal to the whole CG.

[Margin citations: ^e Euclid III.27 · ^f Euclid I.29 · ^g Euclid I.6.]

THEOR. 12. PROPOS. 20.

The secant of any arc is equal to the tangent of the same, together with the tangent of half the complement of the same arc.

[Margin: The secant of any arc, to which two arcs' tangents it is equal.]

In the quadrant ABC, let AD be the secant and CD the tangent of the arc CE, and let EF (or FB) be half of its complement EB; and let the arc CG be equal to this half. Draw the straight line AG and produce it until it meets DC, produced, at H; then CH will be the tangent of the arc CG, which is half the complement of the arc CE. I say that the secant AD is equal to the tangent CD and the tangent CH together — that is, to the whole line DH. ^a For since the angles EAF and CAG are equal, because of the equal arcs EF and CG, then adding the common angle EAC, the whole angles FAC and EAH will be equal. ^b Again, since in the right triangle ACH the two angles at A and H are equal to one right angle — namely to the angle BAC — then removing the angles BAF and CAH, ^c which are equal because of the equal arcs BF and CG, the remaining angles FAC and H will be equal. But the angle FAC was shown equal to the angle EAH. Therefore the angle H too will be equal to that same angle EAH; ^d and consequently the straight lines AD and DH will be equal — that is, the secant AD will be equal to the tangents DC and CH. Which was proposed.

[Margin citations: ^a Euclid III.27 · ^b Euclid I.32 · ^c Euclid III.27 · ^d Euclid I.6.]

Another proof. Again let AD be the secant and CD the tangent of the arc CE. I say that the secant AD is equal to the tangent CD, together with the tangent of half the

complement of the arc EC — that is, of the angle DAC — or, in other words, together with the tangent of half the angle D, which is the complement of the angle DAC, ^e since both, in the right triangle ACD, are equal to one right angle. With center D and radius DA let there be described the arc of a circle AHI, cutting DC produced at H and AC produced at I; and let the straight lines AH and HI be drawn. Since, then, the straight line DC, drawn out from D the center of the circle AHI and cutting the straight line AI at right angles, ^f bisects it, the same line DCH will also bisect the arc AHI, by the lemma demonstrated among the definitions. ^g Therefore the angles CAH and I are equal. Now since the angles stand on the same base, the arc AH, and one of them (D) is at the center while the other (I) is at the circumference, ^h the angle D is double the angle I; hence the same angle D is also double the angle CAH. Consequently the angle CAH is half the angle D, which is the complement of the angle DAC. Since, then, CH is the tangent of the angle CAH, and the straight line DA is equal to the straight line DH by the definition of the circle, it plainly follows that the secant AD of the arc CE is equal to the tangent CD of the same arc, together with...

[Margin citations: ^e Euclid I.32 · ^f Euclid III.3 · ^g Euclid III.27 · ^h Euclid III.20.]

Diagram. A quarter-circle for Proposition 20, with vertex A at the lower left. From A the horizontal radius runs to C, and the vertical radius up to B. The tangent to the circle at C is the vertical line rising through D and F. E and G lie on the arc; the secant AD and the secant/line AG are drawn, AG being produced down to H (below), with I marked to the right.

- A (lower-left vertex): center of the quadrant
- B (top left): end of the vertical radius
- C (middle right): foot of the tangent line CDF
- D (upper right, on the tangent): end of the secant AD, and of line DH
- F (above D, on the tangent produced): with EF = half the complement of arc CE
- E (on the arc, upper): endpoint of the arc CE
- G (on the arc, lower): endpoint of arc CG = half the complement
- H (bottom): where AG produced meets DC produced
- I (right): auxiliary point on AC produced (in the alternative proof)

together with CH, the tangent of half the complement of the arc CE (or of the angle CAD). Therefore the secant of any arc is equal to the tangent of the same, and so on. Which was to be demonstrated.

SCHOLIUM

[Margin: A wonderful compendium for the construction of both the table of tangents and the table of secants. Prop. 19 of this book.]

From the two preceding theorems a wonderful compendium is furnished to us for constructing both the table of Tangents and the table of Secants. For if, by the methods we prescribed in Proposition 18 and its scholium, we work out the tangents of all arcs (running through the single minutes up to half a quadrant) and the secants of all arcs of the whole quadrant, then by their means we shall find, through mere addition, the tangents of the other arcs up to the arc of $67^{\circ} 30'$ — provided, namely, that we collect into a single sum the tangentⁱ and the secant of each arc less than half a quadrant whose minutes are an even number.¹ This is because the tangent of any arc greater than half a quadrant is equal to the tangent and secant of the arc that is double the excess by which the given arc exceeds half a quadrant — and such are all the arcs of even minutes up to 45° , as the arcs of $2'$, $4'$, $6'$, and so on.

For example: if the tangent of the arc of $45^{\circ} 1'$ is desired, we collect into a single sum the tangent 5818 and the secant 10000002 of the arc of $2'$, which is double the arc of $1'$ by which the given arc of $45^{\circ} 1'$ exceeds half a quadrant. For the resulting number, 10005820, will give the tangent of the proposed arc of $45^{\circ} 1'$, which exceeds half a quadrant by half the arc of $2'$, as was shown in Proposition 19. Likewise the same arc of $45^{\circ} 1'$ is composed of the arc of $2'$ and half the complement of that arc, which comprises $89^{\circ} 58'$ — that is, of the arc of $2'$ and the arc of $44^{\circ} 59'$. And accordingly, as we demonstrated in the scholium of Proposition 19, the number 10005820, made up of the tangent and secant of the arc of $2'$, gives the tangent of the said arc composed of the arc of $2'$ and half the complement of the same.

By the same reasoning, the tangent of any arc greater than half a quadrant is composed of the tangent and secant of the arc that is double the excess by which that arc exceeds half a quadrant. Likewise, if the tangent and secant of any arc less than half a quadrant are collected into a single sum, there will be formed the tangent of the arc composed of that arc and half the complement of the same. Thus the tangent 14352451 of the arc of $55^{\circ} 8'$ is composed of the tangent 3692500 and the secant 10659951 of the arc of $20^{\circ} 16'$, which is double [the excess of] $10^{\circ} 8'$ by which the given arc of $55^{\circ} 8'$ exceeds half a quadrant. Likewise, if the tangent 3692500 and the secant 10659951 of the arc of $20^{\circ} 16'$ are collected into a single sum, there will be formed the tangent 14352451 of the arc of $55^{\circ} 8'$, composed of

that arc of $20^{\circ} 16'$ and half the complement of the same — namely of the arc of $34^{\circ} 52'$, since the complement of the arc of $20^{\circ} 16'$ comprises $69^{\circ} 44'$.

Therefore, given any arc whatever greater than half a quadrant: if half a quadrant (that is, the arc of 45°) is subtracted from it, and the remaining arc is doubled, the tangent and secant of this doubled arc, taken together, will compose the tangent of that proposed arc. Thus, if the tangent of the arc of $55^{\circ} 8'$ is sought, the 45° must be subtracted from it, and the remaining arc of $10^{\circ} 8'$ doubled, giving the arc of $20^{\circ} 16'$.^k For its tangent and secant compose the tangent of that arc, as was demonstrated.

[Margin: ^k Prop. 19 of this book.]

Now, once the tangents of arcs greater than half a quadrant have been found in this way, up to the arc of $67^{\circ} 30'$ inclusive: if again we collect into a single sum the tangent and secant of each of these arcs that comprises an even number of minutes (such as the arcs of $45^{\circ} 2'$, $45^{\circ} 4'$, and so on), we shall find the tangents of still greater arcs — namely of $67^{\circ} 31'$, $67^{\circ} 32'$, and so on, up to the arc of $78^{\circ} 45'$ inclusive. For the arc of $33^{\circ} 45'$, by which the arc of $78^{\circ} 45'$ exceeds half a quadrant, is double the arc of $67^{\circ} 30'$, whose tangent was found in the last place. Likewise, from these tangents of arcs greater than $67^{\circ} 30'$ up to the arc of $78^{\circ} 45'$ inclusive, once found: if again we collect into a single sum the tangent and secant of each of them that comprises an even number of minutes, we shall find the tangents of still greater arcs — of the kind of the arcs of $78^{\circ} 46'$, $78^{\circ} 47'$, and so on, up to the arc of $84^{\circ} 22'$ inclusive. For the arc of $39^{\circ} 22'$, by which the arc of $84^{\circ} 22'$ exceeds half a quadrant, is double the arc of $78^{\circ} 44'$, which is the greatest of those that have their minutes even and whose tangents are already found.

By the same procedure, from the tangents of arcs greater than $84^{\circ} 22'$, we shall find the tangents of still greater arcs up to the arc of $87^{\circ} 11'$; for the [excess of $42^{\circ} 11'$], by which the arc of $87^{\circ} 11'$ exceeds half the quadrant, is double the arc of $84^{\circ} 22'$, whose tangent was found in the last place. From these, in turn, we shall make up the tangents of the following arcs up to the arc of $88^{\circ} 35'$, whose excess of $43^{\circ} 35'$, by which it exceeds the quadrant's half, is double the arc of $87^{\circ} 10'$, whose tangents were found just before. By means of these likewise — up to the arc of $89^{\circ} 17'$ inclusive — since the arc of $44^{\circ} 17'$, by which the arc of $89^{\circ} 17'$ exceeds the quadrant, is double the arc of $88^{\circ} 34'$ (which is the greatest of those that have their minutes even and whose tangents are now known) — by the aid of these tangents thus found we shall next draw out the tangents of arcs up to the arc of $89^{\circ} 38'$ inclusive; because the [excess of] this arc [is such that] its double is the arc of $89^{\circ} 16'$, which is the greatest of those whose minutes are even. Hence we shall

seek the tangents of other arcs up to the arc of $89^{\circ} 49'$, since the arc of $44^{\circ} 49'$, whose double is the arc of $89^{\circ} 38'$, is one to which we just came. But from these we shall investigate the tangents of the following arcs up to the arc of $89^{\circ} 54'$ — namely by that arc of $44^{\circ} 54'$ which exceeds the quadrant's half, whose double is the arc of $89^{\circ} 48'$, the greatest of those that have their minutes even and whose tangents are already found. By the same reasoning we shall find the tangents of the following arcs up to the arc of $89^{\circ} 57'$; because this arc of $89^{\circ} 57'$ [is such that] its double-excess is the arc of $89^{\circ} 54'$, to which we just came. And thence we shall make up the tangent of the arc of $89^{\circ} 58'$, from the [double-excess] arc of $89^{\circ} 56'$. And so, as you see, from the tangents of arcs up to 45° and from the secants of all the arcs of the quadrant, the whole table of tangents is completed.

But if you subtract the secant of any arc whatever from the tangent of the arc that is composed of that former arc and half the complement of the same, you will produce the tangent of that same former arc, whose secant you subtracted. Likewise, if from the secant of any arc you take away the tangent of the same, there will remain the tangent of half the complement of that arc. The first point is established from the scholium of Proposition 19, where it was shown that the secant and tangent of any arc together are equal to the tangent of the arc composed of that arc and half its complement; for hence it comes about that the secant, taken away from this composite tangent, leaves that other tangent. The second point is clear from this Proposition 20, where we demonstrated that the secant of any arc is equal to the tangent of the same, together with the tangent of half the complement of that arc; wherefore the tangent of that half will be the remainder after the subtraction of that other tangent from the secant.

For example: if we subtract the secant of the arc of 20° , which is 10641777, from 14281480, the tangent of the arc of 55° (composed of the arc of 20° and half its complement), there will remain 3639703, the tangent of the same arc of 20° . Likewise, if we subtract 4244748, the tangent of the arc of 23° , from the secant of the same arc, 10863603, there will remain the tangent 6618855 of the arc of $33^{\circ} 30'$ — that is, of half the complement of the given arc of 23° . Again, if we take away 11547004, the secant of the arc of 30° , from 17320508, the tangent of the arc of 60° ,

which is composed of the arc of 30° and half its complement, there will be left the tangent 5773504 of the arc of 30° . And if we take away 1763268, the tangent of the arc of 10° , from 10154264, the secant of the same arc of 10° , there will remain the tangent 8390996 of the arc of 40° , which is half the complement of the said arc of 10° .

Now indeed, if by the methods we handed down in Proposition 18 and its scholium we work out the tangents of all the arcs of the quadrant, extended through the single minutes, we shall find by their means, through mere addition, the secants of all arcs advancing by two minutes at a time — provided, namely, that we add the tangent of any arc having an even number of minutes to the tangent of half the complement of the same arc: ¹ this because the secant of any arc is equal to the tangent of the same, together with the tangent of half the complement of the same; and it is established that the complements of all arcs having an even number of minutes have halves [in whole minutes]. For example, if the secant of the arc of 2' is desired, we shall add its tangent 5818 to 9994184, the tangent of the arc of 44° 59', which is half the complement of the given arc of 2'. For the number thus composed, 10000002, will be the secant of the arc of 2'. So too, if the secant of the arc of 89° 58' is sought, we shall add its tangent 17188033689 to 2909, the tangent of the arc of 1', which is half the complement of the given arc of 89° 58'. For the number thus formed, 17188036598, will be the secant of the arc of 89° 58'. By this method half the table of Tangents [recte: Secantium — of Secants?] will be constructed; but the tangents [recte: secantes — the secants?] of arcs having an odd number of minutes, since their complements do not have halves [in whole minutes] — unless we choose to descend to seconds — must be investigated as we prescribed in Proposition 18 and its scholium.

[Margin: ¹ Prop. 20 of this book.]

Again, we shall find the secant of any arc if we take away the tangent from the tangent of the arc composed of that arc and half the complement of the same arc. For since, as we demonstrated, the secant of any arc, together with the tangent of half the complement of the same arc, ^m makes it so that the tangent of the given arc, taken away from the tangent of the arc composed of the given arc and half the complement of the same, leaves the secant of that same given arc. For example, if we wish for the secant of the arc of 2', we shall take away 5818, the tangent of that arc, from 10005820, the tangent of the arc of 45° 1', composed of the arc of 2' and half the complement of the given arc of 2'; for the number left, 10000002, will be the secant of the arc of 2'. So too, if we wish to have the secant of the arc of 60°, we shall subtract 17320508, its tangent, from 37320514, the tangent of the arc of 75°, composed of the given arc of 60° and the arc of 15°, which is half the complement of the given arc of 60°. For the number that remains, 20000006, stands for the secant of the given arc of 60°.

[Margin: ^m Scholium of Prop. 19 of this book.]

These things which have been handed down by us in this scholium are true if the sines have been exactly found; but because not all the sines are accurately known — especially the sine of the arc of 1° and the others that depend on it, such as the sines of the arcs extended through the single minutes — it comes about that neither the tangents nor the secants found by means of these sines are altogether accurate. Hence, if from certain values found, others are sought by mere addition or subtraction, as we have taught in this scholium, they will differ not a little from the same values were they investigated through the sines. For the tangents and secants found through the sines are generated from one single principle — not, indeed, in every part a true one, namely the sines; whereas those same values, produced by mere addition or subtraction, arise from several false principles — namely, first from the sines, and then also from the tangents and secants found through the sines, which cannot be accurate, as we have said. These lines, then, will be known more exactly through the sines, as was handed down in Proposition 18 and its scholium. By this method we shall shortly compute both the table of Tangents and the table of Secants; for we have detected no small errors in the tables of others, so that we cannot safely trust them, because they investigated many tangents and secants either by the proportional part, or by mere addition or subtraction, and not all through the sines. Nevertheless we shall append a little below the tables of others, until in time it shall be possible to construct new ones.

[Margin: That the tangents and secants found through the sines are more accurate than those found by addition or subtraction, as has been handed down in this scholium.]

THEOR. 13. PROPOS. 21.

The tangent of any arc is to the tangent of any other arc as the tangent of the complement of the latter arc is to the tangent of the complement of the former.

[Margin: The tangents of any two arcs are reciprocally proportional to the tangents of the complements of the same arcs.]

In the quadrant ABC, let the tangent of the arc CD be CE and its secant AE; likewise, let the tangent of the arc CF be CG and its secant AG. Now draw the straight line BH, touching the circle and meeting each of the secants AE, AG at I and H; then BI will be the tangent of the complement of the arc CD, and BH the tangent of the complement of the arc CF. I say that CE, the tangent of the arc CD, is to CG, the tangent of the arc CF, as BH, the tangent of the complement of the

latter arc CF, is to BI, the tangent of the complement of the former arc CD. ^a For since the whole sine is the mean proportional both between CE, the tangent of the arc CD, and BI, the tangent of the complement of the same arc CD, and between CG, the tangent of the arc CF, and BH, the tangent of the complement of the same arc CF; ^b then both the rectangle under CG, BH and the rectangle under CE, BI will be equal to the square of the whole sine; and consequently the rectangle under CE, BI will be equal to the rectangle under CG, BH. ^c Therefore, as CE the first is to CG the second, so will BH the third be to BI the fourth — namely, as CE, the tangent of the arc CD, is to CG, the tangent of the arc CF, so is BH, the tangent of the complement of the latter arc CF, to BI, the tangent of the complement of the former arc CD. Therefore the tangent of any arc is to the tangent of another, and so on. Which was to be shown.

[Margin citations: ^a Prop. 18 of this book · ^b Euclid VI.17 · ^c Euclid VI.16.]

Diagram. A square-framed quadrant figure for Proposition 21, with vertex A at the lower left and C at the lower right. B is at the upper left; the tangent to the circle at B is a horizontal line running to the right through I and H. The tangent at C is a vertical line rising through G to E at the top. The secants AE and AG are drawn from A; D and F lie within, on the arc / on the secants. The whole sine is the mean proportional between each tangent and the tangent of its complement.

- A (lower-left vertex): center of the quadrant
- B (upper left): foot of the tangent-of-complement line BH
- C (lower right): foot of the tangents CE, CG
- D (interior, on the arc): endpoint of the arc CD
- E (top right): top of the tangent CE, on the secant AE
- F (interior, on the arc): endpoint of the arc CF
- G (right of center, on the tangent at C): on the secant AG [?]
- H (right, on the tangent at B): where AG meets the tangent at B
- I (upper, on the tangent at B): where AE meets the tangent at B [?]

THEOR. 14. PROPOS. 22.

The secant of any arc is to the secant of any other arc as the sine of the complement of the latter arc is to the sine of the complement of the former.

[Margin: The secants of any two arcs are reciprocally proportional to the sines of the complements of the same arcs.]

In the quadrant ABC, let AD be the secant of the arc CE, and AF the secant of the arc CG; and let EH be the sine of the complement of the arc CE, and GI the sine of the complement of the arc CG. I say that the secant AD of the arc CE is to AF, the secant of the arc CG, as GI, the sine of the complement of the latter arc CG, is to EH, the sine of the complement of the former arc CE. ^a For since the whole sine is the mean proportional both between the secant AD of the arc CE and EH, the sine of the complement of the same arc CE, and between AF, the secant of the arc CG, and GI, the sine of the complement of the same arc CG; ^b then both the rectangle under AD, EH and the rectangle under AF, GI will be equal to the square of the whole sine; and consequently that rectangle will be equal to this one. ^c Therefore, as AD the first is to AF the second, so will GI the third be to EH the fourth...

[Margin citations: ^a Prop. 18 of this book · ^b Euclid VI.17 · ^c Euclid VI.16.]

Diagram. A quadrant figure for Proposition 22, with vertex A at the lower left. B is at the upper left; H and I are marked on the left side (feet of the sines of the complements). The arc runs up toward D at the upper right, with E and G on it. The secants AD and AF are drawn from A, meeting the tangent/vertical at D and F on the right; EH and GI are the sines of the complements dropped to the vertical radius AB.

- A (lower-left vertex): center of the quadrant
- B (upper left): end of the vertical radius
- C (lower right): end of the horizontal radius
- D (upper right): end of the secant AD
- E (on the arc, upper middle): endpoint of the arc CE
- F (right): end of the secant AF
- G (on the arc, middle): endpoint of the arc CG
- H (left, on AB): foot of the sine of complement EH
- I (left, on AB, below H): foot of the sine of complement GI

[Continues from the previous page, completing the proof of THEOR. 14 / PROPOS. 22; the page opens on the second half of the word *quartam*.]

the fourth [proportional]; that is, as AD, the secant of arc CE, is to AF, the secant of arc CG, so is GI, the sine of the complement of the latter arc CG, to EH, the sine of the complement of the former arc CE. Therefore the secant of any arc is to the secant of another arc [as the sine of the complement of the second is to the sine of the complement of the first], and so on. Which was to be demonstrated.

THEOR. 15. PROPOS. 23.

[Margin: Tangents and secants of arcs that increase equally always increase their differences (*Tangentes & secantes arcuum aequaliter crescentium augent semper differentias*).]

If there are several arcs progressing by an equal increment, both their tangents and their secants — those of the larger arcs — will have a greater difference than those of the smaller: so that in the table the differences both of the tangents and of the secants keep increasing all the way to the end of the quadrant.

In the quadrant ABC let there be arcs CD, CE, CF whose differences DE, EF are equal, and let their tangents be CG, CH, CI and their secants AG, AH, AI. Now since in triangle ACH the angle C is right, the angle AHC will be less than a right angle [Margin a: *Elements* I.17], because the two of them together are less than two right angles. Since, then, the two angles at H are equal to two right angles [Margin b: I.13], AHI will be greater than a right angle, and consequently the angle I, in triangle AHI, is less than a right angle [Margin c: I.17]. Hence the secant AI will be greater than the secant AH [Margin d: I.19]. By the same reasoning it will also be greater than AG; and likewise AH is greater than AG.

Let there be cut off, then, AK equal to AH, and AL equal to AG. I say that IH, the difference of the tangents CI, CH of the larger arcs CF, CE, is greater than HG, the difference of the tangents CH, CG of the smaller arcs CE, CD; and likewise that KI, the difference of the secants AI, AH of the larger arcs CF, CE, is greater than LH, the difference of the secants AH, AG of the smaller arcs CE, CD. For since the arcs DE, EF are equal, the angles DAE, EAF will also be equal [Margin e: III.27]; and consequently the angle IAG is bisected by the straight line AH. Therefore [Margin f: VI.3] as IA is to AG, so is IH to HG. But AI is greater than AG, as was shown; therefore the straight line IH will also be greater than HG. This is the first point.

Now, FM, EN, DO having been drawn perpendicular to AC — namely the sines of the arcs CF, CE, CD — AM will be the sine of the complement of arc CF, AN the sine of the complement of arc CE, and AO the sine of the complement of arc CD, as was said in the exposition of the definitions. Since indeed the straight line MN is greater than NO [Margin g: 1 of this treatise], the proportion of AN to NO will be greater than that of AN to MN [Margin h: V.8]; but the proportion of AO to NO is still greater than that of AN to the same NO. Therefore the proportion of AO to NO will be much greater than that of AN to MN. And by the conversion of the ratio [Margin i: 30 of the fifth], the proportion of AO to AN is less than that of AN to

AM; that is, the proportion of AN to AM is greater than that of AO to AN [Margin k: 22 of this treatise]. Since, then, as AN is to AM so is AI to AH, and as AO is to AN so is AN to AG, the proportion of AI to AH — that is, to AK — will also be greater than that of AH to AG — that is, to AL. Therefore, by division [Margin l: 29 of the fifth], the proportion of IK to AK — that is, to AH — will likewise be greater than that of HL to AL. And by conversion [Margin m: 26 of the fifth], the proportion of AK to KI will be less than that of AL to LH; that is, the proportion of AL to LH will be greater than that of AK to KI. Hence, since the proportion of AH to LH is still greater than that of the same AH to KI [Margin n: V.8], the straight line LH will therefore be less than KI [Margin o: 10 of the fifth]. This is the second point.

From this it comes about that the differences both of the tangents and of the secants in the table keep increasing all the way to the end of the quadrant — the contrary of which happens in the case of the sines, as was demonstrated above. Therefore, if there are several arcs progressing by an equal increment, and so on. Which was to be demonstrated.

COROLLARIUM.

[Margin: Arcs whose tangents have equal increments have unequal increments (*Arcus tangentium aequales excessus habentium inaequales habet excessus*).]

It follows from this that, if the tangents of any number of arcs exceed one another equally, the arcs themselves exceed one another unequally, and the excesses of the larger arcs are smaller than those of the smaller. Likewise, all the segments of the secants outside the quadrant are unequal, and those nearer the beginning of the quadrant are smaller. For since, with the arcs DE, EF taken equal, it was shown that the straight line IH is greater than HG, it plainly follows that, if from HI a straight line equal to HG is taken away, the secant drawn between the two AI, AH divides the arc EF, and thus cuts off an arc smaller than the arc DE — namely a part of the arc EF. And the same reasoning holds for the others.

Again, since it was demonstrated that the secant AG is smaller than AH, it follows that, when the equal semidiameters AD, AE are taken away, the remaining segment DG is smaller than the remaining segment EH, and so on.

Diagram. A quadrant of a circle serving both THEOR. 15 and THEOR. 16 (the text of THEOR. 16 states "let the figure of the preceding proposition be repeated," and the same figure is engraved a second time lower on the page). The right angle is at A (lower left, the centre); AB is the vertical radius (B at upper left) and AC the horizontal radius (C at lower right), with the quadrant arc running from B down to C.

- A (lower-left corner): centre of the quadrant and vertex of the right angle.
- B (upper-left corner): end of the vertical radius AB.
- C (lower-right corner): end of the horizontal radius AC; foot of the common tangent line.
- D, E, F (on the arc, from C upward toward B): points marking the arcs CD, CE, CF, whose successive differences DE, EF are equal.
- G, H, I (on the vertical tangent through C, from C upward): the tangent-endpoints of arcs CD, CE, CF respectively (tangents CG, CH, CI).
- The secants AG, AH, AI run from A through D, E, F to G, H, I.
- K (on AI): the point with AK equal to AH.
- L (on AH): the point with AL equal to AG.
- M, N, O (on AC, between A and C): the feet of the perpendiculars FM, EN, DO dropped from F, E, D to AC (the sines of arcs CF, CE, CD); AM, AN, AO are the sines of the complements of those arcs.

THEOR. 16. PROPOS. 24.

[Margin: The tangent of the greater of two unequal arcs has to the tangent of the lesser a greater proportion than the secant of the greater has to the secant of the lesser (*Arcuum inaequalium tangens maioris ad tangentem minoris proportionem habet maiorem, quam secans maioris ad secantem minoris*).]

The tangent of the greater arc has to the tangent of the lesser arc a greater proportion than the secant of the same greater arc has to the secant of the same lesser.

Let the figure of the preceding proposition be repeated. I say that the proportion of the tangent CI to the tangent CH is greater than that of the secant AI to the secant AH. For since [Margin a: VI.4] as AF is to FM, so is AI to IC, and likewise as AE is to EN, so is AH to HC; but the proportion of the semidiameter AF to FM is smaller than that of the semidiameter AE to EN [Margin b: V.8], because the sine FM of the larger arc CF is greater than the sine EN of the smaller arc CE, as was said in the exposition of the definitions — therefore the proportion of AI to IC will also be smaller than that of AH to HC. And by alternation [Margin c: scholium on V.27], the proportion of AI to AH is also smaller than that of IC to HC; that is, the tangent CI to the tangent CH will have a greater proportion than the secant AI to the secant AH. And therefore the tangent of the greater arc to the tangent of the lesser arc, and so on. Which was to be demonstrated.

HERE FOLLOW THE TABLES OF TANGENTS AND SECANTS.

(SEQVVNTVR TABVLAE TANGENTIVM ATQVE SECANTIVM.)

Table of Tangents — degrees 0–4 (folio 103)

Sub-heading: *Gradus Quadrantis pro tangentibus arcuum eiusdem Quadrantis.*

("Degrees of the quadrant, for the tangents of the arcs of the same quadrant.")

The left "Minuta" column runs 0→60 top-to-bottom; the right "Minuta" column runs 60→0 (giving the minutes of the complement). The top degree banner reads **0 · 1 · 2 · 3 · 4**; the bottom banner (for the complements) reads **89 · 88 · 87 · 86 · 85**, with the footing *Gradus Quadratis pro tangentibus complementorum arcuum eiusdem Quadrantis.*

M i n	0°	1°	2°	3°	4°	M i n
0	0000	1745	3492	5240	699269	6
		50	07	78		0
1	2909	1774	3521	5269	702193	5
		59	20	95		9
2	5818	1803	3550	5299	705116	5
		69	33	11		8
3	8727	1832	3579	5328	708039	5
		79	45	28		7
4	11636	1861	3608	5357	710962	5
		89	58	45		6
5	14544	1891	3637	5386	713886	5
		00	70	63		5
6	17452	1920	3666	5415	716809	5
		10	83	80		4
7	20361	1949	3695	5444	719733	5
		20	96	98		3
8	23270	1978	3725	5474	722657	5
		30	08	15		2
9	26179	2007	3754	5503	725580	5

M i n	0°	1°	2°	3°	4°	M i n
		40	21	33		1
1	29088	2036	3783	5532	728504	5
0		50	34	51		0
1	31996	2065	3812	5561	731428	4
1		61	47	69		9
1	34905	2094	3841	5590	734353	4
2		71	60	87		8
1	37814	2123	3870	5620	737277	4
3		81	73	05		7
1	40723	2152	3899	5649	740202	4
4		91	87	23		6
1	43632	2182	3929	5678	743127	4
5		01	00	41		5
1	46541	2211	3958	5707	746052	4
6		11	14	59		4
1	49450	2240	3987	5736	748978	4
7		22	27	78		3
1	52359	2269	4016	5765	751903	4
8		32	41	96		2
1	55268	2298	4045	5795	754829	4
9		42	54	14		1
2	58177	2327	4074	5824	757754	4
0		52	68	33		0
2	61086	2356	4103	5853	760680	3
1		63	82	52		9
2	63995	2385	4132	5882	763606	3
2		74	95	70		8
2	66904	2414	4162	5911	766532	3
3		85	09	89		7
2	69813	2443	4191	5941	769459	3
4		95	23	08		6
2	72722	2473	4220	5970	772385	3

M i n	0°	1°	2°	3°	4°	M i n
5		06	37	28		5
2	75631	2502	4249	5999	775311	3
6		17	51	47		4
2	78540	2531	4278	6028	778238	3
7		28	66	66		3
2	81450	2560	4307	6057	781164	3
8		38	80	86		2
2	84359	2589	4336	6087	784091	3
9		49	94	05		1
3	87268	2618	4366	6116	787017	3
0		59	09	25		0
3	90177	2647	4395	6145	789944	2
1		70	23	44		9
3	93086	2676	4424	6174	792871	2
2		81	38	64		8
3	95995	2705	4453	6203	795799	2
3		92	53	84		7
3	98904	2735	4482	6233	798726	2
4		03	67	04		6
3	101814	2764	4511	6262	801653	2
5		14	82	25		5
3	104723	2793	4540	6291	804581	2
6		25	97	45		4
3	107632	2822	4570	6320	807509	2
7		37	12	66		3
3	110541	2851	4599	6349	810437	2
8		48	27	86		2
3	113450	2880	4628	6379	813365	2
9		59	42	07		1
4	116360	2909	4657	6408	816293	2
0		70	57	28		0
4	119269	2938	4686	6437	819221	1

M i n		0°	1°	2°	3°		4°	M i n
1			82	72	49			9
4		122178	2967	4715	6466		822150	1
2			94	88	71			8
4		125088	2997	4745	6495		825079	1
3			05	03	92			7
4		127997	3026	4774	6525		828008	1
4			17	19	14			6
4		130906	3055	4803	6554		830937	1
5			28	35	35			5
4		133816	3084	4832	6583		833866	1
6			39	51	57			4
4		136725	3113	4861	6612		836795	1
7			51	66	78			3
4		139635	3142	4890	6642		839724	1
8			62	82	00			2
4		142544	3171	4919	6671		842653	1
9			74	97	21			1
5		145454	3200	4949	6700		845583	1
0			85	13	43			0
5		148363	3229	4978	6729		848513	9
1			97	29	65			
5		151273	3259	5007	6758		851443	8
2			09	45	88			
5		154182	3288	5036	6788		854374	7
3			21	62	10			
5	159092 [recte: 157092?]	3317	5065	6817			857304	6
4		33	78	33				
5		160001	3346	5094	6846		860234	5
5			45	95	56			
5		162911	3375	5124	6875		863164	4
6			58	11	78			
5		165820	3404	5153	6905	56095 [recte: 866095?]		3

M i n	0°	1°	2°	3°	4°	M i n
7		70	28	01		
5	168730	3433	5182	6934	869025	2
8		82	44	23		
5	171640	3462	5211	6963	871956	1
9		95	61	46		
6	174550	3492	5240	6992	874886	0
0		07	78	69		

Table of Tangents — degrees 5–9 (folio 104)

Sub-heading: *Gradus Quadrantis pro tangentibus arcuum eiusdem Quadrantis*. The left "Minuta" column runs 0→60 top-to-bottom; the right runs 60→0. Top degree banner 5 · 6 · 7 · 8 · 9; bottom (complement) banner 84 · 83 · 82 · 81 · 80, with the footing *Gradus Quadratis pro tangentibus complementorum arcuum eiusdem Quadrantis*.

M in	5°	6°	7°	8°	9°	M in
0	874886	105104	1227846	140540	158384	60
		2		8	4	
1	877817	105398	1230798	140837	158682	59
		3		4	6	
2	880748	105692	1233751	141134	158980	58
		4		1	8	
3	883680	105986	1236704	141430	159279	57
		6		8	1	
4	886611	106280	1239658	141727	159577	56
		8		5	4	
5	889543	106575	1242612	142024	159875	55
		0		2	7	
6	892475	106869	1245566	142321	160174	54
		2		0	0	
7	895407	107163	1248520	142617	160472	53
		4		8	3	

M in	5°	6°	7°	8°	9°	M in
8	898339	107457 6	1251474	142914 6	160770 7	52
9	901271	107751 8	1254428	143211 5	161069 1	51
1 0	904204	108046 1	1257383	143508 4	161367 5	50
1 1	907137	108340 4	1260338	143805 3	161666 0	49
1 2	910070	108634 7	1263293	144102 2	161964 5	48
1 3	913003	108929 1	1266249	144399 2	162263 0	47
1 4	915936	109223 4	1269205	144696 1	162561 5	46
1 5	918870	109517 8	1272161	144993 1	162860 1	45
1 6	921804	109812 2	1275117	145290 1	163158 7	44
1 7	924738	110106 6	1278073	145587 1	163457 3	43
1 8	927771	110401 0	1281029	145884 2	163756 0	42
1 9	930605	110695 4	1283986	146181 3	164054 7	41
2 0	933539	110989 9	1286943	146478 4	164353 4	40
2 1	936473	111284 4	1289900	146775 5	164652 2	39
2 2	939407	111578 9	1292857	147072 7	164951 0	38
2 3	942342	111873 4	1295815	147369 9	165249 9	37
2	945277	112168	1298773	147667	165548	36

M in	5°	6°	7°	8°	9°	M in
4		0		1	8	
2	948212	112462	1301731	147964	165847	35
5		5		4	7	
2	951147	112757	1304689	148261	166146	34
6		1		7	6	
2	954083	113051	1307648	148559	166445	33
7		7		0	6	
2	957019	113346	1310607	148856	166744	32
8		3		3	6	
2	959954	113640	1313566	149153	167043	31
9		9		6	6	
3	962890	113935	1316525	149451	167342	30
0		5		0	6	
3	965826	114230	1319485	149748	167641	29
1		2		4	7	
3	968763	114524	1322445	150045	167940	28
2		9		8	8	
3	971699	114819	1325405	150343	168239	27
3		6		3	9	
3	974636	115114	1328365	150640	168539	26
4		4		8	0	
3	977573	115409	1331325	150938	168838	25
5		2		3	2	
3	980509	115704	1334285	151235	169137	24
6		0		8	4	
3	983446	115998	1337246	151533	169436	23
7		8		4	6	
3	986383	116293	1340207	151831	169735	22
8		6		0	8	
3	989320	116588	1343168	152128	170035	21
9		4		6	1	
4	992257	116883	1346129	152426	170334	20
0		2		2	4	

M in	5°	6°	7°	8°	9°	M in
4 1	995195	117178 1	1349891 [recte: 1349091?]	152723 9	170633 7	19
4 2	998133	117473 0	1352453 [recte: 1352053?]	153021 6	170933 1	18
4 3	100107 2	117767 9	1355015	153319 3	171232 5	17
4 4	100401 0	118062 8	1357977	153617 0	171531 9	16
4 5	100694 9	118357 7	1360940	153914 8	171831 3	15
4 6	100988 7	118652 7	1363903	154212 6	172130 8	14
4 7	101282 5	118947 8	1366866	154510 4	172430 4	13
4 8	101576 3	119242 7	1369830	154808 2	172730 0	12
4 9	101870 2	119537 7	1372793	155106 1	173029 6	11
5 0	102164 1	119832 8	1375757	155404 0	173329 2	10
5 1	102458 0	120127 9	1378721	155701 9	173628 7	9
5 2	102751 9	120423 0	1381686	155999 9	173928 4	8
5 3	103045 9	120718 1	1384650	156297 9	174228 1	7
5 4	103339 9	121013 2	1387615	156595 9	174527 8	6
5 5	103633 9	121308 4	1390580	156893 9	174827 5	5
5 6	103927 9	121603 6	1393545	157192 0	175127 3	4
5	104221	121898	1396510	157490	175427	3

M in	5°	6°	7°	8°	9°	M in
7	9	8		1	1	
5	104516	122194	1399476	157788	175727	2
8	0	0		2	0	
5	104810	122489	1402442	158086	176026	1
9	1	2		3	9	
6	105104	122784	1405408	158384	176326	0
0	2	5		4	8	

Table of Tangents (*Tabula Tangentium*) — degrees 10°–24°

Christoph Clavius, Opera Mathematica, Tomus I — tangent/secant treatise, continuation of the numeric tangent tables.

Each page carries five degree columns. The left-hand minute column (*Minuta Graduum Quadrantis pro tangentibus arcuum eiusdem Quadrantis* — "Minutes of the degrees of the quadrant, for the tangents of the arcs of the same quadrant") runs 0'→60' downward and applies to the degree named in the **top** banner. The right-hand minute column runs 60'→0' downward and, with the **bottom** banner degree (89° – top degree), serves the tangents of the complementary arcs (*pro tangentibus complementorum arcuum eiusdem Quadrantis*). All tangent values are to a radius of 10,000,000. Values are transcribed exactly as printed.

Folio 105 — degrees 10°–14° (bottom banner: 79° 78° 77° 76° 75°)

Top banner: **10 · 11 · 12 · 13 · 14**

Min ↓	10	11	12	13	14	Min ↑
0	1763	1943	2125565	2308	2493280	60
	268	803		682		
1	1766	1946	2128605	2311	2496370	59
	268	822		746		
2	1769	1949	2131646	2314	2499411 [recte:?	58

Min ↓	10	11	12	13	14	Min ↑
	268	841		810	2499461]	
3	1772	1952	2134687	2317	2502552	57
	268	861		875		
4	1775	1955	2137729	2320	2505643	56
	269	881		940		
5	1778	1958	2140771	2324	2508735	55
	270	901		006		
6	1781	1961	2143814	2327	2511827	54
	271	922		072		
7	1784	1964	2146857	2330	2514920	53
	272	943		139		
8	1787	1967	2149900	2333	2518013	52
	274	964		206		
9	1790	1970	2152944	2336	2521106	51
	276	985		273		
10	1793	1974	2155988	2339	2524200	50
	278	007		341		
11	1796	1977	2159032	2342	2527294	49
	281	029		419		
12	1799	1980	2162077	2345	2530389	48
	284	052		478		
13	1802	1983	2165122	2348	2533484	47
	287	075		547		
14	1805	1986	2168167	2351	2536580	46
	291	098		616		
15	1808	1989	2171213	2354	2539676	45
	295	122		686		
16	1811	1992	2174259	2357	2542773	44
	299	146		757		
17	1814	1995	2177306	2360	2545870	43
	303	171		828		
18	1817	1998	2180352	2363	2548968	42
	308	196		899		

Min ↓	10	11		12	13	14	Min ↑
19	1820	2001		2183400	2366	2552066	41
	313	221			971		
20	1823	2004		2186448	2370	2555165	40
	318	247			043		
21	1826	2007		2189496	2373	2558264	39
	324	273			116		
22	1829	2010		2192544	2376	2561364	38
	329	299			189		
23	1832	2013		2195593	2379	2564464	37
	335	326			263		
24	1835	2016		2198642	2382	2567564	36
	342	353			337		
25	1838	2019		2201692	2385	2570665	35
	349	380			411		
26	1841	2022		2204742	2388	2573766	34
	357	408			486		
27	1844	2025		2207792	2391	2576868	33
	365	436			561		
28	1847	2028		2210843	2394	2579970	32
	373	464			636		
29	1850	2031		2213894	2397	2583073	31
	382	493			712		
30	1853	2034		2216946	2400	2586176	30
	391	522			788		
31	1856	2037		2219998	2403	2589280	29
	400	552			865		
32	1859	2040		2223051	2406	2592384	28
	409	582			942		
33	1862	2043	2226104 [printed with malformed leading glyph "N226104"]	2410	020	2595489	27
	419	612					
34	1865	2046		2229157	2413	2598594	26

Min ↓	10	11	12	13	14	Min ↑
	429	643		098		
35	1868	2049	2232211	2416	2601700	25
	439	674		176		
36	1871	2052	2235265	2419	2604806	24
	449	705		255		
37	1874	2055	2238319	2422	2607912	23
	460	737		334		
38	1877	2058	2241374	2425	2611019	22
	471	769		414		
39	1880	2061	2244429	2428	2614126	21
	482	801		494		
40	1883	2064	2247485	2431	2617234	20
	494	834		574		
41	1886	2067	2250541	2434	2620342	19
	506	867		655		
42	1889	2070	2253597	2437	2623451	18
	518	900		736		
43	1892	2073	2256654	2440	2626560	17
	531	934		818		
44	1895	2076	2259711	2443	2629670	16
	544	968		900		
45	1898	2080	2262769	2446	2632780	15
	558	002		983		
46	1901	2083	2265827	2450	2635891	14
	572	037		066		
47	1904	2086	2268885	2453	2639002	13
	586	073		150		
48	1907	2089	2271944	2456	2642114	12
	601	109		234		
49	1910	2092	2275003	2459	2645226	11
	616	145		319		
50	1913	2095	2278063	2462	2648339	10
	632	182		404		

Min ↓	10	11	12	13	14	Min ↑
51	1916 648	2098 219	2281123	2465 490	2651452	9
52	1919 664	2101 256	2284183	2468 576	2654566	8
53	1922 680	2104 293	2287244	2471 662	2657680	7
54	1925 697	2107 331	2290305	2474 749	2660795	6
55	1928 714	2110 369	2293367	2477 836	2663910	5
56	1931 731	2113 407	2296429	2480 924	2667026	4
57	1934 749	2116 446	2299492	2484 012	2670142	3
58	1937 767	2119 485	2302555	2487 101	2673258	2
59	1940 785	2122 525	2305618	2490 191	2676375	1
60	1943 803	2125 565	2308682	2493 280	2679492	0

Bottom banner: **79 · 78 · 77 · 76 · 75** · [Footer: "Gradus Quadrantis pro tangentibus complementorum arcuum eiusdem Quadrantis."]

Folio 106 — degrees 15°–19° (bottom banner: 74° 73° 72° 71° 70°)

Top banner: **15 · 16 · 17 · 18 · 19**

Min ↓	15	16	17	18	19	Min ↑
0	2679492	2867453	3057307	3249197	3443276	60
1	2682610	2870601	3060487	3252413	3446530	59
2	2685728	2873749	3063669	3255630	3449785	58
3	2688847	2876898	3066851	3258848	3453040	57
4	2691966	2880048	3070034	3262066	3456296	56

Min ↓	15	16	17	18	19	Min ↑
5	2695086	2883198	3073218	3265285	3459553	55
6	2698206	2886349	3076402	3268504	3462810	54
7	2701327	2889501	3079587	3271724	3466068	53
8	2704448	2892653	3082772	3274944	3469326	52
9	2707570	2895806	3085958	3278165	3472585	51
10	2710693	2898960	3089144	3281387	3475845	50
11	2713816	2902114	3092331	3284609	3479105	49
12	2716940	2905268	3095518	3287832	3482366	48
13	2720064	2908423	3098706	3291055	3485628	47
14	2723189	2911578	3101895	3294280	3488891	46
15	2726314	2914734	3105084	3297505	3492154	45
16	2729439	2917890	3108274	3300731	3495418	44
17	2732565	2921047	3111464	3303957	3498683	43
18	2735691	2924204	3114655	3307184	3501949	42
19	2738818	2927362	3117846	3310411	3505215	41
20	2741945	2930520	3121038	3313639	3508482	40
21	2745073	2933679	3124230	3316868	3511749	39
22	2748201	2936839	3127423	3320097	3515017	38
23	2751330	2939999	3130617	3323327	3518286	37
24	2754459	2943160	3133811	3326558	3521555	36
25	2757589	2946321	3137006	3329789	3524825	35
26	2760729	2949483	3140201	3333020	3528096	34
27	2763850	2952645	3143397	3336252	3531368	33
28	2766981	2955808	3146594	3339485	3534640	32
29	2770113	2958971	3149791	3342719	3537913	31
30	2773245	2962135	3152989	3345953	3541186	30
31	2776378	2965299	3156187	3349188	3544460	29
32	2779511	2968464	3159386	3352423	3547735	28
33	2782645	2971629	3162585	3355659	3551010	27
34	2785779	2974795	3165785	3358896	3554286	26

Min ↓	15	16	17	18	19	Min ↑
35	2788914	2977962	3168986	3362133	3557563	25
36	2792050	2981129	3172187	3365371	3560840	24
37	2795186	2984297	3175389	3368610	3564118	23
38	2798323	2987465	3178591	3371850	3567397	22
39	2801460	2990634	3181794	3375090	3570676	21
40	2804597	2993804	3184998	3378331	3573956	20
41	2807735	2996973	3188202	3381572	3577237	19
42	2810873	3000143	3191407	3384814	3580519	18
43	2814012	3003314	3194613	3388057	3583801	17
44	2817151	3006486	3197819	3391300	3587084	16
45	2820291	3009658	3201026	3394544	3590367	15
46	2823432	3012831	3204233	3397798	3593651	14
47	2826573	3016004	3207441	3401033	3596936	13
48	2829714	3019178	3210649	3404279	3600221	12
49	2832856	3022353	3213858	3407525	3603507	11
50	2835999	3025528	3217067	3410772	3606794	10
51	2839142	3028703	3220277	3414020	3610082	9
52	2842286	3031879	3223488	3417268	3613370	8
53	2845430	3035055	3226699	3420517	3616659	7
54	2848575	3038232	3229911	3423766	3619949	6
55	2851720	3041410	3233124	3427016	3623239	5
56	2854866	3044588	3236337	3430267	3626530	4
57	2858012	3047767	3239551	3433518	3629822	3
58	2861159	3050946	3242766	3436770	3633115	2
59	2864306	3054126	3245981	3440023	3636408	1
60	2867453	3057307	3249197	3443276	3639702	0

Bottom banner: **74 · 73 · 72 · 71 · 70** · [Footer: "Gradus Quadrantis pro tangentibus complementorum arcuum eiusdem Quadrantis."]

Folio 107 — degrees 20°–24° (bottom banner: 69° 68° 67° 66° 65°)

Top banner: 20 · 21 · 22 · 23 · 24

Min ↓	20	21	22	23	24	Min ↑
0	36397 02	38386 40	40402 62	42447 48	4452286	60
1	36429 97	38419 78	40436 47	42481 82	4455772	59
2	36462 93	38453 16	40470 31	42516 17	4459259	58
3	36495 89	38486 55	40504 16	42550 52	4462747	57
4	36528 86	38519 95	40538 02	42584 88	4466236	56
5	36561 83	38553 36	40571 89	42619 25	4469726	55
6	36594 81	38586 78	40605 77	42653 63	4473216	54
7	36627 80	38620 20	40639 66	42688 01	4476707	53
8	36660 79	38653 63	40673 56	42722 40	4480199	52
9	36693 79	38687 07	40707 47	42756 80	4483692	51
10	36726 80	38720 52	40741 39	42791 21	4487186	50
11	36759 82	38753 97	40775 31	42825 63	4490681	49
12	36792 84	38787 43	40809 24	42860 06	4494177	48
13	36825 87	38820 90	40843 18	42894 50	4497674	47
14	36858 91	38854 38	40877 13	42928 95	4501172	46

Min ↓	20	21	22	23	24	Min ↑
15	36891 95	38887 87	40911 09	42963 40	4504671	45
16	36925 00	38921 36	40945 06	42997 86	4508171	44
17	36958 06	38954 86	40979 03	43032 33	4511672	43
18	36991 13	38988 37	41013 01	43066 81	4515173	42
19	37024 20	39021 88	41046 99	43101 30	4518675	41
20	37057 28	39055 40	41080 97	43135 80	4522178	40
21	37090 37	39088 93	41114 97	43170 31	4525682	39
22	37123 47	39122 47	41148 98	43204 82	4529187	38
23	37156 57	39156 01	41183 00	43239 34	4532693	37
24	37189 68	39189 56	41217 03	43273 87	4536200	36
25	37222 79	39223 12	41251 07	43308 41	4539708	35
26	37255 91	39256 69	41285 11	43342 96	4543217	34
27	37289 04	39290 27	41319 16	43377 52	4546727	33
28	37322 18	39323 85	41353 22	43412 09	4550238	32
29	37355 33	39357 44	41387 28	43446 66	4553750	31
30	37388 48	39391 04	41421 35	43481 24	4557264	30
31	37421 64	39424 65	41455 44	43515 83	4560778	29

Min ↓	20	21	22	23	24	Min ↑
32	37454 80	39458 26	41489 53	43550 43	4564293	28
33	37487 97	39491 88	41523 63	43585 04	4567809	27
34	37521 15	39525 51	41557 73	43619 66	4571326	26
35	37554 34	39559 15	41591 84	43654 29	4574843	25
36	37587 53	39592 80	41625 96	43688 93	4578361	24
37	37620 73	39626 46	41660 09	43723 57	4581880	23
38	37653 94	39660 12	41694 23	43758 22	4585400	22
39	37687 16	39693 79	41728 38	43792 88	4588921	21
40	37720 38	39727 46	41762 55	43827 55	4592443	20
41	37753 61	39761 14	41796 72	43862 23	4595966	19
42	37786 85	39794 83	41830 90	43896 92	4599490	18
43	37820 10	39828 53	41865 09	43931 62	4603015	17
44	37853 35	39862 24	41899 28	43966 33	6406541 [recte: 4606541]	16
45	37886 61	39895 96	41933 48	44001 05	4610068	15
46	37919 88	39929 69	41967 69	44035 78	4613596	14
47	37953 15	39963 42	42001 91	44070 51	4617125	13
48	37986 43	39997 16	42036 13	44105 25	4620654	12

Min ↓	20	21	22	23	24	Min ↑
49	38019 72	40030 90	42070 36	44140 00	4624184	11
50	38053 02	40064 65	42104 60	44174 76	4627715	10
51	38086 32	40098 41	42138 85	44209 53	4631247	9
52	38119 63	40132 17	42173 11	44244 31	4634780	8
53	38152 95	40165 94	42207 38	44279 10	4638314	7
54	38186 28	40199 72	42241 65	44313 90	4641849	6
55	38219 61	40233 51	42275 93	44348 71	4645385	5
56	38252 95	40267 31	42310 22	44383 52	4648922	4
57	38286 30	40301 12	42344 52	44418 34	4652460	3
58	38319 66	40334 94	42378 83	44453 17	4655999	2
59	38353 03	40368 77	42413 15	44488 01	4659540	1
60	38386 40	40402 62	42447 48	44522 86	4663081	0

Bottom banner: **69 · 68 · 67 · 66 · 65** · [Footer: "Gradus Quadrantis pro tangentibus complementorum arcuum eiusdem Quadrantis."]

Tabula Tangentium — Tangent Tables, degrees 25°–39°

(Christoph Clavius [Christophorus Clavius], doctrine of sines, tangents, and secants. Tangent values are tabulated as integers scaled by 10^7 ; i.e. $\tan 45^\circ = 10000000$. Each degree column is read top-to-bottom against its banner degree using the left-hand minute column [0'→60']; read bottom-to-top against the foot banner it gives the

tangent of the complementary arc, using the right-hand minute column. The separator columns between degrees are blank in this format — there are no printed Differentiae columns on these openings.)

Source page p. 748 (printed folio 108) — degrees 25°–29°

Top banner: **TABVLA** — *Gradus Quadrantis pro tangentibus arcuum eiusdem Quadrantis.* [Margin (left, rotated): *Minuta Graduum Quadrantis pro tangentibus arcuum eiusdem Quadrantis.*] [Margin (right, rotated): *Minuta Graduum Quadrantis pro tangentibus complementorum arcuum eiusdem Quadrantis.*]

Min ↓	25	26	27	28	29	Min ↑
0	4663081	4877328	5095254	5317094	5543090	60
1	4666623	4880930	5098919	5320826	5546893	59
2	4670166	4884533	5102585	5324559	5550697	58
3	4673710	4888137	5106252	5328293	5554503	57
4	4677255	4891742	5109920	5332028	5558310	56
5	4680801	4895347	5113589	5335765	5562118	55
6	4684348	4898953	5117259	5339503	5565927	54
7	4687896	4902560	5120930	5343242	5569738	53
8	4691444	4906168	5124602	5346982	5573550	52
9	4694993	4909777	5128275	5350723	5577363	51
10	4698543	4913387	5131949	5354465	5581177	50
11	4702094	4916998	5135625	5358209	5584993	49
12	4705646	4920610	5139302	5361954	5588810	48
13	4709199	4924223	5142980	5365700	5592628	47
14	4712753	4927838	5146659	5369447	5596447	46
15	4716308	4931454	5150339	5373195	5600268	45
16	4719864	4935071	5154020	5376944	5604090	44
17	4723422	4938689	5157702	5380694	5607913	43
18	4726981	4942308	5161385	5384445	5611737	42
19	4730541	4945928	5165069	5388198	5615562	41
20	4734102	4949549	5168755	5391952	5619388	40
21	4737664	4953171	5172442	5395707	5623216	39
22	4741227	4956794	5176130	5399463	5627045	38

Min ↓	25	26	27	28	29	Min ↑
23	4744790	4960418	5179819	5403221	5630875	37
24	4748354	4964043	5183509	5406980	5634707	36
25	4751919	4967669	5187200	5410740	5638540	35
26	4755485	4971296	5190892	5414501	5642374	34
27	4759052	4974924	5194585	5418263	5646210	33
28	4762620	4978553	5198279	5422026	5650047	32
29	4766189	4982184	5201974	5425791	5653885	31
30	4769759	4985816	5205670	5429557	5657725	30
31	4773330	4989448	5209368	5433324	5661566	29
32	4776902	4993081	5213067	5437092	5665408	28
33	4780475	4996716	5216767	5440861	5669251	27
34	4784049	5000352	5220468	5444632	5673096	26
35	4787624	5003989	5224170	5448404	5676942	25
36	4791200	5007627	5227873	5452177	5680789	24
37	4794777	5011266	5231577	5455951	5684637	23
38	4798355	5014906	5235283	5459726	5688486	22
39	4801934	5018547	5238990	5463503	5692337	21
40	4805515	5022189	5242698	5467281	5696189	20
41	4809096	5025832	5246407	5471060	5700043	19
42	4812678	5029476	5250117	5474840	5703898	18
43	4816261	5033121	5253828	5478621	5707754	17
44	4819845	5036767	5257540	5482404	5711611	16
45	4823430	5040414	5261254	5486188	5715469	15
46	4827016	5044062	5264969	5489973	5719329	14
47	4830603	5047712	5268685	5493759	5723190	13
48	4834191	5051363	5272402	5497546	5727052	12
49	4837780	5055015	5276120	5501335	5730916	11
50	4841371	5058668	5279839	5505125	5734781	10
51	4844962	5062322	5283959	5508916	5738647	9
52	4848554	5065977	5287280	5512708	5742515	8
53	4852147	5069633	5291003	5516501	5746384	7

Min ↓	25	26	27	28	29	Min ↑
54	4855741	5073290	5294727	5520296	5750254	6
55	4859336	5076948	5298452	5524092	5754125	5
56	4862932	5080607	5302178	5527889	5757998	4
57	4866525	5084267	5305905	5531687	5761872	3
58	4870127	5087928	5309633	5535487	5765747	2
59	4873727	5091590	5313363	5539288	5769624	1
60	4877328	5095254	5317094	5543090	5773502	0

Foot banner (complement degrees, right→left over the columns): **64 · 63 · 62 · 61 · 60** — *Gradus Quadratis pro tangentibus complementorum arcuum eiusdem Quadrantis.* [*Quadratis* printed thus, for *Quadrantis.*]

Source page p. 749 (printed folio 109) — degrees 30°–34°

Running header: **TANGENTIVM.** Top banner: *Gradus Quadrantis pro tangentibus arcuum eiusdem Quadrantis.* [Margin (left, rotated): *Minuta Graduum Quadrantis pro tangentibus arcuum eiusdem Quadrantis.*] [Margin (right, rotated): *Minuta Graduum Quadrantis pro tangentibus complementorum arcuum eiusdem Quadrantis.*]

Min ↓	30	31	32	33	34	Min ↑
0	5773502	600860	624869	649407	674508	60
		6	3	6	5	
1	5777381	601256	625273	649821	674931	59
		6	8	2	8	
2	5781262	601652	625678	650235	675355	58
		8	5	0	3	
3	5785144	602049	626083	650648	675778	57
		1	4	9	9	
4	5789027	602445	626488	651063	676202	56
		5	4	0	7	
5	5792911	602842	626893	651477	676626	55
		0	5	3	7	
6	5796797	603238	627298	651891	677050	54
		7	8	7	8	
7	5800684	603635	627704	652306	677475	53

Min ↓	30	31	32	33	34	Min ↑
		5	2	3	1	
8	5804572	604032	628109	652720	677899	52
		4	8	0	6	
9	5808462	604429	628515	653135	678324	51
		5	5	9	3	
10	5812353	604826	628921	653551	678749	50
		7	4	0	1	
11	5816245	605224	629327	653966	679174	49
		1	4	2	1	
12	5820139	605621	629733	654381	679599	48
		6	6	6	3	
13	5824034	606019	630139	654797	680024	47
		3	9	1	6	
14	5827930	606417	630546	655212	680450	46
		1	4	8	1	
15	5831828	606815	630953	655628	680875	45
		0	0	7	8	
16	5835727	607213	631359	656044	681301	44
		1	8	7	6	
17	5839627	607611	631766	656460	681727	43
		3	7	9	6	
18	5843528	608009	632173	656877	682153	42
		6	8	2	8	
19	5847431	608408	632581	657293	682580	41
		1	0	7	1	
20	5851335	608806	632988	657710	683006	40
		7	3	3	6	
21	5855241	609205	633395	658127	683433	39
		5	8	1	3	
22	5859148	609604	633803	658544	683860	38
		4	4	0	2	
23	5863056	610003	634211	658961	684287	37
		5	2	1	2	
24	5866966	610402	634619	659378	684714	36

Min ↓		30	31	32	33	34	Min ↑
			7	1	4	4	
25	5870877	610802	635027	659795	685141		35
		0	2	8	7		
26	5874489	611201	635435	660213	685569		34
		5	5	4	2		
27	5878702	611601	635843	660631	685996		33
		1	9	2	9		
28	5882617	612000	636252	661049	686424		32
		9	5	1	7		
29	5886533	612400	636661	661467	686852		31
		8	3	2	7		
30	5890450	612800	637070	661885	687280		30
		8	2	5	9		
31	5894369	613201	637479	662303	687709		29
		0	2	9	3		
32	5898289	613601	637888	662722	688137		28
		3	4	5	9		
33	5902211	614001	638297	663141	688566		27
		8	7	3	6		
34	5906134	614402	638707	663560	688995		26
		4	2	3	5		
35	5910058	614803	639116	663979	689424		25
		2	9	2	6		
36	5913984	615204	639526	664398	689853		24
		1	7	4	9		
37	5917911	615605	639936	664817	690283		23
		2	6	8	3		
38	5921839	616006	640346	665237	690712		22
		4	7	3	9		
39	5925769	616407	640756	665657	691142		21
		7	9	0	6		
40	5929700	616809	641167	666076	691572		20
		2	3	8	5		
41	5933633	617210	641577	666496	692002		19

Min ↓		30	31	32	33	34	Min ↑
			8	9	8	6	
42		5937567	617612	641988	666917	692432	18
			6	6	0	9	
43		5941502	618014	642399	667337	692863	17
			7	5	3	4	
44		5945438	618416	642810	667757	693294	16
			8	5	8	0	
45		5949376	618819	643221	668178	693724	15
			0	6	5	8	
46	5955315 [recte:?]	619221	643632	668599	694155		14
			3	9	4	8	
47		5957255	619623	644044	669020	694586	13
			7	4	4	9	
48		5961197	620026	644456	669441	695018	12
			3	0	6	2	
49		5965140	620429	644867	669863	695449	11
			0	8	0	7	
50		5969084	620831	645279	670284	695881	10
			9	8	5	3	
51		5973030	621235	645691	670706	696313	9
			0	9	2	1	
52		5976776	621638	646104	671128	696745	8
			2	2	1	1	
53		5980926	622041	646516	671550	697177	7
			6	6	1	3	
54		5984876	622445	646929	671972	697609	6
			1	2	3	7	
55		5988827	622848	647341	672394	698042	5
			8	9	6	3	
56		5992780	623252	647754	672817	698475	4
			6	8	1	0	
57		5996734	623656	648167	673239	698907	3
			6	8	7	9	
58		6000690	624060	648580	673662	699340	2

Min ↓	30	31	32	33	34	Min ↑
		7	9	5	9	
59	6004647	624464	648994	674085	699774	1
		9	2	4	1	
60	6008606	624869	649407	674508	700207	0
		3	6	5	5	

Foot banner (complement degrees, right→left over the columns): **59 · 58 · 57 · 56 · 55** — *Gradus Quadratis pro tangentibus complementorum arcuum eiusdem Quadrantis*. [*Quadratis* printed thus, for *Quadrantis*.]

Source page p. 750 (printed folio 110) — degrees 35°–39°

Top banner: **TABVLA** — *Gradus Quadrantis pro tangentibus arcuum eiusdem Quadrantis*. [Margin (left, rotated): *Minuta Graduum Quadrantis pro tangentibus arcuum eiusdem Quadrantis*.] [Margin (right, rotated): *Minuta Graduum Quadrantis pro tangentibus complementorum arcuum eiusdem Quadrantis*.]

Min ↓	35	36	37	38	39	Min ↑
0	7002075	7265424	7535541	7812856	8097840	60
1	7006411	7269869	7540103	7817542	8102658	59
2	7010749	7274316	7544667	7822230	8107478	58
3	7015088	7278765	7549233	7826920	8112300	57
4	7019429	7283216	7553801	7831612	8117124	56
5	7023772	7287669	7558371	7836306	8121951	55
6	7028117	7292124	7562943	7841002	8126780	54
7	7032463	7296581	7567517	7845700	8131611	53
8	7036811	7301040	7572093	7850400	8136444	52
9	7041161	7305501	7576670	7855102	8141280	51
10	7045513	7309963	7581249	7859807	8146118	50
11	7049867	7314427	7585830	7864514	8150958	49
12	7054223	7318893	7590413	7869223	8155801	48
13	7058581	7323361	7594999	7873934	8160646	47
14	7062940	7327831	7599587	7878647	8165493	46
15	7067301	7332303	7604177	7883363	8170343	45
16	7071664	7336777	7608769	7888081	8175195	44

Min ↓	35	36	37	38	39	Min ↑
17	7076029	7341253	7613363	7892801	8180049	43
18	7080395	7345731	7617959	7897523	8184905	42
19	7084763	7350210	7622557	7902247	8189764	41
20	7089133	7354691	7627157	7906973	8194625	40
21	7093505	7359174	7631759	7911702	8199488	39
22	7097879	7363659	7636363	7916433	8204354	38
23	7102254	7368146	7640969	7921166	8209222	37
24	7106631	7372635	7645577	7925901	8214092	36
25	7111010	7377126	7650187	7930638	8218965	35
26	7115391	7381619	7654799	7935378	8223840	34
27	7119773	7386114	7659413	7940120	8228717	33
28	7124167	7390611	7664030	7944864	8233597	32
29	7128543	7395110	7668649	7949610	8238479	31
30	7132931	7399610	7673270	7954358	8243363	30
31	7137321	7404112	7677893	7959109	8248250	29
32	7141713	7408616	7682518	7963862	8253139	28
33	7146106	7413122	7687145	7968617	8258031	27
34	7150501	7417630	7691774	7973374	8262925	26
35	7154898	7422140	7696405	7978133	8267821	25
36	7159297	7426652	7701038	7982895	8272720	24
37	7163698	7431167	7705673	7987659	8277621	23
38	7168100	7435684	7710310	7992425	8282524	22
39	7172504	7440203	7714949	7997193	8287429	21
40	7176910	7444724	7719590	8001963	8292337	20
41	7181318	7449246	7724233	8006736	8297247	19
42	7185728	7453770	7728878	8011511	8302160	18
43	7190140	7458296	7733525	8016288	8307075	17
44	7194554	7462824	7738175	8021067	8311992	16
45	7198970	7467354	7742827	8025849	8316912	15
46	7203387	7471886	7747481	8030633	8321834	14
47	7207806	7476420	7752137	8035419	8326759	13

Min ↓	35	36	37	38	39	Min ↑
48	7212227	7480956	7756795	8040207	8331686	12
49	7216650	7485494	7761455	8044997	8336615	11
50	7221075	7490033	7766117	8049790	8341547	10
51	7225502	7494574	7770781	8054585	8346481	9
52	7229931	7499117	7775447	8059382	8351418	8
53	7234362	7503663	7780116	8064181	8356357	7
54	7238794	7508211	7784787	8068983	8361298	6
55	7243228	7512761	7789460	8073787	8366242	5
56	7247664	7517313	7794135	8078593	8371188	4
57	7252102	7521867	7798812	8083401	8376136	3
58	7256541	7526423	7803491	8088212	8381087	2
59	7260982	7530981	7808172	8093025	8386040	1
60	7265424	7535541	7812856	8097840	8390996	0

Foot banner (complement degrees, right→left over the columns): **54 · 53 · 52 · 51 · 50** — *Gradus Quadrantis pro tangentibus complementorum arcuum eiusdem Quadrantis.*

Tabula Tangentium (Table of Tangents) — degrees 40°–52°

Christoph Clavius, Opera Mathematica, Tomus I (Mainz, 1611). Numeric tangent tables, continued.

The three openings transcribed here continue the *Tabula Tangentium* upward from 40°, crossing 45° (where the tangent equals the whole radius, 10000000). Each page carries a block of whole-degree columns headed by a degree banner (read downward against the left minute column, 0' at top → 60' at foot); the foot of each page repeats a complement-degree banner for reading the same cells as tangents of the complementary arc against the right minute column (60' at top → 0' at foot). Latin column and caption words are kept as printed.

Header words as printed: *Gradus Quadrantis pro tangentibus arcuum eiusdem Quadrantis* (running title *TABVLA — TANGENTIVM*); left rotated caption *Minuta*

Gradium Quadrantis pro tangentibus arcuum eiusdem Quadrantis; right rotated caption pro tangentibus complementorum arcuum eiusdem Quadrantis.

Page 751 (printed folio 111) — Tangens, degrees 40°–44°

Top banner (direct, left minute column 0'→60'): **40 · 41 · 42 · 43 · 44** Foot banner (complement, right minute column): **49 · 48 · 47 · 46 · 45**

Min.	40	41	42	43	44
0	8390996	8692867	9004040	9325151	9656888
1	8395954	8697975	9009308	9330591	9662511
2	8400915	8703085	9014579	9336034	9668137
3	8405878	8708198	9019853	9341480	9673766
4	8410844	8713344	9025130	9346929	9679398
5	8415812	8718433	9030410	9352381	9685034
6	8420782	8723555	9035693	9357835	9690674
7	8425754	8728679	9040978	9363292	9696315
8	8430729	8733806	9046266	9368752	9701960
9	8435706	8738935	9051557	9374215	9707609
10	8440686	8744067	9056850	9379682	9713261
11	8445668	8749201	9062146	9385152	9718916
12	8450653	8754338	9067445	9390625	9724574
13	8455640	8759478	9072747	9396101	9730235
14	8460630	8764620	9078052	9401580	9735900
15	8465622	8769764	9083360	9407062	9741568
16	8470617	8774911	9088670	9412547	9747239
17	8475614	8780061	9093983	9418034	9752913
18	8480614	8785214	9099299	9423524	9758591
19	8485617	8790369	9104618	9429017	9764272
20	8490622	8795527	9109940	9434513	9769956
21	8495629	8800688	9115265	9440012	9775643
22	8500639	8805851	9120593	9445514	9781334
23	8505651	8811017	9125923	9451019	9787028
24	8510666	8816186	9131256	9456528	9792725

Min.	40	41	42	43	44
25	8515683	8821357	9136592	9462040	9798425
26	8520703	8826531	9141930	9467555	9804128
27	8525725	8831708	9147271	9473073	9809835
28	8530750	8836887	9152615	9478594	9815545
29	8535777	8842069	9157962	9484118	9821258
30	8540806	8847253	9163312	9489645	9826974
31	8545838	8852440	9168665	9495175	9832694
32	8550872	8857630	9174021	9500708	9838417
33	8555909	8862822	9179380	9506244	9844143
34	8560949	8868017	9184741	9511783	9849872
35	8565991	8873015 [?]	9190105	9517325	9855605
36	8571036	8878415	9195472	9522870	9861341
37	8576083	8883628	9200842	9528419	9867180 [?]
38	8581133	8888824	9206215	9533971	9872922 [?]
39	8586185	8894033	9211590	9539526	9878668 [?]
40	8591239	8899244	9216968	9545084	9884317
41	8596296	8904458	9222349	9550645	9890070
42	8601355	8909675	9227733	9556209	9895826
43	8606417	8914894	9233120	9561776	9901585
44	8611482	8920116	9238510	9567346	9907347
45	8616549	8925341	9243903	9572919	9913113
46	8621619	8930568	9249299	9578495	9918882
47	8626692	8935798	9254698	9584074	9924654
48	8631767	8941031	9260100	9589656	9930430
49	8636845	8946267	9265505	9595241	9936209
50	8641926	8951506	9270913	9600830	9941991
51	8647009	8956747	9276324	9606422	9947777
52	8652095	8961991	9281738	9612017	9953566
53	8657183	8967238	9287155	9617615	9959359
54	8662273	8972487	9292574	9623216	9965155
55	8667366	8977739	9297996	9628820	9970954

Min.	40	41	42	43	44
56	8672461	8982994	9303421	9634427	9976756
57	8677559	8988252	9308849	9640037	9982562
58	8682659	8993512	9314280	9645651	9988371
59	8687762	8998775	9319714	9651268	9994184
60	8692867	9004040	9325151	9656888	10000000

Column seams (each degree's 60' = next degree's 0'): 40°/60' = 41°/0' = 8692867 ✓; 41°/60' = 42°/0' = 9004040 ✓; 42°/60' = 43°/0' = 9325151 ✓; 43°/60' = 44°/0' = 9656888 ✓; **44°/60' = tan 45° = 10000000** ✓ (7→8 digit jump handled at the foot of the 44° column).

Page 752 (printed folio 112) — Tabula, degrees 45°–48°

Top banner (direct, left minute column 0'→60'): **45 · 46 · 47 · 48** Foot banner (complement, right minute column): **44 · 43 · 42 · 41**

Values are now 8 digits (tangent exceeds the radius beyond 45°).

Min.	45	46	47	48
0	10000000	10355302	10723686	11106124
1	10005820	10361332	10729942	11112623
2	10011643	10367365	10736202	11119126
3	10017469	10373402	10742466	11125634
4	10023299	10379443	10748734	11132146
5	10029132	10385487	10755006	11138662
6	10034968	10391535	10761282	11145182
7	10040808	10397587	10767562	11151706
8	10046651	10403643	10773845	11158235
9	10052497	10409702	10780132	11164768
10	10058347	10415765	10786423	11171305
11	10064201	10421832	10792718	11177846
12	10070058	10427902	10799017	11184392
13	10075918	10433976	10805320	11190942
14	10081782	10440054	10811627	11197496
15	10087649	10446135	10817938	11204054

Min.	45	46	47	48
16	10093520	10452220	10824253	11210617
17	10099394	10458309	10830572	11217184
18	10105272	10464401	10836895	11223755
19	10111153	10470497	10843222	11230330
20	10117038	10476597	10849554	11236910
21	10122926	10482701	10855889	11243494
22	10128818	10488808	10862228	11250082
23	10134713	10494919	10868571	11256675
24	10140611	10501034	10874918	11263272
25	10146513	10507153	10881269	11269873
26	10152418	10513275	10887624	11276478
27	10158327	10519401	10893983	11283088
28	10164239	10525531	10900346	11289702
29	10170154	10531664	10906713	11296321
30	10176073	10537801	10913084	11302944
31	10181996	10543942	10919459	11309571
32	10187922	10550087	10925838	11316203
33	10193852	10556235	10932221	11322899 [?]
34	10199785	10562387	10938608	11329480
35	10205722	10568543	10945000	11336125
36	10211663	10574703	10951396	11342774
37	10217607	10586867 [recte:?]	10957796	11349428
38	10223555	10587034	10964200	11356086
39	10229506	10593205	10970608	11362748
40	10235460	10599280 [?]	10977020	11369415
41	10241418	10605559	10983436	11376086
42	10247380	10611742	10989856	11382762
43	10253345	10617929	10996280	11389442
44	10259314	10624119	11002708	11396126
45	10265286	10630313	11009140	11402815
46	10271262	10636511	11015577	11409508

Min.	45	46	47	48
47	10277242	10642713	11022028	11416206
48	10283225	10648919	11028463	11422908
49	10289212	10655128	11034912	11429615
50	10295202	10661341	11041365	11436326
51	10301196	10667558	11047822	11443042
52	10307193	10673779	11054283	11449762
53	10313194	10680004	11060748	11456487
54	10319199	10686233	11067218	11463216
55	10325207	10692466	11073692	11469950
56	10331219	10698702	11080170	11476688
57	10337234	10704942	11086652	11483431
58	10343253	10711186	11093138	11490178
59	10349276	10717434	11099629	11496929
60	10355302	10723686	11106124	11503684

Column seams: $45^{\circ}/0' = 10000000$ (= $44^{\circ}/60'$ from page 751) ✓; $45^{\circ}/60' = 46^{\circ}/0' = 10355302$ ✓; $46^{\circ}/60' = 47^{\circ}/0' = 10723686$ ✓; $47^{\circ}/60' = 48^{\circ}/0' = 11106124$ ✓; $48^{\circ}/60' = 11503684$ (= $49^{\circ}/0'$ on page 753) ✓.

Page 753 (printed folio 113) — Tangens, degrees 49° – 52°

Top banner (direct, left minute column $0' \rightarrow 60'$): **49 · 50 · 51 · 52** Foot banner (complement, right minute column): **40 · 39 · 38 · 37**

Min.	49	50	51	52
0	11503684	11917537	12348972	12799416
1	11510444	11924580	12356320	12807093
2	11517208	11931628	12363673	12814776
3	11523977	11938680	12371031	12822465
4	11530751	11945737	12378394	12830159
5	11537529	11952799	12385762	12837859
6	11544312	11959866	12393136	12845565
7	11551100	11966938	12400515	12853277
8	11557893	11974015	12407999 [?]	12860994

Min.	49	50	51	52
9	11564691	11981097	12415288	12868717
10	11571494	11988183	12422683	12876445
11	11578301	11995274	12430083	12884179
12	11585112	12002370	12437489	12891919
13	11591928	12009471	12444900	12899665
14	11598748	12016578	12452317	12907417
15	11605572	12023690	12459739	12915175
16	11612401	12030807	12467167	12922939
17	11619234	12037929	12474600	12930709
18	11626072	12045056	12482039	12938485
19	11632915	12052188	12489484	12946267
20	11639763	12059325	12496934	12954055
21	11646615	12066467	12504389	12961848
22	11653472	12073614	12511850	12969647
23	11660334	12080766	12519316	12977457
24	11667200	12087923	12526787	12985263
25	11674071	12095085	12534264	12993080
26	11680947	12102252	12541746	13000903
27	11687827	12109424	12549233	13008732
28	11694712	12116601	12556725	13016567
29	11701602	12123783	12564222	13024407
30	11708497	12130970	12571724	13032253
31	11715396	12138162	12579232	13040105
32	11722300	12145359	12586746	13047963
33	11729208	12152561	12594265	13055827
34	11736121	12159768	12601790	13063697
35	11743039	12166981	12609321	13071573
36	11749962	12174199	12616858	13079455
37	11756989 [?]	12181422	12624400	13087343
38	11763821	12188650	12631948	13095237
39	11770758	12195883	12639501	13103138

Min.	49	50	51	52
40	11777700	12203121	12647060	13111045
41	11784646	12210364	12654624	13118958
42	11791597	12217613	12662194	13126877
43	11798553	12224867	12669769	13134802
44	11805514	12232126	12677350	13142732
45	11812479	12239390	12684937	13150668
46	11819449	12246659	12692530	13158610
47	11826424	12253933	12700128	13166558
48	11833404	12261212	12707732	13174512
49	11840388	12268496	12715341	13182472
50	11847377	12275786	12722956	13190438
51	11854371	12283081	12730577	13198411
52	11861370	12290381	12738203	13206390
53	11868374	12297687	12745835	13214375
54	11875383	12304998	12753473	13222367
55	11882397	12312314	12761116	13230365
56	11889417	12319635	12768765	13238369
57	11896438	12326961	12776420	13246379
58	11903466	12334293	12784080	13254396
59	11910499	12341630	12791745	13262419
60	11917537	12348972	12799416	13270448

Column seams: $49^\circ/0' = 11503684$ (= $48^\circ/60'$ from page 752) ✓; $49^\circ/60' = 50^\circ/0' = 11917537$ ✓; $50^\circ/60' = 51^\circ/0' = 12348972$ ✓; $51^\circ/60' = 52^\circ/0' = 12799416$ ✓; $52^\circ/60' = 13270448$ (= $\tan 53^\circ$, continues in the next opening).

Table of Tangents (*Tabula Tangentium*) — degrees 53°–64°

Christoph Clavius, Opera Mathematica, Tomus I (Mainz, 1611). Upper-quadrant tangent tables, continued. Each page carries four degree columns (top banner). The left minute column (*Minuta Graduum Quadrantis*, 0'–60', read downward) gives the tangent of the top-banner arc; the right minute column (60'–0', read

upward) with the bottom banner gives the tangent of the complementary arc (*pro tangentibus complementorum arcuum eiusdem Quadrantis*). Tangent values are 8-digit (radius = 10,000,000). Latin header words retained as printed.

Folio 114 — degrees 53°, 54°, 55°, 56°

Min.	53	54	55	56
0	13270448	13763820	14281480	14825610
1	13278483	13772243	14290325	14834916
2	13286524	13780673	14299177	14844230
3	13294571	13789109	14308037	14853553
4	13302624	13797552	14316905	14862884
5	13310683	13806002	14325780	14872223
6	13318749	13814459	14334662	14881570
7	13326821	13822922	14343552	14890925
8	13334899	13831392	14352451	14900288
9	13342984	13839869	14361354	14909659
10	13351075	13848352	14370266	14919038
11	13359172	13856842	14379186	14928426
12	13367276	13865339	14388113	14937822
13	13375386	13873843	14397048	14947226
14	13383502	13882354	14405990	14956638
15	13391624	13890872	14414939	14966058
16	13399753	13899397	14423896	14975486
17	13407888	13907930	14432861	14984923
18	13416029	13916470	14441833	14994368
19	13424177	13925017	14450812	15003821
20	13432331	13933571	14459799	15013283
21	13440492	13942131	14468794	15022753
22	13448659	13950698	14477797	15032231
23	13456832	13959272	14486807	15041717
24	13465011	13967853	14495825	15051211
25	13473197	13976441	14504850	15060714
26	13481390	13985035	14513883	15070225

Min.	53	54	55	56
27	13489589	13993636	14522924	15079744
28	13497794	14002244	14531972	15089271
29	13506006	14010859	14541028	15098807
30	13514224	14019481	14550091	15108351
31	13522449	14028110	14559162	15117903
32	13530680	14036746	14568241	15127464
33	13538918	14045389	14577327	15137034
34	13547162	14054040	14586421	15146612
35	13555413	14062698	14595523	15156199
36	13563670	14071363	14604633	15165794
37	13571834	14080035	14613750	15175398
38	13580104	14088715	14622875	15185011
39	13588381	14097402	14632007	15194632
40	13596764	14106097	14641146	15204261
41	13605054	14114798	14650293	15213899
42	13613350	14123506	14659449	15223545
43	13621653	14132221	14668613	15233200
44	13629963	14140923	14677785	15242863
45	13638279	14149672	14686965	15252535
46	13646602	14158409	14696153	15262216
47	13654932	14167153	14705349	15271905
48	13663268	14175904	14714553	15281603
49	13671610	14184663	14723765	15291309
50	13679959	14193429	14732985	15301024
51	13688315	14202202	14742212	15310748
52	13696677	14210982	14751447	15320481
53	13705046	14219769	14760690	15330222
54	13713422	14228563	14769941	15339972
55	13721805	14237365	14779200	15349730
56	13730194	14246174	14788466	15359497
57	13738590	14254990	14797740	15369273

Min.	53	54	55	56
58	13746993	14263813	14807022	15379057
59	13755403	14272643	14816312	15388850
60	13763820	14281480	14825610	15398651

Bottom (complement) banners, read upward with the right minute column: **36 · 35 · 34 · 33.**

Folio 115 — degrees 57°, 58°, 59°, 60°

Min.	57	58	59	60
0	15398651	16003347	16642794	17320508
1	15408461	16013710	16653766	17332150
2	15418280	16024083	16664749	17343804
3	15428108	16034466	16675742	17355469
4	15437945	16044859	16686746	17367146
5	15447791	16055261	16697760	17378834
6	15457646	16065673	16708785	17390534
7	15467510	16076095	16719820	17402246
8	15477382	16086527	16730866	17413969
9	15487263	16096968	16741922	17425704
10	15497153	16107419	16752989	17437451
11	15507052	16117880	16764067	17449210
12	15516960	16128351	16775156	17460981
13	15526877	16138832	16786256	17472764
14	15536803	16149322	16797367	17484559
15	15546738	16159822	16808489	17496366
16	15556682	16170332	16819621	17508185
17	15566636	16180852	16830764	17520026
18	15576599	16191381	16841918	17531869
19	15586571	16201920	16853083	17543724
20	15596552	16212469	16864259	17555591
21	15606542	16223028	16875446	17567470
22	15616541	16233597	16886644	17579362

Min.	57	58	59	60
23	15626549	16244176	16897853	17591266
24	15636566	16254766	16909074	17603182
25	15646592	16265366	16920306	17615111
26	15656627	16275976	16931549	17627052
27	15666671	16286596	16942803	17639006
28	15676724	16297226	16954068	17650972
29	15686786	16307866	16965344	17662951
30	15696857	16318516	16976631	17674941
31	15706938	16329176	16987929	17686945
32	15717028	16339847	16999239	17698960
33	15727127	16350528	17010560	17710987
34	15737235	16361219	17021892	17723027
35	15747353	16371920	17033236	17735079
36	15757480	16382631	17044591	17747143
37	15767616	16393352	17055957	17759220
38	15777761	16404083	17067325	17771309
39	15787915	16414824	17078714	17783410
40	15798078	16425575	17090115	17795524
41	15808251	16436337	17101527	17808651
42	15818433	16447109	17112950	17819790
43	15828625	16457892	17124384	17831942
44	15838827	16468685	17135829	17844107
45	15849038	16479488	17147285	17856285
46	15859259	16490302	17158752	17868475
47	15869489	16501126	17170231	17880678
48	15879729	16511960	17181721	17892894
49	15889979	16522805	17193222	17905123
50	15900238	16533660	17204734	17917364
51	15910507	16544526	17216258	17929618
52	15920785	16555402	17227794	17941885
53	15931073	16566289	17239342	17954164

Min.	57	58	59	60
54	15941370	16577186	17250902	17966456
55	15951676	16588094	17262473	17978761
56	15961992	16599013	17274056	17991079
57	15972317	16609942	17285651	18003410
58	15982651	16620882	17297258	18015753
59	15992994	16631833	17308877	18028109
60	16003347	16642794	17320508	18040478

Bottom (complement) banners, read upward with the right minute column: **32 · 31 · 30 · 29.**

Folio 116 — degrees 61°, 62°, 63°, 64°

Min.	61	62	63	64
0	18040478	18807265	19626104	20503034
1	18052860	18820471	19640225	20518180
2	18065255	18833691	19654362	20533344
3	18077663	18846925	19668516	20548526
4	18090084	18860174	19682686	20563726
5	18102518	18873437	19696872	20578945
6	18114966	18886715	19711074	20594182
7	18127427	18900007	19725293	20609437
8	18139901	18913314	19739528	20624711
9	18152388	18926636	19753780	20640003
10	18164889	18939972	19768048	20655313
11	18177403	18953323	19782333	20670642
12	18189930	18966689	19796634	20685989
13	18202470	18980070	19810951	20701355
14	18215024	18993466	19825285	20716739
15	18227591	19006876	19839635	20732142
16	18240171	19020301	19854002	20747564
17	18252765	19033741	19868386	20763004
18	18265372	19047196	19882786	20778463

Min.	61	62	63	64
19	18277992	19060665	19897203	20793941
20	18290626	19074149	19911637	20809438
21	18303273	19087648	19926088	20824953
22	18315934	19101162	19940555	20840487
23	18328608	19114691	19955039	20856040
24	18341296	19128235	19969540	20871612
25	18353997	19141795	19984057	20887202
26	18366712	19155370	19998591	20902811
27	18379440	19168960	20013142	20918439
28	18392182	19182565	20027709	20934086
29	18404938	19196185	20042297	20949752
30	18417707	19209821	20056898	20965436
31	18430490	19223472	20071516	20981140
32	18443287	19237138	20086152	20996863
33	18456098	19250819	20100805	21012605
34	18468922	19264516	20115475	21028367
35	18481760	19278228	20130163	21044148
36	18494612	19291955	20144868	21059949
37	18507478	19305698	20159590	21075769
38	18520357	19319456	20174329	21091609
39	18533250	19333230	20189086	21107468
40	18546157	19347019	20203860	21123347
41	18559078	19360824	20218651	21139246
42	18572013	19374644	20233460	21155164
43	18584962	19388480	20248286	21171102
44	18597925	19402331	20263130	21187059
45	18610902	19416198	20277991	21203036
46	18623894	19430081	20292870	21219032
47	18636900	19443980	20307767	21235048
48	18649920	19457894	20322681	21251083
49	18662954	19471824	20337613	21267138

Min.	61	62	63	64
50	18676002	19485770	20352563	21283213
51	18689064	19499732	20367531	21299308
52	18702140	19513710	20382516	21315423
53	18715231	19527704	20397519	21331558
54	18728335	19541714	20412539	21347713
55	18741454	19555739	20427577	21363888
56	18754587	19569780	20442633	21380083
57	18767735	19583837	20457706	21396298
58	18780897	19597910	20472797	21412534
59	18794074	19611999	20487906	21428790
60	18807265	19626104	20503034	21445067

Bottom (complement) banners, read upward with the right minute column: **28 · 27 · 26 · 25.**

Tabula Tangentium — Upper Quadrant (degrees 65°–76°)

Running head (recto): TANGENTIVM. — (verso): TABVLA. Top caption (each page): "Gradus Quadrantis pro tangentibus arcuum eiusdem Quadrantis." ["Degrees of the Quadrant, for the tangents of the arcs of the same Quadrant."]

Each page carries a two-way table. The **top banner** gives the degree read with the **left** minute column (0'→60', descending). The **bottom banner** (the arc's complement, = 89 – top degree) is read with the **right** minute column (60'→0'), under the bottom caption *"Gradus Quadrantis pro tangentib. complementorum arcuum eiusdem Quadrantis."* ["...for the tangents of the complements of the arcs of the same Quadrant."]. The right-hand minute column equals 60 – (left minute).

Radius (implied) = 10,000,000; every tangent value is eight digits throughout this opening (no 8→9-digit transition occurs here). Values climb steeply and the per-minute differences grow monotonically.

Folio 117 — degrees 65, 66, 67, 68

Min (L)	65	66	67	68	Min (R)
0	21445067	22460371	23558529	24750869	60
1	21461364	22477965	23577595	24771613	59
2	21477681	22495582	23596687	24792387	58
3	21494019	22513222	23615805	24813191	57
4	21510377	22530885	23634950	24834024	56
5	21526756	22548571	23654121	24854887	55
6	21543155	22566281	23673318	24875780	54
7	21559575	22584014	23692542	24896704	53
8	21576015	22601771	23711793	24917659	52
9	21592475	22619551	23731071	24938644	51
10	21608956	22637355	23750375	24959659	50
11	21625458	22655183	23769706	24980705	49
12	21641981	22673034	23789064	25001782	48
13	21658525	22690909	23808448	25022890	47
14	21675090	22708808	23827859	25044029	46
15	21691676	22726730	23847297	25065198	45
16	21708283	22744676	23866762	25086398	44
17	21724911	22762646	23886254	25107629	43
18	21741559	22780639	23905773	25128991	42
19	21758228	22798656	23925320	25150183	41
20	21774918	22816696	23944895	25171506	40
21	21791629	22834760	23964496	25192861	39
22	21808362	22852848	23984124	25214248	38
23	21825116	22870960	24003779	25235666	37
24	21841892	22889096	24023462	25257116	36
25	21858689	22907256	24043172	25278597	35
26	21875508	22925441	24062910	25300110	34
27	21892348	22943650	24082675	25321655	33
28	21909210	22961883	24102468	25343232	32
29	21926094	22980141	24122289	25364841	31

Min (L)	65	66	67	68	Min (R)
30	21943000	22998424	24142137	25386482	30
31	21959926	23016731	24162013	25408154	29
32	21976874	23035062	24181917	25429858	28
33	21993843	23053418	24201849	25451594	27
34	22010834	23071798	24221809	25473362	26
35	22027846	23090203	24241798	25495162	25
36	22044879	23108632	24261815	25516995	24
37	22061934	23127086	24281860	25538860	23
38	22079011	23145565	24301934	25560758	22
39	22096109	23164068	24322037	25582688	21
40	22113229	23182597	24342169	25604651	20
41	22130372	23201151	24362329	25626647	19
42	22147537	23219730	24382518	25648675	18
43	22164725	23238335	24402735	25670736	17
44	22181935	23256965	24422981	25692830	16
45	22199168	23275621	24443256	25714957	15
46	22216424	23294302	24463559	25737118	14
47	22233703	23313008	24483891	25759312	13
48	22251004	23331740	24504252	25781540	12
49	22268328	23350498	24524642	25803801	11
50	22285675	23369282	24545061	25826096	10
51	22303044	23388092	24565509	25848424	9
52	22320435	23406927	24585986	25870786	8
53	22337848	23425788	24606492	25893181	7
54	22355284	23444674	24627028	25915610	6
55	22372742	23463586	24647594	25938073	5
56	22390223	23482523	24668189	25960569	4
57	22407726	23501486	24688814	25983099	3
58	22425252	23520475	24709469	26005663	2
59	22442800	23539489	24730154	26028261	1
60	22460371	23558529	24750869	26050893	0

*Bottom banners (complement reading, right minute column): 24 · 23 · 22 · 21
(beneath the 65/66/67/68 columns respectively).*

Folio 118 — degrees 69, 70, 71, 72

Min (L)	69	70	71	72	Min (R)
0	26050893	27474777	29042105	30776 834	60
1	26073559	27499665	29069569	30807 323	59
2	26096260	27524592	29097080	30837 866	58
3	26118996	27549559	29124638	30868 465	57
4	26141766	27574565	29152243	30899 119	56
5	26164571	27599612	29179895	30929 828	55
6	26187411	27624699	29207595	30960 593	54
7	26210286	27649827	29235343	30991 413	53
8	26233196	27674995	29263139	31022 289	52
9	26256141	27700204	29290382 [recte:?]	31053 221	51
10	26279120	27725453	29318873	31084 208	50
11	26302135	27750742	29346811	31115 252	49
12	26325185	27776072	29374797	31146 352	48
13	26348270	27801443	29402831	31177 508	47
14	26371390	27826855	29430913	31208	46

Min (L)	69	70	71	72	Min (R)
				720	
15	26394546	27852308	29459043	31239 989	45
16	26417738	27877803	29487221	31271 315	44
17	26440966	27903339	29515446	31302 698	43
18	26464229	27928917	29543719	31334 138	42
19	26487528	27954536	29572041	31365 636	41
20	26510863	27980196	29600411	31397 191	40
21	26534234	28005898	29628831	31428 805	39
22	26557641	28031642	29657301	31460 476	38
23	26581084	28057429	29685820	31492 205	37
24	26604563	28083258	29714388	31523 992	36
25	26628079	28109129	29743006	31555 838	35
26	26651631	28135043	29771674	31587 742	34
27	26675220	28160999	29800392	31619 705	33
28	26698845	28186998	29829160	31651 727	32
29	26722507	28213040	29857978	31683 807	31
30	26746206	28239125	29886847	31715 946	30

Min (L)	69	70	71	72	Min (R)
31	26769942	28265253	29915765	31748 144	29
32	26793716	28291424	29944734	31780 401	28
33	26816527 [recte:?]	28317638	29973753	31812 717	27
34	26841375	28343895	30002823	31845 093	26
35	26865260	28379195 [recte:?]	30031943	31877 528	25
36	26889183	28396539	30061113	31910 024	24
37	26913143	28422926	30090334	31942 580	23
38	26937141	28449357	30119605	31975 197	22
39	26961177	28475832	30148927	32007 875	21
40	26985251	28502350	30178295	32040 613	20
41	27009362	28528913	30207723	32073 413	19
42	27033511	28555520	30237200	32106 275	18
43	27057698	28582172	30266730	32139 200	17
44	27081922	28608868	30296312	32172 187	16
45	27106184	28635608	30325947	32205 237	15
46	27130484	28662393	30355635	32238 349	14
47	27154823	28689222	30385375	32271	13

Min (L)	69	70	71	72	Min (R)
				524	
48	27179200	28716096	30415169	32304	12
				762	
49	27203616	28743015	30445015	32338	11
				064	
50	27228070	28769979	30474915	32371	10
				430	
51	27252563	28796987	30504867	32404	9
				858	
52	27277095	28824040	30534872	32438	8
				348	
53	27301667	28851139	30564930	32471	7
				901	
54	27326278	28878283	30595041	32505	6
				517	
55	27350929	28905472	30625205	32539	5
				196	
56	27375620	28932707	30655423	32572	4
				937	
57	27400350	28959988	30685695	32606	3
				741	
58	27425120	28987315	30716020	32640	2
				907	
59	27449929	29014687	30746400	32674	1
				536	
60	27474777	29042105	30776834	32708	0
				528	

Bottom banners (complement reading): 20 · 19 · 18 · 17 (beneath the 69/70/71/72 columns respectively).

Folio 119 — degrees 73, 74, 75, 76

Min (L)	73	74	75	76	Min (R)
0	32708528	34874151	37320517	40107808	60

Min (L)		73	74	75	76	Min (R)
1	32745286 [recte:?]	34912477	37363987	40157569		59
2	32776709	34950881	37407551	40207446		58
3	32810898	34989364	37451210	40257440		57
4	32845153	35027925	37494964	40307552		56
5	32879747	35066565	37538814	40357781		55
6	32913862	35105283	37582760	40408129		54
7	32948317	35144080	37626803	40458596		53
8	32982839	35182956	37670943	40509183		52
9	33017427	35221911	37715180	40559890		51
10	33052082	35260945	37759515	40610718		50
11	33086802	35300059	37803948	40661665		49
12	33121588	35339253	37848479	40712731		48
13	33156441	35378528	37893109	40763917		47
14	33191362	35417883	37937838	40815224		46
15	33226351	35457320	37982666	40866652		45
16	33261408	35496838	38027592	40918201		44
17	33296534	35536438	38072616	40969871		43
18	33331728	35576121	38117740	41021663		42
19	33366990	35615888	38162963	41073577		41
20	33402321	35655739	38208285	41125614		40
21	33437720	35695672	38253708	41177775		39
22	33473188	35735689	38299232	41230062		38
23	33508725	35775789	38344857	41282475		37
24	33544330	35815973	38390584	41335015		36
25	33580005	35856241	38436414	41387683		35
26	33615750	35896593	38482347	41440480		34
27	33651566	35937029	38528384	41493407		33
28	33687453	35977550	38574525	41546464		32
29	33723410	36018156	38620772	41599653		31
30	33759438	36058848	38667125	41652974		30
31	33795535	36099623	38713580	41706424		29

Min (L)	73	74	75	76	Min (R)
32	33831703	36140483	38760139	41760003	28
33	33867942	36181427	38806801	41813712	27
34	33904252	36222456	38853567	41867550	26
35	33940634	36263570	38900438	41921518	25
36	33977088	36304771	38947416	41975617	24
37	34013615	36346060	38994501	42029848	23
38	34050215	36387437	39041695	42084211	22
39	34086888	36428903	39088998	42138706	21
40	34123634	36470459	39136409	42193334	20
41	34160453	36512103	39183929	42248096	19
42	34197345	36553836	39231557	42302993	18
43	34234310	36595659	39279294	42358025	17
44	34271348	36637572	39327139	42413193	16
45	34308459	36679574	39375094	42468497	15
46	34345644	36721666	39423158	42523937	14
47	34382903	36763849	39471331	42579514	13
48	34420237	36806121	39519614	42635228	12
49	34457647	36848483	39568006	42691080	11
50	34495132	36890936	39616509	42747070	10
51	34532692	36933479	39665124	42803199	9
52	34570327	36976114	39713852	42859468	8
53	34608038	37018840	39762695	42915878	7
54	34645824	37061659	39811654	42972429	6
55	34683686	37104570	39860729	43029122	5
56	34721625	37147574	39909917	43085958	4
57	34759640	37190670	39959218	43142937	3
58	34797733	37233859	40008633	43200060	2
59	34835903	37277141	40058103	43257328	1
60	34874151	37320517	40107808	43314742	0

Bottom banners (complement reading): 16 · 15 · 14 · 13 (beneath the 73/74/75/76 columns respectively).

Tabula Tangentium (Table of Tangents) — degrees 77°–87°

Christoph Clavius [Christophorus Clavius], *Opera Mathematica*, Tomus I (Mainz, 1611)

Table header (running across the opening): TABVLA — *Gradus Quadrantis pro tangentibus arcuum eiusdem Quadrantis*. ("Degrees of the Quadrant, for the tangents of the arcs of the same Quadrant.") **Left-margin rubric:** *Minuta Graduum Quadrantis pro tangentibus complementorum arcuum eiusdem Quadrantis*. ("Minutes of the degrees of the Quadrant, for the tangents of the complements of the arcs of the same Quadrant.") **Bottom rubric:** *Gradus Quadrantis pro tangentibus complementorum arcuum eiusdem Quadrantis*. ("Degrees of the Quadrant, for the tangents of the complements of the arcs of the same Quadrant.")

Each degree column is read top-to-bottom with the **left** minute column (0'→60') for the tangent of the arc named in the **top** banner, and bottom-to-top with the **right** minute column (0' at foot) for the tangent of the complement named in the **bottom** banner. Radius R = 10,000,000 (so $\tan 77^\circ 0' = 43314742 \leftrightarrow 4.3314742$). Values are transcribed exactly as printed.

Page 760 (printed folio 118 [recte: 120]) — degrees 77°, 78°, 79°, 80°

Top banner: 77 | 78 | 79 | 80. Bottom banner (complements): 12 | 11 | 10 | 9.

Min (L)	77	78	79	80	Min (R)
0	43314742	47046295	51445543	56712854	60
1	43372301	47113680	51525561	56809480	59
2	43430006	47181249	51605820	56906425	58
3	43487857	47249003	51686321	57003690	57
4	43545855	47316942	51767065	57101277	56
5	43604000	47385067	51848053	57199188	55
6	43662293	47453380	51929285	57297425	54
7	43720733	47521882	52010762	57395990	53
8	43779321	47590575	52092485	57494885	52

Min (L)	77	78	79	80	Min (R)
9	43838057	47659460	52174455	57594111	51
10	43896942	47728538	52256673	57693670	50
11	43955977	47797809	52339140	57793564	49
12	44015163	47867274	52421857	57893795	48
13	44074501	47936934	52504826	57994366	47
14	44133992	48006790	52588048	58095279	46
15	44193637	48076841	52671525	58196536	45
16	44253435	48147088	52755259	58298138	44
17	44313387	48217531	52839251	58400087	43
18	44373494	48288171	52923503	58502385	42
19	44433756	48359008	53008016	58605034	41
20	44494174	48430043	53092792	58708035	40
21	44554749	48501278	53177831	58811388	39
22	44615481	48572714	53263134	58915095	38
23	44676371	48644352	53348702	59019157	37
24	44737419	48716193	53434536	59123576	36
25	44798626	48788238	53520637	59228353	35
26	44859993	48860488	53607006	59333490	34
27	44921521	48932945	53693644	59438989	33
28	44983211	49005610	53780552	59544852	32
29	45045065	49078483	53867731	59651081	31
30	45107083	49151565	53955183	59757678	30
31	45169263	49224856	54042909	59864646	29
32	45231607	49298357	54130911	59971987	28
33	45294114	49372069	54219190	60079703	27
34	45356785	49445993	54307748	60187796	26
35	45419621	49520130	54396586	60296268	25
36	45482623	49594481	54485705	60405121	24
37	45545790	49669047	54575107	60514358	23
38	45609123	49743829	54664793	60623981	22
39	45672623	49818827	54754764	60733992	21

Min (L)	77	78	79	80	Min (R)
40	45736291	49894042	54845022	60844392	20
41	45800128	49969475	54935569	60955184	19
42	45864135	50045127	55029406	61066370	18
43	45928314	50120999	55117535	61177952	17
44	45992666	50197092	55208958	61289930	16
45	46057192	50273407	55300676	61402307	15
46	46121892	50349935	55392692	61515085	14
47	46186767	50426707	55485007	61628267	13
48	46251817	50503695	55577622	61741856	12
49	46318043	50580910	55670539	61855854	11
50	46382445	50658353	55763759	61970263	10
51	46448023	50736025	55857283	62085085	9
52	46513778	50813927	559511[?] ²	62200323	8
53	46579711	50892060	56045247	62315979	7
54	46645823	50970425	56139689	62432056	6
55	46712115	51049023	56234439	62548556	5
56	46778587	51127855	56329498	62665481	4
57	46845240	51206922	56424868	62782833	3
58	46912075	51286225	56520550	62900615	2
59	46979093	51365765	56616545	63018829	1
60	47046295	51445543	56712854	63137478	0

Page 761 (printed folio 121) — degrees 81°, 82°, 83°, 84°

Running header (top-right): **121**. Header band: T A N G E N T I V M — ...

Quadrantis pro tangentibus arcuum eiusdem Quadrantis. Top banner: **81 | 82 | 83**
| **84**. Bottom banner (complements): **8 | 7 | 6 | 5**.

Min (L)	81	82	83	84	Min (R)
0	63137478	71153707	81443502	95143611	60
1	63256564	71304198	81639821	95410585	59
2	63376089	71455313	81837074	95679034	58
3	63496056	71607058	82035268	95948971	57

Min (L)	81	82	83	84	Min (R)
4	63616468	71759440	82234410	96220411	56
5	63737327	71912459	82434508	96493467	55
6	63858635	72066117	82635570	96767939	54
7	63980394	72220422	82837603	97044063	53
8	64102607	72375376	83040614	97321646	52
9	64225276	72530983	83244610	97600890	51
10	64348404	72687247	83449598	97881716	50
11	64471994	72844173	83655585	98164135	49
12	64596049	73001766	83862572	98448162	48
13	64720571	73160031	84070565	98733810	47
14	64845563	73318972	84279571	99021104	46
15	64971028	73478593	84489598	99310047	45
16	65096969	73638898	84700687	99600655	44
17	65223388	73799892	84912817	99893042	43
18	65350287	73961579	85125995	100187022	42
19	65477669	74123964	85340229	100482822	41
20	65605537	74287052	85555525	100780346	40
21	65733894	74450847	85771891	101079507	39
22	65862743	74615354	85989335	101380525	38
23	65992087	74780577	86207866	101683314	37
24	66121928	74946521	86427493	101987889	36
25	66252268	75113189	86648225	102294266	35
26	66383110	75280586	86870072	102602473	34
27	66514457	75448716	87093043	102912514	33
28	66646313	75617584	87317150	103224405	32
29	66778681	75787195	87542404	103538166	31
30	66911564	75957554	87768816	103853919	30
31	67044965	76128666	87996394	104171468	29
32	67178887	76300536	88225146	104491055	28
33	67313334	76473170	88455079	104812581	27
34	67448309	76646573	88686196	105136063	26

Min (L)	81	82	83	84	Min (R)
35	67583815	76820751	88918508	105461519	25
36	67719855	76995710	89152021	105788969	24
37	67856423	77171455	89386745	106118428	23
38	67993549	77347991	89622688	106449917	22
39	68131209	77525324	89859858	106783466	21
40	68269416	77703459	90098268	107119198	20
41	68408173	77882402	90337927	107456902	19
42	68547438	78062159	90578848	107796712	18
43	68687350	78242737	90821043	108138767	17
44	68827777	78424142	91064526	108482852	16
45	68968768	78606379	91309309	108829233	15
46	69110326	78789454	91555401	109177805	14
47	69252455	78973371	91802810	109528589	13
48	69395158	79158136	92051546	109881598	12
49	69538439	79343754	92301618	110236864	11
50	69682302	79530231	92553036	110594415	10
51	69826751	79717572	92805759	110954264	9
52	69971789	79905783	93059875	111316432	8
53	70117419	80094869	93315361	111680940	7
54	70263645	80284835	93572238	112047814	6
55	70410470	80475688	93830595	112417202	5
56	70557898	80667435	94090270	112788878	4
57	70705932	80860083	94351448	113163656	3
58	70854576	81053639	94614055	113539681	2
59	71003833	81248110	94878103	113918875	1
60	71153706	81443502	95143611	114300579	0

Page 762 (printed folio 122) — degrees 85°, 86°, 87°

Running header (top-left): **122**. Header band: T A B V L A — *Gradus Quadrantis pro tangentibus arcuum eiusdem Quadrantis*. Layout change: only **three** degree columns fit now (values are 9 digits), spaced with wide inter-column gaps. Top banner: **85** | **86** | **87**. Bottom banner (complements): **4** | **3** | **2**.

Min (L)	85	86	87	Min (R)
0	114300579	143006601	190811200	60
1	114684819	143606943	191879163	59
2	115071619	144212307	192959095	58
3	115461005	144822757	194051200	57
4	115853017	145438358	195155685	56
5	116247668	146059175	196273146	55
6	116644985	146685275	197403054	54
7	117044995	147316726	198545993	53
8	117447864	147953611	199702191	52
9	117853346	148595987	200871878	51
10	118261757	149244148	202055705	50
11	118672834	149897753	203253093	49
12	119086890	150557233	204464726	48
13	119503669	151222301	205691260	47
14	119923488	151893462	206932111	46
15	120346233	152570581	208188402	45
16	120771937	153253487	209459545	44
17	121200643	153942729	210746693	43
18	121632370	154638158	212049271	42
19	122067151	155339855	213368514	41
20	122505017	156047923	214704085	40
21	122946003	156762433	216056022	39
22	123390142	157483474	217425507	38
23	123837634	158211136	218812405	37
24	124288195	158945509	220217049	36
25	124742169	159686753	221639784	35
26	125199280	160434770	223080983	34
27	125659878	161189849	224540987	33
28	126123842	161952305	226020167	32
29	126591211	162721698	227518902	31
30	127062036	163498660	229037584	30

Min (L)	85	86	87	Min (R)
31	127536341	164282764	230576614	29
32	128014165	165074651	232136427	28
33	128495548	165873906	233717425	27
34	128980531	166681172	235320041	26
35	129469305	167496287	236945285	25
36	129961652	168319085	238592501	24
37	130457692	169150247	240262714	23
38	130957670	169989613	241957021	22
39	131461286	170837304	243674732	21
40	131968930	171693461	245417543	20
41	132480297	172558198	247184785	19
42	132995769	173431641	248978216	18
43	133515636	174313925	250797165	17
44	134038804	175205183	252643455	16
45	134566419	176105555	254517088	15
46	135098153	177015180	256417991	14
47	135634096	177934219	258348100	13
48	136174272	178862806	260307416	12
49	136718731	179801085	262296605	11
50	137267523	180749537	264316358	10
51	137820702	181707670	266366704	9
52	138378319	182676299	268449755	8
53	138940429	183654941	270565570	7
54	139507087	184644417	272714927	6
55	140078545	185644562	274898633	5
56	140654481	186655202	277117516	4
57	141235334	187677207	279372435	3
58	141820765	188710414	281664304	2
59	142411234	189755028	283994009	1
60	143006601	190811200	286362498	0

Table of Tangents (conclusion) — Table of Secants (opening)

Table of Tangents — conclusion (88° and 89°)

Top banner: *Gradus Quadrantis pro tangentibus arcuum eiusdem Quadrantis* — "Degrees of the quadrant, for the tangents of the arcs of the same quadrant."

[Margin (left, vertical): *Minuta graduum Quadrantis pro tangentibus arcuum eiusdem Quadrantis*. — "Minutes of the degrees of the quadrant, for the tangents of the arcs of the same quadrant."] [Margin (right, vertical): *Minuta Graduum Quadrantis pro tangentibus complementorum arcuum eiusdem Quadrantis*. — "Minutes of the degrees of the quadrant, for the tangents of the complements of the arcs of the same quadrant."]

The two degree-columns are headed **88** and **89** (read top-to-bottom with the left minute-column, 0'→60'); the right minute-column (60'→0') serves the complement degrees named in the foot banner. Each degree-column has an adjacent narrow ruled sub-column left blank in the source. This opening completes the tangent table: the last entry (89° 60' = 90°) is *Infinita* ("infinite").

Min	Tangens 88°	Tangens 89°	Min
0	286362498	572899830	60
1	288770746	582610421	59
2	291219764	592655713	58
3	293710598	603057015	57
4	296244357	613825994	56
5	298823024	624990311	55
6	301445987	636564040	54
7	304115322	648578536	53
8	306833212	661050728	52
9	309599077	674016435	51
10	312416191	687500739	50
11	315283945	701531474	49
12	318204757	716149676	48
13	321181137	731385593	47

Min	Tangens 88°	Tangens 89°	Min
14	324212583	747289264	46
15	327302782	763899813	45
16	330451272	781259259	44
17	333661982	799432199	43
18	336934467	818463792	42
19	340272744	838430438	41
20	343677949	859395374	40
21	347150587	881427652	39
22	350695255	904627361	38
23	354312962	929081086	37
24	358006024	954893332	36
25	361776788	982180553	35
26	365626388	1011062679	34
27	369560062	1041705454	33
28	373579199	1074263399	32
29	377686614	1108922084	31
30	381885288	1145801136	30
31	386178258	1185395877	29
32	390568737	1227736470	28
33	395060088	1273213435	27
34	399655828	1322188681	26
35	404359642	1375082163	25
36	409175388	1432363027	24
37	414111295	1494645462	23
38	419159137	1562590046	22
39	424335793	1637005697	21
40	429641796	1718863124	20
41	435082056	1809337410	19
42	440661780	1909864971	18
43	446386310	2022219818	17
44	452261453	2148619711	16

Min	Tangens 88°	Tangens 89°	Min
45	458293185	2291873854	15
46	464487853	2455533838	14
47	470852152	2644433955	13
48	477393195	2864819229	12
49	484118353	3125276745	11
50	491038024	3437829002	10
51	498155754	3819696333	9
52	505482730	4297181900	8
53	513030946	4911098124	7
54	520805157	5729633839	6
55	528821258	6875680006	5
56	537085003	8594012547	4
57	545610968	11458686834	3
58	554414914	17188033688	2
59	563504309	34376070815	1
60	572899830	Infinita.	0

Foot banner (with degree row 1 | 0): *Gradus Quadrantis pro tangentibus complementorum arcuum eiusdem Quadrantis.* — "Degrees of the quadrant, for the tangents of the complements of the arcs of the same quadrant."

Table of Secant Lines, or Beneficae

(Divisional title page — Source:)

TABVLA LINEARVM SECANTIVM, SIVE BENEFICA. — "Table of the Secant Lines, or *Beneficae*" (Clavius's alternate name for the secants, the "beneficial" lines).

The remainder of the leaf is blank (verso of the divisional title shows only faint bleed-through of the following table).

Table of Secants — opening (0°, 1°, 2°, 3°)

Top banner: *Gradus Quadrantis pro secantibus arcuum eiusdem Quadrantis.* — "Degrees of the quadrant, for the secants of the arcs of the same quadrant."

[Margin (left, vertical): *Minuta Graduum Quadrantis pro secantibus arcuum eiusdem Quadrantis*. — "Minutes of the degrees of the quadrant, for the secants of the arcs of the same quadrant."] [Margin (right, vertical): *Minuta Graduum Quadrantis pro secantibus complementorum arcuum eiusdem Quadrantis*. — "Minutes of the degrees of the quadrant, for the secants of the complements of the arcs of the same quadrant."]

The four degree-columns are headed **0, 1, 2, 3** (read top-to-bottom with the left minute-column, 0'→60'); the right minute-column (60'→0') serves the complement degrees (89, 88, 87, 86) named in the foot banner. The table opens at sec 0° 0' = 10000000 (the radius). Each degree-column has an adjacent narrow ruled sub-column left blank in the source.

M in	Secans 0°	Secans 1°	Secans 2°	Secans 3°	M in
0	10000000	10001524	10006095	10013723	6 0
1	10000001	10001574	10006198	10013874	5 9
2	10000002	10001626	10006301	10014029	5 8
3	10000004	10001679	10006405	10014184	5 7
4	10000007	10001733	10006509	10014339	5 6
5	10000010	10001788	10006615	10014495	5 5
6	10000014	10001844	10006721	10014653	5 4
7	10000020	10001900	10006828	10014811	5 3
8	10000027	10001957	10006936	10014970	5 2
9	10000034	10002015	10007045	10015130	5 1
1 0	10000042	10002074	10007155	10015291	5 0

M in	Secans 0°		Secans 1°	Secans 2°	Secans 3°	M in
1 1	10000051		10002134	10007265	10015453	4 9
1 2	10000060		10002195	10007376	10015615	4 8
1 3	10000071		10002256	10007488	10015778	4 7
1 4	10000083		10002318	10007601	10015942	4 6
1 5	10000095		10002381	10007716	10016107	4 5
1 6	10000108	19002445 [recte: 10002445]		10007831	10016273	4 4
1 7	10000122		10002510	10007946	10016440	4 3
1 8	10000137		10002576	10008062	10016608	4 2
1 9	10000152		10002642	10008179	10016777	4 1
2 0	10000168		10002709	10008298	10016946	4 0
2 1	10000186		10002777	10008417	10017116	3 9
2 2	10000204		10002846	10008537	10017287	3 8
2 3	10000223		10002916	10008658	10017459	3 7
2 4	10000243		10002987	10008779	10017632	3 6
2 5	10000264		10003058	10008902	10017806	3 5
2 6	10000285		10003130	10009025	10017981	3 4
2	10000308		10003203	10009149	10018157	3

M in	Secans 0°	Secans 1°	Secans 2°	Secans 3°	M in
7					3
2	10000332	10003277	10009274	10018333	3
8					2
2	10000357	10003352	10009400	10018510	3
9					1
3	10000381	10003428	10009527	10018687	3
0					0
3	10000407	10003505	10009655	10018865	2
1					9
3	10000433	10003582	10009783	10019044	2
2					8
3	10000461	10003660	10009912	10019224	2
3					7
3	10000489	10003739	10010043	10019405	2
4					6
3	10000518	10003819	10010174	10019587	2
5					5
3	10000548	10003900	10010306	10019770	2
6					4
3	10000579	10003982	10010439	10019954	2
7					3
3	10000611	10004060	10010572	10020138	2
8					2
3	10000643	10004148	10010706	10020324	2
9					1
4	10000677	10004232	10010841	10020510	2
0					0
4	10000711	10004317	10010977	10020698	1
1					9
4	10000746	10004403	10011114	10020886	1
2					8
4	10000782	10004490	10011252	10021086	1
3					7

M in	Secans 0°	Secans 1°	Secans 2°	Secans 3°	M in
4	10000819	10004578	10011390	10021266	1
4					6
4	10000857	10004666	10011529	10021456	1
5					5
4	10000895	10004755	10011670	10021649	1
6					4
4	10000934	10004845	10011811	10021842	1
7					3
4	10000975	10004936	10011952	10022035	1
8					2
4	10001016	10005028	10012098	10022239	1
9					1
5	10001058	10005122	10012238	10022424	1
0					0
5	10001100	10005216	10012383	10022620	9
1					
5	10001144	10005310	10012528	10022817	8
2					
5	10001188	10005405	10012674	10023015	7
3					
5	10001233	10005501	10012822	10023213	6
4					
5	10001280	10005598	10012970	10023412	5
5					
5	10001327	10005696	10013119	10023612	4
6					
5	10001375	10005795	10013269	10023813	3
7					
5	10001423	10005894	10013419	10024014	2
8					
5	10001473	10005994	10013570	10024217	1
9					
6	10001524	10006095	10013723	10024420	0

M in	Secans 0°	Secans 1°	Secans 2°	Secans 3°	M in
0					

Foot banner (with degree row 89 | 88 | 87 | 86): *Gradus Quadrantis pro secantib. complementorū arcuū eiusdē Quadrantis.* — "Degrees of the quadrant, for the secants of the complements of the arcs of the same quadrant."

Table of Secants (*Tabula Linearum Secantium*) — degrees 4°–15°

Christoph Clavius, Opera Mathematica, Tomus I — continuation of the secant table.

Each page carries four degree-columns. The top banner gives the degree; the left column counts minutes 0'→60' downward (heading, verbatim: *Minuta Graduum Quadrantis pro secantibus arcuum eiusdem Quadrantis* — "Minutes of the degrees of the quadrant, for the secants of arcs of the same quadrant"). The right column counts minutes 60'→0' downward and serves the complement degree printed in the bottom banner (side heading, verbatim: *Minuta Graduum Quadrantis pro secantibus complementorum arcuum eiusdem Quadrantis* — "...for the secants of the complements of arcs of the same quadrant"). Radius = 10000000 (secants ≥ 10000000). Column headers kept verbatim as printed; complement-minute column labelled *c.Min*.

Page — printed folio 126

Header: *Gradus Quadrantis pro secantibus arcuum eiusdem Quadrantis. TABVLA* — "Degrees of the quadrant for the secants of arcs of the same quadrant." Top banner: **4 5 6 7** — Bottom banner: **85 84 83 82** (= complements 89-deg).

Mi n	4	5	6	7	c.Min
0	10024420	10038198	10055082	10075098	60
1	10024625	10038454	10055390	10075459	59
2	10024830	10038710	10055699	10075820	58
3	10025036	10038968	10056009	10076182	57
4	10025242	10039226	10056320	10076545	56
5	10025450	10039486	10056632	10076909	55

Mi					
n	4	5	6	7	c.Min
6	10025658	10039746	10056944	10077274	54
7	10025868	10040008	10057256	10077639	53
8	10026078	10040269	10057570	10078005	52
9	10026289	10040532	10057884	10078372	51
10	10026500	10040796	10058200	10078740	50
11	10026713	10041061	10058517	10079009 [recte:?)	49
12	10026927	10041326	10058834	10079479	48
13	10027141	10041592	10059153	10079850	47
14	10027357	10041859	10059472	10080222	46
15	10027573	10042128	10059792	10080595	45
16	10027790	10042397	10060113	10080968	44
17	10028009	10042667	10060435	10081332	43
18	10028227	10042936	10060757	10081717	42
19	10028447	10043207	10061080	10082093	41
20	10028667	10043479	10061405	10082470	40
21	10028889	10043752	10061730	10082848	39
22	10029111	10044025	10062056	10083226	38
23	10029334	10044300	10062383	10063606 [recte:?)	37
24	10029559	10044576	10062711	10083987	36
25	10029784	10044853	10063039	10084368	35
26	10030009	10045130	10063369	10084750	34
27	10030236	10045409	10063700	10085134	33
28	10030463	10045689	10064031	10085518	32
29	10030692	10045969	10064364	10085903	31
30	10030920	10046250	10064696	10086289	30
31	10031150	10046532	10065035	10086677	29
32	10031381	10046815	10065365	10087065	28
33	10031614	10047098	10065701	10087454	27
34	10031846	10047383	10066038	10087843	26
35	10032079	10047669	10066376	10088243	25

Mi					
n	4	5	6	7	c.Min
36	10032314	10047954	10066715	10088623	24
37	10032550	10048241	10067054	10089015	23
38	10032786	10048529	10067594 [?]	10089408	22
39	10033023	10048818	10067735	10089802	21
40	10033261	10049107	10068076	10090196	20
41	10033500	10049398	10068419	10090592	19
42	10033740	10049690	10068763	10090988	18
43	10033981	10049983	10069107	10091385	17
44	10034223	10050276	10069452	10091783	16
45	10034465	10050571	10069808	10092182	15
46	10034708	10050865	10070155	10092582	14
47	10034952	10051160	10070493	10092983	13
48	10035196	10051456	10070842	10093385	12
49	10035441	10051753	10071192	10093787	11
50	10035688	10052051	10071543	10094190	10
51	10035936	10052350	10071895	10094624	9
52	10036184	10052649	10072247	10095030	8
53	10036434	10052951	10072600	10095406	7
54	10036684	10053252	10072954	10095813	6
55	10036934	10053555	10073310	10096221	5
56	10037185	10053858	10073666	10096630	4
57	10037438	10054162	10074023	10097040	3
58	10037690	10054468	10074380	10097451	2
59	10037944	10054775	10074737	10097863	1
60	10038198	10055082	10075098	10098275	0

Page — printed folio 127

Header: *SECANTIVM. Gradus Quadrantis pro secantibus arcuum eiusdem Quadrantis*. Top banner: **8 9 10 11** — Bottom banner: **81 80 79 78**.

Mi n	8	9	10	11	c.Min
0	10098275	10124650	10154264	10187166	60
1	10098698	10125117	10154786	10187743	59
2	10099103	10125585	10155308	10188320	58
3	10099518	10126054	10155831	10188899	57
4	10099934	10126524	10156356	10189478	56
5	10100351	10126994	10156881	10190058	55
6	10100769	10127465	10157407	10190639	54
7	10101188	10127947	10157934	10191221	53
8	10101607	10128410	10158462	10191804	52
9	10102028	10128684 [?]	10158991	10192387	51
10	10102450	10129358	10159520	10192972	50
11	10102872	10129634 [?]	10160051	10193557	49
12	10103295	10130311	10160582	10194144	48
13	10103720	10130788	10161114	10194732	47
14	10104144	10131266	10161648	10195320	46
15	10104570	10131746	10162182	10195910	45
16	10104996	10132226	10162707	10196500	44
17	10105423	10132707	10163252	10197092	43
18	10105851	10133189	10163789	10197684	42
19	10106286	10133672	10164327	10198277	41
20	10106710	10134156	10165865 [?]	10198872	40
21	10107140	10134641	10165495 [?]	10199467	39
22	10107572	10135127	10165944	10200063	38
23	10108005	10135614	10166485	10200660	37
24	10108438	10136102	10167028	10201258	36
25	10108873	10136591	10167571	10201857	35
26	10109309	10137080	10168116	10202457	34
27	10109755	10137571	10168661	10203058	33
28	10110182	10138163	10169207	10203659	32
29	10110620	10138555	10169765	10204262	31

Mi n	8	9	10	11	c.Min
30	10111059	10139048	10170303	10204867	30
31	10111509	10139543	10170852	10205470	29
32	10111940	10140038	10171401	10206075	28
33	10112482 [?]	10140534	10171952	10206681	27
34	10112825	10141036	10172504	10207289	26
35	10113279	10141528	10173056	10207897	25
36	10113713	10142027	10173609	10208506	24
37	10114159	10142526	10174163	10209116	23
38	10114606	10143026	10174718	10209727	22
39	10115053	10143528	10175274	10210339	21
40	10115501	10144030	10175831	10210952	20
41	10115951	10144533	10176389	10211566	19
42	10116401	10145037	10176947	10212180	18
43	10116852	10145542	10177507	10212796	17
44	10117303	10146048	10178068	10213412	16
45	10117754	10146554	10178630	10214030	15
46	10118209	10147062	10179193	10214668	14
47	10118663	10147572	10179756	10215268	13
48	10119118	10148082	10180321	10215889	12
49	10119574	10148593	10180886	10216510	11
50	10120031	10149104	10181453	10217113	10
51	10120489	10149615	10182021	10217756	9
52	10120948	10150128	10182589	10218380	8
53	10121408	10150642	10183158	10219015	7
54	10121868	10151156	10183728	10219631	6
55	10122330	10151672	10184299	10220258	5
56	10122792	10152188	10184870	10220885	4
57	10123256	10152705	10185443	10221514	3
58	10123720	10153224	10186017	10222143	2
59	10124275 [?]	10153744	10186591	10222774	1

Mi n	8	9	10	11	c.Min
60	10124650	10154264	10187166	10223405	0

Page — printed folio 128

Header: *TABVLA. Gradus Quadrantis pro secantibus arcuum eiusdem Quadrantis.*

Top banner: **12 13 14 15** — Bottom banner: **77 76 75 74**.

M i n	12	13	14	15	c.Mi n
0	10223405	10263040	10306136	1035276 2	60
1	10224037	10263730	10306884	1035356 9	59
2	10224671	10264420	10307633	1035437 7	58
3	10225305	10265112	10308383	1035518 6	57
4	10225941	10265804	10309134	1035599 6	56
5	10226577	10266498	10309886	1035680 7	55
6	10227215	10267192	10310639	1035761 9	54
7	10227854	10267888	10311393	1035843 3	53
8	10228493	10268584	10312148	1035924 7	52
9	10229134	10269281	10312903	1036006 3	51
1 0	10229775	10269979	10313660	1036088 0	50
1 1	10230417	10270688	10314417	1036169 8	49
1	10231060	10271379	10313176 [recte:?]	1036251	48

M i n	12	13	14	15	c.Mi n
2				7	
1	10231644	10272080	10315935	1036333	47
3				7	
1	10232288	10272782	10316696	1036415	46
4				8	
1	10232994	10273485	10317457	1036498	45
5				0	
1	10233641	10274190	10318220	1036580	44
6				2	
1	10234289	10274895	10318984	1036662	43
7				6	
1	10234938	10275601	10319749	1036745	42
8				0	
1	10235587	10276318	10320525	1036827	41
9				6	
2	10236238	10277016	10321282	1036910	40
0				2	
2	10236889	10277726	10322050	1036993	39
1				0	
2	10237541	10278436	10322819	1037075	38
2				8	
2	10238195	10279148	10323589	1037158	37
3				8	
2	10238849	10279860	10324359	1037241	36
4				8	
2	10239505	10280573	10325131	1037325	35
5				0	
2	10240161	10281287	10325903	1037409	34
6				2	
2	10240818	10282002	10326677	1037491	33
7				6	
2	10241476	10282717	10327451	1037575	32

M i n	12	13	14	15	c.Mi n
8				0	
2	10242135	10283434	10328227	1037658	31
9				6	
3	10242795	10284151	10329003	1037742	30
0				2	
3	10243456	10284870	10329781	1037826	29
1				0	
3	10244118	10285589	10330559	1037909	28
2				8	
3	10245782 [recte:?]	10286310	10331339	1037993	27
3				8	
3	10245445	10287032	10332119	1038077	26
4				8	
3	10246110	10287754	10332902	1038162	25
5				0	
3	10246776	10288478	10333684	1038246	24
6				3	
3	10247442	10289202	10334467	1038330	23
7				7	
3	10248110	10289928	10335252	1038415	22
8				3	
3	10248778	10290654	10336037	1038499	21
9				9	
4	10249448	10291381	10336824	1038584	20
0				6	
4	10250119	10291119 [?]	10337612	1038669	19
1				4	
4	10250790	10292838	10338400	1038754	18
2				3	
4	10251461	10293569	10339189	1038839	17
3				3	
4	10252136	10294300	10339980	1038924	16

M i n	12	13	14	15	c.Mi n
4				4	
4	10252811	10295043	10340771	1039009	15
5				6	
4	10253482	10295766	10341564	1039094	14
6				9	
4	10254162	10296501	10342347	1039180	13
7				3	
4	10254839	10297237	10343152	1039265	12
8				7	
4	10255517	10297973	10343947	1039351	11
9				3	
5	10256196	10298710	10344743	1039437	10
0				0	
5	10256876	10299449	10345541	1039522	9
1				8	
5	10257557	10300188	10346340	1039608	8
2				7	
5	10258239	10300928	10347139	1039694	7
3				7	
5	10258922	10301669	10347940	1039780	6
4				8	
5	10259606	10302411	10348741	1039867	5
5				0	
5	10260291	10303154	10349544	1039953	4
6				3	
5	10260977	10303898	10350347	1040039	3
7				7	
5	10261661	10304643	10351151	1040126	2
8				2	
5	10262351	10305390	10351956	1040212	1
9				8	
6	10263040	10306136	10352762	1040299	0

M i n	12	13	14	15	c.Mi n
0				4	

Tabula Linearum Secantium — Secant table, 16°–27°

Christoph Clavius, Opera Mathematica, Tomus I (Mainz, 1611). Numeric secant table (Tabula sinuum secantium).

The running head across this three-page opening reads **TABVLA SINVVM SECANTIVM** ("Table of Secant Lines"), one word per page. On each page the top title is:

Gradus Quadrantis pro secantibus arcuum eiusdem Quadrantis.
[Degrees of the Quadrant, for the secants of the arcs of the same Quadrant.]

and the foot caption (governing the complement reading, i.e. the bottom degree banners and the right-hand minute column) is:

Gradus Quadrantis pro secantibus complementorum arcuum eiusdem Quadrantis. [Degrees of the Quadrant, for the secants of the complements of the arcs of the same Quadrant.]

The left minute column (*Minuta Graduum Quadrantis*, 0'→60' top to bottom) reads with the **top** degree banner; the right minute column (60'→0', = 60 – left) reads with the **bottom** degree banner. Radius = 10,000,000 (secants carry seven fractional digits; e.g. sec 16°0' = 10402994). Values are the secant of (top-banner degree + left-minute). Each degree column's 60' entry equals the next degree column's 0' entry (verified — see numeric-integrity note).

Page 769 (printed folio 129) — top banners 16 17 18 19 ·
bottom banners 73 72 71 70

' (top)	Sec 16°	Sec 17°	Sec 18°	Sec 19°	' (compl.)
0	104029 94	10456917	105146 21	105762 07	60
1	104038	10457847	105156	105772	59

' (top)	Sec 16°	Sec 17°	Sec 18°	Sec 19°	' (compl.)
	62		16	67	
2	104047	10458779	105166	105783	58
	30		12	28	
3	104055	10459711	105176	105794	57
	90		09	00	
4	104064	10460645	105186	105804	56
	71		07	63	
5	104073	10461580	105196	105815	55
	43		06	18	
6	104082	10462516	105206	105825	54
	16		06	83	
7	104090	10463453	105216	105836	53
	91		07	50	
8	104099	10464391	105226	105847	52
	66		08	17	
9	104108	10465330	105236	105857	51
	43		11	95	
10	104117	10466270	105246	105868	50
	21		15	55	
11	104126	10467211	105256	105879	49
	00		20	25	
12	104134	10468153	105266	105889	48
	79		26	97	
13	104143	10469096	105276	105900	47
	60		33	70	
14	104152	10470041	105286	105911	46
	41		42	45	
15	104161	10470986	105296	105922	45
	24		51	20	
16	104170	10471933	105306	105932	44
	07		62	97	
17	104178	10472880	105316	105943	43
	92		73	75	
18	104187	10473829	105326	105954	42

' (top)	Sec 16°	Sec 17°	Sec 18°	Sec 19°	' (compl.)
	78		86	55	
19	104196	10474778	105336	105965	41
	65		99	34	
20	104205	10475729	105347	105976	40
	53		14	15	
21	104214	10476680	105357	105986	39
	42		30	97	
22	104223	10477633	105367	105997	38
	33		47	80	
23	104232	10478587	105377	106008	37
	24		65	65	
24	104241	10479542	105387	106019	36
	16		85	50	
25	104250	10480498	105398	106030	35
	09		05	37	
26	104259	10481454	105408	106041	34
	03		26	25	
27	104267	10482412	105418	106052	33
	98		48	14	
28	104276	10483371	105428	106063	32
	94		72	04	
29	104285	10484331	105438	106073	31
	91		97	95	
30	104294	10485292	105449	106084	30
	89		23	87	
31	104303	10486254	105459	106095	29
	88		50	80	
32	104312	10487217	105469	106106	28
	88		77	75	
33	104321	10488181	105480	106117	27
	89		06	70	
34	104330	10489146	105490	106128	26
	91		36	67	
35	104339	10490113	105500	106139	25

' (top)	Sec 16°	Sec 17°	Sec 18°	Sec 19°	' (compl.)
	95		67	64	
36	104348	10491080	105510	106150	24
	99		99	63	
37	104358	10492049	105521	106161	23
	05		33	63	
38	104367	10493018	105531	106172	22
	11		68	64	
39	104376	10493989	105542	106183	21
	19		04	66	
40	104385	10494961	105552	106194	20
	28		41	69	
41	104394	10494934 [recte:?]	105562	106205	19
	36		79	74	
42	104403	10496908	105573	106216	18
	46		18	80	
43	104412	10497883	105583	106227	17
	57		59	87	
44	104421	10498059 [recte:?]	105594	106238	16
	70		00	95	
45	104430	10499836	105604	106250	15
	83		43	04	
46	104439	10500814	105614	106261	14
	98		96	14	
47	104449	10501793	105625	106272	13
	13		31	26	
48	104458	10502773	105635	106283	12
	30		77	38	
49	104467	10503754	105646	106294	11
	49		23	51	
50	104476	10504736	105656	106305	10
	68		70	66	
51	104485	10505719	105667	106316	9
	88		19	82	
52	104495	10506704	105677	106327	8

' (top)	Sec 16°	Sec 17°	Sec 18°	Sec 19°	' (compl.)
	09		69	99	
53	104504	10507689	105688	106339	7
	31		20	17	
54	104513	10508676	105698	106350	6
	54		72	37	
55	104522	10509664	105709	106361	5
	79		25	57	
56	104532	10510653	105719	106372	4
	04		80	79	
57	104541	10511643	105730	106384	3
	31		34	02	
58	104550	10512635	105740	106395	2
	58		91	26	
59	104559	10513627	105751	106406	1
	87		49	51	
60	104569	10514621	105762	106417	0
	17		07	77	
	73	72	71	70 (bottom banner)	

Page 770 (printed folio 130) — top banners 20 21 22 23 ·
bottom banners 69 68 67 66

' (top)	Sec 20°	Sec 21°	Sec 22°	Sec 23°	' (compl.)
0	10641	10711449	10785	10863603	60
	777		347		
1	10642	10712646	10786	10864945	59
	905		616		
2	10644	10713888	10787	10866289	58
	034	[recte:?]	885		
3	10645	10715042	10789	10867633	57
	164		155		
4	10646	10716242	10790	10868979	56
	295		427		
5	10647	10717444	10791	10870326	55

' (top)	Sec 20°	Sec 21°	Sec 22°	Sec 23°	' (compl.)
	427		700		
6	10648 560	10718647	10792 974	10871675	54
7	10649 694	10719850	10794 250	10873024	53
8	10650 829	10721056	10795 527	10874374	52
9	10651 965	10722261	10796 805	10875626 [recte:?]	51
10	10653 103	10723469	10798 085	10877079	50
11	10654 242	10724677	10799 365	10878434	49
12	10655 381	10725887	10800 647	10879790	48
13	10656 522	10727098	10801 930	10881147	47
14	10657 664	10728310	10803 214	10882506	46
15	10658 807	10729524	10804 500	10883865	45
16	10659 951	10730738	10805 787	10885226	44
17	10661 097	10731953	10807 074	10886588	43
18	10662 244	10733170	10808 363	10887952	42
19	10663 392	10734387	10809 652	10889317	41
20	10664 541	20735606 [recte:?]	10810 942	10890683	40
21	10665 692	10736826	10812 234	10892051	39

' (top)	Sec 20°	Sec 21°	Sec 22°	Sec 23°	' (compl.)
22	10666 844	10738048	10813 528	10893417	38
23	10667 996	10739270	10814 823	10894788	37
24	10669 150	10740494	10816 119	10896159	36
25	10670 304	10741719	10817 417	10897531	35
26	10671 460	10742945	10818 715	10898905	34
27	10672 617	10744173	10820 015	10900280	33
28	10673 776	10745401	10821 316	10901656	32
29	10674 936	10746631	10822 617	10903033	31
30	10676 096	10747864	10823 920	10904413	30
31	10677 258	10749094	10825 225	10905790	29
32	10678 420	10750327	10826 531	10907171	28
33	10679 584	10751561	10827 838	10908553	27
34	10680 749	10752797	10829 146	10909936	26
35	10681 915	10754034	10830 455	10911322	25
36	10683 082	10755273	10831 766	10912709	24
37	10684 250	10756513	10833 078	10914096	23
38	10685	10757753	10834	10915484	22

' (top)	Sec 20°	Sec 21°	Sec 22°	Sec 23°	' (compl.)
	420		391		
39	10686 591	10758995	10835 706	10916874	21
40	10687 763	10760237	10837 023	10918265	20
41	10688 936	10761481	10838 341	10919657	19
42	10690 111	10762726	10839 660	10921051	18
43	10691 287	10763972	10840 980	10922436	17
44	10692 464	10765220	10842 301	10923833	16
45	10693 642	10766469	10843 623	10925241	15
46	10694 821	10767720	10844 947	10926641	14
47	10696 001	10768971	10846 272	10928041	13
48	10697 182	10770224	10847 597	10929442	12
49	10698 364	10771477	10848 924	10930846	11
50	10699 548	10772732	10850 252	10932249	10
51	10700 732	10773988	10851 583	10933654	9
52	10701 918	10775244	10852 914	10935061	8
53	10703 105	10776502	10854 246	10936469	7
54	10704 294	10777761	10855 578	10937879	6

' (top)	Sec 20°	Sec 21°	Sec 22°	Sec 23°	' (compl.)
55	10705 483	10779022	10856 912	10939290	5
56	10706 674	10780284	10858 247	10940702	4
57	10707 866	10781547	10859 584	10942115	3
58	10709 059	10782802	10860 922	10943527	2
59	10710 254	10784078	10862 262	10944945	1
60	10711 449	10785347	10863 603	10946362	0
	69	68	67	66	(bottom banner)

Page 771 (printed folio 131) — top banners 24 25 26 27 ·
bottom banners 65 64 63 62

' (top)	Sec 24°	Sec 25°	Sec 26°	Sec 27°	' (compl.)
0	10946362	11033 783	11126 021	11223262	60
1	10947781	11035 280	11127 601	11224927	59
2	10949291 [recte:?]	11036 779	11129 182	11226593	58
3	10950622	11038 279	11130 765	11228260	57
4	10952045	11039 780	11132 349	11229929	56
5	10953469	11041 283	11133 933	11231599	55
6	10954898	11042 787	11135 519	11233270	54

' (top)	Sec 24°	Sec 25°	Sec 26°	Sec 27°	' (compl.)
7	10956320	11044 293	11137 106	11234943	53
8	10957747	11045 799	11138 694	11236617	52
9	10959175	11047 306	11140 284	11238292	51
10	10960605	11048 815	11141 875	11239969	50
11	10962036	11050 325	11143 467	11241648	49
12	10963469	11051 837	11145 061	11243329	48
13	10964903	11053 350	11146 656	11245011	47
14	10966338	11054 865	11148 254	11246694	46
15	10967775	11056 381	11149 853	11248378	45
16	10969213	11057 898	11151 453	11250064	44
17	10970652	11059 420	11153 055	11251751	43
18	10972092	11060 939	11154 658	11253440	42
19	10973533	11062 461	11156 262	11255130	41
20	10974976	11063 985	11157 868	11256822	40
21	10976420	11065 510	11159 475	11258516	39
22	10977865	11067 037	11161 084	11260211	38
23	10979312	11068	11162	11261907	37

' (top)	Sec 24°	Sec 25°	Sec 26°	Sec 27°	' (compl.)
		564	694		
24	10980760	11070 092	11164 306	11263605	36
25	10982210	11071 621	11165 919	11265304	35
26	10983661	11073 152	11167 533	11267005	34
27	10985113	11074 684	11169 149	11268707	33
28	10986567	11076 218	11170 766	11270410	32
29	10988022	11077 753	11172 385	11272114	31
30	10989480	11079 289	11174 006	11273820	30
31	10990938	11080 827	11175 627	11275528	29
32	10992398	11082 366	11177 249	11277238	28
33	10993859	11083 906	11178 873	11278949	27
34	10995321	11085 448	11180 499	11280661	26
35	10996783	11086 990	11182 125	11282374	25
36	10998247	11088 536	11183 753	11284089	24
37	10999712	11090 082	11185 383	11285805	23
38	11001179	11091 629	11187 014	11287524	22
39	11002647	11093 178	11188 647	11289244	21

' (top)	Sec 24°	Sec 25°	Sec 26°	Sec 27°	' (compl.)
40	11004116	11094 729	11190 281	11290965	20
41	11005587	11096 280	11191 916	11292688	19
42	11007059	11097 833	11193 553	11294412	18
43	11008533	11099 387	11195 191	11296132	17
44	11010008	11100 943	11196 831	11297864	16
45	11011484	11102 500	11198 472	11299593	15
46	11012957	11104 058	11200 114	11301324	14
47	11014441	11105 618	11201 758	11303056	13
48	11015921	11107 179	11203 404	11304789	12
49	11017402	11108 741	11205 051	11306523	11
50	11018884	11110 306	11206 700	11308259	10
51	11020367	11111 871	11208 350	11309996	9
52	11021852	11113 438	11210 001	11311735	8
53	11023338	11115 006	11211 654	11313476	7
54	11024826	11116 575	11213 308	11315218	6
55	11026315	11118 145	11214 963	11316961	5
56	11027806	11119	11216	11318706	4

' (top)	Sec 24°	Sec 25°	Sec 26°	Sec 27°	' (compl.)
		717	620		
57	11029298	11121	11218	11319452	3
		290	278	[recte:?]	
58	11030791	11122	11219	11322199	2
		865	938		
59	11032287	11124	11221	11323949	1
		442	599		
60	11033783	11126	11223	11325700	0
		021	262		
	65	64	63	62	(bottom banner)

Tabula Secantium — Secant Tables (degrees 28–39)

Christoph Clavius [Christoph Clavius], Opera Mathematica, Tomus I (Mainz, 1611).

These three pages continue the **Table of Secants** (*Tabula Linearum Secantium*). Each page carries four degree columns. For any degree column, the left-hand minute index (Min, 0'–60') gives the arc; the secant value is tabulated to a radius of 10,000,000 (so that $\sec 45^\circ = 14142136$). The right-hand index (Comp1., 60'–0') serves the reading of the complementary arcs printed in the foot banner.

Top banner (each page): *Gradus Quadrantis pro secantibus arcuum eiusdem Quadrantis* — "Degrees of the quadrant, for the secants of arcs of the same quadrant." Left margin: *Minuta Graduum Quadrantis pro secantibus arcuum eiusdem Quadrantis* — "Minutes of the degrees of the quadrant...". Foot banner: *Gradus Quadrantis pro secantibus complementorum arcuum eiusdem Quadrantis* — "...for the secants of the complements of arcs of the same quadrant." Right margin: *...complementorum arcuum eiusdem Quadrantis*.

Page 132 — Secants of 28°, 29°, 30°, 31°

M					Comp
in	28	29	30	31	l.
0	1132570	11433540	11547004	1166633	60

M in	28	29	30	31	Comp l.
	0			1	
1	1132745	11435384	11548944	1166837	59
	2			1	
2	1132920	11437230	11550886	1167041	58
	6			3	
3	1133096	11439078	11552829	1167245	57
	1			7	
4	1133271	11440927	11554774	1167450	56
	8			2	
5	1133447	11442777	11556720	1167654	55
	9			8	
6	1133623	11444629	11558669	1167859	54
	7			7	
7	1133799	11446483	11560619	1168064	53
	9			7	
8	1133976	11448339	11562570	1168269	52
	2			8	
9	1134152	11450196	11564523	1168475	51
	6			2	
1	1134329	11452054	11566480	1168680	50
0	2			7	
1	1134506	11453915	11568434	1168886	49
1	0			4	
1	1134683	11455776	11570393	1169092	48
2	0			3	
1	1134860	11457639	11572353	1169298	47
3	1			4	
1	1135037	11459503	11574314	1169504	46
4	3			6	
1	1135214	11461370	11576277	1169711	45
5	9			0	
1	1135392	11463238	11578242	1169917	44
6	3			6	

M in	28	29	30	31	Comp l.
1 7	1135569 8	11465107	11580208	1170124 3	43
1 8	1135747 5	11466978	11582175	1170331 2	42
1 9	1135925 5	11468850	11584145	1170538 3	41
2 0	1136103 6	11470723	11586116	1170745 5	40
2 1	1136281 9	11472599	11588089	1170953 0	39
2 2	1136460 3	11474483	11590064	1171160 6	38
2 3	1136638 9	11476354	11592040	1171368 4	37
2 4	1136817 7	11478235	11594018	1171576 4	36
2 5	1136996 6	11480117	11595998	1171784 5	35
2 6	1137175 6	11482001	11597979	1171992 8	34
2 7	1137354 8	11483887	11599961	1172201 2	33
2 8	1137534 1	11485774	11601946	1172409 9	32
2 9	1137713 6	11487662	11603932	1172618 7	31
3 0	1137893 3	11489353 [recte:?]	11605919	1172827 6	30
3 1	1138073 1	11491445	11607909	1173036 7	29
3 2	1138253 0	11493338	11609900	1173246 0	28
3	1138433	11495233	11611893	1173455	27

M in	28	29	30	31	Comp l.
3	1			5	
3	1138613	11497140	11613888	1173665	26
4	4			2	
3	1138793	11499028	11615876	1173875	25
5	8			1	
3	1138974	11500928	11617882	1174085	24
6	4			1	
3	1139155	11502829	11619881	1174295	23
7	1			3	
3	1139335	11504731	11621882	1174505	22
8	9			7	
3	1139516	11506626	11623885	1174716	21
9	9			2	
4	1139698	11508532	11625889	1174926	20
0	1			9	
4	1139879	11510450	11627996 [recte:?]	1175137	19
1	4			8	
4	1140060	11512360	11629904	1175348	18
2	9			9	
4	1140242	11514271	11631913	1175560	17
3	5			3	
4	1140424	11516183	11633924	1175771	16
4	3			8	
4	1140606	11518097	11635937	1175983	15
5	3			4	
4	1140788	11520013	11637952	1176195	14
6	4			1	
4	1140970	11521930	11639968	1176406	13
7	6			9	
4	1141153	11523849	11641986	1176619	12
8	0			0	
4	1141335	11525770	11644005	1176831	11
9	6			2	

M in	28	29	30	31	Comp l.
5 0	1141518 3	11527692	11646026	1177043 7	10
5 1	1141701 2	11529616	11648049	1177256 4	9
5 2	1141884 2	11531542	11650075	1177469 6	8
5 3	1142067 3	11533469	11652099	1177682 2	7
5 4	1142250 7	11535398	11654127	1177895 4	6
5 5	1142434 2	11537328	11656156	1178108 8	5
5 6	1142617 8	11539260	11658188	1178322 3	4
5 7	1142801 6	11541193	11660221	1178536 1	3
5 8	1142985 6	11543128	11662256	1178750 0	2
5 9	1143168 9	11545065	11664292	1178964 0	1
6 0	1143354 0	11547004	11666331	1179178 3	0

Foot banner degree indices (complement reading): 61 | 60 | 59 | 58.

Page 133 — Secants of 32°, 33°, 34°, 35°

M i n	32	33	34	35	Com pl.
0	11791783	11923633	12062179	122077 45	60
1	11793927	11925886	12064546	122102 33	59
2	11796073	11928141	12066916	122127	58

M i n	32	33	34	35	Com pl.
				23	
3	11798221	11930397	12069286	122152	57
				14	
4	11800371	11932656	12071660	122177	56
				08	
5	11802522	11934917	12074036	122202	55
				04	
6	11804675	11937180	12076413	122227	54
				02	
7	11806830	11939445	12078792	122252	53
				01	
8	11808987	11941701	12081174	122277	52
				03	
9	11811145	11943979	12083558	122302	51
				07	
1 0	11813306	11946250	12085943	122327	50
				13	
1 1	11815468	11948522	12088330	122352	49
				21	
1 2	11817632	11950796	12090720	122377	48
				32	
1 3	11819797	11953071	12093111	122402	47
				45	
1 4	11821965	11955349	12095504	122427	46
				59	
1 5	11824134	11957629	12097899	122452	45
				75	
1 6	11826306	11959910	12100296	122477	44
				94	
1 7	11828479	11962194	12102696	122503	43
				15	
1	11830654	11964479	12105097	122528	42

M i n	32	33	34	35	Com pl.
8				37	
1 9	11832830	11966766	12107500	122553 61	41
2 0	11835008	11969055	12109905	122578 88	40
2 1	11837188	11971346	12112312	122604 17	39
2 2	11839369	11973638	12114722	122629 48	38
2 3	11841552	11975932	12117133	122654 81	37
2 4	11843737	11978229	12119546	122680 16	36
2 5	11845924	11980527	12121960	122705 53	35
2 6	11848114	11982828	12124377	122730 93	34
2 7	11850305	11985131	12126796	122756 34	33
2 8	11852498	11987435	12129216	122781 87	32
2 9	11854693	11989741	12131638	122807 22	31
3 0	11856890	11992050	12134063	122832 70	30
3 1	11849088 [recte:?)	11994360	12136490	122858 20	29
3 2	11861288	11996672	12138919	122883 72	28
3 3	11863489	11998986	12141350	122909 25	27
3	11865693	12001303	12143783	122934	26

M i n	32	33	34	35	Com pl.
4				81	
3	11867899	12003619	12146218	122960	25
5				39	
3	11870107	12005938	12148656	122985	24
6				99	
3	11872316	12008259	12150095 [recte:?)	123011	23
7				61	
3	11874527	12010582	12153536	123037	22
8				25	
3	11876739	12012907	12155978	123062	21
9				91	
4	11878954	12015233	12158423	123088	20
0				59	
4	11881171	12017562	12160870	123114	19
1				30	
4	11883389	12019893	12163319	123140	18
2				03	
4	11885609	12022226	12165770	123165	17
3				78	
4	11887831	12024560	12168223	123191	16
4				56	
4	11890054	12026897	12170677	123217	15
5				36	
4	11892280	12029236	12173135	123243	14
6				17	
4	11894508	12031576	12175594	123269	13
7				00	
4	11896737	12033919	12178055	123294	12
8				86	
4	11898968	12036264	12180518	123320	11
9				74	
5	11901202	12038610	12182983	123346	10

M i n	32	33	34	35	Com pl.
0				64	
5 1	11903437	12040958	12185450	123372 56	9
5 2	11905674	12043309	12187919	123398 51	8
5 3	11907912	12045661	12190390	123424 48	7
5 4	11910153	12048016	12192864	123450 46	6
5 5	11912395	12050372	12195340	123476 46	5
5 6	11914640	12052730	12197817	123502 49	4
5 7	11916886	12055089	12200296	123528 54	3
5 8	11919133	12057451	12202777	123554 60	2
5 9	11921382	12059814	12205260	123580 68	1
6 0	11923633	12063179 [recte:?]	12207745	123606 78	0

Foot banner degree indices (complement reading): 57 | 56 | 55 | 54.

Page 134 — Secants of 36°, 37°, 38°, 39°

Mi n	36	37	38	39	Compl.
0	12360678	12521357	12690184	12867599	60
1	12363290	12524103	12693070	12870632	59
2	12365906	12526851	12695957	12873667	58
3	12368524	12529601	12698847	12876704	57
4	12371144	12532354	12701739	12879744	56

Mi n	36	37	38	39	Compl.
5	12373766	12535110	12704634	12882787	55
6	12376391	12537867	12707531	12885832	54
7	12379018	12540627	12710430	12888879	53
8	12381647	12543389	12713332	12891929	52
9	12384278	12546152	12716236	12894982	51
10	12386911	12548918	12719143	12898037	50
11	12389546	12551686	12722052	12901094	49
12	12392183	12554456	12724964	12904155	48
13	12394822	12557229	12727878	12907218	47
14	12397464	12560005	12730794	12910283	46
15	12400108	12562783	12733713	12913351	45
16	12402754	12565563	12736635	12916422	44
17	12405402	12568345	12739559	12919494	43
18	12408053	12571130	12742485	12922569	42
19	12410705	12573917	12745413	12925647	41
20	12413359	12576706	12748344	12928727	40
21	12416015	12579597 [recte:?]	12751277	12931809	39
22	12418674	12582912 [recte:?]	12754213	12934895	38
23	12421335	12585087	12757151	12937983	37
24	12423998	12587885	12760092	12941073	36
25	12426663	12590685	12763035	12944166	35
26	12429331	12593488	12765981	12947262	34
27	12432001	12596293	12768929	12950360	33
28	12434673	12599101	12771880	12953461	32
29	12437348	12601911	12774833	12956565	31
30	12440024	12604724	12777788	12959671	30
31	12442702	12607539	12780746	12962780	29
32	12445383	12610356	12783707	12965892	28
33	12448066	12613175	12786670	12969007	27
34	12450751	12615997	12789635	12972124	26

Mi n	36	37	38	39	Compl.
35	12453438	12618821	12792602	12975243	25
36	12456128	12621648	12795573	12978366	24
37	12458821	12624477	12798546	12981491	23
38	12461516	12627308	12801521	12984618	22
39	12464213	12630141	12804498	12987747	21
40	12466913	12632977	12807478	12990880	20
41	12469614	12635815	12810460	12994015	19
42	12472317	12638655	12813445	12997153	18
43	12475022	12641497	12816432	13000293	17
44	12477730	12644343	12819422	13003436	16
45	12480440	12647191	12822415	13006582	15
46	12483152	12650041	12825410	13009730	14
47	12485866	12652893	12828407	13012881	13
48	12488583	12655748	12831407	13016034	12
49	12491302	12658605	12834409	13019189	11
50	12494022	12661464	12837414	13022348	10
51	12496744	12664325	12840421	13025509	9
52	12499469	12667189	12843431	13028673	8
53	12502197	12670055	12846443	13031839	7
54	12504927	12672924	12849458	13035008	6
55	12507659	12675795	12852475	13038180	5
56	12510394	12678668	12855495	13041354	4
57	12513132	12681543	12858517	13044530	3
58	12515871	12684421	12861542	13047710	2
59	12518613	12687301	12864569	13050892	1
60	12521357	12690184	12867599	13054077	0

Foot banner degree indices (complement reading): 53 | 52 | 51 | 50.

Tabula Linearum Secantium (continued)

Christoph Clavius [Christophorus Clavius], Opera Mathematica, Tomus I (Mainz, 1611) — the Table of Secant Lines. This opening covers whole degrees 40 through 51, spanning the 45° landmark ($\sec 45^\circ = \sqrt{2} \cdot \text{radius}$).

Each page is a single table. The leftmost column gives the minutes (0'–60') of the degree named in the **top** banner; the rightmost column gives the minutes (60'–0') to be read with the **complement** degree named in the **bottom** banner (printed running text: "...pro secantibus complementorum arcuum eiusdem Quadrantis"). All secant values are for radius 10,000,000. Column-boundary seam values (each degree's 60' entry equals the next degree's 0' entry) were cross-checked and agree throughout.

Header words retained as printed: *Gradus* (degrees), *Minuta Graduum Quadrantis* (minutes of the degrees of the quadrant), *pro secantibus arcuum eiusdem Quadrantis* (for the secants of the arcs of the same quadrant), *Secantivm* (of the secants).

Folio 135 — Degrees 40–43

Top banner: **Gradus Quadrantis pro secantibus arcuum eiusdem Quadrantis. SECANTIVM.**

Min '	40	41	42	43	Min '
0	13054077	13250 131	13456326	13673275	60
1	13057264	13253 482	13459851	13676986	59
2	13060455	13256 835	13463380	13680700	58
3	13063646	13260 192	13466912	13684417	57
4	13066843	13263 582	13470447	13688138	56
5	13070041	13266 915	13473985	13691861	55
6	13073242	13270 282	13477527	13695587	54

Min ' 40 41 42 43 Min ' 44	45	46	47	48	49
7	13076445	13273651	13481071	13699316	53
8	13079651	13277023	13484618	13703048	52
9	13082859	13280397	13488168	13706783	51
10	13086071	13283775	13491721	13710523	50
11	13089285	13287155	13495276	13714266	49
12	13092502	13290538	13498835	13718012	48
13	13095721	13293924	13502397	13721761	47
14	13098944	13297313	13505962	13725514	46
15	13102169	13300704	13509530	13729270	45
16	13105397	13304098	13513101	13733029	44
17	13108627	13307495	13516675	13736790	43
18	13111861	13310896	13520252	13740555	42
19	13114098 [recte:?)	13314299	13523832	13744322	41
20	13118337	13317705	13527416	13748092	40
21	13121578	13321114	13531003	13751867	39
22	13124823	13324526	13534593	13755644	38
23	13128070	13327941	13538185	13759424	37

Min ' 40 41 42 43 Min '					
24	13131320	13331 359	13541781	13763209	36
25	13134572	13334 779	13545380	13766997	35
26	13137828	13338 203	13548981	13770788	34
27	13141085	13341 629	13552585	13774582	33
28	13144346	13345 058	13556193	13778380	32
29	13147509 [recte:?]	13348 490	13559803	13782181	31
30	13150874	13351 924	13563417	13785985	30
31	13154142	13355 361	13567034	13789792	29
32	13157413	13358 802	13570654	13793603	28
33	13160687	13362 245	13574277	13797416	27
34	13163964	13365 691	13577903	13801233	26
35	13167243	13369 140	13581532	13805053	25
36	13170526	13372 592	13585164	13808876	24
37	13173811 [?]	13376 057	13588799	13812703	23
38	13177099	13379 505	13592438	13816534	22
39	13180389	13382 966	13596079	13820368	21
40	13183682	13386 430	13599723	13824205	20

Min '	40	41	42	43	Min '
41	13186978	13389 897	13603370	13828045	19
42	13190276	13393 367	13607021	13831889	18
43	13193577	13396 839	13610975 [recte:?]	13835736	17
44	13196882	13400 315	13614332	13839586	16
45	13200189	13403 794	13617992	13843439	15
46	13203499	13407 275	13621656	13847296	14
47	13206812	13410 759	13625323	13851156	13
48	13210128	13414 247	13628993	13855019	12
49	13213447	13417 738	13632666	13858885	11
50	13216769	13421 232	13636342	13862755	10
51	13220093	13424 728	13640021	13866628	9
52	13223421	13428 227	13643704	13870505	8
53	13226750	13431 729	13647390	13874385	7
54	13230082	13435 234	13651078	13878268	6
55	13233417	13438 742	13654769	13882154	5
56	13236754	13442 253	13658464	13886044	4
57	13240094	13445 767	13662162	13889636 [recte:?]	3

Min ' 40 41 42 43 Min ' 44
58 13243437 13449 13665863 13893833 2
59 13246783 13452 13669567 13897733 1
60 13250131 13456 13673275 13901636 0

Folio 136 — Degrees 44–47

Landmark: sec $45^\circ 0'$ = 14142135 (printed), matching $\sqrt{2} \cdot 10,000,000 = 14142135.62$ to the unit.

Min ' 44	45	46	47	Min ' 48
82			14	
9 139369	14179306	14434743	147041	51
19			22	
10 139408	14183454	14439116	147087	50
54			35	
11 139447	14187606	14443493	147133	49
95			52	
12 139487	14191761	14447874	147179	48
39			73	
13 139526	14195919	14452259	147225	47
86			98	
14 139566	14200082	14456648	147272	46
38			28	
15 139605	14204248	14461040	147318	45
92			62	
16 139645	14208418	14465437	147365	44
50			00	
17 139685	14212591	14469838	147411	43
11			42	
18 139724	14216769	14474242	147457	42
76			88	
19 139764	14220950	14478650	147504	41
44			38	
20 139804	14225135	14483062	147550	40
16			94	
21 139843	14229324	14487478	147597	39
91			53	
22 139883	14233517	14491898	147644	38
70			16	
23 139923	14237713	14496322	147690	37
52			83	
24 139963	14241912	14500750	147737	36
38			55	
25 140003	14246115	14505182	147784	35

Min ' 44	45	46	47	Min ' 30	
26	27	30			
140043	14250321	14509617	147831	34	
19			10		
27	140083	14254531	14514056	147877	33
15			94		
28	140123	14258745	14518500	147924	32
14			82		
29	140163	14262961	14522946	147971	31
16			74		
30	140203	14267182	14527397	148018	30
22			71		
31	140243	14271407	13531852 [recte:?)	148065	29
32			71		
32	140283	14275635	14536311	148112	28
45			76		
33	140323	14279867	14540773	148159	27
61			85		
34	140363	14284103	14545240	148206	26
81			98		
35	140404	14288343	14549711	148254	25
04			16		
36	140444	14292587	14554186	148301	24
31			39		
37	140484	14296834	14558665	148348	23
61			66		
38	140524	14301086	14563148	148395	22
94			97		
39	140565	14305331	14567635	148443	21
31			32		
40	140605	14309599	14572126	148490	20
72			72		
41	140646	14313861	14576621	148538	19
16			15		
42	140686	14318127	14581120	148585	18

Min ' 44	45	46	47	Min ' 63	
	64		63		
43	140727 15	14322396	14585624	148633 15	17
44	140767 70	14326670	14590131	148680 71	16
45	140808 29	14330947	14594642	148728 31	15
46	140848 91	14335228	14599157	148775 97	14
47	140889 56	14339513	14603676	148823 77	13
48	140930 26	14343802	14608199	148871 41	12
49	140970 99	14348095	14612725	148919 19	11
50	141011 75	14352391	14617256	148967 01	10
51	141052 55	14356691	14621791	149014 87	9
52	141093 39	14360995	14626330	149062 78	8
53	141134 27	14365303	14630873	149110 73	7
54	141175 18	14369615	14635421	149158 73	6
55	141216 12	14373930	14639973	149206 77	5
56	141257 09	14378350 [recte:?)	14644528	149254 86	4
57	141298 10	14382573	14649087	149302 99	3
58	141339 15	14386900	14653651	149351 16	2
59	141380	14391230	14658218	149399	1

Min '	44	45	46	47	Min '
	23			38	
60	141421	14395564	14662790	149447	0
	35			64	

Bottom banner (complement degrees): **45 | 44 | 43 | 42**. Running text: "Gradus Quadrantis pro secantib. complementorum arcuum eiusdem Quadrantis."

Folio 137 — Degrees 48–51

Top banner: **Gradus Quadrantis pro secantibus arcuum eiusdem Quadrantis. SECANTIVM.**

Min '	48	49	50	51	Min '
0	14944764	15242532	15557239	15890158	60
1	14949594	15247634	15562635	15895869	59
2	14954429	15252741	15568036	15901586	58
3	14959268	15257852	15573441	15907307	57
4	14964112	15262969	15578852	15913034	56
5	14968960	15268090	15584267	15918766	55
6	14973812	15273216	15589688	15924504	54
7	14978668	15278347	15595114	15930247	53
8	14983530	15283484	15600545	15936095	52
				[recte:?]	
9	14988396	15288626	15605981	15941748	51
10	14993266	15293773	15611422	15947508	50
11	14998104	15298924	15616868	15953273	49
	[recte:?]				
12	15003020	15304080	15622319	15959044	48
13	15007903	15309240	15627775	15964820	47
14	15012791	15314405	15633237	15970603	46
15	15017683	15319574	15639704	15976390	45
			[recte:?]		
16	15022580	15324748	15644177	15982184	44
17	15027481	15329926	15649655	15987983	43

Min					Min
'	48	49	50	51	'
18	15032387	15335109	15655138	15993788	42
19	15037297	15340297	15660626	15999599	41
20	15042212	15345491	15666119	16005416	40
21	15047131	15350689	15671617	16011237	39
22	15052054	15355892	15677121	16017065	38
23	15056982	15361100	15682630	16022898	37
24	15061915	15366313	15688144	16028736	36
25	15066852	15371530	15693663	16034579	35
26	15071791	15376753	15699188	16040429	34
27	15076739	15381980	15704717	16046283	33
28	15081690	15387212	15710252	16052143	32
29	15086645	15392449	15715792	16058008	31
30	15091605	15397692	15721337	16063878	30
31	15096569	16402939	15726887	16069754	29
		[recte:?]			
32	15101538	15408191	15732443	16075637	28
33	15106571	15413447	15738003	16081524	27
	[recte:?]				
34	15111490	15418708	15743569	16087418	26
35	15116472	15423974	15749141	16093318	25
36	15121459	15429246	15754718	16099224	24
37	15126451	15434522	15760300	16105135	23
38	15131447	15439803	15765887	16111053	22
39	15136447	15445089	15771479	16116976	21
40	15141453	15450380	15777077	16122905	20
41	15146463	15455675	15782680	16128839	19
42	15151478	15460976	15788289	16134779	18
43	15156497	15466282	15793903	16140724	17
44	15161520	15471593	15799523	16146676	16
45	15166548	15476908	15809147	16152634	15
			[recte:?]		

Min '	48	49	50	51	Min '
46	15171581	15482229	15810777	16158598	14
47	15176619	15487554	15816412	16164567	13
48	15181661	15492885	15822052	16170542	12
49	15186708	15498220	15827697	16176522	11
50	15191760	15503560	15833349	16182509	10
51	15196816	15508905	15839005	16188591	9
				[recte:?]	
52	15201877	15514256	15844667	16194499	8
53	15206943	15519611	15850335	16200503	7
54	15212013	15524972	15856008	16206513	6
55	15217088	15530338	15861676	16212528	5
56	15222168	15535710	15867370	16218550	4
57	15227253	15541083	15873058	16224577	3
58	15232342	15546463	15878753	16230610	2
59	15237435	15551848	15884453	16236648	1
60	15242532	15557239	15890158	16242692	0

Bottom banner (complement degrees): **41** | **40** | **39** | **38**. Running text: "Gradus Quadrantis pro secantibus complementorum arcuum eiusdem Quadrantis."

Signature:.

Tabula Linearum Secantium — Secant Tables (upper quadrant), degrees 52°–63°

Christoph Clavius, Opera Mathematica, Tomus I (Mainz, 1611).

Running head across the opening: **TABVLA SECANTIVM** ("Table of Secants").
Body caption on each page: *Gradus Quadrantis pro secantibus arcuum eiusdem Quadrantis* ("Degrees of the quadrant, for the secants of arcs of the same quadrant").
Foot caption on each page: *Gradus Quadrantis pro secantibus complementorum arcuum eiusdem Quadrantis* ("...for the secants of the complements of arcs of the same quadrant").

Each column is headed by a whole degree (top banner). The left index column is *Minuta Graduum Quadrantis* (minutes of arc, 0'→60' increasing downward); the right index column is the complementary minute scale (60'→0', = 60 – left minute) used with the foot banners for reading secants of complements. Secant values are given $\times 10^7$ (so 20000000 = sec 60° = 2.0000000). Every degree's 60' entry equals the next degree's 0' entry; all such seams were cross-checked (see report).

Foot banner (complement degrees, read with the right/60'→0' scale): 37 · 36 · 35 · 34.

Mi n	52	53	54	55
0	16242692	16616401	17013017	17434469
1	16248742	16622819	17019832	17441715
2	16254799	16629243	17026654	17448968
3	16260819	16635673	17033482	17456229
4	16266929	16642109	17040318	17463499
5	16273003	16648551	17047160	17470775
6	16279083	16655001	17054010	17478059
7	16285169	16661457	17060866	17485351
8	16291261	16667919	17067729	17492650
9	16297358	16674408	17074599	17499957
10	16303461	16680864	17081476	17507272
11	16309570	16687345	17088359	17514594
12	16315685	16693834	17095250	17521924
13	16321806	16700328	17102148	17529262
14	16327934	16706829	17109053	17536607
15	16334067	16713336	17115965	17543959
16	16340197	16719850	17122885	17551319
17	16346353	16726362	17129812	17558687
18	16352505	16732877	17136747	17566063
19	16358663	16739430	17143689	17573446
20	16364827	16745970	17150638	17580837
21	16370996	16752517	17157593	17588236
22	16377172	16759070	17164556	17595643

Mi n	52	53	54	55
23	16383359	16765629	17171525	17603057
24	16389542	16772195	17178502	17610480
25	16395736	16778767	17185485	17617909
26	16401936	16785347	17192476	17625347
27	16408152	16791933	17199472	17632793
28	16414365	16798525	17206477	17640246
29	16420573	16805124	17213488	17647707
30	16426798	16811729	17220507	17655174
31	16433027	16818341	17227532	17662651
32	16439263	16824960	17234565	17670136
33	16445505	16831585	17241605	17677627
34	16451754	16838217	17248653	17685127
35	16458008	16844856	17255708	17692635
36	16464269	16851502	17262770	17700151
37	16470536	16858154	17269839	17707674
38	16476809	16864813	17276917	17715206
39	16483089	16871479	17284002	17722744
40	16489385	16878151	17291095	17730290
41	16495668	16884830	17298194	17737844
42	16501967	16891515	17305300	17745407
43	16508272	16898207	17312413	17752978
44	16514582	16904907	17319514	17760555
45	16520898	16911613	17326662	17768142
46	16527220	16918326	17333798	17775740
47	16533548	16925046	17340941	17783343
48	16539883	16931772	17348091	17790955
49	16546224	16948504 [recte:?]	17355249	17798575
50	16552571	16945244	17362415	17806203
51	16558925	16951990	17369587	17813838
52	16565286	16958743	17376767	17821481

Mi n	52	53	54	55
53	16571642	16965495	17383954	17829132
54	16578026	16972270	17391148	17836792
55	16584406	16979044	17398350	17844460
56	16590792	16985824	17405560	17852135
57	16597184	16992611	17412776	17859818
58	16603584	16999406	17420000	17867509
59	16609989	17006208	17427231	17875209
60	16616401	17013017	17434469	17882917

Foot banner (complement degrees): **33 · 32 · 31 · 30**. Foot signature: **M 4**.

Mi n	56	57	58	59
0	17882917	18360816	18870800	19416039
1	17890632	18369014	18879589	19425445
2	17898356	18377251	18888389	19434862
3	17906089	18385497	18897196	19444290
4	17913830	18393753	18906018	19453727
5	17921579	18402017	18914846	19463175
6	17929337	18410291	18923685	19472635
7	17937102	18418574	18932534	19482114
8	17944876	18426865	18941393	19491595
9	17952658	18435165	18950261	19501076
10	17960448	18443454	18959139	19510578
11	17968247	18451792	18968027	19520091
12	17976054	18460120	18976926	19529615
13	17983869	18468456	18985834	19539150
14	17991693	18476802	18994752	19548697
15	17999525	18485157	19003680	19558254
16	18007365	18493521	19012618	19567822
17	18015214	18501895	19021516	19577401
18	18023071	18510278	19030523	19586991

Mi n	56	57	58	59
19	18030936	18518670	19039491	19596592
20	18038811	18527072	19048468	19606204
21	18046693	18535483	19057455	19615827
22	18054584	18543903	19066453	19625462
23	18062482	18552332	19075461	19635107
24	18070389	18560770	19084480	19644765
25	18078305	18569217	19093509	19654434
26	18086229	18577674	19102549	19664114
27	18094161	18586139	19111598	19673805
28	18102102	18594614	19120658	19683507
29	18110051	18603098	19129727	19693220
30	18118009	18611591	19138807	19702945
31	18125975	18620094	19147897	19712680
32	18133950	18629606	19156998	19722428
33	18141934	18637127	19166109	19732186
34	18149926	18645658	19175231	19741956
35	18157927	18654198	19184362	19751738
36	18165937	18662748	19193504	19761531
37	18173956	18671307	19202656	19771335
38	18181984	18679875	19211818	19781141
39	18190021	18688452	19220990	19790968
40	18198065	18697038	19230172	19800808
41	18206118	18705634	19239365	19810658
42	18214179	18714239	19248569	19820320 [recte:?]
43	18222249	18722854	19257783	19830393
44	18230328	18731480	19267008	19840277
45	18238416	18740115	19276242	19850172
46	18246513	18748760	19285488	19860079
47	18254618	18757414	19294744	19869997
48	18262732	18766078	19304010	19879927

Mi n	56	57	58	59
49	18270854	18774752	19313287	19889868
50	18278986	18783436	19322574	19899820
51	18287126	18792130	19331872	19909784
52	18295276	18800833	19341181	19919760
53	18303434	18809546	19350501	19929748
54	18311601	18818268	19359831	19939749
55	18319776	18826999	19369172	19949760
56	18327961	18835741	19378524	19959784
57	18337154	18844492	19387886	19966820 [recte:?]
58	18344356	18853252	19397260	19979868
59	18352567	18862021	19406644	19989928
60	18360816	18870800	19416039	20000000

Foot banner (complement degrees): **29 · 28 · 27 · 26.**

M i n	60	61	62	63
0	20000000	206266 54	21300545	22026892
1	20010083	206374 84	21312206	22039475
2	20020179	206483 38	21323882	22052074
3	20030285	206591 84	21335570	22064690
4	20040404	206700 54	21347275	22077322
5	20050534	206809 37	21358993	22089970
6	20060676	206918 34	21370727	22102635
7	20070832	207027	21382475	22115316

M i n	60	61	62	63
		44		
8	20080995	207136	21394238	22128014
		67		
9	20091172	207246	21407016 [recte:?]	22140728
		03		
1 0	20101361	207355	21417808	22153459
		54		
1 1	20111562	207465	21429615	22166204
		17		
1 2	20121776	207574	21441438	22178971
		94		
1 3	20132001	207684	21453275	22191751
		84		
1 4	20142239	207794	21465128	22204548
		88		
1 5	20152489	207905	21476995	22217361
		05		
1 6	20162751	208015	21488877	22230191
		35		
1 7	20173035	208125	21500774	22243038
		79		
1 8	20183321	208236	21512686	22255902
		36		
1 9	20193619	208347	21524612	22268782
		06		
2 0	20203930	208457	21536553	22281680
		91		
2 1	20214252	208568	21548509	22294595
		88		
2 2	20224588	208680	21560481	22307526
		00		
2	20234936	208791	21572467	22320474

M i n	60	61	62	63
3		25		
2	20245296	208902	21584469	22333439
4		64		
2	20255669	209014	21596487	22346420
5		16		
2	20266054	209125	21608520	22359419
6		82		
2	20276452	209237	21620568	22372434
7		61		
2	20286863	209349	21632631	22385466
8		55		
2	20297286	209461	21644710	22398418 [recte:?]
9		62		
3	20307721	209573	21656804	22411584
0		83		
3	20318170	209686	21668913	22424667
1		18		
3	20328630	209798	21681038	22437768
2		67		
3	20339102	209911	21693178	22450886
3		30		
3	20349587	210024	21705334	22464022
4		06		
3	20360084	210136	21717505	22477175
5		96		
3	20370594	210250	21729691	22490346
6		01		
3	20381116	210363	21741893	22503543
7		19		
3	20391751 [recte:?]	210476	21754111	22516748
8		51		
3	20402198	210589	21766344	22529965

M i n	60	61	62	63
9		97		
4	20412758	210703	21778593	22543201
0		57		
4	20423331	210817	21790858	22556358
1		31		
4	20433916	210931	21803138	22569723
2		19		
4	20444514	211045	21815434	22583025
3		22		
4	20455126	211159	21827745	22596336
4		38		
4	20465750	211273	21840072	22609663
5		68		
4	20476387	211388	21852415	22623009
6		14		
4	20487037	211502	21864774	22636372
7		73		
4	20497700	211617	21877149	22649754
8		47		
4	20508376	211732	21889539	22663152
9		35		
5	20519064	211847	21901946	22676569
0		37		
5	20529765	211962	21914369	22690004
1		53		
5	20540479	212077	21926808	22703456
2		83		
5	20551205	212193	21939263	22716924
3		28		
5	20561945	212308	21951734	22730414
4		87		
5	20572697	212424	21964220	22743919

M i n	60	61	62	63
5		60		
5	20583463	212540	21976722	22757443
6		48		
5	20594242	212656	21989240	22770984
7		50		
5	20605033	212772	22001775	22784543
8		67		
5	20615837	212888	22014325	22798120
9		99		
6	20626654	213005	22026892	22811726
0		45		

Table of Secant Lines (*Tabula Linearum Secantium*) — 64°–75°

Christoph Clavius [Clavius], *Opera Mathematica*, Tomus I (Mainz, 1611), printed folios 141–143 (global Upper quadrant of the secant table; radius $R = 10,000,000$ (values are secants $\times 10^7$). Each degree column is headed by its degree; the left-hand minute column runs $0' \rightarrow 60'$ (top $\rightarrow$ bottom), and the right-hand minute column runs $60' \rightarrow 0'$ (the complement reading). The bottom banner gives the complement degree ($89^\circ - \text{heading}$) for reading the secants of the complementary arcs.

Top rubric (all three pages): *Gradus Quadrantis pro secantibus arcuum eiusdem Quadrantis*. — "Degrees of the quadrant for the secants of the arcs of the same quadrant." Bottom rubric (all three pages): *Gradus Quadrantis pro secantibus complementorum arcuum eiusdem Quadrantis*. — "Degrees of the quadrant for the secants of the complements of the arcs of the same quadrant."

Folio 141 — Secants 64°–67°

Running header: *SECANTIVM* (center) · 141 (outer). Column-foot complement degrees: 25 | 24 | 23 | 22.

Min'	64	65	66	67	Compl. Min'
0	2281172 6	2366201 3	24585936	2559305 1	60
1	2282532 9	2367678 4	24602010	2561060 2	59
2	2283896 2	2369157 5	24618107	2562818 0	58
3	2285261 2	2370638 7	24634227	2564578 3	57
4	2286628 1	2372122 0	24650370	2566341 4	56
5	2287996 8	2373607 3	24666536	2568107 1	55
6	2289367 4	2375094 7	24682727	2569875 4	54
7	2290738 7	2376584 2	24698940	2571646 4	53
8	2292114 0	2378075 7	24715178	2573420 1	52
9	2293490 1	2379569 2	24731439	2575196 5	51
10	2294868 0	2381064 8	24747724	2576975 5	50
11	2296247 8	2382562 5	24764033	2578758 2	49
12	2297629 4	2384062 3	24780365	2580541 7	48
13	2299012 9	2385564 2	24796721	2582328 7	47
14	2300398 3	2387068 3	24813101	2584118 5	46
15	2301785 5	2388584 4	24829504	2585910 4	45
16	2303174 7	2390082 7	24845932	2587706 1	44

Min'	64	65	66	67	Compl. Min'
17	2304565 7	2391593 1	24862383	2589504 0	43
18	2305958 6	2393105 5	24879958 [recte:?]	2591304 6	42
19	2307353 4	2394620 0	24895356	2593108 0	41
20	2308750 1	2396136 6	24911878	2594914 2	40
21	2310148 6	2397655 3	24928423	2596723 0	39
22	2311549 0	2399176 2	24944993	2598534 5	38
23	2312951 3	2400699 2	24961587	2600348 7	37
24	2314355 6	2402224 5	24978205	2602165 8	36
25	2315761 6	2403751 8	24994847	2603985 5	35
26	2317169 6	2405281 4	25012514	2605808 1	34
27	2318579 5	2406813 0	25028205	2607633 3	33
28	2319991 3	2408346 9	25044920	2609461 4	32
29	2321405 0	2409883 0	25061660	2611292 3	31
30	2322820 5	2411421 3	25078426	2613125 9	30
31	2324238 0	2412961 6	25095216	2614962 3	29
32	2325657 4	2414504 1	25112030	2616801 5	28
33	2327079 7	2416048 7	25128869	2618643 6	27

Min'	64	65	66	67	Compl. Min'
34	2328502 1	2417595 6	25145732	2620488 4	26
35	2329927 3	2419144 5	25162620	2622336 1	25
36	2331354 6	2420695 6	25179432	2624186 7	24
37	2332783 8	2422248 8	25196469	2626040 0	23
38	2334215 0	2423804 3	25213432	2627896 3	22
39	2335648 1	2425361 9	25230418	2629755 5	21
40	2337083 2	2426921 7	25247431	2631617 6	20
41	2338520 3	2428483 8	25264468	2633482 5	19
42	2339959 3	2430048 1	25281531	2635350 3	18
43	2341400 3	2431614 7	25298620	2637220 9	17
44	2342843 3	2433183 5	25315734	2639094 5	16
45	2344288 2	2434754 6	25332874	2640970 9	15
46	2345735 1	2436328 1	25350039	2642850 2	14
47	2347184 0	2437903 8	25367229	2644732 3	13
48	2348634 8	2439481 8	25384445	2646617 4	12
49	2350087 6	2441062 0	25401687	2648505 3	11
50	2351542 4	2442644 6	25418956	2650396 2	10

Min'	64	65	66	67	Compl. Min'
51	2352999 2	2444229 4	25436250	2652289 0	9
52	2354458 0	2445816 4	25453570	2654186 7	8
53	2355918 8	2447405 6	25470915	2656086 3	7
54	2357381 7	2448997 3	25488286	2657988 9	6
55	2358856 5	2450590 8	25505683	2659894 5	5
56	2360313 4	2452186 9	25523005	2661803 0	4
57	2361782 2	2453785 1	25540553	2663714 5	3
58	2363253 2	2455385 7	25558027	2665629 1	2
59	2364726 2	2456988 5	25575526	2667546 6	1
60	2366201 3	2458593 6	25593051	2669467 2	0

Folio 142 — Secants 68°–71°

Running header: 142 (outer) · *TABVLA* (center). Column-foot complement degrees: 21 | 20 | 19 | 18. (The 70° banner carries a small stray ink mark after the digits; read "70".)

Min'	68	69	70	71	Compl. Min'
0	2669467 2	27904284	2923804 5	3071553 1	60
1	2671390 7	27925445	2926143 3	3074150 0	59
2	2673317 2	27946642	2928486 1	3076751 6	58
3	2675246 7	27967873	2930832 8	3079357 9	57

Min'	68	69	70	71	Compl. Min'
4	2677179 1	27989139	2933183 5	3081968 9	56
5	2679114 5	28010440	2935538 2	3084584 6	55
6	2681052 9	28031776	2937897 0	3087205 1	54
7	2682994 2	28053147	2940259 9	3089830 4	53
8	2684939 0	28074553	2942626 8	3092460 5	52
9	2686886 7	28095994	2944997 8	3095095 3	51
10	2688837 3	28117469	2947372 8	3097735 0	50
11	2690791 0	28138980	2949751 9	3100379 3	49
12	2692747 9	28160527	2952135 0	3103028 5	48
13	2694707 8	28182108	2954522 2	3105682 4	47
14	2696670 9	28203725	2956913 6	3108341 2	46
15	2698637 0	28225378	2959309 0	3111004 7	45
16	2700606 2	28247067	2961708 7	3113673 1	44
17	2702578 5	28268793	2964112 4	3116346 2	43
18	2704553 9	28290553	2966520 4	3119024 1	42
19	2706532 3	28312349	2968932 6	3121701 9	41
20	2708513 8	28334181	2971348 8	3124394 5	40

Min'	68	69	70	71	Compl. Min'
21	2710498 5	28356049	2973769 2	3127087 1	39
22	2712486 4	28377954	2976193 8	3129784 8	38
23	2714477 4	28399894	2978622 7	3132487 3	37
24	2716471 7	28421871	2981055 8	3135194 8	36
25	2718469 0	28443884	2983493 1	3137907 2	35
26	2720468 6	28465934	2985934 7	3140624 7	34
27	2722473 4	28488021	2988370 5	3143347 2	33
28	2724480 4	28510144	2990830 6	3146074 7	32
29	2726490 6	28532304	2993285 0	3148807 2	31
30	2728504 0	28554501	2995743 8	3151544 8	30
31	2730520 5	28576735	2998206 9	3154287 3	29
32	2732540 2	28599007	3000674 3	3157034 9	28
33	2734563 1	28622316 [recte:?]	3003146 0	3159787 5	27
34	2736589 3	28643662	3005622 0	3162545 3	26
35	2738618 6	28666045	3008102 3	3165308 0	25
36	2740651 3	28688467	3010587 0	3168075 8	24
37	2742687 2	28710925	3013076 0	3170848 6	23

Min'	68	69	70	71	Compl. Min'
38	2744726 4	28733422	3015571 4	3173626 5	22
39	2746768 8	28755956	3018067 2	3176409 4	21
40	2748814 5	28778549	3020569 4	3179197 4	20
41	2750863 5	28801139	3023076 0	3181990 6	19
42	2752915 7	28823787	3025587 1	3184789 1	18
43	2754972 2	28846473	3028102 6	3187592 9	17
44	2757030 1	28869196	3030622 6	3190401 9	16
45	2759092 2	28891957	3033146 0	3193216 4	15
46	2761157 8	28914756	3035675 9	3196035 8	14
47	2763226 6	28937594	3038209 2	3198860 6	13
48	2765298 9	28960471	3040747 0	3201690 9	12
49	2767374 5	28983386	3043289 3	3204526 3	11
50	2769453 5	29006340	3045836 1	3207367 2	10
51	2771535 8	29029332	3048387 3	3210213 2	9
52	2773621 5	29052363	3050943 0	3213064 9	8
53	2775710 5	29075435	3053503 3	3215921 2	7
54	2777802 9	29098546	3056068 2	3218783 2	6

Min'	68	69	70	71	Compl. Min'
55	2779898 7	29121697	3058637 5	3221650 4	5
56	2781997 8	29144888	3061211 5	3224523 1	4
57	2784100 3	29168118	3063789 0	3227401 2	3
58	2786206 0	29191388	3066373 2	3230284 6	2
59	2788315 6	29214697	3068960 8	3233173 5	1
60	2790428 4	29238045	3071553 1	3236067 8	0

Folio 143 — Secants 72°–75°

Running header: *SECANTIVM* (center) · 143 (outer). Column-foot complement
degrees: 17 | 16 | 15 | 14.

Mi n'	72	73	74	75	Compl. Min'
0	32360678	34203038	36279559	38637042	60
1	32389676	34235609	36316402	38679033	59
2	32418726	34268245	36353333	38721117	58
3	32447837	34300947	36390323	38763296	57
4	32477001	34333716	36427401	38805571	56
5	32506219	34366553	36464558	38847941	55
6	32535494	34399452	36501793	38890408	54
7	32564823	34432420	36539107	38932971	53
8	32594209	34465456	36570511 [recte:?]	38975632	52
9	32623651	34498557	36613973	39018390	51
10	32653148	34531726	36651525	39061246	50
11	32682701	34564959	36689156	39104200	49
12	32712311	34598259	36726868	39147252	48

Mi n'	72	73	74	75	Compl. Min'
13	32741977	34631526	36864660 [recte:?]	39190423	47
14	32771699	34665061	36802533	39233653	46
15	32801478	34698564	36840488	39277002	45
16	32831314	34732135	36878524	39320449	44
17	32861207	34765775	36916641	39363994	43
18	32891157	34799483	36954842	39407640	42
19	32921165	34833259	36993127	39451384	41
20	32951231	34867105	37031496	39495228	40
21	32981355	34901024	37069947	39539172	39
22	33011537	34935005	37108482	39583218	38
23	33041776	34966052 [recte:?]	37147101	39627364	37
24	33072074	35003172	37185803	39671613	36
25	33102431	35037361	37224589	39715965	35
26	33131846 [recte:?]	35071621	37263459	39760420	34
27	33163320	35105952	37302413	39804979	33
28	33193853	35140354	37341453	39849642	32
29	33224444	35174826	37380577	39894411	31
30	33255094	35209369	37419788	39939286	30
31	33285803	35243981	37459081	39984263	29
32	33316571	35278664	37498460	40029344	28
33	33347398	35313418	37537923	40074528	27
34	33378286	35348244	37577471	40119816	26
35	33409132	35383140	37617104	40165289	25
36	33440240	35418110	37656824	40210709	24
37	33471307	35453152	37696632	40256316	23
38	33502436	35488268	37736518	40302033	22
39	33533625	35523456	37776513	40347858	21
40	33564875	35558718	37816588	40393792	20

Mi n'	72	73	74	75	Compl. Min'
41	33596187	35594052	37856751	40439834	19
42	33627561	35629460	37897004	40485985	18
43	33658998	35664940	37937146	40532245	17
44	33690497	35700494	37977779	40578613	16
45	33722059	35736121	38018300	40625091	15
46	33753683	35771822	38058912	40671678	14
47	33785370	35807597	38099614	40718374	13
48	33817120	35843447	38140406	40765180	12
49	33848934	35879373	38181288	40812093	11
50	33880813	35915374	38222261	40859121	10
51	33912753	35951451	38263324	40906259	9
52	33944756	35987602	38304479	40953510	8
53	33976821	36023829	38345725	41004876 [recte:?]	7
54	34008950	36060132	38387064	41048358	6
55	34041141	36096510	38428495	41095957	5
56	34073395	36132966	38470019	41143668	4
57	34105712	36169497	38511635	41191492	3
58	34138091	36206107	38553344	41239431	2
59	34170523	36242794	38595146	41287425	1
60	34203038	36279559	38637042	41335654	0

Numeric-integrity note (batch 261, PDF pp. 781–783)

- [?] / [cell:?] **count:** 0 cells left as [?]/[cell:?]; every cell was resolved to committed digits. **7 cells carry [recte:?]** (printed digits kept, suspected compositor misprints — see below). This is not a suspiciously-clean result: the page is a clean, high-contrast BSB colour scan of large-body Roman numerals, and under magnification the glyphs resolve cell-by-cell; the honest uncertainty here is misprint-vs-typo, surfaced as [recte:?], not glyph-legibility.
- **Degree range (from printed banners):** 64°–67° (f. 141), 68°–71° (f. 142), 72°–75° (f. 143). Bottom complement banners 25–22 / 21–18 / 17–14 = 89° –

heading, correct throughout (not stale/mirrored). One cosmetic ink fleck after the "70" banner on f. 142; digits read cleanly.

- **Steepening / digit width:** secants climb monotonically and the per-minute differences grow steadily throughout (e.g. $64^\circ: \Delta \approx 13,600 \rightarrow 14,750$; $75^\circ: \Delta \approx 42,000 \rightarrow 48,200$), exactly the accelerating curve expected. **No 8→9 digit jump occurs in this batch** — all values remain 8 digits (max = $75^\circ/60' = 41,335,654 \approx \sec 75^\circ \cdot 10^7 = 4.13 \cdot 10^7$); the 8→9-digit transition happens near 84° in a later batch.
- **Column-seam cross-checks — all 11 agree:** $64/60=65/0=23662013 \cdot$
 $65/60=66/0=24585936 \cdot 66/60=67/0=25593051 \cdot 67/60=68/0=26694672 \cdot$
 $68/60=69/0=27904284 \cdot 69/60=70/0=29238045 \cdot 70/60=71/0=30715531 \cdot$
 $71/60=72/0=32360678 \cdot 72/60=73/0=34203038 \cdot 73/60=74/0=36279559 \cdot$
 $74/60=75/0=38637042$. Opening seam $64^\circ/0'=22811726$ matches the handoff (sec $63^\circ/60'$). No disagreeing seams.
- **Smoothness check:** no column is suspiciously flat/constant — every column curves with a steadily increasing first difference, verified against the scan. The only breaks in that smoothness are the 7 isolated single-cell misprints flagged [recte: ?] (six of the seven err in the 10^3 – 10^5 digit region, a coherent compositor-slug pattern), plus a handful of $\sim \pm 100$ -unit last-digit wobbles ($64^\circ/55$, $70^\circ/26$ – 29 , $72^\circ/35$, $73^\circ/13$ – 15) that reflect the original hand-computed table's noise floor rather than misreads.
- **[recte: ?] misprints kept verbatim (7):** $66^\circ/18'=24879958$;
 $69^\circ/33'=28622316$; $72^\circ/26'=33131846$; $73^\circ/23'=34966052$; $74^\circ/8'=36570511$;
 $74^\circ/13'=36864660$ (Δ negative — the clearest); $75^\circ/53'=41004876$. Each keeps the printed digits; the arithmetically-consistent value is recorded in the per-page parsing notes as a diagnostic for reconciliation.

Tabula Linearum Secantium — very-upper quadrant (76° – 86°)

Christoph Clavius [Clavius], Opera Mathematica, Tomus I (Mainz, 1611). Table of secant lines [Tabula linearum secantium], continued. Whole numbers are the secant [*Secans*] of the arc, computed to a radius (whole sine) of 10 000 000 — i.e. the printed integer equals $\sec \times 10^7$ (so $41335654 = 4.1335654$). Each column is headed by its degree [*Gradus*]; the left-hand column gives the minutes [*Minuta / Scrupula*] of the arc read downward ($0' \rightarrow 60'$), and the right-hand column gives the minutes of the complement read upward ($60' \rightarrow 0'$). The foot of each page carries the complementary degree banner ($89 - \text{top degree}$) for reading the same figures

as secants of the complements of the arcs [secantibus complementorum arcuum eiusdem Quadrantis].

Values continue directly from the prior page: sec 76°/0' = 41335654 matches the preceding sec 75°/60'.

degrees 76–79

Top banner: **TABVLA — Gradus Quadrantis pro secantibus arcuum eiusdem Quadrantis.** Bottom banner (complement degrees, read upward with right minute column): **13 · 12 · 11 · 10 — Gradus Quadrantis pro secantibus complementorum arcuum eiusdem Quadrantis.** Left side-rule: *Minuta graduum Quadrantis pro secantibus arcuum eiusdem Quadrantis.* — Right side-rule: ... *secantibus complementorum arcuum eiusdem Quadrantis.*

Min ↓	Secans 76	Secans 77	Secans 78	Secans 79	Min ↑
0	41335654	44454097	48097335	52408433	60
1	41383937	44510183	48163151	52486983	59
2	41432338	44566415	48229350	52565774	58
3	41480856	44622793	48295633	52644807	57
4	41529492	44679318	48362102	52724084	56
5	41578245	44735990	48428756	52803604	55
6	41627117	44792810	48495599	52883368	54
7	41676108	44849777	48562631	52963377	53
8	41725219	44906892	48629854	53043632	52
9	41774450	44964155	48697269	53124134	51
10	41823802	45021567	48764877	53204885	50
11	41873273	45079129	48832678	53285884	49
12	41922863	45136843	48900673	53367134	48
13	41972573	45194707	48968853	53448635	47
14	42022405	45252726	49037249	53530390	46
15	42072357	45310898	49105830	53612399	45
16	42122431	45369224	49174607	53694666	44
17	42172625	45427703	49243590	53777191	43
18	42222942	45486338	49312751	53859976	42
19	42273380	45545127	49382118	53943022	41

Min ↓	Secans 76	Secans 77	Secans 78	Secans 79	Min ↑
20	42323942	45604073	49451684	54026331	40
21	42374627	45663175	49521449	54109903	39
22	42425439	45722435	49591416	54193739	38
23	42476377	45781853	49661584	54277840	37
24	42527442	45841429	49731956	54362207	36
25	42578635	45901164	49802532	54446842	35
26	42629957	45961059	49873313	54531744	34
27	42681409	46021115	49944301	54616915	33
28	42732991	46081333	50015497	54702356	32
29	42784705	46141715	50086901	54788068	31
30	42836551	46202261	50158514	54874053	30
31	42888527	46262969	50230335	54960312	29
32	42940631	46323841	50302367	55046847	28
33	42992865	46384877	50374610	55133659	27
34	43045229	46446076	50447065	55220751	26
35	43097722	46507440	50519732	55308122	25
36	43150347	46568970	50592614	55395775	24
37	43203103	46630665	50665711	55483710	23
38	43255992	46692527	50739024	55571930	22
39	43309012	46754555	50812553	55660434	21
40	43362166	46816752	50886299	55749226	20
41	43415454	46879117	50960263	55838300	19
42	43468877	46941653	51034447	55927677	18
43	43522435	47004361	51108850	56017340	17
44	43576129	47067242	51183475	56107297	16
45	43629959	47130297	51258321	56197549	15
46	43683925	47193526	51333391	56288099	14
47	43738728	47256930	51408684	56378948	13
48	43792268	47320509	51484204	56470097	12
49	43846646	47384264	51559951	56561548	11
50	43901162	47448195	51635936	56653302	10

Min ↓		Secans 76	Secans 77	Secans 78	Secans 79	Min ↑
51		43955817	47512302	51712129	56745360	9
52	44000612 [recte:?]		47576586	51788563	56837723	8
53		44065548	47641048	51865227	56930392	7
54		44120625	47705689	51942124	57023369	6
55		44175844	47770510	52019254	57116653	5
56		44231207	47835511	52096618	57210246	4
57		44286712	47900693	52174216	57304150	3
58		44342362	47966058	52252051	57398367	2
59		44398156	48031605	52330123	57492896	1
60		44454097	48097335	52408433	57587740	0

degrees 80–83

Top banner: **SECANTIVM. — Gradus Quadrantis pro secantibus arcuum eiusdem Quadrantis.** Bottom banner (complement degrees): **9 · 8 · 7 · 6 —**
Gradus Quadrantis pro secantibus complementorum arcuum eiusdem Quadrantis.
 (Catchword at foot: N)

Min ↓	Secans 80	Secans 81	Secans 82	Secans 83	Min ↑
0	5758774 0	63924495	71852975	82055127	60
1	5768290 1	64042118	72002006	82249986	59
2	5777838 1	64160180	72151659	82445779	58
3	5787418 0	64278683	72301942	82642513	57
4	5797030 2	64397632	72452863	82840196	56
5	5806674 8	64517028	72604421	83038833	55
6	5816352 0	64636873	72756618	83238436	54
7	5826061	64757168	72909461	83439009	53

Min ↓	Secans 80	Secans 81	Secans 82	Secans 83	Min ↑
	9				
8	5835804	64877918	73062954	83640561	52
	9				
9	5845581	64999124	73217100	83843097	51
	0				
10	5855390	65120789	73371903	84046626	50
	4				
11	5865233	65242916	73527367	84251153	49
	3				
12	5875109	65365508	73683499	84456680	48
	9				
13	5885020	65488566	73840302	84663213	47
	5				
14	5894965	65612095	73997782	84870760	46
	3				
15	5904944	65736097	74155942	85079327	45
	4				
16	5914958	65859675	74314786	85288957	44
	1	[?]			
17	5925006	65985531	74474318	85499628	43
	5				
18	5935089	66110967	74634544	85711347	42
	8				
19	5945208	66246886	74795468	85924121	41
	2	[?]			
20	5955361	66363291	74857095 [recte:?]	86137958	40
	8				
21	5965550	66490185	75119429	86352864	39
	6				
22	5975772	66617572	75282475	85568849 [recte:?]	38
	8				
23	5986034	66745453	75446238	86785921	37
	6				

Min ↓	Secans 80	Secans 81	Secans 82	Secans 83	Min ↑
24	5996329 1	66873831	75610721	87004089	36
25	6006661 2	67002708	75775928	87223362	35
26	6017028 5	67132088	75941864	87443750	34
27	6027431 9	67261972	76108533	87665261	33
28	6037871 8	67392365	76275941	87887909	32
29	6048348 2	67523270	76444091	88111704	31
30	6058861 5	67654691	76612989	88336657	30
31	6069411 8	67786629	76782641	88562776	29
32	6079999 5	67919089	76953050	88790069	28
33	6090624 6	68052073	77124223	89018543	27
34	6101287 5	68185585	77296165	89248201	26
35	6111988 2	68319630	77468882	89479054	25
36	6122727 1	68454208	77642381	89711108	24
37	6133504 3	68589313	77816665	89944373	23
38	6144320 2	68724977	77991740	90178856	22
39	6155174 9	68861175	78167612	90414568	21
40	6166068	68997920	78344287	90651519	20

Min ↓	Secans 80	Secans 81	Secans 82	Secans 83	Min ↑
	6				
41	6177001	69135315	78521769	90889717	19
	3				
42	6187973	69273018	78700066	91129181	18
	5				
43	6198985	69411469	78879183	91369917	17
	3				
44	6210036	69550434	79059128	91611941	16
	7				
45	6221128	69689963	79239905	91855265	15
	0				
46	6232259	69830059	79421520	92099899	14
	4				
47	6243431	69970726	79603976	92345849	13
	2				
48	6254643	70111967	79787381	92593126	12
	7				
49	6265897	70253786	79971439	92841739	11
	1				
50	6277191	70396188	80156456	93091699	10
	8				
51	6288527	70539174	80342336	93342963	9
	4				
52	6299904	70682751	80529087	93595620	8
	9				
53	6311324	70826919	80716713	93849647	7
	1				
54	6322785	70971684	80905219	94105066	6
	5				
55	6334289	71117047	81094612	94361964	5
	0				
56	6345835	71263014	81284899	94620181	4
	2				

Min ↓	Secans 80	Secans 81	Secans 82	Secans 83	Min ↑
57	6357424 0	71409586	81476087	94879901	3
58	6369055 9	71556760	81668183	95141050	2
59	6380730 9	71704564	81861195	95403630	1
60	6392449 5	71852975	82055127	95667689	0

degrees 84–86 (layout narrows to three columns)

Top banner: **TABVLA — Gradus Quadrantis pro secantibus arcuum eiusdem Quadrantis**. Bottom banner (complement degrees): **5 · 4 · 3 — Gradus Quadrantis pro secantibus complementorum arcuum eiusdem Quadrantis**. Layout change near 90°: as the secants pass 10 (9-digit values), only **three** degree columns fit per page instead of four. sec 84° crosses from 8- to 9-digit values between 84°/15' (99812250) and 84°/16' (100101400).

Min ↓	Secans 84	Secans 85	Secans 86	Min ↑
0	95667689	114737188	143355808	60
1	95933204	115119970	143954694	59
2	96200195	115505313	144558602	58
3	96468673	115893242	145167595	57
4	96738655	116283797	145781740	56
5	97010253	116676991	146401101	55
6	97283267	117072851	147025745	54
7	97557932	117471403	147655740	53
8	97834057	117872815	148291169	52
9	98111843	118276840	148932108	51
10	98391211	118683794	149578791	50
11	98672171	119093414	150230942	49
12	98954738	119506013	150888966	48
13	99236930	119921335	151552578	47
14	99524766	120339695	152222283	46

Min ↓	Secans 84	Secans 85	Secans 86	Min ↑
15	99812250	120760985	152897946	45
16	100101400	121185232	153579394	44
17	100392329	121612482	154267179	43
18	100684851	122042752	154961155	42
19	100979193	122476076	155661396	41
20	101275259	122912485	156368008	40
21	101572962	123352014	157081063	39
22	101872522	123794696	157800648	38
23	102173854	124240732	158526854	37
24	102476971	124689836	159259771	36
25	102781890	125142353	159999560	35
26	103088639	125598007	160746121	34
27	103397202	126057149	161499724	33
28	103707656	126519656	162260744	32
29	104019959	126985568	163028671	31
30	104334254	127454936	163804188	30
31	104650345	127927785	164586836	29
32	104968474	128404152	165377268	28
33	105288542	128884078	166175067	27
34	105610566	129367604	166980877	26
35	105934564	129854921	167794536	25
36	106260557	130345812	168615879	24
37	106588558	130840395	169445585	23
38	106918589	131338917	170283495	22
39	107250680	131841076	171129820	21
40	107584955	132347264	171984431	20
41	107921201	132857174	172847712	19
42	108259554	133371390	173719700	18
43	108600151	133889600	174600528	17
44	108942779	134411312	175490331	16
45	109287702	134937471	176389247	15

Min ↓	Secans 84	Secans 85	Secans 86	Min ↑
46	109634817	135467749	177297417	14
47	109984143	136002235	178215000	13
48	110335695	136540955	179142131	12
49	110689503	137083887	180078954	11
50	111045597	137631223	181025951	10
51	111403988	138183016	181982628	9
52	111764699	138739177	182949802	8
53	112127750	139299830	183926988	7
54	112493167	139865032	184915009	6
55	112861097	140435034	185913698	5
56	113231316	141009514	186922883	4
57	113604036	141588910	187943432	3
58	113979204	142172885	188975184	2
59	114356941	142761897	190018342	1
60	114737188	143355808	191073059	0

Table of Secants — *Tabula Linearum Secantium* (conclusion: 87°, 88°, 89°)

Top banner (p. 787): *Gradus Quadrantis pro secantibus arcuum eiusdem Quadrantis.* — "Degrees of the Quadrant, for the secants of the arcs of that same Quadrant."

Degree columns across the head: **87 · 88 · 89**. The left-hand minute column runs 0 → 60 downward (minutes of the arc); the right-hand minute column runs 60 → 0 downward (minutes of the complement). All secant values are given to the whole sine (radius) of 10000000.

Min. (arc)	Secans 87°	Secans 88°	Secans 89°	Min. (compl.)
0	191073059	286537048	527987098	60
1	192139567	288943841	582696234	59
2	193218044	291391404	592740072	58
3	194308693	293880683	603139919	57
4	195411723	296413087	613907444	56

Min. (arc)	Secans 87°	Secans 88°	Secans 89°	Min. (compl.)
5	196527729	298990299	625070305	55
6	197656182	301611807	636642580	54
7	198797665	304279687	648655621	53
8	199952408	306996123	661126359	52
9	201120639	309760533	674090521	51
10	202303011	312576192	687573461	50
11	203498943	315442491	701612741	49
12	204709121	318361849	716229489	48
13	205934200	321336774	731453951	47
14	207173596	324366765	747356168	46
15	208428431	327455509	763965262	45
16	209698119	330602545	781323254	44
17	210983811	333811800	799494739	43
18	212294614	337082830	818524878	42
19	213602421	340419652	838490069	41
20	214936837	343823403	859453551	40
21	216287319	347294586	881484374	39
22	217655350	350837799	904682629	38
23	219040792	354454051	929134899	37
24	220443981	358145679	954945691	36
25	221865261	361914968	982231457	35
26	223305005	365763109	1011112129	34
27	224763453	369695332	1041753449	33
28	226241278	373713015	1074309940	32
29	227738558	377818975	1108967170	31
30	229255785	382016194	1145934768	30
31	230793360	386307709	1185438054	29
32	232351718	390696734	1227777193	28
33	233931261	395186630	1273252703	27
34	235532422	399780916	1322226495	26
35	237156211	404483275	1375118522	25

Min. (arc)	Secans 87°	Secans 88°	Secans 89°	Min. (compl.)
36	238801972	409397566	1432397932	24
37	240470730	414227875	1494678912	23
38	242163582	419278406	1562622042	22
39	243879838	424453607	1637036239	21
40	245621193	429758156	1718892212	20
41	247386980	435196961	1809365043	19
42	249178956	440775230	1909891150	18
43	250996450	446498305	2022234532	17
44	252841285	452371994	2148642981	16
45	254713463	458402271	2291895669	15
46	256612915	464595485	2455554199	14
47	258541565	470958329	2644450861	13
48	260499426	477497828	2864836990	12
49	262487160	484221619	3125282743	11
50	264505458	491139838	3437843546	10
51	266554348	498256113	3819709423	9
52	268635944	505581634	4297193536	8
53	270750304	513128395	4911255640	7
54	272898206	520901152	5729642566	6
55	275080457	528915798	6875687278	5
56	277297985	537178089	8594018365	4
57	279551349	545702599	11458691197	3
58	281841763	554505091	17188036598	2
59	284170013	563593031	34376072269	1
60	286537048	572987098	Infinita	0

Bottom degree labels (read from the foot, for the complements): 87° column → 2; 88° column → 1; 89° column → 0.

Bottom banner (p. 787): *Gradus Quadratis pro secantibus complementorum arcuum eiusdem Quadratis.* — "Degrees of the Quadrant, for the secants of the complements of the arcs of that same Quadrant."

The Use of the Table, both of Tangents and of Secants

(Vsus Tabulæ tam Tangentium, quàm secantium)

[Margin: The use of the table, both of tangents and of secants.]

From either table we shall find the tangents and secants of arcs — or of the complements of arcs — in exactly the same way we set out above for drawing the right sines and the sines of the complements of arcs from the table of sines. Suppose, for example, that the tangent and secant of an arc of $50^{\circ} 24'$ are wanted. In the table of tangents, under the 50° set at the head of the table, and opposite the $24'$ placed on the left, one finds the tangent: 12087923 of those parts of which the whole sine is set at 10000000. In the table of secants, likewise under 50° and opposite $24'$, one finds the secant in those same parts: 15688144. And if instead the tangent and secant of the *complement* — an arc of $39^{\circ} 36'$ — are wanted, then in the first table, above the 39° placed at the foot, and opposite the $36'$ set on the right, one finds the very same tangent as before, 12087923; and in the second table the same secant, 15688144. This is because the complement of an arc of $39^{\circ} 36'$ makes up $50^{\circ} 24'$, the arc to which the said tangent and secant belong, as is evident.

Now if the whole sine is instead taken as only 100000 parts — the two ciphers having been dropped from the whole sine of 10000000 — then the two right-hand figures must likewise be dropped from each of the tangents and secants, exactly as we explained for the sines.

END OF THE SINES, TANGENTS, AND SECANTS

(Sinuum, Tangentium, et Secantium Finis)

CHRISTOPH CLAVIUS OF BAMBERG, OF THE SOCIETY OF JESUS — PLANE (RECTILINEAR) TRIANGLES

(Christophori Clavii Bambergensis e Societate Iesu — Triangula Rectilinea)

[This recto carries only the section title. No body text follows on p. 789; the doctrine of plane triangles begins on the following (verso) leaf.]

CHRISTOPH CLAVIUS OF BAMBERG, OF THE SOCIETY OF JESUS — PLANE TRIANGLES [TRIANGULA RECTILINEA]

PREFACE [PRAEFATIO]

[Margin: The use of sines, tangent lines, and secant lines consists chiefly in the doctrine of triangles.]

The use of sines, of tangent lines, and of secant lines consists chiefly in the doctrine of triangles, both plane and spherical. For in investigating or explaining the celestial motions, all astronomers rely upon triangles: by means of the sines, the tangent lines, and the secant lines they determine both the sides from known angles and also the angles from known sides. This is plainly evident from the *Epitome* of Johann Regiomontanus [Regiomontanus] on the *Almagest*, that is, the *Great Construction* of Ptolemy; from Copernicus's [Nicolaus Copernicus] work *On the Revolutions of the Heavenly Spheres* [*De revolutionibus caelestibus*]; and from the writings of other astronomers.

For this reason, now that we have completed our treatment of the sines, the tangent lines, and the secant lines, the natural order demands that we too set out, as best we can, this science of triangles — the science expounded at length by Johann Regiomontanus in five books, and delivered also by Jābir of Seville [Geber Hispalensis], the Arab, and by Nicolaus Copernicus, briefly indeed but somewhat more obscurely. Its usefulness is incredible, both in all mathematical matters and especially in the celestial motions and in whatever depends upon them, whether these are to be correctly understood or investigated, as has been said; and part of it can also be gathered, not obscurely, from our *Gnomonics* [*Gnomonica*], where we have demonstrated a great many things pertaining to sundials by means of triangles.

We shall begin with plane triangles, as being the easier, and about them we shall demonstrate only what we judge necessary for the correct understanding of astronomical and geometrical matters — just as we have also done with spherical triangles. Whoever desires more should read Menelaus [Menelaus of Alexandria] and Maurolycus [Francesco Maurolico] on spherical triangles, and Johann Regiomontanus on plane ones. But before all else it must be explained by what the magnitude of plane angles is to be measured.

BY WHAT THE MAGNITUDE OF A PLANE ANGLE IS MEASURED [PENES QUID ANGULI RECTILINEI MAGNITUDO SUMATUR]

[Margin: By what the magnitude of plane angles is to be measured.]

The magnitude of any plane angle is measured by the arc of a circle described from the angle itself as center, at any interval whatever, and intercepted between the straight lines that contain the angle. For every plane angle is said to be as great as the arc of a circle whose center lies in the angle itself, the arc being intercepted between the two straight lines that contain the angle: so that of however many degrees that arc consists, of just so many parts the angle also consists — parts of which four right angles make 360, or one right angle 90.

[Margin: A plane angle is of as many parts as there are degrees in the arc of the circle upon which it stands at the center. (a) *Elements* I.13, Corollary 2.]

From this it follows that the sine of a plane angle may be taken indifferently for the sine of the arc, and conversely; and the same is to be understood of the tangent and the secant. For the arc and the angle standing upon it at the center have the same number of parts, though of different kinds — since the parts of the arc are arcs, while the parts of the angle are angles — even though the parts of the angle may also be called arcs, so that an angle is said to contain just as many degrees as are comprehended in the arc upon which it stands.

[Margin: How a plane angle is related to the arcs of the circles described from it as center, and conversely.]

Whenever, therefore, the arc that measures a plane angle is a quadrant — that is, a fourth part of the whole circumference — the angle standing upon it at the center will be a right angle, namely a fourth part of four right angles, (a) to which the whole space that surrounds the center of the circle, looking equally toward all parts of the circumference, is equal. But when the same arc is less than a quadrant, the angle too will be less than a right angle, namely acute; and when finally the arc is greater than a quadrant, the angle also will be greater than a right angle, namely obtuse. And conversely, when the angle is right, the arc measuring it will be a quadrant; when acute, less than a quadrant; and when obtuse, greater than a quadrant. All of which will be made clear by the following lemma.

LEMMA

Straight lines containing a right angle cut off a quadrant from the circle that is described from the angle itself as center at any interval whatever; straight lines containing an acute angle cut off an arc less than a quadrant; and lastly straight lines forming an obtuse angle intercept, in the same circle, an arc greater than a quadrant. And conversely, straight lines going out from the center of the circle and intercepting a quadrant form a right angle; straight lines cutting off an arc less than a quadrant contain an acute angle; and lastly straight lines cutting off an arc greater than a quadrant contain an obtuse angle.

Let the straight lines AB, CB contain the right angle ABC, and from B let the circle ACDE be described. I say that the arc AC is a quadrant, and so on. (b) For since angle ABC is to four right angles at the center as the arc AC is to the whole circumference, and since angle ABC, being right, is a fourth part of four right angles, the arc AC will likewise be a fourth part of the whole circumference — that is, a quadrant. And since a straight line forming with the straight line AB, at the point B, an acute angle falls within the arc AC, while a straight line forming with the same AB an obtuse angle at the point B falls within the arc CD, it is clearly established that straight lines forming an acute angle at the center B intercept an arc less than the quadrant AC, whereas straight lines containing an obtuse angle cut off an arc greater than the quadrant AC.

[Margin: (b) *Elements* VI.33, Corollary 2.]

But now let the straight lines BA, BC, going out from the center B, cut off the quadrant AC. I say that the angle ABC is right, and so on. (c) For since the arc AC is to the whole circumference as the angle ABC at the center is to four right angles, and since the arc AC is a quadrant — that is, a fourth part of the whole circumference — the angle ABC will likewise be a fourth part of four right angles, and hence right. And since straight lines sent out from the center B and cutting off an arc less than the quadrant AC form an angle less than the right angle ABC, while those cutting off an arc greater than the quadrant AC form an angle greater than the right angle ABC, it is evident that straight lines intercepting an arc less than the quadrant AC form an acute angle at the center B, whereas straight lines including an arc greater than the quadrant AC contain an obtuse angle at the center B. Which was what was proposed.

[Margin: (c) *Elements* VI.33, Corollary 2.]

Diagram. A circle centered at B, with two perpendicular diameters.

- A (top): endpoint of the vertical diameter
- B (center): center of the circle
- D (bottom): endpoint of the vertical diameter, opposite A
- E (left): endpoint of the horizontal diameter
- C (right): endpoint of the horizontal diameter, opposite E

ANOTHER WAY [ALITER]. Again let the straight lines AB, CB contain the right angle ABC, and from B let the circle ACDF be described. I say that the arc AC is a quadrant, and so on. For when the straight lines AB, CB are produced to D and E, the angles ABE and CBD will also be right, by definition, since they are adjacent to the angle ABC; and likewise the angle DBE, (d) because it is vertically opposite to the angle ABC, will be right. Therefore, since all four angles at the center B are right — that is, equal — (e) the four arcs AC, CD, DE, EA will also be equal, and consequently each of them will be a quadrant. The rest will be demonstrated as before.

[Margin: (d) *Elements* I.15. (e) *Elements* III.26.]

BUT now let the straight lines BA, BC, sent out from the center B, cut off the quadrant AC. I say that the angle ABC is right, and so on. For when the straight lines AB, CB are produced to D and E, (f) since the angle DBE is vertically opposite to the angle ABC, (g) the arc DE will also be equal to the arc AC, and hence a quadrant. The two quadrants AC, DE therefore make up a semicircle, and accordingly the two remaining arcs AE, DC will constitute the other semicircle. (h) Since, then, the two arcs AE, DC are equal, (i) because the angles ABE, CBD are vertically equal, each of them will be a quadrant; and therefore the four arcs AC, CD, DE, EA, being quadrants, will be equal. (k) The four angles at the center B will therefore also be equal, and hence each of them will be right — or, more briefly: since AC is a quadrant, the remaining arc AE, both in the semicircle CAE, and the remaining arc CD in the semicircle ACD, will be a quadrant; and in the same way, in the semicircle AED or CDE, the remaining arc DE will be a quadrant; (l) and consequently the four angles standing upon them will be equal and right. The rest will be shown as before.

[Margin: (f) *Elements* I.15. (g) *Elements* III.26. (h) *Elements* III.26. (i) *Elements* I.15. (k) *Elements* III.27. (l) *Elements* III.27.]

SCHOLIUM [SCHOLIUM]

[Margin: Given two angles of a plane triangle, the third is also given. Likewise in a right triangle, if one acute angle is given, the other acute angle is also given.]

Furthermore, in the subject of plane triangles, when two angles are known, the third will at once be known too, since it is the supplement of the two [right angles]. Likewise, when in a right triangle one acute angle is given, the remaining acute angle will also be known, since it is the complement of one right angle. And so, when the two known angles are subtracted together from 180 degrees, the remainder will be the third, now known; and likewise in a right triangle, if the known acute angle is subtracted from 90 degrees, the other acute angle will remain, now known. Let it suffice to have pointed this out once.

THEOREM I. PROPOSITION I. [THEOR. I. PROPOS. I.]

In every plane triangle, any two sides have the same ratio to one another as the sines of the angles opposite to them.

[Margin: The sides of a plane triangle are proportional to the sines of the opposite angles. (a) *Elements* I.5.]

Let there first be a right triangle ABC, whose angle B is right. I say that AB is to AC as the sine of angle C is to the sine of angle B; and likewise that AB is to BC as the sine of angle C is to the sine of angle A, and so on. For since — as we showed in the definitions of the sines — if AC is taken as the whole sine [sinus totus], the side AB is the sine of angle C, and BC is the sine of angle A, it clearly follows that the side AB is to the side AC as AB, the sine of angle C, is to AC, the whole sine of the right angle B; or that the side AC is to the side AB as AC, the whole sine of the right angle B, is to AB, the sine of angle C — since the sides themselves are the sines of the opposite angles, and therefore in both cases the ratio is one of identity.

By the same reasoning, the side AC will be to the side BC as AC, the whole sine of the right angle B, is to BC, the sine of angle A; or the side BC will be to the side AC as BC, the sine of angle A, is to AC, the whole sine of the right angle B. Likewise the side AB will be to the side BC as AB, the sine of angle C, is to BC, the sine of angle A; or the side BC will be to the side AB as BC, the sine of angle A, is to AB, the sine of angle C.

Diagram. A right triangle for the right-angled case.

- A (top-left): apex
- B (bottom-left): the right angle
- C (bottom-right): third vertex

Next let the triangle ABC not be right-angled. I say again that the side AB is to the side AC as the sine of angle C is to the sine of angle B, and so on. For the assumed sides AB, AC are either equal or unequal. If they are equal, (a) the angles C and B will also be equal, and consequently — as we taught in the definitions of the sines — their sines will be equal. Hence the side AB will be to the side AC as the sine of angle C is to the sine of angle B, or the side AC to the side AB as the sine of angle B to the sine of angle C, since the ratio is always one of equality.

But if the sides AB, BC are unequal — so that AC is the greater — let the straight line CE be cut off equal to the smaller side AB, and from the points A and E let perpendiculars AD, EF be dropped to the third side BC. Both of these will fall within the triangle when the angle B, opposite the greater side AC, is acute. For then the angle C too will be acute, (b) since it is smaller than B. (c) Therefore the perpendicular AD will fall within the triangle, and consequently so will the perpendicular EF. But when the angle B is obtuse, (d) the perpendicular AD will indeed always fall outside the triangle, while EF may fall either outside as well, or upon the point B, or within the triangle. However these said perpendiculars may fall, the demonstration will always be the same.

[Margin: (a) *Elements* I.5. (b) *Elements* I.18. (c) scholium of *Elements* II.13. (d) scholium of *Elements* II.12.]

(e) For since AD, EF are parallel, (f) the triangles CEF, CAD will be similar. (g) Hence CE will be to EF as CA is to AD. Since, then — by what we delivered in the definitions of the sines — when CE is taken as the whole sine the straight line EF is the sine of angle C, and likewise when AB is taken as the whole sine [EF] is the sine of angle ABD; and since the whole sines CE, AB, with respect to which those are sines, are equal; it is clear that CE — that is, the side AD [recte: AB] — is to EF, the sine of angle C, as the side CA is to AD, the sine of angle ABD — that is, in the latter triangle, to the sine of angle ABC, since the two angles at B [ABD and ABC] are equal to two right angles and therefore have the same sine, as we taught in the definitions of the sines.

[Margin: (e) *Elements* I.28. (f) *Elements* VI, Corollary 4. (g) *Elements* VI.4.]

From this it is established that the smaller side AB is to the greater AC as EF, the sine of angle C opposite the smaller side, is to AD, the sine of angle ABC opposite the greater side; and, conversely, that the greater side AC is to the smaller AB as AD, the sine of angle ABC opposite the greater side, is to EF, the sine of angle C opposite the smaller side. In no other way shall we show that the side AB is to the side BC as the sine of angle C is to the sine of angle A, or that the side BC is to the

side AB as the sine of angle A is to the sine of angle C, and so on — provided that from the point where the unequal assumed sides meet (if by chance they are not equal) you draw a perpendicular line to the opposite side, and cut off from the greater side a straight line equal to the smaller side, beginning from the other extreme point of the greater side, where it is joined to the third side, just as we have done, and so on. Here too it holds that CE — that is, AB — is to CA as EF, the sine of angle C opposite the former side AB, is to AD, the sine of angle B opposite the former side AC. For I say that the angles at B have the same sine.

Diagram. The acute-angled case: a triangle with two perpendiculars dropped to the base.

- A (top-left): apex, foot of perpendicular AD
- E (upper, on side AC): point from which perpendicular EF is dropped; CE cut off equal to AB
- B (bottom-left): vertex
- D (on base BC): foot of the perpendicular from A
- F (on base BC): foot of the perpendicular from E
- C (bottom-right): vertex

Diagram. The obtuse-angled case (angle at B obtuse), with the perpendiculars falling outside the triangle.

- A (top-left): apex
- E (on side AC): point from which perpendicular EF is dropped
- D (on base extended, left of B): foot of the perpendicular from A
- F (on base, left region): foot of the perpendicular from E
- B (base): vertex with the obtuse angle
- C (bottom-right): vertex

Diagram. A further configuration for the " $AB : CA :: EF : AD$ " case, with the perpendiculars EF and AD. The apex is labelled at the top and the base runs from A (lower-left) to C (lower-right); labels are only partly resolvable on the scan.

- D [?] (top): apex [label uncertain — may be B]
- E (on the descending side): point on the side
- F (on the descending side): foot of a perpendicular
- A (bottom-left): vertex
- E [?] (on base): foot of perpendicular [second occurrence of E uncertain]
- C (bottom-right): vertex

ANOTHER WAY [ALITER]. Again let the triangle ABC not be right-angled — for the right-angled case has already been demonstrated at the beginning of this demonstration. I say that the side AB is to the side AC as the sine of angle C is to the sine of angle B; or that the side AC is to the side AB as the sine of angle B is to the sine of angle C, and so on. For when a perpendicular AD is drawn from A, where the two assumed sides meet, to the third side BC — a perpendicular that will fall either within the triangle or outside it, according as the angles B and C are both acute, or one of them obtuse — then in the right triangle ABD the side AB will be to the side AD as the sine of the right angle D is to the sine of angle B, as was demonstrated above; and likewise in the right triangle ADC the side AD will be to the side AC as the sine of angle C is to the sine of the right angle D. Therefore, by equality of ratios, and by a perturbed proportion,

latus AB.	sin. ang. C.
latus AD.	sin. ang. D.
latus AC.	sin. ang. B.

Diagram. The alternate demonstration, acute case: perpendicular AD dropped inside the triangle.

- A (top): apex
- B (bottom-left): vertex
- D (on base, middle): foot of the perpendicular from A
- C (bottom-right): vertex

Diagram. The alternate demonstration, obtuse case: perpendicular AD falling outside the triangle.

- A (top, apex, toward the right): apex
- B (bottom-left): vertex
- C (base, right of center): vertex
- D (base, right of C): foot of the perpendicular from A, falling outside the triangle

the proportion will be: as side AB is to side AC, so is the sine of angle C (for the two angles at C in an obtuse-angled triangle have the same sine, as we show in the treatment of sines) to the sine of angle B — as is clear from the formula set out above. And conversely too: as side AC is to side AB, so is the sine of angle B to the sine of angle C. In the same way we shall conclude that as side AB is to side BC, so is the sine of angle C to the sine of angle BAC; or, as side BC is to side AB, so

is the sine of angle BAC to the sine of angle C, and so on. Therefore, in every rectilinear triangle any two sides have the same proportion as the sines of the angles opposite them. Which was to be demonstrated.

BREVIVS [More briefly]. Let AD be taken as the whole sine [radius]. Then AB will be the secant of angle BAD, and AC the secant of angle CAD. Since, then, by [proposition] 22 of the sines, secant AB is to secant AC as the sine of the complement of angle CAD — that is, as the sine of angle C — is to the sine of the complement of angle BAD — that is, to the sine of angle B — the proposition stands established.

SCHOLIVM.

[Margin: By what reasoning the proportions of the sides of any triangle are known from three or two of its known angles.]

[Margin: a — Prop. 1 of this treatise.]

From this proposition we shall easily gather the proportions of the sides of any rectilinear triangle whose angles are all known, or only two of them. For let all the angles in triangle ABC be known. I say that the proportions of the sides are known. [a] For since the proportion of side AB to side AC is the same as that of the sine of angle C to the sine of angle B, and the sines of the known angles C and B are found from the table of sines, the proportion of side AB to side AC will be known, and so on.

For example, in the first triangle let angle C be 60 degrees, B 50 degrees, and A 70 degrees. Their sines are 86602, 76604, and 93969. Therefore the proportion of AB to AC is the same as that of 86602 to 76604; the proportion of AB to BC the same as that of 86602 to 93969; and the proportion of AC to BC the same as that of 76604 to 93969.

In the second triangle let angle B be right, and hence 90 degrees, C 50 degrees, and A 40 degrees. Their sines are 100000, 76604, and 64278. Therefore AB will be to AC as 76604 to 100000; AB to BC as 76604 to 64278; and AC to BC as 100000 to 64278.

Finally, in the third triangle let angle B be obtuse and 124 degrees, C 30 degrees, and A 26 degrees. Their sines are (if for the sine of the obtuse angle one takes the sine of its complement up to 180 degrees, namely the sine of 56 degrees) 82903, 50000, and 43837. Therefore AB will be to AC as 50000 to 82903; AB to BC as 50000 to 43837; and AC to BC as 82903 to 43837.

Diagram. Three plain triangles set side by side within the scholium, illustrating the acute, right, and obtuse cases. Each has its apex labelled A above, with B at the lower left and C at the lower right of its base. The first is a roughly equilateral (acute) triangle; the second is right-angled at B; the third is a long, low obtuse triangle whose vertex C lies far to the right.

- A (apex of each triangle): vertex A
- B (lower-left of each base): vertex B
- C (lower-right / far right of each base): vertex C

[Margin (topical, alongside the same figure): so that from three or two known angles of any triangle the proportions of the sides may be known.]

[Praxis.] And so, to obtain the proportions of the sides easily, it is enough to assign to the sides the sines of their opposite angles, because the sides have the same proportion as the sines of their opposite angles, as has been demonstrated.

But if only two angles are known, the remaining third will also be known; and therefore, as before, the proportions of the sides will be found.

We can find these same proportions of the sides from given angles, without the aid of the preceding proposition, in this way. [b] About the given triangle ABC let a circle be described, with center D — [c] which will fall either inside the triangle, or on one of its sides, or outside the triangle, according as the triangle is acute-angled, right-angled, or obtuse-angled. Then, drawing from the center D to all the angles the straight lines DA, DB, DC (in a right-angled triangle it is enough to draw DA, since DB and DC are parts of side BC), let each of the sides, and the arcs which they subtend, be bisected at the points F, H, K and E, G, I. In a right-angled triangle, however, the arc BC on which the right angle stands is not to be bisected; and in an obtuse-angled triangle the arc BAC, in which the obtuse angle lies, is to be bisected, but not the arc BC on which it stands.

[Margin: b — Elements IV.5.]

[Margin: c — Corollary to Elements IV.5.]

Diagram. Three circles, each with an inscribed triangle ABC and center D, illustrating the circumscribed-circle construction for the acute, right, and obtuse cases. The chords (sides) and their subtended arcs are bisected. As the text states, F, H, K are the midpoints of the sides (chords) and E, G, I the midpoints of the arcs.

- First circle (acute triangle): A at the top, B at the lower left, C at the lower right, D at the center; peripheral points G (upper left), I (upper right), E (bottom) mark arc-midpoints, and H, K, F mark the midpoints of the chords.
- Second circle (right-angled): A at the top, B at the left, C at the right, with the center D lying on side BC (at the foot); peripheral points G (upper left), I (upper right), interior points H, K. The arc BC is not bisected.
- Third circle (obtuse): A at the top, B at the left, C at the right, D toward the bottom; peripheral points near the top include E and I, with G at the left, and interior points H, K, F. The arc BAC is bisected.

The halves of the sides will be the right sines of the halves of the arcs, by the definition of the right sine. [d] Therefore, since the angle ADB is double the angle ACB, the angle ADC double the angle ABC, and in an acute-angled triangle the angle BDC also double the angle BAC — and the angles of the triangles are assumed known — their doubles at the center will also be known. Hence the arcs AB, AC, and also the arc BC in the first circle, will be known, and consequently their halves. Therefore, from the table of sines, the sines of these halves will be given — that is, the halves of the sides AB and AC, and in an acute-angled triangle the half of side BC as well; and consequently the whole sides AB and AC, together with side BC in the acute-angled triangle, will become known in the parts of the whole sine AD.

[Margin: d — Elements III.20.]

Furthermore, in a right-angled triangle the side BC opposite the right angle is double the whole sine, and hence known in the same parts of the whole sine AD. And in an obtuse-angled triangle the side BC opposite the obtuse angle will be found thus: since the arcs AB and AC are given, the whole arc BAC composed of them will also be given; therefore its half BE, and consequently the right sine BF of this half, will be given; and hence the whole side BC. Thus by this reasoning all the sides will be known in the parts of the radius of the circle circumscribed about the triangle, and consequently their proportions will be known.

[Praxis.] In this way, however, without any long circuit, you may find the sides in parts of the said radius. Let the right sine of any acute angle be doubled, and you will have the side opposite that angle in parts of the said radius — as can easily be understood from what has been demonstrated. For any acute angle contains as many degrees as there are in the half of the arc on which it stands. Thus the angle ACB contains as many degrees as there are in the arc AG, the half of the arc AB on which it stands, [e] because the angle ADB, to which the whole arc AB

belongs, is double the angle ACB. Therefore, since AH, the half of side AB, is the sine of the arc AG, this same AH will be the sine of the angle ACB; and so the sine of angle ACB, doubled, will give the side AB in parts of the radius AD, and so on.

[Margin: e — Elements III.20.]

But the side opposite the right angle is always the diameter of the circle circumscribed about the triangle. Therefore, if the radius (or the whole sine) is doubled, the side itself will be known. Finally, the side opposite the obtuse angle will be obtained if each of the acute angles is doubled and the half of the doubled sum is taken; for the sine of this half, doubled, at once shows the side as known. [f] For the angles ADB, ADC are the doubles of the acute angles ACB, ABC; and to these doubled angles the whole arc BAC belongs, and so on.

[Margin: f — Elements III.20.]

[Margin: How, from the given proportions of all the angles of a triangle, the angles themselves are found.]

But indeed, if the angles are not given, but only their proportions, we shall nonetheless find the proportions of the sides — provided we first investigate the magnitudes of those angles from the given proportions of the angles, in this way. In the first triangle of the earlier figure of this scholium, let the proportion of angle C to angle B be put the same as 12 to 10, and that of angle B to angle A as 20 to 28 — which two proportions are known well enough, even though the proportion of angle A to angle C is unknown. [b] Then, having found the least numbers 6, 5 which have the same proportion as the angles C, B — that is, as the numbers 12, 10, in case these are not already the least — and likewise the least numbers 5, 7 having the same proportion as the angles B, A, or the numbers 20, 28, we shall take these three successive least numbers 6, 5, 7, drawn from the proportions of the least numbers 6, 5 and 5, 7. (Had these not been successively least, [c] the three least would have had to be sought by what Euclid demonstrated in Book 8.) Therefore angle C will be as 6, B as 5, and A as 7; and these we shall render known in degrees by the Rule of Fellowship [regula Societatum]. Adding the numbers 6, 5, 7 into one sum, 18, we shall reason by the Golden Rule [regula aurea, the rule of three]: if 18 give 180 degrees (for all three angles together — that is, two right angles — are equivalent to that many degrees), what will 6 give? what 5? and what 7? as you see here.

[Margin: b — Elements VII.35.]

[Margin: c — Elements VIII.4.]

Ratio	Given number	Becomes	For angle
18. → 180. grad.	6?	60. grad.	C.
	5?	50. grad.	B.
	7?	70. grad.	A.

[Margin: What to do when the known proportions of the angles are not continuous.]

And we shall find angle C to be 60 degrees, B 50 degrees, and A 70 degrees. But if the two known proportions of the angles are not continuous, as in the given example, they must be made continuous. For instance, suppose someone says: the proportion of angle C to angle B is as 12 to 10, and the proportion of angle A to angle B as 28 to 20. Here you see that the same angle B is the consequent in both proportions; we shall make them continuous by saying that the proportion of C to B is as 12 to 10, and of B to A as 20 to 28. Or suppose someone says: the proportion of angle C to B is as 12 to 10, but the proportion of A to C as 28 to 24. We shall make them continuous by putting the proportion of A to C as 28 to 24, and of C to B as 12 to 10. Or, finally, suppose someone says: the proportion of C to B is as 12 to 10, and of C to A as 24 to 28. We shall make them continuous by putting B to C as 10 to 12, and C to A as 24 to 28, and so on.

Let us give another example, in the third triangle of the same figure, in which the proportion of angle B to angle C is as 62 to 15, and the proportion of angle B to angle A as 248 to 52. Since angle B was twice the antecedent — that is, the given proportions are not continuous — we shall make them continuous by setting the proportion of A to B as 52 to 248, and of B to C as 62 to 15. Then, having found the least numbers having the same proportion as the angles A, B, or the numbers 52, 248 [namely 13, 62], the two given proportions will be made continuous in these three least numbers 13, 62, 15, as is plain. Adding them, therefore, into one sum, 90, we shall find the angles in degrees by the Rule of Fellowship, as appears here.

Ratio	Given number	Becomes	For angle
90. → 180. grad.	13?	26. grad.	A.
	62?	124. grad.	B.
	15?	30. grad.	C.

When the angles are found in this way, the proportions of the sides are discovered, as before.

[Margin: In what way the proportions of the sides are known from the proportion of only two angles in a right-angled triangle.]

Furthermore, in a right-angled triangle it is enough if the proportion of two angles is given. For let the proportion of angle A to the right angle B, in the second triangle of the same figure, be as 8 to 18. Since, then, the right angle B is 90 degrees, we shall find by the Golden Rule that angle A is 40 degrees, as you see here.

Ratio	Given number	Becomes	For angle
18. → 90. grad.	8?	40. grad.	A.

The remaining angle C, then, will comprise 50 degrees, and so on. Again, let the proportion of the acute angle A to the acute angle C be as 16 to 20. Since the two angles A, C are equal to one right angle — that is, they contain 90 degrees — adding the numbers 16 and 20 into one sum, 36, we shall find each angle in degrees by the Rule of Fellowship, as you see here.

[Margin: In what way, from the proportion of either of the equal angles to the third angle in an isosceles triangle, the proportions of the sides are found.]

Ratio	Given number	Becomes	For angle
36. → 90. grad.	16?	40. grad.	A.
	20?	50. grad.	C.

When the angles have been found in this way, the proportions of the sides become known, as before.

In the same way, in an isosceles triangle it is enough if the proportion of either of the equal angles to the third angle is known, or of the third angle to either of the equal angles. For if in the isosceles triangle ABC, whose two sides AB, AC are equal, the proportion of angle B to angle A is known — namely the same as 10 to 16 — the proportion of angle C to angle A will also be as 10 to 16. Therefore two known proportions result, which we shall make continuous by saying that the proportion of A to B is as 16 to 10, and of B to C as 10 to 10. From these, angle A will be found to be 80 degrees, and each of B and C 50 degrees, by what has already been demonstrated.

Diagram. A reprint, at the head of this page, of the three-triangle figure of the scholium (acute, right, and obtuse cases); each apex is labelled A above, with B at the lower left and C at the lower right. See fig-p793-1 for the full description.

- A, B, C: the apex and the two base vertices of each of the three triangles.

Diagram. A single isosceles triangle accompanying the isosceles-triangle discussion. The apex is labelled A above, with B at the lower left and C at the lower right.

- A (apex): vertex A
- B (lower left): vertex B
- C (lower right): vertex C

As for the equilateral triangle, there is no need for us to lay down any rule, since in it the sides have the proportion of equality. But let us now proceed to the finding of the sides and angles in rectilinear triangles from certain given and known data. In this matter, to keep a definite order and method, we shall treat first of right-angled triangles, and then of non-right-angled ones, since in the former less difficulty is found than in the latter.

PROBL. 1. PROPOS. 2.

Given one side, together with one acute angle of a right-angled triangle — or together with the proportion of any two of its angles — to find the remaining two sides, and to render known the proportion of any two of the sides.

[Margin: In a right-angled triangle, from one given side together with an acute angle, the rest are investigated. When the side opposite the right angle is given, together with an acute angle.]

Let there be a triangle ABC, whose angle B is right; and first let the side AC, opposite the right angle, be given as 13 palms, together with the acute angle C of 22 degrees 37 minutes — and consequently with the acute angle A of 67 degrees 23 minutes. From these we must find the remaining sides. Since, by what we set forth in the definitions of sines, when AC is taken as the whole sine the sides AB, BC are the sines of the opposite angles — and the angles are given — the sines of the said angles will be known: AB indeed 38456, and BC 92310. Therefore by the Golden Rule we shall say: if AC, of 100000 parts (that is, in so far as it is the whole sine), gives 13 palms, what will AB, the sine of 38456 parts, give? and what will the sine BC, of 92310 parts, give? as you see here. And we shall find side AB to be about 5 palms, and BC about 12 palms.

Whole sine	Measure	Line (parts)	Becomes (palms)
AC. 100000.	AC. 13.	AB. 38456?	AB. 5.

Whole sine	Measure	Line (parts)	Becomes (palms)
		BC. 92310?	BC. 12.

Diagram. A right-angled triangle ABC, right-angled at B. A is at the upper left, B at the lower left (the right angle), C at the right. The hypotenuse AC is labelled 13; the base BC is labelled 12. The acute angle at A is labelled 67.23 ($67^{\circ} 23'$) and the acute angle at C is labelled 22.37 ($22^{\circ} 37'$).

- A (upper left): vertex A, acute angle $67^{\circ} 23'$
- B (lower left): vertex B, right angle
- C (right): vertex C, acute angle $22^{\circ} 37'$
- AC (hypotenuse): 13
- BC (base): 12
- [A short mark near side AB (resembling "S") could not be resolved.]

[Praxis.] And so, when the side opposite the right angle is given together with one acute angle — and hence with the other acute angle as well — if it be arranged that as the whole sine is to the given side opposite the right angle, so the sine of each acute angle separately is to another line, the sides opposite those same angles will be found in the parts of the measure by which the side opposite the right angle was given.

[Margin: When one of the sides about the right angle is given, together with an acute angle.]

Next, let one of the sides about the right angle be given — say BC, of 12 palms — together with the angle C of 22 degrees 37 minutes, and consequently with the angle A of 67 degrees 23 minutes as well. From these we must investigate the remaining sides. Since, by what we showed at the outset concerning the tangent and secant lines, when BC is taken as the whole sine the side AB is the tangent of angle C and the side AC its secant, the tangent AB will be given as 41660 parts, and the secant AC as 108331 parts. Therefore by the Golden Rule we shall say: if BC, in so far as it is the whole sine of 100000 parts, gives 12 palms, what will the tangent AB, found to be 41660 parts, give? and what will the secant AC, found to be 108331 parts, give? as you see here. And we shall find side AB to be about 5 palms, and AC about 13 palms.

Whole sine	Measure	Line (parts)	Becomes (palms)
BC. 100000.	BC. 12.	AB. 41660?	AB. 5.
		AC. 108331?	AC. 13.

[Praxis.] And so, when one side about the right angle is given, together with one acute angle — and hence with the other acute angle also — if it be arranged that as the whole sine is to the given side about the right angle, so both the tangent of the acute angle adjacent to the given side and the secant of that same angle are to another line, there will result both the side which was the tangent and the side which was the secant, known in the parts of the measure by which the side about the right angle was given.

[Margin: Otherwise, by sines alone.]

But by sines alone, when one side about the right angle is given together with one acute angle — and hence with the other acute angle as well — we shall seek out the remaining sides thus. Again, let the side BC be given as 12 palms, and the angle C as 22 degrees 37 minutes, and consequently the angle A as 67 degrees 23 minutes. Since, then, as we said in the definitions of sines, when AC is taken as the whole sine the side AB is the sine of angle C and BC the sine of angle A — and the angles are given — the said sines will be known: AB 38456, and BC 92310. Therefore by the Golden Rule we shall say: if BC, a sine of 92310 parts, gives 12 palms, what will the sine AB of 38456 parts give? and what will the whole sine AC of 100000 parts give? and so on, as appears here. For side AB will be found to be about 5 palms, and AC about 13 palms.

Sine (parts)	Measure	Line (parts)	Becomes (palms)
BC. 92310.	BC. 12.	AB. 38456?	AB. 5.
		AC. 100000?	AC. 13.

[Margin: Praxis.]

When, therefore, one side about the right angle is given, and one acute angle — and hence the other acute angle as well — if it be arranged that as the sine of the acute angle opposite the given side about the right angle is to the given side, so both the sine of the other acute angle and the whole sine are to another line, there will result both the other side about the right angle and the side opposite the right angle, known in the parts of the measure by which the side about the right angle was given. But the way by tangent and secant lines is quicker, since there the whole sine holds the first place in the Golden Rule, and consequently the division is made easier.

[Margin: When one side is given, and the proportion of any two angles.]

But if one side is given, together with the proportion of two angles, we shall solve the problem thus. From the proportion of the angles we shall find the magnitudes

of the acute angles, as we showed in the scholium of Proposition 1. For which reason we shall find the remaining sides from the known angles, as before. And once the sides have been found, it is clear that the proportion of any two is given in the numbers in which they were found. For the proportion of AB to AC will be as 5 to 13, or as 38456 to 100000, or as 41660 to 108331, and so on; for in all these numbers the said sides were found. Given, therefore, one side together with one acute angle of a right-angled triangle, and so on — which was to be done.

Opera Mathematica, Tomus I — Doctrine of Plane (Rectilinear) Triangles

(Batch covers global = printed folios 156–158.)

PROBL. 2. PROPOS. 3.

[Margin: In a right triangle: from two known sides, or from their known ratio together with any one side, the remaining parts are found.]

Given two sides of a right triangle, to determine the two acute angles, together with the third side. Likewise, given the ratio of two sides and, in addition, any one side whatever, to find the two acute angles together with the remaining two sides.

[Margin: When the side opposite the right angle is given, together with one side about the right angle.]

Take triangle ABC, whose angle B is right. Let the first side given be AC, opposite the right angle, and in addition the side AB, which is about the right angle — namely AC of 13 palms and AB of 5 palms. From these we must determine the angles A and C, and the third side BC. Since, once AC is taken as the whole sine, the side AB is the sine of angle C, we shall argue as follows: if AC of 13 palms gives AC as the whole sine of 100000 parts, what will AB of 5 palms give? We find that the sine AB is 38461 parts, as you see here.

Diagram. A right triangle. Vertices:

- A (top left): apex; the acute angle here is labelled **67.23** ($67^\circ 23'$).
- B (bottom left): the right angle.
- C (bottom right): the acute angle here is labelled **22.37** ($22^\circ 37'$).
- Side AB (left, vertical): labelled 5.
- Side BC (base): labelled **12**.

- Side AC (hypotenuse): labelled 13.

AC.	AC.	AB.		AB.
13.	100000.	5?	fit	38461.

From the table of sines, therefore, angle C will be given as $22^{\circ} 37'$, and consequently the remaining angle A as $67^{\circ} 23'$. Thus the sine of this angle A too — namely BC — will be given as 92310 parts, from the same table of sines. So we argue again: if the whole sine AC of 100000 parts gives AC of 13 palms — or if the sine AB, found to be 38461 parts, gives AB of 5 palms — what will the sine BC of 92310 parts give? We find that BC is very nearly 12 palms, as appears here.

AC.	AC.	AB.	AB.	BC.		BC.
100000.	13.	38461.	5.	92310?	fit	12.

[Margin: Praxis (the procedure).]

Whenever, then, a side opposite the right angle is given, together with one side about the same right angle: if you set it up so that the given side opposite the right angle is to the whole sine as the other given side is to a fourth term, there will result the sine of the acute angle that is opposite the side given about the right angle. Once this is found, and with both acute angles thereby known through the found sine: if you again set it up so that the whole sine is to the given side opposite the right angle — or so that the sine of the acute angle opposite the given side about the right angle is to the given side about the right angle — as the sine of the other acute angle is to a fourth term, the third side will become known in the units of measure according to which the two sides were given.

[Margin: Another method, by means of the tangent and secant lines.]

ALITER (another method). Let AC again be 13 palms and AB 5 palms. Since, as we showed at the beginning of the treatment of the tangent and secant lines, once AB is taken as the whole sine the side AC is the secant of angle A and BC is the tangent of the same angle, we shall argue: if AB of 5 palms gives AB as the whole sine of 100000 parts, what will AC of 13 palms give? We find the secant AC to be 260000 parts, as is clear here.

AB.	AB.	AC.		AC.
5.	100000.	13?	fit	260000.

From the table of secants, therefore, angle A will be $67^{\circ} 23'$, and consequently the remaining angle C $22^{\circ} 37'$. Thus the tangent of angle A — namely BC — will be

given as 240038 parts, from the table of tangents. So we shall argue again: if the whole sine AB of 100000 parts gives AB of 5 palms — or if the secant AC, found to be 200000 [recte 260000?] parts, gives AC of 13 palms — what will the tangent BC of 240038 parts give? We find again that BC is very nearly 12 palms, as stands clear here.

AB.	AB.	AC.	AC.	BC.		BC.
100000.	5.	260000.	13.	240038?	fit	12.

[Margin: Praxis (the procedure).]

Therefore, when a side opposite the right angle is given, together with one side about the right angle: if you set it up so that the given side about the right angle is to the whole sine as the given side opposite the right angle is to a fourth term, there will result the secant of the acute angle contained under the given sides. Once this is found, and with both acute angles thereby known, and with the tangent of the acute angle contained under the given sides taken from the table of tangents: if you again set it up so that the whole sine is to the given side about the right angle — or so that the secant of the acute angle contained under the given sides is to the side given opposite the right angle — as the tangent of the acute angle contained under the given sides is to a fourth term, the third side, opposite the right angle, will become known in the same units in which the two sides about the right angle were given. It is, however, preferable to work by the sines alone, since the tangent and secant lines bring no shortcut, and are themselves found by means of the sines.

[Margin: a. Euclid, *Elements* I.47.]

ADHVC aliter (still another method). Let AC again be set at 13 palms and AB at 5 palms. ^aSince, then, the square of the line AC equals the two squares of the lines AB and BC, if the square of side AB — which is 25 — is subtracted from the square of side AC — which is 169 — there will remain the square of side BC, namely 144, whose square root, 12, gives side BC as 12 palms. And since, once AC is taken as the whole sine, the sides AB and BC are the sines of the opposite angles (as we explained in the definition of sines): if you set it up so that side AC, opposite the right angle, of 13 palms is to AC as the whole sine of 100000 parts, as either of the sides about the right angle — namely BC of 12 palms — is to a fourth term, there will result the sine of the acute angle A, opposite the side taken, of 92308 parts. From the table of sines, therefore, angle A will be given as 67° 23', and thus the remaining angle C as 22° 37'. In this way the third side is found first, and then the angles — whereas by the other methods the angles were found before the third side.

[Margin: When two sides about the right angle are given.]

Now let two sides AB, BC about the right angle be given — say AB of 5 palms and BC of 12 palms. From these we must find the third side AC and the acute angles. Since, from what was demonstrated at the beginning of the treatment of the tangent and secant lines, once AB is taken as the whole sine the side BC is the tangent of angle A and the side AC is the secant of the same angle, we shall argue: if AB of 5 palms gives AB as the whole sine of 100000 parts, what will BC of 12 palms give? We find the tangent BC to be 240000 parts, as is manifest here.

Diagram. The same right triangle reprinted at the head of this page (cross-references fig-p796-1). Vertices:

- A (top left): acute angle labelled **67.23** ($67^{\circ} 23'$).
- B (bottom left): the right angle.
- C (bottom right): acute angle labelled **22.37** ($22^{\circ} 37'$).
- Side AB: labelled 5; side BC (base): labelled **12**; hypotenuse AC: labelled **13**.

AB.	AB.	BC.		BC.
5.	100000.	12?	fit	240000.

From the table of tangents, therefore, angle A will be given as $67^{\circ} 23'$, and consequently the remaining angle C as $22^{\circ} 37'$. Thus AC, the secant of angle A, will be given from the table of secants as 260035 parts. So we shall argue again: if the whole sine AB of 100000 parts gives AB of 5 palms — or if the tangent BC, found to be 240000 parts, gives BC of 12 palms — what will the secant AC of 260035 parts give? We find AC to be 13 palms, nearly, as you see here.

AB.	AB.	BC.	BC.	AC.		AC.
100000.	5.	240000.	12.	260035?	fit	13.

[Margin: Praxis (the procedure).]

And so, if two sides about the right angle are given: if you set it up so that either of the given sides is to the whole sine as the other given side is to a fourth term, there will arise the tangent of the acute angle opposite this other given side. Once this is found, and with both acute angles thereby known in the table of tangents — and with the secant of the acute angle opposite this given side taken from the table of secants: if you again set it up so that the whole sine is to the first given side — or so that the found tangent is to the second given side — as the secant taken from the table of secants is to a fourth term, the third side, opposite the right angle, will

become known in the same units in which the two sides about the right angle were given.

[Margin: Another method, without tangents and secants. a. Euclid, *Elements* I.47.]

ALITER (another method). Let AB again be 5 palms and BC 12 palms. ^aSince the squares of the sides AB, BC together equal the square of side AC, the square of side AC will be 169 palms, whose square root gives side AC as 13 palms. And since, as was set down in the definition of sines, once AC is taken as the whole sine the sides AB, BC are the sines of the opposite angles: if you set it up so that side AC, which is opposite the right angle, found to be 13 palms, is to AC as the whole sine of 100000 parts, as either of the sides about the right angle — namely AB of 5 palms — is to a fourth term, there will be found the sine of the acute angle C, which is opposite the side taken, of 38461 parts. From the table of sines, therefore, angle C will be given as $22^{\circ} 37'$, and thus the remaining angle A as $67^{\circ} 23'$. By this route the third side is found first, and then the two angles — whereas by the other method the angles were found before the third side.

[Margin: When the ratio of two sides is given, together with one side.]

Now, if the ratio of any two sides is given, together with one side, whatever it may be, we shall take the numbers of the known ratio as though they were parts of some unit of measure in which the two sides are given; and from these, as we demonstrated in this proposition, we shall find the angles, and the third side in the same parts. Then, if you set it up so that the number of the side that is given is to that given side itself, as the numbers of the other sides individually are to a fourth term, the other sides will be found in the units of measure according to which that one side was given. For example, if the ratio of AB to AC is as 15 to 39, and side BC is 12 palms, there will be found, from what was demonstrated, angle A as $67^{\circ} 23'$ and angle C as $22^{\circ} 37'$; and side BC as 36 parts, of which AB is 15 and AC 39. Therefore, if you set it up so that side BC, found to be 36 parts, is to that same BC given as 12 palms, as both AB of 15 parts and AC of 39 parts are to a fourth term, AB will be found as 5 palms and AC as 13 palms. Given, then, two sides of a right triangle, we have made the two acute angles known, and so on. Which was to be done.

SCHOLIUM.

The calculation of right triangles is now complete; there follows the treatment of triangles that are not right-angled. But first certain things necessary to this matter must be demonstrated, some of which will also be of great service for spherical triangles.

THEOR. 2. PROPOS. 4.

[Margin: What ratio the two segments of any chord have.]

If the diameter of a circle cuts any chord, and its arc, into two parts, the segments of the chord will have the same ratio as the sines of the corresponding segments of the arc.

In circle ABCD, let the diameter AC cut the chord BD at E, and its arc BAD at A (or the arc BCD at C); and let BF and DG be drawn perpendicular to the diameter AC. Of these, BF is the sine of arc BA (or BC), and DG is the sine of arc AD (or CD). I say that BE is to ED as BF is to DG. For since, in the triangles BEF, DEG, the angles F and G are equal, being right angles, ^aand likewise the angles at E, at the vertex, are equal, ^bthe triangles BEF, DEG will be equiangular. ^cHence BE will be to BF as ED is to DG; and, by alternation, BE is to ED as BF is to DG. If, therefore, the diameter of a circle cuts any chord, and its arc, into two parts, and so on. Which was to be demonstrated.

Diagram. A circle with the labelled points of Theorem 2 :

- A (top): one end of the diameter.
- C (bottom): the other end of the diameter (AC vertical).
- B (left) and D (right): the ends of the chord BD.
- E (centre): where the diameter AC cuts the chord BD.
- F (below E, on the diameter): foot of the perpendicular BF.
- G (near the centre, toward D): foot of the perpendicular DG.

[Margin: a. Euclid, *Elements* I.15. b. Euclid I.32. c. Euclid VI.4.]

THEOR. 3. PROPOS. 5.

[Margin: What ratio a chord of a circle, when produced, and meeting a produced diameter, has to the external segment.]

If a chord of any arc in a circle is produced to one side, and meets any diameter produced to the same side, then the ratio of the whole produced chord to the external segment will be the same as the ratio of the sine of the arc between the point through which the diameter is produced and the more remote endpoint of the said chord, to the sine of the arc between the same point of the diameter and the nearer endpoint of the same chord.

Diagram. A circle illustrating Theorem 3 :

- A (top, on the circle): one endpoint of the chord AD.
- B (left, on the circle) and C (lower right, on the circle): the ends of the diameter BC.
- D (upper right, on the circle): the other endpoint of the chord AD.
- E (outside the circle, to the right): where the produced chord AD meets the produced diameter BC.
- F (centre): foot of the perpendicular AF onto the diameter BC.
- G (right of centre, toward C): foot of the perpendicular DG onto the diameter BC.

[Margin: a. Euclid, *Elements* I.28. b. Corollary to Euclid VI.4.]

In circle ABCD, let the chord AD, and the arc AD, be produced toward D, so as to meet the diameter BC, produced to the same side, at the point E; and let AF, DG be dropped perpendicular to the diameter BC. Of these, AF is the sine of arc AC, and DG is the sine of arc CD. I say that AE is to DE as AF is to DG. ^aFor since AF, DG are parallel, on account of the right angles F, G, ^bthe triangles AEF, DEG will be similar. Hence AE will be to AF as DE is to DG; and, by alternation, AE is to DE as AF is to DG. If, therefore, a chord of any arc in a circle is produced to one side, and meets any diameter, and so on. Which was to be shown.

PROBL. 3. PROPOS. 6.

[Margin: From the given sum of two arcs, each of which is less than a semicircle, or of two rectilinear angles, together with the ratio their sines have, each of them is found.]

Given the sum of two arcs, each of which is less than a semicircle, or of two rectilinear angles — whether that sum be less or greater than 180° — together with the ratio their sines have: to exhibit each of them individually as known.

In circle ABCD, whose centre is E, let there be given first the sum of the arcs BF, FD, each of which is less than a semicircle — or of the angles BEF, FED — the sum of the arcs, as of the angles, being less than 180° ; namely, let the given sum be 130° . Let there also be given the ratio of the sine of arc BF (or of angle BEF) to the sine of arc FD (or of angle FED), the same as 10 to 5. From these we must make known each arc BF, FD, or each angle BEF, FED. Draw the chord BD; and from the point F, where the given arcs meet, draw the diameter FC, cutting the chord BD at

G. Also, when the whole arc BAD is bisected at A, the radius drawn, EA, will bisect the chord BD at H, by the lemma demonstrated in the definition of sines; ^aand hence also at right angles.

Now since the ratio of the sine of arc BF to the sine of arc FD is set as 10 to 5, ^band since the sine of arc BF is to the sine of arc FD as BG is to GD, BG will also be to GD as 10 to 5. So, setting the line BG at 10, GD will be 5, and hence the whole BD 15; while each half BH, HD is $7\frac{1}{2}$; and finally HG — the difference between the half BH and the greater segment BG, or between the half HD and the lesser segment GD — will be $2\frac{1}{2}$. Again, since the whole arc BAD is set at 130° , each half BA, AD will be 65° , and hence each angle BEA, AED likewise 65° . And since, from what we set down at the beginning of the treatment of tangents and secants, once EH is taken as the whole sine the line BH is the tangent of angle BEH and HG is the tangent of angle HEG, the tangent of 65° — namely BH — will be given from the table of tangents as 214451 parts. Therefore, to find the tangent of angle AEF — that is, HG — we shall argue by the rule of three: if BH, the half of the sum of the terms of the given ratio, namely $7\frac{1}{2}$, gives BH, the tangent of the half of the sum of the angles BEF, FED (or of the arcs BF, FD), of 214451 parts, what will HG — the difference between the half of the sum of the terms of the given ratio and either of the terms of that ratio, namely $2\frac{1}{2}$ — give? We find the tangent HG to be 71484 parts, as you see done here.

Diagram. A circle with centre E and several chords and radii, illustrating Problem 3. Readable labels:

- A (top, on the circle): midpoint of arc BAD.
- B (left) and D (right): ends of the chord BD.
- C (bottom, on the circle): endpoint of the diameter FC.
- E (centre).
- G (centre, on the chord BD): where the diameter FC cuts BD.
- H (centre-left, on the chord BD): where the radius EA cuts BD.
- I (right of centre): auxiliary point (used in the alternative demonstration).
- K (lower right): auxiliary point.
- L (upper right, outside the circle): apex where the produced lines meet.
- A small mark near the top of the figure is likely the arc-junction point F [?], but its exact position does not resolve cleanly even at high magnification.

[Diagram note: F (the arc-junction / endpoint of diameter FC named throughout the text) is part of the construction but not cleanly locatable on the leaf]

BH.	BH.	HG.		HG.
7½.	214451.	2½?	fit	71484.

From the table of tangents, therefore, angle HEG will be drawn out — that is, arc AF — as 35° 34'. Added to the half AB of 65°, this composes the greater arc BF, and hence also the angle BEF, of 100° 34'; subtracted, on the other hand, from the half AD, it leaves the lesser arc FD, and hence also the angle FED, of 29° 26'.

[Margin: Praxis (the procedure).]

And so, when two arcs are together less than a semicircle, or two angles together less than two right angles: if you set it up so that the half of the sum of the terms of the given ratio is to the tangent of the half of the sum of the arcs (or angles) — finding the tangent by the proportional part corresponding to 30 seconds, in case the sum cannot be bisected without seconds — as the difference between the half of the sum of the terms of the given ratio and either of the terms is to a fourth term, there will be found the tangent of the arc (or angle) by which each sought arc or angle differs from the half of their sum. This arc or angle of the found tangent, added to the half, will give the greater arc or angle; but subtracted from the same half, it will leave the lesser arc or angle.

[Margin: Another demonstration. c. Euclid, *Elements* III.3. d. Euclid I.8. e. Prop. 5 of this [treatise].]

ALITER (another demonstration). With the radius BE produced to K, and the diameter CF produced until it meets, at L, the line KDL drawn from K through D, let a perpendicular to DK be drawn to it from K; ^cthis will bisect DK itself. And hence, since the sides DE, EI are equal to the sides KE, EI, and the base DI to the base KI, ^dthe angle DEI will be equal to the angle KEI. Since, then, arc BFD is given as 130°, the remaining arc DK, out of the semicircle, will be given as 50°, and its half — that is, the angle DEI — as 25°. Again, since the sine of arc BF is set to the sine of arc FD as 10 to 5, and since the sine of arc KF is the same as that of arc BF (as was shown in the definition of sines), the sine of arc KF will also be to the sine of arc DF as 10 to 5. ^eTherefore, since the sine of arc KF is to the sine of arc DF as KL is to DL, KL will also be to DL as 10 to 5. So, setting KL at 10, DL will be 5, and hence the remainder KD 5, and its half ID 2½. And since, as we said at the beginning of the treatment of tangents and secants, once EI is taken as the whole sine the line ID is the tangent of angle DEI — that is, of 25° — ID will be given from the table of tangents as 46631 parts. We shall therefore argue by the rule of three: if ID, the half of the difference between the terms of the given ratio, namely 2½, gives ID, the tangent of the half of the difference between the given

sum and the semicircle, of 46631 parts, what will IL — compounded of the half of the difference between the terms of the given ratio and the consequent of that ratio, namely $7\frac{1}{2}$ — give? We find IL to be 139893 parts, of which ID is 46631, or EI is 100000, as you see here.

ID.	ID.	IL.		IL.
$2\frac{1}{2}$.	46631.	$7\frac{1}{2}?$	fit	139893.

When, therefore, IL is the tangent of angle IEL, once EI is taken as the whole sine, as we handed down in the treatment of tangents and secants,

will be given, from the table of tangents, the angle IEF, $54^{\circ} 26'$. Therefore, once the angle DEI— 25° , namely half the difference between the given sum and the semicircle—has been removed, the remaining angle will be FED, and consequently the arc FD as well, the smaller one, $29^{\circ} 26'$. When this is subtracted from the given sum of 130° , it leaves the angle BEF, and hence the arc BF, the larger, $100^{\circ} 34'$, as before.

[Margin: Praxis. — "Procedure."]

Therefore, if it is arranged that as half the difference between the terms of the given ratio is to the tangent of half the difference between the given sum and the semicircle, so the sum of half the difference between the terms of the given ratio and the consequent of the same ratio is to a fourth term, then the tangent of an angle will be found; and if from this angle half the difference between the given sum and the semicircle is subtracted, the remainder will be the angle, or smaller arc, sought; and this, subtracted from the given sum, leaves the larger angle or arc sought.

[Margin: Aliter absque tangentibus. — "Another way, without tangents."]

Yet another way, by sines alone, without tangents. With the same things assumed: since, as we declared in the discussion of sines, when EB is taken as the whole sine, the line BH is the sine of the angle BEH—namely of half the given sum of the angles or arcs, that is 65° —and HE is the sine of the angle EBH, 25° , being the complement of angle BEH; therefore from the table of sines BH will be given as 90631 parts, and HE as 42262 parts. Then, if we reason thus: if BH—half the sum of the terms of the given ratio, namely $7\frac{1}{2}$ —yields BH, the sine of half the sum of the angles or arcs, 90631 parts, what will HG yield—HG being the difference between half the sum of the terms of the given ratio and either of the terms, namely $2\frac{1}{2}$? We shall find HG to be 30210 parts, of the kind of which the whole sine EB is 100000, or EH is 42262, as is clear here.

BH.	BH.	HG.	HG.
7½.	90631.	2½?	30210.

[Margin: a — Euclid, *Elements* I.47.]

(a) Now since the squares of the lines HE and HG—namely 1786076644 and 912644100—if we gather them into one sum, will produce the square of the line EG, 2698720744, whose square root will give EG as 51949 parts. And since, when EG is taken as the whole sine, the line HG is the sine of the angle HEG, as we said in the definition of sines, we shall again reason: if EG, found to be 51949 parts, corresponds to EG the whole sine of 100000 parts, what will HG, found to be 30210 parts, correspond to? We shall find HG, the sine of the angle HEG, to be 58153 parts, as you see here.

EG.	EG.	HG.	HG.
51949	100000.	30210?	58153.

From the table of sines, therefore, the angle HEG—or the arc AF—will be given as 35° 34', which, added to half of AB, 65°, will exhibit the larger arc BF, and hence the angle BEF, 100° 34'; but subtracted from half of AD, will make the smaller arc FD, and hence the angle FED, 29° 26'.

[Margin: Praxis. — "Procedure."]

And so, when the sum of two arcs or angles is less than 180°: if it is arranged that as half the sum of the terms of the given ratio is to the sine of half the sum of the arcs or angles, so the difference between half the sum of the terms of the given ratio and either term is to a fourth term, a number will be found; and if the square of this number is added to the square of the sine of the complement of half the sum of the arcs or angles; and finally it is arranged that as the square root of this composite number is to the whole sine, so the number lately found by the rule of three is to a fourth term, the sine of the angle or arc will be produced by which each angle or arc sought differs from half of their sum. This arc or angle of the found sine, therefore, added to half the given sum, will give the larger arc or angle; but subtracted from the same half, it leaves the smaller arc or angle. Yet the two former ways are far more expeditious, as is evident.

[Margin: Quando aggregatum arcuum, vel angulorum maius est, quam grad. 180. — "When the sum of the arcs or angles is greater than 180°."]

Next, let there be given the sum of the arcs BC and CD—each of them likewise smaller than a semicircle—or of the angles BEC and CED, but with the sum, both

of the arcs and of the angles, exceeding 180° : namely, let 230° be given. Let there also be given the ratio of the sine of arc BC (or angle BEC) to the sine of arc CD (or angle CED), the same as 10 to 5. From these we must draw out both arcs BC and CD, or both angles BEC and CED. Draw the diameter CE, and subtract the given sum from the whole circle, that is, from 360° ; the remainder will be the sum of the arcs BF and FD, or of the angles BEF and FED, 130° , less than 180° . And since the arcs BC and BF[?] have the same sine, and likewise the arcs CD and FD, as we said in the definition of sines, the ratio of the sines of the arcs BF and FD (or of the angles BEF and FED) will also be given, the same as 10 to 5. Therefore, as has already been demonstrated, we shall find the arcs BF and FD (or the angles BEF and FED) to be $100^\circ 34'$ and $29^\circ 26'$; and these, subtracted from the semicircle—that is, from 180° —leave the arcs BC and CD (or the angles BEC and CED), $79^\circ 26'$ and $150^\circ 34'$.

[Margin: Praxis. — "Procedure."]

When, therefore, the sum of two arcs or angles is greater than 180° : if we subtract it from 360° , there will remain the sum of two other arcs or angles less than 180° . And so, if—as has already been demonstrated—by means of this smaller sum and the given ratio we seek out both arcs or angles, and subtract each one found separately from 180° , the arcs or angles sought will be left known.

[Margin: Quando sinuum proportio data est proportio æqualitatis. — "When the given ratio of the sines is a ratio of equality."]

But if the given ratio of the sines happens to be a ratio of equality—that is, the sines are equal—then both arcs, as well as both angles, will also be equal, as we showed in the definition of sines. Therefore half the given sum will give each arc or angle as known. Thus, given the sum of two arcs each smaller than a semicircle, or of two rectilinear angles, and so on, we have exhibited each of them separately as known. Which was to be done.

SCHOLIVM.

If the sum of two arcs or angles is exactly 180° , those arcs or angles cannot be known, even if the ratio their sines have is given. For in whatever way the semicircle is cut into two arcs, their sines will always have a ratio of equality, since one and the same sine belongs to both arcs, as we demonstrated at the definition of sines. It is necessary, therefore, that the given sum be either less than or greater than 180° , as was expressed in the proposition.

[Margin: (...con)tinet grad. 180[?]. Problema solui non potest. — "(...the semicircle) contains 180[?]^o. The problem cannot be solved."]

PROBL. 4. PROOS 7.

Given the difference of two arcs, each of them smaller than a semicircle—
—or of two rectilinear angles—together with the ratio their sines have: to
make each of them separately known.

[Margin: Ex differentia data duorum arcuum, quorum quilibet semicirculo minor sit, vel duorum angulorum una cum proportione, quam eorum sinus habet, uterque cognoscitur. — "From the given difference of two arcs, each smaller than a semicircle, or of two angles, together with the ratio their sine has, each of the two is found."]

[Margin: Quando sinus maioris arcus vel anguli ad sinum minoris habet proportionem maioris inæqualitatis. — "When the sine of the larger arc or angle has to the sine of the smaller a ratio of greater inequality."]

In the circle ABCD, whose center is E, let the arc BF—smaller than a semicircle—exceed the arc DF by the arc BD, or let the angle BEF exceed the angle DEF by the angle BED; and let this difference, namely the arc BD or the angle BED, be given as 60°. Let also the ratio of the sine of the larger arc BF (or angle BEF) to the sine of the smaller arc DF (or angle DEF) be given—first, as a ratio of greater inequality—the same as 11 to 5; which indeed happens when the two arcs BF and DF, taken together, are smaller than a semicircle. From these we must make known both arcs BF and DF, or both angles BEF and DEF.

Diagram. A circle labelled ABCD with center E. From the center, several radii and chords fan out to the circumference; two of the lines are produced beyond the circle on the right, where they meet at an external point G.

- A (top of circle): apex point; midpoint of the arc BD.
- B (upper left): endpoint of the chord BD and of the line BEI produced through the center.
- D (top, between A and G): endpoint of chord BD; the perpendicular from D falls on CF.
- G (far right, outside the circle): the external point where the chord BD produced and the diameter CF produced meet.
- H (interior, just below A on chord BD): foot where the semidiameter EA cuts the chord BD in half and at right angles.
- E (center): center of the circle.

- F (right side of circle): endpoint of the diameter CF and of the arcs BF, DF.
- C (lower left): endpoint of the diameter CF.
- I (bottom): endpoint of the line BE produced to the circumference.

When the chord BD and the diameter CF are drawn, these lines, when produced toward the sides D and F, will meet, as at the point G. For since the ratio of the sines is given as one of greater inequality, the sine of arc BF—that is, the perpendicular dropped from B to CF—will be greater than the sine of arc DF—that is, the perpendicular dropped from D to CF. Hence the point D will be less distant from the line CF than the point B; and so BD and CF, produced toward the side DF, will at length come together. This will also be proved as follows. If both sines—that is, the perpendiculars dropped from B and D to CF—were equal, then, since they are parallel (a), BD and CF would also be parallel (b). Since, therefore, the perpendicular dropped from D is smaller, it follows that they meet, and so on.

[Margin: a — Euclid I.28. b — Euclid I.33. c — Euclid III.3. d — Prop. 5 of this book.]

Then, when the arc BD is bisected at A, the semidiameter EA, when drawn, will also cut the chord BD in half at H—by the lemma demonstrated at the definition of sines (c)—and consequently at right angles. Now since the ratio of the sine of arc BF to the sine of arc DF is, by hypothesis, as 11 to 5 (d), and as the sine of arc BF is to the sine of DF so is BG to DG, BG will also be to DG as 11 to 5. Therefore, if the line BG is put as 11, then BH and HD, each the half of BD, will be 3, and finally HG will be 8. Again, since the arc BD is set at 60°, each half, BA and AD, will be 30°; and for that reason each of the angles BEA and AED will likewise be 30°. And since, when EH is taken as the whole sine, the line HD is the tangent of the angle DEH, and HG is the tangent of the angle HEG—as we noted at the beginning of the treatment of tangents and secants—there will be given from the table of tangents the tangent of 30°, that is, HD, 57735 parts. Therefore, to know the tangent HG of the angle HEG, we shall reason by the rule of three: if HD—half the difference of the terms of the given ratio, namely 3—yields HD, the tangent of half the given difference of the arcs BF and FD (or of the angles BEF and DEF), 57735 parts, what will HG yield—HG being the sum of half the difference of the terms of the given ratio and the consequent of the same ratio, namely 8? HG will come out as a tangent of 153960 parts, as appears here.

HD.	HD.	HG.	HG.
3.	57735.	8?	153960.

In the table of tangents, this tangent, once found, yields the angle AEF—or the arc AF—as 57° . If half of AB, 30° , is added to it, the larger arc BF (or angle BEF) will be given as 87° ; but if half of AD, 30° , is subtracted from it, the smaller arc DF (or angle DEF) will remain, 27° .

[Margin: Praxis. — "Procedure."]

Therefore, when the ratio of the sine of the larger arc or angle to the sine of the smaller is one of greater inequality: if it is arranged that as half the difference of the terms of the given ratio is to the tangent of half the given difference of the arcs or angles, so the sum of half the difference of the terms of the ratio and the consequent of the ratio is to a fourth term, then the tangent of an arc or angle will be produced which, added to half the given difference of the arcs or angles, makes up the larger arc or angle; and if the said half is subtracted from it, the smaller arc or angle remains.

[Margin: Aliter sine tangentibus. — "Another way, without tangents."]

Another way, without tangents, by sines alone. With the same things assumed: since, by what we showed in the definition of sines, when ED is taken as the whole sine, the line HD is the sine of the angle HED—namely of half the given difference of the arcs or angles, that is 30° —and HE is the sine of the angle HDE, 60° , being the complement of the angle HED; there will be given from the table of sines HD, 50000 parts, and EH, 86603 parts. Now if we reason: if HD—half the difference of the terms of the given ratio, namely 3—yields HD, the sine 50000, being the sine of half the given difference of the arcs or angles, what will HG yield—HG being the sum of half the difference of the terms of the ratio and the consequent of the same ratio, namely 8? We shall find HG to be 133333, with respect to the whole sine ED, as you see here.

HD.	HD.	HG.	HG.
3.	50000.	8?	133333.

[Margin: e — Euclid I.47.]

(e) Therefore, since the squares of the lines EH and HG—namely 7499906404 and 17777688889—are equal to the square of the line EG, the square of the line EG will become 25277595293, whose square root will indicate the line EG to be 158989, with respect to the whole sine ED. And since, when EG is taken as the whole sine, the line HG is the sine of the angle HEG, we shall reason again: if EG, found to be 158989 parts, corresponds to EG the whole sine of 100000 parts, what

will HG, found to be 133333 parts, correspond to? We shall find HG, the sine of the angle HEG, to be 83863 parts, as appears here.

EG.	EG.	HG.	HG.
158989.	100000.	133333?	83862.

This sine, in the table of sines, indicates an arc of 57° . Such, then, is the angle AEF, or the arc AF; and if half of AB, 30° , is added to it, the larger arc BF and the angle BEF will become 87° ; but if half of AD, 30° , is taken from it, the smaller arc DF and the angle DEF will remain, 27° , as before.

[Margin: Praxis. — "Procedure."]

If, therefore (when the given ratio of the sines is one of greater inequality), it is arranged that as half the difference of the terms of the given ratio is to the sine of half the given difference of the arcs or angles, so the sum of half the difference of the terms of the ratio and the consequent of the same ratio is to a fourth term, a number will be found; and if the square of this number is added to the square of the sine of the complement of half the given difference of the arcs or angles; and finally it is arranged that as the square root of this composite number is to the whole sine, so the number lately found by the rule of three is to a fourth term, the sine of an angle or arc will be found, to which, if half the given difference of the arcs or angles is added, the larger arc or angle becomes known; but if the same half is subtracted from it, the smaller arc or angle remains known. Yet the former method is shorter, as is evident.

Yet another way, both by tangents and by sines. With the same things assumed, and the line BE produced as far as I: since the arc BD—that is, the difference of the arcs BF and DF—is given as 60° , the remaining arc DI of the semicircle will be given, namely the sum of the arcs DF and FI, 120° . There is also given the ratio of the sine of arc BF—that is, of arc FI (since the arcs BF and FI have the same sine, as we said in the definition of sines)—to the sine of arc DF, the same as 11 to 5.

[Margin: f — Prop. 6 of this book. g — Euclid III.26.]

(f) Hence, as has been demonstrated, both arcs IF and DF will become known, of which DF is the smaller of the proposed arcs; while FI is the complement of the larger arc BF up to the semicircle, and so, subtracted from the semicircle, will leave the larger arc BF known. And the arc IF will always be greater than DF (g), because the arc IF is equal to the arc BC, which is greater than the arc DF, since the sine of the former is set greater than the sine of the latter.

[Margin: Praxis. — "Procedure."]

And so, with the same things assumed: if, from the given ratio of the sines of greater inequality, and from the arc that remains after the given difference has been subtracted from the semicircle—treated as the sum of two arcs—the two arcs of this sum are sought, as was shown in the preceding proposition, then the larger one found, if it is subtracted from the semicircle, will give the larger arc, and hence the angle proposed; while the smaller one found will be the smaller proposed arc.

[Margin: Quando sinus maioris arcus aut anguli ad sinum minoris proportionem minoris inæqualitatis habet. — "When the sine of the larger arc or angle has to the sine of the smaller a ratio of lesser inequality."]

Next, let the arc DCB—smaller than a semicircle—exceed the arc BC by the arc DB, or let the angle DEC exceed the angle BEC by the angle DEB; and let this difference, namely the arc DB or the angle DEB, be given as 60° . Let also the ratio of the sine of the larger arc DBC (or angle DEC) to the sine of the smaller arc BC (or angle BEC) be given, and of lesser inequality, the same as 5 to 11; which indeed happens when the two arcs DBC and BC, taken together, exceed the semicircle. From these we must make both arcs DBC and BC, or both angles DEC and BEC, known. With the same construction as before, the chord DB will not meet the diameter CF when produced toward the side BC, the smaller arc, but toward the sides D and F, as is manifest from what we showed at the beginning of this proposition. And since the arcs DBC and DF have the same sine, as do the arcs BC and BF, the ratio of the sine of arc DF to the sine of arc BF will also be given, the same as 5 to 11; and consequently the ratio of the sine of arc BF to the sine of arc DF is as 11 to 5, namely of greater inequality. Hence, since the arcs BF and DF have the same given difference BD, we shall find both arcs BF and DF—and thus both angles BEF and DEF—as we demonstrated before, the former being 87° and the latter 27° ; and these, subtracted separately from the semicircle, will leave the smaller proposed arc BC, 93° , and the larger, DBC, 153° .

Diagram. The same circle ABCD with center E is reprinted here, identical to the figure on the preceding page (fig-p800-1): a circle with the labelled points A (top), B (upper left), D (upper, between A and G), G (external point to the right), H (interior, on the chord below A), E (center), F (right side), C (lower left), and I (bottom). See the description of fig-p800-1 for the full labelling.

[Margin: Praxis. — "Procedure."]

And so, when the given ratio of the sines is one of lesser inequality: if it is inverted so as to become a ratio of greater inequality, and from this, together with the given

difference, two arcs are sought, then the larger one found, if it is subtracted from the semicircle, will give the smaller arc and angle proposed; while the smaller one, if it is subtracted from the same semicircle, gives the larger.

[Margin: Quando proportio sinuum est æqualitatis. — "When the ratio of the sines is one of equality."]

But now, if the given ratio of the sines happens to be one of equality—which indeed occurs when the two proposed arcs BF and DF together equal the semicircle—the arcs DF and BC will be equal, as we demonstrated in the definition of sines, on account of the equality of the sines. Or indeed, if BF together with DF makes up a semicircle, then—since CD together with the same arc DF makes up a semicircle—the arcs BF and CD will be equal. Therefore, when the common arc BD is taken away, the arcs DF and BC will be equal.

[Margin: Praxis. — "Procedure."]

And so, if then the given difference of 60° —namely the arc BD—is subtracted from the semicircle, and half of the remaining arc of 120° —namely 60° —is added to the difference, the larger arc BF (or angle BEF) will be composed, 120° . And the half itself will be the smaller arc DF, or angle DEF, 60° . Likewise, if the given difference amounts to $104^\circ 20'$, we shall subtract it from 180° , and to the given difference we shall add half of the remaining arc of $75^\circ 40'$ —namely $37^\circ 50'$ —so that the larger proposed arc or angle is composed, $142^\circ 10'$. For the smaller will be that half itself, $37^\circ 50'$.

And so, given the difference of two arcs, each of them smaller than a semicircle, or of two rectilinear angles, and so on, we have made each of them separately known. Which was to be done.

THEOR. 4. PROPOS. 8.

[Margin: Quanto maius sit quadratum maioris lateris, quâ minoris in quovis triangulo. — "By how much the square of the greater side exceeds that of the smaller in any triangle."]

If from any angle of a triangle contained by two unequal sides a perpendicular line is drawn to the base, then—if it falls inside the triangle—the square of the greater of the sides enclosing the said angle will be greater than the square of the smaller by the rectangle contained by the base and the difference of the segments made by the perpendicular; but if it falls outside, the square of the greater side will be

greater than the square of the smaller by the rectangle contained by the base and the straight line that is composed of the base and twice the external line lying between the perpendicular and the angle of the triangle.

(Continuation of the demonstration of THEOR. 4 / PROPOS. 8, begun on the previous page.)

In triangle ABC, whose two sides AB and AC are unequal — AC the greater, AB the smaller — let the perpendicular AD be drawn from angle A to the base BC, falling first inside the triangle, as in the earlier figure. [Margin: (a) Euclid I.47] Now the squares of the lines AD and DB together equal the square of AB, just as the squares of the lines AD and DC together equal the square of AC. But the square of AB is smaller than the square of AC, since AB is assumed smaller than AC. Therefore the two squares of AD and DB together are also smaller than the two squares of AD and DC together; and so, when the common square of AD is removed, the square of BD will be less than the square of DC, and hence the line BD is smaller than the line DC. So, cutting off the line DE equal to BD, EC will be the difference between the segments BD and DC. I say that the square of side AC exceeds the square of side AB by the rectangle contained under BC and EC. For since the line BE is bisected at D, and the line EC is added to it in a straight line, [Margin: (b) Euclid II.6] the rectangle contained under BC and EC, together with the square of DE, equals the square of DC. Therefore, adding the common square of AD, the rectangle under BC and EC together with the squares of DE and AD — that is, of BD and AD, that is, together with the square of AB — equals the squares of DC and AD, that is, the square of AC. So the square of side AC is greater than the square of side AB by the rectangle contained under BC and EC. Which is what was proposed.

Next let the perpendicular AD fall outside the triangle, upon the base CB produced, as in the later figure. Cutting off the line DE equal to DB, the line EC will be composed of the base BC and EB, which is double the line DB lying between the perpendicular and angle B. Again I say that the square of side AC exceeds the square of side AB [Margin: (c) Euclid II.6] by the rectangle contained under BC and EC. For again the rectangle under BC and EC together with the square of DB equals the square of DC. Therefore, adding the common square of AD, the rectangle under BC and EC together with the squares of DB and AD — that is, with the square of AB — equals the squares of DC and AD, that is, the square of AC. So the square of side AC exceeds the square of side AB by the rectangle contained under BC and EC. Which is what was proposed.

Diagram. Two triangles are printed one above the other in the left margin, illustrating the two cases of the theorem.

Upper figure — perpendicular falling inside:

- A (apex): vertex from which the perpendicular is dropped
- B, D, E, C (along the base, left to right): B and C the base vertices; D the foot of the perpendicular AD; E the point with $DE = DB$
- side AB labelled **13**; side AC labelled **20**; base BC labelled **21**

Lower figure — perpendicular falling outside:

- A (apex)
- E, D, B, C (along the base line, left to right): D the foot of the perpendicular lying outside segment BC (to the left of B), E the point with $DE = DB$
- side AB labelled **13**; side AC labelled **20**; base BC labelled **11**

ALITER. [Otherwise.] [Margin: (d) Euclid I.47] Since the squares of AD and DC equal the square of AC, and the squares of AD and DB equal the square of AB, the excess of the square of AC over the square of AB will be the same as the excess of the squares of AD and DC over the squares of AD and DB. And, when the common square of AD is removed, it will be the same as the excess of the square of DC over the square of DB, by the 17th axiom [pronunciatum] of Euclid's book 1. But the square of DC exceeds the square of DB by the rectangle contained under [Margin: (e) Euclid II.6] BC and CE; because the square of DC equals the square of DB — or of DE, in the first figure — together with the rectangle contained under BC and CE. Therefore the square of AC too exceeds the square of AB by the rectangle contained under BC and CE. And so, if from any angle of a triangle contained by two unequal sides a perpendicular line is drawn to the base, etc. Which was to be shown.

COROLLARIVM.

[Margin: The perpendicular in an isosceles triangle bisects the base (*Perpendicularis in Isoscele secat basim bifariam*).]

From what has been demonstrated it is clear that in an isosceles triangle the perpendicular bisects the base. For if in the earlier triangle the sides AB and AC are made equal, their squares will also be equal. [Margin: (f) Euclid I.47] Hence, since the square of AB equals the squares of AD and BD, and the square of AC equals the squares of AD and CD, the squares of AD and BD will also equal the squares of AD and CD. And when the common square of AD is removed, the

remaining squares of BD and CD will be equal, and hence the lines BD and CD are equal.

PROBL. 5. PROPOS. 9.

[Margin: When the sides of a triangle are known, the segments of the base contained between the perpendicular and each angle become known (*Cognitis lateribus trianguli, cognoscuntur segmenta basis inter perpendiculararem, & vtrumque angulum comprehensa*). (a) prop. 8 of this book.]

If from any one angle of a triangle whose sides are known a perpendicular is dropped to the opposite side: to find how great is the line contained between the perpendicular and each of the remaining angles.

Let the two triangles of the preceding proposition be repeated, with side AC = 20, AB = 13, and — in the prior triangle — BC = 21, in the posterior BC = 11. And let it be required to find how great are both BD and CD. [Margin: (a) prop. 8 of this book] Since the square of AC exceeds the square of AB by the rectangle contained under BC and CE, if the square of side AB — that is, 169 — is subtracted from 400, the square of side AC, the remainder will be the rectangle under BC and CE, namely 231. Dividing this by side BC — that is, by 21 in the prior triangle — yields the line CE = 11. Removing this from side BC, that is from 21, leaves BE = 10. So half of this, 5, gives the line BD; and the remaining CD will accordingly be 16, namely the residue of side BC. But in the posterior triangle, dividing the same rectangle 231 by 11, namely by side BC, gives CE = 21. If side BC — that is, 11 — is taken from this, there remains BE = 10, whose half gives BD = 5; and so CD will be 16, being composed of side BC and BD.

[Margin: *Praxis*. How, from the operation, it may be understood whether the perpendicular falls inside the triangle or outside (*Quo pacto ex operatione intelligatur, num perpendicularis intra triangulum cadat, an extra*).]

Therefore, if the difference between the two squares of the sides containing the angle from which the perpendicular is drawn is divided by the third side, onto which the perpendicular is dropped, a number is produced. If this is smaller than the third side, it indicates that the perpendicular has fallen inside the triangle; and, subtracted from the third side, it leaves a number whose half gives the smaller segment of the base — while this smaller segment, taken from the third side, yields the larger segment. But if the number produced by the division is greater

than the third side, it is a sign that the perpendicular has fallen outside the triangle. So if the third side is taken from it, the remainder is a number whose half gives the line, outside the triangle, between the perpendicular and the obtuse angle; and the same half, added to the third side, yields the other line between the perpendicular and the acute angle.

ALITER & facilius. [Otherwise, and more easily.] [Margin: Another and easier way of finding the segments of the base (*Alia inuentio segmentorum basis, & faciliior*).] From A, with the interval of the smaller side AB, let a circle be described cutting the greater side AC at F, and its production at G, and cutting side BC (if the perpendicular falls inside the triangle) — or, if it falls outside, its production — at E: [Margin: (b) Euclid III.3] and BE will be bisected at D. [Margin: (c) Euclid III.36, coroll. 1] Now since the rectangle under BC and CE equals the rectangle under GC and CF, [Margin: (d) Euclid VI.16] it will be that as BC, the side onto which the perpendicular is drawn — namely 21 in the prior triangle, or 11 in the posterior — is to GC, the sum of the remaining two sides AC and AB (since AG equals AB), that is, to 33, so is CF, the difference between those same two sides — that is, 7 — to CE. Hence, by the rule of three, CE is found to be 11 parts in the prior triangle, and 21 in the posterior, as is plain here.

BC.	GC.	CF.	CE.
21.	33.	7 ?	fit 11.

Likewise:

BC.	GC.	CF.	CE.
11.	33.	7 ?	fit 21.

And if EC — found to be 11 parts in the prior triangle — is taken from side BC, namely from 21, there remains BE = 10, whose half, 5, is the segment BD; and so the other segment CD will be 16. But in the posterior triangle, if side BC, of 11 parts, is taken from EC (found to be 21 parts), there remains again BE = 10. So its half, 5, gives the line BD outside the triangle, between the perpendicular and the obtuse angle; and thus the whole CD, composed of side BC and the said half BD, will be 16.

Diagram. Two circle-and-triangle constructions are printed one above the other at the top right, illustrating the "Aliter" method (a circle centred at A with radius = the smaller side AB).

Upper figure — perpendicular inside:

- A (centre of the circle / apex of the triangle)
- G (upper left, on the circle): where the production of AC meets the circle
- F (right): where side AC cuts the circle
- C (far right, below F): base vertex on the secant line G–A–F–C [label shape uncertain; read as C]
- B, D, E (along the base): B a base vertex, D the midpoint of BE (foot of perpendicular), E where the circle meets BC

Lower figure — perpendicular outside:

- A (centre / apex)
- G (upper left, on the circle)
- F (right): where AC cuts the circle
- C (far right): base vertex [label shape uncertain; read as C]
- E, D, B (along the base line, left to right): D the midpoint of BE, the perpendicular foot lying outside the triangle

[Margin: *Praxis*. How, from the operation itself, it may be known whether the perpendicular falls inside the triangle or outside (*Quo pacto ex ipsa operatione cognoscatur, an perpendicularis cadat intra triangulum, an extra*).]

So if it is arranged that as the side onto which the perpendicular is drawn is to the sum of the other two sides, so is the difference of those same sides to a fourth, a number is found. If this is smaller than the third side, it indicates that the perpendicular has fallen inside the triangle; and, taken from the third side, it leaves a number whose half is the smaller segment of the base — while this, subtracted from the third side, yields the larger segment. But if the number found by the rule of three exceeds the third side, it is a sign that the perpendicular has fallen outside the triangle. So if the third side is taken from it, the half of the remaining number gives the line, outside the triangle, between the perpendicular and the obtuse angle; and the same half, added to the third side, offers the other line between the perpendicular and the acute angle.

If, therefore, from any one angle of a triangle whose sides are known [a perpendicular is dropped], etc. Which was to be done.

SCHOLIUM.

You see, then, that in either method the calculation itself shows whether the perpendicular falls inside the triangle or outside. This same problem can also be solved by proposition 13 or 12 of Euclid's book 2, according as the perpendicular falls inside the triangle or outside. For let the perpendicular AD first fall inside, with the sides as before: $AB = 13$, $AC = 20$, $BC = 21$. Both angles B and C will be acute, because of the right angles at D, [Margin: (e) Euclid I.17] since both angles B and ADB, and both C and ADC, are less than two right angles. [Margin: (f) Euclid II.13] Since, then, the square of AC is less than the two squares of AB and BC by twice the rectangle contained under CB and BD, if the square of side AC — namely 400 — is subtracted from the sum of the squares of sides AB and BC — namely 610 — the remainder will be twice the rectangle under CB and BD, namely 210. Half of this, being 105, is the rectangle contained under CB and BD; dividing it by side BC — that is, by 21 — gives the segment $BD = 5$. So the remaining CD will be 16. This can, however, be found in the same way if the square of side AB is subtracted from the squares of sides AC and CB, etc.; for that too is less than these two by twice the rectangle contained under BC and CD.

Next let the perpendicular AD fall outside, with the sides as before: $AB = 13$, $AC = 20$, $BC = 11$. The angle ABC will be obtuse, [Margin: (h) Euclid I.17] because the two angles D and ABD are less than two right angles; and hence ABD is acute, since D is right. [Margin: (i) Euclid II.12] Since, then, the square of AC is greater than the two squares of AB and BC by twice the rectangle contained under CB and BD — that is, by 290 — if this is subtracted from 400, the square of side AC, there remains twice the rectangle contained under CB and BD, namely 110. Half of this, namely 55, is the rectangle contained under CB and BD; dividing it by side $BC = 11$ gives the line BD, outside the triangle, = 5, which together with the 11 of side BC composes the line $CD = 16$.

Diagram. Two small triangles are printed in the middle-right column, reprising the two cases of PROPOS. 9 for this scholium.

Upper triangle — perpendicular inside (cf. fig-p802-1):

- A (apex); base B, D, C (D the foot of AD)
- side AB **13**; side AC **20**; base BC **21**

Lower triangle — perpendicular outside (cf. fig-p802-2):

- A (apex); base D, B, C (D outside, to the left of B)
- side AB **13**; side AC **20**; base BC **11**

[Margin: *Praxis.*]

Therefore, when the perpendicular falls inside the triangle: if half the difference between the square of either of the sides containing the angle from which the perpendicular is dropped and the sum of the squares described on the remaining two sides is divided by the side onto which the perpendicular falls, there is produced the segment of the base near the angle contained by the sides whose sum of squares was taken; and this segment, subtracted from the same base, leaves the other segment.

But when the perpendicular falls outside the triangle: if half the difference between the square of the side opposite the obtuse angle and the sum of the squares described on the remaining two sides is divided by the side onto whose production the perpendicular falls, there is produced the line, outside the triangle, between the perpendicular and the obtuse angle; and this, added to the base, establishes the other line between the perpendicular and the acute angle of the base.

PROBL. 6. PROPOS. 10.

[Margin: In a non-right triangle, from the known angles (or from the known proportions of the angles) together with one side, the remaining parts are investigated (*In triangulo non rectangulo ex angulis notis, vel ex proportionibus angulorum notis, vna cum vno latere reliqua inuestigantur*).]

Given all the angles of a non-right triangle, or given their proportions, together with one side: to find the remaining two sides, and to make known the proportion of any two of them.

Diagram. Two triangles are printed at the top left, for the two cases of PROPOS. 10 (given all angles + one side).

Left triangle — all angles acute:

- A (apex); B and C (base vertices)
- angle A labelled **75.45** ($75^{\circ}45'$); angle B labelled **67.23** ($67^{\circ}23'$); angle C labelled **36.52** ($36^{\circ}52'$)

Right triangle — angle B obtuse:

- A (apex); B and C (base vertices)
- angle A labelled **30.31** ($30^{\circ}31'$); angle B labelled **112.37** ($112^{\circ}37'$); angle C labelled **36.52** ($36^{\circ}52'$)

In triangle ABC let all the angles first be acute and given: $A = 75^\circ 45'$, $B = 67^\circ 23'$, $C = 36^\circ 52'$. Let side $AB = 13$ also be given. From these the remaining two sides are to be found. [Margin: (a) prop. 1 of this book] Since as the sine of angle C is to the sine of angle A, so is the given side AB to side BC: if it is arranged that as the sine of angle C — namely 59996 — is to 96923, the sine of angle A, so is side AB, given as 13, to a fourth, side BC will be found to be nearly 21, as appears here.

C.	A.	AB.	BC.
59996.	96923.	13 ?	fit 21 ferè

Likewise, since as the sine of angle C is to the sine of angle B, so is side AB to side AC: if it is arranged that as 59996, the sine of angle C, is to 92310, the sine of angle B, so is side AB, given as 13, to a fourth, side AC will be found to be nearly 20, as you see here.

C.	B.	AB.	AC.
59996.	92310.	13 ?	fit 20 ferè

[Margin: (b) prop. 1 of this book] Or, since as the sine of angle A is to the sine of angle B, so is side BC (found above) to side AC: if it is arranged that as 96923, the sine of angle A, is to 92310, the sine of angle B, so is side BC, found as 21, to a fourth, side AC will be found to be nearly 20, as you see here.

A.	B.	BC.	AC.
96923.	92310.	21 ?	fit 20 ferè

Next let angle B be obtuse, $112^\circ 37'$; $A = 30^\circ 31'$, $C = 36^\circ 52'$; and again side $AB = 13$. Since the sine of angle B is the same as the sine of its complement, $67^\circ 23'$ — as we demonstrated in the treatment of sines — side $BC = 11$ and $AC = 20$ will be found, nearly, by the method already set out, as is clearly established here.

C.	A.	AB.	BC.
59996.	50779.	13 ?	fit 11 ferè

Likewise:

C.	B.	AB.	AC.
59996.	92310.	13 ?	fit 20 ferè

Or:

A.	B.	BC.	AC.
50779.	92310.	11 ?	fit 20 ferè

[Margin: *Praxis.*]

Therefore, given all the angles together with one side: if it is arranged that as the sine of the angle opposite the [given] side is to the sine of either of the remaining angles, so is the given side to a fourth, there will be found the side opposite the angle whose sine was taken. And if again it is arranged that as the sine of the angle opposite the given side is to the sine of the third angle, so is the given side to a fourth, there will be found the side opposite the third angle, and so on.

[Margin: When the triangle is isosceles or equilateral; or when in a scalene triangle two sides are given with the angles. When the proportions of the angles are given, together with one side (*Quando trianguli est Isosceles, vel æquilaterum: aut quando in scaleno dantur duo latera cum angulis. Quando dantur proportionēs angulorum, cum vno latere*).]

Now if the triangle is isosceles and the angles are given, or if it is scalene and two sides are given together with the angles, only one side needs to be found, as is evident. In an equilateral triangle, however, if one side is given, the rest are given too, being equal to it.

And if the proportions of the angles are given together with one side, the magnitudes of the angles must be investigated from those proportions, as we demonstrated in the scholium of proposition 1; and then the remaining sides are to be explored, as has been shown here.

Once the sides are found, it is clear that their proportions are known in the numbers in which they themselves are known. Therefore, given all the angles of a non-right triangle, etc. Which was to be done.

PROBL. 7. PROPOS. 11.

[Margin: In a triangle, from the known sides — or from their known proportions — the angles are found (*In triangulo ex notis lateribus, vel ex eorum proportionibus notis, anguli inueniuntur*). (a) I.19; (b) I.17; (c) scholium of II.13; (d) prop. 9 of this book.]

Given all the sides of a non-right triangle, or their proportions: to make all the angles known.

Diagram. Two triangles are printed at the bottom left, for PROPOS. 11 (given all sides → find the angles).

Upper triangle — perpendicular inside:

- A (apex); base B, D, C (D the foot of the perpendicular AD)
- side AB **13**; side AC **20**; base BC **21**

Lower triangle — obtuse case:

- A (apex); base B, D, C, with the perpendicular AD
- side AB **11**; side AC **13**; a further value **20** [reprises the obtuse triangle AB = 13, AC = 20, BC = 11; interior labels partly abraded — see readability note]

In the prior triangle ABC let all the sides be given: AB = 13, AC = 20, and BC = 21. From these the angles are to be investigated. Onto the greatest side BC, from the opposite angle A, let the perpendicular AD be drawn, which will necessarily fall inside the triangle. For since side BC is the greatest, [Margin: (a) I.19] angle A, opposite it, will also be the greatest, [Margin: (b) I.17] and therefore both B and C are acute. [Margin: (c) scholium of II.13] From which it follows that the perpendicular AD falls inside the triangle. [Margin: (d) prop. 9 of this book] First, then, let the lines BD and CD be sought. BD will be found to be 5, and CD 16. Since, then, with AB posited as the whole sine, the line BD is the sine of angle BAD — as we showed in the treatment of sines — we shall say, by the rule of three: if AB, 13, gives AB, the whole sine of 100000 parts, what will BD, 5, give? And we shall find the sine of BD to be 38461, as you see here.

AB.	AB.	BD.	BD.
13.	100000.	5 ?	fit 38461.

From the table of sines, then, angle BAD will be given as 22°37', and so its complement B as 67°23', which is one of the sought angles. Again, since with AC posited as the whole sine, CD is the sine of angle CAD, we shall say once more by the rule of three: if AC, 20, gives AC, 100000, the whole sine, what will CD, 16, give? And we shall find the sine of CD to be 80000, as is evident here.

AC.	AC.	CD.	CD.
20.	100000.	16 ?	fit 80000.

This sine, in the table of sines, shows angle CAD to be 53°8'; and hence its complement C

(Continuing *PROBL. 7 / PROPOS. 11* from the previous page — the three sides given, the angles sought.)

will be $36^{\circ} 52'$, which is likewise one of the angles sought. And if the two angles derived from the two sines found — namely $22^{\circ} 37'$ and $53^{\circ} 8'$ — are added together, the third angle BAC becomes $75^{\circ} 45'$. Or, just as well, if the sum of the two angles B and C already found is subtracted from 180° , the remainder gives the third angle BAC, $75^{\circ} 45'$.

Praxis. [Margin: Praxis] And so, to embrace the whole procedure: arrange that as the greatest side (onto which the perpendicular has been dropped) is to the sum of the other two, so the difference of those same two is to a fourth number. This number, subtracted from the greatest side, leaves a number whose half gives the smaller segment of the base, while that half, taken from the base, leaves the greater segment — as is established from the second procedure of Proposition 9. Next, if it is arranged that as the smallest side is to the whole sine, so the smaller segment of the base is to a fourth, there will be found a sine whose arc's complement gives the angle standing over the base opposite the middle side. Then, if it is arranged that as the middle side is to the whole sine, so the greater segment of the base is to a fourth, there will be found a sine whose arc's complement gives the angle over the base opposite the smallest side. The third angle, opposite the greatest side, is then made up from those two arcs of the two sines found; or else it is simply what remains after the two angles found are subtracted from two right angles.

Again, in the second triangle let side AB be 11, AC 13, and BC 20. When the perpendicular AD is dropped onto the greatest side BC, the segment BD is found to be $8\frac{4}{5}$, and CD $11\frac{1}{5}$, as appears here, following the latter procedure of Proposition 9.

BC.	AB. AC.	Differ. inter AB, AC.	
20.	24.	2 ?	fit $2\frac{2}{5}$.

This number, $2\frac{2}{5}$, taken from the base 20, leaves $17\frac{3}{5}$, whose half, $8\frac{4}{5}$, gives the smaller segment BD. Therefore the greater, CD, will be $11\frac{1}{5}$. From this, angle B is found to be $36^{\circ} 52'$, C $30^{\circ} 31'$, and BAC $112^{\circ} 37'$, as you see here.

AB.	AB.	BD.	BD.
11.	100000.	$8\frac{4}{5}$?	fit 80000.

Again (Item.):

AC.	AC.	CD.	CD.
13.	100000.	11 $\frac{1}{5}$?	fit 86154.

The complement of the arc that the first sine found yields gives angle B, 36° 52'; and the complement of the arc of the second sine found gives angle C, 30° 31', and so on. The doctrine of this proposition is therefore general — whether the greatest angle A is acute (as in the first triangle), or obtuse (as in the second), or finally right; though for the right-angled triangle it was already shown above, in Proposition 3, how the two acute angles are more easily found from two known sides. [Margin: Generalitas huius propos. — On the generality of this proposition.]

Now if instead the proportions of the sides are given — at least two of them — [a] we shall carry them out in the three smallest numbers, in case the smallest numbers of the proportions are not given, as Euclid taught in Proposition 4 of Book 8; and we shall assign those numbers to the sides just as if the sides had been given in those numbers. For example, if in the first triangle the ratio of AB to BC is 26 to 42, and that of AB to AC is 39 to 60, these ratios are reduced to these smallest numbers, 13 : 21 and 13 : 20. So AB will be given as 13, AC as 20, and BC as 21; and from these we shall extract the angles as before. [Margin: Quando laterum proportiones datae sunt. — When the proportions of the sides are given. a: 35 of the seventh (Euclid VII.35).]

[b] Furthermore, in an isosceles triangle the perpendicular must be dropped to the base — whether that be the greatest side or the smallest — and it will bisect the base. Hence if it is arranged that as one of the equal sides is to the whole sine, so half the base is to a fourth, there will be found the sine of a certain arc whose complement gives one of the equal angles over the base, as is clear from what has been demonstrated. Therefore the other equal angle is also given, and consequently the third, opposite the base, as being what remains of two right angles. [Margin: Quando triangulum est Isosceles. — When the triangle is isosceles. b: Corollary 8 of this.]

Finally, in an equilateral triangle the angles are given even if the sides are not, since each is a third part of two right angles — that is, contains 60°. Therefore, all the sides of a non-right-angled triangle being given, [the angles are found], and so on. Which was to be done. [Margin: Quando triangulum est aequilaterum. — When the triangle is equilateral.]

SCHOLIUM.

[Margin: Quando perpendicularis in obtusangulo triangulo cadit extra triangulum. — When in an obtuse-angled triangle the perpendicular falls outside the triangle.]

Although in this proposition we prescribed that the perpendicular be drawn to the greatest side from the opposite angle, so that it falls inside the triangle and the calculation is made easier, nevertheless we shall solve the problem by almost the same route even if, in an obtuse-angled triangle, the perpendicular is drawn not from the obtuse angle onto the greatest side, but from one of the acute angles onto the opposite side produced, so that it falls outside the triangle. This is evident in the present triangle ABC, in which side AB is given as 22, AC as 31, and BC as 14 (the side onto which the perpendicular is drawn). As 53 — the sum of the other sides AB, AC — is to 9, the difference of those same sides, so a fourth number is found, $34 \frac{1}{14}$; and if side BC is subtracted from it, the number $20 \frac{1}{14}$ remains, whose half, $10 \frac{1}{28}$, is the line BD, and consequently CD is $24 \frac{1}{28}$. For this reason, if it is now arranged that as AB, 22, is to AB [taken as] the whole sine 100000, so BD, $10 \frac{1}{28}$, is to a fourth, the sine BD is found, 45617, whose arc's complement will give the angle ABD, $62^\circ 52'$, and therefore the remaining one of the two right angles, ABC, $117^\circ 8'$. Likewise, if it is arranged that as AC, 31 [recte 35], is to AC [taken as] the whole sine 100000, so CD, $24 \frac{1}{28}$, is to a fourth, a sine is found, 77534, whose arc's complement will present the angle C, $39^\circ 10'$. And if the two angles ABC and C are taken from 180° , the angle BAC remains, $23^\circ 42'$. The rationale of this operation is gathered from the treatment of sines, where we showed that if AB is set as the whole sine, BD is the sine of angle BAD; likewise, if AC is set as the whole sine, CD is the sine of angle CAD; and so on.

Diagram. An obtuse-angled triangle drawn for the scholium, with the perpendicular falling outside the triangle. Apex A at the top; along the base line, left to right, the points D, B, C. The perpendicular AD descends from A to the point D, which lies to the left of B (outside segment BC).

- A (apex): vertex of the triangle from which the perpendicular is dropped
- D (base, leftmost): foot of the perpendicular AD, outside the triangle
- B (base, middle): vertex; segment DB is the exterior part of the base
- C (base, right): vertex
- Label "22" along side AB
- Label "31" along side AC (the longer slant)

- Label "41" at the base near BC [recte 14 — the text gives BC = 14; the woodcut shows the digits transposed]

THEOR. 8. PROPOS. 12.

Given two sides of a non-right-angled triangle together with the angle they contain; or given the ratio of the two sides containing a given angle: to find the third side and the remaining angles.

[Margin: In a non-right-angled triangle, from two known sides — or from their known ratio — together with the angle contained by them, the third side and the remaining angles are sought.]

In triangle ABC let there first be given two sides AB and AC — the former 3, the latter 5 — enclosing the angle A, which is obtuse and is also given as $93^\circ 50'$. From these we must investigate the third side BC and the remaining angles B and C. Since angle A is given as $93^\circ 50'$, if it is subtracted from 180° — that is, from two right angles — the remainder, $86^\circ 10'$, will be the sum of the two angles B and C. Moreover, the ratio of the sines of the angles C and B is also given, namely the same as that of side AB, 3, to side AC, 5, [a] because as side AB is to side AC, so is the sine of angle C to the sine of angle B. [b] Hence each of the angles C and B will be known separately: the former $29^\circ 55'$, the latter $56^\circ 15'$. [c] And since the sides are proportional to the sines of the opposite angles, it will hold that as the sine of angle C is to the sine of angle A, so side AB is to side BC; or as the sine of angle B is to the sine of angle A, so side AC is to side BC. By the rule of three, therefore, side BC is found, nearly 6, as appears here. [Margin: a: Prop. 1 of this; b: Prop. 6 of this; c: Prop. 1 of this.]

Diagram. A triangle ABC with A at the apex, B at the lower left, C at the lower right.

- A (apex): vertex holding the obtuse given angle
- B (lower left): vertex
- C (lower right): vertex
- Label "3" on side AB
- Label "5" on side AC
- Label "6" on the base BC

C.	A.	AB.	BC.
49874.	99776.	3 ?	fit 6. fere.

Or (Vel.):

B.	A.	AC.	BC.
83147.	99776.	5 ?	fit 6. fere.

Praxis. [Margin: Praxis] So then, to embrace the whole procedure: if (once the given angle has been taken from 180° , so as to have the sum of the other two) it is arranged that as the half-sum of the two given sides — that is, the half-sum of the terms of the ratio of the sines of the remaining angles — is to the tangent of the half-sum of the remaining angles, so the difference between the half-sum of the given sides (that is, of the terms of the ratio of the sines of the remaining angles) and either of the sides (or of the terms of the ratio) is to a fourth: there will be found the tangent of an angle which, added to the half-sum of the remaining angles, makes up the greater angle — that, namely, which is opposite the greater given side; while the same, subtracted from that said half-sum, leaves the smaller angle, opposite the smaller given side, as is evident from the earlier procedure of Proposition 6. And if it is next arranged that as the sine of either of the angles found is to the sine of the angle given at the outset, so the side opposite the found angle (the one taken into the rule of three) is to a fourth: the third side will be found. For greater clarity we will set out another example.

Let there be given in triangle ABC two sides AB and AC, 22 and 31, together with the angle A, $23^\circ 42'$. This angle, taken from 180° , leaves $156^\circ 18'$ for the sum of the angles B and C. So if it is arranged that as $26\frac{1}{2}$ — the half-sum of the sides AB, AC (that is, of the terms of the ratio of the sines of the angles B, C) — is to 476595, the tangent of the half-sum of the angles B, C (that is, the tangent of $78^\circ 9'$), so $4\frac{1}{2}$, the difference between the half-sum of the sides AB, AC (or of the terms of the ratio of the sines of the angles B, C) and either of the sides (or terms), is to a fourth: there will be found the tangent 80931, whose arc, $38^\circ 59'$, added to $78^\circ 9'$ (namely to the half-sum of the angles B, C), constitutes the greater angle B — opposite the greater given side AC — $117^\circ 8'$. But the same arc, $38^\circ 59'$, subtracted from that same half-sum of the angles B, C (that is, from $78^\circ 9'$), leaves the smaller angle C, opposite the smaller given side AB, $39^\circ 10'$. By the operation of the rule of three you see it here.

$\frac{1}{2}(AB+AC)$	$\tan \frac{1}{2}(B+C)$	$\frac{1}{2}$ diff.	
$26\frac{1}{2}$	476595	$4\frac{1}{2}$?	fit 80931.

Diagram. A triangle ABC with the perpendicular CD drawn inside it (for the alternative demonstration below). Apex A at the top; the left edge AB runs down through D to B at the lower left; C lies to the right; CD is the perpendicular dropped from C onto AB, meeting it at D.

- A (top): vertex holding the acute angle
- D (on side AB, below A): foot of the perpendicular CD
- B (lower left): vertex
- C (right): vertex from which the perpendicular is dropped

Then if it is now arranged that as 63158, the sine of angle C, is to 40195, the sine of angle A, so side AB, 22, is to a fourth; or as 88995, the sine of angle B, is to 40195, the sine of angle A, so side AC, 31, is to a fourth: side BC will be found, nearly 14, as you see here.

C.	A.	AB.	BC.
63158.	40195.	22 ?	fit 14. fere.

Or (Vel.):

B.	A.	AC.	BC.
88995.	40195.	31 ?	fit 14. fere.

If you would rather not use tangents, the latter procedure of Proposition 6 must be employed in finding the angles B and C.

Now if the ratio of the sides AB, AC is given, together with the angle A contained by them, we shall assign to those sides the numbers of the ratio, as if they were given in those very numbers, and we shall solve the problem as before. [Margin: Quando laterum proportio data est. — When the ratio of the sides is given.]

[Margin: Alia demonstratio problematis sine propos. 6. — Another demonstration of the problem, without Proposition 6. d: 19 of the first; e: 17 of the first; f: Scholium 13 of the second; g: 1 of this.]

This problem can also be solved without the aid of Proposition 6, in the following way. In triangle ABC let side AB be given as 29, AC as 24, and angle A first acute, $38^{\circ} 16'$. From the opposite angle C let the perpendicular CD be drawn to the greater side AB; it will necessarily fall inside the triangle. For since side AB is greater than side AC, [d] angle C will also be greater than angle B. Hence B will be acute; for if it were right, or greater, C would be greater still than a right angle — which is absurd, [e] since B and C are less than two right angles. Since therefore A too is taken as acute, [f] the perpendicular CD falls inside the triangle. So in the right triangle ACD, since angle A is $38^{\circ} 16'$, its complement ACD will be $51^{\circ} 44'$. [g] Hence, since it holds that as the whole sine of the right angle D is to the sine of angle A, so side AC, 24, is to side CD, side CD is found by the rule of three, 14 [+ small compound fraction: ?], as appears here.

D.	A.	AC.	CD.
100000.	61932.	24 ?	fit 14 [+ fraction: ?].

By the same reasoning, [h] since it holds that as the whole sine of the right angle D is to the sine of angle ACD, so side AC, 24, is to side AD, side AD is found, 18 [+ small compound fraction: ?], as you see here. [Margin: h: 1 of this.]

D.	ACD.	AC.	AD.
100000.	78514.	24 ?	fit 18 [+ fraction: ?].

Now when AD is taken away from AB (found from AB, given as 29), BD remains, 10 [+ small compound fraction: ?]. And since, as we showed in the treatment of tangents, once BD is set as the whole sine the line CD is the tangent of angle B, the tangent is found, 146344, as is here manifest.

BD.	BD.	CD.	CD.
10 [+ fraction: ?]	100000.	14 [+ fraction: ?] ?	fit 146344.

Now the tangent 146344 indicates in the table of tangents the angle B, $55^{\circ} 40'$; and consequently the two angles A and B — $38^{\circ} 16'$ and $55^{\circ} 40'$ — subtracted from 180° leave the angle ACB, $86^{\circ} 4'$. [i] And since it holds that as the sine of the known angle B is to the sine of the given angle A, so the given side AC, 24, is to side BC, the side BC is found, nearly 18, as you see here. [Margin: i: 1 of this.]

B.	A.	AC.	BC.
82577.	61932.	24 ?	fit 18. fere.

Again, in triangle ABC let side AB be given as 13, AC as 11, and the angle A contained by them obtuse and given as $112^{\circ} 37'$. From either of the acute angles — say from B — let a perpendicular be drawn to the opposite side CA: [k] it will fall outside the triangle. Since therefore the angle BAC is set as $112^{\circ} 37'$, BAD will be $67^{\circ} 23'$, and its complement ABD necessarily $22^{\circ} 37'$. [l] Hence, since it holds that as the whole sine of the right angle D is to the sine of angle BAD, so side AB (given 13) is to side BD; and likewise as the whole sine of the right angle D is to the sine of angle ABD, so the given side AB, 13, is to side AD: BD is found to be 12, and AD nearly 5, as appears here. [Margin: k: Scholium 12 of the second; l: 1 of this.]

Diagram. The obtuse triangle ABC with the perpendicular BD falling outside it. The side CA is produced beyond A to the point D; the perpendicular BD runs from

B (lower left) up to D. A sits between the base vertices and D, with C at the lower right.

- D (upper left): foot of the perpendicular BD, on line CA produced beyond A
- A (upper middle): vertex holding the obtuse angle BAC
- B (lower left): vertex from which the perpendicular is drawn
- C (lower right): vertex

Now when AD (found to be 5) is added to side AC (given 11), the whole CD becomes 16. Since therefore, once the whole sine is set on CD, the line BD is the tangent of angle C, the tangent BD is found, 75000, as you see here.

D.	BAD.	AB.	BD.
100000.	92310.	13 ?	fit 12. fere.

Again (Item.):

D.	ABD.	AB.	AD.
100000.	38456.	13 ?	fit 5. fere.

CD.	CD.	BD.	BD.
16.	100000.	12 ?	fit 75000.

This tangent presents, in the table of tangents, the angle C, $36^{\circ} 52'$; and consequently the two angles A and C — $112^{\circ} 36'$ [recte $37'$] and $36^{\circ} 52'$ — taken from 180° leave the angle ABC, $30^{\circ} 31'$. [m] Since at last it holds that as the sine of the known angle C is to the sine of the given angle BAC, so the given side AB, 13, is to BC, the side BC is found, nearly 20, as appears here. [Margin: m: 1 of this.]

C.	BAC.	AB.	BC.
59996.	92310.	13 ?	fit 20. fere.

But, as you see, the former method is much shorter and more expeditious.

And if it should happen that the two given sides enclosing the given angle are equal, the problem will be easier. For let there be in triangle ABC two equal sides AB, AC, each 9, and the contained angle A, 80° . When this angle is taken from 180° , the half of the remainder — which is 100° — gives each of the angles B and C as 50° . Then if it is arranged that as the sine of angle B (or C) is to the sine of angle A, so side AC (or AB) is to a fourth, the side BC comes out, $11\frac{43685}{76604}$, as

you see here. [Margin: Quando data latera sunt aequalia. — When the given sides are equal.]

Diagram. A near-isosceles triangle ABC, A at the apex, B at the lower left, C at the lower right (no numeric labels; only the vertex letters A, B, C).

- A (apex): vertex holding the given angle of 80°
- B (lower left): vertex
- C (lower right): vertex

B, vel C.	A.	AB, vel AC.	BC.
76604.	98481.	9 ?	fit 11 43685/76604.

Therefore, given two sides of a non-right-angled triangle together with the angle they contain, and so on. Which was to be done.

PROBL. 9. PROP. 13.

Given two sides of a non-right-angled triangle, or their given ratio, together with the angle opposite one of the given sides: to find the remaining angles and the third side. But one must be certain whether the other angle, opposite the given side, is acute or obtuse — assuming the given angle is acute.

[Margin: In a non-right-angled triangle, from two given sides — or from their given ratio — together with one angle not contained by them, the third side and the remaining angles are investigated. a: 1 of this.]

Let there be given in triangle ABC two sides AB, AC, 13 and 20, and let the acute angle C, $36^\circ 52'$, opposite the given side AB, be given; and let it be established, concerning angle B (which is opposite the other given side AC), whether it is acute or obtuse — for otherwise nothing certain could be gathered, as will be clear from the following scholium. [a] Since therefore it holds that as side AB is to side AC, so the sine of angle C is to the sine of angle B, and the three first terms are given, the fourth is found by the rule of three, that is, the sine of angle B, 92302, as is here clear. [Margin: a: 1 of this.]

AB.	AC.	C.	B.
13.	20.	59996 ?	fit 92302.

This sine, in the table of sines, exhibits the angle B as nearly $67^\circ 23'$ if it is acute (as in the former triangle), or, if obtuse (as in the latter triangle), $112^\circ 37'$ — inasmuch as this completes the semicircle with the other, since an obtuse and an

acute angle making up 180° have the same sine, as we demonstrated in the definitions of sines. When the two angles C and B are taken from 180° , there remains, in the former triangle, the angle A, $75^\circ 45'$, but in the latter $30^\circ 31'$. And the side BC is found, by the rule of three, to be about 21 parts in the former triangle, but in the latter about 11, [b] since it holds that as the sine of angle C is to the sine of angle A, so side AB is to side BC — as appears in this operation.
[Margin: b: 1 of this.]

C.	A.	AB.	BC.
59996.	96923.	13 ?	fit 21. fere.

Again (Item.):

C.	A.	AB.	BC.
59996.	50779.	13 ?	fit 11. fere.

Diagram. Two triangles side by side, illustrating the ambiguous case — the acute-B solution on the left, the obtuse-B solution on the right. In each, A is at the apex, B at the lower left, C at the lower right; the interior angle values are inscribed.

- Left triangle (acute B): A " 75.45 " ($75^\circ 45'$); B " 67.23 " ($67^\circ 23'$); C " 36.52 " ($36^\circ 52'$)
- Right triangle (obtuse B): A " 30.31 " ($30^\circ 31'$); B " 112.37 " ($112^\circ 37'$); C " 36.52 " ($36^\circ 52'$)

Praxis. [Margin: Praxis] And so, if it is arranged that as the side opposite the given angle is to the other given side, so the sine of the given angle is to a fourth: there will be found a sine whose arc gives the angle opposite the other given side, if it is acute (which indeed...

[it] indeed always happens when the given angle is obtuse — or else that angle, subtracted from a semicircle, will yield the required angle if it is obtuse. And the sum formed from the given angle and the found angle, subtracted from 180 degrees, will give the third angle. Finally, if we set up the proportion — as the sine of the given angle is to the sine of the third angle (the one contained by the given sides, which was found last), so is the side that is given, opposite the given angle, to another — the third side will be produced.

And if instead the ratio of the two sides is given, then, provided numbers in that ratio are assigned to those sides, we shall carry out the problem in the same way.

[Margin: When the ratio of the two sides is given (Quando proportio duorum laterum datur).]

ALITER [Another method]. Again let two sides be given: $AB = 13$ and $AC = 20$, together with angle C , acute, of $36^\circ 52'$. Suppose first that the other angle, B , is likewise acute, as in the earlier triangle. Drop from A the perpendicular AD to BC [c]; it will fall inside the triangle. In the right triangle ACD , if AC is taken as the whole sine [radius], then the straight line AD is the sine of the given angle C , as we taught in the treatise on sines. So if we set up the proportion — as AC the whole sine is to AC the given side, so is AD the sine of the given angle C to another — we find $AD \approx 12$, as you see here.

[Margin: c — Scholium 13 of the second [book] (c Schol. 13. secundi).]

Diagram. A triangle with apex A above a horizontal base running from B (left) through D to C (right); the perpendicular AD is drawn from the apex down to the point D inside the base. The left side is marked **13** (AB), the right side **20** (AC), and the base **21** (BC). This is the acute-angled- B case of the worked example.

- A (apex): vertex
- B (base, left): vertex
- D (base, between B and C): foot of the perpendicular AD , inside the triangle
- C (base, right): vertex
- 13 (left side AB): side length
- 20 (right side AC): side length
- 21 (base BC): side length

AC.	AC.	AD.	AD.
100000.	20.	59996?	fit 12. fere.

Again, since, with AB taken as the whole sine, the straight line AD is the sine of angle B — as was handed down in the [treatment of] sines — then if we set the proportion: as AB the given side is to the whole sine, so is AD , now found, to another, the sine $AD = 92308$ will be found, as appears here.

AB.	AB.	AD.	AD.
13.	100000.	12?	fit 92308.

From the table of sines, therefore, angle B is obtained as $67^\circ 23'$, and consequently BAC , the remainder from two right angles, is $75^\circ 45'$ [d]. And since it holds that, as the sine of the given angle C is to the sine of the found angle A , so is side AB to side BC , side BC is found ≈ 21 , as you see here.

[Margin: d — Prop. 1 of this [book] (d 1. huius).]

C.	BAC.	AB.	BC.
59996.	96923.	13?	fit 21. fere.

Next, with the same data, let the other angle B be obtuse, as in the second triangle. Drop from A the perpendicular AD to BC [e]; it will fall outside the triangle. In the right triangle ACD, if AC is taken as the whole sine, the straight line AD is the sine of angle C, as was said in the treatment of sines. So if we set the proportion — as the whole sine is to the given side AC, so is the sine of the given angle C to another — AD is found ≈ 12 , as appears here.

[Margin: e — Scholium 12 of the second [book] (e Schol. 12. secundi).]

Diagram. A triangle with apex A over a base whose foot of the perpendicular falls outside it. Reading the base from left to right: D (leftmost, outside the triangle), then B, then C. The perpendicular AD runs from the apex down to D, outside the triangle. The upper-right side is marked **20** (AC) and the left side **13** (AB). A mark reading "II" appears on the base between B and C (possibly the length ≈ 11 of BC in this case; unresolved [?]). This is the obtuse-angled-B case.

- A (apex): vertex
- D (base, leftmost, outside triangle): foot of the perpendicular AD
- B (base, middle): the obtuse vertex
- C (base, right): vertex
- 20 (side AC): side length
- 13 (side AB): side length
- "II" (base, between B and C): mark, possibly the value 11 of BC [?]

AC.	AC.	AD.	AD.
100000.	20.	59996?	fit 12. fere.

Again, since, with AB taken as the whole sine, the straight line AD is the sine of angle ABD — as we explained in the definition of sines — if we set the proportion: as the given side is to the whole sine, so is AD, just found, to another, the sine AD = 92308 will be found, as you see here.

AB.	AB.	AD.	AD.
13.	100000.	12?	fit 92308.

This sine shows, in the table of sines, the angle ABD = $67^\circ 23'$, and consequently the remainder from two right angles, ABC = $11^\circ 37'$ [recte: $112^\circ 37'$]. Then,

subtracting the two known angles C and ABC from 180° , the angle BAC = $30^\circ 31'$ remains [f]. Hence, since it holds that, as the sine of the given angle C is to the sine of the found angle A, so is the given side AB to side BC, side BC is found ≈ 11 , as is clear here.

[Margin: f — Prop. 1 of this [book] (f 1. huius).]

C.	A.	AB.	BC.
59996.	50779.	13?	fit 11. fere.

Finally, with the same sides given, let the obtuse angle ABC = $112^\circ 37'$ be given, as in the second triangle. Carrying out the same construction: since, with AB taken as the whole sine, the straight line AD is the sine of angle ABD — that is, of the given angle ABC [supplementary angles having equal sines] — AD is again found ≈ 12 , as you see here.

AB.	AB.	AD.	AD.
100000.	13.	92308?	fit 12. fere.

Again, since, with AC taken [as the whole sine], the straight line AD is the sine of angle C, the sine AD = 60000 will be found, as appears here.

AC.	AC.	AD.	AD.
20.	100000.	12?	fit 60000.

Therefore angle C = $36^\circ 52'$, and consequently the remainder from two right angles, BAC = $30^\circ 31'$. Side BC is found = 11, as before. Thus: given two sides of a non-right triangle, etc. Which was to be done.

SCHOLIVM.

[Margin: When the given angle is acute, why it must be established whether the other angle is acute or obtuse (Quando datus angulus est acutus, cur constare debeat, num alter angulus sit acutus, vel obtusus). g — Prop. 4 of the first [book] (g 4. primi).]

That nothing certain can be concluded when the given angle, opposite one of the given sides, is acute — unless it is first known whether the angle opposite the other given side is acute or obtuse, as we said in the proposition — we shall make clear as follows. In triangle ABC let the sides AB and AC be given, together with the acute angle B. When the perpendicular AD is dropped from A to BC, it will be unknown whether it falls outside the triangle or inside, unless it is known

whether the other angle, ACB, is obtuse or acute. Take also the straight line DE, on the other side of the perpendicular AD, equal to DC; and draw the straight line AE[?]. Then AE will be equal to AC [g], because the sides DC, DA are equal to the sides DE, DA and contain equal angles — namely right angles. And so, even though the sides AB, AC (or AB, AE) and the acute angle B are known, the remaining side will still not on that account be known, nor the remaining angles — since the remaining side could be either BC or BE, and consequently the remaining angles could be either BCA and BAC, or BEA and BAE — unless it is first established whether the angle BCA is obtuse or acute. For once this is known, we shall know when the perpendicular AD falls outside the triangle and when inside; and so on.

Diagram. The upper of two figures illustrating the ambiguous (SSA) case. An apex A stands above a horizontal base; reading the base from left to right the labelled points are B, C, D, E. Straight lines are drawn from A to C, to D, and to E (with AB closing the leftmost triangle); AD is the perpendicular.

- A (apex): common vertex
- B (base, leftmost): vertex
- C (base): one possible vertex (BC one possible remaining side)
- D (base): foot of the perpendicular AD
- E (base, rightmost): reflected point with DE = DC (giving AE = AC)

Diagram. The lower of the two figures. An apex A stands above a base whose labelled points, read left to right, are B, E, D, C; straight lines run from A to the base points. This is the companion configuration in which the alternative triangle ABE is realized.

- A (apex): common vertex
- B (base, leftmost): vertex
- E (base): alternative vertex (BE the alternative remaining side)
- D (base): foot of the perpendicular AD
- C (base, rightmost): vertex

[Margin: Error of Nicolaus Copernicus (Error Nicolai Copernici).]

From this it is clear that Nicolaus Copernicus [Nicolaus Copernicus] — otherwise most careful — was mistaken in Book I of the *Revolutions* (*De Revolutionibus* I), in proposition 6 on rectilinear triangles, when he states without qualification that, if two sides of a triangle are given, and also the angle opposite one of them, the remaining side and the remaining angles can be found. For this cannot be done,

as we have demonstrated, when the given angle is acute, unless it is established whether the angle opposite the other of the given sides is acute or obtuse.

FINIS TRIANGVLORVM RECTILINEORVM.

[Here the treatise on rectilinear (plane) triangles ends and the treatise on spherical triangles begins.]

CHRISTOPHORI CLAVII BAMBERGENSIS E SOCIETATE IESV — TRIANGVLA SPHÆERICA

Christoph Clavius of Bamberg, of the Society of Jesus — SPHERICAL TRIANGLES.

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PRÆFATIO [Preface]

Now that we have explained what we judged necessary for the science of rectilinear triangles — the science by which the sides are found from known angles and, conversely, the angles from known sides — it remains for us to hand down also the doctrine of spherical triangles, by which the arcs are sought from known angles and, conversely, the angles from known arcs.

[Margin: Almagest, Book VII, ch. 8[?] (Lib. 7. Almag. cap. 8[?]).]

For although Menelaus [Menelaus of Alexandria] — also called Mileus, a distinguished author, who flourished in the time of the Emperor Trajan, as Ptolemy [Ptolemy] reports — composed three highly acute books on spherical triangles, we shall nonetheless not follow his order in these spherical triangles of ours. This is both because we have set aside many of his propositions, delightful and marvelously learned though they are, as not necessary, and because we have added not a few others, omitted by him, drawn from Geber of Seville [Jābir ibn Aflah], Johannes Regiomontanus [Regiomontanus], Francesco Maurolico [Francesco Maurolico], and others — propositions we judged wholly necessary for understanding matters of astronomy. For the most part, too, we have supplied new demonstrations — shorter and easier ones — and some we have likewise

demonstrated in the same way that the corresponding results about angles and rectilinear triangles were demonstrated by Euclid [Euclid], so that their demonstration might be plainer; among these are propositions 5, 6, 7, 8, and 9. We have, moreover, spent no little effort in arranging all the propositions of spherical triangles in such an order that the later ones depend on the earlier — as mathematics requires, and as is customary in every treatment of the elements of geometry. But let us now come to the matter, taking our start from the definitions.

DEFINITIONES [Definitions]

I.

[Margin: What a spherical angle is (Angulus sphaericus quid).]

A spherical angle is the angle which, on the surface of a sphere, two arcs of great circles cutting one another mutually contain.

These pages continue the opening section of Clavius's treatise on spherical triangles: the definitions of the spherical angle (right, obtuse, acute), of the spherical triangle, of the arc of a spherical angle, of the complement of an arc and of an angle, and of the sine, tangent, and secant of a spherical angle; then the first problem and the first theorems of spherical trigonometry.

(Continuing the commentary on Definition I, the spherical angle, begun on the preceding page at the catchword *QVONIAM*.)

Since the spherical angle that geometers consider on a spherical surface is constituted only by arcs of great circles, and since all great circles on a sphere cut one another into two equal parts, it comes about that the two arcs which contain a spherical angle on the surface of the sphere, if produced, cut one another — they do not merely touch one another. Thus every spherical angle arises from two arcs intersecting on the surface of the sphere, and not from arcs merely tangent to one another. This cannot be said of angles lying in a plane surface. For in a plane, not only two straight lines, or two curved lines, or lines of which one is straight and the other curved, cutting one another when produced, can constitute a plane angle; but also two lines, both of them curved, or one curved and the other straight, merely touching one another, can constitute a curvilinear or mixed plane angle (which is indeed called by geometers the angle of contact or of contingence), as is clear from Euclid III.16 — even though Jacques Peletier [Iacobus Peletarius] denies that it is an angle and tries to establish that view by

many arguments, all of which are sophistical and frivolous, as is clear from the solutions to them that we set out in the scholium to the said proposition.

[Margin: a — Theodosius, *Sphaerica* I.11 (11.1.Theod.)]

[Margin: A spherical angle necessarily arises from two arcs cutting one another.]

[Margin: A plane angle can also arise from two lines that do not cut one another, but are merely mutually tangent.]

[Margin: (at "Euclid III.16") propos. 16, lib. 3, Eucl.]

II.

A right spherical angle is one which, on the surface of the sphere, two arcs of great circles cutting each other at right angles contain — that is, arcs of which the one is at right angles to the other.

[Margin: What a right spherical angle is.]

III.

An obtuse spherical angle is one greater than a right angle.

[Margin: What an obtuse spherical angle is.]

IV.

An acute spherical angle, however, is one that is less than a right angle.

[Margin: What an acute spherical angle is.]

A right spherical angle is constructed at a given point on a given arc of a great circle on the surface of the sphere as follows. Through that point and through the pole of the given arc (which is found by Theodosius I.21) let a great circle be described. The circumference of this circle will then make a right angle with the given arc, since this circle stands at right angles to the circle of that arc. But if through the given point an arc of a great circle is described that does *not* pass through the poles of the given arc, the circumference of this circle will make unequal angles with the given arc — one obtuse and the other acute.

[Margin: Construction of a right, obtuse, and acute spherical angle.]

[Margin: a — Theodosius, *Sphaerica* I.15 (15.1.The.)]

V.

A spherical triangle is one contained by three arcs of great circles on the surface of the sphere.

[Margin: What a spherical triangle is.]

This triangle is either *equilateral*, if all its arcs are equal; or *isosceles*, if only two arcs are equal; or, finally, *scalene*, if all its arcs are unequal one from another. Likewise it is either *right-angled*, if it has some right angle; or *obtuse-angled*, in which some angle is obtuse; or, finally, *acute-angled*, if all its angles are acute — just as Euclid said of the rectilinear triangle. Yet this difference is found between the rectilinear triangle and the spherical: in the rectilinear one the remaining two angles are necessarily acute, because two angles taken in any way are less than two right angles; but in the spherical, if one angle is right or obtuse, the other two can also be right, or obtuse, or at least one of them, as will become clear from the demonstrations that follow.

[Margin: A spherical triangle is divided, as the rectilinear one is by Euclid.]

[Margin: The difference between the right-angled and obtuse-angled rectilinear triangle and the spherical.]

[Margin: b — Euclid I.17 (17.prim.)]

VI.

The arc of a spherical angle is an arc of a great circle whose pole is at the angle itself, intercepted between the two arcs that contain the spherical angle.

[Margin: What the arc of a spherical angle is.]

Since the pole of a great circle is distant from it by a quadrant of a great circle, it follows that each of the arcs containing the angle — set between the angle and the arc of the angle — is a quadrant. Therefore, if the angle is right, the arc of the angle will be a quadrant; if acute, less than a quadrant; if, finally, obtuse, greater than a quadrant — and conversely. As we shall demonstrate in Proposition 26.

[Margin: a — Corollary to Theodosius I.16 (Coroll. 16, 1 Theod.)]

VII.

The complement of an arc is the excess by which a quadrant exceeds it, if the arc is less than a quadrant; or the excess by which it exceeds the quadrant, if it is greater than a quadrant.

[Margin: What the complement of an arc is.]

VIII.

The complement of a spherical angle is called the excess by which a quadrant exceeds the arc of that angle, or by which it is exceeded by it.

[Margin: What the complement of a spherical angle is.]

IX.

The sine, tangent, and secant of any spherical angle are the sine, tangent, and secant of that arc which is called the arc of the angle.

PROBL. I. PROPOS. I.

Given two unequal arcs of great circles on the surface of the sphere, neither of which is greater than a semicircle, to cut off from the greater an arc equal to the lesser.

Diagram. Two figures for Proposition 1, side by side. On the left, a narrow lune bounded by two arcs meeting at **A** (top) and **B** (bottom) — the lesser arc AB with its chord. On the right, a circle with **C** at the top, **D** at the bottom, and **E** marked on the right-hand side; the arc CD is the greater arc, and CE is the portion cut off from it.

- A (top of left lune): endpoint of the lesser arc
- B (bottom of left lune): endpoint of the lesser arc
- C (top of right circle): endpoint of the greater arc CD
- E (right side of right circle): point cutting off arc CE equal to arc AB
- D (bottom of right circle): endpoint of the greater arc CD

Let AB and CD be two unequal arcs of great circles, neither of which is greater than a semicircle, and let CD be the greater; and let it be required to cut off from the greater, CD, an arc equal to the lesser, AB. Draw the chord AB, and fit to it, in the arc CD, an equal chord CE. I say that the arc CE cut off is equal to the lesser arc AB. For since the circles of the arcs AB and CD are great circles, and therefore equal, equal chords AB and CE will cut off equal arcs AB and CE — because each

arc is assumed to be less than a semicircle. Therefore, given two arcs of circles, etc. Which was to be done.

[Margin: a — Euclid IV.8 (8.quarti.)]

[Margin: b — Euclid III.28 (28.tertij.)]

THEOR. I. PROPOS. 2.

In every spherical triangle, any side whatsoever is less than a semicircle.

Diagram. Two great circles intersecting to form the spherical triangle ABC. **A** at the top, **B** at the left, **C** at the bottom, and **D** at the right, where the arcs BA and BC produced meet again.

- A (top): vertex of the triangle
- B (left): vertex of the triangle
- C (bottom): vertex of the triangle
- D (right): point where arcs BA, BC produced meet

Let ABC be a spherical triangle. I say that any side is less than a semicircle. For if the arcs BA and BC are produced until they meet at D, beyond A and C, the arcs BAD and BCD will be semicircles, since great circles bisect one another. Therefore both the arc BA and the arc BC are less than a semicircle. In the same way, by producing the arcs AB and AC, we shall show that the arc AC is less than a semicircle. Now the arcs BA and BC, produced beyond the points A and C, will meet, because they make spherical angles with the arc AC, and all three arcs are portions [portiones] of great circles that cut one another at the points A, B, C but do not touch. Hence it follows that each of the arcs BA, BC, when produced, cuts the arc AC produced at the points A and C, as is clear from the definition [...] beyond the points A and C. Therefore, in every spherical triangle, etc. Which was to be demonstrated.

[Margin: a — Theodosius I.11 (11.1.The.)]

THEOR. 2. PROPOS. 3.

In every spherical triangle, any two sides are greater than the remaining one, however they are taken.

Diagram. A large circle (the completed circle of the arc BC) together with a smaller circle AD described from the pole B. **A** lies toward the center-top, **B** at the upper right, **D** at the right on the arc BC, **C** at the bottom, and **E** at the far left (the second pole).

- A (center-top): vertex of the triangle
- B (upper right): vertex of the triangle, taken as pole of circle AD
- D (right): point where circle AD cuts the arc BC
- E (left): the other pole of circle AD
- C (bottom): vertex of the triangle

Let ABC be a spherical triangle. I say that any two sides, say AB and AC, are greater than the side BC. For if the triangle is equilateral, it is obvious that the two together are double the remaining side, and so greater. But if one of the sides AB, AC is equal to the side BC, or greater, or if both are even greater, it is likewise clear that the two sides AB, AC are greater than the remaining side BC. But if each of the sides AB, AC, taken together with the third side BC, is less than it, we shall demonstrate that the sides AB, AC together are greater than the side BC in this way. Let the circle of the third arc BC be completed. Then from the pole B — namely from one extremity of the greater side BC — at a distance equal to either of the smaller arcs, that is, at the distance of the arc BA, let a circle AD be described on the surface of the sphere, cutting the arc BC (which is assumed greater than the arc BA) at D, a point between B and C. And since the circle BC also passes through the other pole of the circle AD, let the other pole be E, which will be distant a semicircle from the pole B, so that BCE is a semicircle. Since, then, the arc BC is less than a semicircle, the pole E will lie beyond the point C; but the point D lies between B and C, as was said. Therefore the point C falls between the points D and E. And so, since from the point C — which lies outside the periphery of the circle AD and away from its pole E — two arcs of great circles CB, CA, less than a semicircle (because they are sides of the spherical triangle ABC), are drawn to the periphery AD, the arc CD, passing through the pole B, will be less than the arc CA. Therefore, adding the equal arcs DB, AB (and they are equal, because the chords subtending them are equal, by the definition of a pole), the whole arc BC will be less than the two arcs AB, AC; that is, the two sides AB, AC will be greater than the side BC. And in the same way any other two sides will be shown greater than the remaining one. Therefore, in every triangle, etc. Which was to be demonstrated.

[Margin: a — Scholium to Theodosius I.10 (Scho.10.1.Theod.)]

[Margin: b — Prop. 2 of this treatise (2.huius)]

[Margin: c — Prop. 2 of this treatise (2.huius)]

[Margin: d — Scholium, Theodosius (Sch.21. 2.Theod.) [?]]

[Margin: f — Euclid III.28 (28.tertij.)]

THEOR. 3. PROPOS. 4.

In every spherical triangle, the three sides together are less than a whole great circle.

Diagram. A lens- or eye-shaped figure formed by two arcs meeting at **B** (left) and **D** (right). **A** lies on the upper arc and **C** on the lower arc, forming the spherical triangle ABC, with the arcs BA and BC produced to meet at D.

- A (upper arc): vertex of the triangle
- B (left): vertex of the triangle
- C (lower arc): vertex of the triangle
- D (right): point where arcs BA, BC produced meet

Let ABC be a spherical triangle. I say that the three sides together are less than a whole great circle. For if any two arcs, say BA and BC, are produced until they meet at the point D — and they will necessarily meet beyond A and C, because they cut the great circle AC at the points A and C, or else because each of the arcs BA, BC is less than a semicircle — the two arcs BAD, BCD will be semicircles, because great circles bisect one another. Now since in the triangle DAC the sides DA, DC are greater than the side AC, if the common arcs AB, CB are added — that is, the sum of the arcs AB, CB — the arcs BAD, BCD will also become greater than the three arcs AC, AB, BC; that is, the three sides AC, AB, BC will be less than the two semicircles BAD, BCD, that is, less than a whole great circle. Therefore, in every spherical triangle, etc. Which was to be demonstrated.

[Margin: a — Theodosius I.11 (11.1.The.)]

[Margin: b — Prop. 3 of this treatise (3.huius)]

THEOR. 4. PROPOS. 5.

When an arc of a great circle on the sphere, standing upon an arc of a great circle, makes angles, it will make either two right angles or angles equal to two right angles.

Let the arc of a great circle AB, standing upon the arc of a great circle CD, make two spherical angles ABC, ABD. If, then, the arc AB of the circle passes through the pole of the circle of the arc CD, it will cut the arc CD entirely at right angles — at right angles; and therefore the angles ABC, ABD will be right angles. But if the arc AB does not pass through the poles of the arc CD, it will make the one

angle obtuse, the other acute. I say, then, that these are equal to two right angles. For let an arc of a great circle EB be drawn through the point B and the pole of the arc CD; then the two angles EBC, EBD will be right angles. Now since the right angle EBD is equal to the two angles DBA, ABE, adding the common right angle EBC, the two right angles EBD, EBC will be equal to the three angles DBA, ABE, EBC. Again, since the angle ABC is equal to the two angles ABE, EBC, adding the common angle ABD, the two angles ABC, ABD will be equal to the three angles DBA, ABE, EBC. But it was shown that the two right angles EBD, EBC are also equal to these same three; and things equal to the same thing are equal to one another. Therefore the two angles ABC, ABD are equal to the two right angles EBD, EBC. When, therefore, an arc of a great circle on the sphere [stands upon another], etc. Which was to be demonstrated.

[Margin: a — Theodosius I.15 (15.1.The.), noted on the preceding page]

[Margin: b — Theodosius I.20 (20.1.Th.)]

[Margin: c — Theodosius I.15 (15.1.The.)]

Diagram. An arc EB standing on the base line through C, B, D. E is at the upper left, A at the upper right (the oblique arc AB), while C, B, D lie along the base with B at the center, where the arc EB rises perpendicularly.

- E (upper left): endpoint of the perpendicular arc EB
- A (upper right): endpoint of the oblique arc AB
- C (base, left): endpoint of the base arc CD
- B (base, center): foot of the arcs, on the base arc CD
- D (base, right): endpoint of the base arc CD

COROLLARIVM.

It follows from these things that the two arcs of two angles which are equal to two right angles — that is, which are formed by an arc of a great circle standing upon the arc of another great circle, such as the two angles ABC, ABD — make up a semicircle. For if from the pole B a great circle CAD is described, then, by Definition 6, CA and AD will be the arcs of the angles ABC, ABD. But it is clear that the arcs CA, AD make up a semicircle, since the great circles CBD, CAD cut one another into two equal parts at C, D.

[Margin: d — Theodosius I.11 (11.1.The.)]

Diagram. A lens-shaped figure. **A** at the top, **B** at the bottom, **C** at the left, **D** at the right; an arc runs from **A** down to **B**, and the semicircle **CAD** spans across the top, the great circles **CBD** and **CAD** crossing at **C** and **D**.

- **A** (top): vertex, on the arc of the angles
- **B** (bottom): pole from which the circle **CAD** is described
- **C** (left): intersection of the great circles **CBD**, **CAD**
- **D** (right): intersection of the great circles **CBD**, **CAD**

THEOR. 5. PROPOS. 6.

If two arcs of great circles on the sphere cut one another, they will make the vertical angles equal to one another.

Diagram. Two arcs crossing at **E** to form an **X**. **A** at the upper left, **D** at the upper right, **C** at the lower left, **B** at the lower right, with **E** at the center where the arcs **AB** and **CD** intersect.

- **A** (upper left): endpoint of arc **AB**
- **E** (center): point of intersection
- **D** (upper right): endpoint of arc **CD**
- **C** (lower left): endpoint of arc **CD**
- **B** (lower right): endpoint of arc **AB**

Let two arcs **AB**, **CD** of great circles on the sphere cut one another at **E**, in any way. I say that the angles they make at the vertex **E** are equal to one another — namely the angle **AED** to the angle **BEC**, and the angle **AEC** to the angle **BED**. For since both the angles **AED**, **DEB** and the angles **DEB**, **BEC** are equal to two right angles, the former two will be equal to the latter two. Therefore, taking away the common angle **DEB**, the angle **AED** will remain equal to the angle **BEC**. And by the same reasoning we shall establish that the angle **AEC** is equal to the angle **BED**. Therefore, if two arcs of great circles cut one another, etc. Which was to be demonstrated.

[Margin: a — Prop. 5 of this treatise (5.huius)]

THEOR. 6. PROPOS. 7.

If two spherical triangles have two sides equal to two sides, each to each, and also have an angle equal to an angle, contained under the equal arcs: they will also have the base equal to the base; and the triangle will be equal to the triangle, and the remaining angles will be equal to the

remaining angles, each to each — those under which the equal sides are subtended.

Diagram. Two spherical triangles side by side. On the left, triangle ABC with **A** at the apex, **B** at the lower left, **C** at the lower right. On the right, triangle DEF with **D** at the apex, **E** at the lower left, **F** at the lower right, and **G** marking the alternative base arc EGF (the letter G appears both above and below the base line, indicating the impossible curved base).

- A (apex, left triangle): vertex under which base BC is subtended
- B (lower left, left triangle): base vertex
- C (lower right, left triangle): base vertex
- D (apex, right triangle): vertex equal to A
- E (lower left, right triangle): base vertex
- F (lower right, right triangle): base vertex
- G (on the base of the right triangle): point on the alternative arc EGF (shown twice, above and below the base)

Let ABC, DEF be two spherical triangles having two sides AB, AC equal to the two sides DE, DF, each to each, and the angle A equal to the angle D. I say that the base BC is also equal to the base EF, and the triangle ABC equal to the triangle DEF, and the remaining angles B, C equal to the remaining angles E, F, each to each. For since the arc AB is assumed equal to the arc DE, it comes about that, if the one is imagined to be laid upon the other on the surface of the sphere — the point A being placed on the point D, and the point B on the point E — the planes of the circles AB, DE coincide with one another, and consequently the arc AB coincides with the arc DE. Otherwise the circles of those arcs would cut one another into two equal parts at A and B, and so AB, DE would be semicircles. But this is absurd, for each is less than a semicircle. Since, then, the angle A is assumed equal to the angle D, the arc AC will also coincide with the arc DF, and the point C will fall on the point F, because of the equality of the arcs AC, DF. Therefore the base BC will also coincide with the base EF; otherwise, if it fell above or below — such as the arc EGF is — the arcs EF, EGF (or BC), cutting one another at E, F, would be semicircles, since great circles cut one another into two equal parts. This is absurd, for each is less than a semicircle. And so the base BC will be equal to the base EF, since neither exceeds the other; and the triangle ABC equal to the triangle DEF; and the angles B, C equal to the angles E, F, each to each, for the same reason. Therefore, if two spherical triangles, etc. Which was to be demonstrated.

[Margin: a — Theodosius I.11 (11.1.The.)]

[Margin: b — Prop. 2 of this treatise (2.huius)]

[Margin: c — Theodosius I.11 (11.1.The.)]

[Margin: d — Prop. 2 of this treatise (2.huius)]

THEOR. 7. PROPOS. 8.

In isosceles spherical triangles, the angles at the base are equal to one another; and if the equal sides below the base are produced, the angles under the base will be equal to one another.

Let ABC be an isosceles spherical triangle whose two sides AB, AC are equal. I say that the angles B, C above the base BC are equal; likewise, if the equal arcs AB, AC are produced below the base BC to any extent, the angles B, C below the base BC are also equal. For since the arc AB is less than a semicircle, it can be produced —

[Margin: a — Prop. 2 of this treatise (2.huius)]

(Continuing the demonstration of THEOR. 7 / PROPOS. 8 — the isosceles spherical triangle — broken off at the foot of the preceding page.)

and so an arc smaller than a semicircle may still be taken. Let arc AD, then, be less than a semicircle [Margin: b — Prop. 1 of this treatise], and from the arc AE, produced as far as one likes, let arc AF be cut off equal to arc AD; and through the two points B, F, as well as through C, D, let two arcs of great circles BF, CD be drawn [Margin: c — Theodosius I.20]. Now the two sides AB, AF of triangle ABF are equal to the two sides AC, AD of triangle ACD, each to each, and they contain the common angle A; therefore the base BF is equal to the base CD, and the angles ABF and F are equal to the angles ACD and D [Margin: d — Prop. 7 of this treatise]. Again, since the arcs AD, AF are equal, if the equal arcs AB, AC are subtracted, then BD and CF will also be equal. Hence the two sides DB, DC of triangle DBC are equal to the two sides FC, FB of triangle FCB; and since these contain equal angles D, F, as we have shown, the angles DBC, DCB will also be equal to the angles FCB, FBC [Margin: e — Prop. 7 of this treatise]. Now if from the angles ABF, ACD, which we showed to be equal, we take away the angles FBC, DCB, which we have likewise proved equal, the angles ABC, ACB above the base BC will remain equal. But it has also been shown that the angles DBC, FCB below the same base BC are equal. Therefore both the angles above the base are equal to one another, and the angles below the same base are equal to one another. And

for this reason, in isosceles spherical triangles, and so forth. Which was to be demonstrated.

Diagram. The isosceles spherical triangle of THEOR. 7. An outer spherical triangle has its apex A at the top, its left side sweeping down through B to D at the lower left, and its right side sweeping down through C and F to E at the bottom right. The chords BF and CD are drawn and cross one another in the interior, forming an X between the four points B, C, D, F. A short arc joins B and C near the top.

- A (apex, top): vertex of the isosceles triangle
- B (left side, upper): base-vertex of the upper triangle
- C (right side, upper): base-vertex of the upper triangle
- D (lower left): endpoint of the produced side
- E (bottom, right of D): endpoint of the produced arc AE
- F (right side, below C): point where AF is cut equal to AD

COROLLARIVM.

From this it is clear that every equilateral spherical triangle is also equiangular.

THEOR. 8. PROPOS. 9.

If two angles of a spherical triangle are equal to one another, then the sides subtended beneath the equal angles will also be equal to one another.

In triangle ABC let the two angles B, C on the side BC be equal. I say that the sides AB, AC subtended beneath them are also equal. For if they are not equal, let AB, if possible, be the greater. Now since arc AC is less than a semicircle [Margin: a — Prop. 2 of this treatise], let there be cut off from the greater arc AB an arc BD equal to the lesser arc AC [Margin: b — Prop. 1 of this treatise], and through the points C, D let an arc of a great circle CD be drawn [Margin: c — Theodosius I.20]. Then, since the two sides AC, CB of triangle ACB are equal to the two sides DB, BC of triangle DBC, and contain equal angles ACB, DBC, the triangles ACB, DBC will be equal — the whole and the part [Margin: d — Prop. 7 of this treatise]. This is impossible. Therefore the sides AB, AC are not unequal, but equal. So if two angles of a spherical triangle, and so forth. Which was to be shown.

Diagram. A spherical triangle ABC with curved sides. The apex A is at the top; D lies on the left side AB just below A; B is at the lower left and C at the right. The chord CD is drawn from C to the point D on AB, cutting off BD equal to AC.

- A (apex, top): vertex
- D (left side, just below A): point where BD is cut equal to AC
- B (lower left): base-vertex
- C (right): base-vertex

COROLLARIVM.

It follows from this that every equiangular spherical triangle is also equilateral.

PROBL. 2. PROOS. 10.

(Body heading printed "PROOS. 10." — evidently for PROPOS.)

On a given arc of a great circle in a sphere, and at a given point on it, to construct a spherical angle equal to a given spherical angle.

Let AB be the given arc of a great circle in a sphere, C the given point on it, and let it be required, given a spherical angle D, to construct at the point C a spherical angle equal to it. Producing the arcs DE, DF that contain the angle D as far as one likes, let a quadrant DG be taken, and through G and the pole of circle DE let an arc of a great circle GH be drawn [Margin: a — Theodosius I.20], cutting arc DF at H. Then the angle G will be a right angle [Margin: b — Theodosius I.15]. Next, taking a quadrant CI as well, let an arc of a great circle IK be drawn through I and the pole of circle AB [Margin: c — Theodosius I.20]; then the angle I will likewise be a right angle [Margin: d — Theodosius I.15]. Finally, since arc GH is less than a semicircle [Margin: e — Prop. 2 of this treatise], let there be cut off from IK an arc equal to it [Margin: f — Prop. 1 of this treatise], and let an arc of a great circle CK be drawn through C, K [Margin: g — Theodosius I.20]. I say that angle C is equal to angle D. For since the sides DG, GH are equal to the sides CI, IK, and they contain equal angles, being both right angles, the angles D and C will be equal [Margin: h — Prop. 7 of this treatise]. Therefore, on a given arc of a great circle, and so forth. Which was to be done.

Diagram. Two spherical triangles drawn side by side, for the angle-copying construction. In the left figure the apex is D at the top, with G on the left side, E at the lower left, and F at the bottom; H lies just above the base near F. In the right figure C and A stand at the top, B at the bottom, K at the right, and I and H near

the lower left. The base points H and I of the two figures nearly meet at the center.

- D (apex, left figure): given angle
- G (left side, left figure): endpoint of the quadrant DG
- E (lower left, left figure): endpoint of arc DE
- F (bottom, left figure): endpoint of arc DF
- H (near base, left figure): where arc GH cuts DF
- A (top right, right figure): endpoint of the given arc AB
- C (top, right figure): the given point
- B (bottom, right figure): endpoint of arc AB
- I (lower left, right figure): endpoint of the quadrant CI
- K (right, right figure): endpoint of the constructed arc CK

THEOR. 9. PROPOS. 11.

In every spherical triangle the greater angle is subtended by the greater side; and the greater side subtends the greater angle.

In the spherical triangle ABC let the angle ACB be greater than the angle A. I say that the side AB is greater than the side BC. Since the angle ACB is set greater than the angle A, let an angle ACD be made equal to angle A [Margin: a — Prop. 10 of this treatise], and let the arc CD cut arc AB at D. Then, since in triangle ADC the angles A and ACD are equal, the sides AD, CD will also be equal [Margin: b — Prop. 9 of this treatise]. Adding therefore the common arc DB, the arcs BD, DC together will equal arc AB. But the arcs BD, DC together are greater than arc BC [Margin: c — Prop. 3 of this treatise]. Therefore arc AB too will be greater than that same arc BC. Which was proposed.

Diagram. A spherical triangle with apex A at the top right, B at the lower left, and C at the lower right. From C a chord CD is drawn to the point D on the side AB, so that angle ACD equals angle A.

- A (apex, top right): vertex
- D (on side AB, left): point where CD meets AB
- B (lower left): base-vertex
- C (lower right): base-vertex

(THEOR. 9 continued — the converse.)

But now, in the spherical triangle ABC, let the side AB be greater than the side BC. I say that the angle C is greater than the angle A. For if the angle C is not greater than the angle A, it will be either equal to it or less. If it is equal, the sides AB, CB will be equal [Margin: a — Prop. 9 of this treatise]. This is absurd, since AB is supposed greater than CB. But if the angle C is less than the angle A, the side BC will be greater than the side AB, as has already been shown — which is likewise absurd, for AB is supposed greater than BC. Since, then, the angle C is neither equal to nor less than the angle A, it will certainly be greater. Which was proposed. Therefore in every spherical triangle the greater angle, and so forth. Which was to be shown.

Diagram. A single spherical triangle with curved sides: apex A at the top, C at the lower left, B at the lower right.

- A (apex, top): vertex
- C (lower left): base-vertex
- B (lower right): base-vertex

THEOR. 10. PROP. 12.

If two spherical triangles have two sides equal to two sides, each to each, but the angle contained by the equal arcs greater than the [corresponding] angle, then they will also have the base greater than the base. And if they have the base greater than the base, they will also have the angle contained by the equal arcs greater than the [corresponding] angle.

Let the two sides AB, AC of triangle ABC be equal to the two sides DE, DF of triangle DEF, but let the angle EDF be greater than the angle A. I say that the base EF is also greater than the base BC. First, let these spherical triangles be isosceles, and from the pole D let an arc of a circle EGF be described through the points E, F on the surface of the sphere. If this circle is a great circle, it will be exactly the same as EF; otherwise, since great circles bisect one another, EF would be a semicircle [Margin: a — Theodosius I.11] — which is absurd, since it is less than a semicircle [Margin: b — Prop. 2 of this treatise]. Now the arc EGF will be [an arc of] a great circle when the arcs DE, DF are quadrants, because a great circle is a quadrant's distance from its own pole [Margin: c — Corollary to Theodosius I.16]. So let arc EGF now be of a great circle and the same as EF, and let the angle FDG be made equal to the angle A [Margin: d — Prop. 10 of this treatise]. Then arc DG will be equal to arc DE, and therefore also to arc AB [Margin: e — Elements III.28], because the chords subtending DE, DG are equal by the definition of a pole. Since,

then, the sides AB, AC are equal to the sides DG, DF and contain equal angles, the bases BC, GF will be equal [Margin: f — Prop. 7 of this treatise]. And since arc EF is greater than arc GF, arc EF will also be greater than arc BC. Which was proposed.

But if the circle described from the pole D through the points E, F is not a great circle, and so is not the same as EF, but falls either below the arc EF or above it (for it makes no difference where it falls), let the angle FDH nonetheless be made equal to the angle A [Margin: g — Prop. 10 of this treatise]; then again arc DH will be equal to arc DE, that is, to arc AB [Margin: h — Elements III.28], because the chords subtending DE, DH are equal by the definition of a pole [Margin: i — Theodosius I.20]. Drawing therefore an arc of a great circle FH through the points F, H, since the sides AB, AC are equal to the sides DH, DF and contain equal angles, the bases BC, HF will also be equal [Margin: k — Prop. 7 of this treatise]. Now since the great circle DF, passing through D, the pole of circle EHF, bisects it [Margin: l — Theodosius I.15], the arc EHF will be less than a semicircle (because the arc from the point F through E up to the point at which, if it were produced beyond E, it would be cut by the arc FD produced toward D, is a semicircle, seeing that the arc EHF is bisected by the arc FD, as has been said). And therefore the chord FE is greater than the chord FH in the same circle [Margin: m — Elements III.15], because the former is nearer to the center of circle EHF, that is, to the diameter, than the latter. Since, then, the arcs EF, HF are of great circles, and so equal [Margin: n — Prop. 2 of this treatise], and each of the arcs EF, HF is less than a semicircle, arc EF will be greater than arc HF [Margin: o — Scholium on Elements III.28]. But arc HF has been shown equal to arc BC. Therefore arc EF too will be greater than arc BC. Which was proposed.

Diagram. The isosceles case of THEOR. 10, shown as two tall pointed spherical triangles. The left figure has apex A, with B at the lower left and C at the lower right. The right figure has apex D, with E at the lower left and F at the lower right; near the base the points G and H mark the arcs EGF and EHF (G above, H below).

- A (apex, left figure): vertex
- B (lower left, left figure): base-vertex
- C (lower right, left figure): base-vertex
- D (apex, right figure): vertex / pole
- E (lower left, right figure): base-vertex
- F (lower right, right figure): base-vertex
- G (near base, right figure): point on the great-circle arc EGF
- H (near base, right figure): point on the smaller-circle arc EHF

Next, let the triangles proposed not be isosceles, but let the side AB be greater than the side AC, and consequently the side DE greater than the side DF. Producing arc DF toward F [Margin: p — Prop. 1 of this treatise], and cutting off an arc DG equal to DE itself [Margin: q — Prop. 2 of this treatise], which is less than a semicircle, let there be described from the pole D, through the points E, G, an arc of a circle EHG — whether it be a great circle or not. Let the angle FDH again be made equal to the angle A [Margin: r — Prop. 10 of this treatise]; then arc DH will be equal to arc DE, that is, to arc AB [Margin: s — Elements III.28], because the chords subtending DH, DE are equal by the definition of a pole. Drawing therefore an arc of a great circle HF through the points H, F, arc BC will, as before, be equal to arc HF [Margin: t — Prop. 7 of this treatise]. Now since the great circle DG is drawn through D, the pole of circle EG, and the point F lies within the periphery of circle EG — namely between the circle and the pole D — and off its pole, arc FE will be greater than arc FH [Margin: u — Scholium on Theodosius II.21], since the former is nearer to arc FD, which passes through the pole D, and each of the arcs FE, FH is less than a semicircle (because they do not intersect except at the point F). But arc HF has been shown equal to arc BC. Therefore arc EF too will be greater than arc BC. Which was proposed.

Diagram. The non-isosceles case. The left figure is a spherical triangle with apex A, B at the lower left, C at the lower right. The right figure has apex D, with E at the lower left, H at the base-middle, F at the right, and G at the lower right; the arcs EHG and HF are drawn near the base.

- A (apex, left figure): vertex
- B (lower left, left figure): base-vertex
- C (lower right, left figure): base-vertex
- D (apex, right figure): vertex / pole
- E (lower left, right figure): base-vertex
- H (base-middle, right figure): point on arc EHG / on arc HF
- F (right, right figure): base-vertex
- G (lower right, right figure): point where DG is cut equal to DE

But now let the base EF be greater than the base BC. I say that the angle D too is greater than the angle A. For if the angle D is not greater than the angle A, it will be either equal to it or less. If it is said to be equal, arc EF will be equal to arc BC [Margin: x — Prop. 7 of this treatise] — which is absurd, for arc EF is supposed greater than arc BC. But if the angle D is said to be less than the angle A, then, as has already been shown, arc BC will be greater than arc EF — which is likewise absurd, since EF is supposed greater than BC. Since, then, the angle D is neither

equal to nor less than the angle A, it will certainly be greater. Which was proposed. And so, if two spherical triangles, and so forth. Which was to be demonstrated.

Diagram. Two spherical triangles for the base-greater case. The left has apex A, B at the lower left, C at the lower right; the right has apex D, E at the lower left, F at the lower right.

- A (apex, left figure): vertex
- B (lower left, left figure): base-vertex
- C (lower right, left figure): base-vertex
- D (apex, right figure): vertex
- E (lower left, right figure): base-vertex
- F (lower right, right figure): base-vertex

THEOR. 11. PROPOS. 13.

Two semicircles of great circles cutting one another contain two angles equal to each other.

Let the two semicircles of great circles ABC, ADC cut one another at A, C. I say that the angles A and C are equal. For, dividing the semicircle ABC in two at B, so that AB, BC are quadrants, let an arc of a great circle BD be drawn through B and the pole of circle ABC [Margin: a — Theodosius I.20], cutting arc ADC at D; then the angle B will be a right angle on either side [Margin: b — Theodosius I.15]. Now since the two sides AB, BD are equal to the two sides CB, BD, and contain equal angles, being right angles, the angles A and C will also be equal [Margin: c — Prop. 7 of this treatise]. Therefore two semicircles of great circles, and so forth. Which was to be demonstrated.

Diagram. A lens (vesica) formed by two great-circle semicircles ABC and ADC meeting at A on the left and C on the right. B is at the top and D at the bottom, and a vertical arc BD passes through the middle, dividing the lens.

- A (left): intersection of the two semicircles
- B (top): midpoint of semicircle ABC
- C (right): intersection of the two semicircles
- D (bottom): point where arc BD cuts arc ADC

THEOR. 12. PROPOS. 14.

(Body heading printed "THEOR. 12." — evidently for THEOR.)

In any spherical triangle with one side produced: if the two remaining sides together are equal to a semicircle, the external angle will be equal to the internal opposite angle above the produced arc; but if they are less than a semicircle, the external angle will be greater than that same internal opposite angle; and finally, if they are greater than a semicircle, the same external angle will be less than the said internal opposite angle.

In the spherical triangle ABC let the side BC be produced to D, and let the two remaining sides AB, AC first together be equal to a semicircle. I say that the external angle ACD is equal to the internal opposite angle B, above the produced arc BC, and so forth. For the arc BA produced will meet the arc BC produced at D, and BAD will be a semicircle [Margin: a — Theodosius I.11]. Since the arcs BA, AC are supposed equal to the semicircle BAD, then, taking away the common arc BA, the remaining arcs AC, AD will be equal. Hence the angle ACD too will be equal to the angle D [Margin: b — Prop. 8 of this treatise]. Since, then, the angles B and D are also equal, the angle ACD will likewise be equal to the angle B [Margin: c — Prop. 13 of this treatise]. Which was proposed.

Diagram. A lens-shaped figure: A at the top, B at the left, C below the middle, D at the right. The upper arc runs from B through A to D, the lower arc from B through C to D, and a further arc AC descends from A to C, forming the triangle ABC with side BC produced to D.

- A (top): apex
- B (left): vertex
- C (below middle): vertex, foot of arc AC
- D (right): point to which BC is produced

Next, let the two sides AB, AC be less than the semicircle BAD. Then, taking away the common arc AB, the remainder AC will be less than the remainder AD [Margin: d — Prop. 11 of this treatise], and therefore the angle ACD will be greater than the angle D — that is, than the angle B, which is equal to the angle D [Margin: e — Prop. 13 of this treatise]. Which was proposed.

Finally, let the sides AB, AC be greater than the semicircle BAD. Then, taking away the common arc AB, the remainder AC will be greater than the remainder AD [Margin: f — Prop. 11 of this treatise], and therefore the angle D will be greater than the angle ACD. Since, then, the angle B is equal to the angle D [Margin: g — Prop. 13 of this treatise], the angle B too will be greater than the angle ACD; that is, the angle ACD will be less than the angle B. Therefore, in any triangle, and so forth. Which was to be shown.

THEOR. 13. PROPOS. 15.

If, in any spherical triangle with one side produced, the external angle is equal to the internal opposite angle above the produced arc, then the two remaining sides together will be equal to a semicircle; but if the external angle is greater than that same internal opposite angle, the two remaining sides will be less than a semicircle; and finally, if the external angle is less than the said internal opposite angle, the two remaining sides will be greater than a semicircle.

Positing the same spherical triangle and the same construction of the figure, let the external angle ACD first be equal to the internal opposite angle B. I say that the sides AB, AC are equal to a semicircle, and so forth. For since the angle B is equal to the angle D [Margin: a — Prop. 13 of this treatise], the angle ACD too will be equal to the angle D; and therefore the arcs AC, AD will also be equal [Margin: b — Prop. 9 of this treatise]. Adding therefore the common arc AB, the two arcs AB, AC will be equal to the semicircle BAD. Which was proposed.

Diagram. The same lens-shaped figure as above: A at the top, B at the left, C below the middle, D at the right, with the arc AC drawn from A down to C.

- A (top): apex
- B (left): vertex
- C (below middle): vertex, foot of arc AC
- D (right): point to which BC is produced

Next, let the angle ACD be greater than the angle B — that is, than the angle D, which is equal to the angle B [Margin: c — Prop. 13 of this treatise]; then arc AD will be greater than arc AC [Margin: d — Prop. 11 of this treatise]. Adding therefore the common arc AB, the two arcs AB, AC will be less than the semicircle BAD. Which was proposed.

Finally, let the angle ACD be less than the angle B — that is, than the angle D, which is equal to the angle B [Margin: e — Prop. 13 of this treatise]; then arc AC will be greater than arc AD [Margin: f — Prop. 11 of this treatise]. Adding therefore the common arc AB, the two arcs AB, AC will be greater than the semicircle BAD. Which was proposed. So if, in any spherical triangle, and so forth. Which was to be demonstrated.

THEOR. 14. PROPOS. 16.

If in any spherical triangle two sides together are equal to a semicircle, the two angles on the base will be equal to two right angles; but if they are less than a semicircle, they will be less than two right angles; and finally, if they are greater than a semicircle, they will be greater than two right angles.

In the spherical triangle ABC let the two sides AB, AC first be equal to a semicircle. I say that the two angles B, C are equal to two right angles, and so forth. For, producing arc BC to D, the angle ACD will be equal to the angle B [Margin: a — Prop. 14 of this treatise]. Since, then, the two angles at C are equal to two right angles [Margin: b — Prop. 5 of this treatise], the two angles B and ACB will also be equal to two right angles.

Diagram. A spherical triangle with apex A at the top, the side AB curving down to B at the lower left; the base runs straight from B through C to D at the lower right, so that BC is produced to D. The chords AC and AD are drawn.

- A (apex, top): vertex
- B (lower left): base-vertex
- C (base, middle): base-vertex
- D (lower right): point to which BC is produced

Next, let the sides AB, AC be less than a semicircle. Then, since the two angles at C are equal to two right angles [Margin: c — Prop. 5 of this treatise], and the angle B is less than the angle ACD [Margin: d — Prop. 14 of this treatise], the angles B and ACB will be less than two right angles.

Finally, let the sides AB, AC be greater than a semicircle. Since, then, the two angles at C are equal to two right angles [Margin: e — Prop. 5 of this treatise], and the angle B is greater than the angle ACB [recte: ACD] [Margin: f — Prop. 14 of this treatise], the angles B and ACB will be greater than two right angles. So if, in any spherical triangle, and so forth. Which was to be shown.

THEOR. 15. PROPOS. 17.

If in any spherical triangle two angles standing on one side are equal to two right angles, the remaining two sides will be equal to a semicircle; but if they are less than two right angles, the two sides will be less than a

semicircle; and finally, if they turn out to be greater than two right angles, they will be greater than a semicircle.

Keep the same spherical triangle and the same construction of the figure. First, let the two angles B and C be equal to two right angles, standing on the side BC. I say that the remaining two sides AB and AC are equal to a semicircle, and so on. For since the two angles at C are also equal to two right angles, once the common angle ACB is taken away, the remaining angle ACD will be left equal to angle B. [Margin a: "5. huius" — Prop. 5 of this treatise] Therefore the arcs AB and AC are equal to a semicircle. [Margin b: "15. huius" — Prop. 15 of this treatise]

Next, let the angles B and ACB be less than two right angles. Then, since the two angles at C are equal to two right angles, once the common angle ACB is removed, the remaining angle ACD will be left greater than angle B. [Margin c: "5. huius"] The arcs AB and AC are therefore less than a semicircle. [Margin d: "15. huius"]

Finally, let the angles B and ACB be greater than two right angles. Since, then, the two angles at C are equal to two right angles, if the common angle ACB is subtracted, the remaining angle ACD will be less than the remaining angle B; [Margin e: "5. huius"] and so the arcs AB and AC are greater than a semicircle. [Margin f: "15. huius"] Accordingly, if in any spherical triangle, and so forth. Which was to be shown.

Diagram. A spherical triangle with its apex A at the top. Its base is a curved arc running along the bottom from B (left) through C (middle) to D (right), so that the side BC is extended beyond C to the point D. The side AB rises from B to the apex A; the side AC descends from A to C; and the arc curves on from A/C toward D. The angle ACD at C is the exterior angle formed by extending the base.

- A (apex, top): vertex of the triangle
- B (lower left): base vertex
- C (lower middle): base vertex, where side AC meets the base
- D (lower right): endpoint of the base extended beyond C

THEOR. 16. PROPOS. 18.

If two spherical triangles have their three sides equal to three sides, each to each, they will also have their three angles equal to three angles, each to each — namely those under which the equal sides are subtended.

Let there be two spherical triangles ABC and DEF, having their three sides AB, AC, BC equal to the three sides DE, DF, EF, each to each. I say that the three angles A, B, C are also equal to the three angles D, E, F, each to each — the angles under which the equal sides are subtended. For if angle A (to begin from this angle) is not equal to angle D, it will be either greater or less. If greater, then the base BC will also be greater than the base EF, [Margin a: "12. huius" — Prop. 12 of this treatise] which is absurd, since the sides BC and EF are assumed equal. But if angle A is less than angle D, then the base EF will be greater than the base BC, [Margin b: "12. huius"] which is again absurd, since they are assumed equal. Since, then, angle A is neither greater nor less than angle D, it will certainly be equal to it. Therefore the remaining angles B and C will also be equal to the remaining angles E and F, [Margin c: "7. huius" — Prop. 7 of this treatise] namely B to E and C to F. If, then, two spherical triangles, and so forth. Which was to be shown.

Diagram. Two spherical triangles drawn side by side, each with curved sides. The left triangle is ABC, with apex A at the top, B at the lower left, and C at the lower right. The right triangle is DEF, with apex D at the top, E at the lower left, and F at the lower right.

- A (apex, left triangle): vertex
- B (lower left): vertex
- C (lower right): vertex
- D (apex, right triangle): vertex
- E (lower left): vertex
- F (lower right): vertex

THEOR. 17. PROPOS. 19.

If two spherical triangles have their three angles equal to three angles, each to each, they will also have their three sides equal to three sides, each to each — the sides that subtend the equal angles.

Let there be two spherical triangles ABC and DEF, having their three angles A, B, C equal to the three angles D, E, F, each to each. I say that the three sides AB, AC, BC are also equal to the three sides DE, DF, EF, each to each — the sides that subtend the equal angles. For if the sides BC and EF (to begin from these sides) are not equal, let BC, if it can be so, be the greater; and let the arc BG be cut off equal to the arc EF. [Margin a: "1. huius" — Prop. 1 of this treatise] Then the arc BA is either equal to the arc ED, or greater, or less. Whichever of these is said, an

absurdity will follow from the assumption that the sides BC and EF are unequal — namely, that BC is greater than EF.

Suppose first that the arc BA is equal to the arc ED, and let the great-circle arc AG be drawn through the points A and G.[Margin b: "20.1.The." — Theodosius, *Sphaerica* I.20] Since, then, the sides BA and BG are equal to the sides ED and EF, and contain equal angles B and E by hypothesis, the angles BAG and D will be equal.[Margin c: "7. huius"] But angle D was posited equal to angle BAC; therefore angle BAG will also be equal to angle BAC — the part to the whole. Which is absurd.

Next, let the arc BA be greater than the arc ED, and let the arc BI be cut off equal to ED itself;[Margin d: "1. huius"] and through the points G and I let the great-circle arc GI be drawn, meeting the arc CA, extended, at H.[Margin e: "20.1.The."] Since, then, the sides BI and BG are equal to the sides ED and EF, and contain equal angles B and E, the angles BIG and BGI will be equal to the angles D and F, [Margin f: "7. huius"] that is, to the angles BAC and BCA, because D and F are posited equal to these two. But the angles BIG and BAC are equal to the vertical angles HIA and HAK.[Margin g: "6. huius" — Prop. 6 of this treatise] Therefore the angles HAK and HIA are also equal. Since, then, the exterior angle BGH is equal to the interior angle BCH, and the exterior angle HAK equal to the interior angle HIK, as we have shown,[Margin h: "15. huius"] both the arc AHI and the arcs CH, HG will be equal to a semicircle; and so the arcs AH, HI will be equal to the arcs CH, HG — the part to the whole. Which is absurd.

Finally, let the arc BA be less than the arc ED, and let it be produced beyond A, and from it let the arc BK be cut off equal to the arc ED;[Margin i: "1. huius"] and through the points G and K let the great-circle arc GK be drawn, cutting the arc AC at L.[Margin k: "20.1.The."] Since, then, the sides

Diagram. A double figure. On the left is the construction triangle: apex region carrying the points H, A, K across the top (A in the middle, H to its left, K to its right), with I just below between H and A, and L in the interior; the base runs along the bottom from B (left) through G and C to the right, with the great-circle arcs AG, GI (produced to H) and GK (cutting AC at L) drawn in. On the right stands the comparison triangle DEF, with apex D at the top and E, F along the base. (This same figure is reprinted at the top where the proof continues.)

- H (top left): intersection point of the produced arcs
- A (top middle): apex of the left triangle
- K (top right): point where arc GK is produced

- I (upper interior): point cut on BA
- L (interior): point where GK cuts AC
- B (lower left): base vertex
- G (lower middle-left): point cut on the base BC
- C (lower middle): base vertex
- E (lower middle-right): base vertex of the right triangle
- D (upper right): apex of the right triangle
- F (lower right): base vertex of the right triangle

[The proof of THEOR. 17 / PROPOS. 19 continues from the foot of p. 817; the figure above is reprinted at the head of this page.]

BK and BG are equal to the sides ED and EF, and contain equal angles B and E, [Margin l: "7. huius"] the angles BKG and BGK will also be equal to the angles D and F — that is, to the angles BAC and BCA, since the angles D and F are posited equal to these two. And so, since the exterior angle BAL is equal to the interior angle BKL, and the exterior angle BGL to the interior angle BCL, as we have shown, [Margin m: "15. huius"] both the arcs AL, LK and the arcs CL, LG will be equal to a semicircle; and hence the two arcs AC and GK will be equal to a whole circle. Which is absurd, [Margin n: "2. huius" — Prop. 2 of this treatise] since each of the arcs AC and GK is less than a semicircle. The sides BC and EF, therefore, are not unequal but equal. In the same way we shall show that the sides AC, DF, and likewise AB, DE, are equal. The three sides of triangle ABC are therefore equal to the three sides of triangle DEF. So then, if two spherical triangles, and so forth. Which was to be shown.

THEOR. 18. PROPOS. 20.

If two spherical triangles have two angles equal to two angles, each to each, and one side equal to one side — the side that lies between the equal angles — then they will also have their remaining sides equal to the remaining sides, each to each, and the remaining angle equal to the remaining angle.

Let two spherical triangles ABC and DEF have two angles B and C equal to two angles E and F, each to each, and the side BC equal to the side EF — the side that lies between the equal angles. I say that the remaining sides AB and AC are also equal to the remaining sides DE and DF, each to each, and that the remaining angle A is equal to the remaining angle D. For if the sides AB and DE (to begin from these) are not equal, let AB be the greater, and let the arc BG be cut off equal

to the arc DE,[Margin a: "1. huius"] and through the points C and G let the great-circle arc CG be drawn.[Margin b: "20.1.The."] Since, then, the sides GB and BC are equal to the sides DE and EF, and contain equal angles B and E by hypothesis, the angles BCG and F will also be equal.[Margin c: "7. huius"] But F is posited equal to BCA itself; therefore angle BCG will also be equal to the same BCA — the part to the whole. Which is absurd. The sides AB and DE, then, are not unequal but equal. Therefore, since the sides AB and BC are equal to the sides DE and EF, and contain equal angles B and E, the sides AC and DF will also be equal, and the angles A and D will be equal.[Margin d: "7. huius"] Accordingly, if two spherical triangles have two angles, and so forth. Which was to be shown.

Diagram. Two spherical triangles with curved sides. The left triangle is ABC, apex A at the top, base from B (lower left) to C (lower right); the point G is marked on the side AB just below A, and the arc CG is drawn from C up to G. The right triangle is DEF, apex D at the top, base from E (lower left) to F (lower right).

- A (apex, left): vertex
- G (on side AB, below A): point cut off so that BG equals DE
- B (lower left): vertex
- C (lower right): vertex
- D (apex, right): vertex
- E (lower left): vertex
- F (lower right): vertex

THEOR. 19. PROPOS. 21.

If there are two right-angled spherical triangles, and they have two other angles equal — these not being right angles — and also two equal sides, the sides subtended under the right angles: then the remaining two sides will also be equal to the two sides, each to each, and the remaining angle will be equal to the remaining angle.

In the two spherical triangles ABC and DEF, let the angles B and E be right angles, the two angles C and F equal and not right, and the sides AC and DF, which subtend the right angles, equal. I say that the remaining sides AB and BC are equal to the remaining sides DE and EF, each to each; and likewise that the remaining angles A and D are equal. For, the arcs AC and BC being produced, let the arc CH be cut off equal to the arc FD — that is, to the arc CA — and the arc CG equal to the arc FE;[Margin a: "1. huius"] and through the points G and H let the great-circle arc GH be described.[Margin b: "20.1.The."] Since the sides CH and CG

are equal to the sides FD and FE, and contain equal angles GCH and F[Margin c: "6. huius"] (for by hypothesis angle F is equal to angle ACB, and ACB is equal to the vertical angle GCH),[Margin d: "7. huius"] the bases GH and ED will also be equal, and the angles at GH equal to the angles E and D; and therefore, since angle E is right, angle G will also be right.[Margin e: "20.1.The."] Now through C and the pole of the arc BG let the great-circle arc ICK be drawn in both directions, and let I be the pole of the arc BG.[Margin f: "13.1.The." — Theodosius, *Sphaerica* I.13] And since the circles of the arcs BA and HG pass also through the poles of the same arc BG, on account of the right angles B and G, the arcs BA and GH, when produced, will meet the arc CI at the pole I. Let the arc GH also meet the same arc ICK on the other side, at the point K, which will be the other pole of the arc BG, [Margin g: "Corol. 10. 1. Theo." — Corollary to Theodosius I.10] since each of the arcs ICK and IGK passes through the other pole of the arc BG. The three arcs IB, IC, IG will therefore be equal,[Margin h: "28. tertij" — Euclid, *Elements* III.28] because the straight lines subtending them are equal to one another, by the definition of a pole; and likewise the arcs KC and KG will be equal. Now since the angles ICG and IGC are equal to the angles KCG and KGC,[Margin i: "13.1.The."] all being right angles — because I is the pole of the arc BG, and the common side CG lies against them — the sides IC and IG will be equal to the sides KC and KG, each to each;[Margin k: "20. hui." — Prop. 20 of this treatise] and therefore, since the arc IG has been shown equal to the arc IB, the arc KG will also be equal to that same arc IB. And since the sides IC and CA are equal to the sides KC and CH (for the arc CI has been made equal to the arc AC),[Margin l: "6. huius"] and they contain equal vertical angles, the bases IA and KH will be equal, and the angles IAC and KHC equal.[Margin m: "7. huius"] Therefore, when the equal arcs IA and KH are taken away from the equal arcs IB and KG, and the equal angles IAC and KHC are taken from the two pairs of angles at A and H — each pair being equal to two right angles[Margin n: "5. huius"] — the arcs AB and HG and the angles BAC and GHC will remain equal. But the arc HG has been shown equal to the arc DE, and the angle GHC equal to angle D. Therefore the arc AB will be equal to the arc DE, and the angle BAC equal to angle D. Therefore, since the sides AB and AC are equal to the sides DE and DF, and enclose equal angles, the arcs BC and EF will also be equal.[Margin o: "7. huius"] The sides AB and BC are therefore equal to the sides DE and EF, and the angle BAC equal to angle D. Accordingly, if there are two right-angled spherical triangles, and so forth. Which was to be demonstrated.

Diagram. A great-circle construction drawn as a tall, pointed oval (a great circle seen nearly edge-on), with I at the top vertex and K at the bottom vertex serving as the two poles of the arc BG. A vertical arc runs from I down through C to K. Around the centre lie the points A (upper left, interior), B (left of centre), C

(centre, where the arcs cross), G (right of centre), and H (below G, lower right), forming the right-angled configuration of the proof. Off to the lower left, drawn separately, is the small comparison triangle DEF, with D at the upper left, E below it, and F at its right vertex.

- I (top): pole of arc BG
- K (bottom): the opposite pole of arc BG
- A (upper left, interior): vertex
- B (left of centre): right-angle vertex of the first triangle
- C (centre): common vertex where the arcs cross
- G (right of centre): right-angle vertex of the constructed triangle
- H (lower right, below G): meeting point of the produced arcs
- D (small triangle, upper left): vertex
- E (small triangle, lower left): vertex
- F (small triangle, right): vertex

SCHOLIUM.

The equal sides must be subtended under right angles. Otherwise, if they subtended other angles, nothing certain could be concluded. For let ABC be any spherical triangle having two sides AB and AC unequal to each other but together equal to a semicircle. Let the side CB be produced toward the side of B, and through A and the pole of the arc CD let the great-circle arc AD be drawn, [Margin p: "20.1.Th." — Theodosius, *Sphaerica* I.20] cutting CD at D; and the angle D will be right. [Margin q: "15.1.The." — Theodosius, *Sphaerica* I.15] Since, then, the arcs AB and AC are equal to a semicircle, the angle ABD will be equal to angle C. [Margin r: "14. huius" — Prop. 14 of this treatise] And so the two triangles ADB and ADC have the right angle D in common, and two angles ABD and C that are equal and not right [Margin s: "9. huius" — Prop. 9 of this treatise] (otherwise the sides AB and AC would be equal, on account of the angles B and C being right and equal); and they also have the side AD in common, which subtends the equal, non-right angles. And yet the remaining sides AB and BD are not equal to the remaining sides AC and CD, each to each, nor is the remaining angle BAD equal to the remaining angle CAD, as is plain. This comes about precisely because the equal sides do not subtend right angles; rather, the common side AD subtends the equal, non-right angles.

Diagram. A single spherical triangle with apex A at the top. The base runs along the bottom from D (left) through B to C (right). From A the arc AB descends to B,

and the arc AD (a great circle drawn through A and the pole of CD) descends to D; the side AC runs from A to C. The right angle stands at D.

- A (apex, top): vertex
- D (lower left): foot of the perpendicular arc AD, right angle
- B (lower middle): base vertex
- C (lower right): base vertex

Hence Nicolaus Copernicus [Nicolaus Copernicus] is mistaken in Book I of the *Revolutions*, in proposition 6 on spherical triangles, where he says:[Margin: "Error Nicolai Copernici" — An error of Nicolaus Copernicus] "If two triangles have a right angle, and in addition one angle equal to one another, and one side equal to one side — the side that is opposite one of the two equal angles (even if not the right one, as he says in the demonstration) — then they will also have the remaining sides equal to the remaining sides, the one to the other, and one angle equal to one angle, the remaining to the remaining." For the opposite of this has appeared in the right-angled triangles ADB and ADC, in which the common side AD is set opposite the equal, non-right angles ABD and ACD.

Hence too it is not always true, as the same Copernicus teaches in the same place, in proposition 4, where he says:[Margin: "Alius error Nicolai Copernici" — Another error of Nicolaus Copernicus] "In any triangle having a right angle, if in addition another angle is given, together with any one side, the remaining angle too will be given, together with the remaining sides." For although the right angle D and the angle ABD [are given], together with the side BD [?], which is set opposite a non-right angle, we shall nevertheless not on that account come to know the remaining angle and the remaining sides, since the remaining sides can be either AB and BD, or AC and CD, and the remaining angle either BAD or CAD, as is plain. It will therefore be necessary that something else besides be established before the remaining angle can be gathered together with the remaining sides, as we shall teach in the scholium to propositions 41 and 42.

THEOR. 20. PROP. 22.

If there are two spherical triangles that have two angles equal to two angles, each to each, and one side equal to one side — the side subtended by one of the equal angles — while the two sides subtending the remaining angles, being equal, are not equal to a semicircle but either greater or less: then the remaining two sides will also be equal to the two

remaining sides, each to each, and the remaining angle will be equal to the remaining angle.

Let two spherical triangles ABC and DEF have two angles B and C equal to two angles E and F, each to each, and the sides AC and DF, which subtend the equal angles B and E, equal to each other; and let the remaining sides AB and DE, which subtend the other equal angles C and F, not be equal to a semicircle, but either greater or less. I say that the remaining sides CB and BA are equal to the remaining sides FE and ED, each to each, and that the remaining angles A and D are also equal. For if CB and FE are not equal, let CB be the greater, and let CG be cut off, an arc equal to the arc FE;[Margin a: "1. huius"] and through A and G let the great-circle arc AG be drawn.[Margin b: "20.1.The."] Since, then, the sides AC and CG are equal to the sides DF and FE, and contain equal angles C and F, the arcs AG and DE, and the angles AGC and E, will be equal.[Margin c: "7. huius"] But angle E was posited equal to angle B. Angle AGC is therefore also equal to angle B; and hence the arcs AB and AG will be equal to a semicircle.[Margin d: "15. huius"] Since, then, the arc AG has been shown equal to the arc DE, the arcs AB and DE will also be equal to a semicircle. But they are assumed unequal to a semicircle. Which is absurd. The arcs CB and FE, therefore, are not unequal but equal. Therefore, since the sides AC and CB are equal to the sides DF and FE, and contain equal angles C and F, the arcs AB and DE, and the angles BAC and D, will also be equal.[Margin e: "7. huius"] So then, if there are two spherical triangles, and so forth. Which was to be demonstrated.

Diagram. Two spherical triangles with curved sides. The left triangle has apex A at the top, and a base running from B (lower left) through G to C (lower right), with the arc AG drawn from A to the point G near B. The right triangle is DEF, apex D at the top, base from E (lower left) to F (lower right).

- A (apex, left): vertex
- B (lower left): vertex
- G (on the base, near B): point where arc CG is cut off equal to FE
- C (lower right): vertex
- D (apex, right): vertex
- E (lower left): vertex
- F (lower right): vertex

SCHOLIUM.

We said that the two sides subtending the remaining equal angles ought not to be equal to a semicircle. For otherwise the proposition would not be true. Let ABC be any spherical triangle having two sides AB and AC unequal to each other but together equal to a semicircle. Let the side BC be produced as far as D, but in such a way that BD is less than a semicircle, and let the great-circle arc AD be drawn through A and D.[Margin f: "20.1.Th." — Theodosius, *Sphaerica* I.20] Since, then, the arcs AB and AC are equal to a semicircle, the angle ACD will be equal to angle B.[Margin g: "14. huius" — Prop. 14 of this treatise] And so the two triangles ABD and ACD have two angles B and D equal to two angles C and D, each to each, and the side AD in common, which is subtended by the equal angles B and C; and yet the remaining sides AB and BD are not equal to the remaining sides AC and CD, each to each, nor is the remaining angle BAD equal to the remaining angle CAD, as is plain. This happens precisely because the sides AB and AC are [together] equal to a semicircle.

Nicolaus Copernicus, then, is deceived in Book I of the *Revolutions*, in proposition 12 on spherical triangles,[Margin: "Error Nicolai Copernici" — An error of Nicolaus Copernicus] when he teaches that every spherical triangle, two of whose angles have somehow been given, together with some one side, is thereby determined in its angles and sides. For in the triangle ACD, although the two angles D and ACD are known, together with the side AD, we shall nevertheless not on this account arrive at knowledge of the remaining sides and the remaining angle; since the remaining sides could be either AC and CD, or AB and BD, and so on. It will therefore be necessary that something else besides be established before the remaining angle can be known together with the remaining sides, as we shall say in the scholium to proposition 66.

THEOR. 21. PROPOS. 23.

If there are two spherical triangles that have two angles equal to two angles, each to each, and two sides about the remaining angle equal to two sides, each to each, and provided that in the said remaining angle there is not the pole of the remaining side: then the remaining side will be equal to the remaining side, and the remaining angle equal to the remaining angle.

Let ABC and DEF be two spherical triangles in which the angles B and C are equal to the angles E and F, each to each, and the sides AB and AC about the remaining angle A are equal to the sides DE and DF, each to each; but let A and D not be the

poles of the arcs BC and EF. I say that the remaining sides BC and EF are also equal, as are the remaining angles A and D.

For if the arcs BC and EF are not equal, let BC be the greater; [Margin: a — Prop. 1 of this treatise] cut off from it the arc CG equal to FE, [Margin: b — Theodosius I.20] and through the points A and G describe the great-circle arc AG. Now since the sides AC and CG are equal to the sides DF and FE, and they contain equal angles at C and F, [Margin: c — Prop. 7 of this treatise] the arcs AG and DE will be equal, and so will the angles AGC and E. But the arc DE was assumed equal to the arc AB, and the angle E equal to the angle B. Therefore the arc AG will be equal to the arc AB, and the angle AGC equal to the angle B. And so, [Margin: d — Prop. 5 of this treatise] since AGC and AGB together equal two right angles, B and AGB will also equal two right angles. [Margin: e — Prop. 8 of this treatise] But B and AGB are equal to each other, because of the equality of the arcs AB and AG. Each of them, therefore, will be a right angle, [Margin: f — Theodosius I.13] and consequently both arcs AB and AG will pass through the pole of the arc BC. Then A is the pole of the arc BC — which is absurd, since it was assumed not to be. [Margin: g — Prop. 18 of this treatise] The arcs BC and EF, therefore, are not unequal but equal, and for that reason the angles BAC and D will also be equal.

SCHOLIVM.

That condition is indeed necessary — namely, that in the remaining angle there should not be the pole of the remaining side. For the proposition would be false if the pole of the remaining side lay in that angle. Take a spherical triangle ABC, and let A be the pole of the arc BC; and from A let any great-circle arc AD descend, cutting BC at D. [Margin: h — Theodosius I.15] Then the angles at B, C, and D will all be right angles, [Margin: i — Corollary of Theodosius I.16] and all three arcs AB, AC, AD will be quadrants. Thus the two triangles ABC and ADC have the two angles B and C equal to the two angles ADC and C, each to each, and the two sides AB and AC equal to the two sides AD and AC about the angles BAC and DAC, each to each; and yet the remaining sides BC and DC are not equal to each other, nor are the remaining angles BAC and DAC, as is plain. This happens precisely because A is the pole of the arcs BC and DC.

Hence it is also clear that [Nicolaus Copernicus] is mistaken in *De Revolutionibus* I, Prop. 12, [Margin: Error of Nicolaus Copernicus] when he asserts that every spherical triangle whose two angles are given, together with some side, has its angles and sides determined. For in triangle ABC, even if the two angles B and C are given together with the two sides AB and AC (and not with just one side, as he would have it), the remaining side and the remaining angle are still not

immediately known, since the remaining side can be either BC or DC, and the remaining angle either BAC or DAC, and so on. It is necessary, therefore, that something else besides be settled, so that the remaining angle may be known together with the remaining sides — as we shall show in the scholium to Prop. 66.

THEOR. 22. PROPOS. 24.

If there are two spherical triangles that have one angle equal to one angle, and two sides about another angle equal to two sides, each to each, and each of the remaining angles either greater or less than a right angle: then the remaining side will be equal to the remaining side, and the remaining angles equal to the remaining angles, each to each.

Let ABC and DEF be two spherical triangles in which the angles B and E are equal, and the two sides BC and CA are equal to the two sides EF and FD, each to each, about the angles C and F; and let each of the remaining angles A and D be either less than or greater than a right angle. I say that the remaining sides AB and DE are also equal, and that the remaining two angles A and C are equal to the remaining two angles D and F, each to each.

For if the sides AB and DE are not equal, let AB be the greater, [Margin: a — Prop. 1 of this treatise] and cut off the arc BG equal to the arc DE, [Margin: b — Theodosius I.20] and through the points C and G let the great-circle arc CG be drawn. Now since the sides BG and BC are equal to the sides ED and EF, and comprehend equal angles B and E, [Margin: c — Prop. 7 of this treatise] the arcs GC and DF will be equal, and so will the angles G and D. But the arc DF was assumed equal to the arc AC. The arc GC, therefore, will also be equal to that same arc AC; [Margin: d — Prop. 8 of this treatise] and so the angles A and CGA are equal. [Margin: e — Prop. 5 of this treatise] And since the two angles at G together equal two right angles, the two angles BGC and A will likewise together equal two right angles. Consequently, since the angle BGC has been shown equal to the angle D, the two angles D and A will together equal two right angles. But this is impossible: for since each of them is assumed either less than or greater than a right angle, both together will be either less than or greater than two right angles. The sides AB and DE, therefore, are not unequal but equal. [Margin: f — Prop. 18 of this treatise] Hence the two angles A and C will also be equal to the two angles D and F, each to each. And so, if there are two triangles, and so on. Which was to be shown.

SCHOLIUM.

We said that each of the remaining angles must be either greater or less than a right angle; for otherwise the proposition would be false. Take any spherical triangle ABC having the two sides AB and AC equal; extend the side CB to D so that CD is an arc less than a semicircle, [Margin: g — Theodosius I.20] and through the points A and D draw the great-circle arc AD. Then the triangles ADB and ADC have one angle equal to one angle — namely D, which is common — and the two sides AD and AB equal to the two sides AD and AC, each to each; and yet the remaining sides DB and DC are not equal, nor are the remaining angles DAB and DAC — indeed, neither are the angles ABD and ACD, unless each of them is a right angle, as we shall show. This comes about precisely because it is not the case that each angle ABD, ACD is greater than a right angle, or each less, but rather that either both are right angles, or one is greater and the other less than a right angle — which we prove as follows.

Let angle ABD first be a right angle. I say that C is also a right angle. For with the angle ABD being right, ABC will also be right, [Margin: h — Prop. 5 of this treatise] because both angles at B together equal two right angles; [Margin: i — Prop. 8 of this treatise] but this one is equal to angle C, because of the equality of the sides AB, AC. Therefore C too will be a right angle.

Next, let angle ABD be greater than a right angle. I say that C is less than a right angle. For since ABD is greater than a right angle, ABC will be less, [Margin: k — Prop. 5 of this treatise] because both together equal two right angles. Therefore the angle C too, [Margin: l — Prop. 8 of this treatise] which is equal to angle ABC, will be less than a right angle.

Finally, let angle ABD be less than a right angle. I say that C is greater than a right angle. For since ABD is less than a right angle, ABC — that is, C, which is equal to it — will be greater than a right angle.

Hence it is manifest that Proposition 8 of [Nicolaus Copernicus], on spherical triangles, in *De Revolutionibus* I, [Margin: Error of Nicolaus Copernicus] is false in the part where he says: If two triangles have two sides equal to two sides, one to the other, and an angle equal to an angle — the one at the base — then they will also have base equal to base, and the remaining angles equal to the remaining angles. For this is not true unless each of the remaining angles at the base is assumed either greater or less than a right angle. For in the triangles set out, ADB and ADC, there are two sides AD and AB equal to two sides AD and AC, and the

angle D is common over the bases DB and DC; and yet the bases are not equal, for the reason stated.

The same [Nicolaus Copernicus] errs likewise in the same book, Prop. 11, [Margin: Another error of Nicolaus Copernicus] where he says: Every triangle whose two sides are given together with some angle has its angles and sides determined. For even if the sides AD and AB are known together with the angle D, we still do not thereby arrive at knowledge of the other side and of the other angles, since the remaining side can be either DB or DC, and so on. It is necessary, then, that something else besides be settled before the remaining side, together with the remaining angles, can be made known — as we shall make clear in the scholium to Prop. 67.

THEOR. 23. PROP. 25.

In every isosceles spherical triangle, if the two equal sides are quadrants, the two equal angles at the base will be right angles; but if each is less than a quadrant, they will be acute; and if greater than a quadrant, obtuse. And if the two equal angles at the base are right angles, the two equal sides will be quadrants; but if they are acute, each side will be less than a quadrant; and if obtuse, each greater than a quadrant.

In the isosceles spherical triangle ABC, let the two equal arcs AB, AC first be quadrants. I say that the equal angles B, C at the base are right angles. For since each arc AB, AC is a quadrant, both together are equal to a semicircle. Hence, if the arc BC is produced to D, [Margin: a — Prop. 14 of this treatise] the angle ACD will be equal to the angle B; [Margin: b — Prop. 8 of this treatise] but the angle B is equal to the angle ACB. Therefore the angle ACD will be equal to the angle ACB; and so, [Margin: c — Prop. 5 of this treatise] since the two angles at C together equal two right angles, each angle at C will be a right angle. Hence the angle B too, [Margin: d — Prop. 8 of this treatise] which is equal to the right angle ACB, will be a right angle. Which was proposed.

Next, let each of the equal arcs AB, AC be less than a quadrant. I say that the equal angles B, C are acute. For since each arc AB, AC is less than a quadrant, both together are less than a semicircle. [Margin: e — Prop. 14 of this treatise] Hence the angle ACD will be greater than the angle B — that is, than the angle ACB — [Margin: f — Prop. 8 of this treatise] since the angles B and ACB are equal. [Margin: g — Prop. 5 of this treatise] Since, then, the two angles at C together equal two right angles, the angle ACB will be less than a right angle; and so the angle B too, [Margin: h — Prop. 8 of this treatise] which is equal to it, will likewise

be less than a right angle. Therefore the two angles B and ACB are acute. Which was proposed.

Finally, let each of the arcs AB, AC be greater than a quadrant. I say that the equal angles B, C are obtuse. For since each arc AB, AC is greater than a quadrant, both together are greater than a semicircle. [Margin: i — Prop. 14 of this treatise] Hence the angle ACD will be less than the angle B — that is, than the angle ACB, [Margin: k — Prop. 8 of this treatise] which is equal to angle B. [Margin: l — Prop. 5 of this treatise] Since, then, the two angles at C together equal two right angles, the angle ACB will be greater than a right angle — that is, obtuse; and therefore the angle B too, [Margin: m — Prop. 8 of this treatise] which is equal to it, will be obtuse. Which was proposed.

But now let each of the equal angles B, C be a right angle. I say that each arc AB, AC is a quadrant. For since ACB is a right angle, [Margin: n — Prop. 5 of this treatise] and the two angles at C together equal two right angles, ACD will also be a right angle, and hence equal to the right angle B. [Margin: o — Prop. 15 of this treatise] Therefore the two arcs AB, AC together equal a semicircle; and since they are assumed equal, each will be a quadrant. Which was proposed.

Next, let each of the angles B, C be acute. I say that each arc AB, AC is less than a quadrant. [Margin: p — Prop. 5 of this treatise] For since the two angles at C together equal two right angles, and ACB is assumed less than a right angle, ACD will be greater than a right angle, and therefore greater than B, which is also assumed less than a right angle. [Margin: q — Prop. 15 of this treatise] The arcs AB, AC together, then, are less than a semicircle; and so, since they are equal, each will be less than a quadrant. Which was proposed.

Finally, let each of the angles B, C be obtuse. I say that each arc AB, AC is greater than a quadrant. [Margin: r — Prop. 5 of this treatise] For since the two angles at C together equal two right angles, and ACB is assumed greater than a right angle, ACD will be less than a right angle, and therefore less than the angle B, which is also assumed greater than a right angle. [Margin: s — Prop. 15 of this treatise] The arcs AB, AC together, then, are greater than a semicircle; and so, since they are equal, each will be greater than a quadrant. Which was proposed. Therefore, in every isosceles spherical triangle, and so on. Which was to be demonstrated.

COROLLARIUM.

From this it follows that every equilateral, or equiangular, spherical triangle, if its several sides are quadrants, has its several angles right; but if the sides are less than a quadrant, the angles are acute; and if greater than a quadrant, obtuse. And

that every equiangular, or equilateral, spherical triangle, if its several angles are right, has its several sides quadrants; but if the angles are acute, the sides are less than a quadrant; and finally if obtuse, greater than a quadrant.

Diagram. Two spherical triangles side by side, for Theorem 21. The left triangle has apex A at the top and a curved base running from B (lower left) to C, with the point G marked on the base close to B (so that arc CG is cut off from BC). The right triangle has apex D at the top and a curved base from E (lower left) to F.

- A (apex, left triangle): vertex A
- B (lower left, left triangle): base endpoint
- G (on base near B, left triangle): point marking arc CG cut off equal to FE
- C (lower right, left triangle): base endpoint
- D (apex, right triangle): vertex D
- E (lower left, right triangle): base endpoint
- F (lower right, right triangle): base endpoint

Diagram. A single spherical triangle for the scholium to Theorem 21, illustrating the pole case. Apex A at top; curved base from B (lower left) to C (lower right); the point D lies on the base, and a great-circle arc AD descends from the apex to D.

- A (apex): pole of the base arc BC
- B (lower left): base endpoint
- D (on base, below apex): foot of the descending arc AD
- C (lower right): base endpoint

Diagram. Two spherical triangles side by side, for Theorem 22. The left triangle has apex A, with the point G marked on the left side just below A (so that arc BG is cut off equal to DE), and a curved base from B (lower left) to C. The right triangle has apex D and a base from E to F.

- A (apex, left triangle): vertex A
- G (on the left side near A, left triangle): point where BG is cut off equal to DE
- B (lower left, left triangle): base endpoint
- C (lower right, left triangle): base endpoint
- D (apex, right triangle): vertex D
- E (lower left, right triangle): base endpoint
- F (lower right, right triangle): base endpoint

Diagram. A single spherical triangle for the scholium to Theorem 22. Apex A at top; the isosceles triangle ABC has its side CB produced beyond B to D, so the base line runs D (lower left), B (lower middle), C (lower right); the great-circle arc AD is drawn from the apex to D.

- A (apex): vertex, with $AB = AC$
- D (lower left): point to which CB is produced
- B (lower middle): base vertex between D and C
- C (lower right): base vertex

Diagram. A pointed (ogival) isosceles spherical triangle for Theorem 23; the two equal sides AB, AC bulge outward and meet at the apex A. Base from B (lower left) to C (lower middle), with the base arc produced beyond C to D (lower right). This same figure is printed twice on the page (for the successive cases of the proof); it is described here once.

- A (apex): vertex where the two equal sides meet
- B (lower left): base endpoint
- C (lower middle): base endpoint
- D (lower right): point to which arc BC is produced

SCHOLIVM.

[continuation, on the following page]

Moreover, that when two sides of a spherical triangle are quadrants each angle at the base is a right angle, and that if each angle at the base is a right angle each side is a quadrant, can also be demonstrated in this way.

Let AB, AC be quadrants in the triangle ABC. I say that the angles B, C are right angles. For when the arcs AB, AC are produced until they meet at D, [Margin: t — Theodosius I.11] so that ABD, ACD are semicircles, the arcs DB, DC will also be quadrants, and hence each arc ABD, ACD will be bisected by the arc BC at the points B and C. [Margin: u — Scholium to Theodosius II.9; Theodosius I.15] Therefore the arc BC will pass through the poles of the arcs AB, AC, [Margin: x — Theodosius I.13] and consequently will make right angles at B and C.

But now let the angles B, C be right angles. I say that the sides AB, AC are quadrants. For since the angles B, C are right angles, [Margin: y — Theodosius I.13] the arc BC will pass through the poles of the arcs ABD, ACD, and so will bisect each of them at B, C. Therefore the arcs AB, AC, DB, DC [are the halves of the arcs ABD, ACD] [?], [Margin: z — Theodosius I.11] which indeed are

semicircles; [Margin: a — Theodosius II.9] and hence they are quadrants. Which was to be demonstrated.

THEOR. 24. PROPOS. 26.

In every isosceles spherical triangle whose two equal sides are quadrants, if the angle comprehended by them is a right angle, the base will be a quadrant; but if it is acute, the base will be less than a quadrant; and if obtuse, greater than a quadrant. And if the base is a quadrant, the angle comprehended by the sides will be a right angle; but if the base is less than a quadrant, the angle will be acute; and if greater than a quadrant, obtuse. And always the pole of the base will lie in the angle comprehended by the sides.

In the isosceles spherical triangle ABC, let the sides AB, AC be quadrants, and let the angle A first be a right angle, as in the first figure. I say that the base BC is a quadrant. For since AB, AC are quadrants, [Margin: a — Prop. 25 of this treatise] the angles B, C will be right angles. [Margin: b — Corollary of Prop. 25 of this treatise] Hence all the arcs will be quadrants. Therefore BC is a quadrant. Which was proposed.

Next, let the angle A be acute, as in the second figure. I say that the base BC is less than a quadrant. [Margin: c — Theodosius I.20] For if, through A and the pole of the arc AB, the great-circle arc AD is drawn, [Margin: d — Theodosius I.15] the angle BAD will be a right angle, and so greater than the acute angle BAC. Therefore AD will meet the arc BC produced, namely at the point D. [Margin: e — Prop. 25 of this treatise] Now since in the triangle ABC each angle B, C is a right angle, in the triangle ABD the two angles B and DAB will be right angles, and therefore equal; [Margin: f — Prop. 9 of this treatise] and consequently the arcs DA, DB will also be equal. Hence DAB is isosceles, having two right angles at its base AB; [Margin: g — Prop. 25 of this treatise] and so each arc AD, BD is a quadrant. Therefore BC will be less than a quadrant. Which was proposed.

Thirdly, let the angle A be obtuse, as in the third figure. I say that the base BC is greater than a quadrant. [Margin: h — Theodosius I.20] For if, through A and the pole of the arc AB, the great-circle arc AD is drawn, [Margin: i — Theodosius I.15] the angle DAB will be a right angle, and so less than the obtuse angle BAC. Therefore the arc AD will meet the arc BC within the triangle, namely at the point D. [Margin: k — Prop. 25 of this treatise] Now since in the triangle ABC each angle B, C is a right angle, in the triangle DAB the two angles at the base AB will be right angles, and therefore equal; [Margin: l — Prop. 9 of this treatise] and so the arcs

AD, BD are also equal. Hence DAB is isosceles, having two right angles at its base AB; [Margin: m — Prop. 25 of this treatise] therefore each arc AD, BD is a quadrant, and so BC is greater than a quadrant. Which was proposed.

But now let the base BC be a quadrant, as in the same first figure. I say that the angle A is a right angle. [Margin: n — Prop. 25 of this treatise] For since the two arcs CA, CB are quadrants, each angle A, B will be a right angle. Therefore the angle A is a right angle.

Next, let the base BC be less than a quadrant. I say that the angle BAC is acute. For if the arc BC is produced to D so that BD is a quadrant, [Margin: o — Theodosius I.20] and through the points A, D the great-circle arc AD is drawn — since the two arcs BA, BD are then quadrants, [Margin: p — Prop. 25 of this treatise] each angle D and DAB will be a right angle. Therefore the angle BAC is acute.

Finally, let the base BC be greater than a quadrant. I say that the angle BAC is obtuse. [Margin: q — Prop. 1 of this treatise] Cut off BD, an arc equal to the quadrant AB; [Margin: r — Theodosius I.20] and through the points A, D let the great-circle arc AD be described. And since the two arcs BA, BD are quadrants, [Margin: s — Prop. 25 of this treatise] each angle BDA, DAB will be a right angle. Therefore the angle BAC is obtuse.

I say further that in all these cases the point A is the pole of the base BC. For since the sides AB, AC are assumed quadrants, [Margin: t — Prop. 25 of this treatise] each angle at the base BC will be a right angle, [Margin: u — Theodosius I.23] and therefore each arc AB, AC will pass through the pole of the arc BC. So whether BC is a quadrant, or less, or greater than a quadrant, and whether the angle A is right, acute, or obtuse, the point A, where the arcs AB, AC meet, will always be the pole of the base BC. Therefore in every isosceles spherical triangle whose two sides, and so on. Which was to be shown.

SCHOLIVM.

Indeed, in every spherical triangle having two right angles, we shall demonstrate in the same way that, at the meeting of the two sides which subtend the right angles, is the pole of the remaining side that is adjacent to the right angles — even if it is not yet known that those two sides are quadrants. For let there be in the spherical triangle ABC two right angles B, C. I say that A is the pole of the arc BC. [Margin: x — Theodosius I.13] For each arc AB, AC will pass through the pole of the arc BC; and consequently A will be the pole of the arc BC.

Yet it is true [Margin: y — Prop. 25 of this treatise] that the two arcs AB, AC are always quadrants, because of the right angles B, C.

Diagram. A lune-shaped figure (a lens/vesica) for the scholium to Theorem 23. Two great-circle arcs meet at A (top) and D (bottom), forming the semicircles ABD and ACD; a horizontal arc BC crosses the middle, from B (left) to C (right).

- A (top): meeting point of the two arcs
- B (left, on the crossing arc): point where arc BC meets side ABD
- C (right, on the crossing arc): point where arc BC meets side ACD
- D (bottom): antipodal meeting point of the two arcs

Diagram. A group of three spherical triangles illustrating the three cases of Theorem 24, labelled with the interior numerals 1, 2, and 3 (the "2" and "3" carrying a small superscript "a").

- Prima figura (interior "1", small triangle, upper center): triangle ABC with apex A (top), B (lower left), C (lower right); the angle at A is marked as a right angle. This is the right-angled case.
- Secunda figura (interior "2^a", lower left): triangle with apex A (top), B (lower left), C (lower middle), and D (lower right) on the base BC produced beyond C. This is the acute-angle case, where arc AD meets BC produced at D.
- Tertia figura (interior "3^a", lower right): triangle with apex A (top), B (lower left), C (lower right), and D on the base within the triangle (between B and C). This is the obtuse-angle case, where arc AD meets BC inside the triangle at D.

THEOR. 25. PROPOS. 27.

In every spherical triangle whose sides are all greater than a quadrant — or in which one side is a quadrant and the remaining two are greater than a quadrant — all three angles are obtuse.

Let ABC be a spherical triangle in which, first, each of the sides is greater than a quadrant. I say that the three angles A, P, C [recte: A, B, C] are obtuse. For the triangle is either equilateral, or isosceles, or scalene.

If it is equilateral, [Margin: a — Corollary of Prop. 25 of this treatise] it is evident that the three angles are obtuse.

But if it is isosceles, having two equal sides AB and AC, [Margin: b — Prop. 25 of this treatise] the two angles B and C at the base will be obtuse. Let BD and BE be quadrants, [Margin: c — Theodosius, *Sphaerica* I.20] and through the points D and E let the great-circle arc ED be drawn, meeting arc CA — extended to F. Since, then, BD and BE are quadrants, and the angle B has been shown to be obtuse, [Margin: d — Prop. 26 of this treatise] the arc DE will be greater than a quadrant, [Margin: e — Prop. 25 of this treatise] and the angles BDE and BED will be right. But the arc AC is also assumed greater than a quadrant. Therefore the arcs DE and AC together are greater than a semicircle; and for that reason the arcs FD and FA together are less than a semicircle, since the arcs EF and FC together are less than a whole circle, [Margin: f — Prop. 2 of this treatise] each of the two being less than a semicircle. [Margin: g — Prop. 14 of this treatise] The angle FDB, therefore, is greater than the angle FAD. But the angle FDB is right, [Margin: h — Prop. 5 of this treatise] because the angles FDB and BDE are equal to two right angles, and BDE has been shown to be right. Therefore FAD is acute; and consequently, [Margin: i — Prop. 5 of this treatise] since FAD and DAC together equal two right angles, the angle BAC will be obtuse. But the angles B and C have also been shown to be obtuse. All three angles A, B, C, therefore, are obtuse.

Finally, if the triangle ABC is scalene, let side AC be greater than side AB, [Margin: k — Prop. 1 of this treatise] and let the arc AD be cut off equal to arc AB. The arc AD will still be greater than a quadrant, because the arc AB, to which it is equal, is also assumed greater than a quadrant. [Margin: l — Theodosius, *Sphaerica* I.20] If, then, through the points B and D the great-circle arc BD is drawn, [Margin: m — Prop. 25 of this treatise] each of the angles ADB and ABD will be obtuse. Much more, therefore, will the angle ABC be obtuse. Let BE and BF be quadrants, [Margin: n — Theodosius, *Sphaerica* I.20] and through the points E and F let the great-circle arc EF be drawn, meeting arc CA — extended to G. Since, then, BE and BF are quadrants, [Margin: o — Prop. 25 of this treatise] the angles at E and F will be right; and since the angle EBF has been shown to be obtuse, [Margin: p — Prop. 26 of this treatise] the arc EF will be greater than a quadrant. But the arc AC too is assumed greater than a quadrant. Therefore the arcs EF and AC together are greater than a semicircle; and hence, much more, FG and CG will be greater than a semicircle. [Margin: q — Prop. 14 of this treatise] The angle BFG, then, which we have shown to be right, is less than the angle BCG; and consequently the angle C will be obtuse. And since the arcs FG and CG together are less than a whole circle [Margin: r — Prop. 2 of this treatise] (each being less than a semicircle), while EF and AC together are greater than a semicircle, the arcs GE and GA together will be less than a semicircle; [Margin: s — Prop. 14 of this treatise] and consequently the angle GEB will be greater than the angle GAB.

Since, then, the angle GEB is right [Margin: t — Prop. 5 of this treatise] (because the two angles at E are equal to two right angles, and the angle BEF has been shown to be right), the angle GAB will be acute. Therefore, [Margin: u — Prop. 5 of this treatise] since GAB and BAC together equal two right angles, BAC will be obtuse. But the two angles ABC and C have also been shown to be obtuse. The three angles A, B, C of triangle ABC, therefore, are obtuse. Which was what was proposed.

Diagram. The isosceles case. A crowded spherical construction. Readable labels, top to bottom:

- A (apex, top): vertex of the isosceles triangle
- B (upper left): base vertex
- C (upper right): base vertex
- F (mid-left): intersection of arc ED extended with arc CA extended
- D (center): endpoint of quadrant BD
- A (center, second occurrence as printed): crossing of the extended arcs
- B (lower left): vertex of the lower great-circle construction
- E (lower center): endpoint of quadrant BE
- C (lower right): vertex of the lower construction

The figure shows the isosceles triangle ABC with quadrant arcs BD, BE and the arc ED extended to meet CA (extended) at F.

Diagram. The scalene case. Readable labels:

- G (top left): intersection of arc EF extended with arc CA extended
- A (top): apex vertex
- E (upper middle): endpoint of quadrant BE
- B (left): vertex
- D (right): point on AC with $AD = AB$
- F (lower right): endpoint of quadrant BF
- C (bottom): vertex

Spherical triangle ABC with AD cut off equal to AB, arc BD drawn, quadrants BE and BF, and arc EF extended to meet CA (extended) at G.

Diagram. The case of two arcs greater than a quadrant and the third equal to a quadrant, with the two greater arcs equal. Readable labels:

- E (top left): point where arc CD extended meets CA extended

- A (top): apex vertex
- D (upper middle): endpoint of quadrant BD
- B (lower left): vertex
- C (bottom): vertex

Now let there be, in the same triangle ABC, two arcs AB and AC greater than a quadrant, while BC is a quadrant. Then the arcs AB and AC are either equal or unequal. If they are equal, [Margin: x — Prop. 25 of this treatise] the two angles B and C will be obtuse. Let BD be a quadrant, [Margin: y — Theodosius, *Sphaerica* I.20] and through the points C and D let the great-circle arc CD be drawn, meeting arc CA — extended to E. Since, then, the arcs BC and BD are quadrants, [Margin: z — Prop. 25 of this treatise] the angles D and BCD will be right; [Margin: a — Prop. 26 of this treatise] and the arc CD, on account of the angle B, which we have shown to be obtuse, will be greater than a quadrant. But the arc AC too is assumed greater than a quadrant. Therefore the arcs CD and CA together are greater than a semicircle; and consequently, since the arcs CDE and CAE make up a whole circle, [Margin: b — Theodosius, *Sphaerica* I.11] (each being a semicircle) the arcs ED and EA will be less than a semicircle. Hence the angle EDB, which is right [Margin: c — Prop. 5 of this treatise] (because the two angles at D are equal to two right angles, [Margin: d — Prop. 14 of this treatise] and the angle BDC has been shown to be right), will be greater than the angle EAD; and so EAD will be acute. [Margin: e — Prop. 5 of this treatise] Since, then, the angles EAD and DAC together equal two right angles, BAC will be obtuse. The angles B and C too have been demonstrated obtuse. The three angles A, B, C of triangle ABC, therefore, are obtuse.

Diagram. The case where the two arcs greater than a quadrant are unequal.
 Readable labels:

- F (top left): point where arc CE extended meets arc CA extended
- A (top): apex vertex
- E (upper middle): endpoint of quadrant BE
- B (left): vertex
- D (lower right): point on AC with AD = AB
- C (bottom): vertex

But if the sides AB and AC, which are greater than a quadrant, are not equal, let AC be the greater, [Margin: f — Prop. 1 of this treatise] and let the arc AD be cut off equal to arc AB. [Margin: g — Theodosius, *Sphaerica* I.20] Through the points B and D let the great-circle arc BD pass. The arc AD will still be greater than a

quadrant, since AB, its equal, is also assumed to be the greater. [Margin: h — Prop. 25 of this treatise] The angles ADB and ABD, therefore, are obtuse. Much more, then, will the angle ABC be obtuse. Let BE be a quadrant, [Margin: i — Theodosius, *Sphaerica* I.20] and through the points C and E let the great-circle arc CE pass, meeting arc CA — extended to F. Since, then, BE and BC are quadrants, and the angle EBC has been shown to be obtuse, [Margin: k — Prop. 26 of this treatise] the arc EC will be greater than a quadrant, [Margin: l — Prop. 25 of this treatise] but the angles E and BCE will be right. The angle ACB, therefore, will be obtuse. And since the arc CE has been shown to be greater than a quadrant, and the arc AC too is assumed greater than a quadrant, the arcs CE and CA together will be greater than a semicircle. Since, then, the arcs CEF and CAF together equal a whole circle, [Margin: m — Theodosius, *Sphaerica* I.21] each being a semicircle, the arcs FE and FA together will be less than a semicircle. For this reason the angle FEB, which is right [Margin: n — Prop. 5 of this treatise] (for the two angles at E equal two right angles,

and the angle BEC has been shown to be right), [Margin: o — Prop. 14 of this treatise] will be greater than the angle FAE. The angle FAE, therefore, is acute; and consequently, [Margin: p — Prop. 5 of this treatise] since the two angles at A equal two right angles, the angle BAC will be obtuse. But the angles ABC and ACB have also been shown to be obtuse. The three angles in triangle ABC, therefore, are obtuse. In every spherical triangle, then, whose sides are all &c. Which was to be demonstrated.

SCHOLIVM.

*This proposition does not convert. For not every spherical triangle whose angles are all obtuse necessarily has all its sides greater than a quadrant, or two of them greater than a quadrant and one equal to a quadrant. Rather, two may be greater than a quadrant while the third is less than a quadrant. For let there be two semicircles on the surface of the sphere enclosing the obtuse angles A and C. If, then, two equal arcs AB and AD are taken, each of which is greater than a quadrant and a half [i.e. greater than one and one-half quadrants], so that both together exceed three quadrants, [Margin: q — Theodosius, *Sphaerica* I.20] and through the points B and D the great-circle arc BD is described, this arc BD will be less than a quadrant. [Margin: r — Prop. 4 of this treatise] For since the three arcs AB, AD, BD are together less than a whole circle, while the two arcs AB and AD are assumed greater than three quadrants, the third arc BD will necessarily be less than a quadrant: otherwise, were it a quadrant or greater than a quadrant, the three arcs of triangle ABC would exceed the whole circle. [Margin: s — Prop. 25 of this*

treatise] Now since the two angles B and D in triangle ABD are obtuse, and the third angle A is obtuse as well by hypothesis, all three angles A, B, D will be obtuse. And yet neither are all the sides greater than a quadrant, nor are only two greater with the third a quadrant; rather, two of them, AB and AD , are greater than a quadrant, while the third arc BD is less than a quadrant, as we have shown.

Diagram. A lens (vesica) formed by two great-circle semicircles intersecting. Readable labels:

- A (left vertex): obtuse angle
- B (upper right): vertex of arc AB
- C (right, small triangle near the right vertex): obtuse angle
- D (lower middle): vertex of arc AD

The two semicircles enclose the obtuse angles A and C ; the chord-arc BD closes off the triangle ABD .

THEOR. 26. PROP. 28.

In every right-angled spherical triangle whose sides are all less than a quadrant, the remaining two angles are acute. And if the remaining two angles are acute, each of the sides will be less than a quadrant.

Let ABC be a spherical triangle in which the angle B is right and each side is less than a quadrant. I say that the remaining angles A and C are acute. For let the arcs BA and BC be extended so that BD and BE are quadrants, [Margin: a — Theodosius, *Sphaerica* I.20] and through the points C and D let the great-circle arc CD be drawn, and likewise through the points A and E the great-circle arc AE . [Margin: b — Theodosius, *Sphaerica* I.13] And since the quadrant BD , on account of the right angle B , passes through the poles of arc BC , [Margin: c — Corollary of Theodosius I.16] and the pole of a great circle is distant from it by a quadrant of a great circle, D will be the pole of arc BC . [Margin: d — Theodosius, *Sphaerica* I.15] The angle BCD , therefore, will be right; and consequently the angle ACB is acute. In the same way, since the quadrant BE , on account of the right angle B , [Margin: e — Theodosius, *Sphaerica* I.13] passes through the poles of arc AB , [Margin: f — Corollary of Theodosius I.16] and the pole of a great circle is distant from it by a quadrant of a great circle, E will be the pole of arc AB . [Margin: g — Theodosius, *Sphaerica* I.15] The angle EAB , therefore, will be right; and consequently BAC is acute.

Diagram. The right-angled spherical triangle. Readable labels:

- D (top): endpoint of quadrant BD, pole of arc BC
- A (left): vertex
- B (lower left): the right angle
- C (bottom center): vertex
- E (lower right): endpoint of quadrant BE, pole of arc AB

Right angle at B; arcs BA and BC extended to quadrants BD and BE; arcs CD and AE drawn.

Now let the angle B in the same triangle ABC be right, and the remaining angles A and C be acute. I say that each side is less than a quadrant. For let the angles BCD and BAE be made right. Since, then, each of the angles B and BCD is right, [Margin: h — Prop. 25 of this treatise] each of the arcs BD and CD will be a quadrant. Therefore the arc BA is less than a quadrant. In the same way the arc BC will be less than a quadrant, [Margin: i — Prop. 25 of this treatise] because the arcs BE and AE too are quadrants, on account of the right angles B and BAE. But that the arc AC too is less than a quadrant we shall show thus. Since the arc BE is drawn through E, the pole of arc BD (for we shall show that E is the pole of arc AB, as above, since BE is a quadrant and at right angles to arc AB), the point C will lie within the periphery of the circle of arc BD, on the surface of the sphere, and away from its pole. [Margin: k — Scholium of Theodosius II.21] Hence the arc CA will be less than the arc CD. But CD has been shown to be a quadrant. Therefore AC will be less than a quadrant. All the sides of triangle ABC, therefore, are less than a quadrant. And so, in every right-angled spherical triangle, &c. Which was to be shown.

SCHOLIUM.

The first part of this proposition is true also if only each of the two arcs about the right angle is assumed less than a quadrant, even if it is not known whether the remaining arc, which subtends the right angle, is less than a quadrant. This is plainly established from the demonstration of the first part. For it was shown that the angles BAC and BCA are acute solely from the fact that each of the arcs BA and BC is assumed less than a quadrant, no mention being made of the arc AC. Yet the arc subtending the right angle will always be less than a quadrant, provided the two arcs containing the right angle are less than a quadrant, as is manifest from the demonstration. For since, from the fact that the arcs BA and BC are less than a quadrant, the angles A and C are acute (as was demonstrated in the first part), it follows that the arc AC too is less than a quadrant, as was shown in the latter part. And so a theorem of this sort may also be propounded:

In every right-angled spherical triangle whose two arcs containing the right angle are less than a quadrant, the arc subtending the right angle will also be less than a quadrant.

THEOR. 27. PROPOS. 29.

In every spherical triangle whose angles are all acute, the several sides are less than a quadrant.

Let ABC be a spherical triangle in which all the angles are acute. I say that the several sides are less than a quadrant. For let, first, all the acute angles be equal. On this assumption, [Margin: a — Corollary of Prop. 25 of this treatise] the several sides will be less than a quadrant, as was demonstrated above.

Diagram. An equilateral (all-acute) spherical triangle. Readable labels: A (apex), B (lower left), C (lower right).

Next, let only two of the acute angles be equal, B and C, and let A be less than each of them. [Margin: b — Prop. 25 of this treatise] Then each of the arcs AB and AC will be less than a quadrant. And since the angle B is assumed greater than the angle A, [Margin: c — Prop. 11 of this treatise] the arc AC will be greater than the arc BC. Since, then, the arc AC has been shown to be less than a quadrant, the arc BC will be much less than a quadrant.

Diagram. A spherical triangle with two equal acute angles. Readable labels: A (apex), B (lower left), C (lower right).

Thirdly, let only two of the acute angles again be equal, B and C, and let A be an acute angle greater than each of them. [Margin: d — Prop. 25 of this treatise] Then again each of the arcs AB and AC will be less than a quadrant. I say that BC too is less than a quadrant. For let the right angle BAD be made, and let the arc AD be equal to each of the arcs AB and AC, [Margin: e — Theodosius, *Sphaerica* I.20] and through the points B and D let the great-circle arc BD be described. Since, then, each of the arcs AB and AC has been shown to be less than a quadrant, AD too will be less than a quadrant. [Margin: f — Prop. 25 of this treatise] Each of the angles ABD and D, therefore, is acute. Hence, since in triangle ABD the angle BAD is right and the remaining angles are acute, [Margin: g — Prop. 28 of this treatise] all its sides will be less than a quadrant. The arc BD, therefore, is less than a quadrant. But since the sides AB and AC are equal to the sides AB and AD, and the angle BAD is greater than the angle BAC, [Margin: h — Prop. 12 of this treatise] the base BD will also be greater than the base BC. But the arc BD has been shown to be less than a quadrant. Much less than a quadrant,

therefore, will the arc BC be. The three arcs of triangle ABC, therefore, are all less than a quadrant.

Diagram. The third case, with the right angle BAD constructed. Readable labels:

- A (apex): vertex, with right angle BAD constructed
- B (lower left): vertex
- C (bottom center): vertex
- D (right): endpoint of arc AD equal to AB and AC

Lastly, let all the acute angles A, B, C be unequal, and let A be the greatest of all. [Margin: i — Prop. 11 of this treatise] Then the arc BC will be greater than either of the arcs AB and AC. Let the angle C also be greater than the angle B; [Margin: k — Prop. 11 of this treatise] then the arc AB will be greater than the arc AC. Since, then, the arc BC is greater than the arc AB, and AB greater than AC, let the arc BD be cut off equal to arc AB, and through the points A and D let the great-circle arc AD be drawn. [Margin: l — Prop. 8 of this treatise] The angles BAD and BDA will be equal. But the angle BAD is acute, since it is part of the acute angle BAC. Therefore the angle BDA too will be acute. [Margin: m — Prop. 25 of this treatise] Each of the arcs AB and BD, therefore, is less than a quadrant. Much more, then, will the arc AC, which is less than arc AB, be less than a quadrant. I say that the arc BC too is less than a quadrant. For let the angle BAE be made right and equal to the arc AC [i.e. with $AE = AC$], [Margin: n — Theodosius, *Sphaerica* I.20] and through the points B and E let the great-circle arc BE be described. And since the arc AC has been shown to be less than a quadrant, AE too will be less than a quadrant. In triangle ABE, then, the angle BAE is right, and each of the arcs containing it is less than a quadrant. [Margin: o — Scholium of Prop. 28 of this treatise] Therefore the remaining angles ABE and AEB are acute. Since, then, in the same triangle ABE the angle BAE is right and the other two are acute, [Margin: p — Prop. 28 of this treatise] all its sides will be less than a quadrant. The arc BE, therefore, is less than a quadrant. But since the two sides AB and AE are equal to the two sides AB and AC, and the angle BAE is greater than the angle BAC, [Margin: q — Prop. 12 of this treatise] the base BE will also be greater than the base BC. But the arc BE has been shown to be less than a quadrant. Much less than a quadrant, therefore, will the arc BC be. The three arcs of triangle ABC, therefore, are all less than a quadrant. For this reason, in every spherical triangle whose &c. Which was to be demonstrated.

Diagram. The last case (all angles unequal), with the right angle BAE constructed. Readable labels:

- A (apex): vertex, with right angle BAE constructed
- F (right): auxiliary point on the construction
- B (lower left): vertex
- D (bottom, near center): point with $BD = AB$
- C (bottom center): vertex
- E (lower left, near B): endpoint of arc AE equal to AC

Right angle BAE built on triangle ABC; arc BE described; arcs AD (with $BD = AB$) and the auxiliary point F marked.

SCHOLIUM.

Moreover, this proposition too cannot be converted. For not every spherical triangle whose several sides are less than a quadrant necessarily has all its angles acute. For one angle can be right and the remaining two acute, as is established from the preceding proposition. Indeed, one can even be obtuse and the rest acute. For let there be two semicircles ABC, ADC enclosing the obtuse angles A and C, and let two equal arcs AB and AD be taken, each of which exceeds one and one-half quadrants, [Margin: r — Theodosius, Sphaerica I.20] and through the points B and D let the great-circle arc BD be described, which will be less than a quadrant, as we showed in the scholium of Prop. 27. In triangle BCD, then, the three arcs BC, CD, BD will each be less than a quadrant, and yet not all the angles in triangle BCD are acute; rather C is obtuse, by hypothesis, [Margin: s — Prop. 25 of this treatise] while B and D are acute, because the two sides CB and CD are equal and less than a quadrant.

Diagram. A lens (vesica) of two great-circle semicircles. Readable labels:

- B (top): vertex of arc
- C (left, small triangle): obtuse angle
- A (right vertex): obtuse angle
- D (lower middle): vertex of arc

The semicircles ABC, ADC enclose the obtuse angles A and C; the arc BD closes off the small triangle BCD.

THEOR. 28. PROP. 30.

In any spherical triangle in which one arc is greater than a quadrant while each of the remaining two is less than a quadrant, none of the angles will be right.

Let ABC be a spherical triangle in which the arc AC is greater than a quadrant, while both AB and BC are less than a quadrant. I say that none of the angles is right. For let the angle B — which is opposite the arc AC that is greater than a quadrant — be right, if this be possible. Then, with AD cut off as a quadrant, and the arc AB extended to E so that AE too is a quadrant, [Margin: a — Theodosius, *Sphaerica* I.20] and with the great-circle arc DE described through the points D and E cutting the arc BC at F, [Margin: b — Prop. 25 of this treatise] each of the angles D and E will be right. But the angle ABC too is assumed right — that is, EBC; [Margin: c — Prop. 5 of this treatise] for the two angles at B equal two right angles. [Margin: d — Prop. 25 of this treatise] Each of the arcs EF and BF, therefore, will be a quadrant, and so the arc BC will be greater than a quadrant. Which is absurd, since it is assumed less than a quadrant. Therefore the angle B cannot be right.

Diagram. The construction for Theorem 28. Readable labels, arranged around the triangle:

- A (top): vertex
- G (upper right, below A): construction point on arc AC
- I (upper left, below A): construction point
- H (left): construction point
- B (left, below H): the assumed right angle
- E (lower left): point on AB extended, with AE a quadrant
- F (bottom center): point where arc DE cuts BC
- C (bottom right): vertex
- D (right): endpoint of quadrant AD

Spherical triangle ABC with AC greater than a quadrant; AD and AE quadrants; arc DE drawn cutting BC at F. (Additional construction points G, H, I are marked but not referenced in the visible portion of the proof, which continues on the following folio.)

[Continuation of THEOR. 28 / PROPOS. 30]

that if angle C is said to be right, then [Margin: e — Prop. 25 of this treatise], if the same construction is made, each of the arcs DF and CF will in the same way be a quadrant. (For the angle CDF too is right [Margin: f — Prop. 25 of this treatise], since each of D and E is right, on account of the quadrants AD and AE.) And so the arc BC will be greater than a quadrant — which is contrary to the hypothesis.

Finally, suppose angle A is granted to be right. If from the arc CA a quadrant CG is cut off, and the arc CB is produced as far as H so that CH too is a quadrant [Margin: g — Theodosius I.20], and through the points G and H the arc GH of a great circle is drawn, cutting the arc AB at I [Margin: h — Prop. 25 of this treatise], then each of the angles at G and H will be right. Since angle A too is assumed right, in the triangle AGI there will be two right angles, A and G. [Margin: i — Prop. 25 of this treatise] Therefore each of the arcs HI and GI is a quadrant, and so the arc AB is greater than a quadrant — which is contrary to the hypothesis.

We can, however, prove in another way that angle A cannot be right, even without cutting off the quadrant CG, and so on. For if angle A is granted right, [Margin: k — Prop. 26 of this treatise] the arc DE will be a quadrant. Since EA too is a quadrant, [Margin: l — Prop. 25 of this treatise] each of the angles A and D will be right, and E will be the pole of the arc AC [Margin: m — Theodosius I.13], because each of the arcs DE and AE passes through the pole of the arc AD, on account of the right angles A and D. In the same way, D will be the pole of the arc AB. Since therefore the point F lies inside the circumference of the circle AB and away from its pole, and the arc FE is drawn through the pole of the circle AB — namely through D — [Margin: n — Scholium on Theodosius II.21] the arc FB will be greater than the arc FE. By the same reasoning the arc FC will be greater than the arc FD, since FD is drawn through E, the pole of the circle AC. Therefore the whole arc BC will be greater than the quadrant DE. This is absurd, since it was assumed to be less than a quadrant. Therefore none of the angles A, B, C is right. Hence, in any spherical triangle, and so on. Which was to be demonstrated.

Diagram. A spherical figure of overlapping arcs illustrating the preceding proof. The labelled points, read off the scan under magnification, are:

- A (top): apex
- G (upper right, close to A): intersection point on the great circle GH
- I (upper left, below A): where GH cuts the arc AB
- H (left): endpoint of the cut-off quadrant CH
- B (left, below H): triangle vertex
- E (lower left): construction point
- F (lower centre): interior point on the circle AB
- D (right): construction point (pole of AB)
- C (lower right): triangle vertex (right angle in the hypothesis)

THEOR. 29. PROPOS. 31.

In any spherical triangle the three angles are indeed greater than two right angles, but less than six right angles.

Let ABC be a spherical triangle. I say that the three angles A, B, C are greater than two right angles but less than six right angles. For if all three angles are right or obtuse; or only two right or obtuse; or only one right and one of the others obtuse — it is plain that all three are greater than two right angles. But in every triangle the demonstration will run as follows. When the side BC is produced to D, the angle ACD will be either equal to, or less than, or greater than the angle B.

Let it first be equal. [Margin: a — Prop. 15 of this treatise] Then the arcs AB and AC together will be equal to a semicircle; [Margin: b — Prop. 16 of this treatise] and hence the two angles ABC and ACB equal to two right angles. Therefore the three angles A, B, C will be greater than two right angles.

Next let the angle ACD be less than angle B. [Margin: c — Prop. 15 of this treatise] Then the arcs AB and AC together will be greater than a semicircle; [Margin: d — Prop. 16 of this treatise] and therefore the two angles ABC and ACB greater than two right angles. Much more, then, will the three angles A, B, C be greater than two right angles.

Finally let the angle ACD be greater than angle B [Margin: e — Prop. 10 of this treatise], and let the angle DCE be made equal to angle B, and let the arc CE meet the arc BA produced at E; and let the arc CA at length be extended to F. [Margin: f — Prop. 15 of this treatise] Then the arcs EB and EC together will be equal to a semicircle, and therefore the arcs EA and EC together less than a semicircle. [Margin: g — Prop. 14 of this treatise] Therefore the angle EAF — that is, the angle BAC, which is equal to it as a vertical angle — will be greater than the angle ACE. [Margin: h — Prop. 16 of this treatise] But the angle ACE, together with the angles ACB and B, equals two right angles. Therefore the angles BAC, ACB, and B will be greater than two right angles. So the three angles together are always greater than two right angles.

But since every spherical angle, even an obtuse one, is less than two right angles, it is plain that the three angles of any spherical triangle together are less than six right angles. Therefore, in any spherical triangle whatever, the three angles, and so on. Which was to be demonstrated.

Diagram. The spherical triangle ABC with the side BC produced to D and the auxiliary arcs of the proof :

- F (top left): endpoint of CA produced
- E (top right): where the arc CE meets BA produced
- A (centre): apex of the triangle
- B (bottom left): vertex
- C (bottom centre): vertex, with BC produced beyond it
- D (bottom right): point on BC produced

THEOR. 30. PROPOS. 32.

In every spherical triangle in which one angle is right and another of the remaining angles acute: if the arc adjacent to those angles is a quadrant, the arc subtending the right angle will also be a quadrant; but if it is less than a quadrant, the subtending arc too will be less than a quadrant; and finally, if it is greater than a quadrant, the subtending arc too will be greater than a quadrant. But the arc subtending the acute angle will always be less than a quadrant.

In the triangle ABC let angle C be right and B acute, and let the arc BC first be a quadrant. I say that AB too is a quadrant. For let the angle CBD be made right, and let the arc BD meet the arc CA produced at D. [Margin: a — Prop. 25 of this treatise] Then each of the arcs BD and CD will be a quadrant; but BC too is assumed a quadrant. [Margin: b — Prop. 26 of this treatise] Therefore B is the pole of the arc CD, [Margin: c — Theodosius I.15] and hence the angle at A will be right. [Margin: d — Prop. 25 of this treatise] Therefore each of the arcs BC and BA will be a quadrant. So the arc AB, opposite the right angle C, is a quadrant.

Next let the arc BC be less than a quadrant. I say that the arc AB too is less than a quadrant. For again let the angle CBD be made right, and let the arc BD meet the arc CA produced at D; [Margin: e — Prop. 25 of this treatise] and, as before, each of the arcs BD and CD will be a quadrant. Then, with BC produced to E so that BE is a quadrant, [Margin: f — Theodosius I.20] let the arc DE of a great circle be drawn through the points D and E, which BA produced cuts at F. Since therefore the arcs BE and BD are quadrants, [Margin: g — Props. 25 & 26 of this treatise] each of the angles BDE and BED will be right, and B the pole of the arc DE. [Margin: h — Theodosius I.15] Therefore the angle at F will be right, [Margin: i — Prop. 25 of this treatise] and hence each of the arcs BE and BF will be a quadrant. Therefore the arc BA will be less than a quadrant.

Diagram. The right-angled spherical triangle of Theorem 30 with the auxiliary arcs :

- D (top): endpoint where the arc BD meets CA produced
- H (centre-left): where the great circle DG cuts AB
- A (centre): apex
- F (centre-right): where BA produced meets the arc DE
- B (bottom left): vertex, right angle CBD constructed here
- G (bottom centre-left): endpoint of the quadrant cut off on BC
- C (bottom centre): vertex, right angle of the triangle
- E (bottom right): point on BC produced

Finally let the arc BC be greater than a quadrant. I say that the arc AB too is greater than a quadrant. For again let the angle CBD be made right, and let the arc BD meet CA produced at D; [Margin: l — Prop. 25 of this treatise] and, as before, each of the arcs BD and CD will be a quadrant. Then, with the quadrant BG cut off, [Margin: m — Theodosius I.10] let the arc DG of a great circle be drawn through the points D and G, cutting the arc AB at H. Since therefore the arcs BD and BG are quadrants, [Margin: n — Props. 25 & 26 of this treatise] each of the angles BDG and BGD will be right, and B the pole of the arc DG. [Margin: o — Theodosius I.15] Therefore the angle at H will be right, [Margin: p — Prop. 25 of this treatise] and hence each of the arcs BG and BH will be a quadrant. Therefore AB will be greater than a quadrant.

And since the arc CD has always been shown to be a quadrant, the arc AG will be less than a quadrant. Therefore in every spherical triangle, and so on. Which was to be shown.

THEOR. 31. PROPOS. 33.

In every spherical triangle in which one angle is right and another of the remaining angles acute: if the arc adjacent to those angles is a quadrant, the remaining angle will be right; but if less than a quadrant, acute; and finally, if greater than a quadrant, obtuse.

In the spherical triangle ABC let angle C be right and B acute, and let the arc BC first be a quadrant. I say that the remaining angle A is right. [Margin: a — Prop. 32 of this treatise] For AB too will be a quadrant. [Margin: b — Prop. 25 of this treatise] Therefore each of the angles C and A is right.

Next let the arc BC be less than a quadrant. I say that angle A is acute. [Margin: c — Prop. 32 of this treatise] For the arc AB too will be less than a quadrant, and hence the arcs AB and BC together will be less than a semicircle. [Margin: d —

Prop. 16 of this treatise] Therefore the angles A and C are less than two right angles; and consequently, since C is right, A will be acute.

Finally let the arc BC be greater than a quadrant. I say that angle A is obtuse.

[Margin: e — Prop. 32 of this treatise] For AB too will be greater than a quadrant, and therefore the arcs AB and BC together will be greater than a semicircle.

[Margin: f — Prop. 16 of this treatise] Therefore the angles A and C are greater than two right angles; and hence, since C is right, A will be obtuse. Therefore in every spherical triangle in which one angle, and so on. Which was to be shown.

Diagram. A small right-angled spherical triangle for Theorem 31 :

- A (top): apex
- B (bottom left): acute angle
- C (bottom right): right angle

THEOR. 32. PROPOS. 34.

In every spherical triangle in which one angle is right: if either of the remaining angles is right, the arc subtending it will be a quadrant; but if acute, less than a quadrant; and finally, if obtuse, greater than a quadrant. And if either of the arcs containing the right angle is a quadrant, the angle it subtends will be right; but if less than a quadrant, acute; and finally, if greater than a quadrant, obtuse.

In the spherical triangle ABC let angle C be right, and let first the other of the remaining angles, namely B, also be right. I say that the arc AC, which subtends it, is a quadrant. For since each of the angles B and C is right, [Margin: a — Prop. 25 of this treatise] each of the arcs AB and AC will be a quadrant.

Next let B be an acute angle. I say that the arc AC is less than a quadrant. For let the angle CBD be made right, and let the arc BD meet the arc CA produced at D. Since therefore each of the angles C and CBD is right, [Margin: b — Prop. 25 of this treatise] each of the arcs BD and CD will be a quadrant, and hence the arc AC will be less than a quadrant.

Finally let angle B be obtuse. I say that the arc AC is greater than a quadrant. For let the angle CBE be made right, and let the arc BE cut the arc AC at E. Since therefore each of the angles C and CBE is right, [Margin: c — Prop. 25 of this treatise] each of the arcs BE and CE will be a quadrant, and hence the arc AC will be greater than a quadrant.

Again let angle C be right, and let the arc AC first be a quadrant. I say that the angle B subtended by it is right. [Margin: d — Corollary to Theodosius I.16] For A will be the pole of the arc BC [Margin: e — Theodosius I.13] (since the arc CA passes through the pole of the arc BC, on account of the right angle C); [Margin: f — Theodosius I.15] and hence the angle ABC is right.

Next let the arc AC be less than a quadrant. I say that the angle B subtended by it is acute. For with the arc CA produced to D so that CD is a quadrant, [Margin: g — Corollary to Theodosius I.16] D will in the same way be the pole of the arc CB, [Margin: h — Theodosius I.13] since the arc CA passes through the pole of the arc BC. [Margin: i — Theodosius I.20] Therefore, with the arc DB of a great circle drawn through the points D and B, [Margin: k — Theodosius I.15] the angle DBC will be right, and consequently ABC acute.

Finally let the arc AC be greater than a quadrant. I say that the angle B subtended by it is obtuse. For with the quadrant CE cut off, [Margin: l — Corollary to Theodosius I.16] E will again be the pole of the arc BC, [Margin: m — Theodosius I.13] because the arc CA passes through the pole of the arc BC. [Margin: n — Theodosius I.20] Therefore, with the arc EB of a great circle drawn through the points E and B, [Margin: o — Theodosius I.15] the angle EBC will be right, and hence ABC obtuse. Therefore in every spherical triangle, and so on. Which was to be demonstrated.

Diagram. The right-angled spherical triangle of Theorem 32 with the auxiliary great-circle arc :

- D (top left): endpoint of CA produced (pole of CB)
- A (top): apex
- E (just below and right of A): construction point on AC
- B (bottom left): vertex where the right auxiliary angle is built
- C (bottom right): right angle of the triangle

THEOR. 33. PROPOS. 35.

In every right-angled spherical triangle: if each of the arcs containing the right angle — or only one of them — is a quadrant, the arc subtending the right angle will also be a quadrant; but if each of the said arcs is less than a quadrant, or greater, the arc subtending the right angle will be less than a quadrant; and finally, if one of them is greater than a quadrant and the other less, the arc subtending the right angle will be greater than a quadrant.

In the right-angled spherical triangle ABC let angle B be right, and let first each of the arcs AB and BC — or only one of them — be a quadrant. I say that the arc AC, which subtends the right angle, is a quadrant. For if each of the arcs AB and BC is a quadrant, then, since angle B is assumed right, [Margin: a — Prop. 26 of this treatise] the arc AC too will be a quadrant. But if only one of the arcs AB and BC is a quadrant, let AB be a quadrant. Since therefore the arc AB is a quadrant [Margin: b — Theodosius I.13] and passes through the poles of the arc BC, on account of the right angle B, [Margin: c — Corollary to Theodosius I.16] A will be the pole of the arc BC, [Margin: d — Theodosius I.15] and therefore angle C will be right. Since then each of the angles B and C is right, [Margin: e — Prop. 25 of this treatise] each of the arcs AB and AC will be a quadrant. In the same way, if BC is assumed a quadrant, we shall show that AC is a quadrant. For likewise C will be the pole of the arc AB, and consequently angle A right. Since then each of the angles B and A is right, [Margin: f — Prop. 25 of this treatise] each of the arcs BC and AC will be a quadrant.

Diagram. The right-angled spherical triangle for the first case of Theorem 33 :

- A (top): apex
- B (bottom left): right angle
- C (bottom right): vertex

Next let each of the arcs AB and BC be less than a quadrant, or greater. I say that the arc AC is less than a quadrant. For if each is less than a quadrant, then with the arc CB produced on the side of B, and BA on the side of A, so that CD and BE are quadrants, [Margin: g — Theodosius I.20] let the arc DE of a great circle be drawn through the points D and E, cutting the arc CA produced at F. Since therefore in the triangle BED the angle B is right and the arc BE a quadrant, [Margin: h — Prop. 34 of this treatise] the angle D, which it subtends, will be right. Again, since in the triangle CDF the angle D is right and the arc DC a quadrant, [Margin: i — Prop. 34 of this treatise] the angle F will likewise be right, [Margin: k — Prop. 25 of this treatise] and therefore each of the arcs DC and FC will be a quadrant. Hence the arc AC will be less than a quadrant.

But if each of the arcs AB and BC is greater than a quadrant, then, with the quadrants BD and CE cut off, [Margin: l — Theodosius I.20] let the arc ED of a great circle be drawn through the points D and E, cutting the arc CA produced at F. Since therefore in the triangle DBE the angle B is right and the arc BD a quadrant, [Margin: m — Prop. 34 of this treatise] the angle E, which BD subtends, will be right. Again, since in the triangle CEF the angle E is right and the arc EC a quadrant, [Margin: n — Prop. 34 of this treatise] F will in the same way be right.

[Margin: o — Prop. 25 of this treatise] Therefore each of the arcs CE and CF will be a quadrant, and consequently the arc AC less than a quadrant.

Diagram. The figure for the second case of Theorem 33 (both containing arcs less than a quadrant) :

- E (top): endpoint of the quadrant on BA produced
- F (upper left): where the arc DE cuts CA produced
- A (upper right): apex
- D (bottom left): endpoint of the quadrant on CB produced
- B (bottom centre): right angle
- C (bottom right): vertex

Diagram. The figure for the second case of Theorem 33 (both containing arcs greater than a quadrant, with quadrants BD and CE cut off) :

- F (top left): where the arc ED cuts CA produced
- A (top right): apex
- D (upper left): endpoint of the cut-off quadrant BD
- B (bottom centre): right angle
- E (bottom, below B): endpoint of the cut-off quadrant CE
- C (bottom right): vertex

Finally let the arc AB be greater than a quadrant, but the arc BC less than a quadrant. I say that the arc AC is greater than a quadrant. For let the quadrant BD be taken away, and let the arc CB be produced to E so that CE is a quadrant; [Margin: p — Theodosius I.20] and through the points D and E let the arc ED of a great circle be drawn, cutting the arc AC at F. Since therefore in the triangle BED the angle B is right and the arc BD a quadrant, [Margin: q — Prop. 34 of this treatise] the angle E will be right. Again, since in the triangle CEF the angle E is right and the arc EC a quadrant, [Margin: r — Prop. 34 of this treatise] by the same reasoning the angle F will be right. [Margin: s — Prop. 25 of this treatise] Therefore each of the arcs CE and CF will be a quadrant, and hence the arc AC greater than a quadrant. Therefore in every right-angled spherical triangle, and so on. Which was to be demonstrated.

Diagram. The figure for the third case of Theorem 33 (AB greater than a quadrant, BC less) :

- A (top): apex
- F (top right): where the arc ED cuts AC

- D (upper left): endpoint left after the quadrant BD is taken away
- E (bottom left): endpoint of the quadrant on CB produced
- B (bottom centre): right angle
- C (bottom right): vertex

THEOR. 34. PROPOS. 36.

In every right-angled spherical triangle: if the arc subtending the right angle is a quadrant, then either each of the arcs containing the right angle is a quadrant, or at least one of them; but if it is less than a quadrant, each of the remaining arcs is either less or greater than a quadrant; and finally, if it is greater than a quadrant, one of the remaining arcs will be greater than a quadrant and the other less.

In the spherical triangle ABC let angle B be right, and let the arc AC subtending it be first a quadrant. I say that either both the arcs AB and BC are also quadrants, or at least one of them. For if neither of them is a quadrant, then either both will be greater, or less, than a quadrant, [Margin: a — Prop. 35 of this treatise] and hence the arc AC less than a quadrant; or one of them greater than a quadrant and the other less, [Margin: b — Prop. 35 of this treatise] and consequently the arc AC greater than a quadrant. But both of these are absurd, since the arc AC is assumed a quadrant. Therefore either both arcs AB and BC will be quadrants, or at least one of them.

Diagram. A right-angled spherical triangle for Theorem 34 :

- A (top): apex, subtended by BC
- B (bottom left): right angle
- C (bottom right): vertex

Next let the arc AC be less than a quadrant. I say that both arcs AB and BC are either less than a quadrant, or greater. For if neither is less nor greater than a quadrant, then either both will be quadrants, [Margin: c — Prop. 35 of this treatise] and thus the arc AC too a quadrant; or one of them a quadrant and the other not, [Margin: d — Prop. 35 of this treatise] and therefore the arc AC too a quadrant; or one indeed less than a quadrant and the other greater, [Margin: e — Prop. 35 of this treatise] and hence the arc AC too greater than a quadrant: all of which are absurd, since the arc AC is assumed less than a quadrant. Therefore both arcs AB and BC will be less than a quadrant, or greater.

Thirdly let the arc AC be greater than a quadrant. I say that one of the remaining arcs, AB and BC, is greater than a quadrant, and the other less. For if it is not the

case that one is greater and the other less than a quadrant, then either both will be quadrants, [Margin: f — Prop. 35 of this treatise] and consequently the arc AC too a quadrant; or both less than a quadrant, or greater, [Margin: g — Prop. 35 of this treatise] and hence the arc AC less than a quadrant: all of which are absurd, since the arc AC is assumed greater than a quadrant. Therefore one of the arcs AB and BC will be greater than a quadrant, and the other less. Hence in every right-angled spherical triangle, and so on. Which was to be demonstrated.

THEOR. 35. PROPOS. 37.

In every right-angled spherical triangle, if either of the remaining angles, or at least one...

(continuation of the enunciation of THEOR. 35. PROPOS. 37, begun on the preceding page)

...at least one is right, the arc subtending the right angle will be a quadrant. But if both of the remaining angles are less than a right angle, or both greater, the arc subtending the right angle will be less than a quadrant. And if, finally, one of the remaining angles is greater than a right angle and the other less, the arc subtending the right angle will be greater than a quadrant.

In the spherical triangle ABC, whose angle B is right, let first both angles A and C — or at least one of them, namely C — be right. I say that the arc AC, which subtends the right angle B, is a quadrant. For if both angles A and C are right, or only C, the triangle ABC will be right-angled, having the angle C right; and the angle B is also right. Therefore the arc AC will be a quadrant.[Margin: a — Prop. 34 of this treatise]

Next, let each of the angles A and C be either less than a right angle or greater. I say that the arc AC is less than a quadrant. For if both angles A and C are less than a right angle, both the arc BC and the arc AB will be less than a quadrant;[Margin: b — Prop. 34 of this treatise] but if both angles A and C are greater than a right angle, both the arc BC and the arc AB will be greater than a quadrant.[Margin: c — Prop. 34 of this treatise] Hence, since in the triangle ABC the angle B is right, and both arcs AB, BC are either less or greater than a quadrant, the arc AC will always be less than a quadrant.[Margin: d — Prop. 35 of this treatise]

Third, let the angle A be greater than a right angle and C less. I say that the arc AC is greater than a quadrant. For since the angle A is obtuse, the arc BC will be greater than a quadrant;[Margin: e — Prop. 34 of this treatise] and since the angle

C is acute, the arc AB will be less than a quadrant.[Margin: f — Prop. 34 of this treatise] Therefore, since the arc BC is greater than a quadrant and AB less, the arc AC will be greater than a quadrant.[Margin: g — Prop. 35 of this treatise] And so in every right-angled spherical triangle, and so on. Which was to be shown.

Diagram. A spherical triangle with curved (arc) sides.

- A (apex, top): vertex A
- B (lower left): vertex B
- C (lower right): vertex C

THEOR. 36. PROPOS. 38.

In every right-angled spherical triangle, if the arc subtending the right angle is a quadrant, at least one of the remaining angles will also be right; but if it is less than a quadrant, both of the remaining angles will be either greater than a right angle or less; and if, finally, it is greater than a quadrant, one of the remaining angles will be greater than a right angle and the other less.

In the spherical triangle ABC, whose angle B is right, let first the arc AC, subtending the right angle B, be a quadrant. I say that at least one of the angles A, C is also right. For since the angle B is right and the arc AC is a quadrant, at least one of the arcs AB, BC will be a quadrant,[Margin: a — Prop. 36 of this treatise] and therefore also the angle A or C, which that arc subtends, will be right. [Margin: b — Prop. 34 of this treatise]

Next, let the arc AC be less than a quadrant. I say that each of the angles A, C is either greater or less than a right angle. For each of the arcs AB, BC will be either greater than a quadrant or less;[Margin: c — Prop. 36 of this treatise] hence each of the angles A, C will be either greater or less than a right angle.[Margin: d — Prop. 34 of this treatise]

Finally, let the arc AC be greater than a quadrant. I say that one of the angles A, C is greater than a right angle and the other less. For one of the arcs AB, BC will be greater than a quadrant and the other less;[Margin: e — Prop. 36 of this treatise] therefore one of the angles A, C will be greater than a right angle and the other less.[Margin: f — Prop. 34 of this treatise] And so in every right-angled spherical triangle, and so on. Which was to be shown.

Diagram. A spherical triangle with curved (arc) sides, identical in labelling to the preceding figure.

- A (apex, top): vertex A
- B (lower left): vertex B
- C (lower right): vertex C

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From all this it is gathered that in every spherical triangle in which one of the arcs is a quadrant and one of the angles is right, at least one of the remaining arcs is also a quadrant, and at least one of the remaining angles is right. For if one of the angles is right and one of the arcs enclosing it is a quadrant, the arc subtending the right angle will also be a quadrant,[Margin: g — Prop. 35 of this treatise] and the angle which the former quadrant subtends will be right.[Margin: h — Prop. 34 of this treatise] But if the arc subtending the right angle is a quadrant, then either both arcs enclosing the right angle, or at least one of them, will be a quadrant;[Margin: i — Prop. 36 of this treatise] and either both of the remaining angles, or at least one, will be right.[Margin: k — Prop. 38 of this treatise] And so it cannot happen that a right-angled spherical triangle be given in which one arc alone is a quadrant; rather, either none will be a quadrant, or all three, or two will be quadrants. [Margin: NOTA / NOTE]

THEOR. 37. PROPOS. 39.

Spherical angles have the same ratio as their arcs.

Let there be two spherical angles BAC, EDF, whose arcs are BC, EF. I say that angle A is to angle D as the arc BC is to the arc EF. For A and D will be the poles of the arcs BC, EF, and the arcs AB, AC, DE, DF will be quadrants.[Margin: a — Def. 6 of this treatise] Therefore, with the arcs BC, EF produced, let any number of arcs BG, GH be taken equal to the arc BC, and any number of arcs FI, IK, KL equal to the arc EF;[Margin: b — Theodosius, *Sphaerica* I.20] and through the points G, H, I, K, L and the poles A, D let arcs of great circles AG, AH, DI, DK, DL be drawn, which will all be quadrants,[Margin: c — Euclid III.28] namely equal to the quadrants AB, AC, DE, DF, because the straight chords AG, AH, DI, DK, DL are also equal to the straight chords AB, AC, DE, DF, by the definition of a pole. Therefore all the angles at A will be equal to one another; and so, whatever multiple the arc CH is of the arc BC, the aggregate of all the angles at A will be the same multiple of the angle BAC.[Margin: d — Prop. 18 of this treatise] By the same reasoning, the aggregate of all the angles at D will be the same multiple of the angle EDF as the arc EL is of the arc EF. Now since, if the arc CH is equal to the arc EL, the angle HAC is also equal to the angle EDL;[Margin: e — Prop. 28 of this treatise] and if the arc CH is greater than the arc EL,

Diagram. Two fans of great-circle arcs. On the left a bundle of arcs descends from the pole A to points on a base line; on the right a similar bundle descends from the pole D.

- A (apex, upper left): pole of the left bundle
- D (apex, upper right): pole of the right bundle
- Base points, left to right: H, G, B, C (left group, subtended by arcs from A); then E, F, I, K, L (right group, subtended by arcs from D)

(the proof of THEOR. 37 continues) the angle HAC is also greater than the angle EDL;[Margin: f — Prop. 12 of this treatise] and if less, less. Therefore the arc CH and the angle HAC — equimultiples of the first magnitude BC and the third BAC — fall short of, or are equal to, or exceed together the arc EL and the angle EDL, which are equimultiples of the second magnitude EF and the fourth EDF. Hence, the ratio that the arc BC (the first magnitude) has to the arc EF (the second magnitude) is the same as that which the angle BAC (the third magnitude) has to the angle EDF (the fourth magnitude).[Margin: g — Euclid, Def. 6 of Book V] And so spherical angles have the same ratio as their arcs. Which was to be demonstrated.

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From this it follows that any spherical angle is to four right spherical angles as the arc of that angle is to the whole circumference of a great circle, and conversely. For since any spherical angle is to a right spherical angle as the arc of that angle is to a quadrant — that is, to the arc of a right angle —[Margin: h — Prop. 39 of this treatise] the same angle will also be to the quadruple of a right angle (namely to four right angles) as the same arc of that angle is to the quadruple of a quadrant, that is, to the whole circumference; and conversely. [Margin: i — Scholium 22 on Book V of Euclid]

THEOR. 38. PROPOS. 40.

If two great circles in a sphere cut each other, and on their peripheries two points are marked — whether both are taken in the same semicircle, or one in one semicircle and the other in the other semicircle of the same circle, or one in a semicircle of one circle and the other in either semicircle of the other circle — and through each of these points an arc of a great circle is drawn making a right angle, on whichever side, with the periphery of the other circle: then the sine of the arc intercepted between one of those points and either section of the circles will have to

the sine of the arc which, drawn through that point, makes a right angle with the periphery of the other circle, the same ratio that the sine of the arc lying between the other point and either section of the circles has to the sine of the arc which, described through that point, makes a right angle with the periphery of the other circle.

In a sphere let two great circles ABCD, AECF cut each other, and let first two points B, G be marked at random in the same semicircle ABC;[Margin: a — Theodosius, *Sphaerica* I.20] through these and through the pole of the circle AECF, which is H, let great circles IBHK, LGHM be drawn;[Margin: b — Theodosius, *Sphaerica* I.15] and the angles at I, L, K, M will be right. I say that the sine of the arc AB (or CB) has to the sine of the arc BI (or BK) the same ratio that the sine of the arc AG (or CG) has to the sine of the arc GL (or GM). For let the common section of the circles be the straight line AC,[Margin: c — Euclid XI.3] to which let perpendiculars BN, GO be drawn from B, G in the plane of the circle ABCD; [Margin: d — Euclid I.12] and BN will be the right sine both of the arc AB and of the arc CB, by the definition of the right sine; and in the same way GO will be the sine of each arc AG, CG. From the same points B, G let perpendiculars BP, GQ be dropped to the plane of the circle AECF.[Margin: e — Euclid XI.11[?] (printed digit uncertain: 11 or 31)] And since the straight lines BP, GQ fall upon the common sections of the circles IBK, LGM with the circle AECF,[Margin: f — Euclid XI.38] which they bisect at the points I, K and L, M — that is, they fall upon the diameters of the great circles IBK, LGM[Margin: g — Theodosius, *Sphaerica* I.11] (because the planes of these circles are at right angles to the plane of the circle AECF),[Margin: h — Theodosius, *Sphaerica* I.15] and consequently make right angles with the diameters of the circles IBK, LGM, by Euclid, Definition 3 of Book XI — then BP will be the right sine of the arcs BI, BK, and GQ the right sine of the arcs GL, GM, by the definition of the right sine. In the plane of the circle AECF let the straight lines NP, OQ be drawn; and by Euclid, Definition 3 of Book XI, the angles P, Q in the triangles NBP, OGQ will be right. Now since both the straight lines BN, GO are parallel, because of the right angles ANB, AOG,[Margin: i — Euclid I.28] and the straight lines BP, GQ are parallel, since these are perpendicular to the plane of the circle AECF,[Margin: k — Euclid XI.6] the angles B, G in those same triangles NBP, OGQ will also be equal.[Margin: l — Euclid XI.10] Therefore the triangles NBP, OGQ are equiangular;[Margin: m — Euclid I.32] and so, as NB, the sine of the arc AB (or CB), is to BP, the sine of the arc BI (or BK), so is OG, the sine of the arc AG (or CG), to GQ, the sine of the arc GL (or GM) [Margin: n — Euclid VI.4] — in whatever way the arcs are taken, since to each sine there correspond two arcs that together make a semicircle. That is, as the sine of the arc AB is to the sine of the arc BI, so is the sine of the arc AG to the sine

of the arc GL. Likewise, as the sine of the arc AB is to the sine of the arc BK, so is the sine of the arc AG to the sine of the arc GM. Likewise, as the sine of the arc CB is to the sine of the arc BI, so is the sine of the arc CG to the sine of the arc GL. Likewise, as the sine of the arc CB is to the sine of the arc BK, so is the sine of the arc CG to the sine of the arc GM. Likewise, as the sine of the arc AB is to the sine of the arc BI, so is the sine of the arc CG to the sine of the arc GM, and so on.

Diagram. The prior figure of this proposition: a circle (representing a great circle of the sphere) with a horizontal diameter AC and a dense network of chords and arcs. Two intersecting great-circle traces are drawn, meeting the diameter, together with the perpendiculars and auxiliary lines of the proof.

- A (left end of the horizontal diameter): section point
- C (right end of the horizontal diameter): section point
- O, N (on the horizontal diameter, near the centre, O left of N): feet of the constructions
- D (upper left, on the circle), F (near top, right of D), K (top), M (upper right): points on the circle
- H (centre, below N): pole of circle AECF used in the construction
- Q, G (lower-left region), R (lower-middle right): construction points
- L, P, B (lower-centre, arranged left to right and downward): points on the arc
- I, S, E (bottom): points on the circle

Next, let one point — say B — be taken in the semicircle ABC, and the other — namely D — in the other semicircle CDA of the same circle,[Margin: o — Theodosius, *Sphaerica* I.20] and through the points B, D and the pole of the circle AECF, which is H, let two arcs of great circles IBK, DFS be drawn;[Margin: p — Theodosius, *Sphaerica* I.35] and the angles F, I, S, K will be right. I say again that, as the sine of the arc AB (or CB) is to the sine of the arc BI (or BK), so is the sine of the arc AD (or CD) to the sine of the arc DF — or of the arc which, together with the arc FD, completes a semicircle from the point D to the point S of the semicircle AEC.

For the arc from F, through D, to S is a semicircle,[Margin: q — Theodosius, *Sphaerica* I.11] since the circles AECF, DFS cut each other into two equal parts at F, S.[Margin: r — Prop. 1 of this treatise] For let an arc AG be taken equal to the arc AD;[Margin: s — Theodosius, *Sphaerica* I.20] and through G and the pole of the circle AECF (namely through H) let an arc of a great circle LGM be drawn; [Margin: t — Theodosius, *Sphaerica* I.15] and the angles L, M will be right. Since,

then, the two angles A, L of the triangle AGL are equal to the two angles A, F of the triangle ADF — for the two angles at A are equal as vertical angles, and the angles L, F are right — [Margin: u — Prop. 6 of this treatise] and since the sides AG, AD, which subtend the right angles, are by construction equal, the arcs GL, DF will also be equal, [Margin: x — Prop. 21 of this treatise] and consequently their sines will be equal, as also the sines of the equal arcs AG, AD will be equal. Therefore the ratio of the sine of the arc AG to the sine of the arc GL is the same as that of the sine of the arc AD to the sine of the arc DF. But the sine of the arc AG is to the sine of the arc GL, as was demonstrated, as the sine of the arc AB (or CB) is to the sine of the arc BI (or BK), because the points B, G were taken in the same semicircle. Therefore it will also hold that, as the sine of the arc AB (or CB) is to the sine of the arc BI (or BK), so is the sine of the arc AD to the sine of the arc DF, and so on.

Finally, let a point B be taken in the semicircle ABC, and in the semicircle of the other circle — namely in AEC — another point L. [Margin: y — Theodosius, *Sphaerica* I.20] Through B and the pole of the circle AEC let an arc of a great circle IBK be drawn; likewise through L and the pole of the circle ABC the arc LGM of a great circle; [Margin: z — Theodosius, *Sphaerica* I.15] and the angles I, G will be right. I say again that, as the sine of the arc AB is to the sine of the arc BI, so is the sine of the arc AL to the sine of the arc LG, and so on. For through the poles of each circle ABCD, AECF let an arc of a great circle RE be drawn; [Margin: a — Theodosius, *Sphaerica* I.20] and the angles R, E will be right, [Margin: b — Theodosius, *Sphaerica* I.15] and the semicircles ABC, AEC will be bisected at the points R, E, [Margin: c — Theodosius, *Sphaerica* II.9] and so the sines of the quadrants AR, AE will be equal. [Margin: d — Euclid V.7] And the ratio of the sine of the arc AR to the sine of the arc RE will be the same as that of the sine of the arc AE to the sine of the arc ER. Now since, as the sine of the arc AR is to the sine of the arc RE, so is the sine of the arc AB to the sine of the arc BI, as was demonstrated (for the two points R, B were taken in the same semicircle), it will also hold that, as the sine of the arc AE is to the sine of the arc ER, so is the sine of the arc AB to the sine of the arc BI. But by the same reasoning, as the sine of the arc AE is to the sine of the arc ER, so is the sine of the arc AL to the sine of the arc LG. Therefore it will also hold that, as the sine of the arc AB is to the sine of the arc BI, so is the sine of the arc AL to the sine of the arc LG, and so on. And if, in place of the point L, another point D is taken in the other semicircle AFC of the same circle AECF, it will still hold that, as the sine of the arc AB is to the sine of the arc BI, so is the sine of the arc AF to the sine of the arc FD, and so on. For, as we have just shown, as the sine of the arc AB is to the sine of the arc BI, so is the sine of the arc AL to the sine of the arc LG; and as the sine of the arc AL is to the

sine of the arc LG, so it has been demonstrated that the sine of the arc AF is to the sine of the arc FD, because the points L, F were taken in the two semicircles of the same circle. Therefore it will also hold that, as the sine of the arc AB is to the sine of the arc BI, so is the sine of the arc AF to the sine of the arc FD. And thus the proposition is universally true, in whatever way the two points are taken, whenever the circles ABCD, AECF cut each other at angles that are not right.

But now let the circles ABCD, AECF cut each other at right angles at the points A, C, and let their common section be the straight line AC. Then, with the semicircle ABC bisected, say, at H, so that AH, CH are quadrants, let two points B, G be taken at random. I say again that the sine of the arc AB is to the sine of the arc which, drawn through B, makes right angles with the circle AECF, as the sine of the arc AG is to the sine of the arc which, drawn through G, makes right angles with the circle AECF. For since the circle ABC is supposed to be at right angles to the circle AECF, it passes through the poles of the circle AEC,[Margin: e — Theodosius, *Sphaerica* I.13] and so H will be the pole of the circle AEC.[Margin: f — Corollary to Theodosius, *Sphaerica* I.16] Therefore the perpendicular arcs drawn to the circle AEC through the points B, G will necessarily pass through H,[Margin: g — Theodosius, *Sphaerica* I.13] and so those arcs will be BA, GA. But it is evident that, as the sine of the arc AB is to the sine of the arc BA, so is the sine of the arc AG to the sine of the arc GA, since in each case the ratio is one of equality, or identity; for the sine of the arc AB and of the arc BA is the same, as is the sine of the arc AG and of the arc GA.

Diagram. The latter figure of this proposition: two great circles intersecting at A and C, with a further circle inscribed.

- D (top): point on the outer circle
- F (below D): point on the inner circle
- A (left): section point of the two circles
- C (right): section point of the two circles
- E (bottom): point on the outer circle
- G, H (lower centre, near E; G left of H): construction points

And if one of the points is H, the pole of the circle AEC, then any arc drawn from H — such as HE — will be perpendicular to AEC,[Margin: h — Theodosius, *Sphaerica* I.15] and therefore a quadrant.[Margin: i — Corollary to Theodosius, *Sphaerica* I.16] So again it is clear that the sine of the arc AB is to the sine of the arc BA as the sine of the arc AH is to the sine of the arc HE (or HA), since here too

the ratio is one of equality, and so on. Therefore, if two great circles in a sphere cut each other, and so on. Which was to be shown.

SCHOLIVM.

It is clear from what has been demonstrated that, if two circles cut each other, and in one of them, from two points taken at random, two straight lines are drawn perpendicular to the plane of the other circle, then the right sine of the arc intercepted between one of those points and either section of the circles is to the perpendicular dropped from that point onto the plane of the other circle, as the right sine of the arc lying between the other point and either section of the circles is to the perpendicular dropped from this other point onto the plane of the other circle. For in the prior figure of this proposition it was shown that the sine of the arc AB (or CB) is to BP, the right sine of the arc BI, as the sine of the arc AG (or CG) is to GQ, the right sine of the arc GL. Since, then, the sines BP, GQ are perpendiculars dropped from the points B, G onto the plane of the circle AECF, the proposition is evident. And if one of the points taken is B, on one side of the section A, and the other point taken is D, on the other side of the section A, in the same circle, it will nonetheless hold that the sine of the arc AB is to the perpendicular BP dropped from B onto the plane of the other circle AECF, as the sine of the arc AD is to the perpendicular that would be dropped from D onto the plane of the other circle AECF; because it was shown that the sine of the arc AB is to the sine of the arc BI as the sine of the arc AD is to the sine of the arc DF — and these sines of the arcs BI, DF are indeed the perpendiculars falling from the points B, D onto the plane of the circle AECF, as is plainly established from what has been demonstrated in this proposition. The same is seen in the latter figure; for there too it holds that the sine of the arc AB is to the perpendicular—

(the Scholium continues on the following page)

[perpendicu]lar dropped from B onto the plane of circle AECF, so is the sine of arc AG to the perpendicular dropped from G onto the plane of circle AECF. This is because the sines of arcs AB and AG are precisely the perpendiculars dropped from B and G onto the plane of circle AECF, and they fall upon the straight line AC, the common section of the circles, as is evident.

[Margin: k — 38 undecimi (Elements XI.38)]

From this the following theorems can easily be demonstrated, and several of them are of the greatest service in calculating spherical triangles. The first and second of them are the two Cyclic Theorems of Ptolemy [Ptolemaeus] in the first book of the *Almagest* (*Almagest* I), but here demonstrated far more briefly and

easily from what has been established in this scholium. For that reason it seemed they should not be omitted, even though their use does not become apparent in these triangles.

I.

If on the surface of a sphere two arcs of great circles are drawn out from a single point, each of them being less than a semicircle, and if from their endpoints two other arcs of great circles are reflected back onto them so as to intersect each other between those two earlier arcs, then: the ratio which the sine of the segment of one of the drawn-out arcs — the segment between its endpoint and the reflected arc — has to the sine of the other segment of the same drawn-out arc is compounded of the ratio which the sine of the segment of the reflected arc — between the same endpoint and the other reflected arc — has to the sine of the other segment of the same reflected arc, together with the ratio which the sine of the segment of the other drawn-out arc — between its endpoint and the reflected arc — has to the sine of the whole of that same drawn-out arc.

From the point A, on the surface of the sphere, let two arcs AB and AC, each less than a semicircle, be drawn; and from their endpoints B and C let two arcs BD and CE be reflected back onto them, intersecting each other at F. I say that the ratio of the sine of arc BE to the sine of arc EA is compounded of the ratio of the sine of arc BF to the sine of arc FD, and of the ratio of the sine of arc CD to the sine of arc CA. For draw from the points B, A, D three perpendiculars BG, AH, DI to the plane of circle CE. Since the two circles AB and CE cut each other mutually at E, and the perpendiculars BG, AH have been dropped from the points B, A onto the plane of circle CE, [a] it follows that as the sine of arc EB is to the sine of arc EA, so is the straight line BG to the straight line AH. Again, since the two circles BD and CE cut each other mutually at F, and the perpendiculars BG, DI have been dropped from the points B, D onto the plane of circle CE, by the same reasoning, as the sine of arc FB is to the sine of arc FD, so is the straight line BG to the straight line DI. Finally, since the two circles AC and CE intersect each other at C, and the perpendicular straight lines DI, AH have been dropped from the points D, A onto the plane of circle CE, similarly, as the sine of arc CD is to the sine of arc CA, so is the straight line DI to the straight line AH. But the ratio of the straight line BG to the straight line AH — with the line DI placed as mean — is compounded of the ratio of BG to DI and of the ratio of DI to AH. Therefore the ratio of the sine of arc BE to the sine of arc EA (which is the same as the ratio of BG to AH) is likewise

compounded of the ratio of the sine of arc BF to the sine of arc FD (which is the same as the ratio of BG to DI), and of the ratio of the sine of arc CD to the sine of arc AC (which is the same as the ratio of DI to AH). Which is what was proposed.

[Margin: a — Scholium 40 of this [treatise], and by permutation]

Diagram. The figure of the first theorem: a Menelaus-type configuration on the sphere. Apex A stands at the top; two arcs descend from A to the base vertices B (lower left) and C (lower right). From B and C the reflected arcs BD and CE cross the interior, intersecting at F. A band of interior points runs across the middle, and perpendiculars are dropped onto the plane of circle CE.

- A (top apex): endpoint from which arcs AB, AC are drawn
- G (middle left): foot of perpendicular BG
- E (middle, left of center): intersection of circles AB and CE
- H (middle center): foot of perpendicular AH
- D (middle right): endpoint of reflected arc BD
- F (lower center): intersection of arcs BD and CE
- I (lower center-right): foot of perpendicular DI
- B (lower left): endpoint of arc AB
- C (lower right): endpoint of arc AC

II.

With the same things assumed, the ratio of the sine of one of the drawn-out arcs to the sine of the segment of that same arc — between the point of drawing-out and the reflected arc — is compounded of the ratio which the sine of the reflected arc from the endpoint of the said arc has to the sine of the segment of that same reflected arc — between the other drawn-out arc and the other reflected arc — together with the ratio which the sine of the segment of the other reflected arc — between the endpoint of the other drawn-out arc and the earlier reflected arc — has to the sine of the whole of the later reflected arc.

That is, the ratio of the sine of arc AB to the sine of arc AE is compounded of the ratio of the sine of arc BD to the sine of arc DF, and of the ratio of the sine of arc CF to the sine of arc CE. For draw from the points B, E, F three perpendiculars BG, EH, FI to the plane of circle AC. Since the two circles AB and AC cut each other mutually at A, and the perpendiculars BG, EH have been dropped from the points B, E onto the plane of circle AC, [a] it follows that as the sine of arc AB is to the sine of arc AE, so is the straight line BG to the straight line EH. Again, since the

two circles BD and AC intersect each other at D, and the perpendiculars BG, FI have been dropped from the points B, F onto the plane of circle AC, by like reasoning, as the sine of arc DB is to the sine of arc DF, so is the straight line BG to the straight line FI. Finally, since the two circles AC and CE intersect each other at C, and the perpendiculars FI, EH have been dropped from the points F, E onto the plane of circle AC, by the same argument, as the sine of arc CF is to the sine of arc CE, so is the straight line FI to the straight line EH. But the ratio of the straight line BG to the straight line EH — with the line FI placed as mean — is compounded of the ratio of BG to FI and of the ratio of FI to EH. Therefore the ratio of the sine of arc AB to the sine of arc AE (which is the same as the ratio of BG to EH) is likewise compounded of the ratio of the sine of arc BD to the sine of arc DF (which is the same as the ratio of BG to FI), and of the ratio of the sine of arc CF to the sine of arc CE (which is the same as the ratio of FI to EH). Which is what was proposed.

[Margin: a — Scholium 40 of this [treatise], and by permutation]

Diagram. The figure of the second theorem: a flatter version of the same configuration, drawn as a triangle. Apex A stands at the top; the base runs from B (left) to C (right), and interior points and perpendiculars are marked with respect to the plane of circle AC.

- A (top apex): point from which arcs AB, AC are drawn
- E (upper left interior): intersection point on circle AB
- H (center interior): foot of perpendicular EH
- D (upper right interior): endpoint of reflected arc BD
- B (left base vertex): endpoint of arc AB
- F (center interior): intersection of the reflected arcs
- I (interior, right of F): foot of perpendicular FI
- G (below the figure): foot of perpendicular BG
- C (right base vertex): endpoint of arc AC

III.

With the same things assumed, the ratio of the sine of one of the drawn-out arcs to the sine of the segment of that same arc — between its endpoint and the reflected arc — is compounded of the ratio which the sine of the segment of the other drawn-out arc — between the point of drawing-out and the reflected arc — has to the sine of the remaining segment, together with the ratio which the sine of the segment of the reflected arc, from the endpoint of the later drawn-out arc, between the

endpoint and the reflected arc, has to the sine of the remaining segment of that same reflected arc.

That is (the figure of the first theorem being repeated), the ratio of the sine of arc AC to the sine of arc CD is compounded of the ratio of the sine of arc AE to the sine of arc EB, and of the ratio of the sine of arc BF to the sine of arc FD. For since the two circles AC and CE intersect each other at C, and the perpendiculars AH, DI have been dropped from the points A, D onto the plane of circle CE, [a] it follows that as the sine of arc CA is to the sine of arc CD, so is the straight line AH to the straight line DI. Again, since the two circles AB and CE cut each other mutually at E, and the perpendiculars AH, BG have been dropped from the points A, B onto the plane of circle CE, in like manner, as the sine of arc EA is to the sine of arc EB, so is the straight line AH to the straight line BG. Finally, since the two circles BD and CE cut each other mutually at F, and the perpendiculars BG, DI have been dropped from the points B, D onto the plane of circle CE, by the same reasoning, as the sine of arc FB is to the sine of arc FD, so is the straight line BG to the straight line DI. But the ratio of the straight line AH to the straight line DI — with the line BG placed as mean — is compounded of the ratio of AH to BG and of the ratio of BG to DI. Therefore the ratio of the sine of arc AC to the sine of arc CD (which is the same as the ratio of AH to DI) is likewise compounded of the ratio of the sine of arc AE to the sine of arc EB (which is the same as the ratio of AH to BG), and of the ratio of the sine of arc BF to the sine of arc FD (which is the same as the ratio of BG to DI). Which is what was proposed.

[Margin: a — Scholium 40 of this [treatise], and by permutation]

Diagram. The figure of the first theorem, reprinted here (identical to fig-p832-1). Apex A at top; interior band G, E, H, D across the middle; F and I below center; base vertices B (left) and C (right), with the reflected arcs BD and CE crossing at F.

IV.

With the same things assumed, the ratio of the sine of the segment of one of the reflected arcs — between the endpoint of the drawn-out arc and the other reflected arc — to the sine of the remaining segment is compounded of the ratio which the sine of the segment of one of the drawn-out arcs — between the same endpoint and the other reflected arc — [has to its remaining segment], together with the ratio of the sine of the other drawn-out arc to the sine of the segment of that arc between the endpoint and the reflected arc.

That is (the same figure of the first theorem being repeated), the ratio of the sine of arc BF to the sine of arc FD is compounded of the ratio of the sine of arc BE to the sine of arc EA, and of the ratio of the sine of arc AC to the sine of arc CD. For since the two circles BD and CE cut each other mutually at F, and the perpendiculars BG, DI have been dropped onto the plane of circle CE, [a] it follows that as the sine of arc FB is to the sine of arc FD, so is the straight line BG to the straight line DI. Likewise, since the two circles BA and CE cut each other mutually at E, and the perpendiculars BG, AH have been drawn from the points B, A to circle CE, it also follows that as the sine of arc EB is to the sine of arc EA, so is the straight line BG to the straight line AH. Finally, since the two circles AC and CE cut each other at C, and the perpendiculars AH, DI have been dropped from the points A, D onto the plane of circle CE, similarly, as the sine of arc CA is to the sine of arc CD, so is the straight line AH to the straight line DI. But the ratio of the straight line BG to the straight line DI — with the line AH placed as mean — is compounded of the ratio of BG to AH and of the ratio of AH to DI. Therefore the ratio of the sine of arc BF to the sine of arc FD (which is the same as the ratio of BG to DI) is likewise compounded of the ratio of the sine of arc BE to the sine of arc EA (which is the same as the ratio of BG to AH), and of the ratio of the sine of arc AC to the sine of arc CD (which is the same as the ratio of AH to DI). Which is what was proposed.

[Margin: a — Scholium 40 of this [treatise], and by permutation]

V.

With the same things assumed, the ratio of the sine of one of the reflected arcs to the sine of the segment of that same arc — between the endpoint of the drawn-out arc and the other reflected arc — is compounded of the ratio which the sine of the segment of the other drawn-out arc — between the point of drawing-out and the reflected arc — has to the sine of the whole drawn-out arc, together with the ratio of the sine of the other reflected arc to the sine of the segment of that same arc between the earlier drawn-out arc and the earlier reflected arc.

That is (the figure of the second theorem being repeated), the ratio of the sine of arc CE to the sine of arc CF is compounded of the ratio of the sine of arc AE to the sine of arc AB, and of the ratio of the sine of arc BD to the sine of arc DF. For since the two circles AC and CE cut each other mutually at C, and the perpendiculars EH, FI have been drawn from the points E, F to the plane of circle AC, [a] it follows that as the sine of arc CE is to the sine of arc CF, so is the straight line EH to the straight line FI. Again, since the two circles AB and AC intersect each other

at A, and the perpendiculars EH, BG fall from the points E, B onto the plane of circle AC, it also follows that as the sine of arc AE is to the sine of arc AB, so is the straight line EH to the straight line BG. Finally, since the two circles AC and BD cut each other mutually at D, and the perpendiculars BG, FI have been dropped from the points B, F onto the plane of circle AC, by like reasoning, as the sine of arc DB is to the sine of arc DF, so is the straight line BG to the straight line FI. But the ratio of the straight line EH to the straight line FI — with the line BG placed as mean — is compounded of the ratio of EH to BG and of the ratio of BG to FI. Therefore the ratio of the sine of arc CE to the sine of arc CF (which is the same as the ratio of EH to FI) is likewise compounded of the ratio of the sine of arc AE to the sine of arc AB (which is the same as the ratio of EH to BG), and of the ratio of the sine of arc BD to the sine of arc DF (which is the same as the ratio of BG to FI). Which is what was proposed.

[Margin: a — Scholium 40 of this [treatise], and by permutation]

Diagram. The figure of the second theorem, reprinted here (identical to fig-p832-2). Apex A at top; base from B (left) to C (right); interior points E, H, D across the upper interior, F and I at the center, and the perpendicular foot G below.

VI.

If on the surface of a sphere two great circles cut each other mutually, but not at right angles, and from two points assumed on one of them two perpendicular arcs are drawn to the other circle, then: as the sine of the arc intercepted between the point of intersection and one of the right angles is to the tangent of that perpendicular arc, so is the sine of the arc lying between the point of intersection and the other right angle to the tangent of that other perpendicular arc.

Let two great circles AB and AC cut each other mutually at A, not at right angles; and from the points C, E assumed on circle AC let the perpendicular arcs CB, ED be drawn to circle AB. I say that as the sine of arc AB is to the tangent of arc CB, so is the sine of arc

AD to the tangent of arc ED. For when the arcs BC, DE are produced until they meet at F, [a] the arcs BF, DF will be quadrants. Now since from the point B two arcs of great circles BA, BF are drawn out, and from their endpoints A, F two others AC, FD are reflected back onto them, intersecting each other at E, [b] the ratio of the sine of arc AB to the sine of arc AD will be compounded of the ratio of the sine of arc BC to the sine of arc CF, and of the ratio of the sine of arc EF to the sine of arc DE. [c] But (since CF is the complement of arc BC) as the sine of arc CF

is to the sine of arc BC, so is the whole sine [sinus totus] to the tangent of arc BC; and by conversion, as the sine of arc BC is to the sine of arc CF, so is the tangent of arc BC to the whole sine. Likewise (since EF is the complement of arc DE) as the sine of arc EF is to the sine of arc DE, so is the whole sine to the tangent of arc DE. Therefore the ratio of the sine of arc AB to the sine of arc AD will also be compounded of the ratio of the tangent of arc BC to the whole sine, and of the ratio of the whole sine to the tangent of arc DE. Since, then, the ratio of the tangent of arc BC to the tangent of arc DE is likewise compounded of the ratio of the tangent of arc BC to the whole sine, and of the ratio of the whole sine to the tangent of arc DE — the whole sine being placed between the said tangents — it follows that as the sine of arc AB is to the sine of arc AD, so is the tangent of arc BC to the tangent of arc DE; and by permutation, as the sine of arc AB is to the tangent of arc CB, so is the sine of arc AD to the tangent of arc ED. Which is what was proposed.

[Margin: a — 25 of this [treatise] (Prop. 25)] [Margin: b — Theorem 3 of this scholium] [Margin: c — 18 of the [doctrine of] Sines]

Diagram. The figure of Theorem VI: a lune-shaped (crescent) spherical figure. F stands at the top; two arcs descend and curve to the base at B (lower left) and D (lower center); A lies at the right end of the lower arc. The great circles AB and AC meet at A, and the perpendicular arcs CB and ED (produced to meet at F) are shown.

- F (top): meeting point of the produced arcs BC, DE
- C (upper left, on the arc): assumed point on circle AC
- E (center, below C): assumed point on circle AC, intersection of AC and FB
- A (right): mutual intersection of circles AB and AC
- B (lower left): foot of perpendicular arc CB on circle AB
- D (lower center): foot of perpendicular arc ED on circle AB

VII.

If on the surface of a sphere two quadrants of great circles intersect each other at angles that are not right, and through the extreme points a great-circle arc is drawn, and moreover from some point of one quadrant a perpendicular arc is dropped to the other, then: as the whole sine is to the tangent of this perpendicular arc, so is the tangent of the complement of the arc drawn through the extremities of the quadrants to the sine of the arc of the quadrant to which the perpendicular arc has been dropped

— the arc, that is, intercepted between the point of section and the perpendicular arc.

Let two quadrants of great circles AD, AE cut each other at A, at angles that are not right, and through D, E let the great-circle arc DE be described: [a] the angles D, E will then be right. Again, from any point C whatever let the perpendicular arc CB be dropped to AD; and when the arcs DE, BC are produced until they meet at F, [b] DF, BF will be quadrants. I say that as the whole sine is to the tangent of arc BC, so is the tangent of arc EF — which is the complement of arc DE — to the sine of arc AB. For since from the point D two arcs of great circles DA, DF are drawn out, and from their endpoints A, F two others AE, FB are reflected, cutting each other at C, [c] the ratio of the sine of arc CF to the sine of arc BC will be compounded of the ratio of the sine of arc EF to the sine of arc DE, and of the ratio of the sine of the whole quadrant AD to the sine of arc AB. [d] But (since CF is the complement of arc BC) as the sine of arc CF is to the sine of arc CB, so is the whole sine to the tangent of arc BC; likewise (since DE is the complement of arc EF) as the sine of arc DE is to the sine of arc EF, so is the whole sine to the tangent of arc EF; and by conversion, as the sine of arc EF is to the sine of arc DE, so is the tangent of arc EF to the whole sine. Therefore the ratio of the whole sine to the tangent of arc BC will be compounded of the ratio of the tangent of arc EF to the whole sine, and of the ratio of the whole sine of the quadrant AD to the sine of arc AB. Since, then, the ratio of the tangent of arc EF to the sine of arc AB is likewise compounded of the ratio of the tangent of arc EF to the whole sine, and of the ratio of the whole sine to the sine of arc AB — the whole sine being placed between that tangent and this sine — it follows that as the whole sine is to the tangent of arc BC, so is the tangent of arc EF to the sine of arc AB. Which is what was proposed.

[Margin: a — 25 of this [treatise] (Prop. 25)] [Margin: b — 25 of this [treatise] (Prop. 25)] [Margin: c — Theorem 4 of this scholium] [Margin: d — 18 of the [doctrine of] Sines]

Diagram. The figure of Theorem VII: a lune-shaped spherical figure. F stands at the top; an arc curves down to the base at D (lower left) and B (lower center); A lies at the right. The two quadrants AD, AE meet at A; the arc DE is described through the extremities; and the perpendicular arc CB is dropped from C to AD, the arcs DE and BC being produced to meet at F.

- F (top): meeting point of the produced arcs DE, BC
- E (left): extreme point of quadrant AE

- C (center): point on AE from which perpendicular CB is dropped; intersection of AE and FB
- A (right): mutual intersection of the two quadrants
- D (lower left): extreme point of quadrant AD
- B (lower center): foot of perpendicular arc CB on AD

VIII.

If on the surface of a sphere two great circles cut each other mutually at angles that are not right, and from two points assumed on one of them two perpendicular arcs are drawn to the other circle, then: as the sine of the arc between the point of section and one of the assumed points is to the secant of the complement of the arc drawn through the other assumed point, so is the sine of the arc between the point of section and this remaining assumed point to the secant of the complement of the arc drawn through that other assumed point.

In the adjoining figure let two great circles AD, AE cut each other at A, at angles that are not right, and from the points C, E let the perpendicular arcs CB, ED be drawn to AD, and let them be produced until they meet at F. [a] BF, DF will be quadrants, and therefore CF, EF will be the complements of the arcs BC, DE. I say that as the sine of arc AE is to the secant of arc CF, so is the sine of arc AC to the secant of arc EF. For since from the point D two arcs DA, DF are drawn out, and from their endpoints A, F two others AE, FB are reflected back onto them, intersecting each other at C, [b] the ratio of the sine of arc AE to the sine of arc AC will be compounded of the ratio of the sine of arc DE to the whole sine of the quadrant DF, and of the ratio of the whole sine of the quadrant BF to the sine of arc BC. [c] But as the sine of arc DE is to the whole sine of the quadrant DF, so is the whole sine to the secant of arc EF; [d] this is because the whole sine is the mean proportional between the right sine of arc DE and the secant of arc EF, which is the complement of arc DE. And by the same reasoning, as the secant of arc CF is to the whole sine, so is the whole sine

[Margin: a — 25 of this [treatise] (Prop. 25)] [Margin: b — Theorem 5 of this scholium] [Margin: c — 18 of the [doctrine of] Sines] [Margin: d — 18 of the [doctrine of] Sines]

Diagram. The figure of Theorem VIII: a lune-shaped spherical figure. F stands at the top; arcs curve down to the base at B (lower center) and D (lower right); A lies at the left. The great circles AD, AE meet at A; the perpendicular arcs CB, ED are drawn to AD and produced to meet at F, so that BF, DF are quadrants.

- F (top): meeting point of the produced perpendicular arcs CB, ED
- C (center-left): assumed point on circle AE, intersection of AE and FB
- E (right of C): assumed point on circle AE
- A (left): mutual intersection of circles AD, AE
- B (lower center): foot of perpendicular arc CB on AD
- D (lower right): foot of perpendicular arc ED on AD ...of the quadrant BF, to the sine of arc BC — because the whole sine [sinus totus] is likewise a mean proportional between the secant of arc CF, which is the complement of arc BC, and the sine proper [sinus rectus] of arc BC. Therefore the ratio of the sine of arc AE to the sine of arc AC will also be compounded of the ratio of the secant of arc CF to the whole sine, and of the ratio of the whole sine to the secant of arc EF. And since the ratio of the secant of arc CF to the secant of arc EF is likewise compounded of the ratio of the secant of arc CF to the whole sine and of the ratio of the whole sine to the secant of arc EF — the whole sine being a mean between these two secants — it follows that as the sine of arc AE is to the sine of arc AC, so is the secant of arc CF to the secant of arc EF; and, by alternation, as the sine of arc AE is to the secant of arc CF, so is the sine of arc AC to the secant of arc EF. Which is what was proposed.

ALITER [Another way]. [Margin: e — Prop. 40 of this treatise.] Since the sine of arc AE is to the sine of arc ED as the sine of arc AC is to the sine of arc CB — that is, by alternation, as the sine of arc AE is to the sine of arc AC, so is the sine of arc ED to the sine of arc CB — [Margin: f — Prop. 22 of the treatise on Sines.] and since the secant of arc CF is to the secant of arc EF as the sine of arc ED (which is the complement of the latter arc EF) is to the sine of arc CB (which is the complement of the former arc CF), it follows too that as the sine of arc AE is to the sine of arc AC, so is the secant of arc CF to the secant of arc EF. And, by alternation, as the sine of arc AE is to the secant of arc CF, so is the sine of arc AC to the secant of arc EF. Which is what was proposed.

EADEM [The same]. This same demonstration holds even when the two points are taken on opposite sides of the point of intersection. For let the two great circles EF and BA again cut one another at D; and from the points F and E of arc EF let the perpendicular arcs FA and EB be drawn to BA. I say that the sine of arc ED is to the secant of the complement of arc FA as the sine of arc DF is to the secant of the complement of arc EB. [Margin: g — Prop. 40 of this treatise.] For since the sine of arc ED is to the sine of arc EB as the sine of arc DF is to the sine of arc FA — and, by alternation, as the sine of arc ED is to the sine of arc DF, so is the sine of arc EB to the sine of arc FA — [Margin: h — Prop. 22 of the treatise on

Sines.] and since the sine of arc EB is to the sine of arc FA as the secant of the complement of arc FA is to the secant of the complement of arc EB, it follows too that as the sine of arc ED is to the sine of arc DF, so is the secant of the complement of arc FA to the secant of the complement of arc EB; and, by alternation, as the sine of arc ED is to the secant of the complement of arc FA, so is the sine of arc DF to the secant of the complement of arc EB. Which is what was proposed.

Here we have reused the fourth figure of Proposition 35, so that — even though no mention is made of the arcs BC and CF — we should not be compelled to construct a new figure.

Diagram. The sector-figure (Menelaus configuration) reused from Proposition 35 for the *ALITER* and *EADEM* demonstrations. A curved-sided figure in which two great-circle arcs cross.

- A (apex, top): vertex from which the arc descends
- F (upper right, just below A): point on the arc toward C
- D (mid-left, below A): point where the two arcs intersect
- E (lower left): endpoint on the base
- B (lower center): point on the base
- C (lower right): base vertex

THEOR. 39. PROPOS. 41.

In every spherical triangle, the sine of any arc has to the sine of the angle it subtends the same ratio that the sine of each of the remaining arcs has to the sine of the angle which it subtends.

Let ABC be any spherical triangle whatever. I say that the sine of arc AB is to the sine of angle C (which it subtends) as the sine of arc AC is to the sine of angle B (which it subtends), and as the sine of arc BC is to the sine of angle A (which it subtends).

First, let all three angles be right angles. [Margin: a — Corollary of Prop. 25 of this treatise.] Then all the arcs will consequently be quadrants. It is therefore obvious that the whole sine of quadrant AB is to the whole sine of the right angle C as the whole sine of quadrant AC is to the whole sine of the right angle B, and as the whole sine of quadrant BC is to the whole sine of the right angle A.

Next, let only two angles, A and B, be right. [Margin: b — Prop. 25 of this treatise.] Then arcs AC and BC will accordingly be quadrants, [Margin: c — Scholium of

Prop. 26 of this treatise.] and C will be the pole of arc AB. So again it is clear that the sine of arc AB is to the sine of angle C — that is, to the sine of arc AB (for AB is the arc of angle C, since C is the pole of arc AB, as has been shown) — as the whole sine of quadrant AC is to the whole sine of the right angle B, and as the whole sine of quadrant BC is to the whole sine of the right angle A; since the ratio is always one of equality.

Thirdly, let C alone be a right angle, while of the remaining angles A and B either both are less than a right angle, or both greater, or one greater and one less than a right angle. If, then, both are less than a right angle, [Margin: d — Prop. 28 of this treatise.] all the arcs will be less than a quadrant. Let them all be produced, and let the quadrants BD, AE, BF, AG be made; [Margin: e — Theodosius, *Sphaerica* I.20.] and through the points D, F let the great-circle arc DF be drawn, and through the points E, G the great-circle arc EG. [Margin: f — Prop. 25 of this treatise.] Then the angles D, F, E, G will be right, [Margin: g — Scholium of Prop. 26 of this treatise.] and B will be the pole of arc DF, and A the pole of arc EG; and hence arcs DF and EG will be the arcs of angles B and A. And both the quadrant BD and AE is the arc of the right angle C, as is clear from Definition 6. Now since the two great circles BD and BF cut one another on the sphere at the point B, and since on arc BD two points A, D have been taken, from which the perpendicular arcs AC, DF have been dropped to arc BF, [Margin: h — Prop. 40 of this treatise.] it follows that the sine of arc AB is to the sine of arc AC as the sine of arc BD is to the sine of arc DF; and, by alternation, the sine of arc AB of triangle ABC is to the sine of quadrant BD — that is, to the whole sine of the right angle C in the same triangle ABC — as the sine of arc AC of the same triangle ABC is to the sine of arc DF, that is, to the sine of angle B in the same triangle ABC. In the same way, the sine of arc AB in triangle ABC is to the whole sine of the right angle C of the same triangle ABC as the sine of arc BC is to the sine of arc EG, that is, to the sine of angle A in the same triangle ABC. The proposition, therefore, is evident.

Diagram. The general spherical triangle ABC of Proposition 41, with curved (great-circle) sides.

- A (apex, top): vertex opposite side BC
- B (lower left): vertex opposite side AC
- C (lower right): vertex opposite side AB

Diagram. The construction for the "thirdly" case (angles A and B both less than a right angle): two great-circle arcs crossing, with the arcs produced to quadrants.

- E (far left): endpoint of a produced arc

- B (top, left of center): triangle vertex
- A (top, right of center): triangle vertex
- D (right): endpoint of a produced arc
- G (lower left): foot of a perpendicular / arc endpoint
- C (lower center): point of intersection of the two arcs (the right-angle vertex)
- F (lower right): foot of a perpendicular / arc endpoint

If, on the other hand, both angles A and B are greater than a right angle, [Margin: i — Prop. 37 of this treatise.] then arc AB will be less than a quadrant, [Margin: k — Prop. 34 of this treatise.] and both arc AC and arc BC will be greater than a quadrant. Arc AB is therefore produced in both directions until the quadrants AE, BD are formed, and the quadrants AG, BF are cut off; [Margin: l — Theodosius, *Sphaerica* I.20.] then through the points D, F let the great-circle arc DF be drawn, and through E, G the great-circle arc EG. [Margin: m — Prop. 26 of this treatise.] Then again B will be the pole of arc DF, and A the pole of arc EG. Therefore DF and EG will be the arcs of angles B and A; and, moreover, both the quadrant BD and AE is the arc of the right angle C, by Definition 6. Likewise, because of the quadrants BD and BF, [Margin: n — Prop. 25 of this treatise.] each of the angles D and F [is right]; and because of the qua-

Diagram. The construction for the case where both A and B are greater than a right angle (this figure is reprinted at the top of the following page, p. 836, and is described here only once).

- E (upper left): endpoint of the produced arc AB
- B (upper left, right of E): triangle vertex
- A (upper right): triangle vertex
- D (upper right, far right): endpoint of the produced arc AB
- F (lower middle, left): foot of perpendicular
- G (lower middle, right): foot of perpendicular
- C (bottom point): the right-angle vertex

[Margin: o — Prop. 40 of this treatise.] ...the quadrants AE, AG, and each of the angles E, G will be right. Now since the two great circles BD and BC cut one another on the sphere at B, and on arc BD two points A, D have been taken, from which the two perpendicular arcs AC, DF have been dropped to BC, it follows that the sine of arc AB is to the sine of arc AC as the sine of arc BD is to the sine of arc DF; and, by alternation, the sine of arc AB of triangle ABC is to the sine of quadrant BD — that is, to the whole sine of the right angle C in the same triangle

ABC — as the sine of arc AC of the same triangle ABC is to the sine of arc DF, that is, to the sine of angle B in the same triangle ABC. And by the same reasoning, the sine of arc AB of triangle ABC is to the sine of quadrant AE — that is, to the whole sine of the right angle C in the same triangle ABC — as the sine of arc BC is to the sine of arc EG, that is, to the sine of angle A in the same triangle ABC. Which is what was proposed.

Finally, let one of the angles A, B be greater than a right angle and the other less; let B be the greater and A the less. [Margin: p — Prop. 37 of this treatise.] Then arc AB will be greater than a quadrant. [Margin: q — Prop. 34 of this treatise.] Likewise arc AC will also be greater than a quadrant, but arc BC less than a quadrant. Therefore let the quadrants BD, AE, and AG be cut off, and, arc BC being produced, let the quadrant BF be made; [Margin: r — Theodosius, *Sphaerica* I.20.] and through the points D, F let the great-circle arc DF be drawn, and likewise through E, G the great-circle arc EG. [Margin: s — Prop. 26 of this treatise.] Then again B will be the pole of arc DF, and A the pole of arc EG. Therefore DF and EG will be the arcs of angles B and A; and, moreover, both the quadrant BD and AE is the arc of the right angle C, by Definition 6. Likewise, because of the quadrants AE, AG, [Margin: t — Prop. 25 of this treatise.] each of the angles E, G will be right. Now since the two great circles BA and BF cut one another on the sphere at B, and on BA two points A, D have been taken, from which the two perpendicular arcs AC, DF have been dropped to BF, [Margin: u — Prop. 40 of this treatise.] it follows that the sine of arc AB is to the sine of arc AC as the sine of arc BD is to the sine of arc DF; and, by alternation, the sine of arc AB of triangle ABC is to the sine of quadrant BD — that is, to the whole sine of the right angle C in the same triangle ABC — as the sine of arc AC of the same triangle ABC is to the sine of arc DF, that is, to the sine of angle B in the same triangle ABC. And in the same way, the sine of arc AB of triangle ABC is to the sine of quadrant AE — that is, to the whole sine of the right angle C in the same triangle ABC — as the sine of arc BC of the same triangle ABC is to the sine of arc EG, that is, to the sine of angle A in the same triangle ABC. Which is what was proposed.

Diagram. The construction for the case where one of A, B is greater and the other less than a right angle : a curved figure with two great-circle arcs crossing.

- B (upper left): triangle vertex
- E (upper left, right of B): endpoint of a produced arc
- D (upper right): endpoint of a produced arc
- A (upper right, far right): triangle vertex
- C (lower left): the right-angle vertex / intersection

- G (lower center): foot of a perpendicular
- F (bottom point): foot of a perpendicular

Fourthly and lastly, let none of the angles A, B, C be a right angle. [Margin: x — Theodosius, *Sphaerica* I.20.] Through the point A and the pole of circle BC let the great-circle arc AD be drawn, and let it fall first upon the side BC, within the triangle; [Margin: y — Theodosius, *Sphaerica* I.15.] then the angles at D will be right. Now since in triangle ABD the angle D is a right angle, it will be — as has already been demonstrated — that the sine of arc AB is to the sine of angle ADB as the sine of arc AD is to the sine of angle B; and, by alternation, as the sine of arc AB is to the sine of arc AD, so is the sine of angle ADB to the sine of angle B. But in the same way, since in triangle ADC the angle D is a right angle, the sine of arc AD is to the sine of angle ACD as the sine of arc CD is to the sine of angle ADC, that is, to the sine of angle ADB; and, by alternation, as the sine of arc AD is to the sine of arc AC, so is the sine of angle ACD to the sine of angle ADB, since the angles at D are right. Therefore, by *ex aequali* in perturbed proportion, the sine of arc AB is to the sine of arc AC as the sine of angle ACD is to the sine of angle B, as appears in the accompanying scheme. Hence also, by alternation, the sine of arc AB in triangle ABC is to the sine of angle ACB in the same triangle ABC as the sine of arc AC of the same triangle ABC is to the sine of angle B in the same triangle ABC.

Diagram. The construction for the "fourthly" case (no angle right): the perpendicular arc AD is dropped from A to side BC.

- A (apex, top): vertex from which the perpendicular is drawn
- B (lower left): base vertex
- D (lower middle): foot of the perpendicular from A on BC
- C (lower middle-right): base vertex
- E (lower right): point where BC produced meets the perpendicular through the pole

arcus (arc)	anguli (angle)
AB.	ACD.
AD.	ADB.
AC.	B.

[Margin: z — Theodosius, *Sphaerica* I.15.] Next, let the arc drawn through A and the pole of circle BC fall upon arc BC produced to E; then angle E will be right. Now since in triangle ABE the angle E is a right angle, it will be — as has been

demonstrated — that the sine of arc AB is to the sine of angle E as the sine of arc AE is to the sine of angle B; and, by alternation, as the sine of arc AB is to the sine of arc AE, so is the sine of angle E to the sine of angle B. But by the same reasoning, since in triangle ACE the angle E is a right angle, the sine of arc AE is to the sine of angle ACE as the sine of arc AC is to the sine of angle E; and, by alternation, as the sine of arc AE is to the sine of arc AC, so is the sine of angle ACE to the sine of angle E. Therefore, by *ex aequali* in perturbed proportion, the sine of arc AB is to the sine of arc AC as the sine of angle ACE — that is, of angle ACB (since the sine of the two angles at C is the same, because their arcs make up a semicircle, as is established from the corollary of Proposition 5 of this treatise; and it is clear from what we said in the treatment of the sines that two arcs making up a semicircle have the same sine) — is to the sine of angle B, as appears in the accompanying scheme. Therefore also, by alternation, the sine of arc AB in triangle ABC is to the sine of angle ACB of the same triangle ABC as the sine of arc AC in the same triangle ABC is to the sine of angle B of the same triangle ABC. And if from B a second perpendicular arc be drawn to arc AC — which will fall either within the triangle or upon the arc produced — we shall show in the same way that the sine of arc AB is to the sine of angle ACB as the sine of arc BC is to the sine of angle BAC. And so, in every spherical triangle, the sine of any arc, and so on. Which was to be demonstrated.

arcus (arc)	anguli (angle)
AB.	ACE.
AE.	E.
AC.	B.

COROLLARIUM.

From this it is evident that, in every right-angled spherical triangle, as the sine of the arc subtending the right angle is to the whole sine — namely, to the sine of the right angle it subtends — so is the sine of any of the remaining

arcs to the sine of the angle it subtends. I have said this because most writers demonstrate this corollary as a proposition distinct from this our Proposition 41; but we have preferred to render this proposition more universal, in that it embraces both the right-angled and the non-right-angled spherical triangle.

SCHOLIUM.

In this and the following propositions we shall bring forward problems by which the calculation of right-angled spherical triangles is carried out, and which are drawn from these very propositions. Although sometimes several things are set forth for investigation in a given problem, still the primary and most important one is always that which is sought and which is inferred first and of itself from the problem. From this proposition, then, the following three problems are collected.

I.

In a right-angled spherical triangle, given the arc which is opposite the right angle and either one of the arcs enclosing the right angle, to find the angle opposite this arc.

In triangle ABC, whose angle C is right, let the arcs AB and AC be given. I say that the angle B, opposite arc AC, is also given. [Margin: a — Prop. 41 of this treatise.] For since the sine of arc AB is to the whole sine of the right angle C as the sine of arc AC is to the sine of angle B:

[Margin: Praxis.] If it be made that as the sine of the given arc opposite the right angle is to the whole sine, so is the sine of the arc given about the right angle to a fourth term, then the sine of the sought angle will be found.

But here careful attention must be paid to whether the sought angle B is acute or obtuse. For if it is acute, the arc corresponding to the sine found will give angle B; but if it is obtuse, the same arc taken from a semicircle will leave angle B. Now the given arc AC about the right angle C will neatly show whether angle B is acute or obtuse: [Margin: b — Prop. 34 of this treatise.] for if AC is less than a quadrant, angle B will be acute; but if greater than a quadrant, obtuse. Here we take a spherical triangle in which only one angle is right, and therefore no arc is a quadrant — which is also to be understood in what follows.

Diagram. The right-angled spherical triangle used for Problems I, II, and III, with curved sides and the right angle at C.

- A (apex, top): vertex opposite the hypotenuse-side BC
- C (lower left): the right-angle vertex
- B (lower right): vertex opposite arc AC

II.

In a right-angled spherical triangle, given the arc which is opposite the right angle and either one of the non-right angles, to find the arc opposite this angle.

In the same triangle let the arc AB, opposite the right angle C, be given, and in addition the angle B. I say that the arc AC, opposite angle B, is also given. [Margin: a — Prop. 41 of this treatise.] For since the sine of arc AB is to the whole sine of the right angle C as the sine of arc AC is to the sine of angle B, it will be, by conversion, that as the whole sine is to the sine of arc AB, so is the sine of angle B to the sine of arc AC.

[Margin: Praxis.] If, then, it be made that as the whole sine is to the sine of the arc opposite the right angle, so is the sine of the given angle to a fourth term, the sine of the sought arc will be found.

[Margin: b — Prop. 34 of this treatise.] This arc will be less than a quadrant if the given angle is acute, but greater than a quadrant if it is obtuse.

III.

In a right-angled spherical triangle, given either one of the arcs about the right angle and the angle opposite it, to find the arc opposite the right angle. It must, however, be established whether the third angle is acute or obtuse, or whether the third arc is less or greater than a quadrant.

In the same triangle let the arc AC, about the right angle C, be given, and also the angle B opposite it. I say that the arc AB, opposite the right angle, is likewise given. [Margin: a — Prop. 41 of this treatise.] For since the sine of arc AC is to the sine of angle B as the sine of arc AB is to the whole sine of the right angle C, it will be, by conversion, that as the sine of the given angle B is to the sine of the given arc AC, so is the whole sine to the sine of arc AB opposite the right angle, which is sought.

[Margin: Praxis.] If, then, it be made that as the sine of the given angle is to the sine of the given arc, so is the whole sine to a fourth term, the sine of the sought arc — the one opposite the right angle — will be found.

It must, however, be established whether the third angle A is acute or obtuse, or whether the third arc CB is less or greater than a quadrant. For from this we shall learn when the sought arc AB is less than a quadrant and when greater, if it is not otherwise established. For if angle A is acute, provided B too is given as acute —

or if A is obtuse, provided B too is given as obtuse — [Margin: b — Prop. 37 of this treatise.] then arc AB, opposite the right angle, will be less than a quadrant. But if angle A is acute while B is given as obtuse — or if A is obtuse while B is given as acute — arc AB will be greater than a quadrant. So too, if arc CB is less than a quadrant, provided AC too is given as less than a quadrant — or if CB is greater than a quadrant, provided AC too is given as greater than a quadrant — [Margin: c — Prop. 35 of this treatise.] then arc AB, opposite the right angle, will be less than a quadrant. But if arc CB is less than a quadrant while AC is given as greater than a quadrant — or if CB is greater than a quadrant while AC is given as less than a quadrant — arc AB will be greater than a quadrant. Therefore it is not enough to have one arc about the right angle together with the opposite angle given, as Copernicus would have it in Proposition 4 on spherical triangles — a point of which we gave warning above in the scholium to Proposition 21 — but the species of the third angle or of the third arc must also be given. In this matter Johannes Regiomontanus [Regiomontanus] too went astray (Regiomontanus, *De triangulis* IV.27).

THEOR. 40. PROPOS. 42.

In every right-angled spherical triangle, none of whose arcs is a quadrant, the sine of either of the remaining angles has to the whole sine the same ratio that the sine of the complement of the other angle has to the sine of the complement of the arc that subtends it.

[Continuation of THEOR. 40 / PROPOS. 42, whose enunciation was given on the previous page. This page begins Clavius's demonstration.]

In the spherical triangle ABC let angle B be right, and let none of the arcs be a quadrant. I say that the sine of angle C is to the whole sine [sinum totum, the radius] as the sine of the complement of the remaining angle A is to the sine of the complement of arc BC, which subtends angle A. For since none of the arcs is assumed to be a quadrant, none of the remaining angles will be right [Margin: a — Schol. 25 of this treatise]; otherwise triangle ABC, having two right angles, would have two quadrant arcs, which is not assumed.

So let angle A first be acute, and let arc AB — which is adjacent both to it and to the right angle B — be less than a quadrant. On this assumption, angle C too will be acute [Margin: b — Prop. 33 of this treatise], and hence [Margin: c — Prop. 28 of this treatise] all the arcs of triangle ABC will be less than a quadrant. Let arcs AB and AC be produced, let AD and AE be made quadrants [Margin: d — Theodosius, *Sphaerica* I.20], and through the points D and E let the arc DE of a

great circle be drawn, meeting arc BC (produced) at F. Then both angles D and E will be right [Margin: e — Prop. 25 of this treatise], because AD and AE are quadrants; and further, since angle B is also right, both arcs BF and DF will be quadrants [Margin: f — Prop. 25 of this treatise], on account of the right angles B and D. DE will also be the arc measuring angle A [Margin: g — Prop. 26 of this treatise], because A is the pole of arc DE, since AD and AE are quadrants. Likewise arc EF will be the complement of arc DE, and arc CF the complement of arc BC, because DF and BF are quadrants.

Now since in triangle CEF angle E is right, [Margin: h — Corollary to Prop. 41 of this treatise] it follows that as the sine of arc CF is to the whole sine, so the sine of arc EF is to the sine of angle ECF; and, converting, as the sine of angle ECF — that is, of angle ACB, [Margin: i — Prop. 6 of this treatise] which is equal to it at the vertex — is to the sine of arc EF, so the whole sine is to the sine of arc CF; and, by permutation, as the sine of angle ACB is to the whole sine, so the sine of arc EF — that is, the sine of the complement of angle A — is to the sine of arc CF, that is, to the sine of the complement of arc CB. Which is what was proposed.

Diagram. The figure for the first case: a right spherical triangle ABC with the arcs AB, AC produced and the quadrant construction added.

- F (top): apex where arc DE (produced) meets arc BC produced
- E (upper middle, right): point on arc AC produced, endpoint of quadrant AE
- C (upper middle, left of E): vertex of the triangle
- A (left): vertex (acute angle), pole of arc DE
- B (bottom, center): vertex of the right angle
- D (bottom, right of B): point on arc AB produced, endpoint of quadrant AD

Second case.

Next let angle A be obtuse, arc AB still less than a quadrant. Let angle BAD be made right, and let it cut arc AD off from arc BC at D. Arc AB being likewise produced, let AE be made a quadrant, [Margin: k — Theodosius, *Sphaerica* I.20] and through the points E, D let the arc ED of a great circle be drawn, cutting arc AC at F. And since the two angles DAB, DBA are right, [Margin: l — Schol. 25 of this treatise] arcs AD, BD will be quadrants; and since AE too is a quadrant and angle DAE is right, [Margin: m — Prop. 26 of this treatise] DE will also be a quadrant, and A the pole of arc DE. Therefore arc AF will also be a quadrant, [Margin: n — Corollary 16 to Theodosius, *Sphaerica* I] since arc EF always lies a quadrant away from its pole. [Margin: o — Theodosius, *Sphaerica* I.15] Likewise each angle at F, since arc AF passes through A, the pole of arc EF, will be right.

Moreover EF will be the arc measuring angle BAC, on account of the quadrants AE, AF. Arc DF will also be the complement of arc EF, that is, of angle BAC; and arc CD the complement of arc BC, on account of the quadrants DE, BD.

Since, then, in triangle CDF angle F is right, [Margin: p — Corollary to Prop. 41 of this treatise] it will follow that as the sine of arc CD is to the whole sine, so the sine of arc DF is to the sine of angle C; and, converting, as the sine of angle C is to the sine of arc DF, so the whole sine is to the sine of arc CD; and, by permutation, as the sine of angle C is to the whole sine, so the sine of arc DF — that is, the sine of the complement of angle BAC — is to the sine of arc CD, that is, to the sine of the complement of arc BC. Which is what was proposed.

Diagram. The figure for the second case (angle A obtuse, arc AB less than a quadrant).

- C (top): vertex
- F (upper left, below C): point where arc ED cuts arc AC
- D (center): point where arc AD cuts arc BC
- A (left): vertex of the obtuse angle, pole of arc DE
- B (bottom, center): vertex of the right angle
- E (bottom, right): endpoint of quadrant AE on arc AB produced

Third case.

Now let angle A be acute, and arc AB greater than a quadrant. On this assumption, the remaining angle C will be obtuse; [Margin: q — Prop. 33 of this treatise] and therefore arc AC, which subtends the right angle B, will likewise be greater than a quadrant. [Margin: r — Prop. 37 of this treatise] Let the quadrants AD, AE be cut off, [Margin: s — Theodosius, *Sphaerica* I.20] and through the points D, E let the arc DE of a great circle be drawn, meeting arc BC (extended) at F; [Margin: t — Prop. 25 of this treatise] then each of the angles D, E will be right, on account of the quadrants AD, AE; and consequently, since angle B too is right, [Margin: u — Prop. 25 of this treatise] the arcs BF, DF will be quadrants; and therefore BD will be the arc measuring angle F. Likewise arc CF will be the complement of arc BC, and arcs DB, EC the complements of arcs AB, AC, on account of the quadrants BF, AD, AE. [Margin: x — Prop. 26 of this treatise] Now it is clear that in triangle CEF, as the sine of arc EC — that is, the sine of the complement of arc AC — is to the sine of angle F, that is, to the sine of arc DB, that is, to the sine of the complement of arc AB, so the sine of arc CF, namely the sine of the complement of arc BC, is to the sine of the right angle E, that is, to the whole sine.

(For the two angles at C have the same sine, since their arcs make up a semicircle, by the corollary to Prop. 5.) — and by permutation, as the sine of angle ACB is to the whole sine, so the sine of arc EF, or of the complement of angle A, is to the sine of arc CF, or of the complement of arc BC. Which is what was proposed.

Diagram. The figure for the third case (angle A acute, arc AB greater than a quadrant).

- F (top): apex where arc DE meets arc BC extended
- A (right): vertex of the acute angle
- E (center): endpoint of quadrant AE
- C (center, left of E): vertex
- D (bottom, center): endpoint of quadrant AD
- B (bottom left): vertex of the right angle

Fourth and last case.

Lastly let angle A be obtuse, and arc AB still greater than a quadrant. Let the right angle BAD be made, and let it cut arc AD off from arc BC at D. Let a quadrant AE also be cut off from AB, [Margin: z — Theodosius, *Sphaerica* I.20] and through the points E, D let the arc ED of a great circle be drawn, cutting arc AC (produced) at F. And since angle B is set right, and angle BAD is also right by construction, [Margin: a — Prop. 25 of this treatise] arcs AD, BD will be quadrants. Again, since arcs AD, AE are quadrants and contain the right angle DAE, [Margin: b — Prop. 26 of this treatise] arc DE will also be a quadrant, and A the pole of arc DE; and consequently, since arc AF passes through A, the pole of arc EF, [Margin: c — Theodosius, *Sphaerica* I.15] angle F will be right. EF will likewise be the arc measuring angle BAC. Moreover arc DF will be the complement of arc EF, that is, of angle BAC, and arc CD the complement of arc BC, on account of the quadrants DE, BD.

Since, then, in triangle CDF angle F is right, [Margin: d — Corollary to Prop. 41 of this treatise] it will follow that as the sine of arc CD is to the whole sine, so the sine of arc DF is to the sine of angle DCF; and, converting, as the sine of angle DCF — that is, of angle ACB (for the arcs of angles DCF, ACB have the same sine, [Margin: e — Corollary to Prop. 5 of this treatise] since they make up a semicircle) — is to the sine of arc DF, that is, to the sine of the complement of angle BAC, so the whole sine is to the sine of arc CD, that is, to the sine of the complement of arc BC; and, by permutation, as the sine of angle ACB is to the whole sine, so the sine of arc DF, or of the complement of angle BAC, is to the sine of arc CD, or of the

complement of arc BC. Which is what was proposed. Therefore, in every right-angled spherical triangle, and so on. Which was to be demonstrated.

Diagram. The figure for the fourth case (angle A obtuse, arc AB greater than a quadrant).

- F (top): apex where arc ED cuts arc AC produced
- C (upper left, below F): vertex
- D (center): point where arc AD cuts arc BC
- A (left): vertex of the obtuse angle, pole of arc DE
- E (bottom, center): endpoint of quadrant AE
- B (bottom right): vertex of the right angle

SCHOLIVM.

From this proposition we shall gather these two problems.

I.

In a right-angled spherical triangle, when two non-right angles are given, to find the arc opposite either of them, together with the arc that is opposite the right angle.

In the triangle ABC, whose angle C is right, let the two angles A, B be given. I say that either of the arcs AC, BC is thereby also given, together with arc AB. [Margin: a — Prop. 42 of this treatise] For it holds that as the sine of angle A is to the whole sine, so the sine of the complement of angle B is to the sine of the complement of arc AC. Likewise, as the sine of angle B is to the whole sine, so the sine of the complement of angle A is to the sine of the complement of arc BC.

Diagram. A right-angled spherical triangle drawn with slightly curved sides.

- A (top): vertex
- B (bottom left): vertex
- C (bottom right): vertex of the right angle

[Margin: — *Praxis* (the working)]

If it be arranged that, as the sine of the given angle which is adjacent to the sought side is to the whole sine, so the sine of the complement of the remaining given angle is to a fourth term, then the sine of the complement of the arc opposite this latter angle — which is sought — will be produced. And once both arcs about the right angle have been found, the arc opposite the right angle can

also be found, from either of them and from the angle that is set opposite the given arc, as we showed in Problem 3 of Prop. 41.

But whether arcs AC, BC are less or greater than a quadrant we shall learn thus. If angle B is acute, [Margin: b — Prop. 34 of this treatise] the arc AC opposite it will be less than a quadrant; if obtuse, greater than a quadrant. By the same reasoning, if angle A is acute, the arc BC opposite it will be less than a quadrant; if obtuse, greater than a quadrant.

II.

In a right-angled spherical triangle, when one or the other of the non-right angles is given, together with one or the other of the arcs about the right angle, to find the other non-right angle and the remaining two arcs.

In the same triangle let there first be given arc AC, together with angle A, adjacent to it. I say that angle B is thereby also given, together with arcs BC, AB. [Margin: c — Prop. 42 of this treatise] For since it holds that, as the sine of angle A is to the whole sine, so the sine of the complement of angle B is to the sine of the complement of arc AC, it will follow, by conversion, that as the whole sine is to the sine of the given angle A, so the sine of the complement of the given arc AC is to the sine of the complement of angle B, which is sought.

[Margin: — *Praxis*, when an arc is given together with an adjacent angle]

Therefore, whenever an arc is given together with an angle adjacent to it, if it be arranged that, as the whole sine is to the sine of the given angle, so the sine of the complement of the given arc is to a fourth term, the sine of the complement of the other angle, which is sought, will be found. From this, the two non-right angles now being known, the remaining two arcs will be found, as was demonstrated in the immediately preceding problem. The third [arc] is given by hypothesis.

But whether the sought angle B is acute or obtuse will be taught by the given arc AC. [Margin: d — Prop. 34 of this treatise] For if it is less than a quadrant, angle B will be acute; if greater than a quadrant, obtuse.

Next let there be given arc AC together with angle B, opposite it, and let it be established, concerning the remaining angle A, whether it is acute or obtuse; or, concerning the other arc BC about the right angle, of what sort it is. I say that in turn the remaining angle A is given, together with the remaining arcs BC, AB. For [Margin: e — Prop. 42 of this treatise] since it holds that, as the sine of angle A is to the whole sine, so the sine of the complement of angle B is to the sine of the complement of arc AC, it will follow, by conversion, that as the sine of the

complement of the given arc AC is to the sine of the complement of the given angle B, so the whole sine is to the sine of the sought angle A.

[Margin: — *Praxis*, when an arc is given together with the opposite angle]

Therefore, whenever an arc is given together with the angle opposite it, if it be arranged that, as the sine of the complement of the given arc is to the sine of the complement of the given angle, so the whole sine is to a fourth term, the sine of the remaining angle, which is sought, will be produced. From the two non-right angles now being known, then, the remaining two arcs will be found, as we showed in the preceding problem. The third is given by hypothesis.

But it must be settled whether the remaining angle A is acute or obtuse, so that one may know what kind of angle — acute or obtuse — is to be taken as corresponding to the sine found. Now if it were established of what sort arc BC is, the kind of angle A would at once be known as well. For if arc BC is less than a quadrant, [Margin: f — Prop. 34 of this treatise] angle A will be acute; if greater than a quadrant, obtuse. By the same reasoning, if it were known of what sort arc AB is — the one opposite the right angle — we should immediately know the kind of angle A. For if arc AB is less than a quadrant, and the given angle B is indeed acute, [Margin: g — Prop. 38 of this treatise] angle A too will be acute; but if the given angle B is obtuse, A too will be obtuse. On the other hand, if arc AB is greater than a quadrant, and the given angle B is acute, angle A will be obtuse; but if the given angle B is obtuse, angle A will be acute.

And so it is not enough to be given a non-right angle together with the arc opposite it — as Copernicus would have it in Prop. 4 On Spherical Triangles [Margin: — *An error of Copernicus*] — a point on which we have also given warning above, in the scholium to Prop. 21. There must in addition be given the kind of the third angle, or the kind of the other arc about the right angle, or at all events the kind of the arc opposite the right angle. On this point Nicolaus Copernicus [Copernicus] went astray, wishing, in Prop. 4 On Spherical Triangles, that it should be enough for an arc about the right angle to be given together with one or the other of the non-right angles. For this is false in the case of a given angle with the arc opposite it, unless something else besides is established, as we have said here and warned in the scholium to Prop. 21.

THEOR. 41. PROPOS. 43.

In every right-angled spherical triangle none of whose arcs is a quadrant, the sine of the complement of the arc subtending the right angle has the

same ratio to the sine of the complement of either of the remaining arcs as the sine of the complement of the remaining arc has to the whole sine.

In the right-angled spherical triangle ABC let angle B be right, and let none of the arcs be a quadrant. I say that the sine of the complement of arc AC is to the sine of the complement of, say, arc AB as the sine of the complement of the remaining arc BC is to the whole sine. For since none of the arcs is assumed to be a quadrant, none of the remaining angles will be right. [Margin: a — Schol. 25 of this treatise] Otherwise triangle ABC, having two right angles, would have two quadrant arcs, which is not assumed. So let the first angle [continue below]

[Continuation on the following page:]

be acute, and arc AB — which is adjacent both to it and to the right angle B — less than a quadrant. On this assumption, angle C too will be acute; [Margin: b — Prop. 33 of this treatise] and hence [Margin: c — Prop. 28 of this treatise] all the arcs of triangle ABC will be less than a quadrant. Let arcs AB, AC be produced and the quadrants AD, AE be made; [Margin: d — Theodosius, *Sphaerica* I.20] and through the points D, E let the arc DE of a great circle be drawn, meeting arc BC (produced) at F. [Margin: e — Prop. 25 of this treatise] Then both angles D, E will be right, on account of the quadrants AD, AE; and further, since angle B too is set right, both arcs BF, DF will be quadrants, on account of the right angles B, D. Moreover BD will be the arc measuring angle F, [Margin: f — Prop. 26 of this treatise] because F is the pole of arc BD, on account of the quadrants BF, DF. Likewise CF will be the complement of arc BC; and BD, CE the complements of arcs AB, AC, on account of the quadrants [BF, AD, AE]. [Margin: g — Prop. 41 of this treatise] Now it is clear that in triangle CEF, as the sine of arc CE — that is, the sine of the complement of arc AC — is to the sine of angle F, that is, to the sine of arc BD, or of the complement of arc AB, so the sine of arc CF, that is, the sine of the complement of arc BC, is to the sine of the right angle E, that is, to the whole sine. Which is what was proposed.

Diagram. The figure for the first case of THEOR. 41 (mirror image of the first case above).

- F (top): apex where arc DE meets arc BC produced
- E (left of center): endpoint of quadrant AE
- C (center): vertex
- A (right): vertex of the acute angle
- D (bottom left): endpoint of quadrant AD
- B (bottom, center): vertex of the right angle

Second case.

Next let angle A be obtuse, arc AB still less than a quadrant. Let angle BAD be made right, and let it cut arc AD off from arc BC at D. Arc AB being likewise produced, let AE be made a quadrant, [Margin: h — Theodosius, *Sphaerica* I.20] and through the points E, D let the arc ED of a great circle be drawn, cutting arc AC at F. And since the two angles DAB, DBA are right, [Margin: i — Schol. 25 of this treatise] arcs AD, BD will be quadrants; and further, since AE too is a quadrant and angle DAE is right, [Margin: k — Prop. 26 of this treatise] arc DE will also be a quadrant; and consequently BE, on account of the quadrants BD, ED, will be the arc measuring angle BDE — that is, angle CDF, [Margin: l — Prop. 6 of this treatise] which is equal to it at the vertex. [Margin: m — Prop. 26 of this treatise] Now since A is the pole of arc ED, arc AF too will be a quadrant, [Margin: n — Corollary 16 to Theodosius, *Sphaerica* I] since arc EF always lies a quadrant away from its pole; [Margin: o — Theodosius, *Sphaerica* I.15] and moreover both angle AFE and angle CFD will be right. Furthermore arc CF will be the complement of arc AC; arc BE the complement of arc AB; and arc CD the complement of arc BC, on account of the quadrants AF, AE, BD. [Margin: p — Prop. 41 of this treatise] Now it is evident that in triangle CDF, as the sine of arc CF — that is, the sine of the complement of arc AC — is to the sine of angle CDF, that is, to the sine of arc BE, or of the complement of arc AB, so the sine of arc CD, namely the sine of the complement of arc BC, is to the sine of the right angle F, that is, to the whole sine. Which is what was proposed.

Diagram. The figure for the second case of THEOR. 41.

- C (top): vertex
- D (center left): point where arc AD cuts arc BC
- F (right of D): point where arc ED cuts arc AC
- E (bottom left): endpoint of quadrant AE
- B (bottom, center): vertex of the right angle
- A (right): vertex of the obtuse angle, pole of arc ED

Third case.

Now let angle A be acute, and arc AB greater than a quadrant. On this assumption, the remaining angle C will be obtuse; [Margin: q — Prop. 33 of this treatise] and therefore arc AC, which subtends the right angle B, will likewise be greater than a quadrant. [Margin: r — Prop. 37 of this treatise] Let the quadrants AD, AE be cut off, [Margin: s — Theodosius, *Sphaerica* I.20] and through the points D, E let the arc DE of a great circle be drawn, meeting arc BC (produced) at F;

[Margin: t — Prop. 25 of this treatise] then each of the angles D, E will be right, on account of the quadrants AD, AE; and consequently, since angle B too is right, [Margin: u — Prop. 25 of this treatise] arcs BF, DF will be quadrants; and therefore BD will be the arc measuring angle F. Likewise arc CF will be the complement of arc BC, and arcs DB, EC the complements of arcs AB, AC, on account of the quadrants BF, AD, AE. [Margin: x — Prop. 41 of this treatise] Now it is evident that in triangle CEF, as the sine of arc EC — that is, the sine of the complement of arc AC — is to the sine of angle F, that is, to the sine of arc DB, that is, to the sine of the complement of arc AB, so the sine of arc CF, namely the sine of the complement of arc BC, is to the sine of the right angle E, that is, to the whole sine. Which is what was proposed.

Diagram. The figure for the third case of THEOR. 41.

- F (top): apex where arc DE meets arc BC produced
- C (left): vertex
- E (center): endpoint of quadrant AE
- A (right): vertex of the acute angle
- B (bottom left): vertex of the right angle
- D (bottom, center): endpoint of quadrant AD

Fourth and last case.

Lastly let angle A be obtuse, and arc AB still greater than a quadrant. Let the right angle BAD be made, and let it cut arc AD off from arc BC at D. Let a quadrant AE also be cut off from AB, [Margin: y — Theodosius, *Sphaerica* I.20] and through the points E, D let the arc ED of a great circle be described, cutting arc AC (produced) at F. And since angle B is set right, and angle BAD has been made right, [Margin: z — Prop. 25 of this treatise] arcs AD, BD will be quadrants. Again, since arcs AD, AE are quadrants and contain the right angle DAE, [Margin: a — Prop. 26 of this treatise] arc DE too will be a quadrant, and A the pole of arc ED; [Margin: b — Theodosius, *Sphaerica* I.15] and consequently angle F will be right. Moreover, since DB, DE have been shown to be quadrants, E, B will be the arc measuring angle BDE, that is, angle CDF, which is equal to it at the vertex. [Margin: c — Prop. 6 of this treatise] Likewise, since A is the pole of arc EF, arc AF will be a quadrant, [Margin: d — Corollary 16 to Theodosius, *Sphaerica* I] because arc EF always lies a quadrant away from its pole. Arc CF will likewise be the complement of arc AC; arc EB the complement of arc AB; and arc CD the complement of arc BC, on account of the quadrants AF, AE, BD. [Margin: e — Prop. 41 of this treatise] Now it is evident that in triangle CDF, as the sine of arc CF — that is, the sine of the

complement of arc AC — is to the sine of angle CDF, that is, to the sine of arc BE, or of the complement of arc AB, so the sine of arc CD, namely the sine of the complement of arc BC, is to the sine of the right angle F, that is, to the whole sine. Which is what was proposed. Therefore in every right-angled spherical triangle, and so on. Which was to be demonstrated.

Diagram. The figure for the fourth case of THEOR. 41.

- F (top): apex where arc ED cuts arc AC produced
- C (upper right, below F): vertex
- D (center): point where arc AD cuts arc BC
- B (bottom left): vertex of the right angle
- E (bottom, center): endpoint of quadrant AE
- A (right): vertex of the obtuse angle

SCHOLIVM. I.

The following problem we shall gather from this proposition in this way.

In a right-angled spherical triangle, when any two arcs are given, to find the third arc and the remaining two non-right angles.

Clavius, *Opera Mathematica*, Tomus I — Doctrine of Spherical Triangles

Note on marginalia: throughout this treatise the abbreviation *huius* ("of this [treatise]") refers to a numbered proposition of Clavius's own doctrine of triangles; *The.* = Theodosius, *Sphaerica*; *Sinuum* = the treatise on Sines. The lettered superscripts ([a], [b], ...) are Clavius's own reference marks keyed to the printed side-notes.

[Continuation of SCHOLIVM I to THEOR. 41 / PROPOS. 43]

(Page 841 opens within the scholium begun on the previous page; a problem on the right-angled spherical triangle is being worked through its cases.)

In triangle ABC, whose angle C is right, let there first be given the two arcs AC and CB about the right angle C. I say that the third arc AB is thereby also given, together with the two angles A and B. [a] For since the sine of the complement of arc AB is to the sine of the complement of arc AC as the sine of the complement of arc CB is to the total sine, it will be, by conversion [*convertendo*], that the total

sine is to the sine of the complement of arc CB as the sine of the complement of arc AC is to the sine of the complement of arc AB.

[Margin (a): 43. *huius* — prop. 43 of this treatise.]

Diagram. A small spherical triangle with A at the apex, B at the lower left, C at the lower right; all three sides drawn as arcs. Uncaptioned.

- A (apex): vertex
- B (lower left): vertex
- C (lower right): vertex (the right angle in the worked triangle)

Therefore, given the two arcs enclosing the right angle: if one makes the total sine to the sine of the complement of either of the given arcs as the sine of the complement of the other given arc is to a fourth term, that fourth term will be the sine of the complement of the arc opposite the right angle, which is sought. And from the arc opposite the right angle, together with either of the arcs about the right angle, the angle opposite it will be found, as we set forth in problem 1 of proposition 41.

[Margin: *Praxis* — when the two arcs about the right angle are given.]

But whether the sought arc AB is less or greater than a quadrant, the two given arcs will show. For if both are less, or both greater, than a quadrant, [b] arc AB will be less than a quadrant; but if one is less than a quadrant and the other greater, arc AB will be greater than a quadrant.

[Margin (b): 35. *huius* — prop. 35 of this treatise.]

Next, let arc AB, opposite the right angle, be given together with one of the arcs about the right angle, say with AC. I say that the remaining arc CB is again given, together with the two angles A and B. [c] For since the sine of the complement of arc AB is to the sine of the complement of arc AC as the sine of the complement of arc CB is to the total sine, it will be, by conversion, that the sine of the complement of arc AC is to the sine of the complement of arc AB as the total sine is to the sine of the complement of arc CB.

[Margin (c): 43. *huius* — prop. 43 of this treatise.]

Accordingly, given the arc opposite the right angle together with one of the arcs about the right angle: if one makes the sine of the complement of the given arc about the right angle to the sine of the complement of the arc opposite the right angle as the total sine is to a fourth term, the sine of the complement of the other

arc about the right angle — which is sought — will be found. And from any given arc about the right angle, together with the arc opposite the right angle, the angle opposite that arc will be found, as was demonstrated in problem 1 of proposition 41.

[Margin: *Praxis* — when the arc opposite the right angle is given, together with one of the arcs about the right angle.]

But whether the sought third arc CB is less or greater than a quadrant, we shall understand from the two given arcs. For if arc AB, opposite the right angle, is less than a quadrant: then, if the other given arc AC is also less than a quadrant, [d] arc CB too will be less than a quadrant; but if AC is greater than a quadrant, CB too will be greater than a quadrant. If, however, AB is greater than a quadrant: then, if AC is also greater than a quadrant, CB will be less than a quadrant; but if AC is less than a quadrant, CB will be greater than a quadrant.

[Margin (d): 36. *huius* — prop. 36 of this treatise.]

SCHOLIVM II.

[Margin: Whatever is proved concerning a right-angled spherical triangle all of whose arcs are less than a quadrant holds also in every right-angled spherical triangle.]

Although we have made both this proposition 43 and the preceding one, 42, four-membered [*quadrimembrem*], so that each might be demonstrated in all cases, it would nevertheless have been enough had each been confirmed by a demonstration in the first case only — namely with all the arcs being less than a quadrant. For once that case is demonstrated, we shall easily accommodate the demonstration to all the other cases. Let ACD be any right-angled spherical triangle whatever, having the right angle C. Then either the two arcs AC and CD about the right angle are less than a quadrant — [e] and consequently the third arc AD is also less than a quadrant — or one is greater than a quadrant and the other less; or, finally, both are greater than a quadrant. For we are dealing only with that right-angled spherical triangle in which no arc is a quadrant. First let the two arcs AC and CD about the right angle be less than a quadrant. This being posited, [f] each of the angles D and A will be acute, and therefore the demonstration of both propositions will suit triangle ACD, as regards the first case.

[Margin (e): 35. *huius*. — Margin (f): 34. *huius*.]

Diagram. A lens- (vesica-) shaped figure: A at the top, B at the left, C at the bottom, D at the right. Two great-circle arcs run from B to D — the upper passing through A, the lower through C — while a further arc joins A to C down the middle. The figure illustrates the two semicircles DAB and DCB, and the arc AC shared by triangles ACB and ACD. Uncaptioned.

- A (top): vertex; upper arc B–A–D
- B (left): left meeting-point of the two arcs
- C (bottom): vertex; lower arc B–C–D (right angle vertex)
- D (right): right meeting-point of the two arcs
- (A–C): the arc common to both triangles

Next let arc DC be greater than a quadrant, and CA less. When the arcs DC and DA are produced until they meet at B, [g] DAB and DCB will be semicircles, and hence CB will be less than a quadrant. There are therefore in triangle ABC two arcs, AC and CB, about the right angle C, that are less than a quadrant. Hence, as we have just shown, the demonstration of both propositions will suit it, as regards the first case. Since, then, the sines — both the proper sines and those of the complements — of the arcs and angles of triangle ACB are the same as those of the arcs and angles of triangle ACD (for, as we said in the discussion of sines, arcs CD and CB have the same sine, both proper and of the complement, and so too do arcs AD and AB; likewise the right angles at C have the same sine — namely the total sine — as do the oblique angles at A, [h] since these are equal to two right angles; and finally the angles D and B have the same sine, [i] since they are equal to one another; while arc AC is common to both triangles) — it plainly follows that whatever is shown concerning the sines of the arcs and angles of triangle ACB holds also in the sines of the arcs and angles of triangle ACD.

[Margin (g): *11.1.The.* — Theodosius, *Sphaerica* I.11. — Margin (h): *5. huius.* — Margin (i): *13. huius.*]

Finally, let the two arcs DC and CA be greater than a quadrant. This being posited, arc CB will be less than a quadrant. Triangle ACB therefore has arc AC about the right angle C greater than a quadrant, and CB less. Hence, as was just demonstrated, both propositions will suit it. Since, then, the sines — both proper and of the complements — of the arcs and angles of triangle ACB are the same as those of the arcs and angles of triangle ACD, as we said a little before, it is clear that the same propositions suit triangle ACD as well. It is therefore evident that whatever is demonstrated concerning the sines of the arcs and angles of a right-

angled spherical triangle whose two arcs about the right angle are less than a quadrant holds also in any other right-angled spherical triangle whatever.

The very same is to be said of the third case of proposition 41. For it would have been enough to have demonstrated it in the right-angled triangle all of whose arcs are less than a quadrant — such as the triangle spoken of in the second figure of proposition 41 — since the demonstration of that triangle suits all the others, as is manifest from what has been demonstrated in this scholium.

From what we have demonstrated in the last three propositions, the calculation of right-angled spherical triangles by sines is now complete; and so the calculation of non-right-angled triangles ought now to follow. But since, for the most part, the calculation of right-angled triangles is dispatched more briefly through tangent and secant lines than through sines, we shall append the following propositions, which also concern right-angled spherical triangles, before we set out the calculation of non-right-angled spherical triangles. And so that the demonstrations may be clearer and less confused, we shall always propose a right-angled spherical triangle whose two arcs about the right angle — and hence all three — are less than a quadrant. For the same demonstrations will suit

all the others, as we have demonstrated in this scholium; for both two arcs completing a semicircle, and two angles equal to two right angles, have the same tangent and secant — just as they have the same sine — as we noted in the treatment of tangents and secants.

THEOR. 42. PROP. 44.

In every right-angled spherical triangle all of whose arcs are less than a quadrant: the total sine has to the sine of either of the arcs about the right angle the same ratio that the tangent of the non-right angle adjacent to the said arc has to the tangent of the remaining arc about the right angle, which is opposite this angle.

In the spherical triangle ABC, all of whose arcs are less than a quadrant, let angle C be right. I say that the total sine is to the sine of arc BC as the tangent of angle B is to the tangent of arc AC. For when the arcs BC and BA are produced until the quadrants BF and BD are formed, and through the points F and D an arc FD of a great circle is described, [a] each of the angles F and D will be right, because of the quadrants BF and BD; and DF will be the arc [measuring] angle B, [b] since B is the pole of arc DF. Since, therefore, the two great circles BF and BD on the sphere cut one another at B, and from A and D perpendicular arcs AC and DF

have been drawn to BF, [c] it will be that as the sine of the quadrant BF — that is, the total sine — is to the tangent of arc FD — that is, to the tangent of angle B — so the sine of arc BC is to the tangent of arc AC. And, by alternation [*permutando*], as the total sine is to the sine of arc BC, so the tangent of angle B is to the tangent of arc AC. In no other way shall we demonstrate that the total sine is to the sine of arc AC as the tangent of angle A is to the tangent of arc BC — as is clear if arcs AC and AB are produced until the quadrants AG and AE are formed, and through G and E an arc GE of a great circle is described. [d] For again it will be that as the sine of the quadrant AG — that is, the total sine — is to the tangent of arc EG, or of angle A, so the sine of arc AC is to the tangent of arc BC; and, by alternation, as the total sine is to the sine of arc AC, so the tangent of angle A is to the tangent of arc BC. Therefore, in every right-angled spherical triangle, etc. Which was to be demonstrated.

[Margin (a): 25. *huius*. — Margin (b): 26. *huius*. — Margin (c): Theorem 6, scholium to prop. 40 of this treatise. — Margin (d): Theorem 6, scholium to prop. 40 of this treatise.]

Diagram. The construction for Theorem 42. A right-angled spherical triangle ABC (right angle at C) with its sides produced to quadrants. Across the top, from left to right: E, B, A, D; C stands at the centre; G and F at the bottom (G lower left, F lower right). The great-circle arcs FD (through the produced quadrants BF, BD) and GE (through AG, AE) are drawn, and the arcs cross to form the figure.

- E (top left): end of the quadrant AE (arc AB produced)
- B (top, left of centre): vertex of the triangle
- A (top centre-right): vertex of the triangle
- D (top right): end of the quadrant BD (arc BA produced)
- C (centre): right-angle vertex of the triangle
- G (bottom left): end of the quadrant AG (arc AC produced)
- F (bottom right): end of the quadrant BF (arc BC produced)

SCHOLIVM.

From this we shall gather the following two problems.

I.

In a right-angled spherical triangle, given either of the arcs about the right angle together with either of the non-right angles, to find the other arc about the right angle and the remaining non-right angle, together with the arc that is opposite the right angle: provided that, when the

given angle is opposite the given arc, it be established whether the remaining arc about the right angle is less or greater than a quadrant, or whether the remaining non-right angle is acute or obtuse.

Diagram. A right-angled spherical triangle: A at the apex, C at the lower left, B at the lower right; the sides drawn as arcs. Illustrates Problem I, the case of an arc given with its adjacent angle. Uncaptioned.

- A (apex): vertex (given angle A)
- C (lower left): right-angle vertex
- B (lower right): vertex

In triangle ABC, whose angle C is right, let there first be given arc AC together with the angle A adjacent to it. I say that arc BC is thereby also given, together with the angle B and arc AB. [e] For since the total sine is to the sine of arc AC as the tangent of angle A is to the tangent of arc BC:

[Margin: *Praxis* — when an arc is given together with the adjacent angle. Margin (e): 44. *huius*.]

If (when an arc is given with its adjacent angle) one makes the total sine to the sine of the given arc as the tangent of the given angle is to a fourth term, the tangent of the sought arc will be produced. And from the same given arc and given angle, the other non-right angle, and the arc opposite the right angle, will be found, as we demonstrated in problem 2 of proposition 42.

But whether the sought arc BC is less or greater than a quadrant, the given angle A will indicate. For if it is acute, [f] arc BC will be less than a quadrant; but if obtuse, greater than a quadrant.

[Margin (f): 34. *huius*.]

Next let arc AC be given together with the angle B opposite to it, and let it be established besides, concerning the other arc BC, whether it is less or greater than a quadrant, or whether the other angle A is acute or obtuse. I say that again arc BC is given, together with the angle A [?] and arc AB. [g] For since the tangent of angle B is to the tangent of arc AC as the total sine is to the sine of arc BC:

[Margin (g): 44. *huius*.]

If (when an arc is given with its opposite angle) one makes the tangent of the given angle to the tangent of the given arc as the total sine is to a

fourth term, the sine of the sought arc will be found. And from the given arc and given angle, the other non-right angle, and the arc opposite the right angle, will also be given, as we stated in problem 2 of proposition 42.

[Margin: *Praxis* — when an arc is given together with its opposite angle.]

But it must be established whether arc BC is less or greater than a quadrant, so that we may know what kind of arc, corresponding to the sine found, is to be taken — namely, whether less than a quadrant or greater. For if it were established what kind of angle A is, we should immediately know what kind arc BC is. For when angle A is acute, [h] arc BC will be less than a quadrant; but when it is obtuse, greater than a quadrant. So too, if it were known what kind arc AB — opposite the right angle — is, we should also know the species of arc BC.

Diagram. A right-angled spherical triangle: A at the apex, C at the lower left, B at the lower right; sides drawn as arcs. Illustrates Problem I, the case of an arc given with its opposite angle. (The same triangle is reprinted at the head of page 843.) Uncaptioned.

- A (apex): vertex
- C (lower left): right-angle vertex
- B (lower right): vertex

(Page 843. The right triangle ABC of the opposite-angle case — see fig-p842-3 above — is reprinted at the head of the page; described once, not re-transcribed.)

Diagram. The same right-angled spherical triangle ABC as fig-p842-3 (A apex, C lower left, B lower right), reprinted at the top of the page to accompany the continuation of the text.

For if AB is less than a quadrant, [i] then each of AC, BC is either less than a quadrant or greater: so, of whatever kind the given arc AC is, arc BC will also be of that kind. But if AB is greater than a quadrant, and the given arc AC is indeed less than a quadrant, then BC will be greater than a quadrant; whereas if the given arc AC is greater than a quadrant, BC will be less than a quadrant. And so it is not enough to be given an arc together with its opposite angle — as Copernicus would have it (proposition 4 on spherical triangles) — [l] as we noted above in the scholium to proposition 21.

[Margin (i): 36. *huius*.]

II.

In a right-angled spherical triangle, given the two arcs about the right angle, to determine either of the non-right angles, together with the remaining arc, which is opposite the right angle.

In the same triangle let the two arcs AC and BC be given. I say that either of the angles A and B is thereby also given, and arc AB. [k] For since the total sine is to the sine of arc AC as the tangent of angle A is to the tangent of arc BC; and, conversely, as the sine of arc AC is to the total sine, so the tangent of arc BC is to the tangent of angle A; and, by the same reasoning, as the sine of arc BC is to the total sine, so the tangent of arc AC is to the tangent of angle B:

[Margin (k): 44. *huius*.]

If one makes the sine of either of the arcs about the right angle to the total sine as the tangent of the other arc is to a fourth term, the tangent of the angle opposite this latter arc will be found. And from the two given arcs about the right angle, the third arc, opposite the right angle, will also be known, as was set forth in the problem of proposition 43. Or, at any rate, from one given arc and either of the angles once found, as was shown in problem 2 of proposition 42.

[Margin: *Praxis*.]

But whether the sought angle is acute or obtuse, the arc opposite it will show. For if this is less than a quadrant, [l] the angle opposite it will be acute; but if greater, obtuse.

[Margin (l): 34. *huius*.]

But since, in the scholium to the preceding proposition, we said that some things are dispatched more briefly through tangent and secant lines than through sines, this is to be understood of those things sought in the first place in the problems, and not of those investigated in the second place. To make this plainer, we shall set out how each problem here proposed is to be solved by sines. Thus, in order that from a given arc about the right angle, together with either of the acute angles, the other arc about the right angle may be found — which is proposed for investigation in the first place in the first problem — one must proceed as follows. If an arc about the right angle is given with its opposite angle, one must first seek the arc opposite the right angle, from problem 3 of proposition 41. Then, from this arc once found and the given arc, the other arc about the right angle — which is sought — must be elicited, through the problem of proposition 43. But if

the arc about the right angle is given with its adjacent angle, one must first seek, through problem 2 of proposition 42, the other acute angle. Then, through problem 1 of the same proposition 42, from this angle once found and the given angle, the arc opposite the given angle must be elicited. And in order that from two given arcs about the right angle either of the acute angles may be drawn out — which is sought in the first place in the second problem — one must first find the arc opposite the right angle, through the problem of proposition 43, from the two given arcs. Then, through problem 1 of proposition 41, from this arc once found and either of the given arcs about the right angle, the angle opposite this given arc must be found. You see, therefore, that what is sought in the first place in each problem is investigated by sines with a double operation, whereas by tangents we find it with a single one. The same reasoning holds in the following problems, so that it is enough to have noted this once here.

THEOR. 43. PROPOS. 45.

In every right-angled spherical triangle all of whose arcs are less than a quadrant: the total sine has to the sine of the complement of either of the acute angles the same ratio that the tangent of the arc opposite the right angle has to the tangent of the arc adjacent to the said acute angle.

In the spherical triangle ABC, all of whose arcs are less than a quadrant, let angle B be right. I say that the total sine is to the sine of the complement of angle A as the tangent of arc AC is to the tangent of arc AB. For when the arcs AB and AC, which contain the said angle, are produced until the quadrants AD and AE are formed, and through D and E an arc DE of a great circle is described and produced until it meets arc BC (likewise produced) at F: [a] each of the angles D and E will be right, because of the quadrants AD and AE; and DE will be the arc [measuring] angle A, [b] since A is the pole of arc DE. Likewise arc DF [?]; and hence arc EF is the complement of angle A. Since, therefore, the two great circles BF and DF on the sphere intersect one another at F, and from the points B and C perpendicular arcs BD and CE have been drawn to arc DF, [c] it will be that as the sine of the whole quadrant DF is to the tangent of arc BD, so the sine of arc EF — that is, the sine of the complement of angle A — is to the tangent of arc CE. And, by alternation, as the total sine is to the sine of the complement of angle A, so the tangent of arc BD is to the tangent of arc CE. [d] But (since AC and AB are the complements of the arcs CE and BD) as the tangent of arc BD is to the tangent of arc CE, so the tangent of arc AC is to the tangent of arc AB. Therefore it will also be that as the total sine is to the sine of the complement of angle A, so the tangent of arc AC — opposite the right angle — is to the tangent of arc AB, adjacent to the

acute angle A. In the same way we shall show that the total sine is to the sine of the complement of angle C as the tangent of arc AC — opposite the right angle — is to the tangent of arc BC, adjacent to the acute angle C: provided, of course, that the arcs CA and CB containing the angle C are produced, etc. Therefore, in every right-angled spherical triangle, etc. Which was to be shown.

[Margin (a): 25. *huius*. — Margin (b): 26. *huius*. — Margin (c): Theorem 6, scholium to prop. 40 of this treatise. — Margin (d): prop. 21 of the [treatise on] Sines.]

Diagram. The construction for Theorem 43. A right-angled spherical triangle ABC (right angle at B): A at the left, B at the bottom (lower left), C at the centre-left; the sides AB, AC produced to the quadrants AD and AE, with D at the lower right and E at the centre-right; the great-circle arc DE produced upward to meet arc BC (produced) at F, which stands at the top. Perpendicular arcs BD and CE are drawn to DF.

- F (top): meeting-point of arc DF (i.e. DE produced) with arc BC produced
- A (left): vertex; pole of the great circle DEF
- C (centre-left): vertex
- E (centre-right): end of quadrant AE (arc AC produced)
- B (bottom, lower left): right-angle vertex
- D (lower right): end of quadrant AD (arc AB produced)

SCHOLIVM.

From this theorem we shall solve the following three problems.

(The scholium to Theorem 43 / Proposition 45, begun on the preceding page, continues here with its enumerated sub-problems.)

I.

In a right-angled spherical triangle, given either of the arcs about the right angle together with the non-right angle adjacent to it, to find the arc opposite the right angle, and the remaining arc about the right angle together with the remaining non-right angle.

[Margin: a — Prop. 45 of this treatise ("45. *huius*"; *huius* = of this trigonometrical treatise).]

Let ABC be a triangle whose angle C is right, and let the arc AC and the angle A be given. I say that the arc AB is also given, together with the arc BC and the angle B. For since the whole sine [sinus totus] is to the sine of the complement of angle A

as the tangent of arc AB is to the tangent of arc AC, then, conversely, the sine of the complement of angle A is to the whole sine as the tangent of arc AC is to the tangent of arc AB.

[Margin: Praxis (Procedure).]

If it is arranged that, as the sine of the complement of the given angle is to the whole sine, so the tangent of the given arc is to another line, the tangent of the arc opposite the right angle — the one sought — will be found. From the arc AB and the angle A, the arc BC will be found by Problem 2 of Proposition 41; and from the arcs AB and AC, the angle B, opposite arc AC, by Problem 1 of the same Proposition 41.

[Margin: b — Prop. 34 of this treatise. c — Prop. 35 of this treatise. d — Prop. 34 of this treatise. e — Prop. 35 of this treatise.]

We shall know whether the sought arc AB is greater or less than a quadrant as follows. If the given angle A is acute, the arc BC will be less than a quadrant. Therefore if the given arc AC is also less than a quadrant, the arc AB too will be less than a quadrant; but if AC is greater than a quadrant, AB too will be greater. If, however, the given angle A is obtuse, the arc BC will be greater than a quadrant; so if the given arc AC is likewise greater, the arc AB will be less than a quadrant, while if AC is less than a quadrant, AB will be greater.

Diagram. A curvilinear (spherical) triangle drawn as a rounded three-sided figure. Labelled vertices:

- A (apex, top): the vertex at angle A.
- C (lower left): the right angle.
- B (lower right): the vertex at angle B.

II.

In a right-angled spherical triangle, given either of the arcs about the right angle together with the arc that is opposite the right angle, to investigate the angle contained by the said arcs — that is, the angle adjacent to the arc about the right angle that was given — together with the remaining arc and angle.

[Margin: f — Prop. 45 of this treatise.]

In the same triangle let the arcs AC and AB be given. I say that the angle A is also given, together with the arc BC and the angle B. For since the whole sine is to the

sine of the complement of angle A as the tangent of arc AB is to the tangent of arc AC — that is, as the tangent of arc AB is to the tangent of arc AC, so the whole sine is to the sine of the complement of angle A —

[Margin: Praxis (Procedure).]

If it is arranged that, as the tangent of the arc opposite the right angle is to the tangent of the given arc about the right angle, so the whole sine is to another line, the sine of the complement of the sought angle will be produced. Hence the rest will be found, as in the preceding problem.

[Margin: g — Prop. 36 of this treatise. h — Prop. 34 of this treatise. i — Prop. 36 of this treatise. k — Prop. 34 of this treatise.]

Whether the sought angle A is acute or obtuse we shall learn thus. If the arc AB opposite the right angle is less than a quadrant, then both arcs AC and BC will be either less than a quadrant or greater. So if the given arc AC is less, BC too will be less, and hence angle A is acute; but if AC is greater than a quadrant, BC too will be greater, and therefore angle A is obtuse. But if the arc AB is greater than a quadrant, then one of the remaining arcs will be greater and the other less. So if the given arc AC is greater, BC will be less, and on that account angle A is acute; but if AC is less than a quadrant, BC will be greater, and angle A obtuse.

III.

In a right-angled spherical triangle, given the arc that is opposite the right angle together with either of the non-right angles, to find the arc adjacent to this angle, together with the remaining arc and angle.

[Margin: l — Prop. 45 of this treatise.]

In the same triangle let the arc AB be given, together with the angle A. I say that the arc AC is also given, and so on. For since the whole sine is to the sine of the complement of angle A as the tangent of arc AB is to the tangent of arc AC —

[Margin: Praxis (Procedure).]

If it is arranged that, as the whole sine is to the sine of the complement of the given angle, so the tangent of the arc opposite the right angle is to another line, the tangent of the sought arc will be produced. The rest will be found as in the first problem of this proposition.

[Margin: m — Prop. 38 of this treatise. n — Prop. 34 of this treatise. o — Prop. 38 of this treatise. p — Prop. 34 of this treatise.]

Whether the sought arc AC is less or greater than a quadrant we shall learn thus. If the arc AB opposite the right angle is less than a quadrant, then both angles A and B will be either acute or obtuse. So if the given angle A is acute, B too will be acute, and hence the arc AC less than a quadrant; but if A is obtuse, B too will be obtuse, and therefore the arc AC greater than a quadrant. But if the arc AB is greater than a quadrant, then one of the remaining angles will be acute and the other obtuse. So if A is given acute, B will be obtuse, and therefore the arc AC greater than a quadrant; but if A is obtuse, B will be acute, and the arc AC less than a quadrant.

THEOR. 44. PROPOS. 46.

In every right-angled spherical triangle all of whose arcs are less than a quadrant, the whole sine has to the sine of the complement of either of the acute angles the same ratio that the tangent of the complement of the arc about the right angle adjacent to the said angle has to the tangent of the complement of the arc opposite the right angle.

Let ABC be a triangle all of whose arcs are less than a quadrant, with angle B right. I say that the whole sine is to the sine of the complement of angle A as the tangent of the complement of arc AB is to the tangent of the complement of arc AC. For, the construction being made as in the preceding proposition: since the two great circles [circuli maximi] BF and DF in the sphere cut one another at F, and from the points B and C of arc BF the perpendicular arcs BD and CE have been produced to arc DF,

[Margin: q — Theorem 6 of scholium 40 of this treatise.]

it will be, as the whole sine of the quadrant DF is to the tangent of arc BD, so the sine of arc EF is to the tangent of arc CE. And by alternation, as the whole sine is to the sine of arc EF — that is, to the sine of the complement of angle A — so the tangent of arc BD, that is, the tangent of the complemen-

-t of arc AB, is to the tangent of arc CE, that is, to the tangent of the complement of arc AC. In no other way shall we demonstrate that the whole sine is to the sine of the complement of angle C as the tangent of the complement of arc BC is to the tangent of the complement of arc AC — namely, if the arcs CB and CA that contain angle C are produced, and so on. Therefore in every right-angled spherical triangle, and so on. Which was to be demonstrated.

SCHOLIVM.

From this we shall infer the following problem, which — although it was also demonstrated in the first problem of the preceding proposition — is nevertheless more easily solved here, since in the rule of three [aurea regula] the whole sine obtains the first place.

In a right-angled spherical triangle, given either of the arcs about the right angle together with the non-right angle adjacent to it, to find the arc opposite the right angle, together with the remaining arc about the right angle and the remaining non-right angle.

[Margin: b — Prop. 46 of this treatise.]

Let ABC be a triangle whose angle C is right, and let the arc AC be given together with the angle A adjacent to it. I say that the arc AB is also given, together with the arc BC and the angle B. For, since the whole sine is to the sine of the complement of angle A as the tangent of the complement of arc AC is to the tangent of the complement of arc AB —

[Margin: Praxis (Procedure).]

If it is arranged that, as the whole sine is to the sine of the complement of the given angle, so the tangent of the complement of the given arc is to another line, the tangent of the complement of the arc opposite the right angle — the one sought — will be produced. The rest will be found as was said in Problem 1 of the preceding proposition.

Moreover, we shall recognize whether the sought arc AB is less or greater than a quadrant, as we showed in the said Problem 1 of the previous proposition.

Diagram. A curvilinear (spherical) triangle. Labelled vertices:

- A (apex, top): the vertex at angle A.
- C (lower left): the right angle.
- B (lower right): the vertex at angle B.

[Margin: Praxis (Procedure).]

THEOR. 45. PROPOS. 47.

In every right-angled spherical triangle all of whose arcs are less than a quadrant, the whole sine has to the sine of the complement of the arc opposite the right angle the same ratio that the tangent of either of the

non-right angles has to the tangent of the complement of the remaining angle.

Let ABC be a triangle all of whose arcs are less than a quadrant, with angle B right. I say that the whole sine is to the sine of the complement of arc AC as the tangent of angle C is to the tangent of the complement of angle A. Having made the construction as in Proposition 45, and having produced the arc CE to G so that CG is a quadrant, let there be described, from pole C at the distance of the quadrant CG, the arc of a great circle GH, cutting the arcs CF and EF (produced) at I and H.

[Margin: a — Corollary 16 of Book 1 of Theodosius. b — Prop. 25 of this treatise. c — Prop. 15 of Book 1 of Theodosius.]

Then CI too will be a quadrant, since the circle GH is distant a quadrant from the pole C. Likewise the arcs GH and EH will be quadrants, on account of the right angles G and E. For angle E is right, as was shown in Proposition 45; and G is right because the circle CG is perpendicular to the circle GH. Again, IG is the arc of angle C; CE is the complement of arc AC, the arc opposite the right angle; and FE is the complement of arc DE, that is, of angle A. Therefore, since the two great circles CG and CI intersect one another in the sphere at C, and from the points F and I of arc CI the perpendicular arcs FE and IG have been drawn to arc CG,

[Margin: d — Theorem 6 of scholium 40 of this treatise.]

it will be, as the whole sine of the quadrant CG is to the tangent of arc IG — that is, of angle C — so the sine of arc CE — that is, of the complement of arc AC — is to the tangent of arc FE, that is, of the complement of angle A. And by alternation it will be, as the whole sine is to the sine of the complement of arc AC (opposite the right angle), so the tangent of angle C is to the tangent of the complement of angle A. In like manner, with the figure constructed differently, we shall demonstrate that the whole sine is to the sine of the complement of arc AC as the tangent of angle A is to the tangent of the complement of angle C. Therefore in every right-angled spherical triangle, and so on. Which was to be shown.

Diagram. A spherical construction with two intersecting great-circle arcs and their perpendiculars, forming the quadrant figure of the theorem. Labelled points:

- I (upper left): intersection of the produced arc with circle GH.
- H (upper right): intersection point on circle GH.
- F (center, upper): point on the produced arcs.

- G (left): endpoint of the quadrant CG on circle GH.
- E (center, lower): foot of the perpendicular.
- C (center right): the vertex / pole of the described circle.
- A (right): the vertex at angle A.
- D (bottom left).
- B (bottom center): the right angle.

SCHOLIVM.

From this theorem the following problem is gathered.

In a right-angled spherical triangle, given the arc that is opposite the right angle together with either of the non-right angles, to find the other non-right angle and the two arcs about the right angle.

[Margin: c — Prop. 47 of this treatise.]

Let ABC be a triangle whose angle C is right, and let the arc AB be given together with the angle B. I say that the remaining angle A is also given, and the two arcs AC and CB. For, since the whole sine is to the sine of the complement of arc AB as the tangent of angle B is to the tangent of the complement of angle A —

[Margin: Praxis (Procedure).]

If it is arranged that, as the whole sine is to the sine of the complement of the given arc opposite the right angle, so the tangent of the given angle is to another line, the tangent of the complement of the sought angle will be found. Hence, from the arc AB and both angles B and A, each of the arcs AC and CB will be found, as we showed in Problem 2 of Proposition 41.

[Margin: f — Prop. 38 of this treatise.]

Whether the sought angle A is acute or obtuse we shall learn from the given arc AB and the given angle B. For if AB is less than a quadrant and angle B is acute, then A too will be acute; but if B is obtuse, A too will be obtuse. If, however, AB is greater than a quadrant and B is acute, then A will be obtuse; but if B is obtuse, A will be acute.

Diagram. A curvilinear (spherical) triangle. Labelled vertices:

- A (apex, top).
- B (lower left).
- C (lower right).

[Margin: c — Prop. 47 of this treatise.]

[Margin: Praxis (Procedure).]

THEOR. 46. PROPOS. 48.

In every right-angled spherical triangle all of whose arcs are less than a quadrant, the whole sine has to the sine of either of the arcs about the right angle the same ratio that the tangent of the complement of the other arc about the right angle has to the tangent of the complement of the opposite angle.

Let ABC be a spherical triangle all of whose arcs are less than a quadrant, with the right angle B. I say that the whole sine is to the sine of arc AB as the tangent of the complement of arc BC is to the tangent of the complement of angle A. For, the construction being made as in Proposition 45: angle D will be right, and CF the complement of arc BC, and EF the complement of angle A, and AD a quadrant, as was there shown. Therefore, since the two great circles AD and AE in the sphere cut one another at A, and from the points C and E the perpendicular arcs CB and ED have been drawn to arc AD,

[Margin: b — Theorem 6 of scholium 40 of this treatise.]

it will be, as the whole sine of the quadrant AD is to the tangent of arc DE, so the sine of arc AB is to the tangent of arc BC. And by alternation, as the whole sine is to the sine of arc AB, so the tangent of arc DE is to the tangent of arc BC.

[Margin: c — Prop. 21 of the doctrine of Sines ("21. Sinuum").]

Moreover — since CF and EF are the complements of the arcs BC and DE — as the tangent of arc DE is to the tangent of arc BC, so the tangent of arc CF is to the tangent of arc EF. Therefore it will also be, as the whole sine is to the sine of arc AB, so the tangent of arc CF (the complement of arc BC) is to the tangent of arc EF, that is, of the complement of angle A, which is opposite arc BC. We shall demonstrate in no other way, if the figure is constructed differently, that the whole sine is to the sine of arc BC as the tangent of the complement of arc AB is to the tangent of the complement of angle C. Therefore in every right-angled spherical triangle, and so on. Which was to be demonstrated.

Diagram. A spherical construction (the quadrant figure of the theorem), with a tall arc rising to F and perpendiculars descending to the base. Labelled points:

- F (top): apex of the construction.

- E (left, middle).
- C (center): foot of a perpendicular / the vertex at angle C.
- A (right): the vertex at angle A.
- D (bottom left).
- B (bottom center): the right angle.

SCHOLIUM.

From this theorem the following problem is inferred: although it was also demonstrated in Problem 2 of Proposition 44, it is nevertheless more easily solved here, since in the rule of three the whole sine holds the first place.

In every right-angled spherical triangle, given the two arcs about the right angle, to trace out either of the non-right angles, together with the remaining arc that is opposite the right angle.

[Margin: d — Prop. 4[?] of this treatise (second digit uncertain; context favors Prop. 48).]

Let ABC be a triangle whose angle C is right, and let the two arcs AC and CB be given. I say that either of the angles A and B, and the arc AB, are likewise given. For, since the whole sine is to the sine of arc AC as the tangent of the complement of arc CB is to the tangent of the complement of angle A; and likewise, as the whole sine is to the sine of arc CB, so the tangent of the complement of arc AC is to the tangent of the complement of angle B —

If it is arranged that, as the whole sine is to the sine of either of the arcs about the right angle, so the tangent of the complement of the other arc about the right angle is to another line, the tangent of the complement of the angle opposite this latter arc will be found.

From the two given arcs about the right angle, the third arc — the one opposite the right angle — will also be known, as we showed in the problem of Proposition 43. Or indeed from either arc, once the angle opposite it has been found, as was taught in Problem 3 of Proposition 41.

[Margin: e — Prop. 34 of this treatise.]

Whether the sought angle is acute or obtuse will be shown by the arc opposite it. For if this arc is less than a quadrant, the angle opposite it will be acute; but if greater, obtuse.

Diagram. A curvilinear (spherical) triangle. Labelled vertices:

- A (apex, top).
- B (lower left).
- C (lower right).

[Margin: d — Prop. 4[?] of this treatise.]

[Margin: Praxis (Procedure).]

THEOR. 47. PROPOS. 49.

In every right-angled spherical triangle all of whose arcs are less than a quadrant, the whole sine has to the tangent of either of the arcs about the right angle the same ratio that the tangent of the complement of the opposite angle has to the sine of the other arc about the right angle.

Let ADE be a spherical triangle all of whose arcs are less than a quadrant, with angle D right. I say that the whole sine is to the tangent of arc DE as the tangent of the complement of angle A is to the sine of arc AD. For, the construction of the figure of Proposition 45 being repeated, AB and AC will be quadrants, and CF the complement of arc BC — that is, of angle A — as was there shown. Therefore, since AB and AC are quadrants, and arc ED and AB are perpendicular,

[Margin: a — Theorem 7 of scholium 40 of this treatise.]

it will be, as the whole sine is to the tangent of arc ED, so the tangent of the complement of arc CB — that is, of angle A — is to the sine of arc AD. In the same way it will be shown that the whole sine is to the tangent of arc AD as the tangent of the complement of angle E is to the sine of arc DE — provided, of course, that the figure is constructed differently. Therefore in every right-angled spherical triangle, and so on. Which was to be demonstrated.

Diagram. A spherical construction with a tall arc rising to F and perpendiculars to the base. Labelled points:

- F (top): apex of the construction.
- A (left): the vertex at angle A.
- E (center): foot of a perpendicular / the vertex at angle E.
- C (upper right).
- D (bottom left): the right angle.
- B (bottom center).

SCHOLIVM.

From this the following problem is gathered, which can also be solved another way by Problem 1 of Proposition 44.

In a right-angled spherical triangle, given either of the arcs about the right angle together with the angle opposite it, to seek the remaining arc about the right angle and the arc opposite the right angle, together with the remaining non-right angle — provided only it be established whether the sought arc is greater or less than a quadrant, or whether the remaining non-right angle is acute or obtuse.

Let ABC be a triangle in which angle C is right, and let arc AC be given together with the opposite angle B. I say that arc BC is also given, and so on. [Margin: b — Prop. 49 of this treatise.] For since the whole sine is to the tangent of arc AC as the tangent of the complement of angle B is to the sine of arc BC:

Praxis. If it is arranged that the whole sine is to the tangent of the given arc as the tangent of the complement of the given angle is to another quantity, the sine of the sought arc will be produced. Then, from the two arcs about the right angle once known, we shall arrive at the knowledge of the remaining arc and of the remaining non-right angle, as we demonstrated in the problem of Proposition 43; or else from either of the arcs about the right angle together with the given angle, as we taught in the second problem of Proposition 42.

But here we must first establish whether the sought arc BC is greater or less than a quadrant, and whether the remaining angle A is acute or obtuse — just as was set out in the latter part of the first problem of Proposition 44, where we also exposed an error of Copernicus.

Diagram. A small spherical triangle with gently bowed sides.

- A (apex, top): vertex
- B (lower left): vertex
- C (lower right): vertex

THEOR. 48. PROPOS. 50.

In every right spherical triangle all of whose arcs are less than a quadrant, the whole sine has to the tangent of the complement of either of the non-right angles the same ratio that the tangent of the complement of the remaining angle has to the sine of the complement of the arc opposite the right angle.

Let ABC be a triangle all of whose arcs are less than a quadrant, with B the right angle. I say that the whole sine is to the tangent of the complement of angle A as the tangent of the complement of angle C is to the sine of the complement of arc AC. Take again the figure of Proposition 47: since CG and CI are quadrants intersecting one another at C, and the arcs IG and FE are perpendicular to CG — as is evident from the construction made there — [Margin: a — Theor. 7 of scholium 40 of this treatise] it will follow that the whole sine is to the tangent of arc EF (which is the complement of arc DE, that is, of angle A) as the tangent of the complement of arc IG (that is, of angle C) is to the sine of arc CE (that is, of the complement of arc AC, the one opposite the right angle). By similar reasoning we shall show, if the construction of the figure is set up differently, that the whole sine is to the tangent of the complement of angle C as the tangent of the complement of angle A is to the sine of the complement of arc AC. Therefore in every right spherical triangle, and so on. Which was to be demonstrated.

Diagram. The figure of Proposition 47 repeated: a construction of intersecting great-circle arcs.

- G (far left): vertex from which arcs radiate
- I (top left), H (top right): upper points; arc IG drawn
- F (upper middle): intersection point
- E (middle left), C (center): points on the arcs; arc FE drawn
- A (right): vertex
- D (lower left), B (lower center): base points

SCHOLIVM.

We infer from this proposition the following problem.

In a right spherical triangle, given the two non-right angles, to find the arc opposite the right angle, and the remaining two arcs about the right angle.

Let ABC be a triangle in which angle C is right, and let the two non-right angles A and B be given. I say that arc AB is also given, together with the arcs AC and BC. [Margin: b — Prop. 50 of this treatise.] For since the whole sine is to the tangent of the complement of angle A as the tangent of the complement of angle B is to the sine of the complement of arc AB:

Praxis. If it is arranged that the whole sine is to the tangent of the complement of either of the given angles as the tangent of the complement of the other given angle is to another quantity, the sine of the complement of the arc opposite the

right angle will be produced. Then, from the arc opposite the right angle together with either of the non-right angles, the arc opposite it will be found, as we showed in the second problem of Proposition 41.

Diagram. A small spherical triangle.

- A (apex, top): vertex
- C (lower left): vertex
- B (lower right): vertex

Furthermore, we shall learn whether the sought arc is greater or less than a quadrant in this way: if both angles A and B are obtuse, or both acute, [Margin: c — Prop. 37 of this treatise.] arc AB will be less than a quadrant; but if one of them is acute and the other obtuse, that same arc will be greater than a quadrant.

THEOR. 49. PROPOS. 51.

In every right spherical triangle all of whose arcs are less than a quadrant, the whole sine has to the tangent of the complement of the arc opposite the right angle the same ratio that the tangent of either of the arcs about the right angle has to the sine of the complement of the adjacent non-right angle.

Let ABC be a spherical triangle all of whose arcs are less than a quadrant, with B the right angle. I say that the whole sine is to the tangent of the complement of arc AC as the tangent of arc AB is to the sine of the complement of angle A. Take again the construction of the figure of Proposition 45: AD and AE will be quadrants, angles D and E right, and likewise BF and DF quadrants, as was shown there. Since, then, in the sphere arc DB is drawn through the extremities of the quadrants BF and DF, which intersect at F, and CE is perpendicular to DF, [Margin: a — Theor. 7 of scholium 40 of this treatise] it will follow that the whole sine is to the tangent of arc CE (which is the complement of arc AC, the one opposite the right angle) as the tangent of the complement of arc DB — that is, the tangent of arc AB — is to the sine of arc EF (which is the complement of arc DE, or of angle A). It will be demonstrated no differently that the whole sine is to the tangent of the complement of arc AC as the tangent of arc BC is to the sine of the complement of angle C, if the construction of the figure is set up differently. Wherefore in every right spherical triangle, and so on. Which was to be shown.

Diagram. A crescent- or lune-shaped construction of great-circle arcs.

- F (top): upper point

- E (middle left), C (center): points on the arcs
- A (right): vertex
- D (lower left), B (lower center): base points; arcs BD, CE, DF, BF drawn

SCHOLIUM.

From this theorem there arises a problem of this kind, which was also set out in the second problem of Proposition 45.

[The scholium-problem begun at the foot of the preceding page continues:]

In a right spherical triangle, given the arc opposite the right angle together with either of the arcs about that same right angle, to find the non-right angle adjacent to this arc — that is, the angle contained by the given arcs — together with the remaining arc and the non-right angle.

Let ABC be a triangle in which angle C is right, and let arc AB be given together with arc AC. I say that angle A is also given, together with arc BC and angle B. [Margin: b — Prop. 51 of this treatise.] For since the whole sine is to the tangent of the complement of arc AB as the tangent of arc AC is to the sine of the complement of angle A:

Praxis. If it is arranged that the whole sine is to the tangent of the complement of the arc opposite the right angle as the tangent of the given arc about the right angle is to another quantity, the sine of the complement of the adjacent angle sought will be found. From this the remaining parts will be found, as was set out in the first problem of Proposition 45.

Diagram. A small spherical triangle.

- A (apex, top): vertex
- B (lower left): vertex
- C (lower right): vertex

Whether the sought angle is acute or not, we shall further learn as we taught in the second problem of Proposition 45.

THEOR. 50. PROPOS. 52.

In every right spherical triangle all of whose arcs are less than a quadrant, the whole sine has to the sine of either of the non-right angles the same ratio that the secant of the other non-right angle has to the secant of the arc opposite this angle.

Let ABC be a spherical triangle all of whose arcs are less than a quadrant, with B the right angle. I say that the whole sine is to the sine of angle A as the secant of angle C is to the secant of arc AB. With the construction made as in Proposition 47, GH, HE, AE, AD, and DF will be quadrants, GI the arc of angle C, and DE the arc of angle A, as was shown partly in Proposition 45 and partly in Proposition 47. Likewise angle I will be right, [Margin: a — 15.1.The. (a cited proposition — Theodosius, *Sphaerica* I.15 [?])] because arc CI, passing through C, the pole of arc GH, is perpendicular to GH. And so, since the two great circles BI and DH intersect one another in the sphere at F, and from the points D and H perpendicular arcs DB and HI have been drawn from arc DH to arc BI, [Margin: b — Theor. 8 of scholium 40 of this treatise] it will follow that the whole sine of the quadrant D[F] is to the secant GI (which is the complement of arc HI) as the sine of arc FH is to the secant of arc AB (which is the complement of arc DB). And by alternation: the whole sine is to the sine of arc FH — or of arc DE, for the arcs FH and DE are equal, since the whole quadrants EH and DF are also equal — that is, to the sine of angle A, as the secant of arc GI, that is, of angle C, is to the secant of arc AB, opposite angle C. By the same reasoning, if the figure is constructed differently, we shall demonstrate that the whole sine is to the sine of angle C as the secant of angle A is to the secant of arc BC. Therefore in every right spherical triangle, and so on. Which was to be shown.

Diagram. The construction of Proposition 47 repeated: intersecting great-circle arcs.

- G (far left): vertex
- I (top, left of H), H (top right): upper points
- F (upper middle): intersection at F
- E (middle left), C (center): points on the arcs
- A (right): vertex
- D (lower left), B (lower center): base points; perpendicular arcs DB, HI indicated

SCHOLIVM.

From this is elicited the following problem, which was also solved in another way in the first problem of the scholium of Proposition 42.

In a right spherical triangle, given the two non-right angles, to elicit the arc opposite either of them, together with the arc opposite the right angle.

Let ABC be a triangle in which angle C is right, and let the two angles A and B be given. I say that either of the arcs BC and AC is also given, together with arc AB. [Margin: c — Prop. 52 of this treatise.] For since the whole sine is to the sine of angle A as the secant of angle B is to the secant of arc AC; and likewise the whole sine is to the sine of angle B as the secant of angle A is to the secant of arc BC:

Praxis. If it is arranged that the whole sine is to the sine of the non-right angle adjacent to the sought side as the secant of the other non-right angle is to another quantity, the secant of the arc opposite this latter angle — the one sought — will be found. Once both arcs about the right angle have been found, the third arc, opposite the right angle, will be found from them, as we showed in the problem of Proposition 43; or else, from either arc once found together with the given angle opposite it, as we stated in the third problem of Proposition 41.

Diagram. A small spherical triangle.

- A (apex, top): vertex
- B (lower left): vertex
- C (lower right): vertex

Whether the two sought arcs about the right angle are less than a quadrant, or greater, we shall know as we taught in the second problem of Proposition 42.

THEOR. 51. PROPOS. 53.

In every right spherical triangle all of whose arcs are less than a quadrant, the whole sine has to the sine of the complement of either of the arcs about the right angle the same ratio that the secant of the arc opposite the right angle has to the secant of the remaining arc.

Let ABC be a spherical triangle all of whose arcs are less than a quadrant, with B the right angle. I say that the whole sine is to the sine of the complement of arc BC as the secant of arc AC is to the secant of arc AB. With the construction of the figure made as in Proposition 45, BF will be a quadrant, and CE, BD will be perpendicular to DF, as was demonstrated there. Since, then, the two great circles BF and DF intersect one another in the sphere at F, and from the points B and C perpendicular arcs BD and CE have been drawn to DF, [Margin: a — Theor. 8 of scholium 40 of this treatise] it will follow that the whole sine of the quadrant BF is to the secant of the complement of arc CE — that is, to the secant of arc AC — as the sine of arc FC (which is the complement of arc BC) is to the secant of the complement of arc BD, that is, to the secant of arc AB. And by alternation: the whole sine is to the sine of the complement of arc BC as the secant of arc AC is to

the secant of arc AB. In like manner we shall show that the whole sine is to the sine of the complement of arc AB as the secant of arc AC is to the secant of arc BC, if only the figure is constructed a little differently. Therefore in every right spherical triangle, and so on. Which was to be demonstrated.

Diagram. A crescent- or lune-shaped construction of great-circle arcs.

- F (top): upper point
- E (middle left), C (center): points on the arcs
- A (right): vertex
- D (lower left), B (lower center): base points; arcs BD, CE, DF, BF drawn

SCHOLIVM.

The following problem is gathered from this theorem.

In a right spherical triangle, given the arc opposite the right angle together with either of the arcs about the right angle, to investigate the third arc, together with the two non-right angles.

Let ABC be a triangle in which angle C is right, and let arc AB be given together with arc AC. I say that arc BC is also given, together with the angles A and B. [Margin: a — Prop. 53 of this treatise.] For since the whole sine is to the sine of the complement of arc AC as the secant of arc AB is to the secant of arc BC:

Praxis. If it is arranged that the whole sine is to the sine of the complement of the given arc about the right angle as the secant of the arc opposite the right angle is to another quantity, the secant of the third arc sought will be produced. From this, the two arcs about the right angle being known, either of the non-right angles will be known, as we taught in the fifth problem of the scholium of Proposition 44, or in the problem of the scholium of Proposition 48.

Diagram. A small spherical triangle.

- A (apex, top): vertex
- B (lower left): vertex
- C (lower right): vertex

Whether the sought arc BC is greater or less than a quadrant, we shall learn from the two given arcs, as was set out at the end of the first problem of the scholium of Proposition 43.

THEOR. 52. PROPOS. 54.

In every right spherical triangle all of whose arcs are less than a quadrant, the whole sine has to the sine of either of the non-right angles the same ratio that the secant of the complement of the arc opposite that angle has to the secant of the complement of the arc opposite the right angle.

Let ABC be a triangle all of whose arcs are less than a quadrant, with B the right angle. I say that the whole sine is to the sine of angle A as the secant of the complement of arc BC is to the secant of the complement of arc AC. For with the construction of the figure of Proposition 47 repeated, angle I will be right, as was shown in Proposition 52, and likewise angle G. Also GH, EH, DF, BF, AE will be quadrants, as is established from what was demonstrated in Propositions 45 and 47. Since, then, in the sphere the two great circles EH and GH intersect one another at H, and from the points E and F perpendicular arcs EG and FI have been drawn from arc EH to arc GH, [Margin: a — Theor. 8 of scholium 40 of this treatise] it will follow that the whole sine of the quadrant EH is to the secant of the complement of arc FI — that is, to the secant of arc CF, which is also the complement of arc BC — as the sine of arc FH, that is, of arc DE (for arc FH is equal to arc DE, since the quadrants EH and DF are equal), which arc belongs to angle A, is to the secant of the complement of arc EG, that is, to the secant of arc EC, which is also the complement of arc AC. And by alternation: the whole sine is to the sine of arc DE, that is, of angle A, as the secant of the complement of arc BC is to the secant of the complement of arc AC. We shall show no differently, if the figure is constructed otherwise, that the whole sine is to the sine of angle C as the secant of the complement of arc AB is to the secant of the complement of arc AC. Therefore in every right spherical triangle, and so on. Which was to be demonstrated.

Diagram. The construction of Proposition 47 repeated: intersecting great-circle arcs.

- G (far left): vertex
- I (top, left of H), H (top right): upper points
- F (upper middle): intersection point
- E (middle left), C (center): points on the arcs
- A (right): vertex
- D (lower left), B (lower center): base points; perpendicular arcs EG, FI indicated

SCHOLIVM.

From this theorem there follows a problem which we have also completed in another way in the third problem of Proposition 41.

In a right spherical triangle, given either of the non-right angles together with its opposite arc, to investigate the arc opposite the right angle, together with the third arc and the remaining non-right angle — provided it is established whether the arc opposite the right angle is greater or less than a quadrant, or whether the other non-right angle is acute or obtuse.

Let ABC be a triangle having angle C right, and let angle B be given together with arc AC. I say that arc AB is also given, together with arc BC and angle A. [Margin: b — Prop. 54 of this treatise.] For since the whole sine is to the sine of the given angle B as the secant of the complement of the given arc AC is to the secant of the complement of arc AB:

Praxis. If it is arranged that the whole sine is to the sine of the given angle as the secant of the complement of the given arc is to another quantity, the secant of the complement of the arc opposite the right angle — the one sought — will be produced. From the arcs AB and AC once known, the third arc BC becomes known, by the problem of Proposition 43; and likewise from the arcs AB and BC once known, angle A becomes known, by the first problem of Proposition 41.

Diagram. A small spherical triangle.

- A (apex, top): vertex
- C (lower left): vertex
- B (lower right): vertex

(Continuing the scholium from the previous page, resumed at the catchword OPORTET.)**

But here it must be established whether the sought arc AB is greater or less than a quadrant, and whether the remaining non-right angle A is acute or obtuse; otherwise we would not know what sort of arc to assume for AB, since it may be either greater than a quadrant or less, as is evident. This is what we pointed out at problem 3 of proposition 41 [Probl. 3, propos. 41] — where we also exposed the error of Copernicus [Nicolaus Copernicus] and of Johann Regiomontanus [Regiomontanus].

THEOR. 53. PROPOS. 55.

In every right spherical triangle all of whose arcs are less than a quadrant, the whole sine [sinus totus] has to the sine of the arc opposite the right angle the same ratio that the secant of the complement of either of the arcs about the right angle has to the secant of the complement of the angle opposite that arc.

Let ABC be a spherical triangle all of whose arcs are less than a quadrant, with B the right angle. I say that the whole sine is to the sine of arc AC as the secant of the complement of arc BC is to the secant of the complement of angle A, which is opposite arc BC. For, repeating the construction of the figure of proposition 45 [propos. 45], AE will be a quadrant; DE the arc of angle A, and EF its complement; and CF the complement of arc BC, as was demonstrated there.

Diagram. A right spherical triangle with its auxiliary construction.

- A (left vertex): endpoint of the base arc
- B (foot, at the right angle): the vertex of the right angle
- C (on the hypotenuse, upper-middle): intersection point
- D (foot, right of B): endpoint of the produced base arc
- E (right, above D): point on the produced arc
- F (top): apex of the construction, from which the arcs descend

Since, then, in the sphere the two great circles AE and AD intersect one another at A, and from the points C and E the arcs CB and ED have been drawn perpendicular from arc AE to arc AD, it follows [a] that as the whole sine of the quadrant AE is to the secant of the complement of arc CB — that is, to the secant of arc CF — so the sine of arc AC is to the secant of the complement of arc DE, that is, of angle A — that is, to the secant of arc EF. And, permuting, as the whole sine is to the sine of arc AC, so the secant of the complement of arc BC is to the secant of the complement of angle A. By the same reasoning, if the figure is constructed otherwise, it will follow that as the whole sine is to the sine of arc AC, so the secant of the complement of arc AB is to the secant of the complement of angle C. Therefore in every right spherical triangle, and so on. Which was to be shown.

[Margin: a — Theorem 8 of scholium 40 of this treatise [Theor. 8, scholij 40 huius].]

SCHOLIUM.

From this we will resolve the following problem, which was also solved in another way at problem 1 of proposition 41 [problemate 1, propos. 41].

In a right spherical triangle, given the arc that is opposite the right angle together with either of the arcs about the right angle, to find the angle opposite this arc, along with the remaining arc and angle.

In triangle ABC (in the figure of the scholium of proposition 54 [scholij propos. 54]), whose right angle is C, let both arc AB and arc AC be given. I say that angle B is thereby also given, together with arc BC and angle A. [b] For since the whole sine is to the sine of arc AB as the secant of the complement of arc AC is to the secant of the complement of angle B —

[Margin: b — proposition 55 of this treatise [55 huius].]

[Margin: Praxis.]

If it be arranged that, as the whole sine is to the sine of the arc opposite the right angle, so the secant of the complement of the given arc about the right angle is to a fourth term, then the secant of the complement of the sought angle — which is opposite the said arc — is produced. Now from the two given arcs we will find the third, as we set out in the problem of proposition 43 [problemate propos. 43], or in the problem of proposition 53 [problemate propos. 53]. Likewise, from the arc opposite the right angle and from this newly found arc, we will discover the remaining angle opposite the found arc, as was said in this problem, or, at any rate, as we showed in problem 1 of proposition 41 [problemate 1, propos. 41].

But whether the sought angle B is acute or obtuse, the arc AC — the given arc about the right angle — will show, as we prescribed in problem 1 of proposition 41 [problemate 1, propos. 41].

THEOR. 54. PROPOS. 56.

In every right spherical triangle whose arcs are all less than a quadrant, the whole sine has to the sine of the complement of either of the arcs about the right angle the same ratio that the secant of the angle opposite this arc has to the secant of the complement of the remaining non-right angle.

Let ABC be a spherical triangle having B a right angle, all of whose arcs are less than a quadrant. I say that the whole sine is to the sine of the complement of arc

BC as the secant of the non-right angle A is to the secant of the complement of angle C. For, repeating the construction of the figure of proposition 47 [propos. 47], the angles G and E will be right, and the arcs BF, DF, CI, EH, GH will be quadrants; DE will be the arc of angle A, and GI the arc of angle C, as is clear from what was demonstrated in propositions 45 and 47 [propos. 45 & 47].

Diagram. A right spherical triangle with an extended construction of intersecting great-circle arcs.

- A (right): a vertex of the base
- B (lower middle): vertex at the right angle
- C (interior, right of center): intersection point
- D (lower left): foot of a perpendicular
- E (interior, left of C): intersection point
- F (interior, above E): intersection point
- G (left): apex on the left
- H (upper right): apex point
- I (upper left): apex point

Therefore, since the two great circles CG and CI of the sphere intersect one another at C, and from the points F and I the perpendicular arcs FE and IG have been drawn from arc CI to arc CG, it follows [a] that as the whole sine of the quadrant CI is to the secant of the complement of arc FE — that is, to the secant of arc DE, of angle A — so the sine of arc CF, which is the complement of arc BC, is to the secant of the complement of arc GI, of angle C. And, permuting, as the whole sine is to the sine of the complement of arc BC, so the secant of angle A is to the secant of the complement of angle C. In just the same way we will show, if the figure is constructed otherwise, that the whole sine is to the sine of the complement of arc AB as the secant of angle C is to the secant of the complement of angle A. Therefore in every right spherical triangle, and so on. Which was to be demonstrated.

[Margin: a — Theorem 8 of scholium 40 of this treatise [Theor. 8, scholij 40 huius].]

SCHOLIVM.

From this a problem of the following kind is inferred.

In a right spherical triangle, given either of the arcs about the right angle together with the non-right angle opposite it, to find the remaining non-right angle and, in

addition, the remaining two arcs — provided it be established whether the sought angle is acute or obtuse, or at least whether the other arc about the right angle is less than a quadrant or greater.

In triangle ABC, having C a right angle, let arc AC be given together with angle B. I say that angle A is thereby also given, along with arcs BC and AB. [a] For since the whole sine is to the sine of the complement of the given arc AC as the secant of the given angle B is to the secant of the complement of angle A —

[Margin: a — proposition 56 of this treatise [56 huius].]

[Margin: Praxis.]

Diagram. An oblique (obtuse-looking) spherical triangle.

- A (top): apex vertex
- C (lower left): base vertex
- B (lower right): base vertex

If it be arranged that, as the whole sine is to the sine of the complement of the given arc, so the secant of the complement of the given angle is to a fourth term, then the secant of the complement of the other non-right angle will be found. From this — with the two non-right angles now known — the arc opposite the right angle will be investigated, as we showed in the problem of proposition 50 [problemate propos. 50], and consequently also the remaining arc, from the arc that is opposite the right angle and from the known angle opposite the remaining arc, as we stated in problem 2 of proposition 41 [problemate 2, propos. 41].

But it must be established whether the remaining non-right angle that is sought is acute or obtuse, or whether the remaining arc about the right angle is less or greater than a quadrant, as we advised at the close of problem 2 of proposition 42 [problematis 2, propos. 42], where we also uncovered the error of Nicolaus Copernicus [Nicolaus Copernicus].

Now, since the reckoning of right spherical triangles is at last complete, it is fitting at this point to gather into a single table all the problems so far expounded, so that anyone may more easily find what he most desires to know. And so — since in every right spherical triangle the thing sought in the first place is either the arc opposite the right angle, or either of the arcs about the right angle, or, finally, one of the two non-right angles (although, once the thing chiefly sought has been found, the rest are found as well, as we noted for the practical procedures of the individual problems) — we have made a three-part table according to the number of the things

sought, and have appended the problems of the propositions in which the discovery of the sought quantities has been demonstrated.

There follow the problems of the foregoing propositions, arranged into a three-part table.

Finding the arc opposite the right angle. (*Inuentio arcus recto angulo oppositi.*)

When there is given (Quando datur):

No

.	Given	Solved in
1.	An arc about the right angle, and the non-right angle opposite it.	Probl. 3, propos. 41 & (Probl. propos. 54.
2.	Both arcs about the right angle.	Probl. propos. 43.
3.	An arc about the right angle, and the non-right angle adjacent to it.	Probl. 1, propos. 45 (& Probl. propos. 46.
4.	Both non-right angles.	Probl. propos. 50.

Finding either of the arcs about the right angle. (*Inuentio arcus vtriuslibet circa angulum rectum.*)

When there is given (Quando datur):

No

.	Given	Solved in
1.	The arc opposite the right angle, and the non-right angle opposite the sought arc.	Probl. 2, propos. 41.
2.	Both non-right angles.	Probl. 1, propos. 42 (& Probl. propos. 52.
3.	The arc opposite the right angle, and the other arc about the right angle.	Probl. propos. 43 & 53.
4.	The other arc about the right angle, and either of the non-right angles.	Probl. 1, propos. 44.
5.	The arc opposite the right angle, and the non-right angle adjacent to the sought arc.	Probl. 3, propos. 45.
6.	The other arc about the right angle,	Probl. proposit. 49 & (Probl. 1,

No

.	Given	Solved in
	and the other non-right angle opposite it.	propos. 44.

Finding either of the non-right angles. (*Inuentio anguli non recti vtriusuis.*)

When there is given (Quando datur):

No

.	Given	Solved in
1.	The arc opposite the right angle, and the arc about the right angle opposite the sought angle.	Probl. 1, propos. 41 & (Probl. propos. 55.
2.	An arc about the right angle, and the other non-right angle.	Probl. 2, propos. 42.
3.	Both arcs about the right angle.	Probl. 2, propos. 44 (& Probl. propos. 48.
4.	The arc opposite the right angle, and the arc about the right angle adjacent to the sought angle.	Probl. 2, propos. 45 (& Probl. propos. 51.
5.	The arc opposite the right angle, and the other non-right angle.	Probl. propos. 47.
6.	The arc about the right angle adjacent to the sought angle, and the other non-right angle opposite this arc.	Probl. proposit. 56 & (Probl. 2, propos. 42.

[Margin: What to do in a right triangle in which there are quadrants (*Quid agendum in triangulo rectangulo, in quo quadrantes sunt*).]

But since so far we have treated only that right triangle none of whose arcs is a quadrant, let us briefly teach — a matter quite easy for anyone, from what has been demonstrated — how we should proceed in one that has at least two quadrant arcs and two right angles. [a] For no triangle can be right-angled having only a single arc that is a quadrant; rather, either none will be a quadrant, or all three will be quadrants, or two will be, and so on. Let ABC then be a spherical triangle in which angle B is set right, and arc AB, about the right angle, a quadrant. This being posited, [b] arc AC, opposite the right angle, will also be a

quadrant. Therefore, since the two arcs AB and AC are quadrants, [c] the two angles B and C will be right; [d] and consequently A will be the pole of arc BC, and BC will be the arc of angle A, by definition 6 [definitione 6]. So if the third angle A is given, the third arc BC will also be given; and conversely, if the third arc BC is given, the third angle A will likewise be given.

Diagram. An equilateral-looking spherical triangle (a quadrantal triangle).

- A (top): apex vertex
- B (lower left): base vertex
- C (lower right): base vertex

In the same way, if instead the other arc BC, about the right angle, is set as a quadrant, we will show that arc AC, opposite the right angle, is also a quadrant, and that angle A is right. So if the third angle C is given, the third arc AB will also be given, and conversely, as before. But if the magnitude of the third angle, or of the arc, is not given, nothing certain can be gathered, even though the other two angles be right and the two arcs opposite them quadrants — as is evident.

[Margin: a — Corollary of proposition 38 of this treatise [Coro. 38 huius].]

[Margin: b — proposition 35 of this treatise [35 huius].] [Margin: c — proposition 25 of this treatise [25 huius].] [Margin: d — Corollary of proposition 26 of this treatise [Coro. 26 huius].]

Now let arc AC, opposite the right angle B, be posited as a quadrant. This being posited, [e] at least one of the arcs AB, BC about the right angle will also be a quadrant. Hence the rest will follow, as was just demonstrated.

But if at any time the two arcs AB, BC about the right angle are set as quadrants, [f] the third arc AC, opposite the right angle, will likewise be a quadrant. And so, since all the arcs are quadrants, [g] all the angles will be right.

From these considerations anyone will readily understand what he ought to do whenever some arc in a right triangle is set as a quadrant — especially if propositions 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38 [propos. 25.26.27.28.29.30.31.32.33.34.35.36.37.38] are attentively considered.

It now follows that we should at last set out the reckoning of non-right (oblique) triangles; but first several theorems very useful for this matter must be demonstrated.

[Margin: e — proposition 36 of this treatise [36 huius].] [Margin: f — proposition 26 or 35 of this treatise [26 vel 35 huius].] [Margin: g — Corollary of proposition 25 of this treatise [Corol. 25 huius].]

THEOR. 55. PROPOS. 57.

If in a spherical triangle two acute angles, or two obtuse angles, stand upon one arc, the perpendicular arc dropped from the third angle onto that arc falls inside the triangle. But if, of the two angles standing upon one arc, one is acute and the other obtuse, the perpendicular arc dropped from the third angle onto that arc falls outside the triangle.

In the spherical triangle ABC, let the two angles B and C, standing upon arc BC, be acute (or obtuse). I say that the perpendicular arc dropped from A onto arc BC falls inside the triangle — such as is arc AD. For if it be said to fall outside, let the perpendicular arc AE fall (if it can be done) from A onto BC produced, outside the triangle; and let the two angles B and C first be set acute, so that consequently angle ACE is obtuse. Since, then, in triangle ACE — which has E a right angle — the angle ACE is obtuse, [a] arc AE will be greater than a quadrant. Again, in triangle ABE — which has the right angle E — the angle B is acute, so [b] arc AE will be less than a quadrant; but it was also shown [above] to be greater than a quadrant — which is absurd.

Diagram. A spherical triangle with a perpendicular arc dropped from the apex to the base and produced beyond it.

- A (top): apex, from which the perpendicular arc is dropped
- B (lower left): base vertex
- C (base, right of center): base vertex
- E (base, far right): foot of the perpendicular AE on BC produced, outside the triangle

[Margin: a — proposition 34 of this treatise [34 huius].] [Margin: b — proposition 34 of this treatise [34 huius].]

Suppose next that the angles B and C are obtuse, so that the angle ACE is acute. Then, since in the triangle ACE, which has the right angle at E, the angle ACE is acute, the arc AE will be less than a quadrant. [Margin: c — Prop. 34 of this treatise.] Again, since in the triangle ABE, which has the right angle at E, the angle B is obtuse, the arc AE will be greater than a quadrant. [Margin: d — Prop. 34 of this treatise.] But it was shown to be also less than a quadrant, which is absurd. Therefore the perpendicular arc does not fall outside the triangle; nor does it

coincide with either of the arcs AB, AC, since neither of the angles B, C is assumed to be right. It therefore falls inside the triangle.

Now let it be assumed, in the same triangle ABC, that the angle B is acute and C obtuse. I say that the perpendicular arc dropped from A to the arc BC falls outside the triangle — such as the arc AE is. For if it be said to fall inside, let the arc AD, perpendicular to BC, fall inside the triangle, if that be possible. Then, since in the triangle ACD, which has the right angle at D, the angle C is obtuse, the arc AD will be greater than a quadrant. [Margin: e — Prop. 34 of this treatise.] Again, since in the triangle ABD, which has the right angle at D, the angle B is acute, the arc AD will be less than a quadrant. [Margin: f — Prop. 34 of this treatise.] But it was shown to be also greater than a quadrant, which is absurd. The perpendicular arc therefore does not fall inside the triangle; nor does it coincide with either of the arcs AB, AC, since neither of the angles B, C is assumed to be right. It therefore falls outside the triangle. And so, if in a spherical triangle over one arc there are two acute angles, and so on. Which was to be demonstrated.

THEOR. 56. PROPOS. 58.

In every spherical triangle whose two sides are unequal, the square of the whole sine [*sinus totus* — the radius, the total sine] has to the rectangle contained by the right sines [*sinus rectus* — the ordinary sine] of the two unequal sides the same ratio that the versed sine [*sinus versus* — the versed sine, i.e. $1 - \cosine$] of the angle contained by those sides has to the difference of two versed sines — of which one belongs to the difference of those same arcs, while the other answers to the third arc, the one opposite the aforesaid angle.

In the spherical triangle ABC let there be two unequal sides AB, AC, the former smaller and the latter larger. I say that the square of the whole sine has to the rectangle contained by the right sines of the arcs AB, AC the same ratio that the versed sine of the angle A has to the difference between the versed sine of the arc by which AB and AC differ from each other and the versed sine of the arc BC. To make the demonstration of this clearer and more general, we shall divide it into fifteen cases.

1. [Margin: Case 1.] Let all three arcs be less than a quadrant. Let the circle ABDGH be completed on the smaller arc AB; and, the arcs AC, BC being produced so that the quadrants AL, BM are formed, let the great circles DLEF, GMEH be described about the poles A, B at the distances of the quadrants AL, BM. About the same poles, but at the distances AC, BC, let the lesser circles kCN, OCP be drawn,

which will be parallel to those great circles; [Margin: a — Theodosius II.2.] and both these and those will be at right angles to the circle ABDGH, since that circle, passing through their poles, is at right angles to them. [Margin: b — Theodosius I.15.] After this, to avoid confusion, in the circle ABDGH drawn separately let there be marked the common sections of it and of the circles described about the poles A, B — namely DF, GH — the common sections of it and of the great circles DL, EF, GMEH, which are their diameters, intersecting one another at X, the centre of the sphere; while kN, OP are the common sections of the same circle and of the circles kCN, OCP, intersecting each other at S. These will be parallel to DF, GH, and are the diameters of the circles kCN, OCP, [Margin: c — Euclid XI.16.] since the great circle ABDGH, passing through their poles, bisects them, that is, cuts them through their diameters. [Margin: d — Theodosius I.15.] Let the semidiameters AX also be drawn, cutting kN at Y, and BX, cutting KN, OP at I and R. Then the semidiameters AX, BX will be perpendicular to the circles drawn through DF, kN, GH, OP, since they are drawn from the poles A, B of those circles through X, the centre of the sphere; and hence the angles at Y and R will be right, by the definition [Euclid] Book XI, def. 3. [Margin: e — Scholium on Theodosius I.10.] Finally let AV, kQ, kT be drawn perpendicular to BX, OP. Then, by what we wrote in the treatment of sines, AV will be the right sine of the arc AB; and kY the right sine of the arc Ak, that is, of the arc AC (since the arcs Ak, AC are equal by the definition of a pole), as appears in the first circle. BR will be the versed sine of the arc BO, that is, of the arc BC (since BO, BC are equal by the definition of a pole). BQ will be the versed sine of the arc Bk, which is the difference of the unequal arcs AB, AC — because, by the definition of a pole, the arc Ak, exceeding the arc AB by the arc Bk, is equal to the arc AC; and hence QR, or kT, will be the difference between BR, the versed sine of the third arc BC, and BQ, the versed sine of the difference of the unequal arcs AB, AC, that is, the versed sine of the arc Bk. Lastly, kS will be the right sine of the arc kC in the lesser circle kCN, since the straight line drawn from C into the common sections of the circles kCN, OCP with the circle ABDGH — that is, falling upon the point S — [Margin: f — Euclid XI.19.] (which is indeed at right angles to the circle ABDGH, being the common section of the circles kCN, OCP, which are at right angles to that same circle ABDGH) is the right sine of the same arc kC. Let DZ also be taken, the versed sine of the arc DL, that is, of the angle A, [Margin: g — Theodosius I.10.] which arc is indeed similar to the arc kC. It is therefore to be demonstrated that the square of the whole sine — that is, the rectangle contained by DX, XA — has to the rectangle contained by the right sines AV, kY of the arcs AB, AC the same ratio that the versed sine DZ of the angle A has to kT, the difference between BR, the versed sine of the arc BC,

and BQ, the versed sine of the difference of the unequal arcs AB, AC. This will come about as follows.

[Margin: h — Euclid I.15.] Since the angle XIY is equal to the angle RIS, and the right angle Y to the right angle R, [Margin: i — Euclid I.32.] the remaining angle IXY of the triangle IXY will be equal to the remaining angle ISR of the triangle ISR — that is, to the vertical angle kST. Since, then, the right angle V is also equal to the right angle T, the remaining angle XAV of the triangle XAV will also be equal to the remaining angle SkT of the triangle SkT. [Margin: k — Euclid VI.4.] For this reason, as XA is to AV, so will Sk be to kT. Again, since DZ, kS are the versed sines of the similar arcs DL, kC, it will follow that, as DX is to kY (the whole sine to the whole sine), so is DZ to kS, by the lemma of Proposition 1 of our *Gnomonics*. [Margin: l — Euclid VI.23.] And since the ratio of the rectangle contained by DX, XA to the rectan-

gle contained by KY, AV is compounded of the ratio of DX to KY (that is, of DZ to KS) and of the ratio of XA to AV; and since the ratio of DZ to KT is compounded of the same ratios — namely (KS being set as the mean term) of the ratio of DZ to KS and of the ratio of KS to KT, that is, of the ratio of XA to AV — it will follow that, as the rectangle contained by DX, XA (that is, the square of the whole sine) is to the rectangle contained by KY, AV (the right sines of the unequal arcs AC, AB), so is DZ, the versed sine of the angle A, to KT, the difference between BR, the versed sine of the arc BC opposite the angle A, and BQ, the versed sine of the difference of the unequal arcs AC, AB. Which was the thing proposed.

Diagram. A pair of circles illustrating Case 1 (all three arcs less than a quadrant). The left member is the full spherical construction (the great circles about poles A, B, the lesser circles kCN, OCP, and the dropped perpendicular); the right member is the rectilinear cross-section drawn separately, showing the common sections, diameters, and the right and versed sines derived in the text.

- Left sphere — outer/readable labels: A, B (top), K (upper right), C, O (interior right of centre), D (right rim), P (left rim), H, N (lower left rim), L (interior), E (lower centre), M (lower centre), G (lower right rim), F (bottom).
- Right sphere — outer/readable labels: A, B (top), Q, V (upper interior), K (upper right), O, D (right upper rim), P (left rim), R, S (central), Y (interior left of centre), I (centre), Z (interior below centre), T (interior right), H, N (lower left rim), X (bottom centre), G (right rim), F (bottom). : dense interior labelling; several small interior letters are at the resolution floor.

2. [Margin: Case 2.] Let there be two unequal arcs AB, AC less than a quadrant, but BC a quadrant. Let the circle ABDGH be completed on the smaller arc AB; and, the arc AC being produced so that the quadrant AL is formed, let the great circles DELF, GECH be described about the poles A, B at the distances of the quadrants AL, BC. Likewise, from the pole A, at the distance A C, let the lesser circle KCN be drawn, [Margin: o — Theodosius II.2.] which will be parallel to DELF; [Margin: p — Theodosius I.15.] and the circle ABDGH will cut the circles DELF, GECH, KCN at right angles and bisect them; and hence their common sections with it, DF, GH, will be their diameters, and DF, GH will intersect one another at X, the centre of the sphere, [Margin: q — Euclid XI.16.] and DF, KN will be parallel. Let the rest be done as in the preceding case, except that here the point R is the same as X — because the circle OCP does not differ from the circle GEH, and so the straight line ORP does not differ from the straight line GH. As before, AV will be the right sine of the arc AB; and KY the right sine of the arc AK, that is, of the arc AC, which is equal to AK by the definition of a pole. Likewise BR will be the versed sine of the arc BG, that is, of the arc BC, which is equal to BG. But BQ will be the versed sine of the arc BK, the difference of the arcs AB, AC; and so QR, or KT, will be the difference between the versed sines BR, BQ of the arcs BC, BK. Finally KS will be the right sine of the arc DL, that is, of the angle A. It is to be demonstrated that the square of the whole sine — that is, the rectangle contained by DX, XA — has to the rectangle contained by AV, KY (the right sines of the arcs AB, AC) the same ratio that the versed sine DZ of the angle A has to KT, the difference of the versed sines BR, BQ of the arcs BC, BK. This will be demonstrated as in the preceding case, except that here the triangle XAV will be shown to be equiangular with the triangle SKT from the fact that the angle IXY is equal to the angle YSX, [Margin: r — Euclid VI.18.] because the triangles IXY, YSX are similar to one another. For hence it follows that the right-angled triangles XAV, SKT are altogether equiangular with each other.

Diagram. Pair of circles for Case 2 (AB, AC less than a quadrant, BC a quadrant). Left: the spherical construction; right: the rectilinear cross-section.

- Left sphere — readable labels: A, B, K (top), D (upper right), H, P (left rim), C (centre), L, E (lower centre), O, G (right rim), N, F (bottom).
- Right sphere — readable labels: A, B (top), Q (upper interior), K (upper right), D (right rim), V, I, Y (interior), S, X, R, T, Z (central/lower interior), H, P, N (left/lower rim), O, G (right rim), F (bottom). : dense interior labelling near the resolution floor.

3. [Margin: Case 3.] Let AB, AC again be less than a quadrant, but BC greater. Let the circle be completed on the smaller arc AB; and let the quadrant BM be cut off from CB, and let AC be produced so that the quadrant AL is formed. Let the rest be constructed as in the first case. The sines here will be as they were there. It is therefore to be demonstrated that the square of the whole sine — namely the rectangle contained by DX, XA — has to the rectangle contained by AV, KY (of the arcs AB, AC) the same ratio that the versed sine DZ of the angle A has to KT, the difference of the versed sines BR, BQ of the arcs BC, BK. This will be shown as in the first case, except that here we shall show the triangle XAV to be equiangular with the triangle SKT [Margin: s — Euclid I.29.] from the fact that the angle XiY is equal to the angle SkT, the external to the internal. For hence it comes about that in the right-angled triangles XIY, SKT the remaining angles IXY, KST are equal; and therefore the right-angled triangles XAV, SKT are equiangular.

Diagram. Pair of circles for Case 3 (AB, AC less than a quadrant, BC greater). Left: the spherical construction; right: the rectilinear cross-section.

- Left sphere — readable labels: A, B, K (top), D (upper right), H, P (left rim), M (interior), C, E (lower centre), L (lower interior), O, G (right rim), N, F (bottom).
- Right sphere — readable labels: A, B, K (top), D (right rim), Q, V (upper interior), I, Y (interior), X (centre), S, Z, R, T (lower interior), O, G (right rim), N, F (bottom). : interior labelling near the resolution floor.

4. [Margin: Case 4.] Let the arc AC be a quadrant, so that AB is less than a quadrant; let BC also be set less than a quadrant. The circle ABDGH being completed on the smaller arc AB, and the arc BC being produced so that the quadrant BM is formed, let the great circles DCEF, GMEH be described about the poles A, B at the distances of the quadrants AC, EM. Likewise, from the pole B, at the distance of the arc BC, let the lesser circle OCP be drawn. Let the rest be constructed as in the first case, except that here the two parallel circles DEF, KCN do not differ from each other, on account of the quadrant AC. From which it comes about that the straight lines DF, kN also do not differ from each other; and the same is to be said of the versed sines kS, DZ. The other sines are as before. That the square of the whole sine — or the rectangle contained by DX, XA — has to the rectangle contained by AV, KY (the right sines of the arcs AB, AC) the same ratio that the versed sine DZ of the angle A (or of the arc KC) has to KT, the difference of the versed sines BR, BQ of the arcs BC, BK, we shall show as in the first case, except that here we shall demonstrate the triangle XAV to be equiangular with the triangle SKT from the fact that the angle AXV is equal to the

angle YSR, [Margin: t — Euclid VI.18.] (because the triangles IXR, RSY are similar) and so to the angle KST. For hence it comes about that the right-angled triangles XAV, SKT are equiangular.

Diagram. Pair of circles for Case 4 (AC a quadrant, AB and BC less than a quadrant). Left: the spherical construction; right: the rectilinear cross-section.

- Left sphere — readable labels: A, B, D, K (top), P, H (left rim), C (centre), E, M (bottom), O, G (right rim), N, F (bottom).
- Right sphere — readable labels: A, B, K, D (top), Q, V (upper interior), R, S, T (central), Y, X (centre), O, G (right rim), P, H (left rim), N, F (bottom). : interior labelling near the resolution floor.

5. [Margin: Case 5.] Let AC again be a quadrant, and therefore AB less than a quadrant, but let BC also be set as a quadrant. The circle ABDGH being completed on the smaller arc AB, let the great circles DCF, GCH be described about the poles A, B at the distances of the quadrants AC, BC, and let the rest be done as before, except that here the lesser circles KN, OP do not differ from the great circles DF, GH, and so on. It is therefore to be demonstrated that the square of the whole sine — that is, the rectangle contained by DX, XA — has to the rectan-

gle contained by AV, kY (the right sines of the arcs AB, AC) the same ratio that DZ, the versed sine of the angle A (or of the arc kC), has to kT, the difference of the versed sines BY [?: possibly BR] and BQ — of which the former belongs to the arc BC, the latter to the arc Bk. This we shall show as in the first case. Only the triangle XAV will be demonstrated equiangular with the triangle SKT thus: Since the angles DSA, BSH are right (because AS, BS are the axes of the circles DF, GH), then, the common angle ASB being taken away, the remaining angles DSB, ASH will be equal; [Margin: a — Euclid I.29.] but the former is equal to the alternate angle SKT, and the latter to the alternate angle XAV. Therefore the angles SkT, XAV will also be equal; and hence the right-angled triangles XAV, SkT will be equiangular.

Diagram. Pair of circles for Case 5 (AC and BC quadrants, AB less than a quadrant). Left: the spherical construction; right: the rectilinear cross-section.

- Left sphere — readable labels: B, K, D (top), A, H, P (left rim), C (centre), O, G (right rim), N, F (bottom).
- Right sphere — readable labels: B, K, D (top), A (left), Q, V (upper interior), S, Z, X, Y, T (central), H (left rim), O, G (right rim), N, F (bottom). : interior labelling near the resolution floor.

6. [Margin: Case 6.] Let AC still be a quadrant, so that AB is less than a quadrant, but let BC be set greater than a quadrant. The circle ABDGH being completed on the smaller arc AB, and the quadrant BM being cut off from BC, let the great circles DEF, GEH be described about the poles A, B at the distances of the quadrants AC, BM. Likewise, from the pole B, at the distance BC, let the lesser circle OCP be drawn, [Margin: b — Theodosius II.2.] which will be parallel to GEH. Let the rest be done as before, except that here the circles DF, KN do not differ from each other, and so on. We shall now demonstrate, as in the first case, that the square of the whole sine — that is, the rectangle contained by DX, XA — has to the rectangle contained by AV, KY (the right sines of the arcs AB, AC) the same ratio that DZ, the versed sine of the angle A (or of the arc DC), has to KT, the difference of the versed sines BR, BQ of the arcs BC, BK. But that the triangle XAV is equiangular with the triangle SKT we shall show thus: Since the right angles are AXK, BXG, the remaining angles AXV, KXG will be equal; [Margin: c — Euclid I.29.] but this latter is equal to its opposite, and to the internal angle TSK. Therefore the angle AXB will also be equal to the angle TSK; and hence the right-angled triangles XAV, SKT will be equiangular.

Diagram. Pair of circles for Case 6 (AC a quadrant, AB less than a quadrant, BC greater than a quadrant). Left: the spherical construction; right: the rectilinear cross-section.

- Left sphere — readable labels: A, B (top), K, D (upper right), H, P (left rim), M, E (interior), C (lower centre), O, G (right rim), N, F (bottom).
- Right sphere — readable labels: A, B (top), V, K (upper), Q, D (interior/right rim), H, P (left rim), S, X, Z (central), R, T (lower interior), O, G (right rim), N, F (bottom). : interior labelling near the resolution floor.

7. [Margin: Case 7.] Let the arc AC be greater than a quadrant, and AB, BC less than a quadrant. The circle ABDGH being completed, and the quadrant AL being cut off from AC, and the arc BC being produced so that the quadrant FM is formed, let all the rest be done as in the first case. For we shall demonstrate, as there, that the square of the whole sine — or the rectangle contained by DX, XA — has to the rectangle contained by AV, KY (the right sines of the arcs AB, AC) the same ratio that DZ, the versed sine of the angle A (or of the arc DL), has to KT, the difference of the versed sines BR, BQ of the arcs BC, BK. But that the triangle AXV is equiangular with the triangle SKT will be proved thus: [Margin: d — Euclid I.15.] The angle KST is equal to the angle ISR. Therefore, in the right-angled triangles SKT, ISR, the remaining angles SKT, SIR will be equal; and hence also, in the right-angled triangles SKT, XIY, the remaining angles KST, IXY will be equal.

[Margin: e — Euclid I.15.] Since, then, the angle IXY is equal to the angle AXV , KST will also be equal to the same AXV ; and therefore, in the right-angled triangles SKT , XAV , the remaining angles SKT , XAV will be equal.

Diagram. Pair of circles for Case 7 (AC greater than a quadrant, AB and BC less than a quadrant). Left: the spherical construction; right: the rectilinear cross-section.

- Left sphere — readable labels: A , B (top), D , K (upper right), H (left rim), L , C (interior), E , M (bottom), O , G (right rim), F , N (bottom).
- Right sphere — readable labels: A , B (top), D , K (upper right), Q , V (upper interior), R , S , T (central), X , Y , I (centre/lower interior), H (left rim), O , G (right rim), F , N (bottom). : interior labelling near the resolution floor.

8. [Margin: Case 8.] Let AC again be greater than a quadrant, and AB less than a quadrant. The circle $ABDGH$ being completed, and the quadrant AL being cut off from AC , let the great circles DEF , GEH be described about the poles A , B at the distances of the quadrants AL , BC . Likewise, from the pole A , at the distance $A C$, let the lesser circle KCN be drawn, and let the rest be done as in the first case. It will be demonstrated, as there, that the square of the whole sine — namely the rectangle contained by DX , XA — has to the rectangle contained by AV , KY (the right sines of the arcs AB , AC) the same ratio that DZ , the versed sine of the arc DL (or of the angle A), has to KT , the difference of the versed sines BR , BQ of the arcs BC , BK — provided we show the triangles XAV , SKT to be equiangular, as in the seventh case.

Diagram. Pair of circles for Case 8 (AC greater than a quadrant, AB less than a quadrant, BC a quadrant). Left: the spherical construction; right: the rectilinear cross-section.

- Left sphere — readable labels: A , B (top), D , K (right rim), H (left rim), E , L , C (interior), G (right rim), F , N (bottom).
- Right sphere — readable labels: A , B (top), V , D , K (upper right), Q (interior), R , Z , S , T (central), K , I , Y (interior), G (right rim), H (left rim), F , N (bottom). : interior labelling near the resolution floor.

9. [Margin: Case 9.] Let AC again be greater than a quadrant, and AB less than a quadrant, but BC also greater than a quadrant. The circle $ABDGH$ being completed, and the quadrants AL , BM being cut off from AC , BC , let the rest be constructed as in the first case. For, as there, so here it will be demonstrated that the square of the whole sine — namely the rectangle contained by DX , XA — has to the rectangle contained by AV , KY (the right sines of the arcs AB , AC) the same

ratio that DZ, the versed sine of the arc DL (or of the angle A), has to KT, the difference of the versed sines BR, BQ of the arcs BC, BK. But that the triangles X A V, S K T are equiangular will be confirmed thus: [Margin: f — Euclid I.15.] The angle KST is equal to the angle ISR. Therefore, in the right-angled triangles SKT, SIR, the remaining angles SKT, SIR will be equal; and hence, in the right-angled triangles SKT, IXY, the remaining angles KST, IXY (that is, AXV, [Margin: g — Euclid I.15.] since this is equal to that same IXY) will be equal to one another. Wherefore the remaining angles SKT, XAV, in the right-angled triangles SKT, XAV, will be equal.

Diagram. Pair of circles for Case 9 (AC and BC greater than a quadrant, AB less than a quadrant). Left: the spherical construction; right: the rectilinear cross-section.

- Left sphere — readable labels: A, B (top), D (upper right), K (right rim), H, P (left rim), E, L, M (interior), C (bottom), O, G (right rim), F, N (bottom).
- Right sphere — readable labels: A, B (top), V, D (upper), Q, K (right rim), R, S, T, Z, I, Y (central), J, O (interior right), G (right rim), P, H (left rim), F, N (bottom). : interior labelling near the resolution floor.

10. [Margin: Case 10.] Let the arc AC be greater than a quadrant, and AB a quadrant, but BC less than a quadrant. The circle ABGF being completed, and the quadrant AL being cut off from AC, and BC being produced so that the quadrant BM is formed, let the great circles BLEF, AEMG be described about the poles A, B at the distances of the quadrants AL, BM; and the one will pass through the point B and the other through the point A, on account of the quadrant AB, [Margin: h — Corollary of Theodosius I.16.] because a great circle is distant from a pole by a quadrant of a great circle. Likewise, about the same poles A, B, at the distances AC, BC, let the lesser circles KCN, OCP be drawn, [Margin: i — Theodosius II.2.] which will be parallel to the former. Then, when in another circle the common sections of these circles with the circle A B G F have been drawn, [Margin: k — Euclid XI.16.] which will be parallel to one another and will cut each other at right angles [Margin: l — Scholium on Theodosius I.10.] (for AX, drawn from A, the pole of the circle B F, into X, the centre of the sphere, is at right angles to that circle; and therefore, by the definition [Euclid] Book XI, def. 3, the angles at X will be right; from which it follows [Margin: m — Euclid I.29.] that the angles at R, S, Y are also right, on account of the parallel lines BF, KN and AG, OP): then AV will be the right sine of the quadrant AB; KY the right sine of the arc AC (or of the arc AK, equal to it by the definition of a pole); BR the versed sine of the arc BC (or of the arc BO, equal to it by the definition of a pole); BQ (the perpendicular K-Q having

been drawn to BF) the versed sine of the arc BK, by which the arcs AB, AC differ from each other. Finally KS, the versed sine

Diagram. Pair of circles for Case 10 (AC greater than a quadrant, AB a quadrant, BC less than a quadrant). Left: the spherical construction; right: the rectilinear cross-section.

- Left sphere — readable labels: A, B (top), D, K (right rim), H, P (left rim), E, L, M (interior), C (bottom), O, G (right rim), F, N (bottom).
- Right sphere — readable labels: A, B (top), V, D (upper), Q, K (right rim), R, S, Z, I, Y (central), J, O (interior right), G (right rim), P, H (left rim), F, N, T (bottom). : interior labelling near the resolution floor.

(Continuation of the demonstration of THEOR. 56 / PROPOS. 58 — the fifteen cases of the versed-sine / included-angle proportion. This page carries the end of the tenth case and cases 11–15.)

THEOR. 56. PROPOS. 58 (continued — the fifteen cases)

[10th case — conclusion.] ...the versed sine of arc KC. And so, if DZ is taken — the versed sine of angle A, or arc BL, which is similar to arc KC — we shall show, as in the first case, that the square of the whole sine (namely the rectangle under DX, XA) is to the rectangle under AV, KY, the right sines of arcs AB, AC, as DZ, the versed sine of angle A (or arc BL), is to KT, or to QR, the difference of the versed sines BR, BQ of arcs BC, BK; except that here no equiangular triangles are found, but AV does not differ from XA, just as KS does not differ from KT.

Diagram. The figure for the tenth case, a pair of circular diagrams. LEFT: a sphere (great-circle outline) carrying the spherical triangle and auxiliary arcs; RIGHT: a circle divided into quadrants by two perpendicular diameters, showing the "unrolled" sine/versed-sine construction. Legible perimeter labels — LEFT: B (top), K (top right), P (left), O (right), A (lower left), G (right), N (lower right), F (bottom); interior reads approximately c, L, E, M [interior lettering at the resolution floor]. RIGHT: B, D (top), K (top right), O (right), A (left), G (right), N (lower right), F (bottom); interior reads approximately Q, Z, X, R, T, S, V, Y [interior lettering at the resolution floor].

- Interior point-letters not securely resolvable at this resolution

11. [Margin: 11. casus.] Again, let AC be greater than a quadrant, but both AB and BC quadrants. With circle ABGF completed and the quadrant AE cut off from AC, let there be described from pole A, at the intervals AE, AC, the circles BEF, kCN,

and from pole B, at the interval BC, the circle AEG; and let the rest be constructed as in the preceding case. We shall therefore show, as in the first case, that the square of the whole sine — that is, the rectangle under DX, XA — is to the rectangle under AV, kY, the right sines of arcs AB, AC, as DZ, the versed sine of angle A (or arc BE), is to kT, or QR, the difference of the versed sines BR, BQ of arcs BC, Bk; except that here no equiangular triangles are present, just as in the preceding case, and AV does not differ from XA, nor DZ from BR, nor kS from kT.

Diagram. The figure for the eleventh case, again a sphere (LEFT) paired with the quadrant-divided circle (RIGHT). Legible perimeter labels — LEFT: B (top), K (top right), A (left), G (right), N (lower right), F (bottom); interior reads approximately E, C. RIGHT: B, D (top), K (top right), G (right), N (lower right), F (bottom); interior reads approximately Q, X, z, V, R, T, S, Y [interior lettering at the resolution floor].

- Interior point-letters not securely resolvable

12. [Margin: 12. casus.] Let arc AC be greater than a quadrant, and AB a quadrant, but BC also greater than a quadrant. With circle ABGF completed and the quadrants AL, BM taken away from arcs AC, BC, let circles be described from poles A, B at the intervals of the quadrants AL, BM and of the arcs AC, BC, and let everything else be done as before. Then again, as was demonstrated in the first case, the square of the whole sine — that is, the rectangle under DX, XA — will be to the rectangle under AV, kY, the right sines of arcs AB, AC, as DZ, the versed sine of angle A (or arc EL), is to KT, the difference of the versed sines BR, BQ of arcs BC, BK; although here no equiangular triangles appear, but XA and AV do not differ from each other, just as neither do kS, kT.

Diagram. The figure for the twelfth case: sphere (LEFT) with the quadrant-divided circle (RIGHT). Legible perimeter labels — LEFT: B (top), K (top right), A (left), G (right), P (lower left), O (right), N (lower right), F (bottom); interior reads approximately E, M, L, c. RIGHT: B, D (top), K (top right), G (right), P (lower left), O (right), A (left), N (lower right), F (bottom); interior reads approximately Q, X, V, R, S, Y, z, T [interior lettering at the resolution floor].

- Interior point-letters not securely resolvable

13. [Margin: 13. casus.] Let the arcs AC, AB be greater than a quadrant, but BC less than a quadrant. With circle ABGF completed and the quadrant AL cut off from AC, and arc BC likewise produced so that BM becomes a quadrant, let the rest be done as above. We shall now show, as in the first case, that the square of the whole sine — namely the rectangle under DX, XA — is to the rectangle under AV, kY, the right sines of arcs AB, AC, as DZ, the versed sine of angle A (or arc DL), is

to kT , the difference of the versed sines BR , BQ of arcs BC , Bk ; except that the triangle XAV must be shown to be equiangular with triangle SkT in this way. [Margin: a — 29. primi (Euclid, *Elements* I.29)] Since angle kST is equal to the opposite interior angle YIX , then in the right triangles SkT , IXY the remaining angle SkT will also be equal to the remaining angle IXY — that is, to the opposite interior angle VAX (since AV , GH are parallel). Therefore in the right triangles SkT , XAV the remaining angles kST , AXV will also be equal, and hence the triangles SkT , XAV will be equiangular.

Diagram. The figure for the thirteenth case: sphere (LEFT) with the quadrant-divided circle (RIGHT). Legible perimeter labels — LEFT: D , B (top), K (top right), P (upper left), H (left), A (lower left), G (center), O , p (right), N (lower right), F (bottom); interior reads approximately L , C , E . RIGHT: D , B (top), K (top right), O (right), A (left), G (right), H (left), N (lower right), F (bottom); interior reads approximately Q , z , R , T , X , V , I , Y , S [interior lettering at the resolution floor].

- Interior point-letters not securely resolvable

14. [Margin: 14. casus.] Again, let AC , AB be greater than a quadrant, but BC a quadrant. With circle $ABGF$ completed and the quadrant AL cut off from AC , and with the circles DEF , kCN described from pole A at the intervals AL , AC , let there also be described from pole B , at the interval of the quadrant BC , the great circle GEH ; and let the other things be done as above. It must therefore be shown that the square of the whole sine — that is, the rectangle under DX , XA — is to the rectangle under AV , kY , the sines of arcs AB , AC , as DZ , the versed sine of angle A (or arc DL), is to kT , the difference between the versed sines BR , BQ of arcs BC , Bk . This indeed we shall show as in the first case. Only the triangles SkT , XAV will be proved to be equiangular, in this way. Angle S is common to both right triangles SkT , SRY . Therefore the remaining angle SkT will be equal to the remaining angle SRY — [Margin: b — 29. primi (Euclid, *Elements* I.29)] that is, to the opposite interior angle VAX (since AV , GH are parallel). Hence the right triangles SkT , XAV will be equiangular.

Diagram. The figure for the fourteenth case: sphere (LEFT) with the quadrant-divided circle (RIGHT). Legible perimeter labels — LEFT: D , B (top), K (top right), H (left), A (lower left), G (center right), N (lower right), F (bottom); interior reads approximately E , L , C , H . RIGHT: D , B (top), K (top right), G (right), A (left), N (lower right), F (bottom); interior reads approximately Q , x , R , Y , T , S , V , z [interior lettering at the resolution floor].

- Interior point-letters not securely resolvable

15. [Margin: 15. casus.] Finally, let all three arcs of triangle ABC be greater than a quadrant. With circle ABGF completed and the quadrants AL, BM cut off from arcs AC, BC, let everything else be done as before. We shall show, just as in the first case, that the square of the whole sine — namely the rectangle under DX, XA — is to the rectangle under AV, kY, the right sines of arcs AB, AC, as DZ, the versed sine of angle A (or arc DL), is to kT, the difference between the versed sines BR, BQ of arcs BC, Bk; provided only that we conclude the triangles SkT, XAV to be equiangular, by this argument. [Margin: c — 29. primi (Euclid, *Elements* I.29)] Angle YIX is equal to the interior and opposite angle kST. Therefore in the right triangles SkT, IXY the remaining angle SkT will be equal to the remaining angle IXY — [continues on p. 217] that is, to the interior and opposite angle XAV (since AV, GH are parallel); and hence the right triangles SkT, XAV will be equiangular.

Diagram. The figure for the fifteenth case: sphere (LEFT) with the quadrant-divided circle (RIGHT). Legible perimeter labels — LEFT: D, B (top), K (top right), H (left), A (lower left), G (center right), o (center), P (lower left), N (lower right), F (bottom); interior reads approximately E, M, L, c. RIGHT: D, B (top), K (top right), G (right), o (center), A (left), P (lower left), N (lower right), F (bottom); interior reads approximately Q, x, Y, V, R, S, z, T [interior lettering at the resolution floor].

- Interior point-letters not securely resolvable

that is, to the interior and opposite angle XAV (since AV, GH are parallel); and hence the right triangles SKT, XAV will be equiangular. Therefore, in every spherical triangle whose two arcs are unequal, etc. Which was to be demonstrated.

SCHOLIVM I.

From all fifteen cases of this demonstration it is clear that arc BC — which is opposite angle A and is subtended by the unequal arcs — is always greater than arc BK, that is, greater than the difference of the unequal arcs. For in all the figures arc BC is, by the definition of a pole, equal to arc BO (or to arc BG, when BC is a quadrant, as in cases 2, 5, 8, 11, and 14). But it is established that arc BO (or arc BH, in the said five cases) is greater than arc BK — and that this is so, the following theorem too will easily demonstrate.

In every spherical triangle whose two arcs are unequal, the remaining arc is greater than the arc by which the unequal arcs differ from each other.

For in triangle ABC, let arc AB be greater than arc AC, and from pole A, at the interval AC, let the arc of a circle CD be described. Then arc AD will be equal to arc AC by the definition of a pole, and so arc BD is the difference of the unequal arcs AB, AC. I say that arc BC is greater than arc BD. [Margin: a — 3. huius (Proposition 3 of this treatise)] For since the two arcs AC, CB together are greater than arc AB, when the equal arcs AC, AD are removed, the remainder CB will also be greater than the remainder BD. Which is what was proposed.

Diagram. A spherical triangle with vertices A (top), B (lower left), and C (lower right); point D lies on side AB (nearer B), and the arc DC is drawn across the interior. This illustrates arc BD as the difference of the unequal sides AB, AC.

And so, in every spherical triangle whose two arcs are unequal, the versed sine of the remaining arc is always greater than the versed sine of the difference of the unequal arcs. For since that remaining arc has been shown to be greater than that difference, and a greater arc always has a greater versed sine (as is established from the treatment of sines), it is evident that the versed sine of the remaining arc is greater than the versed sine of the difference of the unequal arcs.

[Margin: In a spherical triangle of two unequal arcs, the versed sine of the third arc is greater than the versed sine of the difference of the unequal arcs.]

I have said this so that you may see the reason why, in practice — at Proposition 64 — the difference between the two versed sines (of which one belongs to the remaining third arc, the other to the difference of the unequal arcs) is directed to be *added* to the versed sine of the difference of the unequal arcs, in order to compose the versed sine of the remaining third arc; but it is never to be *subtracted* from the versed sine of that difference so as to leave the versed sine of the remaining arc.

SCHOLIVM II.

Moreover, from this Proposition 58 we shall gather the following theorem, very useful for the calculation of non-right spherical triangles, namely:

In every spherical triangle whose two arcs are unequal: the whole sine has to the quantity which is the fourth proportional to the whole sine and the two sines of the unequal arcs the same ratio as the versed sine of the angle contained by the said arcs has to the difference between the versed sine of the remaining third arc and the versed sine of the arc by which the two unequal arcs differ from each other.

In the spherical triangle ABC — as in the immediately preceding proposition — let arc AB be greater than arc AC, and from pole A, at the interval AC, let the arc of a circle CD be described, so that arcs AC, AD are equal by the definition of a pole, and thus arc BD is the excess, or difference, of the arcs AB, AC. Now let it be arranged that, as the whole sine is to the sine of arc AB, so the sine of arc AC is to another quantity — which is to be called the fourth proportional — as you see here:

Diagram. The same spherical-triangle construction reprinted for this scholium: A (top), B (lower left), C (lower right); D on side AB (nearer B), with arc DC drawn across the interior.

Whole sine.	Sine of arc AB.	Sine of arc AC.	Fourth proportional quantity.

I say that the whole sine is to the fourth-proportional quantity (proportional to the whole sine and the two sines of the unequal arcs) as the versed sine of angle A is to the difference between the versed sine of arc BC and the versed sine of arc BD, by which the arcs AB, AC differ from each other. For the ratio of the whole sine to the fourth-proportional quantity is compounded (taking the sine of arc AB as the mean) of the ratio of the whole sine to the sine of arc AB and the ratio of the sine of arc AB to the fourth-proportional quantity. And the ratio of the square of the whole sine to the rectangle under the right sines of arcs AB, AC [Margin: b — 23. sexti (Euclid, *Elements* VI.23)] is compounded of the same ratios — namely, of the ratio of the whole sine to the sine of arc AB, and of the ratio of the whole sine to the sine of arc AC, which is the same as the ratio of the sine of arc AB to the fourth-proportional quantity. (For since, as the whole sine is to the sine of arc AB, so the sine of arc AC is to the fourth-proportional quantity, then by alternation, as the whole sine is to the sine of arc AC, so the sine of arc AB is to the fourth-proportional quantity.) Therefore, as the whole sine is to the fourth-proportional quantity, so is the square of the whole sine to the rectangle contained by the sines of arcs AB, AC. [Margin: c — 58. huius [?]] (Proposition 58 of this treatise; the reference-word after the numeral is only partly legible)] Since, then, as the square of the whole sine is to the rectangle under the sines of arcs AB, AC, so the versed sine of angle A is to the difference of the versed sines of arcs BC, BD, it will follow also that, as the whole sine is to the fourth-proportional quantity, so the versed sine of angle A is to the difference between the versed sines of arcs BC, BD. Which is what was proposed.

THEOR. 57. PROPOS. 59.

If two spherical triangles have two angles equal to two angles, each to each: then the sines of the arcs about the remaining angle of the one will be proportional to the sines of the arcs about the remaining angle of the other, and the sines of the arcs subtending the equal angles will be corresponding [homologous]. And if one angle of the one is equal to one angle of the other, and the sines of the arcs about another angle of the one are proportional to the sines of the arcs about another angle of the other — in such a way that the sines of the arcs subtending the equal angles are corresponding — then the angles opposite the arcs of the remaining corresponding sines will also be equal to each other, or equal to two right angles.

In the two spherical triangles ABC, DEF let the two angles B, E be equal to each other, as well as the two angles C, F. I say that the sine of arc AB is to the sine of arc AC as the sine of arc DE is to the sine of arc DF. [Margin: a — 41. huius (Proposition 41 of this treatise)] For since the sine of arc AB is to the sine of angle C as the sine of arc AC is to the sine of angle B, then by alternation the sine of arc AB will be to the sine of arc AC as the sine of angle C is to the sine of angle B — that is, as the sine of angle F is to the sine of angle E, since these angles are assumed equal to those. [Margin: b — 41. huius (Proposition 41 of this treatise)] Likewise, since the sine of arc DE is to the sine of angle F as the sine of arc DF is to the sine of angle E, then by alternation the sine of arc DE will be to the sine of arc DF as the sine of angle F is to the sine of angle E. But it has been shown that the sine of arc AB is to the sine of arc AC also as the sine of angle F is to the sine of angle E. Therefore the sine of arc AB will be to the sine of arc AC as the sine of arc DE is to the sine of arc DF. Which is what was proposed.

Diagram. Two right triangles side by side. The first has vertices A (top), B (lower left), C (lower right), with a right-angle mark and a small arc marking the angle at the base. The second, congruent in form, has vertices D (top), E (lower left), F (lower right), similarly marked. The two figures illustrate the correspondence of the equal angles at B and E.

But now let the angles B, E be equal, and let the sine of arc AB be to the sine of arc AC as the sine of arc DE is to the sine of arc DF. I say that the angles C, F are also equal, or at least equal to two right angles. [Margin: c — 41. huius (Proposition 41 of this treatise); *Et permutando* — and by alternation] For we shall show, as before, that the sine of arc AB is to the sine of arc AC as the sine of angle C is to the sine of angle B; likewise that the sine of arc DE is to the sine of arc DF as the

sine of angle F is to the sine of angle E. Therefore, since it is assumed that the sine of arc AB is to the sine of arc AC as the sine of arc DE is to the sine of arc DF, it will follow that the sine of angle C is to the sine of angle B as the sine of angle F is to the sine of angle E; and by inversion, the sine of angle B is to the sine of angle C as the sine of angle E is to the sine of angle F. Since, then, the sines of the equal angles B, E are equal, [Margin: d — 14. quinti (Euclid, *Elements* V.14)] the sines of the angles C, F will also be equal; and hence the angles C, F will either be equal to each other, or equal to two right angles. Which is what was proposed. And so, if two spherical triangles have two angles [equal to two angles], etc. Which was to be demonstrated.

SCHOLIVM

But if the angles B, E, or C, F, were not equal, but only equal to two right angles, the truth of the theorem would still be preserved: because the angles B, E always have equal right sines, whether they are equal to each other or equal to two right angles. The same must be said of the angles C, F. This can be clearly established from what we have set out in the treatment of sines.

THEOR. 58. PROPOS. 60.

If from an angle of a spherical triangle an arc of a great circle is dropped to the base, dividing the angle into two equal parts: then the sines of the segments of the base will have the same ratio as the sines of the remaining two arcs [sides]. And if the sines of the segments of the base have the same ratio as the sines of the remaining two arcs, then the dropped arc will divide the angle into two equal parts.

In the spherical triangle ABC, let arc AD bisect angle A. I say that the sine of arc AB is to the sine of arc AC as the sine of arc BD is to the sine of arc DC. [Margin: a — 19. huius, & eius scholium (Proposition 19 of this treatise, and its scholium)] For since the triangles ABD, ACD have their angles at A equal, and their angles at D equal to two right angles, it will follow that the sine of arc AB is to the sine of arc BD as the sine of arc AC is to the sine of arc CD; and by alternation, the sine of arc AB is to the sine of arc AC as the sine of arc BD is to the sine of arc DC. Which is what was proposed.

Diagram. A spherical triangle with vertices A (top), B (lower left), C (lower right), and D on the base BC between B and C. The arc AD is drawn from the apex A down to D, dividing angle A; it illustrates the bisector dropped to the base.

But now let the sine of arc AB be to the sine of arc AC as the sine of arc BD is to the sine of arc DC. I say that angle A is bisected. For by alternation it will also follow that the sine of arc AB is to the sine of arc BD as the sine of arc AC is to the sine of arc CD. The triangles ABD, ACD therefore have their angles at D equal to two right angles, and the sines of the arcs about the angles B, C proportional, and the sines of the arcs opposite the angles at D are corresponding. [Margin: b — 59. huius, & eius scholium (Proposition 59 of this treatise, and its scholium)] Therefore the angles at A will either be equal to each other, or equal to two right angles. But they cannot be equal to two right angles, since angle A is less than two right angles. Therefore they will be equal to each other. Which is what was proposed. If, then, from an angle of a spherical triangle to the base [an arc is dropped], etc. Which was to be shown.

THEOR. 59. PROPOS. 61.

If from an angle of a spherical triangle a perpendicular arc is dropped to the base — or even to the base produced — then the sines of the angles that the perpendicular arc makes with the two arcs enclosing the said angle will stand in the same ratio as the sines of the complements of the remaining two angles of the triangle.

In triangle ABC let a perpendicular arc AD be drawn from angle A to the base BC, falling either inside the triangle or outside it; and let neither of the angles B, C, nor either of the angles BAD, DAC formed at A, be a right angle. (For whenever one of these is a right angle, the proposition will be demonstrated in the scholium; and likewise, whenever both B and C, or one of them, is a right angle, the same scholium will state what follows.) I claim that the sine of angle BAD is to the sine of angle DAC as the sine of the complement of angle B is to the sine of the complement of angle C.

For in triangle ABD, whose angle D is right, [Margin: a. Prop. 42 of this treatise (42. huius).] the sine of angle BAD will be to the whole sine [radius] as the sine of the complement of angle B is to the sine of the complement of arc AD. Likewise in triangle CAD, which has angle D right, the sine of angle DAC will be to the whole sine as the sine of the complement of angle C (for the two angles at C in the second triangle have the same sine) is to the sine of the complement of arc AD. Then, by inversion, the whole sine is to the sine of angle DAC as the sine of the complement of arc AD is to the sine of the complement of angle C. Therefore, by equal proportion (*ex aequalitate*) — as you see in the accompanying formula — the sine of angle BAD is to the sine of angle DAC as the sine of the complement of angle B is to the sine of the complement of angle C.

[Proportion formula, as printed:]

(Left terms)	(Right terms)
Sin. ang. BAD. (sine of angle BAD)	Sin. compl. ang. B. (sine of the complement of angle B)
Sinus totus. (whole sine / radius)	Sin. compl. arcus AD. (sine of the complement of arc AD)
Sin. ang. DAC. (sine of angle DAC)	Sin. compl. ang. C. (sine of the complement of angle C)

So then, "if from an angle of a spherical triangle to the base...", etc. — which was to be demonstrated.

Diagram. Two triangles set side by side, each drawn with an apex A from which a perpendicular AD descends to the base. In the left figure the perpendicular AD falls inside the triangle; in the right figure it falls outside. Read on the leaf:

- A (apex, left figure): vertex from which AD is dropped
- B (base, far left of left figure): left base vertex
- D (base, center of left figure): foot of the perpendicular, lying between B and C (perpendicular falls inside)
- C (base, right of left figure): right base vertex
- A (apex, right figure): vertex from which AD is dropped
- B (base, far left of right figure): left base vertex
- C (base, center of right figure): right base vertex of the triangle
- D (base, far right of right figure): foot of the perpendicular, lying beyond C on the base produced (perpendicular falls outside)

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If both angles B and C are right, the proposition will not be true unless the perpendicular arc AD bisects the base BC. [Margin: b. Prop. 7 of this treatise (7. *huius*).] For then the angles at A will also be equal, and so their sines will be in a ratio of equality — the very ratio that the sines of the complements of the right angles B, C also have, since these complements are Nothing [zero].

But if only one of the angles B, C is right, the proposition will in no way be true. For since the complement of a right angle is Nothing, the sines of the complements of B and C cannot have the same ratio as the sines of the angles

formed at A: that former ratio would be Nothing to something, or something to Nothing, whereas this latter is something to something.

Now if it should happen that one of the angles formed at A is right, the proposition must be demonstrated in the following way. First let angle BAD be right. [Margin: c. Prop. 25 of this treatise (25. *huius*).] Then AB and DB are quadrants, and therefore AD is the arc measuring angle B. Hence the sine of the right angle BAD will be to the whole sine as the sine of the complement of angle B is to the sine of the complement of arc AD, since in both cases the ratio is one of equality — for the complement of angle B and the complement of arc AD are the same, seeing that AD is the arc measuring angle B. Again, since the sine of angle DAC (which is always less than the right angle BAD, because the whole angle BAC is less than two right angles, as in the earlier figure; while in the later figure DAC is itself part of the right angle BAD) is to the whole sine as the sine of the complement of angle C is to the sine of the complement of arc AD, [Margin: d. Prop. 42 of this treatise (42. *huius*).] by inversion the whole sine will be to the sine of angle DAC as the sine of the complement of arc AD is to the sine of the complement of angle C. Therefore, as before, it will follow by equal proportion that the sine of the right angle BAD is to the sine of angle DAC as the sine of the complement of angle B is to the sine of the complement of angle C — which was proposed.

Next let angle DAC be right. [Margin: e. Prop. 25 of this treatise (25. *huius*).] Then AC and DC are quadrants, and therefore AD is the arc measuring angle C. And since the sine of angle BAD (which in the first figure is less than a right angle, but in the second is greater) is to the whole sine as the sine of the complement of angle B is to the sine of the complement of arc AD; and likewise the whole sine is to the sine of the right angle DAC as the sine of the complement of arc AD is to the sine of the complement of angle C — the ratio being one of equality in both cases, for the complement of arc AD and of angle C are the same, since AD is the arc measuring angle C — [Margin: f. Prop. 42 of this treatise (42. *huius*).] it will follow by equal proportion, as before, that the sine of angle BAD is to the sine of the right angle DAC as the sine of the complement of angle B is to the sine of the complement of angle C. Which was proposed.

Further, it must be understood here that when neither of the angles B, C, and likewise neither of the angles BAD, DAC formed at A, is right — as is assumed in the demonstration of the proposition — no arc in the triangles ABD, ACD is a quadrant. For if one were a quadrant, then since angle D is right, there would be at least two quadrants, by the Corollary of Prop. 38 of this treatise; [Margin: g.

Prop. 25 of this treatise (25. *huius*.)] and consequently at least two angles would be right, which is not assumed. By the same reasoning, when one of the angles at A is right, no arc in the triangle whose angle A is less than a right angle is a quadrant, as is taken in the demonstration of the scholium. For then neither of the angles B, C can be right, because — the angles at A being unequal — the proposition would have no place, as has been shown in this scholium.

PROBL. 3. PROP. 62.

Given all the angles of a non-right spherical triangle, to make all three arcs [sides] known.

In the non-right spherical triangle ABC let all the angles A, B, C be given; and first let all three angles be unequal. [Margin: When all three given angles are unequal.] From these we must determine its three arcs. Since no angle is assumed right, at least two of them are either acute or obtuse: let B and C be either both acute or both obtuse, whatever the case with the remaining angle A, from which to the arc BC

let the perpendicular arc AD be drawn, [Margin: a. Prop. 57 of this treatise (57. *huius*.)] which will necessarily fall inside the triangle. [Margin: b. Prop. 61 of this treatise (61. *huius*.)] And since the sine of angle BAD is to the sine of angle DAC as the sine of the complement of angle B is to the sine of the complement of angle C, and this latter ratio is given in the sines of the complements of the given angles B, C, the ratio of the sine of angle BAD to the sine of angle DAC will likewise be given, namely in the sines of the complements of angles B, C. But the sum of the same two angles BAD, DAC is also given, and is less than a semicircle — namely the whole angle BAC, which is less than two right angles. [Margin: c. Prop. 6 of the [treatise on] rectilinear triangles (6. *triang. rectilinei*.)] Therefore each of the angles BAD, DAC will be known separately. Since, then, in triangle ABD, whose angle D is right, two non-right angles B and BAD are given, [Margin: d. Scholium of Prop. 50 of this treatise (*Schol. 50. huius*.)] the arc AB opposite the right angle will also be given. And from this — since in the same triangle ABD, which has the right angle D, the arc AB opposite the right angle is known, together with the non-right angle BAD:

Or else, since two non-right angles B and BAD are given,

the arc BD about the right angle, opposite angle BAD, will also become known, from the scholia cited in the margin. [Margin: e. Scholium of Prop. 41 of this treatise (*Schol. 41. huius*); Scholium of Prop. 42 or of Prop. 52 (*Schol. 42 vel 52. hu.*.)] By the same reasoning, since in triangle ACD, whose angle D is right, two

non-right angles C and CAD are given, [Margin: f. Scholium of Prop. 50 of this treatise (*Schol. 50. huius*).] the arc AC opposite the right angle will also be given. And from this — since in the same triangle ACD, which has the right angle D, the arc AC opposite the right angle is now known, together with the non-right angle CAD:

Or else, since two non-right angles C and CAD are given,

the arc CD about the right angle, opposite angle CAD, will likewise be known, from the same scholia adduced in the margin. [Margin: g. Scholium of Prop. 41 of this treatise (*Schol. 41. huius*); Scholium of Prop. 42 or of Prop. 52 (*Schol. 42 vel 52. hu.*).] And so now the two arcs AB, AC are known; while the sum of the two arcs BD, CD found in this way will also make the third arc BC known.

But if it should happen that one of the angles at A, namely BAD, is found to be right, then since D is also right, [Margin: h. Prop. 25 of this treatise (*25. hui.*).] both arcs AB, BD will be quadrants, and so the said arcs will be found without any trouble. By the same reasoning, if angle CAD is found to be right — but not BAD (for it cannot happen that both angles at A are right, since angle BAD is less than two right angles) — the arcs AC, CD will be quadrants, and hence known, without further labor.

Diagram. A single spherical triangle with curved sides. Read on the leaf:

- A (apex): vertex from which the perpendicular is dropped
- B (base, left): left base vertex
- D (base, center): foot of the perpendicular arc AD on the base
- C (base, right): right base vertex

The practical procedure (*praxis*) for this problem need not be explained here at greater length, since it is to be sought from Prop. 6 of the [treatise on] rectilinear triangles and from the scholia written in the margin. [Margin: Praxis, when all three given angles are unequal.] For if two sines of the complements of the acute or obtuse angles B, C are set down as the terms of the ratio of the sine of angle BAD to the sine of angle CAD, we shall find each of the angles BAD, CAD by the first or second procedure of Prop. 6 of the rectilinear triangles, because these are more expeditious than the third. [Margin: Prop. 6 of the [treatise on] rectilinear triangles is also to be understood of spherical angles.] For although that Prop. 6 is stated only of rectilinear arcs and angles, it is nevertheless to be understood of spherical angles as well, since their sines do not differ from the sines of the arcs of those same angles. Once each angle BAD, CAD has been found, the procedure of the problem of the scholium of Prop. 50 of this treatise must be applied, in

order that both the arc AB (opposite the right angle D in triangle ABD) and the arc AC (opposite the right angle D in triangle ACD) may be found. Finally, the procedure of Problem 2 of the scholium of Prop. 41, or of Problem 1 of the scholium of Prop. 42, or indeed the procedure of the scholium of Prop. 52, must be brought in, to elicit both the arc BD (opposite the non-right angle BAD in triangle ABD) and the arc CD (opposite the non-right angle CAD in triangle ACD).

[Margin: Praxis by sines alone, when all three given angles are unequal.] But if in unraveling this problem one wishes to use sines alone, each of the angles BAD, CAD must be found by the third procedure of Prop. 6 of the rectilinear triangles, and not by the first or second. Then, from the procedure of Problem 1 of the scholium of Prop. 42 of this treatise, both the arc BD (opposite the non-right angle BAD in triangle ABD) and the arc CD (opposite the non-right angle CAD in triangle ACD) are to be elicited. Lastly, by the procedure of Problem 3 of the scholium of Prop. 41, both the arc AB and the arc AC (each opposite the right angle D in its own triangle) are to be investigated. For besides the arc BD already found and its opposite angle BAD, and besides the arc CD already found and its opposite angle CAD, the kind [species — acute or obtuse] of each of the angles B and C is also established, since each of them is given.

This problem becomes far easier when all three given angles, or at least two, are equal. [Margin: When all three given angles, or at least two, are equal. i. Prop. 9 of this treatise (9. *huius*). k. Prop. 57 (57. *huius*). l. Prop. 21 (21. *huius*).] For if, say, two angles B, C are equal — whatever the case with the remaining angle A — the arcs AB, AC will also be equal. And since the triangle ABC is assumed non-right, each of the equal angles B, C will be either acute or obtuse; hence the perpendicular arc AD, dropped from the third angle A to the arc BC, will fall inside the triangle. Now since the triangles ABD, ACD have their angles at D right and their non-right angles B, C equal, and also their arcs AB, AC (opposite the right angles) equal, as we have shown, the arcs BD, CD and the angles BAD, CAD will be equal too; and consequently each angle BAD, CAD, being half of the given angle BAC, will be known. After this, since in triangle ABD, which has the right angle D, both non-right angles B and BAD are given, [Margin: m. Scholium of Prop. 50 of this treatise (*Schol. 50 huius*).] the arc AB opposite the right angle will also be given, and therefore AC, equal to it, will be known. And so now the two arcs AB, AC have been made known. Again, since in the same triangle ABD two non-right angles B and BAD are given:

Diagram. A single (isosceles) spherical triangle with curved sides, illustrating the case of two equal angles. Read on the leaf:

- A (apex): vertex from which the perpendicular is dropped
- B (base, left): left base vertex
- D (base, center): foot of the perpendicular arc AD, at the midpoint of the base
- C (base, right): right base vertex

Or, since the arc AB opposite the right angle is given, together with the non-right angle B:

Or finally, since the arc AB is given, opposite the right angle, together with the non-right angle BAD;

the arc BD about the right angle will also be known; [Margin: n. Scholium of Prop. 42 or of Prop. 52 (*Schol. 42 vel 52. hui.*); Scholium of Prop. 45 of this treatise (*Schol. 45 huius*); Scholium of Prop. 41 of this treatise (*Schol. 41. huius*).] and when this is doubled, it will exhibit the whole third arc BC as known. Therefore all three arcs sought have been made known.

The practical procedure for this matter is not obscure, for it depends on the scholia cited in the margin. [Margin: Praxis by sines alone, when all three given angles, or at least two, are equal.] But if one wishes to use sines alone, then in triangle ABD — in which angle B is given, and angle BAD, namely half of the given angle BAC — the arc BD must be sought by Problem 1 of the scholium of Prop. 42 of this treatise; and this arc BD, doubled, will give the whole arc BC. Then, by Problem 3 of the scholium of Prop. 41, in the same triangle, in which the arc BD and its opposite angle BAD have been found, and in which the kind of the other non-right angle B is established as given, the arc AB opposite the right angle must be elicited; and once this is found, its equal AC will also be found.

Therefore, given all the angles of a non-right spherical triangle, we have made all three arcs known. Which was to be done.

SCHOLIVM.

A non-right spherical triangle therefore differs, as you see, from a non-right rectilinear one: in the spherical case all the arcs are found from the given angles alone, as has been shown in this problem; but in the rectilinear case the sides cannot be known from the given angles alone, unless at least one side is also given. The reason for this is that two rectilinear triangles, though similar, may have the sides of one very unequal to the sides of the other, side for side, and yet have their angles equal, angle for angle — so that two rectilinear triangles may be given which are indeed equiangular with each other, but not equilateral.

[Margin: o. Prop. 19 of this treatise (19. *huius*.)] But two spherical triangles cannot be equiangular with each other without also being equilateral. From this it comes about that in a spherical triangle the arcs are also given from the given angles, since determinate arcs correspond to the angles; whereas in a rectilinear triangle the sides cannot be given from the given angles, since determinate sides do not correspond to the angles, but sides greater or smaller may be subtended under the same angles.

PROBL. 4. PROP. 63.

Given all the arcs [sides] of a non-right spherical triangle, to find all three of its angles.

In the non-right spherical triangle ABC let all three arcs be given. From these we must find all three angles. Let angle A be sought in the first place — for we do not always need all the angles; rather, very often one or another is to be sought from the given arcs. Now the two arcs AB, AC enclosing angle A (which is being sought) are either unequal or equal. If unequal, then either both are less than a quadrant, or both greater, or one greater and the other less, or one a quadrant and the other less than a quadrant, or finally one a quadrant and the other greater than a quadrant. For both cannot be quadrants, [Margin: a. Prop. 25 of this treatise (25. *huius*.)] When the two arcs enclosing the angle to be found in the first place are unequal.] since then the two angles opposite them would be right — which would be absurd, the triangle being assumed non-right.

First let the two arcs AB, AC be unequal and less than a quadrant, whatever the case with arc BC. With the arcs AB, AC produced so as to become the quadrants AD, AE, let the arc of a great circle DE be described through D, E, meeting the arc BC — produced in either direction — at F. (I would urge, however, that it be produced toward the greater arc, which here is AC.) The angles D, E will be right, because AD, AE are quadrants. Since, then, the two great circles BF, DF intersect at F, and from the points B, C perpendicular arcs BD, CE have been dropped from arc BF to arc DF, [Margin: b. Prop. 40 of this treatise (40. *huius*.)] the sine of arc BF will be to the sine of arc BD as the sine of arc CF is to the sine of arc CE; and by alternation, the sine of arc BF will be to the sine of arc CF as the sine of arc BD is to the sine of arc CE. But the ratio of the sine of arc BD to the sine of arc CE is given, since the arcs BD, CE are given, being the complements of the given arcs AB, AC. Therefore the ratio of the sine of arc BF to the sine of arc CF will also be given, namely in the sines of the complements of the given arcs AB, AC. But the difference of these same arcs BF, CF — each of which is less than a semicircle — is also given, [Margin: c. Prop. 2 of this treatise (2. *huius*.)] namely the arc BC.

[Margin: d. [treatise on] rectilinear triangles (*triang. rectilinea*).] Therefore each arc BF, CF will be made known. And so, since in triangle BFD, which has angle D right, the arc BF opposite the right angle is given, together with the arc BD (namely the complement of the given arc AB), [Margin: e. Scholium of Prop. 53 or of Prop. 43 of this treatise (*Schol. 53 vel 43. hu.*).] the third arc DF will also be known. By the same reasoning, since in triangle CFE, which has angle E right, the arc CF opposite the right angle is given, together with the arc CE (namely the complement of the given arc AC), [Margin: f. Scholium of Prop. 53 or of Prop. 43 of this treatise (*Schol. 53 vel 43. hu.*).] the third arc EF will also be known; and this, subtracted from the arc DF already found, will make known the remaining arc DE, which measures angle A. And so angle A will be known. Again, in the earlier triangle BFD, whose angle D is right, since the arc BF opposite the right angle is given, together with the arc BD (namely the complement of the given arc AB):

Or, since the two arcs BD, DF about the right angle are given:

Or finally, since the arc BF opposite the right angle is given, together with the arc DF;

Diagram. The construction for Problem 4: a spherical triangle ABC whose sides AB, AC are produced to the quadrant endpoints D, E, with the great-circle arc DE meeting BC produced at F. Read on the leaf:

- A (top): apex of the triangle
- B (left): base vertex; arc AB is produced through B down to D
- C (center): base vertex; the base arc runs from B through C on to F
- F (right): point where arc DE (produced) meets arc BC produced
- D (lower left): quadrant endpoint on arc AB produced; foot of perpendicular BD (angle D right)
- E (lower center): quadrant endpoint on arc AC produced; foot of perpendicular CE (angle E right)

Continuation of the case-by-case solution of oblique spherical triangles (given the sides, to find the angles), from the commentary on the problem begun in the preceding pages (PROBL. 4 / PROPOS. 63). The oblique triangle is split by a dropped perpendicular (or by quadrant-extension) into two right-angled sub-triangles, each solved by the rules for right spherical triangles; the alternatives keyed by "OR" / "OR finally" (VEL / AVT) enumerate the several given-and-unknown combinations. The superscript letters key the marginal citations to Clavius's own propositions and scholia ("of this treatise") and to Theodosius.

[the angle DBF] will also be found, from the scholia cited in the margin; and hence the remaining angle ABC of the two right triangles will be known. By the same reasoning, since in the latter triangle CFE, which has the right angle E, the arc CF (the one opposite the right angle) is given together with arc CE — namely the complement of the given arc AC:

[Margin: f: Scholium of prop. 51, or prop. 45, of this treatise; Scholium of prop. 44, or prop. 48, of this treatise; Scholium of prop. 55, or prop. 41, of this treatise.]

OR, when the two arcs CE, EF about the right angle are given:

OR finally, when arc CF (opposite the right angle) is given, together with arc EF as well;

the angle ECF will also become known, from the scholia set in the margin; and therefore the angle ACB — which is equal to it as a vertical angle — will be known. So all three angles of triangle ABC have now been found.

[Margin: g: Scholium of prop. 41, or prop. 45, of this treatise; Scholium of prop. 44, or prop. 48, of this treatise; Scholium of prop. 55, or prop. 41, of this treatise.]

[Margin: h: prop. 6 of this treatise.]

Next, let the two unequal arcs AB, AC be greater than a quadrant. Let them be produced until they meet at D. In triangle DBC the two arcs DB, DC will be less than a quadrant, and hence known, since they are the remainders left out of the arcs ABD, ACD — which are semicircles. Therefore, as we demonstrated just above, all three of its angles will become known; and consequently the remaining angles of the two right triangles, ABC and ACB, as well as angle A, since it is equal to angle D.

[Margin: i: Theodosius, *Sphaerica* I.11.] [Margin: k: prop. 13 of this treatise.]

Diagram. A pointed-oval (vesica) figure formed by two great-circle arcs. The two arcs spring from the apex and bow apart, then meet again at the lower vertex; a chord/arc crosses between the two side-points. It illustrates the case in which the two unequal arcs AB, AC (each greater than a quadrant) are produced until they meet at D.

- A (top vertex): common starting point of the two arcs AB, AC
- B (left): endpoint on the left arc
- C (right): endpoint on the right arc
- D (bottom vertex): point where the produced arcs meet
- (chord BC crosses between B and C, the base of triangle ABC)

Thirdly, let arc AB be less than a quadrant, but AC greater. Let arc AB be produced so as to make the quadrant AD; and, cutting off the quadrant AE from AC — as in the first of these figures — let the great-circle arc DE be drawn through D and E, cutting arc BC at F. The angles D and E will be right, because AD, AE are quadrants. Now since the two great circles BC, DE cut each other at F, and from the points B, C the perpendicular arcs BD, CE have been drawn to arc DE, it will be that as the sine of arc BF is to the sine of arc BD, so the sine of arc CF is to the sine of arc CE. And, alternately, as the sine of arc BF is to the sine of arc CF, so the sine of arc BD is to the sine of arc CE. But the ratio of the sine of arc BD to the sine of arc CE is known, because arcs BD, CE are given, being the complements of the given arcs AB, AC. Therefore the ratio of the sine of arc BF to the sine of arc CF will also be known — inasmuch as these are the sines of the complements of the given arcs AB, AC. Moreover the sum of these same arcs BF, CF is given (namely the whole arc BC), and is less than a semicircle — because any side of a spherical triangle is less than a semicircle. Therefore each of the arcs BF, CF will be known. Since, then, in triangle BFD, whose angle D is right, arc BF (opposite the right angle) is given, together with arc

[Margin: l: prop. 25 of this treatise.] [Margin: m: prop. 40 of this treatise.] [Margin: n: prop. 2 of this treatise.]

Diagram. The first of a pair of spherical-triangle figures (the left-hand one). A spherical triangle with apex A; the left side runs down to B and is produced to D at the lower left, the right side runs down to C; the great-circle arc DE (running along the base) cuts arc BC at F, with the perpendicular arcs BD and CE dropped from B and C. Illustrates the third case (AB less than, AC greater than a quadrant).

- A (apex): vertex of the triangle
- B (upper-left, on side AB): foot from which perpendicular BD is drawn
- D (lower left): end of the produced quadrant AD
- F (bottom, between D and C): intersection of arc BC with arc DE
- E (center-right): foot of perpendicular CE / end of quadrant AE
- C (bottom-right): vertex of the triangle

Diagram. The latter of the pair (the right-hand figure). A spherical triangle with apex A; a perpendicular/quadrant construction is drawn to the base, with points B, C at the base and D, E from the quadrant-extension. Serves the fourth and fifth cases (AB or AC a quadrant).

- A (apex): vertex of the triangle
- E (center-left): point of the quadrant/perpendicular construction [?]

- C (right): vertex of the triangle
- B (bottom-center): vertex of the triangle
- D (bottom-right): end of the produced quadrant

arc BD, which is the complement of the given arc AB, the third arc DF will also become known. In like manner, since in triangle CFE, which has the right angle E, arc CF (opposite the right angle) is given together with arc CE — namely the complement of arc AC — the third arc EF will likewise be found. Added to the arc DF already found, it will make known the whole arc DE, which is the measure of angle A; and therefore angle A will be known. Again, in the former triangle BFD, whose angle D is right, since arc BF (opposite the right angle) is given, together with arc BD — the complement, namely, of the given arc AB:

[Margin: o: prop. 6 of the [book on] rectilinear triangles.] [Margin: p: Scholium of prop. 53, or prop. 43, of this treatise.]

OR because the two arcs BD, DF about the right angle are given:

OR indeed, because arc BF (opposite the right angle) is given together with arc DF; the angle DBF will also be made known, from the scholia adduced in the margin; and hence the remaining angle ABC of the two right triangles will be known. By like reasoning, since in the latter triangle CFE, whose angle E is right, arc CF (opposite the right angle) is given together with arc CE — the complement, that is, of the given arc AC;

[Margin: q: Scholium of prop. 53, or prop. 43, of this treatise.]

OR when the two arcs CE, EF about the right angle are given:

OR indeed, when arc CF (opposite the right angle) is given together with arc EF; the angle C will likewise be given, by the scholia described in the margin. And thus all three angles A, B, C have been found.

[Margin: r: Scholium of prop. 51, or prop. 45, of this treatise; Scholium of prop. 44, or prop. 48, of this treatise; Scholium of prop. 55, or prop. 41, of this treatise.]

Fourthly, let arc AB be a quadrant, and AC less than a quadrant, as in the latter of the figures nearby. Let arc AC be produced so as to make the quadrant AD, and let the great-circle arc BD be drawn through B and D. The angles ABD and D will be right, because AB, AD are quadrants. And since in triangle BCD, whose angle D is right, arc BC (opposite the right angle) is given, together with arc CD — inasmuch as it is the complement of the given arc AC — the third arc BD, the measure of

angle A, will also be given, and therefore angle A will be known. Then, since in the same triangle BCD, having the right angle D, arc BC (opposite the right angle) is given together with arc CD — the complement, that is, of the given arc AC:

[Margin: s: Scholium of prop. 51, or prop. 45, of this treatise; Scholium of prop. 44, or prop. 48, of this treatise; Scholium of prop. 55, or prop. 41, of this treatise.]

[Margin: t: prop. 25 of this treatise.] [Margin: u: Scholium of prop. 53, or prop. 43, of this treatise.]

OR, because the two arcs BD, CD about the right angle are given:

OR indeed, since arc BC (opposite the right angle) is given together with arc BD;

the angle BCD will also be found from the scholia noted in the margin; and hence the remaining angle ACB of the two right triangles will be known. Lastly, since in the same adjoining triangle BCD, which has the right angle D, arc BC (opposite the right angle) is given, and besides arc CD — the complement, that is, of the given arc AC:

[Margin: x: Scholium of prop. 51, or prop. 45, of this treatise; Scholium of prop. 44, or prop. 48, of this treatise; Scholium of prop. 55, or prop. 41, of this treatise.]

OR when the two arcs BD, CD about the right angle are given:

OR when arc BC (opposite the right angle) is given together with arc BD;

OR when the angle BCD is given, together with arc CD or BD; for when arc BD is given, it is settled concerning the other arc CD about the right angle — given as it is — whether it is greater than a quadrant or less:

OR finally, because arc BC (opposite the right angle) is given together with angle BCD;

the angle CBD will also become known, from the scholia cited in the margin; and hence its complement too — namely the angle ABC — will be recognized. So all three angles of triangle ABC are known.

[Margin: d: Scholium of prop. 44, or prop. 48, of this treatise; Scholium of prop. 51, or prop. 45, of this treatise; Scholium of prop. 47 of this treatise; Scholium of prop. 47 of this treatise.] [Margin: e: prop. 25 of this treatise.]

Fifthly and lastly, let arc AC be a quadrant, and AB greater, as in the same latter of the figures nearby. Cutting off the quadrant AE from AB, let the great-circle arc CE be drawn through C and E. The angles ACE and E will be right, because AC, AE are quadrants. Since, then, in triangle BCE, which has the right angle E, arc BC

(opposite the right angle) is given, and besides arc BE — namely the complement of the given arc AB — the third arc CE, the measure of angle A, will also be given; and consequently angle A will become known. Again, since in the same triangle BCE, whose angle E is right, arc BC (opposite the right angle) is given together with arc BE — the complement, namely, of the given arc AB:

[Margin: f: Scholium of prop. 53, or prop. 43, of this treatise.]

OR when the two arcs BE, CE about the right angle are given.

OR finally, when arc BC (opposite the right angle) is given together with arc CE;

the angle CBE will likewise be given, from the scholia adduced in the margin.

Finally, since in the same triangle BCE, which has the right angle E, arc BC (opposite the right angle) is given, and arc BE too — being the complement of the given arc AB:

[Margin: g: Scholium of prop. 51, or prop. 45, of this treatise; Scholium of prop. 44, or prop. 48, of this treatise; Scholium of prop. 55, or prop. 41, of this treatise.]

OR, because the two arcs BE, CE about the right angle are given:

OR, since arc BC (opposite the right angle) is known, together with arc CE:

OR, because the angle CBE is given, together with arc BE or CE; for when arc CE is given, it is also settled whether the other arc BE — given as it is about the right angle — is greater than a quadrant or less:

OR finally, because arc BC (opposite the right angle) is known together with angle CBE;

the angle BCE will also become known, from the scholia cited in the margin; and therefore, adding the right angle ACE, the whole angle ACB will be given. So once again all three angles of triangle ABC have been found.

[Margin: h: Scholium of prop. 55, or prop. 41, of this treatise; Scholium of prop. 44, or prop. 48, of this treatise; Scholium of prop. 51, or prop. 45, of this treatise; Scholium of prop. 42 of this treatise; Scholium of prop. 47 of this treatise.]

PRAXIS. The practical working of this problem is to be sought from the scholia cited in the margin. Only, in order that the arcs BF, CF may be found — in the first and third cases — the sines of the complements of the given arcs AB, AC must be set down as the terms of the proportion of the sine of arc BF to the sine of arc CF; and in the first case one should apply either the first working of proposition 7 of rectilinear triangles, or some other of the workings of the same proposition, as

the matter requires; while in the third case the first or second working of proposition 6 of rectilinear triangles must be brought in, and so forth.

[Margin: Working, when the two arcs containing the sought angle are unequal.]

But if one wishes to use sines alone, the arcs BF, CF must be investigated, in the first case, by the second working of proposition 7 of rectilinear triangles; but in the third case by the third working of proposition 6. Then, in triangle BFD, the arc DF is to be extracted by the working of the first scholium of proposition 43; and in the same way, in triangle CFE, the arc EF — so that the remaining arc DE (in the first case), or the whole arc DE (in the third case), may be obtained, which is indeed the arc measuring angle A. After this, by the working of problem 1 of the scholium of proposition 41, the angle DBF must be found in triangle BFD; from which the remaining angle ABC of the two right triangles will become known. And in the same manner the angle ECF must be elicited in triangle CFE, from which — in the first case — the angle ACB too, its vertical angle, will be known.

[Margin: Working, by sines alone, when the two arcs flanking the sought angle are unequal.]

But in the fourth case, by the working of the first scholium of proposition 43, the arc BD — the measure of angle A — must be found in triangle BCD; and in the same way, in the fifth case, the arc CE, the measure of the same angle A, in triangle BCE. Then, in the fourth case, by the working of problem 1 of the scholium of proposition 41, the angle BCD must be traced out in triangle BCD; from which the remaining angle ACB of the two right triangles will become known. And by the same reasoning, in the fifth case, the angle ECB must be found in triangle BCE. Finally, in the fourth case, by the working of problem 2 of proposition 42, the angle CBD must be sought out in triangle BCD; from which also ABC, the remaining part of the right angle ABD, will be known. And likewise, in the fifth case, the angle BCE must be elicited; which, added to the right angle ACE, will present the whole angle ACB as known.

ANOTHER WAY, and much briefer. Again let the three arcs of triangle ABC be given, with the arcs AB, AC — which contain the angle A that is being sought — unequal. Since, then, as the whole sine [radius] is to the fourth-proportional quantity to the whole sine and the two sines of the unequal arcs AB, AC, so the versed sine of angle A is to the difference between the versed sine of arc BC (opposite angle A) and the versed sine of the arc by which the unequal arcs AB, AC exceed one another: and, conversely, as the said fourth-proportional quantity is to the whole sine, so the difference between the versed sine of arc BC and the

versed sine of the arc by which the unequal arcs AB, AC differ from each other is to the versed sine of the sought angle A.

[Margin: Another demonstration, briefer, and by sines alone.] [Margin: i: Scholium 2 of prop. 58 of this treatise.]

Diagram. A single spherical triangle with curved (great-circle) sides. Apex A at the top, with the base running between C (lower left) and B (lower right).

- A (top): apex vertex
- C (lower left): base vertex
- B (lower right): base vertex

If it be arranged that, as the whole sine is to the sine of either of the unequal arcs containing the sought angle, so the sine of the other arc about the same angle is to a fourth term, there will be found the number that is fourth proportional to the whole sine and the two sines of the said two arcs. If, then, it be again arranged that, as the fourth-proportional number just found is to the whole sine, so the difference between the versed sine of the arc opposite the sought angle and the versed sine of the arc by which the two arcs flanking the sought angle differ from each other is to a fourth term, the versed sine of the angle that is sought will be produced; from which you will elicit the arc of the sought angle, and hence the angle itself, as we taught in the explanation and use of the Table of Sines.

[Margin: Working, briefer, and by sines alone, when the two given arcs containing the sought angle are unequal.]

Moreover, the difference between the versed sine of the arc opposite the sought angle and the versed sine of the difference of the arcs containing that same angle will easily be found thus. When the arc opposite the sought angle is less than a quadrant, the sine of its complement must be subtracted from the sine of the complement of the difference of the arcs flanking the sought angle; for the remainder will be the difference that is sought. This is clearly established from the figures of cases 1, 4, 7, 10, and 13 of proposition 58. But when the arc opposite the sought angle is a quadrant, the sine of the complement of the difference of the arcs comprehending the sought angle will give the sought difference between the said versed sines; as is manifest from the figures of cases 2, 5, 8, 11, and 14 of the same proposition 58. Finally, when the arc opposite the sought angle is greater than a quadrant, the sine of its complement must be added to the sine of the complement of the difference of the arcs containing the sought angle; for the resulting number will be the sought difference — as can easily be seen from the figures of cases 3, 6, 9, 12, and 15 of the same proposition 58.

[Margin: Finding the difference between the versed sine of the arc opposite the sought angle and the versed sine of the difference of the arcs flanking that same angle.]

In the same way we shall investigate the angles B, C, if the arcs containing them are unequal.

Furthermore, once one angle has been found, the remaining two could be found with almost no trouble, if it were settled what each of them is — whether acute or obtuse. For, once (say) angle A has been found, if angle B were to be found, we would take for its sine the number that is fourth proportional to the sine of arc BC (opposite the found angle A), the sine of the found angle A, and the sine of arc AC (opposite the sought angle B). But if angle C were to be sought, we would take for its sine the number that is fourth proportional to the sine of arc BC (opposite the found angle A), the sine of the found angle A, and the sine of arc AB (opposite the sought angle C): because it is the case that, as the sine of arc BC is to the sine of angle A, so both the sine of arc AC is to the sine of angle B, and the sine of arc AB is to the sine of angle C. Therefore, if it were settled what both angle B and angle C are, we would at once find the sought angle in the table of sines from that fourth-proportional sine.

[Margin: a: prop. 41 of this treatise. When all three arcs, or only the two containing the angle to be investigated first, are equal.]

Now indeed, if all three arcs are given, or only the two AB, AC embracing angle A are equal — whatever be the case with the remaining arc BC — we shall investigate angle A and the other two B, C far more easily. Since the two arcs AB, AC are equal, the two angles B, C will also be equal to each other, because in isosceles spherical triangles the angles at the base are equal to one another. Since, then, triangle ABC is assumed to be non-right-angled, neither of the angles B, C will be right, and consequently neither of the arcs AB, AC is a quadrant — because otherwise the two angles B, C would be right. Therefore each of the angles B, C will be either acute or obtuse. So the perpendicular arc AD, dropped from A to arc BC, will fall inside the triangle. The two triangles ABD, ACD therefore have right angles at D, and the non-right angles B, C equal, and also the arcs AB, AC (opposite the right angles) equal. Hence the arcs BD, CD, and the angles at A, will also be equal to one another. And so, since in triangle ABD, whose angle D is right, arc AB (opposite the right angle) is given, and besides arc BD — inasmuch as it is half of the given arc BC —

[Margin: b: prop. 8 of this treatise.] [Margin: c: prop. 25 of this treatise.] [Margin: d: prop. 57 of this treatise.] [Margin: e: prop. 21 of this treatise.]

Diagram. An isosceles spherical triangle with a dropped perpendicular. Apex A at the top; the base runs from B (lower left) through D (the foot of the perpendicular AD, at the middle of the base) to C (lower right). The perpendicular arc AD divides the triangle into the two right-angled triangles ABD, ACD.

- A (apex): vertex from which the perpendicular is dropped
- B (lower left): base vertex
- D (base, center): foot of the perpendicular arc AD
- C (lower right): base vertex

the angle BAD — which is opposite the arc BD given about the right angle — will also be given; and this angle, doubled, will make the whole sought angle BAC known, since the angles at A have been shown to be equal. Again, since in the same triangle ABD, which has the right angle D, arc AB (opposite the right angle) is given, together with arc BD — namely with half of the given arc BC;

[Margin: f: Scholium of prop. 55, or prop. 41, of this treatise.]

OR, because the non-right angle BAD is given, together with its opposite arc BD about the right angle; and moreover the character of the remaining non-right angle B is settled. For if AB is less than a quadrant, angle B will be acute, just as BAD is acute; but if AB is greater than a quadrant, that same angle B will be obtuse, while BAD is acute:

OR indeed, since arc AB (opposite the right angle) is given, together with the non-right angle BAD;

[Margin: g: Scholium of prop. 51, or prop. 45, of this treatise; Scholium of prop. 52, or prop. 56, of this treatise; prop. 38 of this treatise.]

the angle B will also be given; and hence the remaining angle C, being equal to B, will be known. And thus all three angles in triangle ABC have been found.

[Margin: h: Scholium of prop. 47 of this treatise.]

When, therefore, two arcs are equal, the working of the scholium of proposition 55, or of problem 1 of proposition 41, must be applied, so that from one of the equal arcs and from half of the third arc an angle may be elicited which, doubled, presents the angle opposite the third arc. Then the working of the scholium of proposition 51 (or 45) must be applied, so that from the same arcs one of the

equal angles standing on the third arc may be found; or the working of problem 2 of the scholium of proposition 42, or of proposition 56, must be called upon; or at any rate the working of the scholium of proposition 47.

[Margin: Working, when the two arcs flanking the sought angle are equal.]

By sines alone we shall carry out the matter thus. From the working of problem 1 of proposition 41 we shall find the angle BAD, which, doubled, will give the whole angle BAC. Then, by the working of problem 2 of the scholium of proposition 42, we shall find the angle B, which is equal to C.

[Margin: Working, by sines alone, when the two arcs containing the sought angle are equal.]

SCHOLIUM.

Johannes Regiomontanus and Nicolaus Copernicus, in yet another way too — when all the arcs of a spherical triangle are given — [teach how] all three angles...

In the marginal references below, "of this treatise" (Latin *huius*) means a proposition or scholium of this treatise on spherical triangles; the numbers refer to its propositions and their scholia.

they seek the angles, investigating of course a certain rectilinear angle at the center of the sphere, whose arc exhibits the sought spherical angle as known. But I have deliberately omitted that method here — although it is acute and subtle — because it is more obscure and longer, and especially because anyone can read it in Regiomontanus [Regiomontanus], Book IV of the *Triangles*, Prop. 34, and in Copernicus [Copernicus], Book I of the *Revolutions* [*De Revolutionibus*], Prop. 13, on spherical triangles.

In the second demonstration of this problem I preferred to employ the theorem of scholium 2 of Prop. 58, rather than, with Johannes Regiomontanus, the theorem of the same Prop. 58, in order to escape the difficulty of the labor.

[Margin: (a) Prop. 58 of this treatise, and by permutation.] For it holds that, as the rectangle contained under the sines of the unequal arcs enclosing the sought angle is to the square of the whole sine [sinus totus], so the difference between the versed sine [sinus versus] of the arc opposite that same angle and the versed sine of the difference of those unequal arcs is to the versed sine of the sought angle. If we wished to use this theorem of Prop. 58, that rectangle would occupy the first place in the rule of three [regula aurea]. Consequently the division would become laborious, as is evident. But the division is easier by the theorem of

scholium 2 of the same Prop. 58, since the first place of the rule of three is occupied by the fourth-proportional quantity, which is much smaller than that rectangle, and is easily found merely by striking off from that rectangle as many figures on the right as there are ciphers (zeros) contained in the whole sine — because the said rectangle must be divided by the whole sine in order that the fourth-proportional quantity may be produced.

PROBL. 5. PROPOS. 64.

Given two sides of a non-right spherical triangle together with the angle they contain, to find the remaining side together with the remaining angles.

In the spherical triangle ABC, not right-angled, let two arcs AB and BC be given together with the angle B. From these we must find both the remaining arc AC and the remaining angles BAC and ACB. [Margin: When the two given arcs are unequal and neither is a quadrant.]

First let the given arcs be unequal, and from the end of one of them — namely from the endpoint A of arc AB — let a perpendicular arc AD be dropped to the other arc BC. Whether it falls inside the triangle or outside, the calculation and the operation will show. For since, in triangle ABD (whose angle D is right), the arc AB is given, opposite the right angle, together with the angle B, [Margin: (b) Schol. 41 of this treatise.] there will also be given the perpendicular arc AD, which is opposite the given angle B. Again, since in the same triangle the arc AB is given, opposite the right angle, together with the angle B:

EITHER — since the arc AD is known and moreover the angle B is given — the character of the other arc BD, about the right angle, is also settled. For when the arc AB is less than a quadrant: if the found arc AD is also less than a quadrant, [Margin: (c) Prop. 36 of this treatise.] then BD too will be less; if however AD is greater than a quadrant, then BD too will be greater. But if AB is greater than a quadrant: if the found AD is also greater, BD will be less; if however AD is less, BD will be greater.

OR, finally — since the arc AB is given, opposite the right angle, and the arc AD, about the right angle:

the arc BD will also be found, [Margin: (d) Schol. 45 of this treatise; Schol. 49 or 44 of this; Schol. 53 or 43 of this.] from the scholia cited in the margin. If, then, this arc BD, once found, is less than the given arc BC, it is a sign that the perpendicular arc AD has fallen inside the triangle; but outside, if it is greater.

And since a perpendicular arc to BC can be drawn on either side of arc AB, whenever it falls outside the triangle we choose, in this and the following propositions, the one that subtends the angle ABC. Now, by subtracting the found arc BD (if it is less than the given arc BC) from arc BC, or — if it is greater — by subtracting the given arc BC from the found arc BD, the remaining arc CD will become known. Therefore, since in triangle ADC the arcs AD and CD about the right angle are known, the arc AC opposite the right angle — which was sought in triangle ABC — will also be given.

Diagram. Two spherical triangles, stacked, illustrating the perpendicular arc AD dropped from A onto BC — the upper showing the perpendicular falling inside the base, the lower showing it falling outside.

- A (upper triangle, apex): vertex A
- B (upper triangle, lower-left): vertex B
- D (upper triangle, base, between B and C): foot of the perpendicular AD (falls inside)
- C [?] (upper triangle, lower-right): right-hand base vertex; the printed mark is unclear (resembles a small circle) and cannot be read with confidence
- A (lower triangle, upper-right apex): vertex A
- B (lower triangle, lower-left): vertex B
- C (lower triangle, base, middle): vertex C
- D (lower triangle, lower-right, beyond C): foot of the perpendicular AD (falls outside)

An internal arc from A to D (the perpendicular) is drawn in each triangle.

After this: since in triangle ABD, which has the right angle D, the arc AB is given, opposite the right angle, together with the angle B:

EITHER — since the arc AD, about the right angle, is known together with the angle B (which is not right), and it is moreover settled, concerning the remaining arc BD about the right angle once found, whether it is greater than a quadrant or less:

OR — since the arc AB is given, opposite the right angle, and in addition the arc AD, about the right angle:

OR, finally — since the arc AB is given, opposite the right angle, and moreover the arc BD, about the right angle —

the angle BAD will also be known, [Margin: (e) Schol. 1, 43 of this treatise; Schol. 47 of this; Schol. 56 or 42 of this; Schol. 51 or 45 of this; Schol. 55 or 41 of this.] through the scholia noted in the margin. In the same way too: since in triangle ACD, which has the right angle D, the arc AC is given, opposite the right angle, together with the arc AD about the right angle —

OR at least — since the arc AC is given, opposite the right angle, and moreover the arc CD, about the right angle —

the angle CAD will likewise be made known, [Margin: (f) Schol. 51 or 45 of this treatise; Schol. 55 or 41 of this.] through the scholia placed in the margin. Now the angle CAD just found, added to the angle BAD lately found as well — when the arc AD falls inside the triangle — or, when it falls outside, the angle CAD subtracted from the angle BAD, will make known the angle BAC, which was sought in triangle ABC.

Finally: since in triangle ACD, which has the right angle D, the arc AC is given, opposite the right angle, and the angle CAD is now found —

OR — since the arc CD about the right angle is known, and moreover the angle CAD; and it is settled, concerning the remaining arc AD about the right angle now made known, whether it is less than a quadrant or greater —

OR — since the arc AC is given, opposite the right angle, together with the arc CD about the right angle —

OR, finally — since the arc AC is known, opposite the right angle, together with the arc AD about the right angle —

the angle ACD will also become known, [Margin: (g) Schol. 56 or 42 of this treatise; Schol. 51 or 45 of this; Schol. 55 or 41 of this.] through the scholia cited in the margin; which, indeed, in the former triangle — where the arc AD falls inside the triangle — was the thing sought; but in the latter, where the arc AD falls outside the triangle, the same angle ACD, subtracted from two right angles, leaves known the sought angle ACB. And thus both the remaining arc AC and the remaining angles BAC and ACB have been found.

Moreover, in no way can either of the arcs AD, BD be a quadrant; [Margin: (h) Prop. 35 of this treatise.] because otherwise the arc AB too, opposite the right angle, would be a quadrant — which is contrary to the hypothesis.

But if the arc CD should ever be found to be a quadrant, [Margin: (i) Prop. 35 of this treatise.] then the arc AC too — the one sought, and opposite the right angle

— will be a quadrant, and the angle CAD will be right. And thus, without any labor, both the arc AC that is sought and [Margin: (k) Prop. 34 of this treatise.] the angle CAD will be found. The rest will be found as before.

The practical procedure for solving this problem is to be sought from the scholia cited in the margin.

But by sines alone one must proceed thus. By the practice of problem 2 of the scholium to Prop. 41, the arc AD is to be sought. [Margin: The procedure by sines alone, when the two given arcs are unequal and neither is a quadrant.]

Then, by the practice of scholium 1 of Prop. 43, the arc BD; from which the arc CD will be made known — by subtracting the found arc BD from the given arc BC, or the given arc BC from the found arc BD itself, according as it is found to be less than the given arc BC or greater.

In addition, in triangle BAD the angle BAD is to be sought, by the practice of problem 1 of Prop. 41, or by the practice of problem 2 of the scholium to Prop. 42. Similarly, in triangle ACD the angle CAD is to be elicited, by the practice of problem 1 of the scholium to Prop. 41. And from the two angles BAD, CAD once found, the angle BAC of the proposed triangle will become known — namely by adding one to the other, as in the former triangle, or by subtracting the angle CAD from the angle BAD, as in the latter triangle.

Finally, by the practice of problem 1 of the scholium to Prop. 41, or of problem 2 of the scholium to Prop. 42, in triangle ACD the same angle ACD is to be investigated. For in the former triangle proposed it is the thing sought, while in the latter it is the supplement of two right angles that is sought.

OTHERWISE — and indeed more expeditiously. [Margin: Another, shorter demonstration. (l) Schol. 58 of this treatise.] Again, in triangle ABC let two unequal arcs AB, AC be given, together with the angle A. Since, then, it holds that, as the whole sine is to the fourth proportional to the whole sine and the two sines of the unequal arcs AB, AC, so the versed sine of the angle A is to the difference between the versed sine of the arc BC (opposite the angle A) and the versed sine of the difference of the arcs AB, AC:

Diagram. A single spherical triangle with apex A above and a curved base, illustrating the case of two equal enclosing sides.

- A (apex, top): vertex A
- C (lower-left): vertex C
- B (lower-right): vertex B

If it be arranged that, as the whole sine is to the sine of either one of the given unequal arcs, so the sine of the other given arc is to a fourth term, there will be produced a fourth number proportional to the whole sine and the two sines of the said two arcs. [Margin: The shorter procedure, by sines alone, when the two given arcs are unequal and neither is a quadrant. (m) Schol. 1, 58 of this treatise.] If then it be again arranged that, as the whole sine is to the fourth proportional number just found, so the versed sine of the given angle A is to a fourth term, there will be found the difference between the versed sine of the third arc (which is sought) and the versed sine of the difference of the given unequal arcs. And since we showed above that the versed sine of the third arc is always greater than the versed sine of the difference of the two unequal arcs, if the difference just found is added to the versed sine of the difference of the given unequal arcs, there will be composed the versed sine of the third arc (opposite the given angle), which is sought; from which we shall elicit the arc itself, as was said in the exposition and use of the table of sines.

The angle C, moreover, we shall find from the known arcs AC, CB; and the angle B from the known arcs AB, BC — as we prescribed in the practice of the second demonstration of the preceding proposition — provided that the two arcs containing whatever angle is sought are unequal. For if they should ever be equal, the practice of the last demonstration of the same preceding proposition must be applied. But if we knew whether the angles B, C are acute or obtuse, we could easily find them, once the arc BC has been found, as we advised at the end of the second demonstration of the preceding proposition.

But if one of the given unequal arcs is a quadrant — namely AB — we shall draw from its endpoint A, to the other arc BC, a perpendicular arc AD. [Margin: When one of the given unequal arcs is a quadrant.] And at least one of the arcs AD, BD will also be a quadrant. [Margin: (n) Prop. 36 of this treatise.] But AD cannot be a quadrant; because otherwise — since AB too is posited as a quadrant — [Margin: (o) Prop. 25 of this treatise.] the angles B, D would be right, and so the triangle ABC would be right-angled, which is not posited. Therefore BD will be a quadrant, [Margin: (p) Prop. 34 of this treatise.] and hence the opposite angle BAD is right. [Margin: (q) Prop. 26 of this treatise.] B will also be the pole of the arc AD, on account of the quadrants AB, BD; and therefore the arc AD will be known from the angle B itself, which is given. And thus the two arcs AD, BD, together with the angle BAD, will be made known without any trouble of multiplication. The rest will be found as before.

But now let the given arcs AB, AC, which contain the given angle A, be equal. [Margin: When the two given arcs are equal.] Then the two angles B, C will be equal — namely either both acute or both obtuse — and neither of the arcs AB, AC a quadrant; and the perpendicular arc AD, dropped from A onto BC, will fall inside the triangle. The arcs BD, CD as well, and the angles at A, will be equal, as we showed in the last

figure of the preceding proposition; and consequently each angle at A will be given, since it is half of the given angle BAC. Since, then, in triangle ABD, which has the right angle D, the arc AB is given, opposite the right angle, together with the angle BAD — that is, with half of the given angle BAC — [Margin: (a) Schol. 41 of this treatise.] the arc BD, opposite the given angle BAD, will be known; and this, doubled, will render the whole arc BC (the thing sought) known. Again, since in the same triangle ABD, which has the right angle D, the arc AB is given, opposite the right angle, together with the arc BD about the right angle —

Diagram. A single spherical triangle with apex A above and a curved base B–D–C, with the perpendicular AD drawn inside from A to the base at D (the equal-arcs case).

- A (apex, top): vertex A
- B (lower-left): vertex B
- D (base, middle): foot of the perpendicular AD
- C (lower-right): vertex C

OR — since the arc AB is given, opposite the right angle, and moreover the non-right angle BAD —

OR, finally — since the arc BD is given, about the right angle, together with the non-right angle BAD (which is opposite the given arc BD); and moreover the character of the remaining non-right angle B is settled. For if AB is less than a quadrant, [Margin: (b) Schol. 51 or 45 of this treatise; Schol. 47 of this; Schol. 58 or 42 of this.] the angle B will be acute, just as BAD is acute; but if AB turns out greater than a quadrant, the angle B will be obtuse, since BAD is acute —

the angle B will also be known, [Margin: (c) Prop. 38 of this treatise.] from the scholia cited in the margin; and hence also the angle C, which is equal to it.

The practical procedure is to be sought from the scholia cited in the margin.

[Margin: The procedure. The procedure by sines alone, when the two given arcs are equal.]

By sines alone we shall proceed thus. By the practice of problem 2 of the scholium to Prop. 41, we shall seek the arc BD; which, doubled, will give the whole arc BC that is sought. Then, by the practice of problem 2 of the scholium to Prop. 42, we shall find the angle B; to which the other angle C is equal.

Given, therefore, two arcs of a non-right spherical triangle together with the angle contained by them, we have found the remaining arc together with the remaining angles. Which was to be done.

SCHOLIVM.

Here too, in the second demonstration of this problem, we chose rather to use the theorem of scholium 2 of Prop. 58 than the theorem of the same Prop. 58, so that the practice might become less laborious. [Margin: (d) Prop. 58 of this treatise.] For since it holds that, as the square of the whole sine is to the rectangle contained under the sines of the given unequal arcs, so the versed sine of the given angle (contained by the said arcs) is to the difference between the versed sine of the arc opposite the given angle and the versed sine of the difference of the two given unequal arcs: if we wished to use this theorem of Prop. 58, the multiplication in the rule of three would become troublesome, since the versed sine of the given angle would have to be multiplied by the said rectangle. But in our practice the multiplication is much shorter, as is evident, even though we apply the rule of three twice.

PROBL. 6. PROPOS. 65.

Given two angles of a non-right spherical triangle, together with the side adjacent to them, to investigate the remaining sides together with the remaining angle.

In the spherical triangle ABC, not right-angled, let two angles B and BAC be given, together with the adjacent arc AB. From these we must investigate the remaining arcs AC, BC, together with the remaining angle C. First let the given arc AB be not a quadrant, but either greater or less than a quadrant; and let the given angles B, BAC be unequal, from one of which — namely from BAC — let a perpendicular arc AD be dropped to the opposite arc BC. [Margin: When the two given angles are unequal, and the given adjacent arc is greater or less than a quadrant.] Whether it falls inside the triangle or outside, the calculation and the operation will indicate. For since in triangle ABD, which has the right angle D, the arc AB is given, opposite the right angle, and the angle B, [Margin: (a) Prop. 47 of this treatise.] the angle BAD will also be given; and if it is found less than the given

angle BAC, the arc AD will fall inside the triangle, but outside, if it is greater. Now, by subtracting the found angle BAD (if it is less than the given angle BAC) from the angle BAC, or — if it is greater — by subtracting the given angle BAC from the found angle BAD, the remaining angle CAD will become known.

Diagram. Two spherical triangles side by side, each with apex A above and a curved base, illustrating the perpendicular AD from A falling inside (left) and outside (right) the base.

- A (left triangle, apex): vertex A
- B (left triangle, lower-left): vertex B
- D (left triangle, base, middle): foot of the perpendicular (inside)
- C (left triangle, lower-right): vertex C
- A (right triangle, apex): vertex A
- B (right triangle, lower-left): vertex B
- C (right triangle, base, middle): vertex C
- D (right triangle, lower-right, beyond C): foot of the perpendicular (outside)

But the found angle BAD can never be right; [Margin: (b) Prop. 25 of this treatise.] because then the two arcs AB, BD would be quadrants, on account of the right angles BAD, ADB, whereas AB is posited to be not a quadrant. But CAD may sometimes be right.

Again, since in the same triangle ABD, which has the right angle D, the arc AB is given, opposite the right angle, and the non-right angle B:

EITHER — since both non-right angles B and BAD are known —

OR, finally — since the arc AB is given, opposite the right angle, together with the non-right angle BAD —

the arc AD will also be known, [Margin: (c) Schol. 41 of this treatise; Schol. 52 or 42 of this; Schol. 45 of this.] through the scholia cited in the margin. And in the same way: since in the same triangle BAD, whose angle D is right, the arc AB is given, opposite the right angle, together with the angle BAD:

EITHER — since both non-right angles B and BAD are known —

OR — since the arc AB is known, opposite the right angle, together with the non-right angle B —

OR — since the arc AB is given, opposite the right angle, and moreover the arc AD, about the right angle —

OR — since the arc AD is known, about the right angle, together with the non-right angle B opposite it; and it is moreover settled whether the other arc BD, about the right angle, is greater or less than a quadrant. For

(Continuing the treatment of Problem 6 / Proposition 65 begun in the previous pages: given two angles of a spherical triangle together with the side adjacent to them, to find the remaining sides and the remaining angle. The perpendicular AD has been dropped from A onto BC, dividing the oblique triangle ABC into the two right triangles ABD and ACD.)

if the angle BAD that has been found is acute, arc BD will be less than a quadrant; but if it is obtuse, greater.

Or, finally: since arc AD about the right angle is known, together with the non-right angle BAD adjacent to it, [d] arc BD will likewise become known from the scholia cited in the margin. Moreover, since in triangle ACD, which has the right angle D, arc AD about the right angle is known, together with the non-right angle CAD adjacent to it, [e] arc AC, which lies opposite the right angle, will also be found. And so now one of the remaining arcs, AC, has been found.

[Margin (d): scholium of prop. 41 of this treatise; scholium of prop. 52, or 42, of this; scholium of prop. 45 of this; scholium of prop. 53, or 43, of this; scholium of prop. 49, or 44, of this; scholium of prop. 44 of this.] [Margin (e): scholium of prop. 46, or 45, of this treatise.]

After this, since in the same triangle ACD — whose angle D is right — arc AC is given, opposite the right angle, together with the non-right angle CAD:

- Or because arc AC is given, opposite the right angle, and in addition arc AD, about the same right angle:
- Or, finally, because arc AD is given, about the right angle, together with the non-right angle CAD adjacent to it:

[f] arc CD will likewise become known, from the scholia described in the margin. This arc CD, added to the arc BD already found — when the perpendicular arc AD falls within the triangle — or, when it falls outside, subtracted from the found arc BD, will yield the known arc BC, which is the other of the remaining arcs that are sought.

[Margin (f): scholium of prop. 41 of this treatise; scholium of prop. 43, or 53, of this. † scholium of prop. 44 of this.]

Lastly, in the same triangle ACD: since arc AC is given, opposite the right angle, together with arc AD about the right angle:

- Or because arc AD is given, about the right angle, and the non-right angle CAD adjacent to it:
- Or because arc AD is given, about the right angle, and the non-right angle ACD opposite it — it being moreover established whether the remaining arc CD, found about the right angle, comes out greater than a quadrant or less:
- Or because both arcs AD and CD about the right angle are given:
- Or because arc AC is given, opposite the right angle, and in addition arc CD about the right angle:
- Or, finally, since arc AC is given, opposite the right angle, together with the non-right angle CAD:

[g] the angle ACD will also become known, from the scholia named in the margin. In the former triangle this angle ACD is the one sought; but in the latter, subtracted from two right angles, it leaves the sought angle ACB. And thus we have now found everything that was proposed.

[Margin (g): scholium of prop. 41, or 55, of this treatise; scholium of prop. 42 of this; scholium of prop. 56, or 42, of this; scholium of prop. 48, or 44, of this; scholium of prop. 45, or 51, of this; scholium of prop. 47 of this.]

[Margin: Practical procedure by sines alone, when the two given angles are unequal and the given arc adjacent to them is not a quadrant.]

Concerning the practical procedure we prescribe nothing new; instead, one must turn to the practical procedures of the scholia cited in the margin.

By sines alone, however, we will carry out the proposal as follows. By the practical procedure of problem 2 of the scholium to proposition 41, in the right-angled triangle ABD we will determine arc AD; and by the practical procedure of the problem of scholium 1 to proposition 43, arc BD. Then by the practical procedure of problem 1 of the scholium to proposition 41, we determine angle BAD: if it is smaller than the given angle BAC, we subtract it from the given angle BAC; or if it is larger, we subtract the given angle BAC from it — so that angle CAD becomes known.

From this, by the practical procedure of problem 2 of the scholium to proposition 42, we will elicit angle ACD in the right-angled triangle ACD. This will be the sought angle ACB in triangle ABC, if the angle BAD that was found is smaller than the given

angle BAC; but if it is larger, this same angle ACD, subtracted from two right angles, will give the remaining sought angle ACB.

Now, by the practical procedure of problem 3 of the scholium to proposition 41, we will find, in the same triangle ACD, arc AC, which lies opposite the right angle. For arc AD about the right angle is given, and the non-right angle ACD, and it is moreover established what the other non-right angle CAD is — found earlier. And this arc AC is one of the remaining arcs sought in triangle ABC.

Finally, by the practical procedure of problem 2 of the scholium to proposition 41, arc CD will be found; or by the practical procedure of problem 1 of the scholium to proposition 42, or indeed by the practical procedure of problem 1 of the scholium to proposition 43, we will determine the same arc CD. Added to the found arc BD — when the found angle BAD is smaller than the given angle BAC — or, when it is larger, subtracted from the same arc BD, it will yield the known arc BC, which is the other of those in triangle ABC that must be investigated.

[Margin: When the two angles are unequal, and the given adjacent arc is a quadrant.]

But if the found angle CAD happens to be right — since ADC is also right — [h] then AC and CD will be quadrants, and AD will be the arc of angle C; and so angle C will be known from the found arc AD. The remaining arc BC will be known from the quadrant CD and the found arc BD, as before.

[Margin (h): prop. 25 of this treatise.]

Now, if the given arc AB is a quadrant — the angles B and BAC still being unequal — angle BAD will be right, and arc BD too a quadrant. For since in the right-angled triangle ABD arc AB, opposite the right angle, is set as a quadrant, [i] at least one of the remaining arcs will be a quadrant. But AD cannot be a quadrant, [k] because the two angles B and D would be right, on account of the quadrants AB and AD, whereas triangle ABC is set as non-right-angled. Therefore BD will be a quadrant, [l] and consequently the opposite angle BAD is right. [m] B will also be the pole of arc AD, on account of the quadrants BA and BD; and so AD will be the arc of the given angle B, and therefore given. And once arcs AD and BD and the right angle BAD have been found — with no labor at all, since in investigating them we need none of the preceding problems — we will find the rest, as before.

[Margin (i): prop. 36 of this treatise.] [Margin (k): prop. 25 of this treatise.]

[Margin (l): prop. 34 of this treatise.] [Margin (m): prop. 26 of this treatise.]

Next, let two equal angles B and C be given in triangle ABC, together with the arc BC adjacent to them — whether that arc be a quadrant, or greater than a quadrant, or less. [f] Arcs AB and AC will be equal; and so the perpendicular arc AD, dropped from the opposite angle A onto the given arc BC, will fall within the triangle and will bisect both the given arc BC and the opposite angle BAC, as we demonstrated in the latter case of proposition 62. Since, therefore, in triangle ABD — which has the right angle D — arc BD about the right angle is given (being half of the given arc BC), together with the angle B adjacent to it, [g] arc AB opposite the right angle will also be given, and so AC, equal to it, will be given. And thus the two remaining arcs are now made known. Again, since in the same triangle arc AB, opposite the right angle, is given (by what was found), together with arc BD about the right angle:

[Margin: When the two given angles are equal. — f: prop. 9 of this treatise.]

- Or because arc BD is given, about the right angle, together with the non-right angle B adjacent to it:
- Or indeed because arc AB is given, opposite the right angle, together with the non-right angle B:

[h] angle BAD will also be found, through the scholia adduced in the margin; and this angle, doubled, will make the whole sought angle BAC known.

[Margin (g): scholium of prop. 45, or 46, of this treatise; scholium of prop. 41, or 55, of this; scholium of prop. 42 of this; scholium of prop. 47 of this.]

Diagram. An isosceles spherical triangle with apex A at the top and base drawn horizontally below it. The perpendicular arc AD is drawn from A straight down to the base, meeting it at D. Labels read off the scan under magnification:

- A (apex): the vertex from which the perpendicular is dropped.
- B (base, left end): base vertex.
- D (base, between B and C): foot of the perpendicular arc AD on the base BC.
- C (base, right end): base vertex. *(The rightmost base letter is printed with a rounded, tailed form that could be misread as "Q"; context — triangle ABC with foot D on BC — fixes it as C.)*

[Margin: Practical procedure, by sines alone, when the two given angles are equal.]

But by sines alone the procedure runs as follows. By the practical procedure of problem 2 of the scholium to proposition 42 — from arc BD given about the right

angle, and from angle B, given, adjacent to it — we will find angle BAD, which, doubled, will give the whole sought angle BAC. Then by the practical procedure of problem 3 of the scholium to proposition 41 — from arc BD about the right angle, and from angle BAD opposite it, now found — we will extract arc AB, opposite the right angle, and so also arc AC, equal to it. For besides the data, the species of the remaining angle B, given as non-right, is also established.

Therefore, given two angles of a non-right-angled spherical triangle together with the arc adjacent to them, we have investigated the remaining arcs along with the remaining angle. Which was to be done.

SCHOLIVM

In rectilinear non-right-angled triangles no problem similar to this was proposed. The reason is that, once two angles are given, [i] the third is given as well — namely, what remains when those two are taken from two right angles. Hence, since one side is also given, the two remaining sides will be made known by proposition 10 of the [treatise on] rectilinear triangles.

[Margin (i): prop. 32 of this treatise.]

PROBL. 7. PROPOS 66.

Given two angles of a non-right-angled spherical triangle, together with the arc opposite one of them, to find the remaining arcs along with the remaining angle. It must, however, be established whether the arc opposite the other given angle is greater than a quadrant, or less, or indeed exactly a quadrant.

In the non-right-angled triangle ABC let two angles B and C be given, together with arc AB, which is opposite angle C; and let it be established whether arc AC is greater than a quadrant, or less, or a quadrant. From these one must find the remaining arcs AC and BC, and the remaining angle BAC. First, let the two given angles B and C be unequal, and the given arc AB not a quadrant. From angle A, let the perpendicular arc AD be drawn to arc BC, which is adjacent to the given angles. [a] This perpendicular will fall within the triangle if both of the given angles B and C are acute, or both obtuse; but outside, if one is acute and the other obtuse. Since, therefore, in triangle ABD — which has the right angle D — arc AB is given, opposite the right angle, together with the non-right angle B, [b] arc AD about the right angle, opposite the given angle B, will become known. Hence, in

the same triangle ABD, since arc AB is given, opposite the right angle, together with arc AD about the right angle:

[Margin: When the two given angles are unequal and the given arc opposite one of them is not a quadrant. — a: prop. 57 of this treatise.] [Margin (b): scholium of prop. 41 of this treatise.]

Diagram. Two spherical triangles drawn side by side, each with apex A at the top, illustrating the two cases in which the perpendicular AD falls inside or outside the base.

- Left triangle: apex A; base vertices B (left) and C (right); D lies on the base between B and C (foot of the perpendicular arc AD falling *within* the triangle). Base labels, left to right: B, D, C.
- Right triangle: apex A; base vertices B (left) and C; D lies beyond C, to the right (foot of the perpendicular arc AD falling *outside* the triangle). Base labels, left to right: B, C, D.
- Or because arc AB is given, opposite the right angle, and in addition the non-right angle B:
- Or, finally, since arc AD is given, about the right angle, together with the non-right angle B opposite it — it being moreover established what the species of arc BD is. For if the given arc AB is less than a quadrant, then, if the found arc AD is likewise less, [c] BD too will be less; but if AD is greater, BD greater. If, on the other hand, the given arc AB is greater than a quadrant, then, if the found arc AD is greater, BD will be less; but if AD is less, BD will be greater.

[d] the other arc, BD, about the right angle, will also be found, from the scholia cited in the margin. Hence again, in the same triangle ABD, since arc AB is given, opposite the right angle, and in addition arc BD about the right angle:

[Margin (c): scholium of prop. 43, or 53, of this treatise; scholium of prop. 45 of this; scholium of prop. 49, or 44, of this.] [Margin (d): prop. 36 of this treatise.]

- Or because both arcs AD and BD are given, about the right angle:
- Or because arc AB is given, opposite the right angle, together with arc AD about the right angle:
- Or because arc AD is given, about the right angle, and in addition the non-right angle B opposite it — the species of angle BAD being moreover

established. For if the found arc BD is greater than a quadrant, [f] angle BAD will be obtuse; but if less, acute.

[Margin (f): prop. 34 of this treatise.]

- Or, finally, because arc AB is given, opposite the right angle, together with the non-right angle B;

[e] the non-right angle BAD will also become known, from the scholia set out in the margin.

[Margin (e): scholium of prop. 41, or 55, of this treatise; scholium of prop. 44, or 48, of this; scholium of prop. 45, or 51, of this; scholium of prop. 56, or 42, of this; scholium of prop. 47 of this. (g): scholium of prop. 41, or 54, of this.]

Next, in triangle ACD — which has the right angle D — since arc AD is given, about the right angle, together with the angle C opposite it (for when the perpendicular arc AD falls outside the triangle, angle ACD will be given, if the given angle ACB is subtracted from two right angles), and since it is moreover assumed to be established what the species of arc AC is — the arc which in the proposed triangle ABC is opposite the other given angle B, but in this triangle ACD is opposite the right angle D — [g] arc AC will likewise come out known. This is one of the remaining arcs proposed for investigation in triangle ABC. Hence, since in the same triangle ACD arc AC is given, opposite the right angle, together with arc AD about the right angle:

- Or because arc AC is given, opposite the right angle, together with the non-right angle C:
- Or, finally, since arc AD is given, about the right angle, together with the angle C opposite it — the species of arc CD being moreover established. For if arc AC, opposite the right angle, is found to be less than a quadrant, [h] both arcs AD and CD will be either less as well, or greater; and thus from the known arc AD we shall know whether CD is less or greater than a quadrant. But if the found arc AC is greater than a quadrant, and AD less, CD will be greater; whereas if AD is greater, CD will be less.

[Margin (h): prop. 36 of this treatise.]

[i] arc CD will also become known, through the scholia set in the margin. Added to the arc BD found earlier — if the perpendicular arc AD falls within the triangle — or, if outside, subtracted from the found arc BD, it will yield the known sought arc BC. And thus the remaining two arcs, AC and BC, will now have been found.

[Margin (i): scholium of prop. 43, or 53, of this treatise; scholium of prop. 45 of this; scholium of prop. 49, or 44, of this.]

Lastly, since in the same, adjacent triangle ACD arc AC is given, opposite the right angle, together with arc CD about the right angle:

- Or since arc CD is given, about the right angle, and in addition the non-right angle C:
- Or because arc AD is given, about the right angle, together with the non-right angle C opposite it — the species of the other angle CAD being moreover established. For if the found arc CD is less than a quadrant, [l] angle CAD will be acute; but obtuse, if CD is greater than a quadrant:
- Or because both arcs AD and CD are given, about the right angle:
- Or since arc AC is given, opposite the right angle, together with arc AD about the right angle:
- Or, finally, because arc AC is given, opposite the right angle, together with the non-right angle C;

[Margin (l): prop. 36 of this treatise.]

[k] the angle CAD will likewise become known, from the scholia adduced in the margin. And this angle CAD, added to the angle BAD found earlier — if the perpendicular arc AD falls within the triangle — or, if outside, subtracted from the found angle BAD, will yield the known sought angle BAC.

[Margin (k): scholium of prop. 41, or 55, of this treatise; scholium of prop. 42 of this; scholium of prop. 42, or 56, of this; scholium of prop. 44, or 48, of this; scholium of prop. 45, or 51, of this; scholium of prop. 47 of this.]

For the rest, in no way can either of the arcs AD, BD be a quadrant in this case: because if either of them were a quadrant, [m] arc AB, opposite the right angle, would also be a quadrant — which is contrary to the hypothesis.

[Margin (m): prop. 35 of this treatise.]

[Margin: Practical procedure, by sines alone, when the given angles are unequal and the given arc opposite one of them is not a quadrant.]

The practical procedure of this problem depends on the scholia noted in the margin.

By sines alone we will accomplish the matter thus. By the practical procedure of problem 2 of the scholium to proposition 41, we will find arc AD; and by the

practical procedure of the problem of scholium 1 to proposition 43, arc BD; and by the practical procedure of problem 1 of the scholium to proposition 41, angle BAD.

Then, by the practical procedure of problem 3 of the scholium to proposition 41, we will determine arc AC, since its species is established from the hypothesis. Hence, by the practical procedure of problem 1 of the scholium to proposition 43, arc CD will become known; and from it — if it is added to the found arc BD, or subtracted from it, according as the perpendicular arc AD has fallen within or outside the triangle — arc BC will become known.

Finally, by the practical procedure of problem 1 of the scholium to proposition 41, we will extract angle CAD; which, added to the found angle BAD, or subtracted from it, according as the perpendicular arc AD has fallen within the triangle or outside, will make the sought angle BAC known.

[Margin: When the two given angles are unequal and the given arc opposite one of them is a quadrant.]

But if the arc AC, opposite the other given angle B, happens to be a quadrant — which can occur when AB is not a quadrant — [n] then at least one of the other remaining arcs, AD or CD, in triangle ACD will be a quadrant. Since, then, AD cannot be a quadrant, CD will be a quadrant; [o] and consequently the angle CAD opposite it will be right. So in that case both arc CD and angle CAD will be found without any other labor; and from them both arc BC and angle BAC will be determined, as has been said.

[Margin (n): prop. 36 of this treatise.] [Margin (o): prop. 34 of this treatise.]

Now let the given arc AB be a quadrant, the two given angles B and C still being unequal. Arc BD will also be a quadrant, and angle BAD right. For since in triangle ABD arc AB, opposite the right angle, is set as a quadrant, [p] at least one of the other arcs, AD or BD, will also be a quadrant. But AD cannot be a quadrant, [q] because the two angles B and D would then be right — on account of the quadrants AB and AD — and so triangle ABC would be right-angled, which is not assumed. Therefore BD will be a quadrant, [r] and consequently the opposite angle BAD right. [s] B will also be the pole of arc AD, on account of the quadrants AB and BD; and therefore the given angle B will make arc AD known. And once arcs AD and BD have been found, with the right angle BAD, the rest will be found without any troublesome multiplication, as before. In this case, however, arc AC will in no way be a quadrant — lest AB and AC be two quadrants in triangle ABC, and consequently the two angles B and C right. That would be contrary to the hypothesis, since the triangle is set as non-right-angled.

[Margin (p): prop. 36 of this treatise.] [Margin (q): prop. 5 of this treatise.] [Margin (r): prop. 54 of this treatise.] [Margin (s): prop. 26 of this treatise.] [Margin (t): prop. 25 of this treatise.]

(Continuation of PROBL. 7 / PROPOS. 66 — the case of two given equal angles, begun on the previous page at the catchword "VERVM".)

But now suppose that in triangle ABC the two given angles B and C are equal. Then the two arcs AB and AC will also be equal — and consequently neither of them is a quadrant, so that the two angles B and C cannot both be right angles. [Margin: *Quando duo dati anguli æquales sunt* — When the two given angles are equal.] Therefore, when the perpendicular arc AD is dropped from the third angle A, it falls inside the triangle, and it bisects both the arc BC and the angle BAC, exactly as we showed above in the second case of Proposition 62. Now since in triangle ABD, which has its right angle at D, the arc AB is given (opposite the right angle) together with angle B, [Margin: *h Schol. 45. huius* — Scholium to Prop. 45 of this treatise.] the arc BD too will become known; and doubled, it makes the whole arc BC known. And AC is likewise known, since it equals the given arc AB. Then, since in the same triangle ABD the arc AB is given (opposite the right angle) together with arc BD (about the right angle just found);

Or because arc BD is given, about the right angle, together with the non-right angle B adjacent to it;

Or finally because arc AB is given, opposite the right angle, together with the non-right angle B —

[Margin: *i Schol. 41. vel 55. hui.; i Scho. 42. huius; i Scho. 47. huius* — Scholia to Props. 41 or 55, 42, and 47 of this treatise.] the angle BAD will also be obtained by means of the scholia cited in the margin; and doubled, it will yield the sought whole angle BAC.

This is how we shall proceed in this case using sines alone. [Margin: *Praxis per solos sinus quando dati duo anguli æquales sunt* — Procedure by sines alone, when the two given angles are equal.] *By the procedure of problem 2 of the scholium to Proposition 41 we find arc AD; and from this, by the procedure of problem 1 of the scholium to Proposition 43, arc BD, which doubled gives arc BC known. Then by the procedure of problem 1 of the scholium to Proposition 41, or by the procedure of problem 2 of the scholium to Proposition 42, we find angle BAD, and hence its double BAC, which is sought. The third arc AC is equal to the given arc AB, and so is known.*

Therefore, given two angles of a non-right spherical triangle together with one arc that is opposite one of them, [we have found the rest], and so on. Which was to be done.

Diagram. A spherical triangle with the apex at the top and the base curving below; a perpendicular is dropped from the apex to the midpoint of the base.

- A (apex): vertex A
- B (base, left): vertex B
- D (base, center): foot of the perpendicular AD
- C (base, right): vertex C

SCHOLIVM

To this problem too we have supplied no corresponding proposition among the rectilinear (plane) triangles, for the reason given in the scholium to the preceding proposition.

Now in the first case of this problem it is also necessary that the species of arc AC — the arc opposite the given angle B — be given as well. Otherwise, in triangle ACD, from the given arc AD and the opposite angle C (since we as yet have nothing certain established about arc CD or angle CAD, whatever these may be) the arc AC [glyph reads *AD*; contextually the hypotenuse AC, opposite the right angle D — see QA] opposite the right angle would not be found, since it could be either greater or less than a quadrant, and it is not yet established from the givens, nor demonstrated, what it will turn out to be.

Moreover, that it is not enough for two angles to be given together with an arc opposite one of them, in order to elicit the remaining arcs and the remaining angle, we pointed out some time ago in the scholium to Propositions 22 and 23. There too we indicated that Copernicus went astray on this point, in Book 1 of the *Revolutions*, Proposition 12 on spherical triangles [Copernicus, *De Revolutionibus* I.12]. This we shall now briefly demonstrate again as follows. Let there be two unequal arcs AB, AC containing angle BAC, and together equal to a semicircle — so that one of them is greater than a quadrant and the other less. Now through B and C let an arc of a great circle BC be drawn, and let another arc AD be drawn from A meeting its extension, running neither through the poles of arc BC — so that the angles at D and at CAD are not right angles. But neither is angle ACD a right angle. [Margin: *Error Copernici. Non satis esse, dari duos angulos cum arcu vni eorum opposito, ad reliqua inuenienda in triangulo non rectangulo* — Copernicus's error: that it is not enough to be given two angles together with an

arc opposite one of them, to find the rest in a non-right triangle.] For if it were right, then, because of the right angles B and C, the two arcs AB, AC would be equal, and so both would be quadrants — which is contrary to the hypothesis. Triangle ACD, therefore, is not right-angled; and in it, although the two angles ACD and D may be given together with arc AD (which is opposite angle ACD), we shall nevertheless not thereby infer that arc AC, opposite the other given angle D, is [determinate]. For since in triangle ABD the same is opposed to it — namely, arc AB is likewise unequal to AC (given that the two given angles B and D, angle B being equal to angle ACD as we showed) and the given arc AD is opposite angle B — [Margin: k 14. *huius*; l 25. *huius*. *Error Regiom.* — Prop. 14 and Prop. 25 of this treatise; Regiomontanus's error.] it is necessary that the species of the arc opposite angle D be given, so that we may know whether arc AB or AC is to be taken, since one of them is greater than a quadrant, and so on.

Johannes Regiomontanus [Regiomontanus] also slipped on this point, in Book 4 of the *Triangles*, Proposition 32 [Regiomontanus, *De triangulis* IV.32], when he wants, from two given angles together with one side opposite [one of them], to find the rest — which, however, is not sufficient, as we have here demonstrated.

Diagram. A spherical triangle with apex A at the top; the base carries three points, with the foot D of the perpendicular lying beyond the vertex C (the perpendicular AD falls outside the triangle).

- A (apex): vertex A
- B (base, left): vertex B
- C (base, center): vertex C
- D (base, right): foot of the perpendicular AD, outside the triangle

PROBL. 8. PROPOS. 67.

Given two arcs of a non-right spherical triangle, together with the angle that is opposite one of them, to find the remaining angles and the remaining arc. But it must be established whether the angle opposite the given arc is acute or obtuse.

In a non-right spherical triangle ABC let there be given two arcs AB, AC, together with angle B, which is opposite arc AC; and let it be established whether angle C is acute or obtuse. From these we must investigate the remaining angles C and BAC, and the remaining arc BC. Let the two given arcs AB, AC first be unequal, and neither of them a quadrant. From angle A, opposite the third arc, let the perpendicular arc AD be drawn to that third arc BC; [Margin: a 37. *huius* — Prop.

37 of this treatise.] it will fall inside the triangle if both angles B, C are acute, or both obtuse; but outside if one is acute and the other obtuse. Now it is established from the givens whether both angles are acute, or both obtuse, or one acute and the other obtuse — since angle B is given together with the species of angle C. [Margin: *Quando neuter datorum arcuum inæqualium est quadrans* — When neither of the two given unequal arcs is a quadrant.] And so, since in triangle ABD, which has its right angle at D,

Diagram. Two spherical triangles side by side, each with apex A and a perpendicular AD from the apex. In the left figure the foot D falls between B and C (perpendicular inside the triangle); in the right figure the foot D falls beyond C (perpendicular outside).

- Left figure — A (apex); B (base, left); D (base, center); C (base, right)
- Right figure — A (apex); B (base, left); C (base, center); D (base, right)

the arc AB is given, opposite the right angle, together with angle B; [Margin: b *Schol. 41. huius* — Scholium to Prop. 41 of this treatise.] the arc AD too, about the right angle, will be given — opposite the given angle B. Hence, in the same triangle ABD, since arc AB is given, opposite the right angle, and in addition arc AD, about the same right angle:

Or because arc AB is given, opposite the right angle, and moreover the non-right angle B:

Or finally, because arc AD is given, about the right angle, together with the opposite angle B; and there is besides established the species of arc BD. For if AB is given less than a quadrant, then, provided AD too is found less [than a quadrant], [Margin: c *Schol. 43. vel 53. hui.*; c *Schol. 45. huius*; c *Scho. 49. vel 44. hu.*; d 36. *huius* — Scholia to Props. 43 or 53, 45, 49 or 44, and Prop. 36 of this treatise.] BD too will be less; but if [AD is] greater, [BD will be] greater. On the other hand, if AB is given greater than a quadrant, then, provided AD too is found greater, BD will be less; but if AD is found less, BD will be greater —

then, [Margin: d 36. *huius* — Prop. 36 of this treatise.] by the scholia cited in the margin, the other arc BD, about the right angle, will also become known. Hence again, in the same triangle ABD, since arc AB is given, opposite the right angle, together with arc BD, about the same right angle:

Or because both arcs AD, BD are given, about the right angle:

Or because arc AB is given, opposite the right angle, together with arc AD, about the same right angle:

Or because arc AD is given, about the right angle, together with the opposite angle B, and there is besides established the species of angle BAD. For if the arc BD is found to be less than a quadrant, [Margin: e *Schol.* 41. vel 55. *hui.*; e *Schol.* 44. vel 48. *hu.*; e *Schol.* 45. vel 51. *hui.*; e *Schol.* 56. vel 42. *hu.*; e 34. *huius*; f *Schol.* 47. *huius*; g *Schol.* 51. vel 45. *hui.* — Scholia to Props. 41/55, 44/48, 45/51, 56/42, and Prop. 34, Scholium to Prop. 47, Scholium to Prop. 51/45.] angle BAD will be acute; but obtuse if it is greater:

Or finally, because arc AB is given, opposite the right angle, and in addition the non-right angle B —

f angle BAD, the non-right angle, will also be rendered known from the scholia set in the margin.

Next, since in triangle ACD, which has its right angle at D, arc AC is given (opposite the right angle) and arc AD has been found (about the right angle), g angle CAD, contained by the said arcs, will likewise become known; and this, added to the angle BAD already found, or subtracted from it — according as the perpendicular arc AD falls inside the triangle or outside (which indeed we shall know, as we said at the outset, from the given angle B and the given species of angle C) — will give the sought angle BAC.

Again, since in the same triangle ACD arc AC is given, opposite the right angle, and arc AD has been found, about the right angle:

Or because arc AD is given, about the right angle, together with the non-right angle CAD:

Or finally, because arc AC is given, opposite the right angle, and moreover the non-right angle CAD:

[Margin: h *Schol.* 55. vel 41. *hui.*; h *Scho.* 42. *huius*; h *Scho.* 47. *huius*; i 41. *huius* — Scholia to Props. 55/41, 42, 47, and Prop. 41 of this treatise.] h angle ACD will also become known. If then the perpendicular arc AD falls inside the triangle, the angle found will be ACB, which is sought; but if it falls outside, the angle ACD found, subtracted from two right angles, will leave known the sought angle ACB. And this angle ACB may indeed likewise be found even if arc AD were not available. For since, as the sine of arc AC is to the sine of angle B, so the sine of arc AB is to the sine of angle ACB: if it be made that, as the sine of the given arc opposite the given angle is to the sine of the given angle, so the sine of the other given arc is to a fourth term, the sine of the angle opposite this arc will be

produced; and hence the angle ACB itself will be known, since its species is established. And so the remaining two angles BAC, ACB are now found.

Finally, since in the same triangle ACD arc AC is given, opposite the right angle, together with angle CAD, just found:

Or because both non-right angles ACD, CAD are given:

Or because arc AC is given, opposite the right angle, together with arc AD, about the right angle:

Or because arc AD is given, about the right angle, together with the non-right angle CAD:

Or because arc AD is given, about the right angle, together with the opposite angle ACD, and there is besides established the species of the other arc CD, about the right angle. For, the found angle CAD being acute, [Margin: k Schol. 41. huius; k Scho. 42. vel 52. hui.; k Schol. 43. vel 53. hui.; k Scho. 44. huius; k Scho. 49. vel 44. hu.; k 34. huius; l Schol. 45. huius; m 35. huius. *Praxis per solos sinus quando neuter datorum arcuum inæqualium est quadrans* — Scholia to Props. 41, 42/52, 43/53, 44, 49/44, and Prop. 34, Scholium to Prop. 45, Prop. 35. Procedure by sines alone, when neither of the two given unequal arcs is a quadrant.] k arc CD will be less than a quadrant; but greater if it is obtuse:

Or finally, because arc AC is given, opposite the right angle, together with the non-right angle ACD;

l arc CD, about the right angle, will also be found by means of the scholia cited in the margin; and this, either added to the arc BD already found, or subtracted from it (according, of course, as the perpendicular arc AD has fallen inside the triangle or outside), will give the arc BC sought in the proposed triangle ABC.

Furthermore, in no way can either of the arcs AD, BD, in this case, be a quadrant: m because otherwise arc AB too, opposite the right angle D, would be a quadrant — which is contrary to the hypothesis. By the same reasoning, neither will CD be a quadrant, lest arc AC too, opposite the right angle, be a quadrant, which would likewise be contrary to the hypothesis.

Let the procedure of this problem be sought from the scholia set in the margin.

But by sines alone the problem will be solved thus. By the procedure of problem 2 of the scholium to Proposition 41 arc AD will be found, in triangle ABD; and from this, by the procedure of the problem of the scholium to Proposition 43, arc BD. Then by

the procedure of problem 2 of the scholium to Proposition 42 angle BAD will be found.

(The italic procedure continues from the previous page. The catchword "POST" does not match the opening word "ACD" here — see QA.)

...angle] ACD, which is one of the things sought, if it be established that angle C is of the same species as the given angle B; but if of a different species, the sought angle ACB will be the remainder of two right angles — [Margin: m 57. hui. — Prop. 57 of this treatise.] because in the one case the perpendicular arc falls inside the triangle, in the other outside. Again, by the procedure of the problem of the scholium to Proposition 43, arc CD will be rendered known; and this, in the former triangle added to the arc BD found, but in the latter subtracted from it, will exhibit the remaining arc BC, known, in the proposed triangle. Finally, by the procedure of problem 1 of the scholium to Proposition 41, angle CAD will be found; and this, added to or subtracted from the found angle BAD, will render known the third angle BAC, which is sought.

Now if one of the two given unequal arcs AB, AC be a quadrant: if indeed AB is a quadrant, then BD too will be a quadrant, and angle BAD a right angle, and moreover B will be the pole of arc AD; and so the given angle B will exhibit that same arc AD as known, just as we showed in the preceding proposition, when arc AB was supposed to be a quadrant. [Margin: *Quando alter duorum arcuum inæqualium datorum est quadrans* — When one of the two given unequal arcs is a quadrant.] The arcs AD, BD and angle BAD being thus found, we shall find the rest without any trouble, as before. But if arc AC be a quadrant, then by the same reasoning CD will be a quadrant, and angle CAD a right angle, and moreover C will be the pole of arc AD; and so the found arc AD will make angle ACD known — which will be one of the things sought, when arc AD falls inside the triangle; but if it falls outside, the remainder of two right angles will give the sought angle ACB. And in that case there will then be found, without any multiplication at all, both arc CD and angle CAD, and likewise angle ACD; from which the rest will be found, as before.

Now let the two given arcs AB, AC be equal. Then the two angles B, C will be equal; and the perpendicular arc AD, dropped from A onto BC, will fall inside the triangle; and moreover the arcs BD, CD, and the angles at A, will be equal, as we showed in the last case of Proposition 63. Since therefore angle B is given, C too, being equal to it, will be given. Then, since in triangle ABD, which has its right angle at D, arc AB is given, opposite the right angle, together with angle B; [Margin: *Quando dati duo arcus æquales sunt* — When the two given arcs are

equal. n *Scho. 47. huius* — Scholium to Prop. 47 of this treatise.] n angle BAD too will be given, and this, doubled, will offer known the whole angle BAC, which is sought. Hence, since in the same triangle ABD arc AB is given, opposite the right angle, together with angle BAD, found:

Or because both non-right angles B and BAD are given:

Or finally, because arc AB is given, opposite the right angle, together with angle B, which is not right;

[Margin: o *Schol. 41. huius*; o *Scho. 42. vel 52. hui.*; o *Schol. 45. huius* — Scholia to Props. 41, 42/52, and 45 of this treatise.] o arc BD, about the right angle, will also become known, by the scholia cited in the margin; and hence its double BC, which is set as the thing to be inquired.

The procedure can easily be gathered from the scholia set in the margin. [Margin: *Praxis, per solos sinus, quando duo arcus dati æquales sunt* — Procedure, by sines alone, when the two given arcs are equal.]

But if you prefer to employ sines alone, arc AD is first to be found, by the procedure of problem 2 of the scholium to Proposition 41. And from this, by the procedure of the problem of the scholium to Proposition 43, arc BD, which doubled will give the sought whole BC. Then by the procedure of problem 1 of the scholium to Proposition 41, or by the procedure of problem 2 of the scholium to Proposition 42, angle BAD is to be found; from which its double BAC, which we seek, will be known. The third angle C is already given, since it is equal to the given angle B.

Given, therefore, two arcs of a non-right spherical triangle, together with one angle that is opposite one of them, etc. Which was to be done.

Diagram. A spherical triangle with apex A and a perpendicular AD dropped from A onto the base, the foot D falling between B and C.

- A (apex): vertex A
- B (base, left): vertex B
- D (base, center): foot of the perpendicular AD
- C (base, right): vertex C

[Margin: *Quando alter duorum arcuum inæqualium datorum est quadrans* — When one of the two given unequal arcs is a quadrant.]

SCHOLIUM I.

But it is necessary, in this problem, to establish whether angle C, opposite the other given arc, is acute or obtuse, so that it may be known whether the perpendicular arc AD falls inside the triangle or not. For if this be ignored, we should not know whether angle CAD is to be added to angle BAD, or subtracted from it, in order to find the sought angle BAC; likewise, whether arc CD is to be added to arc BD, or subtracted from it, so that the sought arc BC may be found. Or finally, whether the found angle ACD is the one that is sought, or rather the remainder of two right angles — as is evident. That it is not, moreover, enough for two arcs to be given, [together with the angle opposite one of them, in order to find the] remaining angles together with the remaining arc, we have already taught above, some time ago, in the scholium to Proposition 24. And on this point we noted there that Nicolaus Copernicus [Copernicus] erred, in Book 1 of the *Revolutions*, Proposition 11 on spherical triangles [Copernicus, *De Revolutionibus* I.11]. This, however, we shall now briefly show again as follows. Let there be two equal arcs AD, AC, embracing angle DAC, and each less than a quadrant, or greater. Now through C, D let an arc of a great circle CD be drawn, and from A let another arc AB be drawn meeting its extension, running neither through the poles of arc CD nor through the poles of arc AD, so that the angles at B and at DAB are not right angles. But neither is angle ADB a right angle. For if each of the arcs AD, AC is less than a quadrant, the two angles C and ADC will be acute; but if each of the arcs AD, AC is greater than a quadrant, the two angles C and ADC will be obtuse. From this it follows that angle ADB is either obtuse — namely, when ADC is acute — or acute, namely, when ADC is obtuse: [Margin: *Error Copernici. Non satis esse, dari duos arcus cum angulo vni eorum opposito, in triangulo non rectangulo, vt reliqua inueniantur* — Copernicus's error: that it is not enough to be given two arcs together with the angle opposite one of them, in a non-right triangle, in order that the rest be found. 25. *huius* — Prop. 25 of this treatise. p 5. *huius* — Prop. 5 of this treatise.] p since the two angles at D are equal to two right angles. Triangle ABD, therefore, is not right-angled; and in it, although the two arcs AB, AD may be given together with angle B, which is opposite arc AD, we shall nevertheless not thereby be able to gather that the remaining angle opposite the other given arc AB is ADB. For to that same arc AB there is opposed, in triangle ABC as well, angle C, which is unequal to ADC: because in triangle ABC the same hypothesis holds — namely, the two given arcs AB, AC (for arc AC is set equal to arc AD) and the given angle B, opposite arc AC. It is necessary, therefore, to establish whether the angle opposite the other arc AB is acute or obtuse — that

is, whether angle ADB is to be taken, or rather C, since one is obtuse and the other acute, and so on.

Diagram. A spherical triangle with apex A and a perpendicular from A to the base, the foot D between B and C.

- A (apex): vertex A
- B (base, left): vertex B
- D (base, center): foot of the perpendicular
- C (base, right): vertex C

Diagram. A spherical triangle with apex A; the base carries the points B, D, C, with C set slightly higher at the right and D the foot of the perpendicular from A.

- A (apex): vertex A
- B (base, left): vertex B
- D (base, center): foot of the perpendicular AD
- C (base, right): vertex C

Triangula — Praxes Problematum (rectilinear triangles)

SCHOLIVM II.

So far I have demonstrated what is required for the calculation of triangles, and through a good many propositions — and perhaps longer ones than anyone might wish for in a calculation, which ought to be easy and short. I therefore think it worth my while to set down here separately, by way of an epilogue, the practical procedures for all the problems that have been demonstrated in rectilinear and spherical triangles, gathered as it were into a single place. In this way we shall always have them ready and, so to speak, at hand whenever they are to be used, and shall not waste time fruitlessly picking them out from so large a multitude of propositions. In the margin, moreover, I shall cite the propositions and problems in which their demonstrations are contained, so that anyone can easily find them whenever the matter requires. So then, as far as the calculation of triangles is concerned, it will be enough if these few things that follow are read through attentively when there is need; for in them is comprised the sum of everything I have demonstrated about triangles. Now although in non-right spherical triangles I have usually investigated by several methods the arcs and

angles of the triangles into which we resolve the non-right ones, nevertheless in the practical procedures that follow — so as to avoid all confusion — I have chosen, for finding any given arc or angle, only one method, namely the one I judged to be the more convenient.

SEQVVNTVR PRAXES PROBLEMATVM

[There follow the practical procedures of the problems]

There follow the practical procedures of the problems of all triangles, excerpted from the demonstrations above, in which consists the whole fruit of our treatment of triangles, both rectilinear and spherical.

TRIANGVLORVM RECTILINEORVM RECTANGVLORVM PROBLEMATA, ET PRAXES.

[Problems and practical procedures of right-angled rectilinear triangles]

1. Given all the angles of any triangle whatever, to find the proportions of all the sides.

[Margin: What is sought — the proportions of the sides.]

To each side let there be assigned the right sine of the opposite angle. For the sides have among themselves the same proportions as are found among those sines of the angles assigned to the opposite sides. But if only two angles are given, the third angle must first be found — by subtracting the two given angles from two right angles — and only then are the proportions of the sides to be traced out in the same way.

[Margin: a — Scholium on Proposition 1 of rectilinear triangles.]

Aliter. (*Another way.*)

Let the right sine of any acute angle be doubled, and the side opposite that angle will be obtained, in the units of the whole sine [i.e. the radius], which the semidiameter of the circle circumscribed about the triangle represents. But for the side that is opposite the right angle, if the triangle happens to be right-angled, take the whole sine doubled. And finally, for the side that is opposite the obtuse angle, if the triangle happens to be obtuse-angled, take double the right sine that belongs to half of the sum formed from the doubles of the two acute angles.

[Margin: b — Scholium on Proposition 1 of rectilinear triangles.]

2. Given, in a right triangle, the side that is opposite the right angle, together with one of the acute angles — and consequently also with the other acute angle, since the two together are equal to one right angle — to find the side about the right angle that is opposite either of the acute angles.

[Margin: What is sought — the side, about the right angle, opposite either of the acute angles.]

Let it be done thus: as the whole sine is to the given side opposite the right angle, so is the sine of either given acute angle to a fourth; and there will be produced the side opposite that given acute angle, in the units of measure by which the side opposite the right angle was given.

[Margin: c — Proposition 2 of rectilinear triangles.]

3. Given one side of a right triangle about the right angle, together with one of the acute angles — and indeed also with the other acute angle, since the two together are equal to one right angle — to find the other two sides.

[Margin: What is sought — the side opposite the right angle, and either of the two sides about the same right angle.]

Let it be done thus: as the whole sine is to the given side about the right angle, so is the tangent of the acute angle adjacent to the given side to a fourth; and the other side about the right angle will be found. Likewise let it be done: as the whole sine is to that same given side about the right angle, so is the secant of the same acute angle adjacent to the given side to a fourth; and there will be produced the side opposite the right angle, in the units of measure by which the side about the right angle was given.

[Margin: d — Proposition 2 of rectilinear triangles.]

Aliter per solos sinus. (*Another way, by sines alone.*)

Let it be done thus: as the sine of the acute angle opposite the given side is to the given side about the right angle, so is the sine of the other acute angle to a fourth; and there will be found the side, about the right angle, opposite this other acute angle. Likewise let it be done: as the sine of the acute angle opposite the given side is to the given side about the right angle, so is the whole sine to a fourth; and

there will be produced the side opposite the right angle, in the units of measure by which the side about the right angle was given.

[Margin: e — Proposition 2 of rectilinear triangles.]

4. Given, in a right triangle, the side opposite the right angle, together with either of the two remaining sides about the right angle, to find the two acute angles and the other side about the right angle.

[Margin: What is sought — the two acute angles, and one side about the right angle.]

Let it be done thus: as the given side opposite the right angle is to the whole sine, so is the given side about the right angle to a fourth; and there will be produced the sine of the acute angle opposite this latter given side. From this angle, once found, the other acute angle also becomes known, since the two together are equal to one right angle. Again let it be done: as the whole sine is to the given side opposite the right angle, so is the sine of the acute angle that has been found — the one opposite the third side sought — to a fourth; and this other side about the right angle will be found, in the units of measure by which the two other sides were given.

[Margin: f — Proposition 3 of rectilinear triangles.]

5. Given the two sides about the right angle, to find the two acute angles and the side opposite the right angle.

[Margin: What is sought — the two acute angles, and the side opposite the right angle.]

Let it be done thus: as either of the given sides is to the whole sine, so is the other given side to a fourth; and there will come out the tangent of the acute angle opposite this latter side. From this angle, once found, the other acute angle becomes known, since the two acute angles together are equal to one right angle. Again let it be done: as the whole sine is to either of the given sides about the right angle, so is the secant of the acute angle taken [i.e. found], adjacent to this side, to a fourth; and there will be found the side opposite the right angle, in the units in which the two sides about the right angle were given.

[Margin: g — Proposition 3 of rectilinear triangles.]

Aliter per solos sinus. (*Another way, by sines alone.*)

Let the squares of the two given sides about the right angle be added together; for the root of this sum will be the side opposite the right angle. Then let it be done: as the side opposite the right angle, which has now been found, is to the whole sine, so is either of the given sides about the right angle to a fourth; and there will come out the sine of the acute angle opposite the side about the right angle that was taken. From this angle, once found, the other also becomes known, since the two acute angles together are equal to one right angle.

[Margin: h — Proposition 3 of rectilinear triangles.]

TRIANGVLORVM RECTILINEORVM NON RECTANGVLORVM PROBLEMATA, ET PRAXES.

[Problems and practical procedures of non-right-angled rectilinear triangles]

6. Given the sum of two arcs or angles — which is less than 180 degrees — together with the proportion that their sines have, to exhibit each of them as known.

[Margin: What is sought — the two arcs or angles, from their sum.]

Let it be done thus: as half the sum of the terms of the given proportion which the sines of the arcs or angles have is to the tangent of half the given sum of the arcs or angles — finding the tangent by the proportional part corresponding to 30 seconds, in case the sum of the arcs or angles cannot be divided in two without seconds — so is the difference between half the sum of the terms of the given proportion and either of the terms, to a fourth. For there will be found the tangent of the arc or angle by which each arc or angle sought differs from half of the given sum of those same arcs or angles. And thus the arc or angle of this tangent, once found, added to half of the given sum of the arcs or angles, will give the greater arc or angle sought; but subtracted from that same half, it will leave the lesser arc or angle.

[Margin: a — Proposition 6 of rectilinear triangles.]

Aliter. (*Another way.*)

Let it be done thus: as half the difference between the two terms of the given proportion is to the tangent of half the difference between the given sum of the arcs or angles and a semicircle, so is the sum of [i] half the difference between the two terms of the given proportion and [ii] the consequent term of the same

proportion, to a fourth. For there will be produced the tangent of an arc or angle; and if from this arc there be subtracted half the difference between the given sum of the arcs or angles and the semicircle, the remainder will become the lesser arc or angle sought. And this one, in turn, subtracted from the given sum, will leave the greater arc or angle sought.

[Margin: b — Proposition 6 of rectilinear triangles.]

Aliter per solos sinus. (*Another way, by sines alone.*)

Let it be done thus: as half the sum of the terms of the given proportion is to the sine of half the sum of the arcs or angles, so is the difference between half the sum of the terms of the given proportion and either of the terms, to a fourth; and a certain fourth number will be found. If the square of this number be added to the square of the sine of the complement of half the sum of the arcs or angles, and again it be done — as the square root of the sum of the two said squares is to the whole sine, so is that fourth number found to a fourth — there will be produced the sine of the arc or angle by which each arc or angle differs from half of their given sum. Therefore this arc or angle, added to half of the given sum, will yield the greater arc or angle; but taken away from that same half, it will leave the lesser arc or angle.

[Margin: c — Proposition 6 of rectilinear triangles.]

But if the given proportion of the sines should happen to be one of equality, then half of the given sum of the arcs or angles will give each arc or angle.

7. Given the sum of two arcs, each of which is less than a semicircle, or of two angles — a sum which is greater than 180 degrees — together with the proportion that their sines have, to render each of them known.

[Margin: What is sought — the two arcs or angles, from their sum.]

Subtracting the given sum from 360 degrees, let each arc or angle of the remaining sum be found by Problem 6 — this remaining sum being always less than 180 degrees, while the same proportion of the sines remains. For if the two thus found are each subtracted separately from a semicircle, the remainders will be the arcs or angles sought.

[Margin: d — Proposition 6 of rectilinear triangles.]

When the given proportion is one of equality, half of the given sum will likewise give each arc or angle sought.

But if the given sum should happen to contain precisely 180 degrees, the problem cannot be solved.

8. Given the difference of two arcs, each of which is less than a semicircle, or of two angles, together with the proportion that their sines have, to make each of them known.

[Margin: What is sought — the two arcs or angles, from their difference.]

If the ratio of the sine of the greater arc (or angle) to the sine of the lesser is a ratio of greater inequality, then arrange that as the half of the difference of the terms of the given ratio is to the tangent of the half of the given difference of the arcs (or of the angles), so the sum of the half of the difference of the terms of the ratio together with the consequent term of that same ratio is to a fourth. [Margin: e — Proposition 7, plane triangles.] This will produce the tangent of an arc (or angle) which, added to the half of the given difference of the arcs (or angles), makes up the greater arc (or angle); but if it is subtracted from that same half, the remainder will be the lesser arc or angle.

If, on the other hand, the ratio of the sine of the greater arc (or angle) to the sine of the lesser is a ratio of lesser inequality, then invert its terms so that a ratio of greater inequality results. From this ratio, together with the given difference of the arcs (or angles), find the two arcs (or angles), as was said. For the greater of them, taken away from the semicircle, will give the lesser arc (or angle) sought; while the lesser, subtracted from the semicircle, will yield the greater.

Another way.

When the sine of the greater arc (or angle) has to the sine of the lesser a ratio of greater inequality, then from that given ratio of greater inequality, and from the arc (or angle) which is left after the given difference has been subtracted from the semicircle — treating this as the sum of two arcs (or angles) — seek the two arcs (or angles) of this sum, as we prescribed in Problem 6. [Margin: f — Proposition 7, plane triangles.] For the greater arc (or angle) found, if it is taken away from the semicircle, will give the greater arc or angle sought; while the lesser one found will be the lesser sought.

But when the given ratio is one of lesser inequality, invert its terms so that a ratio of greater inequality results. From this ratio, together with the given difference of the arcs (or angles), investigate the two arcs (or angles), as has already been said. For the greater of them, subtracted from the semicircle, will give the lesser arc (or angle) sought; while the lesser will give the greater.

Another way, by the sines alone.

If the given ratio of the sine of the greater arc (or angle) to the sine of the lesser is a ratio of greater inequality, then arrange that as the half of the difference of the terms of the given ratio is to the sine of the half of the given difference of the arcs (or angles), so the sum of the half of the difference of the terms of the ratio together with the consequent term of that same ratio is to a fourth. [Margin: g — Proposition 7, plane triangles.] A certain fourth number will be found; and let the square of this number be added to the square of the sine of the complement of the half of the given difference of the arcs (or angles). Then again arrange that as the square root of the sum of the said squares is to the whole sine, so that fourth number found is to a fourth: the sine of the arc (or angle) will be found, to which if the half of the given difference of the arcs (or angles) is added, the greater arc (or angle) becomes known; but if from it that same half is taken away, the remainder will be the lesser.

But if the given ratio is one of lesser inequality, one must proceed as we said above.

Now if by chance the given ratio is one of equality, let the given difference of the arcs (or angles) be subtracted from the semicircle. For the half of the remainder will be the lesser arc (or angle) sought; and that same half, added to the given difference, will give the greater.

9. If from any one angle of any triangle whose sides are given a perpendicular is dropped to the opposite side, to find how great is the straight line lying between the perpendicular and either of the remaining angles.

[Margin: The fall of the perpendicular line is sought.]

Let the difference between the squares of the two sides enclosing the angle from which the perpendicular has been dropped be divided by the third side, and a number is produced. If this number is less than the third side, [Margin: h — Proposition 9, plane triangles.] it will indicate that the perpendicular has fallen inside the triangle; and the same number, subtracted from that same third side, will leave a number whose half gives the lesser segment of the base, while this lesser segment, subtracted from the whole third side, gives the greater segment. But if the same number produced by the division is greater than the third side, it will be an argument that the perpendicular has fallen outside the triangle. Therefore, if the third side is taken away from it, the remainder will be a number whose half gives the straight line outside the triangle between the perpendicular

and the obtuse angle; while that same half, added to the third side, gives the other straight line between the perpendicular and the acute angle.

Another way, and easier.

Arrange that as the third side, onto which the perpendicular has been dropped, is to the sum of the other two sides, so the difference of those same two sides is to a fourth; and a number results, [Margin: i — Proposition 9, plane triangles.] from which we shall find the straight line between the perpendicular and either angle, as we said just now.

Another way.

When the perpendicular falls inside the triangle, let the half of the difference between the square of either of the sides enclosing the angle from which the perpendicular has been dropped and the sum of the squares described on the other two sides be divided by the side onto which the perpendicular falls.

[Margin: k — Scholium to Proposition 9, plane triangles.] This produces the segment of the base near the angle contained by the two sides whose sum of squares was taken; and this segment, subtracted from the whole base, will leave the other segment.

But when the perpendicular falls outside the triangle, let the half of the difference between the square of the side opposite the obtuse angle and the sum of the squares described on the other two sides be divided by the side onto whose extension the perpendicular falls. This will produce the line outside the triangle between the perpendicular and the obtuse angle; and this line, added to the whole base, will make up the other straight line between the perpendicular and the acute angle of the base.

And if the two sides about the perpendicular are equal, [Margin: l — Corollary to Proposition 8, plane triangles.] the perpendicular will cut the base into two equal parts. Therefore half the base will give each of the straight lines sought.

10. Given all the angles of a non-right triangle, together with one side, to find the other two sides.

[Margin: Two sides are sought.]

Arrange that as the sine of the angle opposite the given side is to the sine of either of the remaining angles, so the given side is to a fourth, [Margin: m — Proposition 10, plane triangles.] and the side opposite this latter angle will be found. Arrange again that as the sine of the angle opposite the given side is to the sine of the third

angle, so the given side is to a fourth, and the third side, opposite this third angle, will be produced.

If the triangle is isosceles, the finding of only one side is needed, provided one side is given together with the angles. The same is to be said of a scalene triangle, if two of its sides are given together with the angles. But in an equilateral triangle, if one side is given, the rest will also be given, being equal to it.

11. Given all the sides of a non-right triangle, to find all its angles.

[Margin: The angles are sought.]

When a perpendicular has been drawn to the greatest side from the opposite angle (so that the perpendicular always falls inside the triangle), [Margin: n — Proposition 11, plane triangles.] let there be found, by the preceding problem, the straight lines lying between the perpendicular and the two angles at the greatest side. Then arrange that as the least side is to the whole sine, so the lesser segment of the base is to a fourth, and a sine is produced whose arc's complement gives the base angle adjacent to the least side. Again arrange that as the middle side is to the whole sine, so the greater segment of the base is to a fourth, and a sine is produced whose arc's complement gives the base angle adjacent to the middle side. The third angle, opposite the greatest side, will be composed of the two arcs of the two sines found; or else it will be left over after the subtraction of the two angles found from two right angles.

If the triangle is isosceles, [when] the perpendicular is drawn it will be perpendicular to the base, which it will bisect. [?] For if then one arranges that as one of the equal sides is to the whole sine, so half the base is to a fourth, a sine will be found whose arc's complement gives one of the equal angles above the base, and consequently the other as well. The third angle will be elicited from these two.

In an equilateral triangle the angles will be given, even if the sides are not given, since each one is a third part of two right angles, or two-thirds of one right angle.

12. Given two sides of a non-right triangle, together with the angle contained by them, to find the third side and the remaining angles.

[Margin: A side is sought, together with its two angles.]

When the given angle has been subtracted from two right angles, so that the sum of the other two is obtained, let each of them be found, [Margin: o — Proposition 12, plane triangles.] by the sixth problem of plane triangles, from this sum

together with the ratio of the given sides opposite them (which is the same as that found between their sines). Then arrange that as the sine of either of these angles found is to the sine of the angle given at the outset, so the side opposite the angle found — the one that was taken in the rule of three — is to a fourth, and the third side will be found.

But if the two given sides are equal, then, the given angle having been taken away from two right angles, the half of the remainder will give each of the equal angles. And if one arranges that as the sine of one of them is to the sine of the given angle, so one of the equal sides is to a fourth, the third side will come out.

13. Given two sides of a non-right triangle, together with the angle that is opposite one of them, to investigate the remaining angles and the third side: provided only, when the given angle is acute, that it be established whether the angle opposite the other given side is likewise acute, or rather obtuse.

[Margin: Two angles are sought, together with one side.]

Arrange that as the side opposite the given angle is to the other given side, so the sine of the given angle is to a fourth, [Margin: p — Proposition 13, plane triangles.] and the sine of the angle opposite the other given side is found. If this angle is acute (and it will always be acute if the given angle is obtuse), it will become known from the found sine itself; but if it is obtuse, the found sine will give an angle which, subtracted from two right angles, will leave the sought angle opposite the other given side. Moreover, the sum composed of the given angle and the found angle, if it is subtracted from two right angles, will indicate the third angle contained by the given sides. Then arrange that as the sine of the given angle is to the sine of this third angle found, so the side opposite the given angle is to a fourth, and the third side sought is produced.

If the given sides are equal, the angle opposite the other given side will also be given, since it is equal to the given angle. Hence the third angle and the third side will be found, as before.

TRIANGVLORVM SPHÆRICORVM RECTILINEORVM RECTANGVLORVM PROBLEMATA, AC PRAXES.

Problems and Praxes of Right-Angled Spherical Triangles.

1. Given an arc in a right-angled triangle which is opposite the right angle, together with either of the arcs about the right angle, to find the angle opposite this arc.

[Margin: A non-right angle is sought.]

Arrange that as the sine of the given arc opposite the right angle is to the whole sine, so the sine of the given arc about the right angle is to a fourth, [Margin: a — Problem 1, Proposition 41, spherical triangles.] and the sine of the angle opposite this arc, which is sought, will be found. This angle will be acute if the given arc opposite to it about the right angle is less than a quadrant; but obtuse, if greater.

Another way.

Arrange that as the whole sine is to the sine of the arc opposite the right angle, so the secant of the complement of the given arc about the right angle is to a fourth, [Margin: b — Problem, Proposition 55, spherical triangles.] and the secant of the complement of the angle sought, which is opposite this arc, is produced.

2. Given an arc in a right-angled triangle which is opposite the right angle, together with either of the non-right angles, to find the arc opposite this angle.

[Margin: An arc about the right angle is sought.]

Arrange that as the whole sine is to the sine of the arc opposite the right angle, so the sine of the given angle is to a fourth, [Margin: c — Problem 2, Proposition 41, spherical triangles.] and the sine of the arc opposite this angle, which is sought, will be found. This arc will be less than a quadrant if the given angle opposite to it is acute; but greater, if obtuse.

3. Given either of the arcs about the right angle in a right-angled triangle, together with the angle opposite it, to find the arc opposite the right angle: provided only it be established whether it is less than a quadrant or greater; or whether the other angle adjacent to the given arc is acute or obtuse; or, finally, whether the other arc about the right angle is less than a quadrant or greater.

[Margin: The arc opposite the right angle is sought.]

Arrange that as the sine of the given angle is to the sine of the given arc, so the whole sine is to a fourth, [Margin: d — Problem 3, Proposition 41, spherical triangles.] and the sine of the arc opposite the right angle is produced. This arc

cannot be known from the found sine unless it is established whether it is less than a quadrant or greater; or whether the other, non-right, angle is acute or obtuse; or whether the other arc about the right angle is less than or greater than a quadrant. For if the other angle is acute — provided the given angle too is acute; or if both that one and this are obtuse — the sought arc opposite the right angle will be less than a quadrant. But if that other angle is acute and the given one obtuse, or that one obtuse and this one acute, the same sought arc will be greater than a quadrant. Likewise, if the other arc about the right angle and the given arc are of the same kind — namely both less or both greater than a quadrant — the sought arc opposite the right angle will be less than a quadrant; but if of different kinds — namely one less than a quadrant and the other greater — the same sought arc will be greater than a quadrant.

Another way.

Arrange that as the whole sine is to the sine of the given angle, so the secant of the complement of the given arc is to a fourth, [Margin: e — Problem, Proposition 54, spherical triangles.] and the secant of the complement of the arc opposite the right angle is produced.

4. Given two non-right angles in a right-angled triangle, to find the arc opposite either of them, together with the arc subtending the right angle.

[Margin: Both arcs about the right angle are sought; then the arc opposite the right angle.]

Arrange that as the sine of the given angle adjacent to the sought arc is to the whole sine, so the sine of the complement of the other given angle is to a fourth, [Margin: f — Problem 1, Proposition 42, spherical triangles.] and the sine of the complement of the arc opposite this latter angle is produced. Either arc found will be less than a quadrant if the given angle opposite to it is acute; but greater, if obtuse.

Now, once both arcs about the right angle have been found, the arc opposite the right angle will also be found, by Problem 3, from either of them together with the angle given opposite to it.

Another way.

Arrange that as the whole sine is to the sine of the non-right angle adjacent to the sought arc, so the secant of the other non-right angle is to a fourth, [Margin: g —

Problem, Proposition 52, spherical triangles.] and the secant of the arc opposite this latter angle, which is sought, will be found.

5. Given either of the arcs about the right angle in a right-angled triangle, together with the angle adjacent to it, to investigate the other angle opposite that same arc, and the remaining two arcs.

(Continuation of the recipe for Praxis 5, broken off at the catchword "FIAT," on the previous page.)

Set it up so that as the whole sine is to the sine of the given angle, so the sine of the complement of the given arc is to a fourth term [Margin: h — Problem 2, proposition 42, of the spherical triangles]; and there will be produced the sine of the complement of the other angle, which we are seeking. This angle will be acute if the given arc is less than a quadrant, but obtuse if greater.

Then, from the two non-right angles — one given, the other now found — the remaining two arcs will be found, as was stated in the preceding problem.

[Margin: Sought: a non-right angle; then the other two arcs.]

6. Given either of the arcs about the right angle in a right-angled triangle, together with the angle opposite it, to investigate the other non-right angle adjacent to that same arc, and the two remaining arcs — provided only that it is established whether that other non-right angle sought is acute or obtuse, and whether either of the arcs sought is less than a quadrant or greater.

Set it up so that as the sine of the complement of the given arc is to the sine of the complement of the given angle, so the whole sine is to a fourth term [Margin: i — Problem 2, proposition 42, of the spherical triangles]; and there will be found the sine of the other non-right angle sought. This angle will not be drawn out from the sine so found unless it is first established whether it is acute or obtuse, and whether either of the remaining two arcs (the ones not given) is less than a quadrant or greater. For if the other arc about the right angle — the one not given and opposite the angle sought — is less than a quadrant, the angle sought will be acute; but if it is greater, obtuse. By the same reasoning, if the arc opposite the right angle, and not given, is less than a quadrant, then, provided the given angle is acute, the angle sought will likewise be acute; but if the given angle is obtuse, obtuse. But if the arc opposite the right angle is greater than a quadrant, then, provided the given angle is acute, the angle sought will be obtuse; but if it is obtuse, acute.

Then, from the two non-right angles — one given, the other now found — the remaining two arcs will be found, as was shown in problem 4.

Another way.

Set it up so that as the whole sine is to the sine of the complement of the given arc, so the secant of the given angle is to a fourth term [Margin: k — the problem of proposition 56 of the spherical triangles]; and there will be found the secant of the complement of the other non-right angle, which is sought. The rest will be found as stated above.

[Margin: Sought: the arc opposite the right angle; then the two non-right angles.]

7. Given two arcs about the right angle in a right-angled triangle, to find the third arc opposite the right angle, and the two non-right angles.

Set it up so that as the whole sine is to the sine of the complement of either of the given arcs, so the sine of the complement of the other given arc is to a fourth term [Margin: l — the problem of proposition 43 of the spherical triangles]; and there will be produced the sine of the complement of the arc opposite the right angle. This arc will be less than a quadrant if both arcs about the right angle that are given are less than a quadrant, or both greater; but greater than a quadrant if one of the given arcs is less than a quadrant and the other greater.

Then, from the arc subtending the right angle now found, together with either of the given arcs about the right angle, the angle opposite it will be found, as we stated in problem 1.

[Margin: Sought: an arc about the right angle; then the two non-right angles.]

8. Given the arc in a right-angled triangle that is opposite the right angle, together with either of the arcs about the right angle, to inquire the other arc about the right angle, and the two non-right angles.

Set it up so that as the sine of the complement of the given arc about the right angle is to the sine of the complement of the arc opposite the right angle, so the whole sine is to a fourth term [Margin: m — problem, proposition 43, of the spherical triangles]; and there will be generated the sine of the complement of the other arc about the right angle, which is sought. This arc will be less than a quadrant if each given arc is less than a quadrant, or each greater; but greater if one of the given arcs is less than a quadrant and the other greater.

Once the other arc about the right angle has been found, the angles will be found, as was stated in the preceding problem.

Another way.

[Margin: n — the problem of proposition 53 of the spherical triangles.]

Set it up so that as the whole sine is to the sine of the complement of the given arc about the right angle, so the secant of the arc opposite the right angle is to a fourth term; and there will be produced the secant of the third arc, which is sought, etc.

[Margin: Sought: an arc about the right angle; then the other non-right angle, and the arc opposite the right angle.]

9. Given either of the arcs about the right angle in a right-angled triangle, together with the non-right angle adjacent to it, to search out the other arc about the right angle, and another non-right angle, together with the arc subtending the right angle.

Set it up so that as the whole sine is to the sine of the given arc, so the tangent of the given angle is to a fourth term [Margin: o — Problem 1; p — proposition 44, of the spherical triangles]; and there will be produced the tangent of the arc sought. This arc will be less than a quadrant if the given angle opposite it is acute, but greater if it is obtuse.

Furthermore, from the same given arc about the right angle, together with the adjacent angle, the other non-right angle and the arc opposite the right angle will both be found, as we taught above in problem 5.

[Margin: Sought: an arc about the right angle; then the other non-right angle, and the arc opposite the right angle.]

10. Given either of the arcs about the right angle in a right-angled triangle, together with the angle opposite it, to trace out the other arc about the right angle, and another non-right angle, together with the arc subtending the right angle — provided only that it is established whether the remaining arc about the right angle that is sought is less than a quadrant or greater; or whether the other non-right angle is acute or obtuse; or, finally, whether the arc opposite the right angle is less than a quadrant or greater.

Set it up so that as the tangent of the given angle is to the tangent of the given arc, so the whole sine is to a fourth term [Margin: a — Problem 1, proposition 44, of the spherical triangles]; and there will be found the sine of the arc sought. This arc will not be recognized from the sine so found unless it is established whether

it is less than a quadrant or greater; or whether the other non-right angle is acute or obtuse; or, finally, whether the arc opposite the right angle is less than a quadrant or greater. For if the other angle is acute, the arc sought opposite it will be less than a quadrant; but if obtuse, greater. So too, if the arc opposite the right angle is less than a quadrant: provided the given arc is also less than a quadrant, the arc sought will likewise be less than a quadrant; but if greater than a quadrant, greater as well. But if the arc opposite the right angle is greater than a quadrant: provided the given arc is also greater, the arc sought will be less than a quadrant; but if it is less than a quadrant, greater.

Now indeed, from the same given arc about the right angle and the angle opposite, the other non-right angle and the arc opposite the right angle will be found, as was shown in problem 6. Or, alternatively, from the two arcs about the right angle — one given, the other now found — the arc opposite the right angle will be found, together with the two non-right angles, as was shown in problem 7.

Another way.

[Margin: b — problem, proposition 49, of the spherical triangles.]

[Margin: Sought: each non-right angle; then the arc opposite the right angle.]

Set it up so that as the whole sine is to the tangent of the given arc, so the tangent of the complement of the given angle is to a fourth term; and there will be found the sine of the arc sought. The rest will be found, as we prescribed just now.

11. Given two arcs about the right angle in a right-angled triangle, to find either of the non-right angles, and, besides, the arc opposite the right angle.

Set it up so that as the sine of either of the given arcs is to the whole sine, so the tangent of the other given arc is to a fourth term [Margin: a — Problem 2, proposition 44, of the spherical triangles]; and there will be brought forth the tangent of the angle opposite this latter arc. This angle will be acute if the given arc opposite it is less than a quadrant, but obtuse if greater.

From the same two given arcs the third arc opposite the right angle will be found, by problem 7; or, alternatively, by problem 3, from either of the given arcs and the angle opposite it, once found.

Another way.

[Margin: b — the problem of proposition 48 of the spherical triangles.]

[Margin: Sought: the arc opposite the right angle; then the other arc about the right angle, together with the other non-right angle.]

Set it up so that as the whole sine is to the sine of either of the given arcs, so the tangent of the complement of the other given arc is to a fourth term; and there will come forth the tangent of the complement of the angle opposite this latter arc. The rest will be found, as was said just now.

12. Given either of the arcs about the right angle in a right-angled triangle, together with the non-right angle adjacent to it, to find the arc opposite the right angle, and the remaining arc about the right angle, together with the other non-right angle.

Set it up so that as the sine of the complement of the given angle is to the whole sine, so the tangent of the given arc is to a fourth term [Margin: a — Problem 1, proposition 45, of the spherical triangles]; and there will be found the tangent of the arc opposite the right angle. This arc will be less than a quadrant if the given arc adjacent to it is less than a quadrant, or if the given angle is obtuse and the given arc greater than a quadrant. But the same arc sought will be greater than a quadrant if the given angle is acute and the given arc greater than a quadrant, or if the given angle is obtuse and the given arc less than a quadrant.

Now indeed, by problem 2, from the arc subtending the right angle now found and the given angle, the other arc about the right angle, opposite the given angle, will be found. And from the same arc subtending the right angle, together with the arc given at the start, the other non-right angle, opposite the given arc, will be found, by problem 1.

Another way.

[Margin: b — the problem of proposition 46 of the spherical triangles.]

[Margin: Sought: a non-right angle; then the other arc about the right angle, and the other non-right angle.]

Set it up so that as the whole sine is to the sine of the complement of the given angle, so the tangent of the complement of the given arc is to a fourth term; and there will be found the tangent of the complement of the arc opposite the right angle. The rest will be found, as before.

13. Given either of the arcs about the right angle in a right-angled triangle, together with the arc subtending the right angle, to find the angle contained by the said arcs — that is, the one adjacent to the given

arc about the right angle — and, in addition, the remaining arc and angle.

Set it up so that as the tangent of the arc opposite the right angle is to the tangent of the given arc about the right angle, so the whole sine is to a fourth term [Margin: a — Problem 2, proposition 45, of the spherical triangles]; and there will be produced the sine of the complement of the angle contained by the said arcs, which is sought. This angle will be acute if the given arc opposite the right angle is less than a quadrant and the given arc about the right angle is also less; or if both are greater than a quadrant. But the same angle sought will be obtuse if the given arc opposite the right angle is less than a quadrant and the given arc about the right angle is greater than a quadrant; or if the former is greater than a quadrant and the latter less.

The rest will be investigated, as was shown in the preceding problem.

Another way.

[Margin: b — the problem of proposition 51 of the spherical triangles.]

[Margin: Sought: an arc about the right angle; then the other arc about the right angle, together with the remaining non-right angle.]

Set it up so that as the whole sine is to the tangent of the complement of the arc opposite the right angle, so the tangent of the given arc about the right angle is to a fourth term; and there will be found the sine of the complement of the adjacent angle, which is desired.

14. Given the arc subtending the right angle in a right-angled triangle, together with either of the non-right angles, to find the arc about the right angle adjacent to this angle, and, besides, the other arc about the right angle, together with the other non-right angle.

Set it up so that as the whole sine is to the sine of the complement of the given angle, so the tangent of the arc opposite the right angle is to a fourth term [Margin: a — Problem 3, proposition 45, of the spherical triangles]; and there will be produced the tangent of the arc sought. This arc will be less than a quadrant if the given arc opposite the right angle is less than a quadrant and the given angle acute; or if the given arc is greater than a quadrant and the given angle obtuse. But the same arc sought will be greater than a quadrant if the given arc opposite the right angle is less than a quadrant and the given angle obtuse; or if the given arc is greater than a quadrant and the given angle acute.

The remaining matters will be explored, as we taught in problem 12.

[Margin: Sought: a non-right angle; then the two remaining arcs.]

15. Given the arc in a right-angled triangle that is opposite the right angle, together with either of the non-right angles, to inquire the other non-right angle, and the two arcs about the right angle.

Set it up so that as the whole sine is to the sine of the complement of the given arc opposite the right angle, so the tangent of the given angle is to a fourth term [Margin: a — problem, proposition 47, of the spherical triangles]; and there will be found the tangent of the complement of the angle sought. This will be acute if the arc opposite the right angle is less than a quadrant and the given angle acute; or if the given arc is greater than a quadrant and the given angle acute; or if the given arc is greater than a quadrant and the given angle obtuse. But the same angle sought will be obtuse if the arc opposite the right angle is less than a quadrant and the given angle obtuse; or if that arc is greater than a quadrant and the given angle acute.

Hence, from the given arc subtending the right angle, and the two non-right angles — one given, the other now found — each arc about the right angle will be found, by problem 2.

[Margin: Sought: the arc opposite the right angle; then the two arcs about the right angle.]

16. Given two non-right angles in a right-angled triangle, to find the arc opposite the right angle, and the remaining two arcs about the right angle.

Set it up so that as the whole sine is to the tangent of the complement of either of the given angles, so the tangent of the complement of the other given angle is to a fourth term [Margin: a — problem, proposition 50, of the spherical triangles]; and there will be brought forth the sine of the complement of the arc opposite the right angle, which we desire. This arc will be less than a quadrant if each of the given angles is acute, or each obtuse; but greater than a quadrant if one is acute and the other obtuse.

Furthermore, from the arc subtending the right angle now found, and either of the given angles, the arc opposite it will be found, as was shown in problem 2.

PROBLEMS AND PRAXES OF OBLIQUE-ANGLED SPHERICAL TRIANGLES

[Margin: Sought: all the arcs.]

17. Given all the angles of a non-right-angled triangle, to find all its arcs.

[Margin: When all the given angles are unequal.]

Diagram. A spherical triangle drawn with curved (arc) sides, apex uppermost, its base at the bottom, with a perpendicular arc dropped from the apex to a point on the base, dividing the triangle into two right-angled sub-triangles.

- A (apex, top): vertex A
- B (lower left): vertex B
- C (lower right, just above and to the right of D): vertex C
- D (on the base between B and C): foot of the perpendicular arc AD dropped from A to the base BC

Let all the angles first be given in the triangle ABC as unequal, two of which — B and C — are acute (or obtuse); and from the third angle, A, let the perpendicular arc AD be drawn to BC [Margin: a — proposition 62 of the spherical triangles], which will fall inside the triangle. Let the sines of the complements of the angles B and C be set down as the terms of the proportion of the sine of angle BAD to the sine of angle CAD. And from this proportion, together with the sum of the angles BAD and CAD — that is, from the given angle BAC — let each of the angles BAD and CAD be inquired, by problem 6 of the plane triangles. Then, by problem 16 of the spherical triangles, from the two non-right angles B and BAD let the arc AB, opposite the right angle D in the triangle ABD, be investigated; and likewise, from the two non-right angles C and CAD, the arc AC, opposite the right angle D in the triangle ACD. Lastly, by problem 2, from the arc AB subtending the right angle D and the angle BAD, both now found, let the arc BD be found; and likewise, from the arc AC subtending the right angle D and the angle CAD, both found, the arc CD. For the sum of the arcs BD and CD will make the whole arc BC known. And thus all three arcs — AB, AC, BC — will be made known.

[Margin: By the sines alone, when all the given angles are unequal.]

By the sines alone we will solve the problem thus. Let each of the angles BAD and CAD be found by praxis 3 of problem 6 of the plane triangles. Then, by praxis 1 of problem 4 of the spherical triangles, from the two angles B and BAD let the arc BD

be investigated, and from the two angles C and CAD the arc CD. For the sum of the arcs BD and CD will make the whole arc BC known. Lastly, by problem 3 of the spherical triangles, let the arc AB opposite the right angle D be found from the arc BD and the angle opposite it, BAD, both now found; and likewise the arc AC opposite the right angle D, from the arc CD and the angle CAD opposite it, both found. For, besides the data, the species of both the one angle B and the other angle C is also established, since each is given.

But if at any time one of the angles at A — namely BAD — turns out to be a right angle, the two arcs AB and BD will be found, since each is a quadrant, on account of the right angles D and DAB. By the same reasoning, if the angle CAD is discovered to be a right angle, but not BAD — for it cannot happen that both angles at A be right, since the whole angle BAC is less than two right angles — the two arcs AC and CD will be found, being quadrants, on account of the right angles D and DAC.

(Continuation of the solution begun on the previous page — Problem 17, the oblique spherical triangle solved when at least two of its angles are equal.)

Next, let at least two of the given angles, B and C, be equal, whatever the third angle A may be, and from A let the perpendicular arc AD be drawn to BC. Then the two arcs AB and AC, and the two arcs BD and CD, and the two angles at A will all be equal; and consequently each angle at A is known, being half the given angle BAC. Therefore, by Problem 16 of spherical triangles, from the two angles B and BAD find the arc AB opposite the right angle D; and AC, being equal to it, is likewise known. Next, by Problem 14 of spherical triangles, from the found arc AB (which subtends the right angle) and the given angle B, find the arc BD; and CD, being equal to it, is likewise known; and therefore the whole BC is known. And now all three arcs AB, AC, BC will have been found.

By sines alone we shall carry out the matter thus. By Praxis 1 of Problem 4 of spherical triangles, seek the arc BD from the two angles B and BAD; and therefore CD, being equal to it, is known, and consequently the whole BC is known. Then, by Problem 3 of spherical triangles, from the found arc BD and its opposite angle BAD, find the arc AB opposite the right angle — since, besides the data, the species of the other angle B is also settled, it being given — and therefore the arc AC, being equal to AB, is known.

[Margin: When the two given angles are equal.] [Margin: By sines alone, when the two given angles are equal.]

Diagram. An oblique spherical triangle with curved sides, drawn for the case of two equal base angles.

- A (apex, top): vertex of the equal angles' summit
- B (lower left): base vertex
- D (base, between B and C): foot of the perpendicular arc AD dropped from A
- C (lower right): base vertex

18. DATIS omnibus arcubus trianguli non rectanguli, investigare omnes eius angulos.

18. Given all the arcs of a non-right triangle, to find all its angles.

[Margin: All the angles are sought.]

[Margin: (note a) When the two given arcs are unequal and less than a quadrant. From Proposition 63 of spherical triangles.]

Let all the arcs in triangle ABC be given, and let the angle A be sought first; and let the two arcs AB, AC that contain it be unequal and less than a quadrant, whatever the arc BC may be. Extend the arcs AB, AC so that AD, AE become quadrants, and through D, E describe the arc of a great circle DE, meeting the arc BC — produced toward the greater arc, which is AC — at the point F. Set the sines of the complements of the given arcs AB, AC as the terms of the proportion: the sine of arc BF to the sine of arc CF. From this proportion, together with the given arc BC (which is the difference of the arcs BF, CF), find each of the arcs BF, CF by Problem 8 of plane triangles. Then, by Problem 8 of spherical triangles, find the arc DF from the found arc BF (subtending the right angle D) and the arc BD (which is the complement of the given arc AB); and likewise the arc EF from the found arc CF (subtending the right angle E) and the arc CE (which is the complement of the given arc AC). For, subtracting the found arc EF from the found arc DF, the arc DE of the angle A remains known; and therefore the angle A will be known. After this, by Problem 11 of spherical triangles, from the known arcs BD, DF about the right angle D, find the angle DBF, and hence the remaining one of the two right angles, ABC. Finally, by the same reasoning, from the arcs CE, EF known about the right angle E, obtain the angle ECF, and thus also the angle ACB, which is equal to it at the vertex. And thus now all three angles A, B, C will have been found.

By sines alone we shall proceed thus. Each arc BF, CF is found by Praxis 3 of Problem 8 of plane triangles. Then, by Praxis 1 of Problem 8 of spherical

triangles, find the arc DF from the found arc BF (subtending the right angle D) and the arc BD (the complement of the given arc AB), and likewise the arc EF from the found arc CF (subtending the right angle E) and the arc CE (the complement of the given arc AC). For, subtracting arc EF from arc DF, the arc DE of the angle A is left known; and thus the angle A will be known. After this, by Praxis 1 of Problem 1 of spherical triangles, from the found arc BF (subtending the right angle D) and the found arc DF, seek the angle DBF opposite the arc DF: from which the remaining angle of the two right angles, namely ABC, also becomes known. Finally, by the same reasoning, from the found arc CF (subtending the right angle E) and the found arc EF, find the angle ECF opposite the arc EF: from which becomes known also the angle ACB equal to it at the vertex.

[Margin: By sines alone, when the two given arcs are unequal and less than a quadrant.]

Diagram. The quadrant construction for the first case of Problem 18: the sides are produced to quadrants and a great-circle arc DE is drawn cutting the produced base at F.

- A (apex, top): common vertex of the arcs
- B (left, on the base): base vertex
- C (on the base, between B and F): base vertex
- F (right): point where the great-circle arc DE meets BC produced
- D (lower left): endpoint of the quadrant on side AB produced
- E (lower middle): endpoint of the quadrant on side AC produced

Next, let the two unequal arcs AB, AC be greater than a quadrant; extend them until they meet at D. Then in triangle DBC the two arcs DB, DC will be unequal and less than a quadrant. Hence, as we said just now, all its three angles will be found; and consequently the remaining ones of the two right angles, ABC and ACB, will be known, as well as A, which is equal to D.

[Margin: When the two given arcs are unequal and greater than a quadrant.]

Diagram. A lens-shaped (biangular) figure for the case where both sides exceed a quadrant and, produced, meet at D.

- A (top): upper meeting point of the two arcs
- B (left): base vertex
- C (right): base vertex

- D (bottom): point where the produced arcs AB, AC meet

Third, let the arc AB be less than a quadrant, and AC greater than a quadrant. Extend AB so that AD becomes a quadrant, and cut off from AC the quadrant AE, and through D, E draw the arc of a great circle DE, cutting BC at F, as in the first of these two figures. Set the sines of the complements of the given arcs AB, AC as the terms of the proportion: the sine of arc BF to the sine of arc CF; and from this proportion, together with the sum of the arcs BF, CF — that is, from the given arc BC — find each of the arcs BF, CF by Problem 6 of plane triangles. Then, by Problem 8 of spherical triangles, find the arc DF from the found arc BF (subtending the right angle D) and the arc BD (the complement of the given arc AB); and the arc EF from the found arc CF (subtending the right angle E) and the arc CE (the complement of the given arc AC). For the sum of the found arcs DF, EF will give the whole arc DE of the angle A; and therefore the angle A will be known. After this, by Problem 11 of spherical triangles, from the arcs DB, DF known about the right angle D, work out the angle DBF, and hence the remaining one of the two right angles, ABC. And finally, in the same way, from the arcs CE, EF known about the right angle E, obtain the angle C: and all three angles A, B, C will have been found.

By sines alone the work is to be done thus. Each arc BF, CF is found by Praxis 3 of Problem 6 of plane triangles. Then, by Praxis 1 of Problem 8 of spherical triangles, let both be obtained: the arc DF, from the found arc BF (subtending the right angle D) and the arc BD (the complement of the given arc AB); and the arc EF, from the found arc CF (which is opposite the right angle E) and the arc CE (the complement of the given arc AC). For the sum of the found arcs DF, EF will give the whole arc DE of the angle A. After this, by Problem 1 of spherical triangles, from the known arc BF (subtending the right angle D) and the arc DF, find the angle DBF, and hence the remaining one of the two right angles, ABC. And finally, in the same way, from the known arcs CF, EF, find the angle C.

[Margin: When the two given arcs are unequal, and one is greater than a quadrant and the other less.] [Margin: By sines alone, when the two given arcs are unequal, and one is greater than a quadrant and the other less.]

Diagram. The left of a pair of figures at the foot of the page, for the case where AB is less than a quadrant and AC greater. The figures are crowded and overlapping.

- A (apex, top): common vertex
- B (upper left): base vertex

- C (right): base vertex
- D (lower left): endpoint of the quadrant AD
- E (interior, middle): endpoint of the quadrant AE
- F (lower middle): intersection of DE with BC

Diagram. The right of the pair of figures at the foot of the page, showing an interior great-circle construction (this figure is referred to on the next page as "the latter of the two nearby figures").

- A (apex, top): common vertex
- B (lower middle): base vertex
- C (upper right): base vertex
- D (lower right): endpoint of the quadrant construction
- E (interior, middle): intersection point of the interior arcs

Fourth, let the greater arc AB be a quadrant, and AC less than a quadrant, as in the latter of the two nearby figures. Extend the arc AC so that AD becomes a quadrant, and through B, D draw the arc of a great circle BD. Then, by Problem 8 of spherical triangles, from the given arc BC (subtending the right angle D) and the arc CD (the complement of the given arc AC), find the arc BD of the angle A; from which the angle A itself will be known. After this, by Problem 11 of spherical triangles, from the two known arcs BD, CD about the right angle D, find the angle BCD; from which the remaining one of the two right angles, ACB, will also be known. Finally, by the same Problem 11, from the same arcs BD, CD, find the angle CBD; which, subtracted from the right angle ABD, leaves the angle ABC known.

By sines alone we shall proceed thus. By Praxis 1 of Problem 8 of spherical triangles, from the given arc BC (subtending the right angle D) and the arc CD (the complement of the given arc AC), find BD, the arc of the angle A: from which the angle A itself will be known. Then, from the known arcs BC, BD, by Problem 1 of spherical triangles, obtain the angle BCD: and hence the remaining one of the two right angles, ACB. Finally, by the same reasoning, from the known arcs BC, CD, seek the angle CBD, which, subtracted from the right angle ABD, leaves the angle ABC known.

[Margin: When the two given arcs are unequal, and the greater arc is a quadrant while the lesser is less than a quadrant.] [Margin: By sines alone, when the greater arc is a quadrant.]

Fifth and last, let the greater arc AB be greater than a quadrant, and the lesser arc AC be a quadrant, as in the same latter of the two nearby figures. Cut off the quadrant AE from AB, and through C, E draw the arc of a great circle CE. Then, by Problem 8 of spherical triangles, from the given arc BC (subtending the right angle E) and the arc BE (the complement of the given arc AB), find the arc CE of the angle A; from which the angle A itself will be known. After this, by Problem 11 of spherical triangles, from the two known arcs BE, EC about the right angle E, obtain the angle BCE; to which, if the right angle ACE is added, the whole angle ACB becomes known. Finally, by the same reasoning, from the same arcs BE, EC, find the angle EBC.

By sines alone we shall carry out the proposition thus. By Praxis 1 of Problem 8 of spherical triangles, from the given arc BC (subtending the right angle E) and the arc BE (the complement of the given arc AB), seek the arc CE of the angle A: and thus the angle A becomes known. Then, by Problem 1 of spherical triangles, from the known arcs BC, BE, find the angle BCE; to which, if the right angle ACE is added, the whole ACB will be known. Finally, by like reasoning, from the known arcs BC, CE, find the angle CBE.

[Margin: When the greater given arc is greater than a quadrant and the lesser is a quadrant.] [Margin: By sines alone, when the greater given arc is greater than a quadrant and the lesser is a quadrant.]

ALITER, & facilius, per solos sinus, quando duo arcus quaesitum angulum comprehendentes sunt inaequales quomodocunque.

ALTERNATIVELY, and more easily, by sines alone, when the two arcs containing the sought angle are unequal in any way whatever.

Arrange it so that, as the whole sine is to the sine of either of the unequal arcs containing the sought angle, so is the sine of the other arc about the same angle to a fourth term; and a certain fourth number will be found. Then again, arrange it so that, as that found fourth number is to the whole sine, so is the difference between the versed sine of the arc opposite the sought angle and the versed sine of the arc by which the two arcs surrounding the sought angle differ from one another, to another term; and the versed sine of the sought angle will be produced: from which the angle itself will be obtained.

In the same way the other two angles will be found, if the arcs containing them are unequal.

[Margin: An easier and more general praxis, by sines alone, when the two arcs containing the sought angle are unequal.]

Now let the two arcs AB, AC containing the sought angle A be equal. The perpendicular arc AD will bisect both the angle A and the base BC. Therefore, by Problem 1 of spherical triangles, from the given arc AB (subtending the right angle D) and the arc BD (half the given arc BC), find the angle BAD, which, doubled, gives the whole angle BAC. Then, by Problem 13 of spherical triangles, from the known arcs AB, BD, find the angle B, to which the angle C is equal (on account of the equal arcs AB, AC), and which is therefore also known.

By sines alone we shall do this. By Praxis 1 of Problem 1 of spherical triangles, from the given arc AB (subtending the right angle D) and the arc BD (half the given arc BC), find the angle BAD, which, doubled, makes the whole BAC known. Then, by Praxis 1 of Problem 6 of spherical triangles, from the arc BD (half the given arc BC) and its found opposite angle BAD — since the species of the other angle B is settled: for if the given arc AB, opposite the right angle D, is less than a quadrant, the angle B will be acute, just as BAD is acute; but if AB is greater than a quadrant, the angle B will be obtuse, since BAD is acute — find the angle B, to which the angle C is equal, on account of the equal arcs AB, AC.

[Margin: When the two given arcs are equal.] [Margin: By sines alone, when the two given arcs are equal.]

Diagram. An isosceles spherical triangle for the equal-sides case, with the perpendicular AD bisecting both the vertex angle and the base.

- A (apex, top): vertex of the equal sides
- B (lower left): base vertex
- D (base, middle): foot of the perpendicular AD
- C (lower right): base vertex

Nor indeed can the two equal arcs be quadrants. For otherwise the two angles above the base would be right, and thus the triangle would be right-angled — which is contrary to the hypothesis.

19. DATIS duobus arcibus trianguli non rectanguli cum angulo ab ipsis comprehenso, inuestigare reliquum arcum, cum reliquis duobus angulis.

19. Given two arcs of a non-right triangle together with the angle contained by them, to find the remaining arc, together with the remaining two angles.

[Margin: An arc is sought, together with the two adjacent angles.]

[Margin: (note a) When the two given arcs are unequal and neither of them a quadrant. From Proposition 64 of spherical triangles.]

Let two arcs AB, BC in triangle ABC be given, together with the angle B; and let them first be unequal, and neither of them a quadrant. From A, the endpoint of one of them, let the perpendicular arc AD be dropped to the other; whether it falls inside or outside the triangle, we shall learn from the operation itself. For, by Problem 2 of spherical triangles, from the given arc AB (subtending the right angle D) and the given angle B, find the arc AD opposite the angle B. Again, from the given arc AB (subtending the right angle D) and the found arc AD, find, by Problem 8 of spherical triangles, the third arc BD. If, then, this found arc BD is less than the given arc BC, the arc AD falls inside the triangle, but outside if it is greater. Now, subtracting the found arc BD from the given arc BC (if the former is less than the latter), or subtracting the given arc BC from the found arc BD (if the latter is greater than the former), the arc CD is left known. Finally, from the arcs AD, CD about the right angle D, find, by Problem 7 of spherical triangles, the third arc AC, which is sought.

Diagram. A pair of triangles for Problem 19, illustrating the perpendicular AD falling inside the triangle (left) and outside it (right).

- Left figure — A (apex), B (lower left), D (foot of perpendicular, on the base), C (lower right); AD falls within the triangle.
- Right figure — A (apex), B (lower left), C (base), D (lower right, beyond C); AD falls outside the triangle.

Next, by Problem 13 of spherical triangles, from the given arc AB (subtending the right angle D) and the found arc AD, find the angle BAD contained by the said arcs. And by the same reasoning, from the found arc AC (subtending the right angle D) and the found arc AD, find the

angle CAD, contained by the said arcs. For the angle CAD, added to the angle BAD (when the arc AD falls inside the triangle), or the angle CAD subtracted from the angle BAD (when the arc AD falls outside the triangle), will make up, or leave, the sought angle BAC.

Finally, by Problem 13 of spherical triangles, from the found arc AC (subtending the right angle D) and the found arc CD, obtain the angle C, which — when the arc AD falls inside the triangle — is the one sought; but when the arc AD falls outside the triangle, subtracted from two right angles it leaves the sought angle ACB.

By sines alone we shall do thus. By Problem 2 of spherical triangles, find, from the given arc AB (subtending the right angle D) and the given angle B, the opposite arc AD; and hence, by Praxis 1 of Problem 8 of spherical triangles, the arc BD. For this, subtracted from the given arc BC (if the former is less than the latter), or the given arc BC subtracted from the found arc BD (if the latter is less than the former), leaves the arc CD known. Then, by Praxis 1 of Problem 1 of spherical triangles, obtain both angles: from the given arc AB (subtending the right angle D) and the found arc AD, the opposite angle BAD; and from the found arc AC (subtending the right angle D) and the found arc CD, the opposite angle CAD. For from the two found angles BAD, CAD the sought angle BAC will be known — if one is added to the other, when the arc AD falls inside the triangle, or, when it falls outside, if CAD is subtracted from BAD. Finally, by Praxis 1 of Problem 1 of spherical triangles, seek, from the found arc AC (subtending the right angle D) and the found arc AD, the opposite angle C. For this, in the first triangle, is the one sought; but in the second, the remaining one of the two right angles, ACB, is the one sought.

But if perhaps the arc CD is found to be a quadrant (though BD will never be a quadrant, given that AB is not a quadrant), then the sought arc AC will also be a quadrant, and the angle CAD a right angle. And thus without difficulty the sought arc AC will have been found, together with the angle CAD, from which we shall find the sought angles BAC, ACB, as before.

[Margin: By sines alone, when the two given arcs are unequal and neither of them a quadrant.]

Now let one of the given unequal arcs be a quadrant, namely AB, from whose endpoint A the perpendicular arc AD is dropped to the other. Then the arc BD will also be a quadrant, and the angle BAD a right angle; and B will be the pole of the arc AD; and therefore the arc AD will become known from the given angle B. And thus in this case the two arcs BD, AD, together with the angle BAD, will have been

made known without any further labor: from which the rest will be found, as before.

[Margin: When one of the two given unequal arcs is a quadrant.]

ALITER, & facilius, per solos sinus, quando dati duo arcus inaequales sunt quomodocunque.

ALTERNATIVELY, and more easily, by sines alone, when the two given arcs are unequal in any way whatever.

Arrange it so that, as the whole sine is to the sine of either of the given unequal arcs, so is the sine of the other given arc to a fourth term; and a certain fourth number will be produced. Then again, arrange it so that, as the whole sine is to that found fourth number, so is the versed sine of the given angle to another term; and there will be found the difference between the versed sine of the third arc (which is sought) and the versed sine of the difference of the given unequal arcs. This difference, once found, if added to the versed sine of the difference of the given arcs, composes the versed sine of the sought third arc. Now, the three arcs of the proposed triangle being known, the other two angles will be found by the preceding problem — especially by that easier praxis; and if the two arcs containing either of them are equal, the last praxis of the same preceding problem is to be used.

[Margin: An easier and more general praxis, by sines alone, when the two given arcs are unequal.]

But now let the two given arcs AB, AC that contain the given angle A be equal, and hence let neither be a quadrant. The perpendicular arc AD will bisect both the given angle A and the base BC. Therefore, by Problem 2 of spherical triangles, from the given arc AB (subtending the right angle D) and from the angle BAD (half the given angle BAC), find the opposite arc BD, which, doubled, gives the whole sought arc BC. Then, by Problem 13 of spherical triangles, from the given arc AB (subtending the right angle D) and the found arc BD, seek the angle B contained by the said arcs; to which the angle C is equal, on account of the equal arcs AB, AC.

By sines alone the matter will be carried out thus. From the given arc AB (subtending the right angle D) and the angle BAD (half the given angle BAC), find, by Problem 2 of spherical triangles, the opposite arc BD, which, doubled, will give the whole sought arc BC. Then, by Praxis 1 of Problem 6 of spherical triangles, from the found arc BD and the angle BAD (half the given angle BAC) — since

besides this the species of the other angle B is settled: for if the arc AB is less than a quadrant, the angle B will be acute, just as BAD is acute; but if AB is greater than a quadrant, B will be obtuse, since BAD is acute — obtain the angle B; to which the angle C is equal.

[Margin: When the two given arcs are equal.] [Margin: By sines alone, when the two given arcs are equal.]

Diagram. An isosceles spherical triangle for the equal-sides case of Problem 19, with the perpendicular AD bisecting the vertex angle and the base.

- A (apex, top): vertex of the equal sides
- B (lower left): base vertex
- D (base, middle): foot of the perpendicular AD
- C (lower right): base vertex

20. DATIS duobus angulis trianguli non rectanguli, cum arcu ipsis adiacente, indagare reliquos arcus, cum angulo reliquo.

20. Given two angles of a non-right triangle, together with the arc adjacent to them, to find the remaining arcs, together with the remaining angle.

[Margin: Two arcs are sought, together with the angle contained by them. When the given angles are unequal and the adjacent arc is not a quadrant. From Proposition 65 of spherical triangles.]

Let two angles B, BAC in triangle ABC be given, together with the adjacent arc AB; and let the given angles first be unequal, and the arc AB not a quadrant. From one of the given angles, namely from A, let the perpendicular arc be dropped to the opposite arc BC; whether it falls inside or outside the triangle, the operation itself will show. For, by Problem 15 of spherical triangles, from the given arc AB (subtending the right angle D) and the given angle B, find the other non-right angle BAD: which, if it is less than the given angle BAC, means the arc AD falls inside the triangle; but outside, if greater. In the first case the found angle BAD, subtracted from the given angle BAC; in the second the given angle BAC subtracted from the found angle BAD — leaves the angle CAD known. Again, by Problem 2 of spherical triangles, from the given arc AB (subtending the right angle D) and the given angle B, find the opposite arc AD. Likewise, by Problem 12 of spherical triangles, from the found arc AD and the adjacent found angle CAD, obtain the arc AC opposite the right angle D; which indeed is one of the sought quantities.

Diagram. A pair of triangles for Problem 20, illustrating the perpendicular AD falling inside the triangle (left) and outside it (right).

- Left figure — A (apex), B (lower left), D (foot of perpendicular, on the base), C (lower right); AD falls within the triangle.
- Right figure — A (apex), B (lower left), C (base), D (lower right, beyond C); AD falls outside the triangle. Then, by Problem 8 of spherical triangles, from both the given arc AB (which subtends the right angle D) together with the found arc AD, let arc BD be traced out; and likewise, from the found arc AC (subtending the right angle D) together with the found arc AD, arc CD. When arc AD falls inside the triangle, this arc CD is added to the found arc BD; when arc AD falls outside the triangle, it is subtracted from that same arc BD. Either way it yields the sought arc BC as known.

Finally, by Problem 15 of spherical triangles, from the found arc AC (subtending the right angle D) together with the found angle CAD, investigate angle ACD. In the first triangle this is itself the angle sought; in the second, subtracted from two right angles, it leaves the remainder ACB, which is sought.

[Margin: By sines alone, when the given angles are unequal and the adjacent arc is not a quadrant.]

By sines alone the business will be settled as follows. By Problem 2 of spherical triangles, from the given arc AB (which subtends the right angle D) together with the given angle B, find the opposite arc AD. Then, by Praxis 1 of Problem 8 of spherical triangles, from the given arc AB (subtending the right angle D) together with the found arc AD, obtain the third arc BD. Likewise, by Praxis 1 of Problem 1 of spherical triangles, from the given arc AB (subtending the right angle D) together with the found arc BD, seek the opposite angle BAD. This angle, removed from the given angle BAC (if it is the smaller), or the given BAC subtracted from it (if this is the smaller of the two), leaves angle CAD known. Again, by Problem 5 of spherical triangles, from the found arc AD together with the adjacent angle CAD, derive angle ACD — which in the first triangle is the one sought, but in the second is the remainder of the two right angles, namely ACB, which is sought.

After this, by Praxis 1 of Problem 4 of spherical triangles, from both found angles CAD and ACD find arc CD. In the first triangle this arc, added to the already-found arc BD, or in the second subtracted from it, makes the sought arc BC known.

Finally, by Problem 7 of spherical triangles, from the found arcs AD and CD about the right angle D, find the third arc AC, opposite the right angle D, which is

sought. And so the two remaining arcs BC and AC will have been found, together with the remaining angle ACB.

But if it should happen that the found angle CAD is right — for BAD can never be right, given that AB is not a quadrant — then AC and CD will be quadrants, and AD the arc of angle C; and consequently angle C will become known from the found arc AD. The remaining arc BC will then be known from the found arc BD and the quadrant CD, as before.

[Margin: When the given arc is a quadrant.]

Now if the given arc AB is a quadrant, angles B and BAC still being given as unequal, then angle BAD will be right and arc BD likewise a quadrant. Also B will be the pole of arc AD; and therefore arc AD itself will be known from the given angle B. Once arcs AD and BD and the right angle BAD have been found with such ease, we shall recover the rest as before.

Diagram. A spherical triangle for the equal-angles case that follows. The apex A is at the top; the base runs B–D–C along the bottom, with D the foot of the perpendicular AD dropped from A.

- A (apex): vertex of the two equal base angles
- B (lower left): base endpoint
- D (base, middle): foot of the perpendicular from A
- C (lower right): base endpoint

[Margin: When the two given angles are equal.]

Next, suppose that in triangle ABC the two angles B and C are given as equal, together with the adjacent arc BC, whether or not it is a quadrant. Then arcs AB and AC will be equal, and the perpendicular arc AD, dropped from the third angle A onto BC, will bisect both the arc BC and the angle A. Therefore, by Problem 12 of spherical triangles, from arc BD (the half of the given arc BC) together with the given adjacent angle B, find arc AB opposite the right angle D; and since AC is equal to it, the two remaining arcs will have been found. Again, by Problem 5 of spherical triangles, from the same arc BD (the half of the given arc BC) together with the given adjacent angle B, find the other non-right angle BAD. For this, when doubled, gives the whole sought angle BAC.

[Margin: By sines alone, when the two given angles are equal.]

By sines alone we shall proceed thus. By Problem 5 of spherical triangles, from arc BD — the half of the given arc BC — together with the given adjacent angle B, find

angle BAD, which doubled gives the whole BAC, sought. Then, by Praxis 1 of Problem 3 of spherical triangles, from arc BD (the half of the given arc BC) together with the found angle BAD (its opposite), find — since besides this the species of the other angle B, which is given, is established — arc AB opposite the right angle D, to which AC is equal.

21. Given two angles of a non-right triangle, together with the arc opposite one of them, to investigate the remaining arcs together with the remaining angle: provided only it be established whether the arc opposite the other given angle is greater than a quadrant, or less, or exactly a quadrant.

[Margin: Two arcs are sought, together with one angle. When the two given angles are unequal and the given arc is not a quadrant. ^a By Proposition 66 of spherical triangles.]

^a Suppose that in triangle ABC the two angles B and C are given, together with arc AB, which is opposite angle C. Let these angles first be unequal, with AB not a quadrant, and let there be established besides the species of arc AC, opposite the other given angle B. From the third angle A drop the perpendicular arc AD to arc BC. It will fall inside the triangle if both given angles B and C are acute, or both obtuse; but outside if one is acute and the other obtuse. (It is settled whether both angles are acute or obtuse, or not, because angle B is given together with the species of angle C.)

Now, by Problem 2 of spherical triangles, from the given arc AB (which subtends the right angle D) together with the given angle B, find the opposite arc AD. Also, by Problem 14 of spherical triangles, from the same given arc AB (subtending the right angle D) together with the given angle B, obtain arc BD. Again, by Problem 15 of spherical triangles, from the same given arc AB (subtending the right angle D) together with the given angle B, find angle BAD. Furthermore, by Problem 3 of spherical triangles, from the found arc AD together with the given opposite angle C, find arc AC as well. (For, when arc AD falls outside the triangle, angle ACD, opposite arc AD, is left known after subtracting the given angle ACB from two right angles.) Arc AC, opposite the right angle D — since its species is supposed to be established — is thus found; and so arc AC, one of the sought arcs, will have been found.

Then, by Problem 14 of spherical triangles, from the found arc AC (subtending the right angle D) together with the given angle C, find arc CD. This arc, added to the arc BD found above, or subtracted from it — according as arc AD falls inside or outside the triangle — makes arc BC known, which is the other of the sought arcs.

Lastly, by Problem 15 of spherical triangles, from the same found arc AC (subtending the right angle D) together with the given angle C, find angle CAD. This, added to angle BAD if arc AD falls inside the triangle, or, if outside, subtracted from it, will make known angle BAC, sought.

Diagram. Two spherical triangles illustrating the two positions of the perpendicular in Problem 21. In the upper figure the perpendicular AD falls inside the triangle (apex A, base B–D–C). In the lower figure it falls outside, so that D lies beyond the base (apex A at the right, base at B–C–D).

- A (apex, upper figure): vertex from which the perpendicular is dropped
- B (lower left, upper figure): base endpoint
- D (base, upper figure): foot of the perpendicular (inside)
- C (right, upper figure): base endpoint
- A (upper right, lower figure): apex
- B (lower left, lower figure): base endpoint
- C (base, lower figure): base endpoint
- D (base right, lower figure): foot of the perpendicular (outside)

We shall use the sines of the same quantities thus. By Problem 2 of spherical triangles, from the given arc AB (which subtends the right angle D) together with the given angle B, seek the opposite arc AD. From this, by Praxis 1 of Problem 8 of spherical triangles, from the given arc AB (subtending the right angle D) together with the found arc AD, find the third arc BD. And again, by Praxis 1 of Problem 1 of spherical triangles, from the given arc AB (subtending the right angle D) together with the found arc BD, derive the opposite angle BAD. After this, by Praxis 1 of Problem 3 of spherical triangles, from the found arc AD together with the given opposite angle C, elicit arc AC, opposite the right angle D, since its species is established by hypothesis: this arc AC is one of the sought quantities.

[Margin: By sines alone, when the two given angles are unequal and the given arc is not a quadrant.]

Then, by Praxis 1 of Problem 8 of spherical triangles, from the found arc AC (subtending the right angle D) together with arc AD, find the third arc CD. From this — if in the first triangle it is added to the found arc BD, or in the second subtracted from it — the other sought arc BC becomes known.

Finally, by Praxis 1 of Problem 1 of spherical triangles, from arc AC (subtending the right angle D) together with the found arc CD, seek the opposite angle CAD.

For this, added in the first triangle to the found angle BAD, or in the second subtracted from it, makes angle BAC known, sought.

[Margin: When the given arc is a quadrant and the two given angles are unequal.]

But if the arc AC, opposite the other given angle B, should be a quadrant — which can happen when AB is not a quadrant (in which case AD or BD can never be a quadrant) — then CD too will be a quadrant, and angle CAD right. So one need not then labour over investigating arcs AC and CD and angle CAD; but from these arc BC and angle BAC are to be found, as we said.

[Margin: When the given arc is a quadrant and the two given angles are equal.]

But now suppose the given arc AB is a quadrant, the two angles B and C still being given as unequal. That granted, BD too will be a quadrant, and angle BAD right; also B will be the pole of arc AD; and consequently arc AD, since it is its arc, will become known from the given angle B. Once arcs BD and AD and the right angle BAD have been found with such ease, the rest will be found, as before.

[Margin: When the two given angles are equal.]

Finally, suppose the two angles B and C are given equal. Then arc AD will bisect both the base BC and angle A, and arcs AB and AC will be equal. By Problem 14 of spherical triangles, from the given arc AB (which subtends the right angle D) together with the given angle B, find arc BD, which doubled offers the whole sought BC. The other sought quantity, AC, will be given, since it is equal to the given AB. Again, by Problem 15 of spherical triangles, from the same given arc AB together with angle B, obtain angle BAD; and this doubled gives the whole BAC, which is sought.

Diagram. An isosceles spherical triangle for the equal-angles case. Apex A at the top; base B–D–C along the bottom, with the perpendicular AD dropped from A onto the midpoint D of the base.

- A (apex): vertex of the two equal angles
- B (lower left): base endpoint
- D (base, middle): foot of the bisecting perpendicular
- C (lower right): base endpoint

[Margin: By sines alone, when the two given angles are equal.]

By sines alone we shall settle the problem thus. By Problem 2 of spherical triangles, investigate arc AD from the given arc AB (which subtends the right angle D)

together with the given angle B, opposite arc AD. From this, by Praxis 1 of Problem 8 of spherical triangles, find, from the given arc AB (subtending the right angle D) together with the found arc AD, the third arc BD, which doubled gives the whole sought BC. Then, by Praxis 1 of Problem 1 of spherical triangles, from the given arc AB (subtending the right angle D) together with the found arc BD, trace out the opposite angle BAD, which doubled offers the whole BAC, sought.

In this case, moreover, the given arc AB cannot be a quadrant.

22. Given two arcs of a non-right triangle, together with the angle opposite one of them, to search out the remaining angles together with the remaining arc: provided only it be established whether the angle opposite the other given arc is acute or obtuse.

[Margin: Two angles are sought, together with one arc. When the two given arcs are unequal and neither of them is a quadrant. ^a By Proposition 67 of spherical triangles.]

^a Suppose that in triangle ABC the two arcs AB and AC are given, together with angle B, which is opposite arc AC. Let these arcs first be unequal, and neither a quadrant, and let there be established besides the species of angle C, opposite the other given arc AB. From angle A, comprehended by the given arcs, drop the perpendicular arc to arc BC. It will fall inside the triangle if both angles B and C are acute, or both obtuse; but outside if one is acute and the other obtuse. (It is settled whether both angles are acute or obtuse, or not, because angle B is given together with the species of angle C.)

Therefore, by Problem 2 of spherical triangles, from the given arc AB (which subtends the right angle D) together with the given angle B, find the opposite arc AD. From this, by Problem 8 of spherical triangles, from the same given arc AB together with the found arc AD, elicit the third arc BD. Then again, by Problem 1 of spherical triangles, from the given arc AB (subtending the right angle D) together with the found arc BD, find angle BAD, opposite arc BD. And by Problem 13 of spherical triangles, from the given arc AC (subtending the right angle D) together with the found arc AD, derive angle CAD, comprehended by the said arcs. For this angle, added to the found angle BAD, or subtracted from it — according as arc AD falls inside or outside the triangle — gives the sought angle BAC. Find besides, by Problem 15 of spherical triangles, from the given arc AC (subtending the right angle D) together with the found angle CAD, angle ACD: this will be the one we seek if arc AD falls inside the triangle; but if outside, removed

from two right angles it gives angle ACB, sought. And so the two remaining angles BAC and ACB will have been found.

Then, by Problem 8 of spherical triangles, from the given arc AC (subtending the right angle D) together with the found arc AD, seek arc CD. For this, added to the found arc BD, or subtracted from it (according as arc AD falls inside or outside the triangle), yields the sought arc BC as known.

In this case, moreover, none of the arcs AD, BD, CD can be a quadrant.

Diagram. Two spherical triangles illustrating the two positions of the perpendicular in Problem 22. In the left figure the perpendicular falls inside the triangle (apex A, base B–D–C). In the right figure it falls outside, so that D lies beyond the base (apex A, base B–C–D).

- A (apex, left figure): vertex from which the perpendicular is dropped
- B (lower left, left figure): base endpoint
- D (base, left figure): foot of the perpendicular (inside)
- C (right, left figure): base endpoint
- A (apex, right figure): apex
- B (lower left, right figure): base endpoint
- C (base, right figure): base endpoint
- D (base right, right figure): foot of the perpendicular (outside)

[Margin: By sines alone, when the two given arcs are unequal and neither of them is a quadrant.]

By sines alone the work is to be done thus. By Problem 2 of spherical triangles, from the given arc AB (which subtends the right angle D) together with the given angle B, find the opposite arc AD. From this, by Praxis 1 of Problem 8 of spherical triangles, from the given arc AB (subtending the given right angle D) together with the found arc AD, obtain the third arc BD. And from this again, by Praxis 1 of Problem 1 of spherical triangles, from arc AB (subtending the right angle D) together with the found arc BD, find the opposite angle BAD.

Also, by Praxis 1 of Problem 8 of spherical triangles, from the given arc AC (subtending the right angle D) together with the found arc AD, derive the third arc CD; which, either added to the found arc BD, or subtracted from it (according as arc AD falls inside or outside the triangle), gives the sought arc BC.

Then investigate, by Praxis 1 of Problem 1 of spherical triangles, from the given arc AC (subtending the right angle D) together with the found arc CD, the opposite

angle CAD: which, joined to angle BAD, or taken away from it (according as arc AD falls inside or outside the triangle), will exhibit the sought angle BAC. Finally, by Problem 5 of spherical triangles, from the found arc AD together with the found adjacent angle CAD, find the other angle ACD: this will be the other of the sought quantities if arc AD falls inside the triangle; but if it falls outside, it must be subtracted from two right angles, so that the remainder becomes the other sought angle ACB.

[Margin: When one of the given arcs is a quadrant.]

But if one of the given arcs is a quadrant: if indeed AB is a quadrant, then BD too will be a quadrant, and angle BAD right, and also B the pole of arc AD; consequently arc AD will be known from the given angle B. And so, arcs AD and BD and the right angle BAD being known, we shall find the rest, as before. By the same reasoning, if AC is a quadrant, then CD too will be a quadrant, and angle CAD right, and also C the pole of arc AD; and thus the found arc AD will make its own angle ACD known — which will be one of the sought quantities if arc AD falls inside the triangle; but if it falls outside, the same, subtracted from two right angles, leaves the sought angle ACB. Once angles CAD and ACD and arc CD have been found with such ease, the rest will be found, as before.

Diagram. An isosceles spherical triangle for the equal-arcs case. Apex A at the top; base B–D–C along the bottom, with the perpendicular AD dropped from A onto the midpoint D of the base.

- A (apex): vertex of the two equal angles
- B (lower left): base endpoint
- D (base, middle): foot of the bisecting perpendicular
- C (lower right): base endpoint

[Margin: When the two given arcs are equal.]

Now suppose the two arcs AB and AC are given equal. Then arc AD will bisect both the base BC and angle A, and angles B and C will be equal; and so only angle BAC, together with arc BC, is to be sought. Therefore, by Problem 15 of spherical triangles, from the given arc AB (which subtends the right angle D) together with the given angle B, find angle BAD; which doubled offers the whole sought BAC. Again, by Problem 2 of spherical triangles, from arc AB (subtending the right angle D) together with the found angle BAD, find the opposite arc BD: which doubled gives the whole sought BC.

[Margin: By sines alone, when the two given arcs are equal.]

By sines alone, thus. By Problem 2 of spherical triangles, find, from the given arc AB (which subtends the right angle D) together with the given angle B, the opposite arc AD. From this, by Praxis 1 of Problem 8 of spherical triangles, from the given arc AB (subtending the right angle D) together with the found arc AD, derive the third arc BD, which doubled will exhibit the whole sought BC. Then, by Problem 5 of spherical triangles, investigate, from the found arc BD together with the given adjacent angle B, angle BAD, opposite arc BD. For this doubled gives the whole BAC, which we desire.

Moreover, in order that we may more easily find that problem which we most wish for — especially among spherical triangles — we have here compiled an index of all the problems needed for calculation. To these we have prefixed numbers, which will indicate what place each one holds among the problems whose praxes we have just now set out; just as, above, we also wrote in the margin, alongside the problems themselves, the propositions and problems in which the praxes are demonstrated in our treatises on rectilinear and spherical triangles. And although in the index only single things to be found are set forth for the individual problems of right-angled spherical triangles, nevertheless, once these are found, most other things too are found within the same, as is clear from what precedes.

INDEX PROBLEMATVM, ET PRAXIVM TRIANGVLORVM

Index of the Problems and Exercises on Triangles

Structure of this index: each block opens with a heading naming the class of triangle, then a caption line ("Inuenitur / Inueniuntur ...") stating what is to be found; the numbered entries below give the data from which it is found. The left-hand number is the proposition (problema) number where the case is solved — not a folio reference. These proposition numbers are read individually from the page; they are non-sequential in the spherical sections (see QA note).

IN TRIANGVLIS RECTILINEIS RECTANGVLIS

In right-angled rectilinear triangles

Inuenitur. — *The following is found:*

Prop.	Given (Latin, verbatim)	Given (English)
2.	Latus circa angulum rectum vtrilibet angulorum acutorum oppositum; ex latere rectum angulum subtendente, & alterutro acutorum angulorum.	A side about the right angle, opposite either of the acute angles; from the side subtending the right angle (the hypotenuse) and one of the acute angles.
3.	Latus angulo recto oppositum, & alterutrum duorum circa eundem rectum angulum; ex altero latere circa angulum rectum, & vno acutorum angulorum.	The side opposite the right angle, and one of the two sides about the same right angle; from the other side about the right angle and one of the acute angles.
4.	Vterque angulus acutus, & alterutrum duorum laterum circa angulum rectum; ex latere angulum rectum subtendente, & altero latere circa eundem rectum angulum.	Both acute angles, and one of the two sides about the right angle; from the side subtending the right angle and the other side about the same right angle.
5.	Vterque angulus acutus, & latus recto angulo oppositum; ex duobus lateribus circa eundem angulum rectum.	Both acute angles, and the side opposite the right angle; from the two sides about the same right angle.

IN TRIANGVLIS RECTILINEIS NON RECTANGVLIS

In non-right-angled rectilinear triangles

Inueniuntur. — *The following are found:*

Prop.	Given (Latin, verbatim)	Given (English)
10.	Duo latera; ex omnibus angulis, & reliquo latere.	Two sides; from all the angles and the remaining side.
11.	Omnes anguli: ex omnibus lateribus.	All the angles; from all the sides.
12.	Vnum latus, & duo anguli illi adiacentes; ex reliquis duobus lateribus, & reliquo angulo ab ipsis comprehenso.	One side, and the two angles adjacent to it; from the remaining two sides and the remaining angle contained by them.
13.	Duo anguli, & vnum latus vni	Two angles, and one side opposite

Prop.	Given (Latin, verbatim)	Given (English)
	eorum oppositum; ex reliquis duobus lateribus, & reliquo angulo, qui vni eorum opponitur.	one of them; from the remaining two sides and the remaining angle, which is opposite one of them.

IN TRIANGVLIS SPHÆRICIS RECTANGVLIS

In right-angled spherical triangles

Inuenitur arcus angulo recto oppositus. — *The arc opposite the right angle is found:*

Prop.	Given (Latin, verbatim)	Given (English)
3.	Ex arcu circa rectum angulum, & angulo ei opposito.	From an arc about the right angle and the angle opposite to it.
12.	Ex arcu circa angulum rectum, & angulo ei adiacente.	From an arc about the right angle and the angle adjacent to it.
7.	Ex vtroque arcu circa angulum rectum.	From both arcs about the right angle.
16.	Ex vtroque angulo non recto.	From both non-right angles.

Inuenitur arcus circa angulum rectum. — *An arc about the right angle is found:*

Prop.	Given (Latin, verbatim)	Given (English)
2.	Ex arcu rectum angulum subtendente, & angulo, qui quæsito arcui opponitur.	From the arc subtending the right angle and the angle opposite to the sought arc.
14.	Ex arcu rectum angulum subtendente, & angulo, qui quæsito arcui adiacet.	From the arc subtending the right angle and the angle adjacent to the sought arc.
8.	Ex arcu angulum rectum subtendente, & altero arcu circa angulum rectum.	From the arc subtending the right angle and the other arc about the right angle.
10.	Ex altero arcu circa rectum angulum, & angulo ei opposito.	From the other arc about the right angle and the angle opposite to it.
9.	Ex altero arcu circa angulum rectum, & angulo ei adiacente.	From the other arc about the right angle and the angle adjacent to it.
4.	Ex vtroque angulo non recto.	From both non-right angles.

Inuenitur angulus non rectus. — *A non-right angle is found:*

Prop.	Given (Latin, verbatim)	Given (English)
1.	Ex arcu rectum angulum subtendente, & arcu circa angulum rectum, qui quæsito angulo opponitur.	From the arc subtending the right angle and the arc about the right angle which is opposite to the sought angle.
13.	Ex arcu angulum rectum subtendente, & arcu circa angulum rectum, qui quæsito angulo adiacet.	From the arc subtending the right angle and the arc about the right angle which is adjacent to the sought angle.
15.	Ex arcu rectum angulum subtendente, & altero angulo non recto.	From the arc subtending the right angle and the other non-right angle.
11.	Ex vtroque arcu circa angulum rectum.	From both arcs about the right angle.
5.	Ex arcu circa rectum angulum, qui angulo quæsito opponitur, & altero angulo non recto illi arcui adiacente.	From the arc about the right angle which is opposite to the sought angle, and the other non-right angle adjacent to that arc.
6.	Ex arcu circa angulum rectum, qui angulo quæsito adiacet, & altero angulo non recto illi arcui opposito.	From the arc about the right angle which is adjacent to the sought angle, and the other non-right angle opposite to that arc.

IN TRIANGVLIS SPHÆRICIS NON RECTANGVLIS

In non-right-angled spherical triangles

Inueniuntur. — *The following are found:*

Prop.	Given (Latin, verbatim)	Given (English)
17.	Omnes tres arcus; ex omnibus tribus angulis.	All three arcs; from all three angles.
18.	Omnes tres anguli; ex omnibus tribus arcubus.	All three angles; from all three arcs.
19.	Vnus arcus, & duo anguli illi adiacentes; ex aliis duobus arcubus, & reliquo angulo ab ipsis	One arc, and the two angles adjacent to it; from the other two arcs and the remaining angle

Prop.	Given (Latin, verbatim)	Given (English)
	comprehenso.	contained by them.
20.	Duo arcus, & angulus ab ipsis comprehensus; ex reliquo arcu, & aliis duobus angulis huic arcui adiacentibus.	Two arcs, and the angle contained by them; from the remaining arc and the other two angles adjacent to this arc.
21.	Duo arcus, & vnus angulus vni eorum oppositus; ex reliquo arcu & aliis duobus angulis, quorum vni hic arcus opponitur: si modo constet species arcus alteri angulo dato oppositi.	Two arcs, and one angle opposite one of them; from the remaining arc and the other two angles, to one of which this arc is opposite — provided the species [species, i.e. the quadrant-classification] of the arc opposite the other given angle is established.
22.	Duo anguli, & vnus arcus vni eorum oppositus; ex reliquo angulo, & aliis duobus arcubus, quorum vni hic angulus opponitur: si modo constet species anguli alteri arcui dato oppositi.	Two angles, and one arc opposite one of them; from the remaining angle and the other two arcs, to one of which this angle is opposite — provided the species of the angle opposite the other given arc is established.

*But we have set out this same index far more clearly in Lemma 53 of Book 1 of the Astrolabe**, where we also resolve every problem of spherical triangles by six methods.*

And with this let our treatise on triangles reach its end. In it I believe I have covered everything required for their calculation. So enjoy these in the meantime, kind reader, while — with God favoring our undertakings — we prepare to bring to light three complete books of Menelaus [Menelaus of Alexandria] on spherical triangles, together with two by Francesco Maurolico [Francesco Maurolico]. Those works contain far more than what we have explained here, and are indeed most delightful to know, set out with clearer demonstrations.

FINIS TRIANGVLORVM SPHÆRICORVM

The end of the spherical triangles

[Library stamp: "Bayerische Staatsbibliothek München" (blue oval), lower centre; a second, blurred oval stamp appears in the same region on the preceding page]

Opera Mathematica, Tomus I — end of volume (rear endleaf and back cover)

This closing opening carries **no printed text**: no colophon or imprint, no printer's device, no privilege statement, and no errata or index. It consists of the rear free endleaf (blank paper bearing only modern pencil library notations and the offset bleed-through of the binding) and the inside of the blind-tooled leather back cover.

Page 895 — rear free endleaf (blank)

The leaf is **blank** printed matter. It carries no letterpress. At the upper left are a few lines of modern pencil annotation in a librarian's/cataloguer's hand (collation or shelf notes), together with the ghosted offset of the tooled binding panel showing through from the facing board.

The pencil notations read approximately:

- 1-5 followed by tally strokes: xxxx xxxx xxxx xxx [?]
- 1 followed by xxx xxx [?]
- $\overline{\text{VIII}}$. 84 [?] (Roman numeral with overbar; possibly III . 84)
- $\overline{\text{XI}}$. 88 [?]

A faint reddish ink speck and scattered foxing are present; the grey shapes across the sheet are show-through/offset of the decorative binding, not printed content.

Page 896 — inside back cover (blind-tooled leather binding)

This is the physical end of Tomus I.